Baltic Way

92 - Mathematical Team Contest - Vilnius, November 5-8, 1992

1. Let p, q be two consecutive odd prime numbers. Prove that p+q is a product of at least 3 natural numbers > 1 (not necessarily different).

2. Denote by d(n) the number of all positive divisors of a natural number n (including 1 and n). Prove that there are infinitely many n, such that n=d(n) is an integer.

3. Find an infinite non-constant arithmetic progression of natural numbers such that each term is neither a sum of two squares, nor a sum of two cubes (of natural numbers).

4. Is it possible to draw a hexagon with vertices in the knots of an integer lattice so that the squares of the lengths of the sides are six consecutive positive integers ?

5. It is given that $a^2 + b^2 + (a+b)^2 = c^2 + d^2 + (c+d)^2$. Prove that $a^4 + b^4 + (a+b)^4 = c^4 + d^4 + (c+d)^4$.

6. Prove that the product of the 99 numbers $\frac{k^3 - 1}{k^3 + 1}$, k = 2; 3; …; 100 is greater than 2/3.

7. Let $a = \sqrt[1992]{1992}$. Which number is greater $\left.a^{a^{a^{\cdot^{\cdot^{a}}}}}\right\}1992$ or 1992 ?

8. Find all integers satisfying the equation $2^x (4 - x) = 2x + 4$.

9. A polynomial $f(x) = x^3 + ax^2 + bx + c$ is such that $b < 0$ and $ab = 9c$. Prove that the polynomial f has three different real roots.

10. Find all fourth degree polynomials p(x) such that the following four conditions are satisfied:

(i) $p(x) = p(-x)$ for all x,
(ii) $p(x) \geq 0$ for all x,
(iii) $p(0) = 1$,
(iv) p(x) has exactly two local minimum points x_1 and x_2 such that $|x_1 - x_2| = 2$.

11. Let Q+ denote the set of positive rational numbers. Show that there exists one and only one function f: Q+ → Q+ satisfying the following conditions:

(i) If $0 < q < 1=2$ then $f(q) = 1 + f(q/(1 - 2q))$,
(ii) If $1 < q \leq 2$ then $f(q) = 1 + f(q - 1)$,
(iii) $f(q) f(1=q) = 1$ for all $q \in Q+$.

12. Let N denote the set of natural numbers. Let ϕ: N → N be a bijective function and assume that there exists a finite limit $\lim_{n\to\infty} \frac{\phi(n)}{n} = L$.
What are the possible values of L?

13. Prove that for any positive $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ the inequality

$$\sum_{i=1}^{n} \frac{1}{x_i y_i} \geq \frac{4n^2}{\sum_{i=1}^{n} (x_i + y_i)^2}$$

holds.

14. There is a finite number of towns in a country. They are connected by one direction roads. It is known that, for any two towns, one of them can be reached from another one. Prove that there is a town such that all remaining towns can be reached from it.

15. Noah has 8 species of animals to fit into 4 cages of the ark. He plans to put species in each cage. It turns out that, for each species, there are at most 3 other species with which it cannot share the accomodation. Prove that there is a way to assign the animals to their cages so that each species shares with compatible species.

16. All faces of a convex polyhedron are parallelograms. Can the polyhedron have exactly 1992 faces?

17. Quadrangle ABCD is inscribed in a circle with radius 1 in such a way that the diagonal AC is a diameter of the circle, while the other diagonal BD is as long as AB. The diagonals intesect at P. It is known that the length of PC is 2/5. How long is the side CD?

18. Show that in a non-obtuse triangle the perimenter of the triangle is always greater than two times the diameter of the circumcircle.

19. Let C be a circle in plane. Let C_1 and C_2 be nonintersecting circles touching C internally at points A and B respectively. Let t be a common tangent of C_1 and C_2 touching them at points D and E respectively, such that both C_1 and C_2 are on the same side of t. Let F be the point of intersection of AD and BE. Show that F lies on C.

20. Let $a \leq b \leq c$ be the sides of a right triangle, and let 2p be its perimeter. Show that

$$p(p - c) = (p - a)(p - b) = S,$$

where S is the area of the triangle.

93 - Mathematical Team Contest - Riga, November 11-15, 1993

1. $a_1a_2a_3$ and $a_3a_2a_1$ are two three-digit decimal numbers, with a_1 and a_3 different nonzero digits. Squares of these numbers are five-digit numbers $b_1b_2b_3b_4b_5$ and $b_5b_4b_3b_2b_1$ respectively. Find all such three-digit numbers.

2. Do there exist positive integers $a > b > 1$ such that for each positive integer k there exists a positive integer n for which $an + b$ is a k-th power of a positive integer?

3. Let's call a positive integer interesting if it is a product of two (distinct or equal) prime numbers.
What is the greatest number of consecutive positive integers all of which are interesting?

4. Determine all integers n for which $\sqrt{\frac{25}{2} + \sqrt{\frac{625}{4} - n}} + \sqrt{\frac{25}{2} - \sqrt{\frac{625}{4} - n}}$ is an integer.

5. Prove that for any odd positive integer n, $n^{12} - n^8 - n^4 + 1$ is divisible by 2^9.

6. Suppose two functions f(x) and g(x) are defined for all x with $2 < x < 4$ and satisfy
$2 < f(x) < 4$; $2 < g(x) < 4$; $f(g(x)) = g(f(x)) = x$; $f(x) * g(x) = x^2$ for all $2 < x < 4$.
Prove that $f(3) = g(3)$.

7. Solve the system of equations in integers

$$\begin{aligned} z^x &= y^{2x} \\ 2^z &= 4^x \\ x + y + z &= 20 \end{aligned}$$

8. Compute the sum of all positive integers whose digits form either a strictly increasing or strictly decreasing sequence.

9. Solve the system of equations

$$\begin{aligned} x^5 &= y + y^5 \\ y^5 &= z + z^5 \\ z^5 &= t + t^5 \\ t^5 &= x + x^5 \end{aligned}$$

10. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two finite sequences consisting of 2n real different numbers. Rearranging each of the sequences in the increasing order we obtain $a'_1, a'_2, \ldots, a'_n$ and $b'_1, b'_2, \ldots, b'_n$. Prove that

$$\max_{1 \le i \le n} | a_i - b_i | \ge \max_{1 \le i \le n} | a'_i - b'_i | .$$

11. An equilateral triangle is divided into n² congruent equilateral triangles. A spider stands at one of the vertices, a y at another. Alternately each of them moves to a neighbouring vertex. Prove that the spider can always catch the y.

12. There are 13 cities in a certain kingdom. Between some pairs of the cities a two-way direct bus, train or plane connections are established. What is the least possible number of connections to be established in order that choosing any two means of transportation one can go from any city to any other without using the third kind of vehicle?.

13. An equilateral triangle ABC is divided into 100 congruent equilateral triangles. What is the greatest number of vertices of small triangles that can be chosen so that no two of them lie on a line that is parallel to any of the sides of the triangle ABC.

14. A square is divided into 16 equal squares, obtaining the set of 25 different vertices. What is the least number of vertices one must remove from this set, so that no 4 points of the remaining set are the vertices of any square with sides parallel to the sides of the initial square?

15. On each face of two dice some positive integer is written. The two dice are thrown and the numbers on the top face are added. Determine whether one can select the integers on the faces so that the possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, all equally likely?

16. Two circles, both with the same radius r, are placed in the plane without intersecting each other. A line in the plane intersects the first circle at the points A, B and the other at points C, D, so that | AB | = | BC | = | CD | = 14 cm. Another line intersects the circles at E, F, respectively G, H so that | EF | = | FG | = | GH | = 6 cm. Find the radius r.

17. Let's consider three pairwise non-parallel straight constant lines in the plane. Three points are moving along these lines with different nonzero velocities, one on each line (we consider the movement to have taken place for infinite time and continue infinitely in the future). Is it possible to determine these straight lines, the velocities of each moving point and their positions at some “zero" moment in such a way that the points never were, are or will be collinear?

18. In the triangle ABC, | AB | = 15, | BC | = 12, | AC | = 13. Let the median AM and bisector BK intersect at point O, where $M \in BC$, $K \in AC$. Let $OL \perp AB$, $L \in AB$. Prove that $\angle OLK = \angle OLM$.

19. A convex quadrangle ABCD is inscribed in a circle with center O. The angles AOB, BOC, COD and DOA, taken in some order, are of the same size as the angles of the quadrangle ABCD.
Prove that ABCD is a square.

20. Let Q be a unit cube. We say that a tetrahedron is good if all its edges are equal and all of its vertices lie on the boundary of Q. Find all possible volumes of good tetrahedra.

94 - Mathematical Team Contest - Tartu, November 11, 1994

1. Let $a \bullet b = a + b - ab$. Find all triples (x; y; z) of integers such that

$$(x \bullet y) \bullet z + (y \bullet z) \bullet x + (z \bullet x) \bullet y = 0.$$

2. Let $a_1, a_2, \ldots, a_9$ be any non-negative numbers such that $a_1 = a_9 = 0$ and at least one of the numbers is non-zero. Prove that for some i, $2 \le i \le 8$, the inequality $a_{i-1} + a_{i+1} < 2a_i$ holds. Will the statement remain true if we change the number 2 in the last inequality to 1.9 ?

3. Find the largest value of the expression

$$xy + x\sqrt{1-y^2} + y\sqrt{1-x^2} - \sqrt{(1-x^2)(1-y^2)}$$

4. Is there an integer n such that $\sqrt{n-1} + \sqrt{n+1}$ is a rational number ?

5. Let p(x) be a polynomial with integer coe_cients such that both equations p(x) = 1 and p(x) = 3 have integer solutions. Can the equation p(x) = 2 have two di_erent integer solutions ?

6. Prove that any irreducible fraction p/q, where p and q are positive integers and q is odd, is equal to a fraction $n/(2^k-1)$ for some positive integers n and k.

7. Let p > 2 be a prime number and

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{(p-1)^3} = \frac{m}{n}$$

where m and n are relatively prime. Show that m is a multiple of p.

8. Show that for any integer $a \geq 5$ there exist integers b and c, $c \geq b \geq a$, such that a, b, c are the lengths of the sides of a right-angled triangle.

9. Find all pairs of positive integers (a, b) such that $2^a + 3^b$ is the square of an integer.

10. How many positive integers satisfy the following three conditions:
a) All digits of the number are from the set {1; 2; 3; 4; 5 };
b) The absolute value of the difference between any two consecutive digits is 1;
c) The integer has 1994 digits ?

11. Let NS and EW be two perpendicular diameters of a circle C . A line l touches C at point S. Let A and B be two points on C , symmetric with respect to the diameter EW. Denote the intersection points of l with the lines NA and NB by A' and B', respectively. Show that

$$|SA'| * |SB'| = |SN|^2$$

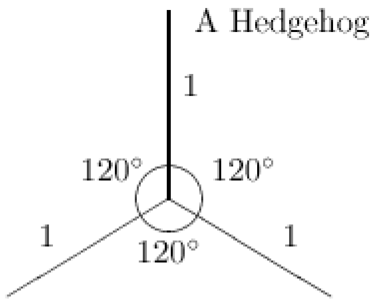
12. The inscribed circle of the triangle $A_1A_2A_3$ touches the sides A_2A_3, A_3A_1, A_1A_2 at points S_1, S_2, S_3, respectively. Let O_1, O_2, O_3 be the centres of the inscribed circles of triangles $A_1S_2S_3$, $A_2S_3S_1$, $A_3S_1S_2$, respectively. Prove that the straight lines O_1S_1, O_2S_2, O_3S_3 intersect at one point.

13. Find the smallest number a such that a square of side a can contain five disks of radius 1, so that no two of the disks have a common interior point.

14. Let α, β, γ be the angles of a triangle opposite to its sides with lengths a, b, c respectively. Prove the inequality

$$a\left(\frac{1}{\beta}+\frac{1}{\gamma}\right)+b\left(\frac{1}{\alpha}+\frac{1}{\gamma}\right)+c\left(\frac{1}{\beta}+\frac{1}{\alpha}\right) \geq 2\cdot\left(\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}\right)$$

15. Does there exist a triangle such that the lengths of all its sides and altitudes are integers and its perimeter is equal to 1995 ?



16. The Wonder Island is inhabited by Hedgehogs. Each Hedgehog consists of three segments of unit length having a common endpoint, with all three angles between them equal to 120° (see Figure). Given that all Hedgehogs are lying at on the island and no two of them touch each other, prove that there is a finite number of Hedgehogs on Wonder Island.

17. In a certain kingdom, the king has decided to build 25 new towns on 13 uninhabited islands so that on each island there will be at least one town. Direct ferry connections will be established between any pair of new towns which are on different islands. Determine the least possible number of these connections.

18. There are n lines ($n > 2$) given in the plane. No two of the lines are parallel and no three of them intersect at one point. Every point of intersection of these lines is labelled with a natural number between 1 and n - 1. Prove that, if and only if n is even, it is possible to assign the labels in such a way that every line has all the numbers from 1 to n - 1 at its points of intersection with the other n - 1 lines.

19. The Wonder Island Intelligence Service has 16 spies in Tartu. Each of them watches on some of his colleagues. It is known that if spy A watches on spy B, then B does not watch on

A. Moreover, any 10 spies can numbered in such a way that the first spy watches on the second, the second watches on the third, …, the tenth watches on the first. Prove that any 11 spies can also be numbered is a similar manner.

20. An equilateral triangle is divided into 9 000 000 congruent equilateral triangles by lines parallel to its sides. Each vertex of the small triangles is coloured in one of three colours. Prove that there exist three points of the same colour being the vertices of a triangle with its sides parallel to the lines of the original triangle.

95 - Mathematical Team Contest – Västeras, November 12, 1995

1. Find all triples (x; y; z) of positive integers satisfying the system of equations

$$x^2 = 2(y+z)$$
$$x^6 = y^6 + z^6 + 31(y^2 + z^2)$$

2. Let a and k be positive integers such that $a^2 + k$ divides $(a - 1)a(a + 1)$. Prove that $k \geq a$.

3. The positive integers a, b, c are pairwise relatively prime, a and c are odd and the numbers satisfy the equation $a^2 + b^2 = c^2$. Prove that b + c is a square of an integer.

4. John is older than Mary. He notices that if he switches the two digits of his age (an integer), he gets Mary's age. Moreover, the difference between the squares of their ages is a square of an integer. How old are Mary and John?

5. Let $a < b < c$ be three positive integers. Prove that among any 2c consecutive positive integers there exist three different numbers x, y, z such that abc divides xyz.

6. Prove that for positive a, b, c, d

$$\frac{a+c}{a+b} + \frac{b+d}{b+c} + \frac{c+a}{c+d} + \frac{d+b}{d+a} \geq 4$$

7. Prove that $\sin^3 18° + \sin^2 18° = 1/8$.

8. The real numbers a, b and c satisfy the inequalities $|a| \geq |b+c|$, $|b| \geq |c+a|$ and $|c| \geq |a+b|$. Prove that $a + b + c = 0$.

9. Prove that

$1995/2 – 1994/3 + 1993/4 - ... – 2/1995 + 1/1996 = 1/999 + 3/1000 + … + 1995/1996.$

10. Find all real-valued functions f de_ned on the set of all non-zero real numbers such that:

(i) $f(1) = 1$,
(ii) $f(1/(x+y)) = f(1/x) + f(1/y)$ for all non-zero x, y, x + y,
(iii) $(x + y) * f(x + y) = xy * f(x) * f(y)$ for all non-zero x, y, x + y.

11. In how many ways can the set of integers {1, 2, …, 1995} be partitioned into three nonempty sets so that none of these sets contains any pair of consecutive integers?

12. Assume we have 95 boxes and 19 balls distributed in these boxes in an arbitrary manner. We take 6 new balls at a time and place them in 6 of the boxes, one ball in each of the six.

Can we, by repeating this process a suitable number of times, achieve a situation in which each of the 95 boxes contains an equal number of balls?

13. Consider the following two person game. A number of pebbles are situated on the table. Two players make their moves alternately. A move consists of taking off the table x pebbles where x is the square of any positive integer. The player who is unable to make a move loses. Prove that there are infinitely many initial situations in which the second player can win no matter how his opponent plays.

14. There are n eas on an infinite sheet of triangulated paper. Initially the eas are in different small triangles, all of which are inside some equilateral triangle consisting of n 2 small triangles. Once a second each flea jumps from its original triangle to one of the three small triangles having a common vertex but no common side with it. For which natural numbers n does there exist an initial configuration such that after a finite number of jumps all the n fleas can meet in a single small triangle?

15. A polygon with 2n+1 vertices is given. Show that it is possible to assign numbers 1, 2, …, 4n+2 to the vertices and midpoints of the sides of the polygon so that for each side the sum of the three numbers assigned to it is the same.

16. In the triangle ABC, let l be the bisector of the external angle at C. The line through the midpoint O of AB parallel to l meets AC at E. Determine | CE | , if | AC | = 7 and | CB | = 4.

17. Prove that there exists a number α such that for any triangle ABC the inequality

$$\max(h_A; h_B; h_C) \le \alpha *\min(m_A; m_B; m_C)$$

holds, where h_A, h_B, h_C denote the lengths of the altitudes and m_A, m_B, m_C denote the lengths of the medians. Find the smallest possible value of α.

18. Let M be the midpoint of the side AC of a triangle ABC and let H be the footpoint of the altitude from B. Let P and Q be orthogonal projections of A and C on the bisector of the angle B. Prove that the four points H, P, M and Q lie on the same circle.

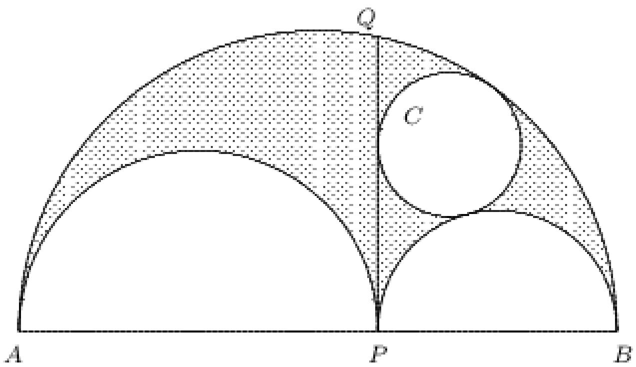
19. The following construction is used for training astronauts:
A circle C_2 of radius 2R rolls along the inside of another, fixed circle C_1 of radius nR; where n is an integer greater than 2. The astronaut is fastened to a third circle C_3 of radius R which rolls along the inside of circle C_2 in such a way that the touching point of the circles C_2 and C_3 remains at maximum distance from the touching point of the circles C_1 and C_2 at all times. How many revolutions (relative to the ground) does the astronaut perform together with the circle C_3 while the circle C_2 completes one full lap around the inside of circle C_1?

20. Prove that if both coordinates of every vertex of a convex pentagon are integers then the area of this pentagon is not less than 5/2.

96 - Mathematical Team Contest – Valkeakoski, 11/3, 1996

1. Let α be the angle between two lines containing the diagonals of a regular 1996-gon, and let β <> 0 be another such angle. Prove that α/β is a rational number.

2. In the figure below, you see three half-circles. The circle C is tangent to two of the half-circles and to the line PQ perpendicular to the diameter AB. The area of the shaded region is 39π, and the area of the circle C is 9π. Find the length of the diameter AB.



3. Let ABCD be a unit square and let P and Q be points in the plane such that Q is the circumcentre of triangle BPC and D be the circumcentre of triangle PQA. Find all possible values of the length of segment PQ.

4. ABCD is a trapezium (AD || BC). P is the point on the line AB such that $\angle$ CPD is maximal. Q is the point on the line CD such that $\angle$ BQA is maximal. Given that P lies on the segment AB, prove that $\angle$ CPD = $\angle$ BQA.

5. Let ABCD be a cyclic convex quadrilateral and let r_a, r_b, r_c, r_d be the radii of the circles inscribed in the triangles BCD, ACD, ABD, ABC, respectively. Prove that $r_a + r_c = r_b + r_d$.

6. Let a, b, c, d be positive integers such that ab = cd. Prove that a + b + c + d is not a prime.

7. A sequence of integers a_1, a_2, … is such that $a_1 = 1$, $a_2 = 2$ and for $n \geq 1$,

$$a_{n+2} = \begin{cases} 5a_{n+1} - 3a_n ; a_n \cdot a_{n+1} \text{even} \\ a_{n+1} - a_n ; a_n \cdot a_{n+1} \text{odd} \end{cases}$$

Prove that $a_n <> 0$ for all n.

8. Consider the sequence: $x_1 = 19$, $x_2 = 95$; $x_{n+2} = \text{lcm}(x_{n+1}, x_n) + x_n$; for $n > 1$, where lcm(a; b) means the least common multiple of a and b. Find the greatest common divisor of x_{1995} and x_{1996}.

9. Let n and k be integers, $1 \leq k < n$. Find an integer b and a set A of n integers satisfying the following conditions:

(i) No product of k - 1 distinct elements of A is divisible by b.
(ii) Every product of k distinct elements of A is divisible by b.
(iii) For all distinct a, a' in A, a does not divide a'.

10. Denote by d(n) the number of distinct positive divisors of a positive integer n (including 1 and n). Let a > 1 and n > 0 be integers such that $a^n + 1$ is a prime. Prove that $d(a^n - 1) \geq n$.

11. Real numbers $x_1, x_2, ..., x_{1996}$ have the following property: For any polynomial W of degree 2 at least three of the numbers $W(x_1), W(x_2), ..., W(x_{1996})$ are equal. Prove that at least three of the numbers $x_1, x_2, ..., x_{1996}$ are equal.

12. Let S be a set of integers containing the numbers 0 and 1996. Suppose further that any integer root of any non-zero polynomial with coe_cients in S also belongs to S. Prove that -2 belongs to S.

13. Consider the functions f defined on the set of integers such that

$$f(x) = f(x^2 + x + 1);$$

for all integer x. Find (a) all even functions, (b) all odd functions of this kind.

14. The graph of the function $f(x) = x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ (where $n > 1$) intersects the line $y = b$ at the points $B_1, B_2, ..., B_n$ (from left to right), and the line $y = c$ ($c <> b$) at the points $C_1, C_2, ... , C_n$ (from left to right). Let P be a point on the line $y = c$, to the right to the point C_n. Find the sum

$$\cot(\angle B_1C_1P) + ... + \cot(\angle B_nC_nP).$$

15. For which positive real numbers a, b does the inequality

$$x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 \geq x_1^a x_2^b x_3^a + x_2^a x_3^b x_4^a + ... + x_n^a x_1^b x_2^a$$

hold for all integers $n > 2$ and positive real numbers $x_1, ..., x_n$?

16. On an infinite checkerboard two players alternately mark one unmarked cell. One of them uses ×, the other •. The first who fills a 2 x 2 square with his symbols wins. Can the player who starts always win ?

17. Using each of the eight digits 1, 3, 4, 5, 6, 7, 8 and 9 exactly once, a three-digit number A, two two-digit numbers B and C, $B < C$, and a one digit number D are formed. The numbers are such that $A + D = B + C = 143$. In how many ways can this be done ?

18. The jury of an olympiad has 30 members in the beginning. Each member of the jury thinks that some of his colleagues are competent, while all the others are not, and these opinions do not change. At the beginning of every session a voting takes place, and those members who are not competent in the opinion of more than one half of the voters are excluded from the jury for the rest of the olympiad. Prove that after at most 15 sessions there will be no more exclusions.
(Note that nobody votes about his own competence.)

19. Four heaps contain 38, 45, 61 and 70 matches respectively. Two players take turn choosing any two of the heaps and take some non-zero number of matches from one heap and some non-zero number of matches from the other heap. The player who cannot make a move, loses. Which one of the players has a winning strategy ?

20. Is it possible to partition all positive integers into disjoint sets A and B such that
(i) no three numbers of A for an arithmetic progression,
(ii) no infinite non-constant arithmetic progression can be formed by numbers of B ?

97 - Mathematical Team Contest – Copenhagen, 11/9, 1997

1. Determine all functions f from the real numbers to the real numbers, different from the zero function, such that $f(x)f(y) = f(x - y)$ for all real numbers x and y.

2. Given a sequence $a_1, a_2, a_3, \ldots$ of positive integers in which every positive integer occurs exactly once. Prove that there exist integers l and m, $1 < l < m$, such that $a_1 + a_m = 2a_l$.

3. Let $x_1 = 1$ and $x_{n+1} = xn + [x_n/n] + 2$, for $n = 1, 2, 3, \ldots$ where $[x]$ denotes the largest integer not greater than x. Determine x_{1997}.

4. Prove that the arithmetic mean a of $x_1, \ldots, x_n$ satisfies

$$(x_1 - a)^2 + ... + (x_n - a)^2 \le (|x_1 - a| + ... + |x_n - a|)^2$$

5. In a sequence u_0, u_1, … of positive integers, u_0 is arbitrary, and for any non-negative integer

$$n,\ u_{n+1} = \begin{cases} \frac{1}{2} u_n ; u_n \text{even} \\ a + u_n ; u_n \text{odd} \end{cases}$$

where a is a fixed odd positive integer. Prove that the sequence is periodic from a certain step.

6. Find all triples (a, b, c) of non-negative integers satisfying a ≥ b ≥ c and

$$1 * a^3 + 9 * b^2 + 9 * c + 7 = 1997.$$

7. Let P and Q be polynomials with integer coefficients. Suppose that the integers a and a + 1997 are roots of P, and that Q(1998) = 2000. Prove that the equation Q(P(x)) = 1 has no integer solutions.

8. If we add 1996 to 1997, we first add the unit digits 6 and 7. Obtaining 13, we write down 3 and "carry" 1 to the next column. Thus we make a carry. Continuing, we see that we are to make three carries in total:

```
   1 1 1
   1 9 9 6
 + 1 9 9 7
   3 9 9 3
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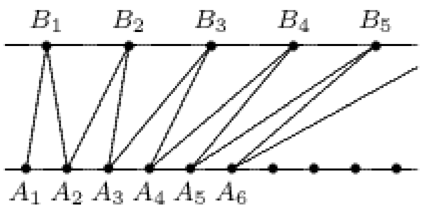
Does there exist a positive integer k such that adding 1996 * k to 1997 * k no carry arises during the whole calculation?

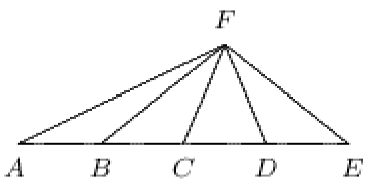
9. The worlds in the Worlds' Sphere are numbered 1, 2, 3, … and connected so that for any integer n ≥ 1, Gandalf the Wizard can move in both directions between any worlds with numbers n, 2n and 3n + 1. Starting his travel from an arbitrary world, can Gandalf reach every other world?

10. Prove that in every sequence of 79 consecutive positive integers written in the decimal system, there is a positive integer whose sum of digits is divisible by 13.



11. On two parallel lines, the distinct points A_1, A_2, A_3, … respectively B_1, B_2, B_3, … are marked in such a way that $|A_iA_{i+1}| = 1$ and $|B_iB_{i+1}| = 2$ for i = 1, 2, … (see figure). Provided that $\angle A_1A_2B_1 = \alpha$, and the infinite sum $\angle A_1B_1A_2 + \angle A_2B_2A_3 + \angle A_3B_3A_4 + \ldots$

12. Two circles C_1 and C_2 intersect in P and Q. A line through P intersects C_1 and C_2 again in A and B, respectively, and X is the midpoint of AB. The line through Q and X intersects C_1 and C_2 again in Y and Z, respectively. Prove that X is the midpoint of Y Z.



13. Five distinct points A, B, C, D and E lie on a line with | AB | = | BC | = | CD | = | DE | . The point F lies outside the line. Let G be the circumcentre of the triangle ADF and H the circumcentre of the triangle BEF.
Show that the lines GH and FC are perpendicular.

14. In the triangle ABC, $| AC |^2$ is the arithmetic mean of $| BC |^2$ and $| AB |^2$. Show that $\cot^2 B \geq \cot A * \cot C$.

15. In the acute triangle ABC, the bisectors of $\angle A$, $\angle B$ and $\angle C$ intersect the circumcircle again in A_1, B_1 and C_1, respectively. Let M be the point of intersection of AB and B_1C_1, and let N be the point of intersection of BC and A_1B_1. Prove that MN passes through the incentre of Δ ABC.

16. On a 5 x 5 chessboard, two players play the following game: The first player places a knight on some square. Then the players alternately move the knight according to the rules of chess, starting with the second player. It is not allowed to move the knight to a square that was visited previously. The player who cannot move loses. Which of the two players has a winning strategy?

17. A rectangle can be divided into n equal squares. The same rectangle can also be divided into n + 76 equal squares. Find n.

18. (i) Prove the existence of two infinite sets A and B, not necessarily disjoint, of non-negative integers such that each non-negative integer n is uniquely representable in the form $n = a + b$ with $a \in A$, $b \in B$.
(ii) Prove that for each such pair (A, B), either A or B contains only multiples of some integer $k > 1$.

19. In a forest each of n animals ($n \geq 3$) lives in its own cave, and there is exactly one separate path between any two of these caves. Before the election for King of the Forest some of the animals make an election campaign. Each campaign-making animal visits each of the other caves exactly once, uses only the paths for moving from cave to cave, never turns from one path to another between the caves and returns to its own cave in the end of its campaign. It is also known that no path between two caves is used by more than one campaign-making animal.
a) Prove that for any prime n, the maximum possible number of campaign-making animals is $(n-1)/2$;
b) Find the maximum number of campaign-making animals for $n = 9$.

20. Twelve cards lie in a row. The cards are of three kinds: with both sides white, both sides black, or with a white and a black side. Initially, nine of the twelve cards have a black side up. The cards 1-6 are turned, and subsequently four of the twelve cards have a black side up. Now cards 4-9 are turned, and six cards have a black side up. Finally, the cards 1-3 and 10-12 are turned, after which five cards have a black side up. How many cards of each kind were there?

98 - Mathematical Team Contest – Warschau, 11/8, 1998

1. Find all functions f of two variables, whose arguments x, y and values f(x; y) are positive integers, satisfying the following conditions (for all positive integers x and y):

$f(x; x) = x;$
$f(x; y) = f(y; x);$
$(x + y)f(x; y) = yf(x; x+y):$

2. A triple (a, b, c) of positive integers is called quasi-Pythagorean if there exists a triangle with lengths of the sides a, b, c and the angle opposite to the side c equal to 120°. Prove that if (a, b, c) is a quasi-Pythagorean triple then c has a prime divisor bigger than 5.

3. Find all pairs of positive integers x, y which satisfy the equation $2x^2 + 5y^2 = 11(xy - 11)$.

4. Let P be a polynomial with integer coefficients. Suppose that for n = 1, 2, 3, …, 1998 the number P(n) is a three-digit positive integer. Prove that the polynomial P has no integer roots.

5. Let a be an odd digit and b an even digit. Prove that for every positive integer n there exists a positive integer, divisible by 2^n, whose decimal representation contains no digits other than a and b.

6. Let P be a polynomial of degree 6 and let a, b be real numbers such that $0 < a < b$. Suppose that $P(a) = P(-a)$, $P(b) = P(-b)$, $P0(0) = 0$. Prove that $P(x) = P(-x)$ for all real x.

7. Let R be the set of all real numbers. Find all functions $f: R \to R$ satisfying for all $x; y \in R$ the equation $f(x) + f(y) = f(f(x)f(y))$.

8. Let $P_k(x) = 1 + x + x^2 + \dots + x^{k-1}$. Show that

$$\sum_{k=1}^{n} \binom{n}{k} P_k(x) = 2^{n-1} P_n(\frac{1+x}{2})\, n$$

for every real number x and every positive integer n.

9. Let the numbers α, β satisfy $0 < \alpha < \beta < \pi/2$ and let γ and δ be the numbers defined by the conditions:
(i) $0 < \gamma < \pi/2$, and tan is the arithmetic mean of tan α and tan β;
(ii) $0 < \delta < \pi/2$, and $1/\cos\delta$ is the arithmetic mean of $1/\cos\alpha$ and $1/\cos\beta$.
Prove that $\gamma < \delta$.

10. Let $n \geq 4$ be an even integer. A regular n-gon and a regular (n-1)-gon are inscribed into the unit circle. For each vertex of the n-gon consider the distance from this vertex to the nearest vertex of the (n-1)-gon, measured along the circumference. Let S be the sum of these n distances. Prove that S depends only on n, and not on the relative position of the two polygons.

11. Let a, b, c be the lengths of the sides of a triangle. Let R denote its circumradius. Prove that $R \geq \dfrac{a^2 + b^2}{2\sqrt{2a^2 + 2b^2 - c^2}}$
When does equality hold ?

12. In a triangle ABC, ∠ BAC = 90°. Point D lies on the side BC and satisfies ∠ BDA = 2 ∠BAD. Prove that 1/AD = ½ (1/BD + 1/CD).

13. In a convex pentagon ABCDE, the sides AE and BC are parallel and ∠ ADE = ∠ BDC. The diagonals AC and BE intersect in P. Prove that ∠ EAD = ∠ BDP and ∠ CBD = ∠ ADP.

14. Given triangle ABC with AB < AC. The line passing through B and parallel to AC meets the external angle bisector of ∠ BAC at D. The line passing through C and parallel to AB meets this bisector at E. Point F lies on the side AC and satisfies the equality FC = AB. Prove that DF = FE.

15. Given acute triangle ABC. Point D is the foot of the perpendicular from A to BC. Point E lies on the segment AD and satisfies the equation

$$AE / ED = CD / DB$$

Point F is the foot of the perpendicular from D to BE. Prove that $\angle AFC = 90°$.

16. Is it possible to cover a 13 x 13 chessboard with fourty-two pieces of dimensions 4 x 1 such that only the central field of the chessboard remains uncovered ? (We assume that each piece covers exactly four fields of the chessboard.)

17. Let n and k be positive integers. There are nk objects (of the same size) and k boxes, each of which can hold n objects. Each object is coloured in one of k different colours. Show that the objects can be packed in the boxes so that each box holds objects of at most two colours.

18. Determine all positive integers n for which there exists a set S with the following properties:
(i) S consists of n positive integers, all smaller than 2^{n-1};
(ii) for any two distinct subsets A and B of S, the sum of the elements of A is different from the sum of the elements of B.

19. Consider a ping-pong match between two teams, each consisting of 1000 players. Each player played against each player of the other team exactly once (there are no draws in ping-pong). Prove that there exist ten players, all from the same team, such that every member of the other team has lost his game against at least one of those ten players.

20. We say that some positive integer m covers the number 1998, if 1, 9, 9, 8 appear in this order as digits of m. (For instance 1998 is covered by 2**1**59**9**36**98** but not by 213326798.) Let k(n) be the number of positive integers that cover 1998 and have exactly n digits ($n \geq 5$), all different from 0. What is the remainder of k(n) in division by 8?

99 - Mathematical Team Contest – Reykjavik, 11/6, 1999

1. Determine all real numbers a, b, c, d that satisfy the following equations

$$abc + ab + bc + ca + a + b + c = 1$$
$$bcd + bc + cd + db + b + c + d = 9$$
$$cda + cd + da + ac + c + d + a = 9$$
$$dab + da + ab + bd + d + a + b = 9$$

2. Determine all positive integers n with the property that the third root of n is obtained by removing its last three decimal digits.

3. Determine all positive integers $n \geq 3$ such that the inequality

$$a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n + a_na_1 \leq 0$$

holds for all real numbers $a_1; a_2; \ldots; a_n$ which satisfy $a_1 + a_2 + \ldots + a_n = 0$.

4. For all positive real numbers x and y let

$$f(x; y) = \min(x; y/(x^2 + y^2)).$$

Show that there exist x_0 and y_0 such that $f(x; y) \leq f(x_0; y_0)$ for all positive x and y, and find $f(x_0; y_0)$.

5. The point (a; b) lies on the circle $x^2 + y^2 = 1$. The tangent to the circle at this point meets the parabola $y = x^2 + 1$ at exactly one point. Find all such points (a; b).

6. What is the least number of moves it takes a knight to get from one corner of an n x n chessboard, where $n \geq 4$, to the diagonally opposite corner?

7. Two squares on an 8 x 8 chessboard are called adjacent if they have a common edge or common corner. Is it possible for a king to begin in some square and visit all squares exactly once in such a way that all moves except the first are made into squares adjacent to an even number of squares already visited?

8. We are given 1999 coins. No two coins have the same weight. A machine is provided which allows us with one operation to determine, for any three coins, which one has the middle weight. Prove that the coin that is the 1000th by weight can be determined using no more than 1000000 operations and that this is the only coin whose position by weight can be determined using this machine.

9. A cube with edge length 3 is divided into 27 unit cubes. The numbers 1, 2, …, 27 are distributed arbitrarily over the unit cubes, with one number in each cube. We form the 27 possible row sums (there are nine such sums of three integers for each of the three directions parallel with the edges of the cube). At most how many of the 27 row sums can be odd?

10. May the points of a disc of radius 1 (including its circumference) be partitioned into three subsets in such a way that no subset contains two points separated by a distance 1?

11. Prove that for any four points in the plane, no three of which are collinear, there exists a circle such that three of the four points are on the circumference and the fourth point is either on the circumference or inside the circle.

12. In a triangle ABC it is given that 2AB = AC + BC. Prove that the incentre of ABC, the circumcenter of ABC, and the midpoints of AC and BC are concyclic.

13. The bisectors of the angles A and B of the triangle ABC meet the sides BC and CA at the points D and E, respectively. Assuming that AE + BD = AB, determine the angle C.

14. Let ABC be an isosceles triangle with AB = AC. Points D and E lie on the sides AB and AC, respectively. The line passing through B and parallel to AC meets the line DE at F. The line passing through C and parallel to AB meets the line DE at G. Prove that

$$[DBCG] / [FBCE] = AD / AE ,$$

where [PQRS] denotes the area of the quadrilateral PQRS.

15. Let ABC be a triangle with $\angle C = 60°$ and AC < BC. The point D lies on the side BC and satisfies BD = AC. The side AC is extended to the point E where AC = CE. Prove that AB = DE.

16. Find the smallest positive integer k which is representable in the form $k = 19^n - 5^m$ for some positive integers m and n.

17. Does there exist a finite sequence of integers $c_1, c_2, \ldots, c_n$ such that all the numbers $a + c_1$, $a + c_2, \ldots, a + c_n$ are primes for more than one but not infinitely many different integers a?

18. Let m be a positive integer such that $m = 2 \pmod 4$. Show that there exists at most one factorization m = ab where a and b are positive integers satisfying $0 < a - b < \sqrt{5 + 4\sqrt{4m+1}}$.

19. Prove that there exist infinitely many even positive integers k such that for every prime p the number $p^2 + k$ is composite.

20. Let a, b, c and d be prime numbers such that $a > 3b > 6c > 12d$ and $a^2 - b^2 + c^2 - d^2 = 1749$. Determine all possible values of $a^2 + b^2 + c^2 + d^2$.

Baltic Way 2000 - Oslo, November 5, 2000

1. Let K be a point inside the triangle ABC. Let M and N be points such that M and K are on opposite sides of the line AB, and N and K are on opposite sides of the line BC. Assume that $\angle MAB = \angle MBA = \angle NBC = \angle NCB = \angle KAC = \angle KCA$. Show that MBNK is a parallelogram.

2. Given an isosceles triangle ABC with $\angle A = 90°$. Let M be the midpoint of AB. The line passing through A and perpendicular to CM intersects the side BC at P. Prove that

$$\angle AMC = 6\ BMP.$$

3. Given a triangle ABC with $\angle A = 90°$ and $AB \neq AC$. The points D, E, F lie on the sides BC, CA, AB, respectively, in such a way that AFDE is a square. Prove that the line BC, the line FE and the line tangent at the point A to the circumcircle of the triangle ABC intersect in one point.

4. Given a triangle ABC with $\angle A = 120°$. The points K and L lie on the sides AB and AC, respectively. Let BKP and CLQ be equilateral triangles constructed outside the triangle ABC. Prove that $PQ \geq \sqrt{3}/2\ (AB + AC)$.

5. Let ABC be a triangle such that $BC / (AB - BC) = (AB + BC) / AC$
Determine the ratio $\angle A : \angle C$.

6. Fredek runs a private hotel. He claims that whenever $n \geq 3$ guests visit the hotel, it is possible to select two guests that have equally many acquaintances among the other guests, and that also have a common acquaintance or a common unknown among the guests. For which values of n is Fredek right? (Acquaintance is a symmetric relation.)

7. In a 40 x 50 array of control buttons, each button has two states: on and off. By touching a button, its state and the states of all buttons in the same row and in the same column are switched. Prove that the array of control buttons may be altered from the all-on state to the all-on state by touching buttons successively, and determine the least number of touches needed to do so.

8. Fourteen friends met at a party. One of them, Fredek, wanted to go to bed early. He said goodbye to 10 of his friends, forgot about the remaining 3, and went to bed. After a while he returned to the party, said goodbye to 10 of his friends (not necessarily the same as before), and went to bed. Later Fredek came back a number of times, each time saying goodbye to exactly 10 of his friends, and then went back to bed. As soon as he had said goodbye to each of his friends at least once, he did not come back again. In the morning Fredek realised that he had said goodbye a di_erent number of times to each of his thirteen friends! What is the smallest possible number of times that Fredek returned to the party?

9. There is a frog jumping on a 2k x 2k chessboard, composed of unit squares. The frog's jumps are $\sqrt{(1 + k^2)}$ long and they carry the frog from the center of a square to the center of another square. Some m squares of the board are marked with an x, and all the squares into which the frog can jump from an x'd square (whether they carry an x or not) are marked with an o. There are n o'd squares. Prove that $n \geq m$.

10. Two positive integers are written on the blackboard. Initially, one of them is 2000 and the other is smaller than 2000. If the arithmetic mean m of the two numbers on the blackboard is an integer, the following operation is allowed: One of the two numbers is erased and replaced by m. Prove that this operation cannot be performed more than ten times. Give an example where the operation is performed ten times.

11. A sequence of positive integers $a_1, a_2, \ldots$ is such that for each m and n the following holds: if m is a divisor of n and $m < n$, then a_m is a divisor of a_n and $a_m < a_n$. Find the least possible value of a2000.

12. Let x1, $x_2, \ldots, x_n$ be positive integers such that no one of them is an initial fragment of any other (for example, 12 is an initial fragment of 12, 125 and 12405). Prove that

$$1/x_1 + 1/x_2 + \ldots + 1/x_n < 3.$$