Olimpiada Matemática de Centroamérica y el Caribe

1st Centromerican 1999

Problem A1

A, B, C, D, E each has a unique piece of news. They make a series of phone calls to each other. In each call, the caller tells the other party all the news he knows, but is not told anything by the other party. What is the minimum number of calls needed for all five people to know all five items of news? What is the minimum for n people?

Answer

8, eg BA, CA, DA, EA, AB, AC, AD, AE.

2n-2

Solution

Consider the case of n people. Let N be the smallest number of calls such that after they have been made at least one person knows all the news. Then $N \geq n-1$, because each of the other n-1 people must make at least one call, otherwise no one but them knows their news. After N calls only one person can know all the news, because otherwise at least one person would have known all the news before the Nth call and N would not be minimal. So at least a further n-1 calls are needed, one to each of the other n-1 people. So at least 2n-2 calls are needed in all. But 2n-2 is easily achieved. First everyone else calls X, then X calls everyone else.

Problem A2

Find a positive integer n with 1000 digits, none 0, such that we can group the digits into 500 pairs so that the sum of the products of the numbers in each pair divides n.

Answer

11...1 2112 2112 ... 2112 (960 1s followed by 10 2112s)

Solution

Suppose we take 980 digits to be 1 and 20 digits to be 2. Then we can take 8 pairs (2,2), 4 pairs (2,1) and 488 pairs (1,1) giving a total of $528 = 16 \cdot 3 \cdot 11$. The sum of the digits is 1020 which is divisible by 3, so n is certainly divisible by 3. We can arrange that half the 1s and half the 2s are in odd positions, which will ensure that n is divisible by 11. Finally, n will be divisible by 16 if the number formed by its last 4 digits is divisible by 16, so we take the last 4 digits to be 2112 ($=16 \cdot 132$). So, for example, we can take n to be 11...1 2112 2112 ... 2112, where we have 960 1s followed by 10 2112s.

Problem A3

A and B play a game as follows. Starting with A, they alternately choose a number from 1 to 9. The first to take the total over 30 loses. After the first choice each choice must be one of the four numbers in the same row or column as the last number (but not equal to the last number):

7	8	9
4	5	6
1	2	3

Find a winning strategy for one of the players.

Answer

A wins

Solution

A plays 9.

Case (1). If B plays 8, then A plays 9. B must now play 3, then A wins with 1.

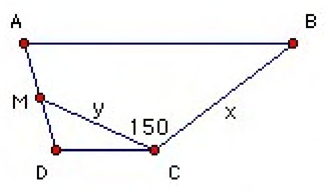
Case (2). If B plays 7, then A plays 9. B must now play 3, then A wins with 2.

Case (3). If B plays 6, then A plays 5. Now if B plays x, A can play 10-x and wins.

Case (4). If B plays 3, then A plays 6. If B plays 9, then A wins with 3 and vice versa. If B plays 5, then A plays 6 and wins. If B plays 4, then A plays 6 and wins.

Problem B1

ABCD is a trapezoid with AB parallel to CD. M is the midpoint of AD, $\angle MCB = 150^o$, BC = x and MC = y. Find area ABCD in terms of x and y.



Answer

xy/2

Solution

Extend CM to meet the line AB at N. Then CDM and NAM are congruent and so area ABCD = area CNB. But CN = 2y, so area CNB = $\frac{1}{2} 2y \cdot x \cdot \sin 150^o = xy/2$.

Problem B2

$a > 17$ is odd and $3a-2$ is a square. Show that there are positive integers $b \neq c$ such that $a+b$, $a+c$, $b+c$ and $a+b+c$ are all squares.

Solution

Let $a = 2k+1$. So we are given that $6k+1$ is a square. Take $b = k^2-4k$, $c = 4k$. Then $b \neq c$ since $a \neq 17$. Also b is positive since $a > 9$. Now $a+b = (k-1)^2$, $a+c = 6k+1$ (given to be a square), $b+c = k^2$, $a+b+c = (k+1)^2$.

Problem B3

$S \in \{1, 2, 3, \dots , 1000\}$ is such that if m and n are distinct elements of S, then m+n does not belong to S. What is the largest possible number of elements in S?

Answer

501 eg $\{500, 501, \dots , 1000\}$

Solution

We show by induction that the largest possible subset of $\{1, 2, \dots , 2n\}$ has n+1 elements. It is obvious for n = 1. Now suppose it is true for n. If we do not include 2n-1 or 2n in the subset, then by induction it can have at most n elements. If we include 2n, then we can include at most one from each of the pairs (1,2n-1), (2,2n-2), ... , (n-1,n+1). So with n and 2n, that gives at most n+1 in all. If we include 2n-1 but not 2n, then we can include at most one from each of the pairs (1,2n-2), (2,2n-3), ... , (n-1,n), so at most n in all.

2nd Centromerican 2000

Problem A1

Find all three digit numbers abc (with $a \neq 0$) such that $a^2 + b^2 + c^2$ divides 26.

Answer

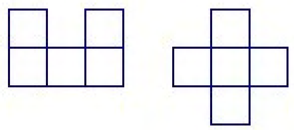
100, 110, 101, 302, 320, 230, 203, 431, 413, 314, 341, 134, 143, 510, 501, 150, 105

Solution

Possible factors are 1, 2, 13, 26. Ignoring order, the possible expressions as a sum of three squares are: $1 = 1^2 + 0^2 + 0^2$, $2 = 1^2 + 1^2 + 0^2$, $13 = 3^2 + 2^2 + 0^2$, $26 = 5^2 + 1^2 + 0^2 = 4^2 + 3^2 + 1^2$.

Problem A2

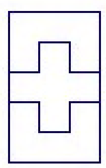
The diagram shows two pentominos made from unit squares. For which $n > 1$ can we tile a 15 x n rectangle with these pentominos?



Answer

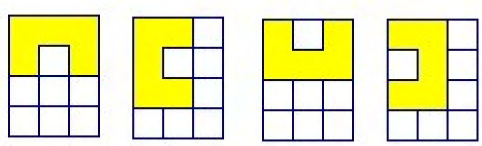
all n except 1, 2, 4, 7

Solution

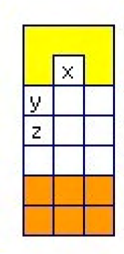


The diagram shows how to tile a 3 x 5 rectangle. That allows us to tile a 15 x 3 rectangle and a 15 x 5 rectangle. Now we can express any integer $n > 7$ as a sum of 3s and 5s, because 8 = 3+5, 9 = 3+3+3, 10 = 5+5 and given a sum for n, we obviously have a sum for n+3. Hence we can tile 15 x n rectangles for any $n \geq 8$. We can also do n = 3, 5, 6.

Obviously n = 1 and n = 2 are impossible, so it remains to consider n = 4 and n = 7.



The diagram shows the are 4 ways of covering the top left square (we only show the 3 left columns of each 15 x 4 rectangle). Evidently none of them work for n = 4. So n = 4 is impossible.

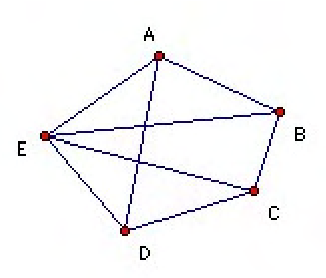


Finally, consider n = 7. As for n = 4, there are only two possibilities for covering the top left square, but two obviously do not work. Consider the possibility in the diagram above. If we cover x using the U shaped piece, then we cannot cover y. So we must cover x with the cross. Then we have to cover z with the U shaped piece. But that leaves the 6 red squares at the bottom, which cannot be covered.

If we cover the top left square with a C, the a similar argument shows that we must cover the top three rows and first 5 columns with cross and two Us. But now we cannot cover the 4 x 6 rectangle underneath.

Problem A3

ABCDE is a convex pentagon. Show that the centroids of the 4 triangles ABE, BCE, CDE, DAE from a parallelogram with whose area is 2/9 area ABCD.



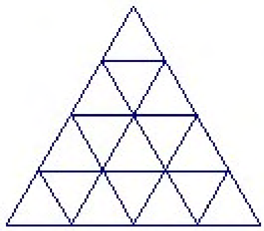
Solution

Use vectors. Take any origin O. Write the vector OA as **a**, OB as **b** etc. Let P, Q, R, S be the centroids of ABE, BCE, CDE, DAE. Then **p** = (**a**+**b**+**e**)/3, **q** = (**b**+**c**+**e**)/3, **r** = (**c**+**d**+**e**)/3, **s** = (**a**+**d**+**e**)/3. So PQ = (**a**-**c**)/3 = SR. Hence PQ and SR are equal and parallel, so PQRS is a parallelogram.

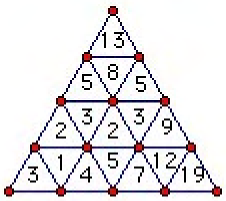
QR = (**d**-**b**)/3, so the area of the parallelogram is PQ x QR = (**a**-**c**) **x** (**d**-**b**)/9. The area of ABCD = area ABC + area ACD = ½ CB X CA + ½ CA x CD = ½ (**b**-**c**) **x** (**a**-**c**) + ½ (**a**-**c**) **x** (**d**-**c**) = ½ (**a**-**c**) **x** (-(**b**-**c**)+**d**-**c**) = (**a**-**c**) **x** (**d**-**b**)/2. So area PQRS = (2/9) area ABCD.

Problem B1

Write an integer in each small triangles so that every triangle with at least two neighbors has a number equal to the difference between the numbers in two of its neighbors.

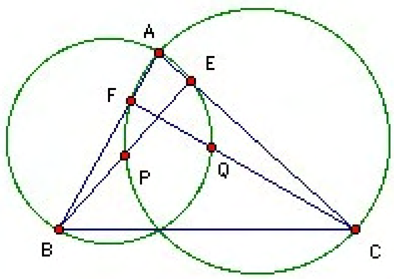


Answer



Problem B2

ABC is acute-angled. The circle diameter AC meets AB again at F, and the circle diameter AB meets AC again at E. BE meets the circle diameter AC at P, and CF meets the circle diameter AB at Q. Show that AP = AQ.



Solution

AB is a diameter, so $\angle AQB = 90^o$. Similarly, AC is a diameter, so $\angle AFQ = \angle AFC = 90^o$. Hence triangles AQB, AFQ are similar, so AQ/AB = AF/AQ, or $AQ^2 = AF\cdot AB$. Similarly, $AP^2 = AE\cdot AC$. But $\angle BEC = \angle BFC = 90^o$, so BFEC is cyclic. Hence $AF\cdot AB = AE\cdot AC$.

Problem B3

A nice representation of a positive integer n is a representation of n as sum of powers of 2 with each power appearing at most twice. For example, 5 = 4 + 1 = 2 + 2 + 1. Which positive integers have an even number of nice representations?

Answer

n = 2 mod 3

Solution

Let f(n) be the number of nice representations of n. We show first that (1) f(2n+1) = f(n), and (2) f(2n) = f(2n-1) + f(n).

(1) is almost obvious because $n = \sum a_i 2^{b_i}$ iff $2n+1 = 1 + \sum a_i 2^{b_i+1}$. (2) is also fairly obvious. There are f(n) representations of 2n without a 1 and f(2n-1) with a 1 (because any nice representation of f(2n-1) must have just one 1).

We now prove the required result by induction. Let S_k be the statement that for $n \le 6k$, f(n) is odd for n = 0, 1 mod 3 and even for n = 2 mod 3. It is easy to check that f(1) = 1, f(2) = 2, f(3) = 1, f(4) = 3, f(5) = 2, f(6) = 3. So S_1 is true. Suppose S_k is true. Then f(6k+1) = f(3k) = odd. f(6k+2) = f(3k+1) + f(6k+1) = odd + odd = even. f(6k+3) = f(3k+1) = odd. f(6k+4) = f(6k+3) + f(3k+2) = odd + even = odd. f(6k+5) = f(3k+2) = odd. So S_{k+1} is true. So the result is true for all k and hence all n.

3rd Centromerican 2001

Problem A1

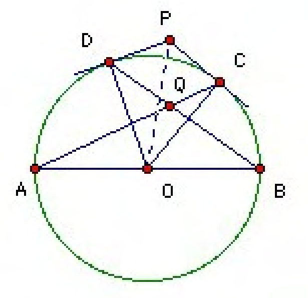
A and B stand in a circle with 2001 other people. A and B are not adjacent. Starting with A they take turns in touching one of their neighbors. Each person who is touched must immediately leave the circle. The winner is the player who manages to touch his opponent. Show that one player has a winning strategy and find it.

Solution

If there is just one person between A and B, then touching that person loses. There are 1999 people who can be touched before that happens, so B is sure to lose provided that A never touches someone who is the only person between him and B.

Problem A2

C and D are points on the circle diameter AB such that ∠AQB = 2 ∠COD. The tangents at C and D meet at P. The circle has radius 1. Find the distance of P from its center.



Answer

$2/\sqrt{3}$

Solution

$\angle AQB = \angle ACB + \angle CBQ = 90^o + \angle CBQ = 90^o + \frac{1}{2}\angle COD = 90^o + \frac{1}{4}\angle AQB$. Hence $\angle AQB = 120^o$, and $\angle COD = 60^o$. So $OP = 1/\cos 30^o = 2/\sqrt{3}$.

Problem A3

Find all squares which have only two non-zero digits, one of them 3.

Answer

36, 3600, 360000, ...

Solution

A square must end in 0, 1, 4, 5, 6, or 9. So the 3 must be the first digit. If a square ends in 0, then it must end in an even number of 0s and removing these must give a square. Thus we need only consider numbers which do not end in 0. The number cannot end in 9, for then it would be divisible by 3 but not 9 (using the sum of digits test). It cannot end in 5, because squares ending in 5 must end in 25. So it remains to consider 1, 4, 6.

36 is a square. But if there are one or more 0s between the 3 and the 6, then the number is divisible by 2 but not 4, so 36 is the only solution ending in 6.

Suppose $3\cdot\ 10^n + 1 = m^2$, so $3\cdot\ 2^n5^n = (m-1)(m+1)$. But m+1 and m-1 cannot both be divisible by 5, so one must be a multiple of 5^n. But 5^n &t; $3\cdot\ 2^n + 2$ for $n > 1$, so that is impossible for $n > 1$. For n = 1, we have 31, which is not a square. Thus there are no squares $3\cdot\ 10^n + 1$.

A similar argument works for $3\cdot\ 10^n + 4$, because if $3\cdot\ 10^n + 4 = m^2$, then 5 cannot divide m-2 and m+2, so 5^n must divide one of them, which is then too big, since $5^n > 3\cdot\ 2^n + 4$ for $n > 1$. For n = 1 we have 34, which is not a square.

Problem B1

Find the smallest n such that the sequence of positive integers $a_1, a_2, \dots , a_n$ has each term ≤ 15 and $a_1! + a_2! + \dots + a_n!$ has last four digits 2001.

Solution

We find that the last 4 digits are as follows: 1! 1, 2! 2, 3! 6, 4! 24, 5! 120, 6! 720, 7! 5040, 8! 320, 9! 2880, 10! 8800, 11! 6800, 12! 1600, 13! 800, 14! 1200, 15! 8000.

Only 1! is odd, so we must include it. None of the others has last 4 digits 2000, so we need at least three factorials. But 13! + 14! + 1! works.

Problem B2

a, b, c are reals such that if p_1, p_2 are the roots of $ax^2 + bx + c = 0$ and q_1, q_2 are the roots of $cx^2 + bx + a = 0$, then p_1, q_1, p_2, q_2 is an arithmetic progression of distinct terms. Show that $a + c = 0$.

Solution

Put $p_1 = h-k$, $q_1 = h$, so $p_2 = h+k$, $q_2 = h+2k$. Then $h^2-k^2 = c/a$, $2h = -b/a$, $h^2+2hk = a/c$, $2h+2k = -b/c$.

So $h = -b/2a$, $k = b/2a - b/2c$ and $b^2/2ac - b^2/4c^2 = c/a$, $b^2/2ac - b^2/4a^2 = a/c$. Subtracting, $(b^2/4)(1/a^2 - 1/c^2) = c/a - a/c$, so $(c^2-a^2)(b^2/4 - ac)/(a^2c^2) = 0$. Hence $a = c$ or $a + c = 0$ or $b^2 = 4ac$. If $b^2 = 4ac$, then $p_1 = p_2$, whereas we are given that p_1, p_2, q_1, q_2 are all distinct. Similarly, if $a = c$, then $\{p_1,p_2\} = \{q_1,q_2\}$. Hence $a + c = 0$.

Problem B3

10000 points are marked on a circle and numbered clockwise from 1 to 10000. The points are divided into 5000 pairs and the points of each pair are joined by a segment, so that each segment intersects just one other segment. Each of the 5000 segments is labeled with the product of the numbers at its endpoints. Show that the sum of the segment labels is a multiple of 4.

Solution

Suppose points i and j are joined. The j-i-1 points on the arc between i and j are paired with each other, with just one exception (the endpoint of the segment that intersects the segment i-j). So we must have j = i+4k+2. Thus the segment i-j is labeled with i(i+4k+2) = i(i+2) mod 4. If i is even, this is 0 mod 4. If i is odd, then it is -1 mod 4. Since odd points are joined to odd points (4k+2 is always even), there are 2500 segments joining odd points. Each has a label = -1 mod 4. So their sum = -2500 = 0 mod 4. All the segments joining even points have labels = 0 mod 4, so the sum of all the segment labels is a multiple of 4.

4th Centromerican 2002

Problem A1

For which $n > 2$ can the numbers 1, 2, ... , n be arranged in a circle so that each number divides the sum of the next two numbers (in a clockwise direction)?

Answer

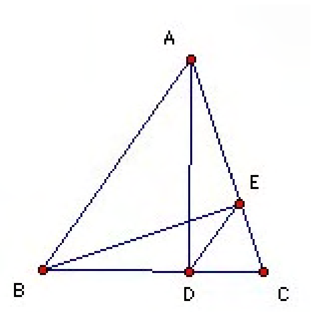
n=3

Solution

Let the numbers be $a_1, a_2, a_3, \ldots$. Where necessary we use cyclic subscripts (so that a_{n+1} means a_1 etc). Suppose a_i and a_{i+1} are both even, then since a_i divides $a_{i+1} + a_{i+2}$, a_{i+2} must also be even. Hence a_{i+3} must be even and so on. Contradiction, since only half the numbers are even. Hence if a_i is even, a_{i+1} must be odd. But $a_{i+1} + a_{i+2}$ must be even, so a_{i+2} must also be odd. In other words, every even number is followed by two odd numbers. But that means there are at least twice as many odd numbers as even numbers. That is only possible for n = 3. It is easy to check that n = 3 works.

Problem A2

ABC is acute-angled. AD and BE are altitudes. area BDE ≤ area DEA ≤ area EAB ≤ ABD. Show that the triangle is isosceles.



Solution

area BDE ≤ area DEA implies that the distance of A from the line DE is no smaller than the distance of B, so if the lines AB and DE intersect, then they do so on the B, D side. But area EAB ≤ area ABD implies that the distance of D from the line AB is no smaller than the distance of E, so if the lines AB and DE intersect, then they do so on the A, E side. Hence they must be parallel. But ABDE is cyclic ($\angle ADB = \angle AEB = 90^o$), so it must be an isosceles trapezoid and hence $\angle A = \angle B$.

Problem A3

Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = A$, $a_{n+1} = a_n + d(an)$, where d(m) is the largest factor of m which is < m. For which integers A > 1 is 2002 a member of the sequence?

Answer

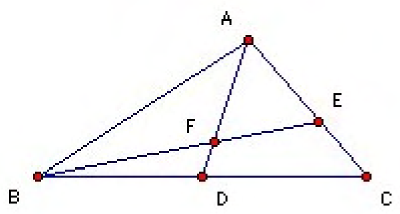
None.

Solution

Let N have largest proper factor m < N. We show that N + m cannot be 2002. Suppose N + m = 2002. Put N = mp. Then p must be a prime (or N would have a larger proper factor than m). So 2002 = m(p+1). Also $p \le m$. Hence p < 44. So k = p+1 is a factor of 2002 smaller than 45 which is 1 greater than a prime. It is easy to check that the only possibility is k = 14. So N = $11 \cdot 13^2$. But this has largest factor 13^2, not $11 \cdot 13$. Contradiction.

Problem B1

ABC is a triangle. D is the midpoint of BC. E is a point on the side AC such that BE = 2AD. BE and AD meet at F and ∠FAE = 60°. Find □FEA.



Solution

Let the line parallel to BE through D meet AC at G. Then DCG, BCE are similar and BC = 2 DC, so BE = 2 DG. Hence AD = DG, so ∠DGA = ∠DAG = 60°. FE is parallel to DG, so ∠FEA = 60°.

Problem B2

Find an infinite set of positive integers such that the sum of any finite number of distinct elements of the set is not a square.

Solution

Consider the set of odd powers of 2. Suppose a1 < a2 < ... < an and odd positive integers. Then $2^{a_1} + 2^{a_2} + ... + 2^{a_n} = 2^{a_1}(1 + 2^{a_2-a_1} + ... + 2^{a_n-a_1})$. Each term in the bracket except the first is even, so the bracket is odd. Hence the sum is divisible by an odd power of 2 and cannot be a square.

Problem B3

A path from (0,0) to (n,n) on the lattice is made up of unit moves upward or rightward. It is *balanced* if the sum of the x-coordinates of its 2n+1 vertices equals the sum of their y-coordinates. Show that a balanced path divides the square with vertices (0,0), (n,0), (n,n), (0,n) into two parts with equal area.

Solution

Denote the vertices of the path as $(0,0) = (x_0,y_0)$, (x_1,y_1), ... , $(x_{2n},y_{2n}) = (n,n)$. Since the path proceeds one step at a time, we have $x_i + y_i = i$. Hence $\sum (x_i + y_i) = \sum i = n(2n+1)$. So if $\sum x_i = \sum y_i$, then $\sum 2x_in = n(2n+1)$. (Note also that n must be even, although we do not use that.) Thus for a balanced path we have $2x_1 + 2x_2 + ... + 2x_{2n} = n(2n+1) = (2n+1)x_{2n}$. Hence $2x_1 + 2x_2 + ... + 2x_{2n-1} = (2n-1)x_{2n}$. Adding $x_2 + 2x_3 + 3x_4 + ... + (2n-2)x_{2n-1}$ to both sides we get $2x_1 + 3x_2 + 4x_3 + ... + 2nx_{2n-1} = x_2 + 2x_3 + 3x_4 + ... + (2n-1)x_{2n}$ or $\sum i\, x_{i-1} = \sum (i-1)x_i$.

Hence $\sum x_{i-1}(i - x_i) = \sum x_i(i-1 - x_{i-1})$ or $\sum x_{i-1}y_i = \sum x_iy_{i-1}$. Hence $\sum (x_i - x_{i-1})y_i = \sum x_i(y_i - y_{i-1})$. But it is easy to see that the lhs is the area under the path and the rhs is the area between the path and the y-axis, in other words the part of the large square that is above the path. So we have established that the path divides the large square into two parts of equal area.

5th Centromerican 2003

Problem A1

There are 2003 stones in a pile. Two players alternately select a positive divisor of the number of stones currently in the pile and remove that number of stones. The player who removes the last stone loses. Find a winning strategy for one of the players.

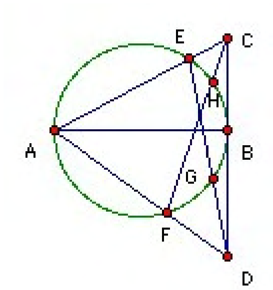
Solution

The second player has a winning strategy: he always takes 1 stone from the pile. One his first move the first player must take an odd number of stones, so leaving an even number. Now the second player always has an even number of stones in the pile and always leaves an odd number. The first player must always take an odd number and hence must leave an even number. Since 0 is not odd, the second player cannot lose.

Problem A2

AB is a diameter of a circle. C and D are points on the tangent at B on opposite sides of B. AC, AD meet the circle again at E, F respectively. CF, DE meet the circle again at G, H respectively. Show that AG = AH.

Solution



AEB, ABC are similar ($\angle A$ common and $\angle AEB = \angle ABC = 90^o$), so $AE \cdot AC = AB^2$. In the same way, AFB and ABD are similar, so $AF \cdot AD = AB^2$, so $AE \cdot AC = AF \cdot AD$. Hence CEFD is cyclic. So $\angle CED = \angle CFD$, in other words, $\angle AEH = \angle AFG$. Hence the corresponding chords are also equal, so AH = AG.

Problem A3

Given integers $a > 1$, $b > 2$, show that $a^b + 1 \ge b(a+1)$. When do we have equality?

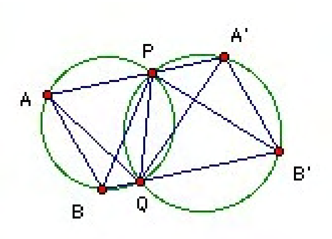
Solution

Induction on b. For b=3 we require $a^3 + 1 \ge 3a + 3$, or $(a-2)(a+1)^2 \ge 0$, which is true, with equality iff a = 2. Suppose the result is true for b. Then $ab+1 + 1 = a(ab + 1) - a + 1 \ge ab(a+1) - a + 1 = a(ab-1) +ab + 1 > a(2b-1) + ab + 1 > a(2b-1) + b + 1 > a(b+1) + b+1 = (a+1)(b+1)$, so the result is true, and a strict inequality, for b+1. Hence the result is true for all b>2 and the only case of equality is b=3, a=2.

Problem B1

Two circles meet at P and Q. A line through P meets the circles again at A and A'. A parallel line through Q meets the circles again at B and B'. Show that PBB' and QAA' have equal perimeters.

Solution



Since AP is parallel to BQ and APQB is inscribed in a circle we must have AQ = PB. Similarly, A'Q = PB'.

Since APQB is cyclic, ∠ABQ = ∠A'PQ. Since A'PQB' is cyclic, ∠A'PQ = 180° - ∠A'B'Q, so AB is parallel to AB. Hence AA'B'B is a parallelogram, so AA' = BB'. So the triangles are in fact congruent.

Problem B2

An 8 x 8 board is divided into unit squares. Each unit square is painted red or blue. Find the number of ways of doing this so that each 2 x 2 square (of four unit squares) has two red squares and two blue squares.

Answer

2^9 - 2.

Solution

We can choose the colors in the first column arbitrarily.

If the squares in the first column alternate in color, then there are 2 choices for the second column, either matching the first column, or opposite colors. Similarly, there are 2 choices for each of the remaining columns. There are 2 ways in which the squares in the first column can alternate in color, so we get 2^8 ways in all with alternating colors in each column.

If the squares in the first column do not alternate in color, then there must be two adjacent squares the same color. Hence the two squares adjacent to them in the second column are determined. Hence all the squares in the second column are determined. It also has two adjacent squares of the same color, so all the squares in the third column are determined, and so on. There are 2^8 ways of coloring the first column. 2 of these ways have alternating colors, so 2^8 - 2 have two adjacent squares the same.

Problem B3

Call a positive integer a *tico* if the sum of its digits (in base 10) is a multiple of 2003. Show that there is an integer N such that N, 2N, 3N, ... , 2003N are all ticos. Does there exist a positive integer such that all its multiples are ticos?

Answer

No.

Solution

Let A = 10001 0001 0001 ... 0001 (with 2003 1s). Then kA is just 2003 repeating groups for k ≤ 9999 and is therefore a tico.

Note that for any N relatively prime to 10 we have (by Euler) $10^{\varphi(9N)} = 1 \bmod 9N$, in other words, 9Nk = 9...9 ($\varphi(9N)$ 9s) for some k. Hence Nk is a repunit divisible by N. Now suppose all multiples of N are ticos. Take k so that Nk is a repunit. Suppose it has h 1s. So it has digit sum h. Then 19Nk = 211..109 with h-2 1s, so its digit sum is h + 9. But h and h+9 cannot both be multiples of 2003. Contradiction.