Inequalities proposed in

"Crux Mathematicorum"

(from vol. 1, no. 1 to vol. 4, no. 2 known as "Eureka")

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(An asterisk (★) after a number indicates that a problem was proposed without a solution.)

2. *Proposed by Léo Sauvé, Algonquin College.*
A rectangular array of m rows and n columns contains mn distinct real numbers. For $i = 1, 2, \ldots, m$, let s_i denote the smallest number of the i^{th} row; and for $j = 1, 2, \ldots, n$, let l_j denote the largest number of the j^{th} column. Let $A = \max\{s_i\}$ and $B = \min\{l_j\}$. Compare A and B.

14. *Proposed by Viktors Linis, University of Ottawa.*
If a, b, c are lengths of three segments which can form a triangle, show the same for $\frac{1}{a+c}$, $\frac{1}{b+c}$, $\frac{1}{a+b}$.

17. *Proposed by Viktors Linis, University of Ottawa.*
Prove the inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{999999}{1000000} < \frac{1}{1000}.$$

23. *Proposed by Léo Sauvé, Collège Algonquin.*
Déterminer s'il existe une suite $\{u_n\}$ d'entiers naturels telle que, pour $n = 1, 2, 3, \ldots$, on ait

$$2^{u_n} < 2n + 1 < 2^{1+u_n}$$

25. *Proposed by Viktors Linis, University of Ottawa.*
Find the smallest positive value of $36^k - 5^l$ where k and l are positive integers.

29. *Proposed by Viktors Linis, University of Ottawa.*
Cut a square into a minimal number of triangles with all angles acute.

36. *Proposed by Léo Sauvé, Collège Algonquin.*
Si m et n sont des entiers positifs, montrer que

$$\sin^{2m}\theta \cos^{2n}\theta \le \frac{m^m n^n}{(m+n)^{m+n}},$$

et dèterminer les valeurs de θ pour lesquelles il y a égalité.

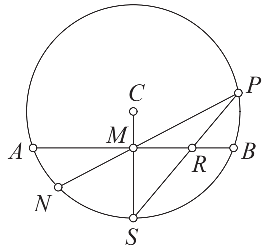
54. *Proposed by Léo Sauvé, Collège Algonquin.*
Si $a, b, c > 0$ et $a < b + c$, montrer que

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

66. *Proposed by John Thomas, University of Ottawa.*
What is the largest non-trivial subgroup of the group of permutations on n elements?

74. *Proposed by Viktors Linis, University of Ottawa.*
Prove that if the sides a, b, c of a triangle satisfy $a^2 + b^2 = kc^2$, then $k > \frac{1}{2}$.

75. *Proposed by R. Duff Butterill, Ottawa Board of Education.*
M is the midpoint of chord AB of the circle with centre C shown in the figure. Prove that $RS > MN$.



79. *Proposed by John Thomas, University of Ottawa.*
Show that, for $x > 0$,

$$\left|\int_x^{x+1} \sin(t^2)\,\mathrm{d}t\right| < \frac{2}{x^2}.$$

84. *Proposed by Viktors Linis, University of Ottawa.*
Prove that for any positive integer n

$$\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}}.$$

98. *Proposed by Viktors Linis, University of Ottawa.*
Prove that, if $0 < a < b$, then

$$\ln \frac{b^2}{a^2} < \frac{b}{a} - \frac{a}{b}.$$

100. *Proposed by Léo Sauvé, Collège Algonquin.*
Soit f une fonction numérique continue et non négative pour tout $x \geq 0$. On suppose qu'il existe un nombre réel $a > 0$ tel que, pout tout $x > 0$,

$$f(x) \leq a \int_0^x f(t)\,\mathrm{d}t.$$

Montrer que la fonction f est nulle.

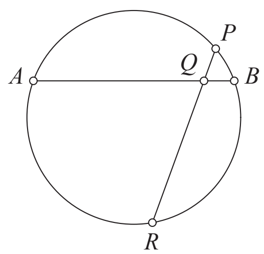
106. *Proposed by Viktors Linis, University of Ottawa.*
Prove that, for any quadrilateral with sides a, b, c, d,

$$a^2 + b^2 + c^2 > \frac{1}{3}d^2.$$

108. *Proposed by Viktors Linis, University of Ottawa.*
Prove that, for all integers $n \geq 2$,

$$\sum_{k=1}^{n} \frac{1}{k^2} > \frac{3n}{2n+1}.$$

110. *Proposed by H. G. Dworschak, Algonquin College.*
(a) Let AB and PR be two chords of a circle intersecting at Q. If A, B, and P are kept fixed, characterize geometrically the position of R for which the length of QR is maximal. (See figure).
(b) Give a Euclidean construction for the point R which maximizes the length of QR, or show that no such construction is possible.



115. *Proposed by Viktors Linis, University of Ottawa.*
Prove the following inequality of Huygens:

$$2\sin\alpha + \tan\alpha \geq 3\alpha, \quad 0 \leq \alpha < \frac{\pi}{2}.$$

119. *Proposed by John A. Tierney, United States Naval Academy.*
A line through the first quadrant point (a, b) forms a right triangle with the positive coordinate axes. Find analytically the minimum perimeter of the triangle.

120. *Proposed by John A. Tierney, United States Naval Academy.*
Given a point P inside an arbitrary angle, give a Euclidean construction of the line through P that determines with the sides of the angle a triangle
(a) of minimum area;
(b) of minimum perimeter.

135. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H. S., Bayside, N. Y.*
How many 3×5 rectangular pieces of cardboard can be cut from a 17×22 rectangular piece of cardboard so that the amount of waste is a minimum?

145. *Proposed by Walter Bluger, Department of National Health and Welfare.*
A *pentagram* is a set of 10 points consisting of the vertices and the intersections of the diagonals of a regular pentagon with an integer assigned to each point. The pentagram is said to be *magic* if the sums of all sets of 4 collinear points are equal.
Construct a magic pentagram with the smallest possible positive primes.

150. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.*
If $\lfloor x\rfloor$ denotes the greatest integer $\le x$, it is trivially true that

$$\left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor > \frac{3^k - 2^k}{2^k} \quad \text{for } k \ge 1,$$

and it seems to be a hard conjecture (see G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Oxford University Press 1960, p. 337, condition (f)) that

$$\left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor \ge \frac{3^k - 2^k + 2}{2^k - 1} \quad \text{for } k \ge 4.$$

Can one find a function $f(k)$ such that

$$\left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor \ge f(k)$$

with $f(k)$ between $\frac{3^k-2^k}{2^k}$ and $\frac{3^k-2^k+2}{2^k-1}$?

160. *Proposed by Viktors Linis, University of Ottawa.*
Find the integral part of $\sum\limits_{n=1}^{10^9} n^{-\frac{2}{3}}$.
This problem is taken from the list submitted for the 1975 Canadian Mathematics Olympiad (but not used on the actual exam).

162. *Proposed by Viktors Linis, University of Ottawa.*
If $x_0 = 5$ and $x_{n+1} = x_n + \frac{1}{x_n}$, show that

$$45 < x_{1000} < 45.1.$$

This problem is taken from the list submitted for the 1975 Canadian Mathematics Olympiad (but not used on the actual exam).

165. *Proposed by Dan Eustice, The Ohio State University.*
Prove that, for each choice of n points in the plane (at least two distinct), there exists a point on the unit circle such that the product of the distances from the point to the chosen points is greater than one.

167. *Proposed by Léo Sauvé, Algonquin College.*
The first half of the Snellius-Huygens double inequality

$$\frac{1}{3}(2\sin\alpha + \tan\alpha) > \alpha > \frac{3\sin\alpha}{2+\cos\alpha}, \quad 0 < \alpha < \frac{\pi}{2},$$

was proved in Problem 115. Prove the second half in a way that could have been understood before the invention of calculus.

173. *Proposed by Dan Eustice, The Ohio State University.*
For each choice of n points on the unit circle ($n \geq 2$), there exists a point on the unit circle such that the product of the distances to the chosen points is greater than or equal to 2. Moreover, the product is 2 if and only if the n points are the vertices of a regular polygon.

179. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H. S., Bayside, N. Y.*
The equation $5x + 7y = c$ has exactly three solutions (x, y) in positive integers. Find the largest possible value of c.

207. *Proposed by Ross Honsberger, University of Waterloo.*
Prove that $\frac{2r+5}{r+2}$ is always a better approximation of $\sqrt{5}$ than r.

219. *Proposed by R. Robinson Rowe, Sacramento, California.*
Find the least integer N which satisfies

$$N = a^{a+2b} = b^{b+2a}, \quad a \neq b.$$

223. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H. S., Bayside, N. Y.*
Without using any table which lists Pythagorean triples, find the smallest integer which can represent the area of two noncongruent primitive Pythagorean triangles.

229. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*
On an examination, one question asked for the largest angle of the triangle with sides 21, 41, and 50. A student obtained the correct answer as follows: Let C denote the desired angle; then $\sin C = \frac{50}{41} = 1 + \frac{9}{41}$. But $\sin 90^\circ = 1$ and $\frac{9}{41} = \sin 12^\circ 40' 49''$. Thus

$$C = 90^\circ + 12^\circ 40' 49'' = 102^\circ 40' 49'',$$

which is correct. Find the triangle of least area having integral sides and possessing this property.

230. *Proposed by R. Robinson Rowe, Sacramento, California.*
Find the least integer N which satisfies

$$N = a^{ma+nb} = b^{mb+na}$$

with m and n positive and $1 < a < b$. (This generalizes Problem 219.)

247★. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.*
On page 215 of *Analytic Inequalities* by D. S. Mitrinović, the following inequality is given: if $0 < b \le a$ then

$$\frac{1}{8}\frac{(a-b)^2}{a} \le \frac{a+b}{2} - \sqrt{ab} \le \frac{1}{8}\frac{(a-b)^2}{b}.$$

Can this be generalized to the following form: if $0 < a_1 \le a_2 \le \cdots \le a_n$ then

$$k\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_n} \le \frac{a_1+\cdots+a_n}{n} - \sqrt[n]{a_1\cdots a_n} \le k\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_1},$$

where k is a constant?

280. *Proposed by L. F. Meyers, The Ohio State University.*
A jukebox has N buttons.
(a) If the set of N buttons is subdivided into disjoint subsets, and a customer is required to press exactly one button from each subset in order to make a selection, what is the distribution of buttons which gives the maximum possible number of different selections?
(b) What choice of n will allow the greatest number of selections if a customer, in making a selection, may press any n distinct buttons out of the N? How many selections are possible then?
(Many jukeboxes have 30 buttons, subdivided into 20 and 10. The answer to part (a) would then be 200 selections.)

282. *Proposed by Erwin Just and Sidney Penner, Bronx Community College.*
On a 6×6 board we place 3×1 trominoes (each tromino covering exactly three unit squares of the board) until no more trominoes can be accommodated. What is the maximum number of squares that can be left vecant?

289. *Proposed by L. F. Meyers, The Ohio State University.*
Derive the laws of reflection and refraction from the principle of least time without use of calculus or its equivalent. Specifically, let L be a straight line, and let A and B be points not on L. Let the speed of light on the side of L on which A lies be c_1, and let the speed of light on the other side of L be c_2. Characterize the points C on L for which the time taken for the route ACB is smallest, if
(a) A and B are on the same side of L (reflection);
(b) A and B are on opposite sides of L (refraction).

295. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*
If $0 < b \le a$, prove that

$$a+b-2\sqrt{ab} \ge \frac{1}{2}\frac{(a-b)^2}{a+b}.$$

303. *Proposed by Viktors Linis, University of Ottawa.*
Huygens' inequality $2\sin\alpha + \tan\alpha \ge 3\alpha$ was proved in Problem 115. Prove the following hyperbolic analogue:

$$2\sinh x + \tanh x \ge 3x, \quad x \ge 0.$$

304. *Proposed by Viktors Linis, University of Ottawa.*
Prove the following inequality:

$$\frac{\ln x}{x-1} \le \frac{1+\sqrt[3]{x}}{x+\sqrt[3]{x}}, \quad x>0,\ x\ne 1.$$

306. *Proposed by Irwin Kaufman, South Shore H. S., Brooklyn, N. Y.*
Solve the following inequality, which was given to me by a student:

$$\sin x \sin 3x > \frac{1}{4}.$$

307. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H. S., Bayside, N. Y.*
It was shown in Problem 153 that the equation $ab = a + b$ has only one solution in positive integers, namely $(a, b) = (2, 2)$. Find the least and greatest values of x (or y) such that

$$xy = nx + ny,$$

if n, x, y are all positive integers.

310. *Proposed by Jack Garfunkel, Forest Hills H. S., Flushing, N. Y.*
Prove that

$$\frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{9a^2+b^2}} + \frac{2ab}{\sqrt{a^2+b^2}\cdot\sqrt{9a^2+b^2}} \le \frac{3}{2}.$$

When is equality attained?

318. *Proposed by C. A. Davis in James Cook Mathematical Notes No. 12 (Oct. 1977), p. 6.*
Given any triangle ABC, thinking of it as in the complex plane, two points L and N may be defined as the stationary values of a cubic that vanishes at the vertices A, B, and C. Prove that L and N are the foci of the ellipse that touches the sides of the triangle at their midpoints, which is the inscribed ellipse of maximal area.

323. *Proposed by Jack Garfunkel, Forest Hills H. S., Flushing, N. Y., and Murray S. Klamkin, University of Alberta.*
If $xyz = (1-x)(1-y)(1-z)$ where $0 \le x, y, z \le 1$, show that

$$x(1-z) + y(1-x) + z(1-y) \ge \frac{3}{4}.$$

344. *Proposed by Viktors Linis, University of Ottawa.*
Given is a set S of n positive numbers. With each nonempty subset P of S, we associate the number

$$\sigma(P) = \text{sum of all its elements.}$$

Show that the set $\{\sigma(P) \mid P \subseteq S\}$ can be partitioned into n subsets such that in each subset the ratio of the largest element to the smallest is at most 2.

347. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum value of

$$\sqrt[3]{4 - 3x + \sqrt{16 - 24x + 9x^2 - x^3}} + \sqrt[3]{4 - 3x - \sqrt{16 - 24x + 9x^2 - x^3}}$$

in the interval $-1 \le x \le 1$.

358. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum of x^2y, subject to the constraints

$$x + y + \sqrt{2x^2 + 2xy + 3y^2} = k \;\; (\text{constant}), \quad x, y \ge 0.$$

362. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.*
In *Crux* 247 [1977: 131; 1978: 23, 37] the following inequality is proved:

$$\frac{1}{2n^2}\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_n}\le\frac{a_1+\cdots+a_n}{n}-\sqrt[n]{a_1\cdots a_n}\le\frac{1}{2n^2}\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_1}.$$

Prove that the constant $\frac{1}{2n^2}$ is best possible.

367★. *Proposed by Viktors Linis, University of Ottawa.*
(a) A closed polygonal curve lies on the surface of a cube with edge of length 1. If the curve intersects every face of the cube, show that the length of the curve is at least $3\sqrt{2}$.
(b) Formulate and prove similar theorems about (i) a rectangular parallelepiped, (ii) a regular tetrahedron.

375. *Proposed by Murray S. Klamkin, University of Alberta.*
A convex n-gon P of cardboard is such that if lines are drawn parallel to all the sides at distances x from them so as to form within P another polygon P', then P' is similar to P. Now let the corresponding consecutive vertices of P and P' be $A_1, A_2, \ldots, A_n$ and $A_1', A_2', \ldots, A_n'$, respectively. From A_2', perpendiculars $A_2'B_1$, $A_2'B_2$ are drawn to A_1A_2, A_2A_3, respectively, and the quadrilateral $A_2'B_1A_2B_2$ is cut away. Then quadrilaterals formed in a similar way are cut away from all the other corners. The remainder is folded along $A_1'A_2', A_2'A_3', \ldots, A_n'A_1'$ so as to form an open polygonal box of base $A_1'A_2'\ldots A_n'$ and of height x. Determine the maximum volume of the box and the corresponding value of x.

394. *Proposed by Harry D. Ruderman, Hunter College Campus School, New York.*
A wine glass has the shape of an isosceles trapezoid rotated about its axis of symmetry. If R, r, and h are the measures of the larger radius, smaller radius, and altitude of the trapezoid, find $r : R : h$ for the most economical dimensions.

395★. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.*
In *Crux* 247 [1977: 131; 1978: 23, 37] the following inequality is proved:

$$\frac{1}{2n^2}\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_n}\le A-G\le\frac{1}{2n^2}\frac{\sum\limits_{1\le i<j\le n}(a_i-a_j)^2}{a_1},$$

where A (resp. G) is the arithmetic (resp. geometric) mean of $a_1, \ldots, a_n$. This is a refinement of the familiar inequality $A \ge G$. If H denotes the harmonic mean of $a_1, \ldots, a_n$, that is,

$$\frac{1}{H}=\frac{1}{n}\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right),$$

find the corresponding refinement of the familiar inequality $G \ge H$.

397. *Proposed by Jack Garfunkel, Forest Hills H. S., Flushing, N. Y.*
Given is $\triangle ABC$ with incenter I. Lines AI, BI, CI are drawn to meet the incircle (I) for the first time in D, E, F, respectively. Prove that

$$(AD+BE+CF)\sqrt{3}$$

is not less than the perimeter of the triangle of maximum perimeter that can be inscribed in circle (I).

402. *Proposed by the late R. Robinson Rowe, Sacramento, California.*
An army with an initial strength of A men is exactly decimeted each day of a 5-day battle and reinforced each night wirh R men from the reserve pool of P men, winding up on the morning of the 6th day with 60 % of its initial strength. At least how large must the initial strength have been if
(a) R was a constant number each day;
(b) R was exactly half the men available in the dwindling pool?

404. *Proposed by Andy Liu, University of Alberta.*
Let A be a set of n distinct positive numbers. Prove that
(a) the number of distinct sums of subsets of A is at least $\frac{1}{2}n(n+1)+1$;
(b) the number of distinct subsets of A with equal sum to half the sum of A is at most $\frac{2^n}{n+1}$.

405. *Proposed by Viktors Linis, University of Ottawa.*
A circle of radius 16 contains 650 points. Prove that there exists an annulus of inner radius 2 and outer radius 3 which contains at least 10 of the given points.

413. *Proposed by G. C. Giri, Research Scholar, Indian Institute of Technology, Kharagpur, India.*
If $a, b, c > 0$, prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{a^8+b^8+c^8}{a^3b^3c^3}.$$

417. *Proposed by John A. Tierney, U. S. Naval Academy, Annapolis, Maryland.*
It is easy to guess from the graph of the folium os Descartes,

$$x^3+y^3-3axy=0, \quad a>0$$

that the point of maximum curvature is $\left(\frac{3a}{2}, \frac{3a}{2}\right)$. Prove it.

423. *Proposed by Jack Garfunkel, Forest Hills H. S., Flushing, N. Y.*
In a triangle ABC whose circumcircle has unit diameter, let m_a and t_a denote the lengths of the median and the internal angle bisector to side a, respectively. Prove that

$$t_a \leq \cos^2\frac{A}{2}\cos\frac{B-C}{2} \leq m_a.$$

427. *Proposed by G. P. Henderson, Campbellcroft, Ontario.*
A corridor of width a intersects a corridor of width b to form an "L". A rectangular plate is to be taken along one corridor, around the corner and along the other corridor with the plate being kept in a horizontal plane. Among all the plates for which this is possible, find those of maximum area.

429. *Proposed by M. S. Klamkin and A. Liu, both from the University of Alberta.*
On a $2n\times 2n$ board we place $n\times 1$ polyominoes (each covering exactly n unit squares of the board) until no more $n\times 1$ polyominoes can be accomodated. What is the number of squares that can be left vacant?
This problem generalizes *Crux* 282 [1978: 114].

440★. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.*
My favourite proof of the well-known result

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

uses the identity

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n+1} = \frac{n(2n-1)}{3}$$

and the inequality

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x, \quad 0 < x < \frac{\pi}{2}$$

to obtain

$$\frac{\pi^2}{(2n+1)^2} \cdot \frac{n(2n-1)}{3} < \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{(2n+1)^2} \left[n + \frac{n(2n-1)}{3} \right],$$

from which the desired result follows upon letting $n \to \infty$.
Can any reader find a new elementary prrof simpler than the above? (Many references to this problem are given by E. L. Stark in *Mathematics Magazine*, 47 (1974) 197–202.)

450★. *Proposed by Andy Liu, University of Alberta.*
Triangle ABC has a fixed base BC and a fixed inradius. Describe the locus of A as the incircle rools along BC. When is AB of minimal length (geometric characterization desired)?

458. *Proposed by Harold N. Shapiro, Courant Institute of Mathematical Sciences, New York University.*
Let $\phi(n)$ denote the Euler function. It is well known that, for each fixed integer $c > 1$, the equation $\phi(n) = n - c$ has at most a finite number of solutions for the integer n. Improve this by showing that any such solution, n, must satisfy the inequalities $c < n \le c^2$.

459. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*
If n is a positive integer, prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \le \frac{\pi^2}{8} \cdot \frac{1}{1 - 2^{-2n}}.$$

468. *Proposed by Viktors Linis, University of Ottawa.*
(a) Prove that the equation

$$a_1 x^{k_1} + a_2 x^{k_2} + \cdots + a_n x^{k_n} - 1 = 0,$$

where $a_1, \ldots, a_n$ are real and $k_1, \ldots, k_n$ are natural numbers, has at most n positive roots.
(b) Prove that the equation

$$ax^k (x+1)^p + bx^l (x+1)^q + cx^m (x+1)^r - 1 = 0,$$

where a, b, c are real and k, l, m, p, q, r are natural numbers, has at most 14 positive roots.

484. *Proposed by Gali Salvatore, Perkins, Québec.*
Let A and B be two independent events in a sample space, and let χ_A, χ_B be their characteristic functions (so that, for example, $\chi_A(x) = 1$ or 0 according as $x \in A$ or $x \notin A$). If $F = \chi_A + \chi_B$, show that at least one of the three numbers $a = P(F = 2)$, $b = P(F = 1)$, $c = P(F = 0)$ is not less than $\frac{4}{9}$.

487. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*
If a, b, c and d are positive real numbers such that $c^2 + d^2 = (a^2 + b^2)^3$, prove that

$$\frac{a^3}{c} + \frac{b^3}{d} \geq 1,$$

with equality if and only if $ad = bc$.

488★. *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*
Given a point P within a given angle, construct a line through P such that the segment intercepted by the sides of the angle has minimum length.

492. *Proposed by Dan Pedoe, University of Minnesota.*
(a) A segment AB and a rusty compass of span $r > \frac{1}{2}AB$ are given. Show how to find the vertex C of an equilateral triangle ABC using, as few times as possible, the rusty compass only.
(b)★ Is the construction possible when $r < \frac{1}{2}AB$?

493. *Proposed by Robert C. Lyness, Southwold, Suffolk, England.*
(a) A, B, C are the angles of a triangle. Prove that there are positive x, y, z, each less than $\frac{1}{2}$, simultaneously satisfying

$$y^2 \cot \frac{B}{2} + 2yz + z^2 \cot \frac{C}{2} = \sin A,$$
$$z^2 \cot \frac{C}{2} + 2zx + x^2 \cot \frac{A}{2} = \sin B,$$
$$x^2 \cot \frac{A}{2} + 2xy + y^2 \cot \frac{B}{2} = \sin C.$$

(b)★ In fact, $\frac{1}{2}$ may be replaced by a smaller $k > 0.4$. What is the least value of k?

495. *Proposed by J. L. Brenner, Palo Alto, California; and Carl Hurd, Pennsylvania State University, Altoona Campus.*
Let S be the set of lattice points (points having integral coordinates) contained ina bounded convex set in the plane. Denote by N the minimum of two measurements of S: the greatest number of points of S on any line of slope $1, -1$. Two lattice points are *adjoining* if they are exactly one unit apart. Let the n points of S be numbered by the integers from 1 to n in such a way that the largest difference of the assigned integers of adjoining points is minimal. This minimal largest difference we call the *discrepancy* of S.
(a) Show that the discrepancy of S is no greater than $N + 1$.
(b) Give such a set S whose discrepancy is $N + 1$.
(c)★ Show that the discrepancy of S is no less than N.

505. *Proposed by Bruce King, Western Connecticut State College and Sidney Penner, Bronx Community College.*
Let $F_1 = F_2 = 1$, $F_n = F_n = F_{n-1} + F_{n-2}$ for $n > 2$ and $G_1 = 1$, $G_n = 2^{n-1} - G_{n-1}$ for $n > 1$. Show that (a) $F_n \leq G_n$ for each n and (b) $\lim\limits_{n \to \infty} \frac{F_n}{G_n} = 0$.

506. *Proposed by Murray S. Klamkin, University of Alberta.*
It is known from an earlier problem in this journal [1975: 28] that if a, b, c are the sides of a triangle, then so are $1/(b+c)$, $1/(c+a)$, $1/(a+b)$. Show more generally that if $a_1, a_2, \ldots, a_n$ are the sides of a polygon then, for $k = 1, 2, \ldots, n$,

$$\frac{n+1}{S-a_k} \geq \sum_{\substack{i=1\\ i\neq k}} \frac{1}{S-a_i} \geq \frac{(n-1)^2}{(2n-3)(S-a_k)},$$

where $S = a_1 + a_2 + \cdots + a_n$.

517★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Given is a triangle ABC with altitudes h_a, h_b, h_c and medians m_a, m_b, m_c to sides a, b, c, respectively. Prove that

$$\frac{h_b}{m_c} + \frac{h_c}{m_a} + \frac{h_a}{m_b} \leq 3,$$

with equality if and only if the triangle is equilateral.

529. *Proposed by J. T. Groenman, Groningen, The Netherlands.*
The sides of a triangle ABC satisfy $a \leq b \leq c$. With the usual notation r, R, and r_c for the in-, circum-, and ex-radii, prove that

$$\operatorname{sgn}(2r + 2R - a - b) = \operatorname{sgn}(2r_c - 2R - a - b) = \operatorname{sgn}(C - 90^\circ).$$

535. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Given a triangle ABC with sides a, b, c, let T_a, T_b, T_c denote the angle bisectors extended to the circumcircle of the triangle. Prove that

$$T_aT_bT_c \geq \frac{8}{9}\sqrt{3}abc,$$

with equality attained in the equilateral triangle.

544. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*
Prove that, in any triangle ABC,

$$2\left(\sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{A}{2} + \sin\frac{A}{2}\sin\frac{B}{2}\right) \leq \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2},$$

with equality if and only if the triangle is equilateral.

552. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*
Given positive constants a, b, c and nonnegative real variables x, y, z subject to the constraint $x + y + z = \pi$, find the maximum value of

$$f(x, y, z) \equiv a\cos x + b\cos y + c\cos z.$$

563. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*
For n a positive integer, let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be two permutations (not necessarily distinct) of $(1, 2, \ldots, n)$. Find sharp upper and lower bounds for

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

570. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*
If $x, y, z > 0$, show that

$$\sum_{\text{cyclic}} \frac{2x^2(y+z)}{(x+y)(x+z)} \le x+y+z,$$

with equality if and only if $x = y = z$.

572★. *Proposed by Paul Erdös, Technion – I.I.T., Haifa, Israel.*
It was proved in *Crux* 458 [1980: 157] that, if ϕ is the Euler function and the integer $c > 1$, then each solution n of the equation

$$\phi(n) = n - c \tag{1}$$

satisfies $c+1 \le n \le c^2$. Let $F(c)$ be the *number* of solutions of (1). Estimate $F(c)$ as well as you can from above and below.

583. *Proposed by Charles W. Trigg, San Diego, California.*
A man, being asked the ages of his two sons, replied: "Each of their ages is one more than three times the sum of its digits." How old is each son?

585. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Consider the following three inequalities for the angles A, B, C of a triangle:

$$\cos\frac{B-C}{2}\cos\frac{C-A}{2}\cos\frac{A-B}{2} \ge 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \tag{1}$$

$$\csc\frac{A}{2}\cos\frac{B-C}{2} + \csc\frac{B}{2}\cos\frac{C-A}{2} + \csc\frac{C}{2}\cos\frac{A-B}{2} \ge 6, \tag{2}$$

$$\csc\frac{A}{2} + \csc\frac{B}{2} + \csc\frac{C}{2} \ge 6.$$

Inequality (3) is well-known (*American Mathematical Monthly* 66 (1959) 916) and it is trivially implied by (2). Prove (1) and show that (1) implies (2).

589. *Proposed by Ngo Tan, student, J. F. Kennedy H. S., Bronx, N. Y..*
In a triangle ABC with semiperimeter s, sides of lengths a, b, c, and medians of lengths m_a, m_b, m_c, prove that:

(a) There exists a triangle with sides of lengths $a(s-a)$, $b(s-b)$, $c(s-c)$.

(b) $\left(\frac{m_a}{a}\right)^2 + \left(\frac{m_b}{b}\right)^2 + \left(\frac{m_c}{c}\right)^2 \ge \frac{9}{4}$, with equality if and only if the triangle is equilateral.

602. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Given are twenty natural numbers a_i such that

$$0 < a_1 < a_2 < \cdots < a_{20} < 70.$$

Show that at least one of the differences $a_i - a_j$, $i > j$, occurs at least four times. (A student proposed this problem to me. I don't know the source.)

606★. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $\sigma_n = A_0A_1\dots A_n$ be an n-simplex in Euclidean space $\mathbb{R}^n$ and let $\sigma_n' = A_0'A_1'\dots A_n'$ be an n-simplex similar to and inscribed in σ_n, and labeled in such a way that

$$A_i' \in \sigma_{n-1} = A_0A_1\dots A_{i-1}A_{i+1}\dots A_n, \quad i = 0, 1, \dots, n.$$

Prove that the ratio of similarity

$$\lambda \equiv \frac{A_i'A_j'}{A_iA_j} \ge \frac{1}{n}.$$

[If no proof of the general case is forthcoming, the editor hopes to receive a proof at least for the special case $n = 2$.]

608. *Proposed by Ngo Tan, student, J. F. Kennedy H. S., Bronx, N. Y..*
ABC is a triangle with sides of lengths a, b, c and semiperimeter s. Prove that

$$\cos^4\frac{A}{2} + \cos^4\frac{B}{2} + \cos^4\frac{C}{2} \le \frac{s^3}{2abc},$$

with equality if and only if the triangle is equilateral.

613. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If $A + B + C = 180^\circ$, prove that

$$\cos\frac{B-C}{2} + \cos\frac{C-A}{2} + \cos\frac{A-B}{2} \ge \frac{2}{\sqrt{3}}(\sin A + \sin B + \sin C).$$

(Here A, B, C are not necessarily the angles of a triangle, but you may assume that they are if it is helpful to achieve a proof without calculus.)

615. *Proposed by G. P. Henderson, Campbellcroft, Ontario.*
Let P be a convex n-gon with vertices $E_1, E_2, \dots, E_n$, perimeter L and area A. Let $2\theta_i$ be the measure of the interior angle at vertex E_i and set $C = \sum \cot\theta_i$. Prove that

$$L^2 - 4AC \ge 0$$

and characterize the convex n-gons for which equality holds.

623★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If PQR is the equilateral triangle of smallest area inscribed in a given triangle ABC, with P on BC, Q on CA, and R on AB, prove or disprove that AP, BQ, and CR are concurrent.

624. *Proposed by Dmitry P. Mavlo, Moscow, U. S. S. R.*
ABC is a given triangle of area K, and PQR is the equilateral triangle of smallest area K_0 inscribed in triangle ABC, with P on BC, Q on CA, and R on AB.
(a) Find ratio

$$\lambda = \frac{K}{K_0} \equiv f(A, B, C)$$

as a function of the angles of the given triangle.
(b) Prove that λ attains its minimum value when the given triangle ABC is equilateral.
(c) Give a Euclidean construction of triangle PQR for an arbitrary given triangle ABC.

626. *Proposed by Andy Liu, University of Alberta.*
A (ν, b, r, k, λ)-configuration on a set with ν elements is a collection of b k-subsets such that
(i) each element appears in exactly r of the k-subsets;
(ii) each pair of elements appears in exactly λ of the k-subsets.
Prove that $k^r \geq \nu^\lambda$ and determine the value of b when equality holds.

627. *Proposed by F. David Hammer, Santa Cruz, California.*
Consider the double inequality

$$6 < 3^{\sqrt{3}} < 7.$$

Using *only* the elementary properties of exponents and inequalities (no calculator, computer, table of logarithms, or estimate of $\sqrt{3}$ may be used), prove that the first inequality implies the second.

628. *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*
Given a triangle ABC with sides a, b, c, let T_a, T_b, T_c denote the angle bisectors extended to the circumcircle of the triangle. If R and r are the circum- and in-radii of the triangle, prove that

$$T_a + T_b + T_c \leq 5R + 2r,$$

with equality just when the triangle is equilateral.

644. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If I is the incenter of triangle ABC and lines AI, BI, CI meet the circumcircle of the triangle again in D, E, F, respectively, prove that

$$\frac{AI}{ID} + \frac{BI}{IE} + \frac{CI}{IF} \geq 3.$$

648. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Given a triangle ABC, its centroid G, and the pedal triangle PQR of its incenter I. The segments AI, BI, CI meet the incircle in U, V, W; and the segments AG, BG, CG meet the incircle in D, E, F. Let ∂ denote the perimeter of a triangle and consider the statement

$$\partial PRQ \leq \partial UVW \leq \partial DEF.$$

(a) Prove the first inequality.
(b)* Prove the second inequality.

650. *Proposed by Paul R. Beesack, Carleton University, Ottawa.*
(a) Two circular cylinders of radii r and R, where $0 < r \leq R$, intersect at right angles (i. e., their central axes intersect at an angle of $\frac{\pi}{2}$). Find the arc length l of one of the two curves of intersection, as a definite integral.
(b) Do the same problem if the cylinders intersect at an angle γ, where $0 < \gamma < \frac{\pi}{2}$.
(c) Show the the arc length l in (a) satisfies

$$l \leq 4r \int_0^{\pi/2} \sqrt{1 + \cos^2\theta}\, \mathrm{d}\theta < \frac{5\pi r}{2}.$$

653. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
For every triangle ABC, show that

$$\sum \cos^2 \frac{B-C}{2} \geq 24 \prod \sin \frac{A}{2},$$

where the sum and product are cyclic over A, B, C, with equality if and only if the triangle is equilateral.

655. *Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, China.*
If $0 < a, b, c, d < 1$, prove that

$$\left(\sum a\right)^3 > 4bcd \sum a + 8a^2bcd \sum \left(\frac{1}{a}\right),$$

where the sums are cyclic over a, b, c, d.

656. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
P is an interior point of a convex region R bounded by the arcs of two intersecting circles C_1 and C_2. Construct through P a "chord" UV of R, with U on C_1 and V on C_2, such that $|PU| \cdot |PV|$ is a minimum.

664. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
An isosceles trapezoid $ABCD$, with parallel bases AB and DC, is inscribed in a circle of diameter AB. Prove that

$$AC > \frac{AB + DC}{2}.$$

665. *Proposed by Jack Garfunkel, Queens College, Flushing, N. Y.*
If A, B, C, D are the interior angles of a convex quadrilateral $ABCD$, prove that

$$\sqrt{2} \sum \cos \frac{A+B}{4} \leq \sum \cot \frac{A}{2}$$

(where the four-term sum on each side is cyclic over A, B, C, D), with equality if and only if $ABCD$ is a rectangle.

673★. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus, Middletown, Pennsylvania.*
Determine for which positive integers n the following property holds: if m is any integer satisfying

$$\frac{n(n+1)(n+2)}{6} \leq m \leq \frac{n(n+1)(2n+1)}{6},$$

then there exist permutations $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ of $(1, 2, \ldots, n)$ such that

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n = m.$$

(See *Crux* 563 [1981: 208].)

682. *Proposed by Robert C. Lyness, Southwold, Suffolk, England.*
Triangle ABC is acute-angled and Δ_1 is its orthic triangle (its vertices are the feet of the altitudes of triangle ABC). Δ_2 is the triangular hull of the three excircles of triangle ABC (that is, its sides are external common tangents of the three pairs of excircles that are not sides of triangle ABC). Prove that the area of triangle Δ_2 is at least 100 times the area of triangle Δ_1.

683. *Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, China.*
Triangle ABC has $AB > AC$, and the internal bisector of angle A meets BC at T. Let P be any point other than T on *line* AT, and suppose lines BP, CP intersect lines AC, AB in D, E, respectively. Prove that $BD > CE$ or $BD < CE$ according as P lies on the same side or on the opposite side of BC as A.

684. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let O be the origin of the lattice plane, and let $M(p,q)$ be a lattice point with relatively prime positive coordinates (with $q > 1$). For $i = 1, 2, \ldots, q-1$, let P_i and Q_i be the lattice points, both with ordinate i, that are respectively the left and right endpoints of the horizontal unit segment intersecting OM. Finally, let $P_iQ_i \cap OM = M_i$.

(a) Calculate $S_1 = \displaystyle\sum_{i=1}^{q-1} \overline{P_iM_i}$.

(b) Find the minimum value of $\overline{P_iM_i}$ for $1 \le i \le q-1$.

(c) Show that $\overline{P_sM_s} + \overline{P_{q-s}M_{q-s}} = 1, \;\; 1 \le s \le q-1$.

(d) Calculate $S_2 = \displaystyle\sum_{i?1}^{q-1} \frac{\overline{P_iM_i}}{\overline{M_iQ_i}}$.

685. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Given is a triangle ABC with internal angle bisectors t_a, t_b, t_c meeting a, b, c in U, V, W, respectively; and medians m_a, m_b, m_c meeting a, b, c in L, M, N, respectively. Let

$$m_a \cap t_b = P, \qquad m_b \cap t_c = Q, \qquad m_c \cap t_a = R.$$

Crux 588 [1980: 317] asks for a proof of the equality

$$\frac{AP}{PL} \cdot \frac{BQ}{QM} \cdot \frac{CR}{RN} = 8.$$

Establish here the inequality

$$\frac{AR}{RU} \cdot \frac{BP}{PV} \cdot \frac{CQ}{QW} \ge 8,$$

with equality if and only if the triangle is equilateral.

689. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Let m_a, m_b, m_c denote the lengths of the medians to sides a, b, c, respectively, of triangle ABC, and let M_a, M_b, M_c denote the lengths of these medians extended to the circumcircle of the triangle. Prove that

$$\frac{M_a}{m_a} + \frac{M_b}{m_b} + \frac{M_c}{m_c} \ge 4.$$

696. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle; a, b, c its sides; and s, r, R its semiperimeter, inradius and circumradius. Prove that, with sums cyclic over A, B, C,

(a) $\displaystyle\frac{3}{4} + \frac{1}{4}\sum \cos\frac{B-C}{2} \ge \sum \cos A$;

(b) $\displaystyle\sum a \cos\frac{B-C}{2} \ge s\left(1 + \frac{2r}{R}\right)$.

697. *Proposed by G. C. Giri, Midnapore College, West Bengal, India.*
Let

$$a = \tan\theta + \tan\phi, \qquad b = \sec\theta + \sec\phi, \qquad c = \csc\theta + \csc\phi.$$

If the angles θ and ϕ such that the requisite functions are defined and $bc \neq 0$, show that $2a/bc < 1$.

700. *Proposed by Jordi Dou, Barcelona, Spain.*
Construct the centre of the ellipse of minimum excentricity circumscribed to a given convex quadrilateral.

706. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let $F(x) = 7x^{11} + 11x^7 + 10ax$, where x ranges over the set of all integers. Find the smallest positive integer a such that $77|F(x)$ for every x.

708. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
A triangle has sides a, b, c, semiperimeter s, inradius r, and circumradius R.
(a) Prove that

$$(2a-s)(b-c)^2 + (2b-s)(c-a)^2 + (2c-s)(a-b)^2 \geq 0,$$

with equality just when the triangle is equilateral.
(b) Prove that the inequality in (a) is equivalent to each of the following:

$$3(a^3+b^3+c^3+3abc) \leq 4s(a^2+b^2+c^2),$$
$$s^2 \geq 16Rr - 5r^2.$$

715. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
Let k be a real number, n an integer, and A, B, C the angles of a triangle.
(a) Prove that

$$8k(\sin nA + \sin nB + \sin nC) \leq 12k^2 + 9.$$

(b) Determine for which k equality is possible in (a), and deduce that

$$|\sin nA + \sin nB + \sin nC| \leq \frac{3\sqrt{3}}{2}.$$

718. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
ABC is an acute-angled triangle with circumcenter O. The lines AO, BO, CO intersect BC, CA, AB in A_1, B_1, C_1, respectively. Show that

$$OA_1 + OB_1 + OC_1 \geq \frac{3}{2}R,$$

where R is the circumradius.

723. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let G be the centroid of a triangle ABC, and suppose that AG, BG, CG meet the circumcircle of the triangle again in A', B', C', respectively. Prove that

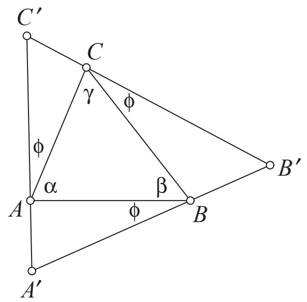
(a) $GA' + GB' + GC' \geq AG + BG + CG$;

(b) $\dfrac{AG}{GA'} + \dfrac{BG}{GB'} + \dfrac{CG}{GC'} = 3$;

(c) $GA' \cdot GB' \cdot GC' \geq AG \cdot BG \cdot CG$.

729. *Proposed jointly by Dick Katz and Dan Sokolowsky, California State University at Los Angeles.*
Given a unit square, let K be the area of a triangle which covers the square. Prove that $K \geq 2$.

732. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Given is a fixed triangle ABC with angles α, β, γ and a variable circumscribed triangle $A'B'C'$ determined by an angle $\phi \in [0, \pi)$, as shown in the figure. It is easy to show that triangles ABC and $A'B'C'$ are directly similar.



(a) Find a formula for the ratio of similitude

$$\lambda = \lambda(\phi) = \frac{B'C'}{BC}.$$

(b) Find the maximal value λ_{m} of λ as ϕ varies in $[0, \pi)$, and show how to construct triangle $A'B'C'$ when $\lambda = \lambda_{\mathrm{m}}$.
(c) Prove that $\lambda_{\mathrm{m}} \geq 2$, with equality just when triangle ABC is equilateral.

733★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
A triangle has sides a, b, c, and the medians of this triangle are used as sides of a new triangle. If r_{m} is the inradius of this new triangle, prove or disprove that

$$r_{\mathrm{m}} \leq \frac{3abc}{4(a^2 + b^2 + c^2)},$$

with equality just when the original triangle is equilateral.

736. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Given is a regular n-gon $V_1V_2 \ldots V_n$ inscribed in a unit circle. Show how to select, among the n vertices V_i, three vertices A, B, C such that
(a) The area of triangle ABC is a maximum;
(b) The perimeter of triangle ABC is a maximum.

743. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle with centroid G inscribed in a circle with center O. A point M lies on the disk ω with diameter OG. The lines AM, BM, CM meet the circle again in A', B', C', respectively, and G' is the centroid of triangle $A'B'C'$. Prove that
(a) M does not lie in the interior of the disk ω' with diameter OG';
(b) $[ABC] \leq [A'B'C']$, where the brackets denote area.

762. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
ABC is a triangle with area K and sides a, b, c in the usual order. The internal bisectors of angles A, B, C meet the opposite sides in D, E, F, respectively, and the area of triangle DEF is K'.
(a) Prove that

$$\frac{3abc}{4(a^3 + b^3 + c^3)} \leq \frac{K'}{K} \leq \frac{1}{4}.$$

(b) If $a = 5$ and $K'/K = 5/24$, determine b and c, given that they are integers.

768. *Proposed by Jack Garfunkel, Flushing, N. Y.; and George Tsintsifas, Thessaloniki, Greece.*
If A, B, C are the angles of a triangle, show that

$$\frac{4}{9}\sum \sin B \sin C \leq \prod \cos \frac{B-C}{2} \leq \frac{2}{3}\sum \cos A,$$

where the sums and product are cyclic over A, B, C.

770. *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*
Let P be an interior point of triangle ABC. Prove that

$$PA \cdot BC + PB \cdot CA > PC \cdot AB.$$

787. *Proposed by J. Walter Lynch, Georgia Southern College.*
(a) Given two sides, a and b, of a triangle, what should be the length of the third side, x, in order that the area enclosed be a maximum?
(b) Given three sides, a, b and c, of a quadrilateral, what should be the length of the fourth side, x, in order that the area enclosed be a maximum?

788. *Proposed by Meir Feder, Haifa, Israel.*
A *pandigital* integer is a (decimal) integer containing each of the ten digits exactly once.
(a) If m and n are distinct pandigital perfect squares, what is the smallest possible value of $|\sqrt{m}-\sqrt{n}|$?
(b) Find two pandigital perfect squares m and n for which this minimum value of $|\sqrt{m}-\sqrt{n}|$ is attained.

790. *Proposed by Roland H. Eddy, Memorial University of Newfoundland.*
Let ABC be a triangle with sides a, b, c in the usual order, and let l_a, l_b, l_c and l'_a, l'_b, l'_c be two sets of concurrent cevians, with l_a, l_b, l_c intersecting a, b, c in L, M, N, respectively. If

$$l_a \cap l'_b = P, \quad l_b \cap l'_c = Q, \quad l_c \cap l'_a = R,$$

prove that, independently of the choice of concurrent cevians l'_a, l'_b, l'_c, we have

$$\frac{AP}{PL} \cdot \frac{BQ}{QM} \cdot \frac{CR}{RN} = \frac{abc}{BL \cdot CM \cdot AN} \geq 8,$$

with equality occuring just when l_a, l_b, l_c are the medians of the triangle.
(This problem extends *Crux* 588 [1981: 306].)

793. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
Consider the following double inequality for the Riemann Zeta function: for $n = 1, 2, 3, \ldots$,

$$\frac{1}{(s-1)(n+1)(n+2)\cdots(n+s-1)} + \zeta_n(s) < \zeta(s) < \zeta_n(s) + \frac{1}{(s-1)n(n+1)\cdots(n+s-2)}, \quad (1)$$

where

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad \text{and} \quad \zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}.$$

Go as far as you can in determining for which of the integers $s = 2, 3, 4, \ldots$ the inequalities (1) hold. (N. D. Kazarinoff asks for a proof that (1) holds for $s = 2$ in his *Analytic Inequalities*, Holt, Rinehart & Winston, 1964, page 79; and Norman Schaumberger asks for a proof of disproof that (1) holds for $s = 3$ in *The Two-Year College Mathematics Journal*, 12 (1981) 336.)

795. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Given a triangle ABC, let t_a, t_b, t_c be the lengths of its internal angle bisectors, and let T_a, T_b, T_c be the lengths of these bisectors extended to the circumcircle of the triangle. Prove that

$$T_a + T_b + T_c \geq \frac{4}{3}(t_a + t_b + t_c).$$

805. *Proposed by Murray S. Klamkin, University of Alberta.*
If $x, y, z > 0$, prove that

$$\frac{x+y+z}{3\sqrt{3}} \geq \frac{yz+zx+xy}{\sqrt{y^2+yz+z^2}+\sqrt{z^2+zx+x^2}+\sqrt{x^2+xy+y^2}},$$

with equality if and only if $x = y = z$.

808★. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*
Find the length of the largest circular arc contained within the right triangle with sides $a \leq b < c$.

815. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let ABC be a triangle with sides a, b, c, internal angle bisectors t_a, t_b, t_c, and semiperimeter s. Prove that the following inequalities hold, with equality if and only if the triangle is equilateral:

$$\text{(a)}\ \sqrt{3}\left(\frac{1}{at_a}+\frac{1}{bt_b}+\frac{1}{ct_c}\right) \geq \frac{4s}{abc};$$

$$\text{(b)}\ 3\sqrt{3}\cdot\frac{\dfrac{1}{at_a}+\dfrac{1}{bt_b}+\dfrac{1}{ct_c}}{at_a+bt_b+ct_c} \geq \sqrt[4]{\frac{2s}{(abc)^3}}.$$

816. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let a, b, c be the sides of a triangle with semiperimeter s, inradius r, and circumradius R. Prove that, with sums and product cyclic over a, b, c,

$$\text{(a)}\ \prod(b+c) \leq 8sR(R+2r),$$

$$\text{(b)}\ \sum bc(b+c) \leq 8sR(R+r),$$

$$\text{(c)}\ \sum a^3 \leq 8s(R^2-r^2).$$

823. *Proposé par Olivier Lafitte, élève de Mathématiques Supérieures au Lycée Montaigne à Bordeaux, France.*
(a) Soit $\{a_1, a_2, a_3, \ldots\}$ une suite de nombres réels strictement positifs. Si

$$v_n = \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n, \quad n = 1, 2, 3, \ldots,$$

montrer que $\lim_{n\to\infty} \sup v_n \geq \mathrm{e}$.
(b) Trouver une suite $\{a_n\}$ pour laquelle intervient l'égalité dans (a).

825★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Of the two triangle inequalities (with sum and product cyclic over A, B, C)

$$\sum \tan^2 \frac{A}{2} \geq 1 \qquad \text{and} \qquad 2 - 8 \prod \sin \frac{A}{2} \geq 1,$$

the first is well known and the second is equivalent to the well-known inequality $\prod \sin(A/2) \leq 1/8$. Prove or disprove the sharper inequality

$$\sum \tan^2 \frac{A}{2} \geq 2 - 8 \prod \sin \frac{A}{2}.$$

826★. *Proposed by Kent D. Boklan, student, Massachusetts Institute of Technology.*
It is a well-known consequence of the pingeonhole principle that, if six circles in the plane have a point in common, the one of the circles must entirely contain a radius of another.
Suppose n spherical balls have a point in common. What is the smallest value of n for which it can be said that one ball *must* entirely contain a radius of another?

832. *Proposed by Richard A. Gibbs, Fort Lewis College, Durango, Colorado.*
Let S be a subset of an $m \times n$ rectangular array of points, with $m, n \geq 2$. A *circuit in* S is a simple (i.e., nonself-intersecting) polygonal closed path whose vertices form a subset of S and whose edges are parallel to the sides of the array.
Prove that a circuit in S always exists for any subset S with $S \geq m + n$, and show that this bound is best possible.

835. *Proposed by Jack Garfunkel, Flushing, N. Y.; and George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle with sides a, b, c, and let R_{m} be the circumradius of the triangle formed by using as sides the medians of triangle ABC. Prove that

$$R_{\mathrm{m}} \geq \frac{a^2 + b^2 + c^2}{2\,(a + b + c)}.$$

836. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
(a) If A, B, C are the angles of a triangle, prove that

$$(1 - \cos A)(1 - \cos B)(1 - \cos C) \geq \cos A \cos B \cos C,$$

with equality if and only if the triangle is equilateral.
(b) Deduce from (a) Bottema's triangle inequality [1982: 296]:

$$(1 + \cos 2A)(1 + \cos 2B)(1 + \cos 2C) + \cos 2A \cos 2B \cos 2C \geq 0.$$

843. *Proposed by J. L. Brenner, Palo Alto, California.*
For integers $m > 1$ and $n > 2$, and real numbers $p, q > 0$ such that $p + q = 1$, prove that

$$(1 - p^m)^n + np^m(1 - p^m)^{n-1} + (1 - q^n - npq^{n-1})^m > 1.$$

846. *Proposed by Jack Garfunkel, Flushing, N. Y.; and George Tsintsifas, Thessaloniki, Greece.*
Given is a triangle ABC with sides a, b, c and medians m_a, m_b, m_c in the usual order, circumradius R, and inradius r. Prove that

(a) $\dfrac{m_a m_b m_c}{m_a^2 + m_b^2 + m_c^2} \geq r$;

(b) $12Rm_a m_b m_c \geq a(b+c)m_a^2 + b(c+a)m_b^2 + c(a+b)m_c^2$;

(c) $4R(am_a + bm_b + cm_c) \geq bc(b+c) + ca(c+a) + ab(a+b)$;

(d) $2R\left(\dfrac{1}{bc} + \dfrac{1}{ca} + \dfrac{1}{ab}\right) \geq \dfrac{m_a}{m_b m_c} + \dfrac{m_b}{m_c m_a} + \dfrac{m_c}{m_a m_b}$.

850. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
Let $x = r/R$ and $y = s/R$, where r, R, s are the inradius, circumradius, and semiperimeter, respectively, of a triangle with side lengths a, b, c. Prove that

$$y \geq \sqrt{x}\,(\sqrt{6} + \sqrt{2-x}),$$

with equality if and only if $a = b = c$.

854. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
For $x, y, z > 0$, let

$$A = \frac{yz}{(y+z)^2} + \frac{zx}{(z+x)^2} + \frac{xy}{(x+y)^2}$$

and

$$B = \frac{yz}{(y+x)(z+x)} + \frac{zx}{(z+y)(x+y)} + \frac{xy}{(x+z)(y+z)}.$$

It is easy to show that $a \leq \frac{3}{4} \leq B$, with equality if and only if $x = y = z$.
(a) Show that the inequality $a \leq \frac{3}{4}$ is "weaker"than $3B \geq \frac{9}{4}$ in the sense that

$$A + 3B \geq \frac{3}{4} + \frac{9}{4} = 3.$$

When does equality occur?
(b) Show that the inequality $4A \leq 3$ is "stronger" than $8B \geq 6$ in the sense that

$$4A + 8B \geq 3 + 6 = 9.$$

When does equality occur?

856. *Proposed by Jack Garfunkel, Flushing, N. Y.*
For a triangle ABC with circumradius R and inradius r, let $M = (R - 2r)/2R$. An inequality $P \geq Q$ involving elements of triangle ABC will be called *strong* or *weak*, respectively, according as $P - Q \leq M$ or $P - Q \geq M$.
(a) Prove that the following inequality is strong:

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} \geq \frac{3}{4}.$$

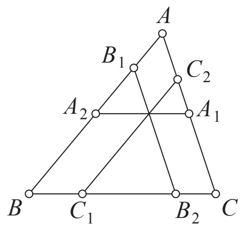
(b) Prove that the following inequality is weak:

$$\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \geq \sin B \sin C + \sin C \sin A + \sin A \sin B.$$

859. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
Let ABC be an acute-angled triangle of type II, that is (see [1982: 64]), such that $A \le B \le \frac{\pi}{3} \le C$, with circumradius R and inradius r. It is known [1982: 66] that for such a triangle $x \ge \frac{1}{4}$, where $x = r/R$. Prove the stronger inequality

$$x \ge \frac{\sqrt{3}-1}{2}.$$

862. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
P is an interior point of a triangle ABC. Lines through P parallel to the sides of the triangle meet those sides in the points A_1, A_2, B_1, B_2, C_1, C_2, as shown in the figure. Prove that



$$\text{(a)}\ \ [A_1B_1C_1] \le \frac{1}{3}[ABC],$$

$$\text{(b)}\ \ [A_1C_2B_1A_2C_1B_2] \le \frac{2}{3}[ABC],$$

where the brackets denote area.

864. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Find all x between 0 and 2π such that

$$2\cos^2 3x - 14\cos^2 2x - 2\cos 5x + 24\cos 3x - 89\cos 2x + 50\cos x > 43.$$

866. *Proposed by Jordi Dou, Barcelona, Spain.*
Given a triangle ABC with sides a, b, c, find the minimum value of

$$a \cdot \overline{XA} + b \cdot \overline{XB} + c \cdot \overline{XC},$$

where X ranges over all the points of the plane of the triangle.

870★. *Proposed by Sidney Kravitz, Dover, New Jersey.*
Of all the simple closed curves which are inscribed in a unit square (touching all four sides), find the one which has the minimum ratio of perimeter to enclosed area.

882. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
The interior surface of a wine glass is a right circular cone. The glass, containing some wine, is first held upright, then tilted slightly but not enough to spill any wine. Let D and E denote the area of the upper surface of the wine and the area of the curved surface in contact with the wine, respectively, when the glass is upright; and let D_1 and E_1 denote the corresponding areas when the glass is tilted. Prove that

$$\text{(a)}\ \ E_1 \ge E, \qquad \text{(b)}\ \ D_1 + E_1 \ge D + E, \qquad \text{(c)}\ \ \frac{D_1}{E_1} \ge \frac{D}{E}.$$

882. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
The interior surface of a wine glass is a right circular cone. The glass, containing some wine, is first held upright, then tilted slightly but not enough to spill any wine. Let D and E denote the area of the upper surface of the wine and the area of the curved surface in contact with the wine, respectively, when the glass is upright; and let D_1 and E_1 denote the corresponding areas when the glass is tilted. Prove that

$$\text{(a)}\ \ E_1 \ge E, \qquad \text{(b)}\ \ D_1 + E_1 \ge D + E, \qquad \text{(c)}\ \ \frac{D_1}{E_1} \ge \frac{D}{E}.$$

883. *Proposed by J. Tabov and S. Troyanski, Sofia, Bulgaria.*
Let ABC be a triangle with area S, sides a, b, c, medians m_a, m_b, m_c, and interior angle bisectors t_a, t_b, t_c. If

$$t_a \cap m_b = F, \qquad t_b \cap m_c = G, \qquad t_c \cap m_a = H,$$

prove that

$$\frac{\sigma}{S} < \frac{1}{6},$$

where σ denotes the area of triangle FGH.

895. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let ABC be a triangle with sides a, b, c in the usual order and circumcircle Γ. A line l through C meets the segment AB in D, Γ again in E, and the perpendicular bisector of AB in F. Assume that $c = 3b$.
(a) Construct the line l for which the length of DE is maximal.
(b) If DE has maximal length, prove that $DF = FE$.
(c) If DE has maximal length and also $CD = DF$, find a in terms of b and the measure of angle A.

896. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Consider the inequalities

$$\sum \sin^2 \frac{A}{2} \geq 1 - \frac{1}{4} \prod \cos \frac{B-C}{2} \geq \frac{3}{4},$$

where the sum and product are cyclic over the angles A, B, C of a triangle. The inequality between the second and third members is obvious, and that between the first and third members is well known. Prove the sharper inequality between the first two members.

897. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
If $\lambda > \mu$ and $a \geq b \geq c > 0$, prove that

$$b^{2\lambda}c^{2\mu} + c^{2\lambda}a^{2\mu} + a^{2\lambda}b^{2\mu} \geq (bc)^{\lambda+\mu} + (ca)^{\lambda+\mu} + (ab)^{\lambda+\mu},$$

with equality just when $a = b = c$.

899. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*
Let $\{a_i\}$ and $\{b_i\}$, $i = 1, 2, \ldots, n$, be two sequences of real numbers with the a_i all positive. Prove that

$$\sum_{i \neq j} a_i b_j = 0 \qquad \Longrightarrow \qquad \sum_{i \neq j} b_i b_j \leq 0.$$

908. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum value of

$$P \equiv \sin^\alpha A \cdot \sin^\beta B \cdot \sin^\gamma C,$$

where A, B, C are the angles of a triangle and α, β, γ are given positive numbers.

914. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
If $a, b, c > 0$, then the equation $x^3 - (a^2 + b^2 + c^2)x - 2abc = 0$ has a unique positive root x_0. Prove that

$$\frac{2}{3}(a + b + c) \leq x_0 < a + b + c.$$

915★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If $x + y + z + w = 180°$, prove or disprove that

$$\sin(x + y) + \sin(y + z) + \sin(z + w) + \sin(w + x) \geq \sin 2x + \sin 2y + \sin 2z + \sin 2w,$$

with equality just when $x = y = z = w$.

922★. *Proposed by A. W. Goodman, University of South Florida.*
Let

$$S_n(z) = \frac{n(n-1)}{2} + \sum_{k=1}^{n-1} (n-k)^2 z^k,$$

where $z = \mathrm{e}^{\mathrm{i}\theta}$. Prove that, for all real θ,

$$\Re\left(S_n(z)\right) = \frac{\sin\theta}{2(1-\cos\theta)^2}\left(n\sin\theta - \sin n\theta\right) \geq 0.$$

939. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Triangle ABC is acute-angled at B, and $AB < AC$. M being a point on the altitude AD, the lines BM and CM intersect AC and AB, respectively, in B' and C'. Prove that $BB' < CC'$.

940. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Show that, for any triangle ABC,

$$\sin B \sin C + \sin C \sin A + \sin A \sin B \leq \frac{7}{4} + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \leq \frac{9}{4}.$$

948. *Proposed by Vedula N. Murty, Pennsylvania State University, Capitol Campus.*
If a, b, c are the side lengths of a triangle of area K, prove that

$$27K^4 \leq a^3 b^3 c^2,$$

and determine when equality occurs.

952. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Consider the following double inequality, where the sum and product are cyclic over the angles A, B, C of a triangle:

$$\sum \sin^2 A \leq 2 + 16 \prod \sin^2\left(\frac{A}{2}\right) \leq \frac{9}{4}.$$

The inequality between the first and third members is well known, and that between the second and third members is equivalent to the well-known $\prod \sin\left(\frac{A}{2}\right) \leq \frac{1}{8}$. Prove the inequality between the first and second members.

954. *Proposed by W. J. Blundon, Memorial University of Newfoundland.*
The notation being the usual one, prove that each of the following is a necessary and sufficient condition for a triangle to be acute-angled:

(a) $IH < r\sqrt{2}$,

(b) $OH < R$,

(c) $\cos^2 A + \cos^2 B + \cos^2 C < 1$,

(d) $r^2 + r_a^2 + r_b^2 + r_c^2 < 8R^2$,

(e) $m_a^2 + m_b^2 + m_c^2 > 6R^2$.

955. *Proposed by Geng-zhe Chang, University of Science and Technology of China, Hefei, Anhui, People's Republic of China.*
If the real numbers A, B, C, a, b, c satisfy

$$A + a \geq b + c, \quad B + b \geq c + a, \quad C + c \geq a + b,$$

show that

$$Q \equiv Ax^2 + By^2 + Cz^2 + 2ayz + 2bzx + 2cxy \geq 0$$

holds for all real x, y, z such that $x + y + z = 0$.

957. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let a, b, c be the sides of a triangle with circumradius R and area K. Prove that

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} \geq \frac{2K}{R},$$

with equality if and only if the triangle is equilateral.

958. *Proposed by Murray S. Klamkin, University of Alberta.*
If A_1, A_2, A_3 are the angles of a triangle, prove that

$$\tan A_1 + \tan A_2 + \tan A_3 \geq \text{ or } \leq 2(\sin 2A_1 + \sin 2A_2 + \sin 2A_3)$$

according as the triangle is acute-angled or obtuse-angled, respectively. When is there equality?

959. *Proposed by Sidney Kravitz, Dover, New Jersey.*
Two houses are located to the north of a straight east-west highway. House A is at a perpendicular distance a from the road, house B is at a perpendicular distance $b \geq a$ from the road, and the feet of the perpendiculars are one unit apart. Design a road system of minimum total length (as a function of a and b) to connect both houses to the highway.

965. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $A_1A_2A_3$ be a nondegenerate triangle with sides $A_2A_3 = a_1$, $A_3A_1 = a_2$, $A_1A_2 = a_3$, and let $PA_i = x_i$ $(i = 1, 2, 3)$, where P is any point in space. Prove that

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} \geq \sqrt{3},$$

and determine when equality occurs.

968. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
For real numbers a, b, c, let $S_n = a^n + b^n + c^n$. If $S_1 \geq 0$, prove that

$$12S_5 + 33S_1S_2^2 + 3S_1^5 + 6S_1^2S_3 \geq 12S_1S_4 + 10S_2S_3 + 20S_1^3S_2.$$

When does equality occur?

970★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let a, b, c and m_a, m_b, m_c denote the side lengths and median lengths of a triangle. Find the set of all real t and, for each such t, the largest positive constant λ_t, such that

$$\frac{m_a m_b m_c}{abc} \geq \lambda_t \cdot \frac{m_a^t + m_b^t + m_c^t}{a+b+c}$$

holds for all triangles.

972★. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*
(a) Prove that two equilateral triangles of unit side cannot be placed inside a unit square without overlapping.
(b) What is the maximum number of regular tetrahedra of unit side that can be packed without overlapping inside a unit cube?
(c) Generalize to higher dimensions.

974. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Consider the following double inequality, where A, B, C are the angles of any triangle:

$$\cos A \cos B \cos C \leq 8 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \leq \frac{1}{8}.$$

The inequality involving the first and third members and that involving the second and third members are both well known. Prove the inequality involving the first and second members.

978. *Proposed by Andy Liu, University of Alberta.*
Determine the smallest positive integer m such that

$$529^n + m \cdot 132^n$$

is divisible by 262417 for all odd positive integers n.

982. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let P and Q be interior points of triangle $A_1A_2A_3$. For $i = 1, 2, 3$, let $PA_i = x_i$, $QA_i = y_i$, and let the distances from P and Q to the side opposite A_i be p_i and q_i, respectively. Prove that

$$\sqrt{x_1y_1} + \sqrt{x_2y_2} + \sqrt{x_3y_3} \geq 2(\sqrt{p_1q_1} + \sqrt{p_2q_2} + \sqrt{p_3q_3}).$$

When $P = Q$, this reduces to the well-known Erdös-Mordell inequality.
(See the article by Clayton W. Dodge in this journal [1984: 274–281].)

987★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If triangle ABC is acute-angled, prove or disprove that

$$\text{(a)}\quad \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \frac{4}{3}\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right),$$

$$\text{(b)}\quad \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geq \frac{4}{\sqrt{3}}\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right).$$

992. *Proposed by Harry D. Ruderman, Bronx, N. Y.*
Let $\alpha = (a_1, a_2, \ldots, a_{mn})$ be a sequence of positive real numbers such that $a_i \leq a_j$ whenever $i < j$, and let $\beta = (b_1, b_2, \ldots, b_{mn})$ be a permutation of α. Prove that

(a) $\displaystyle\sum_{j=1}^{n}\prod_{i=1}^{m} a_{m(j-1)+i} \geq \sum_{j=1}^{n}\prod_{i=1}^{m} b_{m(j-1)+i};$

(b) $\displaystyle\prod_{j=1}^{n}\sum_{i=1}^{m} a_{m(j-1)+i} \leq \prod_{j=1}^{n}\sum_{i=1}^{m} b_{m(j-1)+i}.$

993. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let P be the product of the $n+1$ positive real numbers $x_1, x_2, \ldots, x_{n+1}$. Find a lower bound (as good as possible) for P if the x_i satisfy

(a) $\displaystyle\sum_{i=1}^{n+1} \frac{1}{1+x_i} = 1;$

(b)★ $\displaystyle\sum_{i=1}^{n+1} \frac{a_i}{b_i+x_i} = 1$, where the a_i and b_i are given positive real numbers.

999★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Let R, r, s be the circumradius, inradius, and semiperimeter, respectively, of an acute-angled triangle. Prove or disprove that

$$s^2 \geq 2R^2 + 8Rr + 3r^2.$$

When does equality occur?

1003★. *Proposed by Murray S. Klamkin, University of Alberta.*
Without using tables or a calculator, show that

$$\ln 2 > \left(\frac{2}{5}\right)^{\frac{2}{5}}.$$

1006. *Proposed by Hans Havermann, Weston, Ontario.*
Given a base-ten positive integer of two or more digits, it is possible to spawn two smaller base-ten integers by inserting a space somewhere within the number. We call the left offspring thus created the *farmer* ($\mathcal{F}$) and the value of the right one (ignoring leading zeros, if any) the *ladder* ($\mathcal{L}$). A number is called *modest* if it has an $\mathcal{F}$ and an $\mathcal{L}$ such that the number divided by $\mathcal{L}$ leaves remainder $\mathcal{F}$. (For example, 39 is modest.)
Consider, for $n > 1$, a block of n consecutive positive integers all of which are modest. If the smallest and largest of these are a and b, respectively, and if $a-1$ and $b+1$ are not modest, then we say that the block forms a *multiple berth* of size n. A multiple berth of size 2 is called a set of *twins*, and the smallest twins are $\{411, 412\}$. A multiple berth of size 3 is called a set of *triplets*, and the smallest triplets are $\{4000026, 4000027, 4000028\}$.
(a) Find the smallest quadruplets.
(b)★ Find the smallest quintuplets. (There are none less than 25 million.)

1012. *Proposed by G. P. Henderson, Campbellcroft, Ontario.*
An amateur winemaker is siphoning wine from a carboy. To speed up the process, he tilts the carboy to raise the level of the wine. Naturally, he wants to maximize the height, H, of the surface of the liquid above the table on which the carboy rests. The carboy is actually a circular cylinder, but we will only assume that its base is the interior of a smooth closed convex curve, C, and that the generators are perpendicular to the base. P is a point on C, T is the line tangent to C at P, and the cylinder is rotated about T.

(a) Prove that H is a maximum when the centroid of the surface of the liquid is vertically above T.

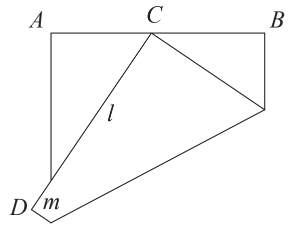
(b) Let the volume of the wine be V and let the area inside C be A. Assume that $V \geq AW/2$, where W is the maximum width of C (i. e., the maximum distance between parallel tangents). Obtain an explicit formula for H_{M}, the maximum value of H. How should P be chosen to maximize H_{M}?

1019. *Proposed by Weixuan Li and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Determine the largest constant k such that the inequality

$$x \leq \alpha \sin x + (1-\alpha)\tan x$$

holds for all $\alpha \leq k$ and for all $x \in \left[0, \frac{\pi}{2}\right)$.
(The inequality obtained when α is replaced by $\frac{2}{3}$ is the Snell-Huygens inequality, which is fully discussed in Problem 115 [1976: 98–99, 111–113, 137–138].)

1025. *Proposed by Peter Messer, M. D., Mequon, Wisconsin.*
A paper square $ABCD$ is folded so that vertex C falls on AB and side CD is divided into two segments of lengths l and m, as shown in the figure. Find the minimum value of the ratio l/m.



1030. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Given are two obtuse triangles with sides a, b, c and p, q, r, the longest sides of each being c and r, respectively. Prove that

$$ap + bq < cr.$$

1036. *Proposed by Gali Salvatore, Perkins, Québec.*
Find sets of positive numbers $\{a, b, c, d, e, f\}$ such that, simultaneously,

$$\frac{abc}{def} < 1, \quad \frac{a+b+c}{d+e+f} < 1, \quad \frac{a}{d}+\frac{b}{e}+\frac{c}{f} > 3, \quad \frac{d}{a}+\frac{e}{b}+\frac{f}{c} > 3,$$

or prove that there are none.

1045. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let P be an interior point of triangle ABC; let x, y, z be the distances of P from vertices A, B, C, respectively; and let u, v, w be the distances of P from sides BC, CA, AB, respectively. The well-known Erdös-Mordell inequality states that

$$x + y + z \geq 2(u + v + w).$$

Prove the following related inequalities:

(a) $\dfrac{x^2}{vw} + \dfrac{y^2}{wu} + \dfrac{z^2}{uv} \geq 12,$ (b) $\dfrac{x}{v+w} + \dfrac{y}{w+u} + \dfrac{z}{u+v} \geq 3,$

(c) $\dfrac{x}{\sqrt{vw}} + \dfrac{y}{\sqrt{wu}} + \dfrac{z}{\sqrt{uv}} \geq 6.$

1046. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*
The *Wallace point* W of any four points A_1, A_2, A_3, A_4 on a circle with center O may be defined by the vector equation

$$\overrightarrow{OW} = \frac{1}{2}\left(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \overrightarrow{OA_3} + \overrightarrow{OA_4}\right)$$

(see the article by Bottema and Groenman in this journal [1982: 126]).
Let γ be a cyclic quadrilateral the Wallace point of whose vertices lies inside γ. Let a_i ($i = 1, 2, 3, 4$) be the sides of γ, and let G_i be the midpoint of the side opposite to a_i. Find the minimum value of

$$f(X) \equiv a_1 \cdot G_1X + a_2 \cdot G_2X + a_3 \cdot G_3X + a_4 \cdot G_4X,$$

where X ranges over all the points of the plane of γ.

1049★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Let ABC and $A'B'C'$ be two nonequilateral triangles such that $A \geq B \geq C$ and $A' \geq B' \geq C'$. Prove that

$$A - C > A' - C' \quad \Longleftrightarrow \quad \frac{s}{r} > \frac{s'}{r'},$$

where s, r and s', r' are the semiperimeter and inradius of triangles ABC and $A'B'C'$, respectively.

1051. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let a, b, c be the side lengths of a triangle of area K, and let u, v, w be positive real numbers. Prove that

$$\frac{ua^4}{v+w} + \frac{vb^4}{w+u} + \frac{wc^4}{u+v} \geq 8K^2.$$

When does equality occur? Some interesting triangle inequalities may result if we assign specific values to u, v, w. Find a few.

1057. *Proposed by Jordi Dou, Barcelona, Spain.*
Let $\mathit{\Omega}$ be a semicircle of unit radius, with diameter AA_0. Consider a sequence of circles γ_i, all interior to $\mathit{\Omega}$, such that γ_1 is tangent to $\mathit{\Omega}$ and to AA_0, γ_2 is tangent to $\mathit{\Omega}$ and to the chord AA_1 tangent to γ_1, γ_3 is tangent to $\mathit{\Omega}$ and to the chord AA_2 tangent to γ_2, etc. Prove that

$$r_1 + r_2 + r_3 + \cdots < 1,$$

where r_i is the radius of γ_i.

1058. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*
Two points X and Y are choosen at random, independently and uniformly with respect to length, on the edges of a unit cube. Determine the probability that

$$1 < XY < \sqrt{2}.$$

1060. *Proposed by Murray S. Klamkin, University of Alberta.*
If ABC is an obtuse triangle, prove that

$$\sin^2 A \tan A + \sin^2 B \tan B + \sin^2 C \tan C < 6 \sin A \sin B \sin C.$$

1064. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Triangles ABC and DEF are similar, with angles $A = D$, $B = E$, $C = F$ and ratio of similitude $\lambda = EF/BC$. Triangle DEF is inscribed in triangle ABC, with D, E, F on the *lines* BC, CA, AB, not necessarily respectively. Three cases can be considered:

$$\text{Case 1:}\quad D \in BC,\ \ E \in CA,\ \ F \in AB;$$

$$\text{Case 2:}\quad D \in CA,\ \ E \in AB,\ \ F \in BC;$$

$$\text{Case 3:}\quad D \in AB,\ \ E \in BC,\ \ F \in CA.$$

For Case 1, it is known that $\lambda \geq \frac{1}{2}$ (see *Crux* 606 [1982: 24, 108]). Prove that, for each of Cases 2 and 3,

$$\lambda \geq \sin\omega,$$

where ω is the Brocard angle of triangle ABC. (This inequality also holds *a fortiori* for Case 1, since $\omega \leq 30°$.)

1065. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.*
The orthocenter H of an orthocentric tetrahedron $ABCD$ lies inside the tetrahedron. If X ranges over all the points of space, find the minimum value of

$$f(X) = \{BCD\} \cdot AX + \{CDA\} \cdot BX + \{DAB\} \cdot CX + \{ABC\} \cdot DX,$$

where the braces denote the (unsigned) area of a triangle.
(This is an extension to 3 dimensions of *Crux* 866 [1984: 327].)

1066★. *Proposed by D. S. Mitrinović, University of Belgrade, Belgrade, Yugoslavia.*
Consider the inequality

$$\begin{aligned}&(y^p + z^p - x^p)(z^p + x^p - y^p)(x^p + y^p - z^p)\\ &\qquad \leq (y^q + z^q - x^q)^r(z^q + x^q - y^q)^r(x^q + y^q - z^q)^r.\end{aligned}$$

(a) Prove that the inequality holds for all real x, y, z if $(p, q, r) = (2, 1, 2)$.
(b) Determine all triples (p, q, r) of natural numbers for each of which the inequality holds for all real x, y, z.

1067. *Proposed by Jack Garfunkel, Flushing, N. Y.*
(a)★ If $x, y, z > 0$, prove that

$$\frac{xyz(x + y + z + \sqrt{x^2 + y^2 + z^2})}{(x^2 + y^2 + z^2)(yz + zx + xy)} \leq \frac{3 + \sqrt{3}}{9}.$$

(b) Let r be the inradius of a triangle and r_1, r_2, r_3 the radii of its three Malfatti circles (see *Crux* 618 [1982: 82]). Deduce from (a) that

$$r \leq (r_1 + r_2 + r_3)\frac{3 + \sqrt{3}}{9}.$$

1075. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle with circumcenter O and incenter I, and let DEF be the pedal triangle of an interior point M of triangle ABC (with D on BC, etc.). Prove that

$$OM \geq OI \quad\Longleftrightarrow\quad r' \leq \frac{r}{2},$$

where r and r' are the inradii of triangles ABC and DEF, respectively.

1077★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
For $i = 1, 2, 3$, let C_i be the center and r_i the radius of the Malfatti circle nearest A_i in triangle $A_1A_2A_3$. Prove that

$$A_1C_1 \cdot A_2C_2 \cdot A_3C_3 \geq \frac{(r_1 + r_2 + r_3)^3 - 3r_1r_2r_3}{3}.$$

When does equality occur?

1079. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let

$$g(a, b, c) = \sum \frac{a}{a + 2b} \cdot \frac{b - 4c}{b + 2c},$$

where the sum is cyclic over the sides a, b, c of a triangle.

(a) Prove that $-\frac{5}{3} < g(a, b, c) \leq -1$.

(b)★ Find the greatest lower bound of $g(a, b, c)$.

1080★. *Proposed by D. S. Mitrinović, University of Belgrade, Belgrade, Yugoslavia.*
Determine the maximum value of

$$f(a, b, c) = \left| \frac{b - c}{b + c} + \frac{c - a}{c + a} + \frac{a - b}{a + b} \right|,$$

where a, b, c are the side lengths of a nondegenerate triangle.

1083★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Consider the double inequality

$$\frac{2}{\sqrt{3}} \sum \sin A \leq \sum \cos \left(\frac{B - C}{2} \right) \leq \frac{2}{\sqrt{3}} \sum \cos \frac{A}{2},$$

where the sums are cyclic over the angles A, B, C of a triangle. The left inequality has already been established in this journal (Problem 613 [1982: 55, 67, 138]). Prove or disprove the right inequality.

1085. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $\sigma_n = A_0A_1 \dots A_n$ be a regular n-simplex in $\mathbb{R}^n$, and let π_i be the hyperplane containing the face $\sigma_{n-1} = A_0A_1 \dots A_{i-1}A_{i+1} \dots A_n$. If $B_i \in \pi_i$ for $i = 0, 1, \dots, n$, show that

$$\sum_{0 \leq i < j \leq n} |\overrightarrow{B_iB_j}| \geq \frac{n + 1}{2} e,$$

where e is the edge length of σ_n.

1086. *Proposed by Murray S. Klamkin, University of Alberta.*
The medians of an n-dimensional simplex $A_0A_1 \dots A_n$ in $\mathbb{R}^n$ intersect at the centroid G and are extended to meet the circumsphere again in the points $B_0, B_1, \dots, B_n$, respectively.

(a) Prove that

$$A_0G + A_1G + \cdots + A_nG \leq B_0G + B_1G + \cdots + B_nG.$$

(b)★ Determine all other points P such that

$$A_0P + A_1P + \cdots + A_nP \leq B_0P + B_1P + \cdots + B_nP.$$

1087. *Proposed by Robert Downes, student, Moravian College, Bethlehem, Pennsylvania.*
Let a, b, c, d be four positive numbers.
(a) There exists a regular tetrahedron $ABCD$ and a point P in space such that $PA = a$, $PB = b$, $PC = c$, and $PD = d$ if and only if a, b, c, d satisfy what condition?
(b) This condition being satisfied, calculate the edge length of the regular tetrahedron $ABCD$.
(For the corresponding problem in a plane, see Problem 39 [1975: 64; 1976: 7].)

1088★. *Proposed by Basil C. Rennie, James Cook University of North Queensland, Australia.*
If R, r, s are the circumradius, inradius, and semiperimeter, respectively, of a triangle with largest angle A, prove or disprove that

$$s \gtreqless 2R + r \quad \text{according as} \quad A \lesseqgtr 90^\circ.$$

1089. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Find the range of the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(\theta) = \sum_{k=1}^{\infty} 3^{-k} \cos k\theta, \quad \theta \in \mathbb{R}.$$

1093★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Prove that

$$\left(\frac{\sum \sin A}{\sum \cos\left(\frac{A}{2}\right)} \right)^3 \geq 8 \prod \sin \frac{A}{2},$$

where the sums and product are cyclic over the angles A, B, C of a triangle. When does equality occur?

1095. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Let $N_n = \{1, 2, \ldots, n\}$, where $n \geq 4$. A subset A of N_n with $|A| \geq 2$ is called an *RC-set* (relatively composite) if $(a, b) > 1$ for all $a, b \in A$. Let $f(n)$ be the maximum cardinality of all RC-sets A in N_n. Determine $f(n)$ and find all RC-sets in N_n of cardinality $f(n)$.

1096. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum and minimum values of

$$S \equiv \cos\frac{A}{4} \cos\frac{B}{4} \cos\frac{C}{4} + \sin\frac{A}{4} \sin\frac{B}{4} \sin\frac{C}{4},$$

where A, B, C are the angles of a triangle. (No calculus, please!)

1098. *Proposed by Jordi Dou, Barcelona, Spain.*
Characterize all trapezoids for which the circumscribed ellipse of minimal area is a circle.

1102. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $\sigma_n = A_0 A_1 \ldots A_n$ be an n-simplex in n-dimensional Euclidean space. Let M be an interior point of σ_n whose barycentric coordinates are $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ and, for $i = 0, 1, \ldots, n$, let p_i be its distances from the $(n-1)$-face

$$\sigma_{n-1} = A_0 A_1 \ldots A_{i-1} A_{i+1} \ldots A_n.$$

Prove that $\lambda_0 p_0 + \lambda_1 p_1 + \cdots + \lambda_n p_n \geq r$, where r is the inradius of σ_n.

1111. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let α, β, γ be the angles of an acute triangle and let

$$f(\alpha,\beta,\gamma)=\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}.$$

(a) Prove that $f(\alpha,\beta,\gamma)>\frac{3}{2}\sqrt[3]{2}$.

(b)★ Prove or disprove that $f(\alpha,\beta,\gamma)>\frac{1}{2}+\sqrt{2}$.

1114. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC, $A'B'C'$ be two triangles with sides a,b,c, a',b',c' and areas F, F' respectively. Show that

$$aa'+bb'+cc'\ge 4\sqrt{3}\sqrt{FF'}.$$

1116. *Proposed by David Grabiner, Claremont High School, Claremont, California.*
(a) Let $f(n)$ be the smallest positive integer which is not a factor of n. Continue the series $f(n), f(f(n)), f(f(f(n))),\ldots$ until you reach 2. What is the maximum length of the series?

(b) Let $g(n)$ be the second smallest positive integer which is not a factor of n. Continue the series $g(n), g(g(n)), g(g(g(n))),\ldots$ until you reach 3. What is the maximum length of the series?

1120★. *Proposed by D. S. Mitrinović, University of Belgrade, Belgrade, Yugoslavia.*
(a) Determine a positive number λ so that

$$(a+b+c)^2(abc)\ge\lambda(bc+ca+ab)(b+c-a)(c+a-b)(a+b-c)$$

holds for all real numbers a,b,c.

(b) As above, but a,b,c are assumed to be positive.

(c) As above, but a,b,c are assumed to satisfy

$$b+c-a>0,\quad c+a-b>0,\quad a+b-c>0.$$

1125★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If A,B,C are the angles of an acute triangle ABC, prove that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\le\frac{3}{2}(\csc 2A+\csc 2B+\csc 2C)$$

with equality when triangle ABC is equilateral.

1126. *Proposed by Péter Ivády, Budapest, Hungary.*
For $0<x\le 1$, show that

$$\sinh x<\frac{3x}{2+\sqrt{1-x^2}}<\tan x.$$

1127★. *Proposed by D. S. Mitrinović, University of Belgrade, Belgrade, Yugoslavia.*
(a) Let a,b,c and r be real numbers >1. Prove or disprove that

$$(\log_a bc)^r+(\log_b ca)^r+(\log_c ab)^r\ge 3\cdot 2^r.$$

(b) Find an analogous inequality for n numbers $a_1,a_2,\ldots,a_n$ rather than three numbers a,b,c.

1129. *Proposed by Donald Cross, Exeter, England.*
(a) Show that every positive whole number ≥ 84 can be written as the sum of three positive whole numbers in at least four ways (all twelve numbers different) such that the sum of the squares of the three numbers in any group is equal to the sum of the squares of the three numbers in each of the other groups.

(b) Same as part (a), but with "three" replaced by "four" and "twelve" by "sixteen".

(c)* Is 84 minimal in (a) and/or (b)?

1130. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Show that

$$a^{\frac{3}{2}} + b^{\frac{3}{2}} + c^{\frac{3}{2}} \leq 3^{\frac{7}{4}} R^{\frac{3}{2}}$$

where a, b, c are the sides of a triangle and R is the circumradius.

1131. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Let $A_1A_2A_3$ be a triangle with sides a_1, a_2, a_3 labelled as usual, and let P be a point in or out of the plane of the triangle. It is a known result that if R_1, R_2, R_3 are the distances from P to the respective vertices A_1, A_2, A_3, then a_1R_1, a_2R_2, a_3R_3 satisfy the triangle inequality, i. e.

$$a_1R_1 + a_2R_2 + a_3R_3 \geq 2a_iR_i, \quad i = 1, 2, 3. \tag{1}$$

For the a_iR_i to form a non-obtuse triangle, we would have to satisfy

$$a_1^2R_1^2 + a_2^2R_2^2 + a_3^2R_3^2 \geq 2a_i^2R_i^2$$

which, however, need not be true. Show that nevertheless

$$a_1^2R_1^2 + a_2^2R_2^2 + a_3^2R_3^2 \geq \sqrt{2}a_i^2R_i^2$$

which is a stronger inequality than (1).

1137*. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove or disprove the triangle inequality

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} > \frac{5}{s},$$

where m_a, m_b, m_c are the medians of a triangle and s is its semiperimeter.

1142. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Suppose ABC is a triangle whose median point lies on its inscribed circle.

(a) Find an equation relating the sides a, b, c of $\triangle ABC$.

(b) Assume $a \geq b \geq c$. Find an upper bound for a/c.

(c) Give an example of a triangle with *integral* sides having the above property.

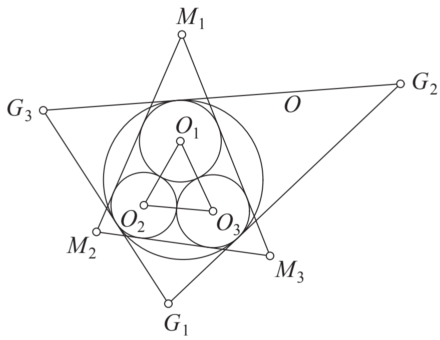
1144. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and P an interior point at distances x_1, x_2, x_3 from the vertices A, B, C and distances p_1, p_2, p_3 from the sides BC, CA, AB, respectively. Show that

$$\frac{x_1x_2}{ab} + \frac{x_2x_3}{bc} + \frac{x_3x_1}{ca} \geq 4\left(\frac{p_1p_2}{ab} + \frac{p_2p_3}{bc} + \frac{p_3p_1}{ca}\right).$$

1145. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Given a plane convex figure and a straight line l (in the same plane) which splits the figure into two parts whose areas are in the ratio $1 : t$ ($t \geq 1$). These parts are then projected orthogonally onto a straight line n perpendicular to l. Determine, in terms of t, the maximum ratio of the lengths of the two projections.

1148. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Nashua, New Hampshire.*
Find the triangle of smallest area that has integral sides and integral altitudes.

1150★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
In the figure, $\triangle M_1M_2M_3$ and the three circles with centers O_1, O_2, O_3 represent the Malfatti configuration. Circle O is externally tangent to these three circles and the sides of triangle $G_1G_2G_3$ are each tangent to O and one of the smaller circles. Prove that



$$\mathcal{P}(\triangle G_1G_2G_3) \geq \mathcal{P}(\triangle M_1M_2M_3) + \mathcal{P}(\triangle O_1O_2O_3),$$

where $\mathcal{P}$ stands for perimeter. Equality is attained when $\triangle O_1O_2O_3$ is equilateral.

1151★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Prove (or disprove) that for an obtuse triangle ABC,

$$m_a + m_b + m_c \leq s\sqrt{3},$$

where m_a, m_b, m_c denote the medians to sides a, b, c and s denotes the semiperimeter of $\triangle ABC$. Equality is attained in the equilateral triangle.

1152. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Prove that

$$\sum \cos\frac{\alpha}{2} \leq \frac{\sqrt{3}}{2}\sum \cos\frac{1}{4}(\beta - \gamma),$$

where α, β, γ are the angles of a triangle and the sums are cyclic over these angles.

1154. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let A, B, and C be the angles of an arbitrary triangle. Determine the best lower and upper bounds of the function

$$f(A, B, C) = \sum \sin\frac{A}{2} - \sum \frac{A}{2}\sin\frac{B}{2}$$

(where the summations are cyclic over A, B, C) and decide whether they are attained.

1156. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*
At any point P of an ellipse with semiaxes a and b ($a > b$), draw a normal line and let Q be the other meeting point. Find the least value of length PQ, in terms of a and b.

1158. *Proposed by Svetoslav Bilchev, Technical University, Russe, Bulgaria.*
Prove that

$$\sum \frac{1}{(\sqrt{2}+1)\cos\frac{A}{8} - \sin\frac{A}{8}} \geq \sqrt{6 - 3\sqrt{2}},$$

where the sum is cyclic over the angles A, B, C of a triangle. When does equality occur?

1159. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and P some interior point with distances $AP = x_1$, $BP = x_2$, $CP = x_3$. Show that

$$(b+c)x_1 + (c+a)x_2 + (a+b)x_3 \geq 8F,$$

where a, b, c are the sides of $\triangle ABC$ and F is its area.

1162. *Proposed by George Tsintsifas, Thessaloniki, Greece.* (Dedicated to Léo Sauvé.)
Let $G = \{A_1, A_2, \ldots, A_{n+1}\}$ be a point set of diameter D (that is, $\max A_iA_j = D$) in $\mathbb{E}^n$. Prove that G can be obtained in a slab of width d, where

$$d \leq \begin{cases} \frac{2D}{\sqrt{2n+2}} & \text{for } n \text{ odd} \\ D \cdot \sqrt{\frac{2(n+1)}{n(n+2)}} & \text{for } n \text{ even.} \end{cases}$$

(A *slab* is a closed connected region in $\mathbb{E}^n$ bounded by two parallel hyperplanes. Its *width* is the distance between these hyperplanes.)

1165★. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated to Léo Sauvé.)
For fixed $n \geq 5$, consider an n-gon P imbedded in a unit cube.

(i) Determine the maximum perimeter of P if n is odd.

(ii) Determine the maximum perimeter of P if it is convex (which implies it is planar).

(iii) Determine the maximum volume of the convex hull of P if also $n < 8$.

1166. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ontario.* (Dedicated to Léo Sauvé.)
Let A and B be positive integers such that the arithmetic progression $\{An+B : n = 0, 1, 2, \ldots\}$ contains at least one square. If M^2 ($M > 0$) is the smallest such square, prove that $M < A + \sqrt{B}$.

1167. *Proposed by Jordan B. Tabov, Sofia, Bulgaria.* (Dedicated to Léo Sauvé.)
Determine the greatest real number r such that for every acute triangle ABC of area 1 there exists a point whose pedal triangle with respect to ABC is right-angled and of area r.

1169. *Proposed by Andy Liu, University of Alberta, Edmonton, Alberta; and Steve Newman, University of Michigan, Ann Arbor, Michigan.* [To Léo Sauvé who, like J. R. R. Tolkien, created a fantastic world.]
(i) *The fellowship of the Ring.* Fellows of a society wear rings formed of 8 beads, with two of each of 4 colours, such that no two adjacent beads are of the same colour. No two members wear indistinguishable rings. What is the maximum number of fellows of this society?
(ii) *The Two Towers.* On two of three pegs are two towers, each of 8 discs of increasing size from top to bottom. The towers are identical except that their bottom discs are of different colours. The task is to disrupt and reform the towers so that the two largest discs trade places. This is to be accomplished by moving one disc at a time from peg to peg, never placing a disc on top of a smaller one. Each peg is long enough to accommodate all 16 discs. What is the minimum number of moves required?
(iii) *The Return of the King.* The King is wandering around his kingdom, which is an ordinary 8 by 8 chessboard. When he is at the north-east corner, he receives an urgent summons to return

to his summer palace at the south-west corner. He travels from cell to cell but only due south, west, or south-west. Along how many different paths can the return be accomplished?

1171★. *Proposed by D. S. Mitrinović and J. E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)
(i) Determine all real numbers λ so that, whenever a, b, c are the lengths of three segments which can form a triangle, the same is true for

$$(b+c)^\lambda, \ (c+a)^\lambda, \ (a+b)^\lambda.$$

(For $\lambda = -1$ we have *Crux* 14 [1975: 281].)
(ii) Determine all pairs of real numbers λ, μ so that, whenever a, b, c are the lengths of three segments which can form a triangle, the same is true for

$$(b+c+\mu a)^\lambda, \ (c+a+\mu b)^\lambda, \ (a+b+\mu c)^\lambda.$$

1172. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Show that for any triangle ABC, and for any real $\lambda \geq 1$,

$$\sum (a+b)\sec^\lambda \frac{C}{2} \geq 4\left(\frac{2}{\sqrt{3}}\right)^\lambda s,$$

where the sum is cyclic over $\triangle ABC$ and s is the semiperimeter.

1175. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Prove that if α, β, γ are the angles of a triangle,

$$-2 < \sin 3\alpha + \sin 3\beta + \sin 3\gamma \leq \frac{3}{2}\sqrt{3}.$$

1181. *Proposed by D. S. Mitrinović and J. E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)
Let x, y, z be real numbers such that

$$xyz(x+y+z) > 0,$$

and let a, b, c be the sides, m_a, m_b, m_c the medians and F the area of a triangle. Prove that

(a) $|yza^2 + zxb^2 + xyc^2| \geq 4F\sqrt{xyz(x+y+z)}$;

(b) $|yzm_a^2 + zxm_b^2 + xym_c^2| \geq 3F\sqrt{xyz(x+y+z)}$.

1182. *Proposed by Peter Andrews and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.* (Dedicated to Léo Sauvé.)
Let $a_1, a_2, \ldots, a_n$ denote positive reals where $n \geq 2$. Prove that

$$\frac{\pi}{2} \leq \tan^{-1}\frac{a_1}{a_2} + \tan^{-1}\frac{a_2}{a_3} + \cdots + \tan^{-1}\frac{a_n}{a_1} \leq \frac{(n-1)\pi}{2}$$

and for each inequality determine when equality holds.

1186. *Proposed by Svetoslav Bilchev, Technical University, and Emilia Velikova, Mathematikalgymnasium, Russe, Bulgaria.*
If a, b, c are the sides of a triangle and s, R, r the semiperimeter, circumradius, and inradius, respectively, prove that

$$\sum (b+c-a)\sqrt{a} \geq 4r(4R+r)\sqrt{\frac{4R+r}{3Rs}}$$

where the sum is cyclic over a, b, c.

1194. *Proposed by Richard I. Hess, Rancho Palos Verdes, California.*
My uncle's ritual for dressing each morning except Sunday includes a trip to the sock drawer where he (1) picks out three socks at random, (2) wears any matching pair and returns the third sock to the drawer, (3) returns the three socks to the drawer if he has no matching pair and repeats steps (1) and (3) until he completes step (2). The drawer starts with 16 socks each Monday morning (8 blue, 6 black, 2 brown) and ends up with 4 socks each Saturday evening.
(a) On which day of the week does he average the longest time at the sock drawer?
(b) On which day of the week is he least likely to get a matching pair from the first three socks chosen?

1199★. *Proposed by D. S. Mitrinović and J. E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)
Prove that for acute triangles,

$$s^2 \leq \frac{27R^2}{27R^2 - 8r^2}(2R+r)^2,$$

where s, r, R are the semiperimeter, inradius, and circumradius, respectively.

1200. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
In a certain game, the first player secretly chooses an n-dimensional vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ all of whose components are integers. The second player is to determine $\boldsymbol{a}$ by choosing any n-dimensional vectors $\boldsymbol{x}_i$, all of whose components are also integers. For each $\boldsymbol{x}_i$ chosen, and before the next $\boldsymbol{x}_i$ is chosen, the first player tells the second player the value of the dot product $\boldsymbol{x}_i \cdot \boldsymbol{a}$. What is the least number of vectors $\boldsymbol{x}_i$ the second player has to choose in order to be able to determine $\boldsymbol{a}$? [Warning: this is somewhat "tricky"!]

1201★. *Proposed by D. S. Mitrinović and J. E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.* (Dedicated to Léo Sauvé.)
Prove that

$$(x+y+z)\left(\frac{xc^2}{a^2} + \frac{ya^2}{b^2} + \frac{zb^2}{c^2}\right) \geq \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)(a^2yz + b^2zx + c^2xy),$$

where a, b, c are the sides of a triangle and x, y, z are real numbers.

1203. *Proposed by Milen N. Naydenov, Varna, Bulgaria.*
A quadrilateral inscribed in a circle of radius R and circumscribed around a circle of radius r has consecutive sides a, b, c, d, semiperimeter s and area F. Prove that

(a) $2\sqrt{F} \leq s \leq r + \sqrt{r^2 + 4R^2}$;

(b) $6F \leq ab + ac + ad + bc + bd + cd \leq 4r^2 + 4R^2 + 4r\sqrt{r^2 + 4R^2}$;

(c) $2sr^2 \leq abc + abd + acd + bcd \leq 2r\left(r + \sqrt{r^2 + 4R^2}\right)^2$;

(d) $4Fr^2 \leq abcd \leq \dfrac{16}{9}r^2(r^2 + 4R^2)$.

1209. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Characterize all positive integers a and b such that

$$a + b + (a, b) \le [a, b],$$

and find when equality holds. Here (a, b) and $[a, b]$ denote respectively the g.c.d. and l.c.m. of a and b.

1210. *Proposed by Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.*
If A, B, C are the angles of an acute triangle, prove that

$$(\tan A + \tan B + \tan C)^2 \ge (\sec A + 1)^2 + (\sec B + 1)^2 + (\sec C + 1)^2.$$

1212. *Proposed by Svetoslav Bilchev, Technical University, and Emilia Velikova, Mathematikalgymnasium, Russe, Bulgaria.*
Prove that

$$\frac{u}{v+w} \cdot \frac{bc}{s-a} + \frac{v}{w+u} \cdot \frac{ca}{s-b} + \frac{w}{u+v} \cdot \frac{ab}{s-c} \ge a + b + c$$

where a, b, c are the sides of a triangle and s is its semiperimeter, and u, v, w are arbitrary positive real numbers.

1213★. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
In *Math. Gazette* 68 (1984) 222, P. Stanbury noted the two close approximations $\mathrm{e}^6 \approx \pi^5 + \pi^4$ and $\pi^9/\mathrm{e}^8 \approx 10$. Can one show without a calculator that (i) $\mathrm{e}^6 > \pi^5 + \pi^4$ and (ii) $\pi^9/\mathrm{e}^8 < 10$?

1214. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let $A_1A_2A_3$ be an equilateral triangle and let P be an interior point. Show that there is a triangle with side lengths PA_1, PA_2, PA_3.

1215. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Let a, b, c be nonnegative real numbers with $a + b + c = 1$. Show that

$$ab + bc + ca \le a^3 + b^3 + c^3 + 6abc \le a^2 + b^2 + c^2 \le 2\,(a^3 + b^3 + c^3) + 3abc,$$

and for each inequality determine all cases when equality holds.

1216★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove or disprove that

$$2 < \frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C} \le \frac{9\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

1218★. *Proposed by D. S. Mitrinović and J. E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*
Let F_1 be the area of the orthic triangle of an acute triangle of area F and circumradius R. Prove that

$$F_1 \le \frac{4F^3}{27R^4}.$$

1221★. *Proposed by D. S. Mitrinović and J. E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*
Let u, v, w be nonnegative numbers and let $0 < t \leq 2$. If a, b, c are the sides of a triangle and if F is its area, prove that

$$\frac{u}{v+w}(bc)^t + \frac{v}{w+u}(ca)^t + \frac{w}{u+v}(ab)^t \geq \frac{3}{2}\left(\frac{4F}{\sqrt{3}}\right)^t.$$

[See Solution II of *Crux* 1051 [1986: 252].]

1224. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
$A_1A_2A_3$ is a triangle with circumcircle Ω. Let ${}_i < X_i$ be the radii of the two circles tangent to A_1A_2, A_1A_3, and arc A_2A_3 of Ω. Let x_2, X_2, x_3, X_3 be defined analogously. Prove that:

$$\text{(a)}\ \sum_{i=1}^{3} \frac{{}_i}{X_i} = 1; \qquad \text{(b)}\ \sum_{i=1}^{3} X_i \geq 3\sum_{i=1}^{3} x_i \geq 12r,$$

where r is the inradius of $\triangle A_1A_2A_3$.

1225★. *Proposed by David Singmaster, The Polytechnic of the South Bank, London, England.*
What convex subset S of a unit cube gives the maximum value for V/A, where V is the volume of S and A is its surface area? (For the two-dimensional case, see *Crux* 870 [1986: 180].)

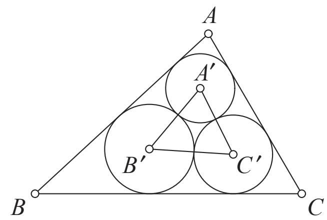
1228. *Proposed by J. Garfunkel, Flushing, New York and C. Gardner, Austin, Texas.*
If QRS is the equilateral triangle of minimum perimeter that can be inscribed in a triangle ABC, show that the perimeter of QRS is at most half the perimeter of ABC, with equality when ABC is equilateral.

1229. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Characterize all positive integers a and b such that

$$(a, b)^{[a,b]} \leq [a, b]^{(a,b)}$$

and determine when equality holds. (As usual, (a, b) and $[a, b]$ denote respectively the g.c.d. and l.c.m. of a and b.)

1234★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Given the Malfatti configuration of three circles inscribed in triangle ABC as shown, let A', B', C' be the centers of the three circles, and let r and r' be the inradii of triangles ABC and $A'B'C'$ respectively. Prove that



$$r \leq (1 + \sqrt{3}\,)r'.$$

Equality is attained when ABC is equilateral.

1236. *Proposed by Gordon Fick, University of Calgary, Calgary, Alberta.*
Prove *without calculus* that if $0 \leq \theta \leq 1$, and $0 \leq y \leq n$ where y and n are integers, then

$$\theta^y(1-\theta)^{n-y} \leq \left(\frac{y}{n}\right)^y \left(1 - \frac{y}{n}\right)^{n-y}.$$

In statistics, this says that the sample proportion is the maximum likelihood estimator of the population proportion. To the best of my knowledge, all mathematical statistics texts prove this result with calculus.

1237★. *Proposed by Niels Bejlegaard, Stavanger, Norway.*
If m_a, m_b, m_c denote the medians to the sides a, b, c of a triangle ABC, and s is the semiperimeter of ABC, show that

$$\sum a\cos A \le \frac{2}{3}\sum m_a \sin A \le s,$$

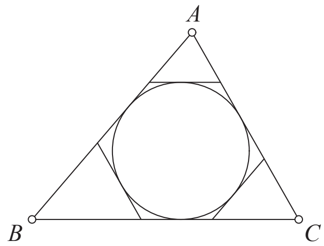
where the sums are cyclic.

1242. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
The following problem appears in a book on matrix analysis: "Show that $\sum\limits_{i,j=1}^{n} a_{ij}x_ix_j$ is positive definite if $\sum\limits_i a_{ii}x_i^2 + \sum_{i\ne j}|a_{ij}|x_ix_j$ is positive definite." Give a counterexample!

1243. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and M an interior point with barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$. The distances of M from the vertices A, B, C are x_1, x_2, x_3 and the circumradii of the triangles MBC, MCA, MAB, ABC are R_1, R_2, R_3, R. Show that

$$\lambda_1R_1 + \lambda_2R_2 + \lambda_3R_3 \ge R \ge \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3.$$

1245. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* (Dedicated to Léo Sauvé.)
Let ABC be a triangle, and let $\mathcal{H}$ be a hexagon created by drawing tangents to the incircle of ABC parallel to the sides of ABC. Prove that



$$\text{perimeter}(\mathcal{H}) \le \frac{2}{3}\text{perimeter}(ABC).$$

When does equality occur?

1247. *Proposed by Robert E. Shafer, Berkeley, California.*
Prove that for $0 \le \phi < \theta \le \pi/2$,

$$\cos^2\frac{\phi}{2}\log\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\log\sin^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}\log\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\log\sin^2\frac{\theta}{2}$$
$$< \frac{3}{4}\left(\sin^{\frac{4}{3}}\theta - \sin^{\frac{4}{3}}\phi\right).$$

1249★. *Proposed by D. S. Mitrinović and J. E. Pecaric, University of Belgrade, Belgrade, Yugoslavia.*
Prove the triangle inequalities

$$\text{(a)}\quad \sum\sin^4 A \le 2 - \frac{1}{2}\left(\frac{r}{R}\right)^2 - 3\left(\frac{r}{R}\right)^4 \le 2 - 5\left(\frac{r}{R}\right)^4;$$

$$\text{(b)}\quad \sum\sin^2 2A \ge 6\left(\frac{r}{R}\right)^2 + 12\left(\frac{r}{R}\right)^4 \ge 36\left(\frac{r}{R}\right)^4;$$

$$\text{(c)}\quad \sum\sin 2B\sin 2C \le 5\left(\frac{r}{R}\right)^2 + 8\left(\frac{r}{R}\right)^3 \le 9\left(\frac{r}{R}\right)^2,$$

where the sums are cyclic over the angles A, B, C of a triangle, and r, R are the inradius and circumradius respectively.

1251. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.* (Dedicated to Léo Sauvé.)

(a) Find all integral n for which there exists a regular n-simplex with integer edge and integer volume.

(b)* Which such n-simplex has the smallest volume?

1252. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and M an interior point with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$. We denote the pedal triangle and the Cevian triangle of M by DEF and $A'B'C'$ respectively. Prove that

$$\frac{[DEF]}{[A'B'C']} \geq 4\lambda_1\lambda_2\lambda_3\left(\frac{s}{R}\right)^2,$$

where s is the semiperimeter and R the circumradius of $\triangle ABC$, and $[X]$ denotes the area of figure X.

1254. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let ABC be a triangle and $n \geq 1$ a natural number. Show that

$$\left|\sum \sin n(B-C)\right| \begin{cases} < 1 & \text{if } n = 1, \\ < \frac{3\sqrt{3}}{2} & \text{if } n = 2, \\ \leq \frac{3\sqrt{3}}{2} & \text{if } n \geq 3, \end{cases}$$

where the sum is cyclic.

1256. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*
Let ABC be a triangle with sides satisfying $a^3 = b^3 + c^3$. Determine the range of angle A.

1258. *Proposed by Ian Witten, University of Calgary, Calgary, Alberta.*
Think of a picture as an $m \times n$ matrix A of real numbers between 0 and 1 inclusive, where a_{ij} represents the brightness of the picture at the point (i, j). To reproduce the picture on a computer we wish to approximate it by an $m \times n$ matrix B of 0's and 1's, such that every "part" of the original picture is "close" to the corresponding part of the reproduction. These are the ideas behind the following definitions:
A *subrectangle* of an $m \times n$ grid is a set of positions of the form

$$\{(i, j) \mid r_1 \leq i \leq r_2,\ s_1 \leq j \leq s_2\}$$

where $1 \leq r_1 \leq r_2 \leq m$ and $1 \leq s_1 \leq s_2 \leq n$ are constants. For any subrectangle R, let

$$d(R) = \left|\sum_{(i,j)\in R} (a_{ij} - b_{ij})\right|,$$

where A and B are as given above, and define

$$d(A, B) = \max d(R),$$

the maximum taken over all subrectangles R.

(a) Show that there exist matrices A such that $d(A, B) > 1$ for every 0-1 matrix B of the same size.

(b)* Is there a constant c such that for every matrix A of any size, there is some 0-1 matrix B of the same size such that $d(A, B) < c$?

1259. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
If $x, y, z \geq 0$, disprove the inequality

$$(yz + zx + xy)^2(x + y + z) \geq 9xyz(x^2 + y^2 + z^2).$$

Determine the largest constant one can replace the 9 with to obtain a valid inequality.

1265. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle with area F and exradii r_a, r_b, r_c, and let $A'B'C'$ be a triangle with area F' and altitudes h'_a, h'_b, h'_c. Show that

$$\frac{r_a}{h'_a} + \frac{r_b}{h'_b} + \frac{r_c}{h'_c} \geq 3\sqrt{\frac{F}{F'}}.$$

1266. *Proposed by Themistocles M. Rassias, Athens, Greece.*
Let $a_1, a_2, \ldots, a_n$ be distinct odd natural numbers, and let $\prod_{i=1}^{n} a_i$ be divisible by exactly k primes, of which p is the smallest. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{I_{p-2}}{I_{p+2k-2}}$$

where

$$I_{2m+1} = \frac{2m(2m-2)\cdots 4\cdot 2}{(2m+1)(2m-1)\cdots 3\cdot 1}.$$

1267. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let $A_1A_2A_3$ be a triangle with inscribed circle I of radius r. Let I_i and J_i, of radii λ_i and μ_i, be the two circles tangent to I and the lines A_1A_2 and A_1A_3. Analogously define circles I_2, J_2, I_3, J_3 of radii λ_2, μ_2, λ_3, μ_3, respectively.

(a) Prove that $\lambda_1\mu_1 = \lambda_2\mu_2 = \lambda_3\mu_3 = r^2$.

(b) Prove that $\displaystyle\sum_{i=1}^{3} \lambda_i + \sum_{i=1}^{3} \mu_i \geq 10r$.

1269★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let ABC be a non-obtuse triangle with circumcenter M and circumradius R. Let u_1, u_2, u_3 be the lengths of the parts of the cevians (through M) between M and the sides opposite to A, B, C respectively. Prove or disprove that

$$\frac{R}{2} \leq \frac{u_1 + u_2 + u_3}{3} < R.$$

1270. *Proposed by Péter Ivády, Budapest, Hungary.*
Prove the inequality

$$\frac{x}{\sqrt{1+x^2}} < \tanh x < \sqrt{1 - \mathrm{e}^{-x^2}}$$

for $x > 0$.

1271. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.* (Dedicated in memoriam to Léo Sauvé.)
Prove that

$$\sqrt{3}\sum \sin\frac{A_i}{2} \geq 4\sum \sin B_i \sin\frac{A_2}{2}\sin\frac{A_3}{2},$$

where $A_1A_2A_3$ and $B_1B_2B_3$ are two triangles and the sums are cyclic over their angles.

1273. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle, M an interior point, and $A'B'C'$ its pedal triangle. Denote the sides of the two triangles by a, b, c and a', b', c' respectively. Prove that

$$\frac{a'}{a}+\frac{b'}{b}+\frac{c'}{c} < 2.$$

1277. *Proposed by Zun Shan and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Determine all possible values of the expression

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1$$

where $n \geq 2$ and $x_i = 1$ or -1 for each i.

1280. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let ABC be a triangle and let A_1, B_1, C_1 be points on BC, CA, AB, respectively, such that

$$\frac{A_1C}{BA_1} = \frac{B_1A}{CB_1} = \frac{C_1B}{AC_1} = k > 1.$$

Show that

$$\frac{k^2-k+1}{k(k+1)} < \frac{\text{perimeter}(A_1B_1C_1)}{\text{perimeter}(ABC)} < \frac{k}{k+1},$$

and that both bounds are best possible.

1281★. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*
Find the area of the largest triangle whose vertices lie in or on a unit n-dimensional cube.

1282. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle, I the incenter, and A', B', C' the intersections of AI, BI, CI with the circumcircle. Show that

$$IA' + IB' + IC' - (IA + IB + IC) \leq 2(R - 2r)$$

where R and r are the circumradius and inradius of $\triangle ABC$.

1283. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Show that the polynomial

$$P(x,y,z) = (x^2+y^2+z^2)^3 - (x^3+y^3+z^3)^2 - (x^2y+y^2z+z^2x)^2 - (xy^2+yz^2+zx^2)^2$$

is nonnegative for all real x, y, z.

1284. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let $A_1A_2A_3A_4$ be a cyclic quadrilateral with $\overline{A_1A_2} = a_1$, $\overline{A_2A_3} = a_2$, $\overline{A_3A_4} = a_3$, $\overline{A_4A_1} = a_4$. Let ρ_1 be the radius of the circle outside the quadrilateral, tangent to the segment A_1A_2 and the extended lines A_2A_3 and A_4A_1. Define ρ_2, ρ_3, ρ_4 analogously. Prove that

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} + \frac{1}{\rho_4} \geq \frac{8}{\sqrt[4]{a_1a_2a_3a_4}}.$$

When does equality hold?

1286. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let x, y, z be positive real numbers. Show that

$$\prod \left[\frac{x(x+y+z)}{(x+y)(x+z)}\right]^x \leq \left[\frac{\left(\sum yz\right)^2}{4xyz(x+y+z)}\right]^{x+y+z},$$

where $\prod$ and $\sum$ are to be understood cyclically.

1288. *Proposed by Len Bos, University of Calgary, Calgary, Alberta.*
Show that for $x_1, x_2, \ldots, x_n > 0$,

$$n(x_1^n + x_2^n + \cdots + x_n^n) \geq (x_1 + x_2 + \cdots + x_n)(x_1^{n-1} + x_2^{n-1} + \cdots + x_n^{n-1}).$$

1289. *Proposed by Carl Friedrich Sutter, Viking, Alberta.*
"To reward you for slaying the dragon", the Queen said to Sir George, "I grant you all the land you can walk around in a day."
She pointed to a pile of wooden stakes. "Take some of these stakes with you", she continued. "Pound them into the ground along the way, and be back at your starting point in 24 hours. All the land in the convex hull of your stakes will then be yours." (The Queen had read a little mathematics.)
Assume that it takes Sir George 1 minute to pound in a stake, and that he walks with constant speed between stakes. How many stakes should he use, to get as much land as possible?

1292*. *Proposed by Jack Garfunkel, Flushing, N. Y.*
It has been shown (see *Crux* 1083 [1987: 96]) that if A, B, C are the angles of a triangle,

$$\frac{2}{\sqrt{3}}\sum \sin A \leq \sum \cos\left(\frac{B-C}{2}\right) \leq \frac{2}{\sqrt{3}}\sum \cos\frac{A}{2},$$

where the sums are cyclic. Prove that

$$\sum \cos\left(\frac{B-C}{2}\right) \leq \frac{1}{\sqrt{3}}\left(\sum \sin A + \sum \cos\frac{A}{2}\right),$$

which if true would imply the right hand inequality above.

1296. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Let r_1, r_2, r_3 be the distances from an interior point of a triangle to its sides a_1, a_2, a_3, respectively, and let R be the circumradius of the triangle. Prove that

$$a_1r_1^n + a_2r_2^n + a_3r_3^n \leq (2R)^{n-2}a_1a_2a_3$$

for all $n \geq 1$, and determine when equality holds.

1297. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* (To the memory of Léo.)
(a) Let $C > 1$ be a real number. The sequence $z_1, z_2, \ldots$ of real numbers satisfies $1 < z_n$ and $z_1 + \cdots + z_n < Cz_{n+1}$ for $n \geq 1$. Prove the existence of a constant $a > 1$ such that $z_n > a^n$, $n \geq 1$.
(b)★ Let conversely $z_1 < z_2 < \cdots$ be a strictly increasing sequence of positive real numbers satisfying $z_n \geq a^n$, $n \geq 1$, where $a > 1$ is a constant. Does there necessarily exist a constant C such that $z_1 + \cdots + z_n < Cz_{n+1}$ for all $n \geq 1$?

1302. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose $a_k > 0$ for $k = 1, 2, \ldots, n$ and $\sum\limits_{k=1}^{n} \tanh^2 a_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sinh a_k} \geq n \sum_{k=1}^{n} \frac{\sinh a_k}{\cosh^2 a_k}.$$

1303. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC and $A_1B_1C_1$ be two triangles with sides a, b, c and a_1, b_1, c_1 and inradii r and r_1, and let P be an interior point of ABC. Set $AP = x$, $BP = y$, $CP = z$. Prove that

$$\frac{a_1x^2 + b_1y^2 + c_1z^2}{a + b + c} \geq 4rr_1.$$

1305. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let $A_1A_2A_3$ be an acute triangle with circumcenter O. Let P_1, Q_1 ($Q_1 \neq A_1$) denote the intersection of A_1O with A_2A_3 and with the circumcircle, respectively, and define P_2, Q_2, P_3, Q_3 analogously. Prove that

(a) $\dfrac{OP_1 \cdot OP_2 \cdot OP_3}{P_1Q_1 \cdot P_2Q_2 \cdot P_3Q_3} \geq 1$;

(b) $\dfrac{OP_1}{P_1Q_1} + \dfrac{OP_2}{P_2Q_2} + \dfrac{OP_3}{P_3Q_3} \geq 3$;

(c) $\dfrac{A_1P_1 \cdot A_2P_2 \cdot A_3P_3}{P_1Q_1 \cdot P_2Q_2 \cdot P_3Q_3} \geq 27$.

1313. *Proposed by Wendel Semenko, Snowflake, Manitoba.*
Show that any triangular piece of paper of area 1 can be folded once so that when placed on a table it will cover an area of less than $\frac{\sqrt{5}-1}{2}$.

1315. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let ABC be a triangle with medians AD, BE, CF and median point G. We denote $\triangle AGF = \triangle_1$, $\triangle BGF = \triangle_2$, $\triangle BGD = \triangle_3$, $\triangle CGD = \triangle_4$, $\triangle CGE = \triangle_5$, $\triangle AGE = \triangle_6$, and let R_i and r_i denote the circumradius and inradius of $\triangle_i$ ($i = 1, 2, \ldots, 6$). Prove that

(i) $R_1R_3R_5 = R_2R_4R_6$;

(ii) $\dfrac{15}{2r} < \dfrac{1}{r_1} + \dfrac{1}{r_3} + \dfrac{1}{r_5} = \dfrac{1}{r_2} + \dfrac{1}{r_4} + \dfrac{1}{r_6} < \dfrac{9}{r}$,

where r is the inradius of $\triangle ABC$.

1318. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*
Find, without calculus, the largest possible value of

$$\frac{\sin 5x + \cos 3x}{\sin 4x + \cos 4x}.$$

1320. *Proposed by Themistocles M. Rassias, Athens, Greece.*
Asumme that $a_1, a_2, a_3, \ldots$ are real numbers satisfying the inequality

$$|a_{m+n} - a_m - a_n| \leq C$$

for all $m, n \geq 1$ and for some constant C. Prove that there exists a constant k such that

$$|a_n - nk| \leq C$$

for all $n \geq 1$.

1327. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let x_1, x_2, x_3 be the distances of the vertices of a triangle from a point P in the same plane. Let r be the inradius of the triangle, and p be the power of the point P with respect to the circumcircle of the triangle. Prove that

$$x_1 x_2 x_3 \geq 2rp.$$

1332. *Proposed by Murray S. Klamkin, University of Alberta.*
It is known that if A, B, C are the angles of a triangle,

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq 1,$$

with equality if and only if the triangle is degenerate with angles $\pi, 0, 0$. Establish the related non-comparable inequality

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq \frac{5r}{R} - 1,$$

where r and R are the inradius and circumradius respectively.

1333. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
If a, b, c and a', b', c' are the sides of two triangles and F, F' are their areas, show that

$$\sum a\,[a' - (\sqrt{b'} - \sqrt{c'})^2] \geq 4\sqrt{3FF'},$$

where the sum is cyclic. (This improves *Crux* 1114 [1987: 185].)

1338. *Proposed by Jean Doyen, Université Libre de Bruxelles, Brussels, Belgium; J. Chris Fisher, University of Regina, Regina, Saskatchewan; and Günter Kist, Technische Universität, Munich, Federal Republic of Germany.*
In a theoretical version of the Canadian lottery "Lotto 6–49", a ticket consists of six distinct integers chosen from 1 to 49 (inclusive). A *t-prize* is awarded for any ticket having t or more numbers in common with a designated "winning" ticket. Denote by $f(t)$ the smallest number of tickets required to be certain of winning a t-prize. Clearly $f(1) = 8$ and $f(6) = \binom{49}{6}$. Show that $f(2) \leq 19$. Can you do better?

1339. *Proposed by Weixuan Li, Changsha Railway Institute, Changsha, Hunan, China, and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Let a, b, m, n denote positive real numbers such that $a \leq b$ and $m \leq n$. Show that

$$(b^m - a^m)^n \leq (b^n - a^n)^m$$

and determine all cases when equality holds.

1341. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*
An ellipse has center O and the ratio of the lengths of the axes is $2+\sqrt{3}$. If P is a point on the ellipse, prove that the (acute) angle between the tangent to the ellipse at P and the radius vector PO is at least 30°.

1344. *Proposed by Florentin Smarandache, Craiova, Romania.*
There are given $mn+1$ points such that among any $m+1$ of them there are two within distance 1 from each other. Prove that there exists a sphere of radius 1 containing at least $n+1$ of the points.

1345. *Proposed by P. Erdos, Hungarian Academy of Sciences, and Esther Szekeres, University of New South Wales, Kensington, Australia.*
Given a convex n-gon $X_1X_2\ldots X_n$ of perimeter p, denote by $f(X_i)$ the sum of the distances of X_i to the other $n-1$ vertices.

(a) Show that if $n \geq 6$, there is a vertex X_i such that $f(X_i) > p$.

(b) Is it true that for n large enough, the *average value* of $f(X_i)$, $1 \leq i \leq n$, is greater than p?

1348★. *Proposed by Murray S. Klamkin, University of Alberta.*
Two congruent convex centrosymmetric planar figures are inclined to each other (in the same plane) at a given angle. Prove or disprove that their intersection has maximum area when the two centers coincide.

1352. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine lower and upper bounds for

$$S_r = \cos^r A + \cos^r B + \cos^r C$$

where A, B, C are the angles of a non-obtuse triangle, and r is a positive real number, $r \neq 1, 2$. (The cases $r=1$ and 2 are known; see items 2.16 and 2.21 of Bottema et al, *Geometric Inequalities.*)

1356★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Show that

$$\frac{x_1}{\sqrt{1-x_1}} + \frac{x_2}{\sqrt{1-x_2}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \frac{\sqrt{x_1}+\cdots+\sqrt{x_n}}{\sqrt{n-1}}$$

for positive real numbers $x_1, \ldots, x_n$ $(n \geq 2)$ satisfying $x_1 + \cdots + x_n = 1$.

1357★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Isosceles right triangles $AA'B$, $BB'C$, $CC'A$ are constructed outwardly on the sides of a triangle ABC, with the right angles at A', B', C', and triangle $A'B'C'$ is drawn. Prove or disprove that

$$\sin A' + \sin B' + \sin C' \geq \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2},$$

where A', B', C' are the angles of $\triangle A'B'C'$.

1361. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Let ABC be a triangle with sides a, b, c and angles α, β, γ, and let its circumcenter lie on the escribed circle to the side a.

(i) Prove that $-\cos\alpha + \cos\beta + \cos\gamma = \sqrt{2}$.

(ii) Find the range of α.

1363★. *Proposed by P. Erdos, Hungarian Academy of Sciences.*
Let there be given n points in the plane, no three on a line and no four on a circle. Is it true that these points must determine at least n distinct distances, if n is large enough? I offer $25 U.S. for the first proof of this.

1365. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Prove that

$$\frac{3}{\pi} < \frac{\sin A}{\pi - A} + \frac{\sin B}{\pi - B} + \frac{\sin C}{\pi - C} < \frac{3\sqrt{3}}{\pi}$$

where A, B, C are the angles (in radians) of an acute triangle.

1366★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove or disprove that

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+z}} + \frac{z}{\sqrt{z+x}} \geq \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{\sqrt{2}}$$

for all positive real numbers x, y, z.

1369. *Proposed by G. R. Veldkamp, De Bilt, The Netherlands.*
The perimeter of a triangle is 24 cm and its area is 24 cm^2. Find the maximal length of a side and write it in a simple form.

1371★. *Proposed by Murray S. Klamkin, University of Alberta.*
In *Math. Gazette* 68 (1984) 222, P. Stanbury noted the close approximation

$$\pi^9/\mathrm{e}^8 \approx 9.999838813 \approx 10.$$

Are there positive integers l, m such that π^l/e^m is closer to a positive integer than for the case given? (See *Crux* 1213 [1988: 116] for a related problem.)

1377. *Proposed by Colin Springer, student, Waterloo, Ontario.*
In right triangle ABC, hypotenuse AC has length 2. Let O be the midpoint of AC and let I be the incentre of the triangle. Show that $OI \geq \sqrt{2} - 1$.

1380. *Proposed by Kee-Wai Lau, Hong Kong.*
Prove the inequality

$$\sin(\tan x) < \tan(\sin x)$$

for $0 < x < \pi$, $x \neq \pi/2$.

1384. *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*
If the center of curvature of every point on an ellipse lies inside the ellipse, prove that the eccentricity of the ellipse is at most $1/\sqrt{2}$.

1386. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $A_1A_2\dots A_n$ be a polygon inscribed in a circle and containing the centre of the circle. Prove that

$$n-2+\frac{4}{\pi}<\sum_{i=1}^{n}\frac{a_i}{\hat{a}_i}\le\frac{n^2}{\pi}\sin\frac{\pi}{n},$$

where a_i is the side A_iA_{i+1} and $\hat{a}_i$ is the arc A_iA_{i+1}.

1389. *Proposed by Derek Chang, California State University, Los Angeles, and Raymond Killgrove, Indiana State University, Terre Haute.*
Find

$$\max_{\pi\in S_n}\sum_{i=1}^{n}|i-\pi(i)|,$$

where S_n is the set of all permutations of $\{1,2,\dots,n\}$.

1390. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*
A, B, C are points on a circle $\mathit{\Gamma}$ such that CM is the perpendicular bisector of AB. P is a point on CM and AP meets $\mathit{\Gamma}$ again at D. As P varies over segment CM, find the largest radius of the inscribed circle tangent to segments PD, PB, and arc DB of $\mathit{\Gamma}$, in terms of the length of CM.

1391. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and D the point on BC so that the incircle of $\triangle ABD$ and the excircle (to side DC) of $\triangle ADC$ have the same radius ρ_1. Define ρ_2, ρ_3 analogously. Prove that

$$\rho_1+\rho_2+\rho_3\ge\frac{9}{4}r,$$

where r is the inradius of $\triangle ABC$.

1392. *Proposed by Angel Dorito, Geld, Ontario.*
An immense spherical balloon is being inflated so that it constantly touches the ground at a fixed point A. A boy standing at a point at unit distance from A fires an arrow at the balloon. The arrow strikes the balloon at its nearest point (to the boy) but does not penetrate it, the balloon absorbing the shock and the arrow falling vertically to the ground. What is the longest distance through which the arrow can fall, and how far from A will it land in this case?

1394. *Proposed by Murray S. Klamkin, University of Alberta.*
If $x, y, z > 0$, prove that

$$\sqrt{y^2+yz+z^2}+\sqrt{z^2+zx+x^2}+\sqrt{x^2+xy+y^2}\ge 3\sqrt{yz+zx+xy}.$$

1399. *Proposed by Sydney Bulman-Fleming and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Prove that

$$\sigma(n!) \le \frac{(n+1)!}{2}$$

for all natural numbers n and determine all cases when equality holds. (Here $\sigma(k)$ denotes the sum of all positive divisors of k.)

1400. *Proposed by Robert E. Shafer, Berkeley, California.*
In a recent issue of the *American Mathematical Monthly* (June-July 1988, page 551), G. Klambauer showed that if $x^s\,\mathrm{e}^{-x} = y^s\,\mathrm{e}^{-y}$ $(x, y, s > 0,\ x \ne y)$ then $x + y > 2s$. Show that if $x^s\,\mathrm{e}^{-x} = y^s\,\mathrm{e}^{-y}$ where $x \ne y$ and $x, y, s > 0$ then $xy(x + y) < 2s^3$.

1401. *Proposed by P. Penning, Delft, The Netherlands.*
Given are a circle C and two straight lines l and m in the plane of C that intersect in a point S inside C. Find the tangent(s) to C intersecting l and m in points P and Q so that the perimeter of $\triangle SPQ$ is a minimum.

1402. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let M be an interior point of the triangle $A_1A_2A_3$ and B_1, B_2, B_3 the feet of the perpendiculars from M to sides A_2A_3, A_3A_1, A_1A_2 respectively. Put $r_i = B_iM$, $i = 1, 2, 3$. R' is the circumradius of $\triangle B_1B_2B_3$, and R, r the circumradius and inradius of $\triangle A_1A_2A_3$. Prove that

$$R'Rr \ge 2r_1r_2r_3.$$

1403★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
For $n \ge 2$, prove or disprove that

$$1 < \frac{x_1 + \cdots + x_n}{n} \le 2$$

for all natural numbers $x_1, x_2, \ldots, x_n$ satisfying

$$x_1 + x_2 + \cdots + x_n = x_1 \cdot x_2 \cdot \ldots \cdot x_n.$$

1406. *Proposed by R. S. Luthar, University of Wisconsin Center, Janesville.*
If $0 < \theta < \pi$, prove without calculus that

$$\cot\frac{\theta}{4} - \cot\theta > 2.$$

1413. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
For $0 < x, y, z < 1$ let

$$u = z(1-y), \quad v = x(1-z), \quad w = y(1-x).$$

Prove that

$$(1-u-v-w)\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right) \ge 3.$$

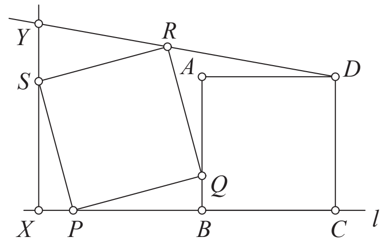
When does equality occur?

1414. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum value of the sum

$$\sqrt{\tan\frac{B}{2}\tan\frac{C}{2}+\lambda}+\sqrt{\tan\frac{C}{2}\tan\frac{A}{2}+\lambda}+\sqrt{\tan\frac{A}{2}\tan\frac{B}{2}+\lambda}$$

where A, B, C are the angles of a triangle and λ is a nonnegative constant. (The case $\lambda = 5$ is item 2.37 of O. Bottema et al, *Geometric Inequalities.*)

1416. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*
In the figure, the unit square $ABCD$ and the line l are fixed, and the unit square $PQRS$ rotates with P and Q lying on l and AB respectively. X is the foot of the perpendicular from S to l. Find the position of point Q so that the length XY is a maximum.



1420. *Proposed by Shailesh Shirali, Rishi Valley School, India.*
If a, b, c are positive integers such that

$$0 < a^2 + b^2 - abc \le c,$$

show that $a^2 + b^2 - abc$ is a perfect square. (This is a generalization of problem 6 of the 1988 I.M.O. [1988: 197].)

1421. *Proposed by J. T. Groenman, Arnhem, The Netherlands, and D. J. Smeenk, Zaltbommel, The Netherlands.*
ABC is a triangle with sides a, b, c. The escribed circle to the side a has centre I_a and touches a, b, c (produced) at D, E, F respectively. M is the midpoint of BC.

(a) Show that the lines I_aD, EF and AM have a common point S_a.

(b) In the same way we have points S_b and S_c. Prove that

$$\frac{\text{area}(\triangle S_aS_bS_c)}{\text{area}(\triangle ABC)} > \frac{3}{2}.$$

1422. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $A_1A_2A_3$ be a triangle and M an interior point; λ_1, λ_2, λ_3 the barycentric coordinates of M; and r_1, r_2, r_3 its distances from the sides A_2A_3, A_3A_1, A_1A_2 respectively. Set $A_iM = R_i$, $i = 1, 2, 3$. Prove that

$$\sum_{i=1}^{3}\lambda_iR_i > 2\left[\lambda_1\cdot\frac{r_2r_3}{r_1}+\lambda_2\cdot\frac{r_3r_1}{r_2}+\lambda_3\cdot\frac{r_1r_2}{r_3}\right].$$

1424. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Show that

$$\sum a\tan A \ge 10R - 2r$$

for any acute triangle ABC, where a, b, c are its sides, R its circumradius, and r its inradius, and the sum is cyclic.

1428★. *Proposed by Svetoslav Bilchev and Emilia Velikova, Technical University, Russe, Bulgaria.*
$A_1A_2A_3$ is a triangle with sides a_1, a_2, a_3, and P is an interior point with distances R_i and r_i ($i = 1, 2, 3$) to the vertices and sides, respectively, of the triangle. Prove that

$$\left(\sum a_1R_1\right)\left(\sum r_1\right) \geq 6\sum a_1r_2r_3$$

where the sums are cyclic.

1429★. *Proposed by D. S. Mitrinović, University of Belgrade, and J. E. Pecaric, University of Zagreb.*
(a) Show that

$$\sup \sum \frac{x_1^2}{x_1^2 + x_2x_3} = n - 1,$$

where $x_1, x_2, \ldots, x_n$ are n positive real numbers ($n \geq 3$), and the sum is cyclic.
(b) More generally, what is

$$\sup \sum \frac{x_1^{r+s}}{x_1^{r+s} + x_2^r x_3^s},$$

for natural numbers r and s?

1430. *Proposed by Mihály Bencze, Brasov, Romania.*
AD, BE, CF are (not necessary concurrent) Cevians in triangle ABC, intersecting the circumcircle of $\triangle ABC$ in the points P, Q, R. Prove that

$$\frac{AD}{DP} + \frac{BE}{EQ} + \frac{CF}{FR} \geq 9.$$

When does equality hold?

1440★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Prove or disprove that if A, B, C are the angles of a triangle,

$$\frac{\sin A}{\sqrt{\sin A + \sin B}} + \frac{\sin B}{\sqrt{\sin B + \sin C}} + \frac{\sin C}{\sqrt{\sin C + \sin A}} \leq \frac{3}{2} \cdot \sqrt[4]{3}.$$

1443. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Given an integer $n \geq 2$, determine the minimum value of

$$\sum_{\substack{1 \leq i,j \leq n \\ i \neq j}} \left(\frac{x_i^2}{x_j}\right)$$

over all positive real numbers $x_1, \ldots, x_n$ such that $x_1^2 + \cdots + x_n^2 = 1$.

1445. *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*
Determine the minimum value of

$$\frac{x^3}{1 - x^8} + \frac{y^3}{1 - y^8} + \frac{z^3}{1 - z^8}$$

where $x, y, z \geq 0$ and $x^4 + y^4 + z^4 = 1$.

1448. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If A, B, C are the angles of a triangle, prove that

$$\frac{2}{3}\left(\sum \sin\frac{A}{2}\right)^2 \geq \sum \cos A,$$

with equality when $A = B = C$.

1449. *Proposed by David C. Vaughan, Wilfrid Laurier University.*
Prove that for all $x \geq y \geq 1$,

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} \geq \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}.$$

1452. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let x_1, x_2, x_3 be positive reals satisfying $x_1 + x_2 + x_3 = 1$, and consider the inequality

$$(1 - x_1)(1 - x_2)(1 - x_3) \geq c_r(x_1x_2x_3)^r. \tag{1}$$

For each real r, find the greatest constant c_r such that (1) holds for all choices of the x_i, or prove that no such constant c_r exists.

1454. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Given a convex pentagon of area S, let $S_1, \ldots, S_5$ denote the areas of the five triangles cut off by the diagonals (each triangle is spanned by three consecutive vertices of the pentagon). Prove that the sum of some four of the S_i's exceeds S.

1457. *Proposed by Colin Springer, student, University of Waterloo.*
In $\triangle ABC$, the sides are a, b, c, the perimeter is p and the circumradius is R. Show that

$$R^2 p \geq \frac{a^2b^2}{a+b-c}.$$

Under what conditions does equality hold?

1460. *Proposed by Mihály Bencze, Brasov, Romania.*
P is an interior point of a convex n-gon $A_1A_2 \ldots A_n$. For each $i = 1, \ldots, n$ let $R_i = \overline{PA_i}$ and w_i be the length of the bisector of $\sphericalangle P$ in $\triangle A_iPA_{i+1}$ ($A_{n+1} = A_1$). Also let $c_1, \ldots, c_n$ be positive real numbers. Prove that

$$2\cos\frac{\pi}{n}\sum_{i=1}^{n} c_i^2 \geq \sum_{i=1}^{n} c_ic_{i+1}w_i\left(\frac{1}{R_i} + \frac{1}{R_{i+1}}\right)$$

($R_{n+1} = R_1$).

1461. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let a, b, c, r, R, s be the sides, inradius, circumradius, and semiperimeter of a triangle and let a', b', c', r', R', s' be similarly defined for a second triangle. Show that

$$\left(4ss' - \sum aa'\right)^2 \geq 4\,(s^2 + r^2 + 4Rr)(s'^2 + r'^2 + 4R'r'),$$

where the sum is cyclic.

1462★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
If A, B, C are the angles of a triangle, prove or disprove that

$$\sqrt{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right) \geq \sqrt{\sin\frac{A}{2}}+\sqrt{\sin\frac{B}{2}}+\sqrt{\sin\frac{C}{2}},$$

with equality when $A = B = C$.

1472. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
For each integer $n \geq 2$, find the largest constant c_n such that

$$c_n \sum_{i=1}^{n} |a_i| \leq \sum_{i<j} |a_i - a_j|$$

for all real numbers $a_1, \ldots, a_n$ satisfying $\sum_{i=1}^{n} a_i = 0$.

1473★. *Proposed by Murray S. Klamkin, University of Alberta.*
Given is a unit circle and an interior point P. Find the convex n-gon of largest area and/or perimeter which is inscribed in the circle and passes through P.

1478★. *Proposed by D. M. Milošević, Pranjani, Yugoslavia.*
A circle of radius R is circumscribed about a regular n-gon. A point on the circle is at distances $a_1, a_2, \ldots, a_n$ from the vertices of the n-gon. Prove that

$$\sum_{i=1}^{n} a_i^3 \geq 2R^3 n\sqrt{2}.$$

1479. *Proposed by Vedula N. Murty, Pennsylvania State University at Harrisburg.*
Given $x > 0$, $y > 0$ satisfying $x^2 + y^2 = 1$, show without calculus that

$$x^3 + y^3 \geq \sqrt{2}xy.$$

1484. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $0 < r, s, t \leq 1$ be fixed. Show that the relation

$$r \cot rA = s \cot sB = t \cot tC$$

holds for exactly one triangle ABC, and that this triangle maximizes the expression

$$\sin rA \sin sB \sin tC$$

over all triangles ABC.

1487. *Proposed by Kee-Wai Lau, Hong Kong.*
Prove the inequality

$$x + \sin x \geq 2\log(1 + x)$$

for $x > -1$.

1488. *Proposed by Avinoam Freedman, Teaneck, N. J.*
Prove that in any acute triangle, the sum of the circumradius and the inradius is less than the length of the second-longest side.

1490$^\bigstar$**.** *Proposed by Jack Garfunkel, Flushing, N. Y.*
This was suggested by Walther Janous' problem *Crux* 1366 [1989: 271].
Find the smallest constant k such that

$$\frac{x}{\sqrt{x+y}}+\frac{y}{\sqrt{y+z}}+\frac{z}{\sqrt{z+x}}\le k\sqrt{x+y+z}$$

for all positive x, y, z.

1493. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Two squares $ABDE$ and $ACFG$ are described on AB and AC outside the triangle ABC. P and Q are on line EG such that BP and CQ are perpendicular to BC. Prove that

$$BP+CQ\ge BC+EG.$$

When does equality hold?

1498$^\bigstar$**.** *Proposed by D. M. Milošević, Pranjani, Yugoslavia.*
Show that

$$\prod_{i=1}^{3}h_i^{a_i}\le(3r)^{2s},$$

where a_1, a_2, a_3 are the sides of a triangle, h_1, h_2, h_3 its altitudes, r its inradius, and s its semi-perimeter.

1504. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $A_1A_2\dots A_n$ be a circumscribable n-gon with incircle of radius 1, and let $F_1, F_2, \dots, F_n$ be the areas of the n corner regions inside the n-gon and outside the incircle. Show that

$$\frac{1}{F_1}+\cdots+\frac{1}{F_n}\ge\frac{n^2}{n\tan\frac{\pi}{n}-\pi}.$$

Equality holds for the regular n-gon.

1508. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
Let $a\le b<c$ be the lengths of the sides of a right triangle. Find the largest constant K such that

$$a^2(b+c)+b^2(c+a)+c^2(a+b)\ge Kabc$$

holds for all right triangles and determine when equality holds. It is known that the inequality holds when $K=6$ (problem 351 of the *College Math. Journal*; solution on p. 259 of Volume 20, 1989).

1509. *Proposed by Carl Friedrich Sutter, Viking, Alberta.*
Professor Chalkdust teaches two sections of a mathematics course, with the same material taught in both sections. Section 1 runs Mondays, Wednesdays, and Fridays for 1 hour each day, and Section 2 runs Tuesdays and Thursdays for 1.5 hours each day. Normally Professor Chalkdust covers one unit of material per hour, but if she is teaching some material for the second time she teaches twice as fast. The course began a Monday. In the long run (i. e. after N weeks as $N\to\infty$) will one section be taught more material than the other? If so, which one, and how much more?

1510★. *Proposed by Jack Garfunkel, Flushing, N. Y.*
P is any point inside a triangle ABC. Lines PA, PB, PC are drawn and angles PAC, PBA, PCB are denoted by α, β, γ respectively. Prove or disprove that

$$\cot\alpha + \cot\beta + \cot\gamma \geq \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2},$$

with equality when P is the incenter of $\triangle ABC$.

1512★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Given $r > 0$, determine a constant $C = C(r)$ such that

$$(1+z)^r(1+z^r) \leq C\,(1+z^2)^r$$

for all $z > 0$.

1515. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
We are given a finite collection of segments in the plane, of total length 1. Prove that there exists a straight line l such that the sum of lengths of projections of the given segments to line l is less than $2/\pi$.

1516. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
ABC is an isosceles triangle in which $AB = AC$ and $\sphericalangle A < 90°$. Let D be any point on segment BC. Draw CE parallel to AD meeting AB produced in E. Prove that $CE > 2CD$.

1523. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $0 < t \leq \frac{1}{2}$ be fixed. Show that

$$\sum \cos tA \geq 2 + \sqrt{2}\cos\left(t + \frac{1}{4}\right)\pi + \sum \sin tA,$$

where the sums are cyclic over the angles A, B, C of a triangle. [This generalizes Murray Klamkin's problem E3180 in the *Amer. Math. Monthly* (solution p. 771, October 1988.]

1524. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
ABC is a triangle with sides a, b, c and area F, and P is an interior point. Put $R_1 = AP$, $R_2 = BP$, $R_3 = CP$. Prove that the triangle with sides aR_1, bR_2, cR_3 has circumradius at least $4F/(3\sqrt{3})$.

1528★. *Proposed by Ji Chen, Ningbo University, China.*
If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, prove or disprove that

$$\frac{a^2}{(a^2+1)^2} + \frac{b^2}{(b^2+1)^2} + \frac{c^2}{(c^2+1)^2} + \frac{d^2}{(d^2+1)^2} \leq \frac{16}{25}.$$

1530★. *Proposed by D. S. Mitrinović, University of Belgrade, and J. E. Pečarić, University of Zagreb.*
Let

$$I_k = \frac{\int_0^{\pi/2} \sin^{2k} x \,\mathrm{d}x}{\int_0^{\pi/2} \sin^{2k+1} x \,\mathrm{d}x}$$

where k is a natural number. Prove that

$$1 \leq I_k \leq 1 + \frac{1}{2k}.$$

1531. *Proposed by J. T. Groenman, Arnhem, The Netherlands.*
Prove that

$$\frac{v+w}{u}\cdot\frac{bc}{s-a}+\frac{w+u}{v}\cdot\frac{ca}{s-b}+\frac{u+v}{w}\cdot\frac{ab}{s-c}\geq 4\,(a+b+c),$$

where a, b, c, s are the sides and semiperimeter of a triangle, and u, v, w are positive real numbers. (Compare with *Crux* 1212 [1988: 115].)

1533. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
For any integers $n \geq k \geq 0$, $n \geq 1$, denote by $p(n,k)$ the probability that a randomly chosen permutation of $\{1, 2, \ldots, n\}$ has exactly k fixed points, and let

$$P(n) = p(n,0)p(n,1)\cdots p(n,n).$$

Prove that

$$P(n) \leq \exp(-2^n n!).$$

1534. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Triangle $H_1H_2H_3$ is formed by joining the feet of the altitudes of an acute triangle $A_1A_2A_3$. Prove that

$$\frac{s}{r}\leq\frac{s'}{r'},$$

where s, s' and r, r' are the semiperimeters and inradii of $A_1A_2A_3$ and $H_1H_2H_3$ respectively.

1539★**.** *Proposed by D. M. Milošević, Pranjani, Yugoslavia.*
If α, β, γ are the angles, s the semiperimeter, R the circumradius and r the inradius of a triangle, prove or disprove that

$$\sum\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}\leq\left(\frac{2R-r}{s}\right)^2,$$

where the sum is cyclic.

1542★**.** *Proposed by Murray S. Klamkin, University of Alberta.*
For fixed n, determine the minimum value of

$$C_n = |\cos\theta| + |\cos 2\theta| + \cdots + |\cos n\theta|.$$

It is conjectured that $\min C_n = [n/2]$ for $n > 2$.

1543. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Show that the circumradius of a triangle is at least four times the inradius of the pedal triangle of any interior point.

1546. *Proposed by Graham Denham, student, University of Alberta.*
Prove that for every positive integer n and every positive real x,

$$\sum_{k=1}^{n}\frac{x^{k^2}}{k}\geq x^{\frac{n(n+1)}{2}}.$$

1550. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $A = [-1, 1]$. Find all functions $f : A \to A$ such that

$$|xf(y) - yf(x)| \geq |x - y|$$

for all $x, y \in A$.

1553. *Proposed by Murray S. Klamkin, University of Alberta.*
It has been shown by Oppenheim that if $ABCD$ is a tetrahedron of circumradius R, a, b, c are the edges of face ABC, and p, q, r are the edges AD, BD, CD, then

$$64R^4 \geq (a^2 + b^2 + c^2)(p^2 + q^2 + r^2).$$

Show more generally that, for n-dimensional simplexes,

$$(n + 1)^4 R^4 \geq 4E_0 E_1,$$

where E_0 is the sum of the squares of all edges emanating from one of the vertices and E_1 is the sum of the squares of all the other edges.

1558. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let P be an interior point of a triangle ABC and let AP, BP, CP intersect the circumcircle of $\triangle ABC$ again in A', B', C', respectively. Prove that the power p of P with respect to the circumcircle satisfies

$$|p| \geq 4rr',$$

where r, r' are the inradii of triangles ABC and $A'B'C'$.

1562. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Let M be the midpoint of BC of a triangle ABC such that $\sphericalangle B = 2\sphericalangle C$, and let D be the intersection of the internal bisector of angle C with AM. Prove that $\sphericalangle MDC \leq 45°$.

1567. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*
Let

$$f(x_1, x_2, \ldots, x_n) = \frac{x_1\sqrt{x_1 + \cdots + x_n}}{(x_1 + \cdots + x_{n-1})^2 + x_n}.$$

Prove that $f(x_1, x_2, \ldots, x_n) \leq \sqrt{2}$ under the condition that $x_1 + \cdots + x_n \geq 2$ and all $x_i \geq 0$.

1568. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Show that

$$\sum \sin A \geq \frac{2}{\sqrt{3}} \left(\sum \cos A\right)^2$$

where the sums are cyclic over the angles A, B, C of an acute triangle.

1571. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle with circumradius R and area F, and let P be a point in the same plane. Put $AP = R_1$, $BP = R_2$, $CP = R_3$, R' the circumradius of the pedal triangle of P, and p the power of P relative to the circumcircle of $\triangle ABC$. Prove that

$$18R^2R' \geq a^2R_1 + b^2R_2 + c^2R_3 \geq 4F\sqrt{3\,|p|}.$$

1574. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine sharp upper and lower bounds for the sum of the squares of the sides of a quadrilateral with given diagonals e and f. For the upper bound, it is assumed that the quadrilateral is convex.

1578. *Proposed by O. Johnson and C. S. Goodlad, students, King Edward's School, Birmingham, England.*
For each fixed positive real number a_n, maximise

$$\frac{a_1 a_2 \cdots a_{n-1}}{(1+a_1)(a_1+a_2)(a_2+a_3)\cdots(a_{n-1}+a_n)}$$

over all positive real numbers $a_1, a_2, \ldots, a_{n-1}$.

1580★. *Proposed by Ji Chen, Ningbo University, China.*
For every convex n-gon, if one circle with centre O and radius R contains it and another circle with centre I and radius r is contained in it, prove or disprove that

$$R^2 \geq r^2 \sec^2 \frac{\pi}{n} + \overline{IO}^2.$$

1581★. *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*
If T_1 and T_2 are two triangles with equal circumradii, it is easy to show that if the angles of T_2 majorize the angles of T_1, then the area and perimeter of T_2 is not greater than the area and perimeter, respectively, of T_1. (One uses the concavity of $\sin x$ and $\log \sin x$ in $(0, \pi)$.) If T_1 and T_2 are two tetrahedra with equal circumradii, and the solid angles of T_2 majorize the solid angles of T_1, is it true that the volume, the surface area, and the total edge length of T_2 are not larger than the corresponding quantities for T_1?

1584. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove that for $\lambda > 1$

$$\left(\frac{\ln \lambda}{\lambda - 1}\right)^3 < \frac{2}{\lambda(\lambda+1)}.$$

1586. *Proposed by Jack Garfunkel, Flushing, N. Y.*
Let ABC be a triangle with angles $A \geq B \geq C$ and sides $a \geq b \geq c$, and let $A'B'C'$ be a triangle with sides

$$a' = a + \lambda, \qquad b' = b + \lambda, \qquad c' = c + \lambda$$

where λ is a positive constant. Prove that $A - C \geq A' - C'$ (i. e., $\triangle A'B'C'$ is in a sense "more equilateral" than $\triangle ABC$).

1588. *Proposed by D. M. Milošević, Pranjani, Yugoslavia.*
Show that

$$\sin B \sin C \leq 1 - \frac{a^2}{(b+c)^2},$$

where a, b, c are the sides of the triangle ABC.

1589. *Proposed by Mihály Bencze, Brasov, Romania.*
Prove that, for any natural number n,

$$\sqrt[n]{n!} + \sqrt[n+2]{(n+2)!} < 2 \cdot \sqrt[n+1]{(n+1)!}.$$

1592. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
If P is a monic polynomial of degree $n > 1$, having n negative roots (counting multiplicities), show that

$$P'(0)P(1) \geq 2n^2 P(0),$$

and find conditions for equality.

1598★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $\lambda > 0$. Determine the maximum constant $C = C(\lambda)$ such that for all non-negative real numbers x_1, x_2 there holds

$$x_1^2 + x_2^2 + \lambda x_1 x_2 \geq C\,(x_1 + x_2)^2.$$

1599. *Proposed by Milen N. Naydenov, Varna, Bulgaria.*
A convex quadrilateral with sides a, b, c, d has both an incircle and a circumcircle. Its circumradius is R and its area F. Prove that

$$abc + abd + acd + bcd \leq 2\sqrt{F}\,(F + 2R^2).$$

1601. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
ABC is a right-angled triangle with the right angle at A. Let D be the foot of the perpendicular from A to BC, and let E and F be the intersections of the bisector of $\sphericalangle B$ with AD and AC respectively. Prove that $DC > 2\,EF$.

1602. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Suppose $x_1, x_2, \ldots, x_n \in [0,1]$ and $\sum_{i=1}^n x_i = m + r$ where m is an integer and $r \in [0,1)$. Prove that

$$\sum_{i=1}^n x_i^2 \leq m + r^2.$$

1606★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
For integers $n \geq k \geq 1$ and real x, $0 \leq x \leq 1$, prove or disprove that

$$\left(1 - \frac{x}{k}\right)^n \geq \sum_{j=0}^{k-1} \left(1 - \frac{j}{k}\right) \binom{n}{j} x^j (1-x)^{n-j}.$$

1609. *Proposed by John G. Heuver, Grande Prairie Composite H. S., Grande Prairie, Alberta.*
P is a point in the interior of a tetrahedron $ABCD$ of volume V, and F_a, F_b, F_c, F_d are the areas of the faces opposite vertices A, B, C, D, respectively. Prove that

$$PA \cdot F_a + PB \cdot F_b + PC \cdot F_c + PD \cdot F_d \geq 9V.$$

1610. *Proposed by P. Penning, Delft, The Netherlands.*
Consider the multiplication $d \times dd \times ddd$, where $d < b - 1$ is a nonzero digit in base b, and the product (base b) has six digits, all less than $b - 1$ as well. Suppose that, when d and the digits of the product are all increased by 1, the multiplication is still true. Find the lowest base b in which this can happen.

1611. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle with angles A, B, C (measured in radians), sides a, b, c, and semiperimeter s. Prove that

$$\text{(i)}\ \sum \frac{b+c-a}{A} \geq \frac{6s}{\pi}; \qquad \text{(ii)}\ \sum \frac{b+c-a}{aA} \geq \frac{9}{\pi}.$$

1612★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let x, y, z be positive real numbers. Show that

$$\sum \frac{y^2 - x^2}{z + x} \geq 0,$$

where the sum is cyclic over x, y, z, and determine when equality holds.

1613. *Proposed by Murray S. Klamkin, University of Alberta.*
Prove that

$$\left(\frac{\sin x}{x}\right)^{2p} + \left(\frac{\tan x}{x}\right)^{p} \geq 2$$

for $p \geq 0$ and $0 < x < \pi/2$. (The case $p = 1$ is problem E3306, *American Math. Monthly*, solution in March 1991, pp. 264–267.)

1619. *Proposed by Hui-Hua Wan and Ji Chen, Ningbo University, Zhejiang, China.*
Let P be an interior point of a triangle ABC and let R_1, R_2, R_3 be the distances from P to the vertices A, B, C, respectively. Prove that, for $0 < k < 1$,

$$R_1^k + R_2^k + R_3^k < (1 + 2^{\frac{1}{k-1}})^{1-k}(a^k + b^k + c^k).$$

1621★. *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to Jack Garfunkel.)
Let P be a point within or on an equilateral triangle and let $c_1 \leq c_2 \leq c_3$ be the lengths of the three concurrent cevians through P. Determine the minimum value of c_2/c_3 over all P.

1622. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Let n be a positive integer.
(a) Prove the inequality

$$\frac{a^{2n} + b^{2n}}{2} \leq \left(\left(\frac{a+b}{2}\right)^2 + (2n-1)\left(\frac{a-b}{2}\right)^2\right)^n$$

for real a, b, and find conditions for equality.
(b) Show that the constant $2n - 1$ in the right-hand expression is the best possible, in the sense that on replacing it by a smaller one we get an inequality which fails to hold for some a, b.

1627. *Proposed by George Tsintsifas, Thessaloniki, Greece.* (Dedicated to Jack Garfunkel.)
Two perpendicular chords MN and ET partition the circle (O, R) into four parts Q_1, Q_2, Q_3, Q_4. We denote by (O_i, r_i) the incircle of Q_i, $1 \leq i \leq 4$. Prove that

$$r_1 + r_2 + r_3 + r_4 \leq 4(\sqrt{2} - 1)R.$$

1629. *Proposed by Rossen Ivanov, student, St. Kliment Ohridsky University, Sofia, Bulgaria.*
In a tetrahedron x and v, y and u, z and t are pairs of opposite edges, and the distances between the midpoints of each pair are respectively l, m, n. The tetrahedron has surface area S, circumradius R, and inradius r. Prove that, for any real number a with $0 \leq a \leq 1$,

$$x^{2a}v^{2a}l^2 + y^{2a}u^{2a}m^2 + z^{2a}t^{2a}n^2 \geq \left(\frac{\sqrt{3}}{4}\right)^{1-a}(2S)^{1+a}(Rr)^a.$$

1630. *Proposed by Isao Ashiba, Tokyo, Japan.*
Maximize

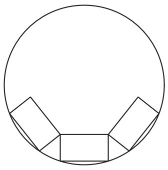
$$a_1a_2 + a_3a_4 + \cdots + a_{2n-1}a_{2n}$$

over all permutations $a_1, a_2, \ldots, a_{2n}$ of the set $\{1, 2, \ldots, 2n\}$.

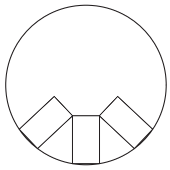
1631★. *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to Jack Garfunkel.)
Let P be a point within or on an equilateral triangle and let c_1, c_2, c_3 be the lengths of the three concurrent cevians through P. Determine the largest constant λ such that $c_1^\lambda, c_2^\lambda, c_3^\lambda$ are the sides of a triangle for any P.

1633. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
In triangle ABC, the internal bisectors of $\sphericalangle B$ and $\sphericalangle C$ meet AC and AB at D and E, respectively. We put $\sphericalangle BDE = x$, $\sphericalangle CED = y$. Prove that if $\sphericalangle A > 60^\circ$ then $\cos 2x + \cos 2y > 1$.

1634. *Proposed by F. F. Nab, Tunnel Mountain, Alberta.*
A cafeteria at a universtity has round tables (of various sizes) and rectangular trays (all the same size). Diners place their trays of food on the table in one of two ways, depending on whether the short or long sides of the trays point toward the centre of the table:



or



Moreover, at the same table everybody aligns their trays the same way. Suppose n mathematics students come in to eat together. How should they align their trays so that the table needed is as small as possible?

1636★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Determine the set of all real exponents r such that

$$d_r(x,y) = \frac{|x-y|}{(x+y)^r}$$

satisfies the triangle inequality

$$d_r(x,y) + d_r(y,z) \ge d_r(x,z) \quad \text{for all } x,y,z > 0$$

(and thus induces a metric on $\mathbb{R}^+$ – see *Crux* 1449, esp. [1990: 224]).

1637. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Prove that

$$\sum \frac{\sin B + \sin C}{A} > \frac{12}{\pi}$$

where the sum is cyclic over the angles A, B, C (measured in radians) of a nonobtuse triangle.

1639. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*
$ABCD$ is a convex cyclic quadrilateral. Prove that

$$(AB+CD)^2 + (AD+BC)^2 \ge (AC+BD)^2.$$

1642. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum value of

$$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2)$$

subject to $yz + zx + xy = 1$ and $x, y, z \ge 0$.

1649$^{\bigstar}$**.** *Proposed by D. M. Milošević, Pranjani, Yugoslavia.*
Prove or disprove that

$$\sum \cot \frac{\alpha}{2} - 2 \sum \cot \alpha \geq \sqrt{3},$$

where the sums are cyclic over the angles α, β, γ of a triangle.

1651. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and A_1, B_1, C_1 the common points of the inscribed circle with the sides BC, CA, AB, respectively. We denote the length of the arc B_1C_1 (not containing A_1) of the incircle by S_a, and similarly define S_b and S_c. Prove that

$$\frac{a}{S_a} + \frac{b}{S_b} + \frac{c}{S_c} \geq \frac{9\sqrt{3}}{\pi}.$$

1652. *Proposed by Murray S. Klamkin, University of Alberta.*
Given fixed constants $a, b, c > 0$ and $m > 1$, find all positive values of x, y, z which minimize

$$\frac{x^m + y^m + z^m + a^m + b^m + c^m}{6} - \left(\frac{x + y + z + a + b + c}{6}\right)^m.$$

1654$^{\bigstar}$**.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let x, y, z be positive real numbers. Show that

$$\sum \frac{x}{x + \sqrt{(x+y)(x+z)}} \leq 1,$$

where the sum is cyclic over x, y, z, and determine when equality holds.

1656. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*
Given a triangle ABC, we take variable points P on segment AB and Q on segment AC. CP meets BQ in T. Where should P and Q be located so that the area of $\triangle PQT$ is maximized?

1662. *Proposed by Murray S. Klamkin, University of Alberta.*
Prove that

$$\frac{x_1^{2r+1}}{s - x_1} + \frac{x_2^{2r+1}}{s - x_2} + \cdots + \frac{x_n^{2r+1}}{s - x_n} \geq \frac{4^r}{(n-1)n^{2r-1}} (x_1x_2 + x_2x_3 + \cdots + x_nx_1)^r,$$

where $n > 3$, $r \geq 1/2$, $x_i \geq 0$ for all i, and $s = x_1 + x_2 + \cdots + x_n$. Also, find some values of n and r such that the inequality is sharp.

1663$^{\bigstar}$**.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let A, B, C be the angles of a triangle, r its inradius and s its semiperimeter. Prove that

$$\sum \sqrt{\cot \frac{A}{2}} \leq \frac{3}{2}\sqrt{\frac{r}{s}} \sum \csc \frac{A}{2},$$

where the sums are cyclic over A, B, C.

1664. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.* (Dedicated to Jack Garfunkel.)
Consider two concentric circles with radii R_1 and R ($R_1 > R$) and a triangle ABC inscribed in the inner circle. Points A_1, B_1, C_1 on the outer circle are determined by extending BC, CA, AB, respectively. Prove that

$$\frac{F_1}{R_1^2} \geq \frac{F}{R^2},$$

where F_1 and F are the areas of triangles $A_1B_1C_1$ and ABC respectively, with equality when ABC is equilateral.

1666. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
(a) How many ways are there to select and draw a triangle T and a quadrilateral Q around a common incircle of unit radius so that the area of $T \cap Q$ is as small as possible? (Rotations and reflections of the figure are not considered different.)
(b)$^\bigstar$ The same question, with the triangle and the quadrilateral replaced by an m-gon and an n-gon, where $m, n \geq 3$.

1672. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Show that for positive real numbers a, b, c, x, y, z,

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) \geq 3\left(\frac{xy+yz+zx}{x+y+z}\right),$$

and determine when equality holds.

1674. *Proposed by Murray S. Klamkin, University of Alberta.*
Given positive real numbers r, s and an integer $n > r/s$, find positive $x_1, x_2, \ldots, x_n$ so as to minimize

$$\left(\frac{1}{x_1^r} + \frac{1}{x_2^r} + \cdots + \frac{1}{x_n^r}\right)(1+x_1)^s(1+x_2)^s \cdots (1+x_n)^s.$$

1676. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*
OA is a fixed radius and OB a variable radius of a unit circle, such that $\sphericalangle AOB \leq 90°$. $PQRS$ is a square inscribed in the sector OAB so that PQ lies along OA. Determine the minimum length of OS.

1678. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Show that

$$\sqrt{s}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}) \leq \sqrt{2}\,(r_a + r_b + r_c),$$

where a, b, c are the sides of a triangle, s the semiperimeter, and r_a, r_b, r_c the exradii.

1680. *Proposed by Zun Shan and Ji Chen, Ningbo University, China.*
If m_a, m_b, m_c are the medians and r_a, r_b, r_c the exradii of a triangle, prove that

$$\frac{r_br_c}{m_bm_c} + \frac{r_cr_a}{m_cm_a} + \frac{r_ar_b}{m_am_b} \geq 3.$$

1691★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $n \geq 2$. Determine the best upper bound of

$$\frac{x_1}{x_2x_3\cdots x_n+1}+\frac{x_2}{x_1x_3\cdots x_n+1}+\cdots+\frac{x_n}{x_1x_2\cdots x_{n-1}+1},$$

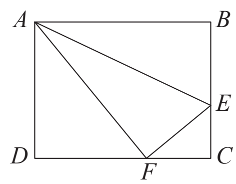
over all $x_1,\ldots,x_n$ with $0 \leq x_i \leq 1$ for $i=1,2,\ldots,n$.

1695. *Proposed by Seung-Jin Bang, Seoul, Republic of Korea.*
Let $p(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5$ with $a_0>0$ and

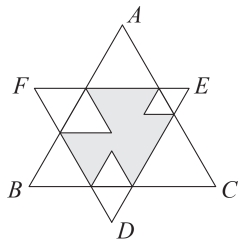
$$a_0+\frac{a_0+a_2}{3}+\frac{a_2+a_4}{5}+\frac{a_4}{7}<0.$$

Prove that there exists at least one zero of $p(x)$ in the interval $(-1,1)$.

1696. *Proposed by Ed Barbeau, University of Toronto.*
An $8\frac{1}{2}$ by 11 sheet of paper is folded along a line AE through the corner A so that the adjacent corner B on the longer side lands on the opposite longer side CD at F. Determine, with a minimum of measurement or computation, whether triangle AEF covers more than half the quadrilateral $AECD$.



1698. *Proposed by Hidetosi Fukagawa, Aichi, Japan.*
ABC is an equilateral triangle of area 1. DEF is an equilateral triangle of variable size, placed so that the two triangles overlap, with $DE \parallel AB$, $EF \parallel BC$, $FD \parallel CA$, and D,E,F not in $\triangle ABC$, as shown. The corners of $\triangle DEF$ sticking outside $\triangle ABC$ are then folded over. Find the maximum possible area of the uncovered (shaded) part of $\triangle DEF$.



1699. *Proposed by Xue-Zhi Yang and Ji Chen, Ningbo University, China.*
Let $R,r,h_a,h_b,h_c,r_a,r_b,r_c$ be the circumradius, inradius, altitudes, and exradii of a triangle. Prove that

$$\sqrt{\frac{2R}{r}+5} \leq \sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b}{h_b}}+\sqrt{\frac{r_c}{h_c}} \leq \sqrt{\frac{4R}{r}+1}.$$

1701★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
If ABC is a triangle, prove or disprove that

$$R \geq 4\max\left\{\frac{h_a\cos A}{1+8\cos^2 A},\frac{h_b\cos B}{1+8\cos^2 B},\frac{h_c\cos C}{1+8\cos^2 C}\right\},$$

where h_a,h_b,h_c are the altitudes of the triangle and R is its circumradius.

1703. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the maximum and minimum values of

$$x^2+y^2+z^2+\lambda xyz,$$

where $x+y+z=1$, $x,y,z \geq 0$, and λ is a given constant.

1707. *Proposed by Allan Wm. Johnson Jr., Washington, D. C.*
What is the largest integer m for which an $m \times m$ square can be cut up into 7 rectangles whose dimensions are $1,2,\ldots,14$ in some order?

1712. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the minimum value of

$$\frac{16\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)+1}{\tan(A/2)\tan(B/2)\tan(C/2)}$$

where A, B, C are the angles of a triangle.

1713. *Proposed by Jeremy Bern, student, Ithaca H. S., Ithaca, N. Y.*
For a fixed positive integer n, let K be the area of the region

$$\left\{z : \sum_{k=1}^{n}\left|\frac{1}{z-k}\right| \geq 1\right\}$$

in the complex plane. Prove that $K \geq \pi(11n^2+1)/12$.

1730. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Prove that

$$\sum bc(s-a)^2 \geq \frac{sabc}{2},$$

where a, b, c, s are the sides and semiperimeter of a triangle, and the sum is cyclic over the sides.

1734. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the minimum value of

$$\sqrt{(1-ax)^2+(ay)^2+(az)^2}+\sqrt{(1-by)^2+(bz)^2+(bx)^2}+\sqrt{(1-cz)^2+(cx)^2+(cy)^2}$$

for all real values of a, b, c, x, y, z.

1742. *Proposed by Murray S. Klamkin, University of Alberta.*
Let $1 \leq r < n$ be integers and $x_{r+1}, x_{r+2}, \ldots, x_n$ be given positive real numbers. Find positive $x_1, x_2, \ldots, x_r$ so as to minimize the sum

$$S = \sum \frac{x_i}{x_j}$$

taken over all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.
(This problem is due to Byron Calhoun, a high school student in McLean, Virginia. It appeared, with solution, in a science project of his.)

1743***.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $0 < \gamma < 180^\circ$ be fixed. Consider the set $\triangle(\gamma)$ of all triangles ABC having angle γ at C, whose altitude through C meets AB in an interior point D such that the line through the incenters of $\triangle ADC$ and $\triangle BCD$ meets the sides AC and BC in interior points E and F respectively. Prove or disprove that

$$\sup_{\triangle(\gamma)}\left(\frac{\mathrm{area}(\triangle EFC)}{\mathrm{area}(\triangle ABC)}\right) = \left(\frac{\cos(\gamma/2)-\sin(\gamma/2)+1}{2\cos(\gamma/2)}\right)^2.$$

(This would generalize problem 5 of the 1988 IMO [1988: 197].)

1749. *Proposed by D. M. Milošević, Pranjani, Yugoslavia.*
Let ABC be a triangle with external angle-bisectors w'_a, w'_b, w'_c, inradius r, and circumradius R. Prove that

$$\text{(i)}\quad \left(\sqrt{\frac{1}{w'_a}}+\sqrt{\frac{1}{w'_b}}+\sqrt{\frac{1}{w'_c}}\right)^2 < \frac{2}{r};$$

$$\text{(ii)}\quad \left(\frac{1}{w'_a}+\frac{1}{w'_b}+\frac{1}{w'_c}\right)^2 < \frac{R}{3r^2}.$$

1750. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*
Pairs of numbers from the set $\{11, 12, \ldots, n\}$ are adjoined to each of the 45 different (unordered) pairs of numbers from the set $\{1, 2, \ldots, 10\}$, to obtain 45 4-element sets $A_1, A_2, \ldots, A_{45}$. Suppose that $|A_i \cap A_j| \le 2$ for all $i \neq j$. What is the smallest n possible?

1754★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let n and k be positive integers such that $2 \le k < n$, and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers satisfying $\sum_{i=1}^n x_i = 1$. Prove or disprove that

$$\sum x_1 x_2 \cdots x_k \le \max\left\{\frac{1}{k^k}, \frac{1}{n^{k-1}}\right\},$$

where the sum is cyclic over $x_1, x_2, \ldots, x_n$. [The case $k = 2$ is known – see inequality (1) in the solution of *Crux* 1662, this issue.]

1756. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*
For positive integers $n \ge 3$ and $r \ge 1$, the *n-gonal number of rank r* is defined as

$$P(n, r) = (n-2)\frac{r^2}{2} - (n-4)\frac{r}{2}.$$

Call a triple (a, b, c) of natural numbers, with $a \le b < c$, an *n-gonal Pythagorean triple* if $P(n, a) + P(n, b) = P(n, c)$. When $n = 4$, we get the usual Pythagorean triple.
(i) Find an n-gonal Pythagorean triple for each n.
(ii) Consider all triangles ABC whose sides are n-gonal Pythagorean triples for some $n \ge 3$. Find the maximum and the minimum possible values of angle C.

1757. *Proposed by Avinoam Freedman, Teaneck, N. J.*
Let $A_1A_2A_3$ be an acute triangle with sides a_1, a_2, a_3 and area F, and let $\triangle B_1B_2B_3$ (with sides b_1, b_2, b_3) be inscribed in $\triangle A_1A_2A_3$ with $B_1 \in A_2A_3$, etc. Show that for any $x_1, x_2, x_3 > 0$,

$$(x_1a_1^2 + x_2a_2^2 + x_3a_3^2)(x_1b_1^2 + x_2b_2^2 + x_3b_3^2) \ge 4F^2(x_2x_3 + x_3x_1 + x_1x_2).$$

1759. *Proposed by Isao Ashiba, Tokyo, Japan.*
A is a fixed point on a circle, and P and Q are variable points on the circle so that $AP + PQ$ equals the diameter of the circle. Find P and Q so that the area of $\triangle APQ$ is as large as possible.

1761. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
ABC is an isosceles triangle with $AB = AC$. Let D be the foot of the perpendicular from C to AB, and let M be the midpoint of CD. Let E be the foot of the perpendicular from A to BM, and let F be the foot of the perpendicular from A to CE. Prove that $AF \le AB/3$.

1762. *Proposed by Steven Laffin, student, École J. H. Picard and Andy Liu, University of Alberta, Edmonton.* (Dedicated to Professor David Monk, University of Edinburgh, on his sixtieth birthday.)
Starship Venture is under attack from a Zokbar fleet, and its Terrorizer is destroyed. While it can hold out, it needs a replacement to drive off the Zokbars. Starbase has spare Terrorizers, which can be taken apart into any number of components, and enough scout ships to provide transport. However, the Zokbars have n Space Octopi, each of which can capture one scout ship at a time. Starship Venture must have at least one copy of each component to reassemble a Terrorizer, but it is essential that the Zokbars should not be able to do the same. Into how many components must each Terrorizer be taken apart (assuming all are taken apart in an identical manner), and how many scout ships are needed to transport them? Give two answers:
(a) assuming that the number of components per Terrorizer is as small as possible, minimize the number of scout ships;
(b) assuming instead that the number of scout ships is as small as possible, minimize the number of components per Terrorizer.

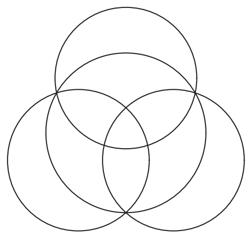
1763. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $t \geq 0$, and for each integer $n \geq 1$ define

$$x_n = \frac{1 + t + t^2 + \cdots + t^n}{n+1}.$$

Prove that $x_1 \leq \sqrt{x_2} \leq \sqrt[3]{x_3} \leq \sqrt[4]{x_4} \leq \cdots$.

1764. *Proposed by Murray S. Klamkin, University of Alberta.*
(a) Determine the extreme values of $a^2b + b^2c + c^2a$, where a, b, c are sides of a triangle of semiperimeter 1.
(b)★ What are the extreme values of $a_1^2a_2 + a_2^2a_3 + \cdots + a_n^2a_1$, where $a_1, a_2, \ldots, a_n$ are the (consecutive) sides of an n-gon of semiperimeter 1?

1765. *Proposed by Kyu Hyon Han, student, Seoul, South Korea.*
There are four circles piled up, making a total of 10 regions. The outer circles each have 5 regions and the central circle has 7 regions. You put one of the numbers $0, 1, \ldots, 9$ in each region, without reusing any number, so that the sum of the numbers in any circle is always the same value, say S. What is the smallest and the largest possible value of S?



1766★. *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*
The sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$, $x_2 = x$, and

$$x_{n+2} = xx_{n+1} + nx_n$$

for $n \geq 0$. Prove or disprove that for each $n \geq 2$, the coefficients of the polynomial $x_{n-1}x_{n+1} - x_n^2$ are all nonnegative, except for the constant term when n is odd.

1771★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let a, b, c be the sides of a triangle and u, v, w be non-negative real numbers such that $u+v+w = 1$. Prove that

$$\sum ubc - s\sum vwa \geq 3Rr,$$

where s, R, r are the semiperimeter, circumradius and inradius of the triangle, and the sums are cyclic.

1772. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*
The equation $x^3 + ax^2 + (a^2 - 6)x + (8 - a^2) = 0$ has only positive roots. Find all possible values of a.

1774★. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine the smallest $\lambda \geq 0$ such that

$$2(x^3 + y^3 + z^3) + 3xyz \geq (x^\lambda + y^\lambda + z^\lambda)(x^{3-\lambda} + y^{3-\lambda} + z^{3-\lambda})$$

for all non-negative x, y, z.

1775. *Proposed by P. Penning, Delft, The Netherlands.*
Find the radius of the smallest sphere (in three-dimensional space) which is tangent to the three lines $y = 1$, $z = -1$; $z = 1$, $x = -1$; $x = 1$, $y = -1$; and whose centre does *not* lie on the line $x = y = z$.

1776. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut.*
Given $0 < x_0 < 1$, the sequence $x_0, x_1, \ldots$ is defined by

$$x_{n+1} = \frac{3}{4} - \frac{3}{2}\left|x_n - \frac{1}{2}\right|$$

for $n \geq 0$. It is easy to see that $0 < x_n < 1$ for all n. Find the smallest closed interval J in $[0, 1]$ so that $x_n \in J$ for all sufficient large n.

1780. *Proposed by Jordan Stoyanov, Queen's University, Kingston, Ontario.*
Prove that, for any natural number n and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$,

$$(1 - \sin^2\alpha_1 \sin^2\alpha_2 \cdots \sin^2\alpha_n)^n + (1 - \cos^2\alpha_1 \cos^2\alpha_2 \cdots \cos^2\alpha_n)^n \geq 1.$$

1781. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $a > 0$ and $x_1, x_2, \ldots, x_n \in [0, a]$ $(n \geq 2)$ such that

$$x_1 x_2 \cdots x_n = (a - x_1)^2 (a - x_2)^2 \cdots (a - x_n)^2.$$

Determine the maximum possible value of the product $x_1 x_2 \cdots x_n$.

1784. *Proposed by Murray S. Klamkin, University of Alberta, and Dale Varberg, Hamline University, St. Paul, Minnesota.*
A point in 3-space is at distances 9, 10, 11 and 12 from the vertices of a tetrahedron. Find the maximum and minimum possible values of the sum of the squares of the edges of the tetrahedron.

1788. *Proposed by Christopher J. Bradley, Clifton College, Bristol, England.*
A pack of cards consists of m red cards and n black cards. The pack is thoroughly shuffled and the cards are then laid down in a row. The number of colour changes one observes in moving from left to right along the row is k. (For example, for $m = 5$ and $n = 4$ the row RRBRBBBRBR exhibits $k = 6$.) Prove that k is more likely to be even than odd if and only if

$$|m - n| > \sqrt{m + n}.$$

1789★. *Proposed by D. M. Milošević, Pranjani, Yugoslavia.*
Let a_1, a_2, a_3 be the sides of a triangle, w_1, w_2, w_3 the angle bisectors, F the area, and s the semiperimeter. Prove or disprove that

$$w_1^{a_1} + w_2^{a_2} + w_3^{a_3} \le (F\sqrt{3})^s.$$

1792. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $x, y \ge 0$ such that $x + y = 1$, and let $\lambda > 0$. Determine the best lower and upper bounds (in terms of λ) for

$$(\lambda + 1)(x^\lambda + y^\lambda) - \lambda(x^{\lambda+1} + y^{\lambda+1}).$$

1793. *Proposed by Murray S. Klamkin, University of Alberta.*
Prove that in any n-dimensional simplex there is at least one vertex such that the n edges emanating from that vertex are possible sides of an n-gon.

1794. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*
Pairs of numbers from the set $\{7, 8, \ldots, n\}$ are adjoined to each of the 20 different (unordered) triples of numbers from the set $\{1, 2, \ldots, 6\}$, to obtain twenty 5-element sets $A_1, A_2, \ldots, A_{20}$. Suppose that $|A_i \cap A_j| \le 2$ for all $i \ne j$. What is the smallest n possible?

1796. *Proposed by Ji Chen, Ningbo University, China.*
If A, B, C are the angles of a triangle, prove that

$$\sum \sin B \sin C \le 3 \sum \sin(B/2)\sin(C/2),$$

where the sums are cyclic.

1801. *Proposed by Murray S. Klamkin, University of Alberta.* (Dedicated to O. Bottema.)
If A_1, A_2, A_3 are angles of a triangle, prove that

$$\sum (1 + 8\cos A_1 \sin A_2 \sin A_3)^2 \sin A_1 \ge 64 \sin A_1 \sin A_2 \sin A_3,$$

where the summation is cyclic over the indices $1, 2, 3$.

1802. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove that, for any real numbers x and y,

$$x^4 + y^4 + (x^2 + 1)(y^2 + 1) \ge x^3(1 + y) + y^3(1 + x) + x + y,$$

and determine when equality holds.

1808. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Three congruent circles that pass through a common point meet again in points A, B, C. $A'B'C'$ is the triangle containing the three circles and whose sides are each tangent to two of the circles. Prove that $[A'B'C'] \ge 9[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

1813★. *Proposed by D. N. Verma, Bombay, India.*
Suppose that $a_1 > a_2 > a_3$ and $r_1 > r_2 > r_3$ are positive real numbers. Prove that the determinant

$$\begin{vmatrix} a_1^{r_1} & a_1^{r_2} & a_1^{r_3} \\ a_2^{r_1} & a_2^{r_2} & a_2^{r_3} \\ a_3^{r_1} & a_3^{r_2} & a_3^{r_3} \end{vmatrix}$$

is positive.

1816. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Given a finite set $\mathcal{S}$ of $n+1$ points in the plane, with two distinguished points B and E in $\mathcal{S}$, consider all polygonal paths $\mathcal{P} = P_0P_1 \ldots P_n$ whose vertices are all points of $\mathcal{S}$, in any order except that $P_0 = B$ and $P_n = E$. For such a path $\mathcal{P}$ define $l(\mathcal{P})$ to be the length of $\mathcal{P}$ and

$$a(\mathcal{P}) = \sum_{i=1}^{n-1} \theta(\overrightarrow{P_{i-1}P_i}, \overrightarrow{P_iP_{i+1}}),$$

where $\theta(\boldsymbol{v}, \boldsymbol{w})$ is the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, $0 \le \theta(\boldsymbol{v}, \boldsymbol{v}) \le \pi$. Prove or disprove that the minimum values of $l(\mathcal{P})$ and of $a(\mathcal{P})$ are attained for the same path $\mathcal{P}$.

1818. *Proposed by Ed Barbeau, University of Toronto.*
Prove that, for $0 \le x \le 1$ and a positive integer k,

$$(1+x)^k[x + (1-x)^{k+1}] \ge 1.$$

1823. *Proposed by G. P. Henderson, Campbellcroft, Ontario.*
A rectangular box is to be decorated with a ribbon that goes across the faces and makes various angles with the edges. If possible, the points where the ribbon crosses the edges are chosen so that the length of the closed path is a local minimum. This will ensure that the ribbon can be tightened and tied without slipping off. Is there always a minimal path that crosses all six faces just once?

1824. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and M a point in its plane. We consider the circles with diameters AM, BM, CM and the circle containing and internally tangent to these three circles. Show that the radius P of this large circle satisfies $P \ge 3r$, where r is the inradius of $\triangle ABC$.

1825. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Suppose that the real polynomial $x^4 + ax^3 + bx^2 + cx + d$ has four positive roots. Prove that $abc \ge a^2d + 5c^2$.

1827. *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D. M. Milošević, Pranjani, Yugoslavia.*
Let a, b, c be the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle.
(i) Prove that

$$\sum \frac{bc}{A(s-a)} \ge \frac{12\,s}{\pi},$$

where the sums here and below are cyclic.
(ii)★ It follows easily from the proof of *Crux* 1611 (see [1992: 62] and the correction in this issue) that also

$$\sum \frac{b+c}{A} \ge \frac{12\,s}{\pi}.$$

Do the two summations above compare in general?

1830. *Proposed by P. Tsaoussoglou, Athens, Greece.*
If $a > b > c > 0$ and $a^{-1} + b^{-1} + c^{-1} = 1$, prove that

$$\frac{4}{c^2} + \frac{1}{(a-b)b} + \frac{1}{(b-c)c} \geq \frac{4}{3}.$$

1831. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let x, y, z be any real numbers and let λ be an *odd* positive integer. Prove or disprove that

$$x(x+y)^\lambda + y(y+z)^\lambda + z(z+x)^\lambda \geq 0.$$

1834. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Given positive numbers A, G and H, show that they are respectively the arithmetic, geometric and harmonic means of some three positive numbers x, y, z if and only if

$$\frac{A^3}{G^3} + \frac{G^3}{H^3} + 1 \leq \frac{3}{4}\left(1 + \frac{A}{H}\right)^2.$$

1837. *Proposed by Andy Liu, University of Alberta.*
A function $f : \mathbb{R} \to \mathbb{R}^+$ is said to be *strictly log-convex* if

$$f(x_1)f(x_2) \geq \left(f\left(\frac{x_1 + x_2}{2}\right)\right)^2$$

for all $x_1, x_2 \in \mathbb{R}$, with equality if and only if $x_1 = x_2$. f is said to be *strictly log-concave* if the inequlaity is reversed.
(a) Prove that if f and g are strictly log-convex functions, then so is $f + g$.
(b)* Does the same conclusion hold for strictly log-concave functions?

1840. *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*
Let $\triangle ABC$ be an acute triangle with area F and circumcenter O. The distances from O to BC, CA, AB are denoted d_a, d_b, d_c respectively. $\triangle A_1B_1C_1$ (with sides a_1, b_1, c_1) is inscribed in $\triangle ABC$, with $A_1 \in BC$ etc. Prove that

$$d_a a_1 + d_b b_1 + d_c c_1 \geq F.$$

1843. *Proposed by Šefket Arslanagić, Trebinje, Yugoslavia, and D. M. Milošević, Pranjani, Yugoslavia.*
Let a, b, c be the sides, A, B, C the angles (measured in radians), and s the semi-perimeter of a triangle.
(i) Prove that

$$\sum \frac{a}{2A(s-a)} \geq \frac{9}{\pi}.$$

(ii)* It is obvious that also

$$\sum \frac{1}{A} \geq \frac{9}{\pi}.$$

Do the two summations compare in general?

1845. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Suppose that x_1, x_2, x_3, x_4, x_5 are real numbers satisfying $x_1 < x_2 < x_3 < x_4 < x_5$ and

$$\sum_i x_i = 10, \quad \sum_{i<j} x_i x_j = 35, \quad \sum_{i<j<k} x_i x_j x_k = 50, \quad \sum_{i<j<k<l} x_i x_j x_k x_l = 25.$$

Prove that

$$\frac{5+\sqrt{5}}{2} < x_5 < 4.$$

1846. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Consider the three excircles of a given triangle ABC. Let $A'B'C'$ be the triangle containing these three circles and whose sides are each tangent to two of the circles. Prove that

$$[A'B'C'] \geq 25[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

1849. *Proposed by Shi-Chang Shi and Ji Chen, Ningbo University, China.*
Let three points P, Q, R be on the sides BC, CA, AB, respectively, of a triangle ABC, such that they cut the perimeter of $\triangle ABC$ into three equal parts; i. e. $QA+AR = RB+BP = PC+CQ$.
(a) Prove that

$$RP \cdot PQ + PQ \cdot QR + QR \cdot RP \geq \frac{1}{12}(a+b+c)^2.$$

(b)$^\bigstar$ Prove or disprove that the circumradius of $\triangle PQR$ is at least half the circumradius of $\triangle ABC$.

1851. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers such that $\sum_{i=1}^n x_i^2 = 1$. Prove that

$$\frac{2\sqrt{n}-1}{5\sqrt{n}-1} \leq \frac{1}{n}\sum_{i=1}^{n} \frac{x_i+2}{x_i+5} \leq \frac{2\sqrt{n}+1}{5\sqrt{n}+1}.$$

1853. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*
Let $\{b_b\}_{n=1}^{\infty}$ be a sequence of positive real numbers which satisfies the condition

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every $n \geq 1$. Prove that either the sequence converges or $\lim_{n\to\infty} b_n = \infty$.

1854. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*
In any convex pentagon prove that the sum of the squares of the diagonals is less than three times the sum of the squares of the sides.

1855. *Proposed by Bernardo Recamán, United World College of Southern Africa, Mbabane, Swaziland.*
Twelve friends agree to eat out once a week. Each week they will divide themselves into 3 groups of 4 each, and each of these groups will sit together at a separate table. They have agreed to meet until any two of the friends will have sat at least once at the same table at the same time. What is the minimum number of weeks this requires?

1856. *Proposed by Jisho Kotani, Akita, Japan.*
Find the rectangular brick of largest volume that can be completely wrapped in a square piece of paper of side 1 (without cutting the paper).

1857. *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*
Prove that, for any positive integer n,

$$1 < \frac{27^n (n!)^3}{(3n+1)!} < \sqrt{2}.$$

1860★. *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*
Prove or disprove that

$$\sum \frac{\cos[(A-B)/4]}{\cos(A/2)\cos(B/2)} \geq 4,$$

where the sum is cyclic over the angles A, B, C of a triangle.

1861. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $f : \mathbb{R}^+ \to \mathbb{R}$ be an increasing and concave function from the positive real numbers to the reals. Prove that if $0 < x \leq y \leq z$ and n is a positive integer then

$$(z^n - x^n) f(y) \geq (z^n - y^n) f(x) + (y^n - x^n) f(z).$$

1864. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Consider the three excircles of a given triangle ABC. Let P be the radius of the circle containing and internally tangent to these three circles. Prove that $P \geq 7r$, where r is the inradius of $\triangle ABC$.

1868. *Proposed by De-jun Zhao, Chengtun High School, Xingchang, China.*
Let $n \geq 3$, $a_1 > a_2 > \cdots > a_n > 0$, and $p > q > 0$. Show that

$$a_1^p a_2^q + a_2^p a_3^q + \cdots + a_{n-1}^p a_n^q + a_n^p a_1^q > a_1^q a_2^p + a_2^q a_3^p + \cdots + a_{n-1}^q a_n^p + a_n^q a_1^p.$$

1870★. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*
In any convex pentagon $ABCDE$ prove or disprove that

$$\begin{aligned} &AC \cdot BD + BD \cdot CE + CE \cdot DA + DA \cdot EB + EB \cdot AC \\ &\qquad > AB \cdot CD + BC \cdot DE + CD \cdot EA + DE \cdot AB + EA \cdot BC. \end{aligned}$$

(*Note*: the first sum involves diagonals, the second sum involves sides.)

1874. *Proposed by Pedro Melendez, Belo Horizonte, Brazil.*
Find the smallest positive integer n such that $n!$ is divisible by 1993^{1994}.

1877. *Proposed by Iliya Bluskov, Technical University, Gabrovo, Bulgaria.*
Let $B_1, B_2, \ldots, B_b$ be k-element subsets of $\{1, 2, \ldots, n\}$ such that $|B_i \cap B_j| \leq 1$ for all $i \neq j$. Show that

$$b \leq \left[\frac{n}{k}\left[\frac{n-1}{k-1}\right]\right],$$

where $[x]$ denotes the greatest integer $\leq x$.

1878★. *Proposed by Jun-hua Huang, The 4th Middle School of Nanxian, Hunan, China.*
Given two triangles ABC and $A'B'C'$, prove or disprove that

$$\frac{\sin A'}{\sin A}+\frac{\sin B'}{\sin B}+\frac{\sin C'}{\sin C}\leq 1+\frac{R}{r},$$

where r, R are the inradius and circumradius of triangle ABC.

1882. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Arthur tosses a fair coin until he obtains two heads in succession. Betty tosses another fair coin until she obtains a head and a tail in succession, with the head coming immediately prior to the tail.

(i) What is the average number of tosses each has to make?

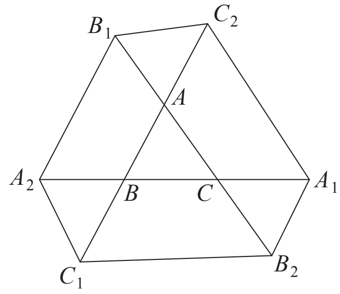
(ii) What is the probability that Betty makes *fewer* tosses than Arthur (rather than the same number or more than Arthur)?

1883. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let ABC be a triangle and construct the circles with sides AB, BC, CA as diameters. $A'B'C'$ is the triangle containing these three circles and whose sides are each tangent to two of these circles. Prove that

$$[A'B'C']\geq\left(\frac{13}{4}+\sqrt{3}\right)[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

1887. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan.*
Given an acute triangle ABC, form the hexagon $A_1C_2B_1A_2C_1B_2$ as shown, where



$$BC = BC_1 = CB_2,$$
$$CA = CA_1 = AC_2,$$

and

$$AB = AB_1 = BA_2.$$

Prove that the area of the hexagon is at least 13 times the area of $\triangle ABC$, with equality when ABC is equilateral.

1890. *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*
Let n be a positive integer and let

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$
$$g(x) = \frac{k}{a_n}x^n + \frac{k}{a_{n-1}}x^{n-1} + \cdots + \frac{k}{a_1}x + \frac{k}{a_0},$$

where k and the a_i's are positive real numbers. Prove that

$$f(g(1))g(f(1)) \geq 4k.$$

When does equality hold?

1892. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Let $n \geq 4$ be an integer. Find the exact upper and lower bounds for the cyclic sum

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1} + x_i + x_{i+1}}$$

(where of course $x_0 = x_n, x_{n+1} = x_1$), over all n-tupels of nonnegative numbers $(x_1, \ldots, x_n)$ without three zeros in cyclic succession. Characterize all cases in which either one of these bounds is attained.

1895. *Proposed by Ji Chen and Gang Yu, Ningbo University, China.*
Let P be an interior point of a triangle $A_1A_2A_3$; R_1, R_2, R_3 the distances from P to A_1, A_2, A_3; and R the circumradius of $\triangle A_1A_2A_3$. Prove that

$$R_1R_2R_3 \leq \frac{32}{27}R^3,$$

with equality when $A_2 = A_3$ and $PA_2 = 2PA_1$.

1901. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous even function such that $f(0) = 0$ and $f(x+y) \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Must f be monotonic on $\mathbb{R}^+$?

1904. *Proposed by Kee-Wai Lau, Hong Kong.*
If m_a, m_b, m_c are the medians of a triangle with sides a, b, c, prove that

$$m_a(bc - a^2) + m_b(ca - b^2) + m_c(ab - c^2) \geq 0.$$

1907. *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*
Find the largest constant k such that

$$\frac{kabc}{a+b+c} \leq (a+b)^2 + (a+b+4c)^2$$

for all $a, b, c > 0$.

1913. *Proposed by N. Kildonan, Winnipeg, Manitoba.*
I was at a restaurant for lunch the other day. The bill came, and I wanted to give the waiter a whole number of dollars, with the difference between what I give him and the bill being the tip. I always like to tip between 10 and 15 percent of the bill. But if I gave him a certain number of dollars, the tip would have been **less** than 10 % of the bill, and if instead I gave him one dollar more, the tip would have been **more** than 15 % of the bill. What was the largest possible amount of the bill? [*Editor's note to non-North American readers*: your answer should be in dollars and cents, where there are (reasonably enough) 100 cents in a dollar.]

1914. *Proposed by K. R. S. Sastry, Addis Ababa, Ethiopia.*
Let $A_1A_2 \ldots A_n$ be a regular n-gon, with $M_1, M_2, \ldots, M_n$ the midpoints of the sides. Let P be a point in the plane of the n-gon. Prove that

$$\sum_{i=1}^{n} PM_i \geq \cos(180°/n) \sum_{i=1}^{n} PA_i.$$

1920. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let a, b, c be the sides of a triangle.
(a) Prove that, for any $0 < \lambda \leq 2$,

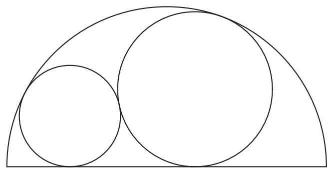
$$\frac{1}{(1+\lambda)^2} < \frac{(a+b)(b+c)(c+a)}{(\lambda a+b+c)(a+\lambda b+c)(a+b+\lambda c)} \leq \left(\frac{2}{2+\lambda}\right)^3,$$

and that both bounds are best possible.
(b)* What are the bounds for $\lambda > 2$?

1924. *Proposed by Jisho Kotani, Akita, Japan.*
A large sphere of radius 1 and a smaller sphere of radius $r < 1$ overlap so that their intersection is a circle of radius r, i. e., a great circle of the small sphere. Find r so that the volume inside the small sphere and outside the large sphere is as large as possible.

1933. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Two externally tangent circles of radii R_1 and R_2 are internally tangent to a semicircle of radius 1, as in the figure. Prove that

$$R_1 + R_2 \leq 2(\sqrt{2}-1).$$



1940. *Proposed by Ji Chen, Ningbo University, China.*
Show that if $x, y, z > 0$,

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

1942. *Proposed by Paul Bracken, University of Waterloo.*
Prove that, for any $a \geq 1$,

$$\left(\sum_{k=0}^{\infty}\frac{1}{(a+k)^2}\right)^2 > 2\sum_{k=0}^{\infty}\frac{1}{(a+k)^3}.$$

1944. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton.*
Find the smallest positive integer n so that

$$(n+1)^{2000} > (2n+1)^{1999}.$$

1945. *Proposed by Murray S. Klamkin, University of Alberta.*
Let $A_1A_2\ldots A_n$ be a convex n-gon.
(a) Prove that

$$A_1A_2 + A_2A_3 + \cdots + A_nA_1 \leq A_1A_3 + A_2A_4 + \cdots + A_nA_2.$$

(b)* Prove or disprove that

$$2\cos\left(\frac{\pi}{n}\right)(A_1A_2 + A_2A_3 + \cdots + A_nA_1) \geq A_1A_3 + A_2A_4 + \cdots + A_nA_2.$$

1948. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Are there any nonconstant differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(f(x))) = f(x) \geq 0$$

for all $x \in \mathbb{R}$?

1949. *Proposed by Federico Ardila, student, Colegio San Carlos, Bogotá, Colombia.*
Let D, E, F be points on the sides BC, CA, AB respectively of triangle ABC, and let R be the circumradius of ABC. Prove that

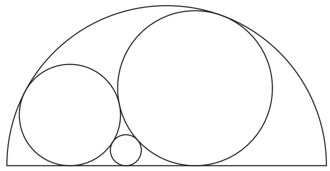
$$\left(\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}\right)(DE+EF+FD) \geq \frac{AB+BC+CA}{R}.$$

1953. *Proposed by Murray S. Klamkin, University of Alberta.*
Determine a necessary and sufficient condition on real constants $r_1, r_2, \ldots, r_n$ such that

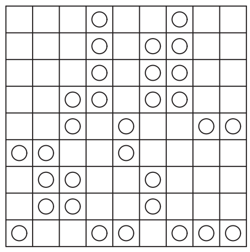
$$x_1^2+x_2^2+\cdots+x_n^2 \geq (r_1x_1+r_2x_2+\cdots+r_nx_n)^2$$

holds for all real $x_1, x_2, \ldots, x_n$.

1956. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
In a semicircle of radius 4 there are three tangent circles as in the figure. Prove that the radius of the smallest circle is at most $\sqrt{2}-1$.



1957. *Proposed by William Soleau, New York.*
A 9 by 9 board is filled with 81 counters, each being green on one side and yellow on the other. Initially, all have their green sides up, except the 31 marked with circles in the diagram. In one move, we can flip over a block of adjacent counters, vertically or horizontally only, provided that at least one of the counters at the ends of the block is on the edge of the board. Determine a shortest sequence of moves which allows us to flip all counters to their green sides.



1958. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Find the tetrahedron of maximum volume given that the sum of the lengths of some four edges is 1.

1961. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
ABC is an isosceles triangle with $AB = AC$. We denote the circumcircle of $\triangle ABC$ by Γ. Let D be the point such that DA and DC are tangent to Γ at A and C respectively. Prove that $\sphericalangle DBC \leq 30°$.

1962. *Proposed by Murray S. Klamkin, University of Alberta.*
If A, B, C, D are non-negative angles with sum π, prove that
(i) $\cos^2 A+\cos^2 B+\cos^2 C+\cos^2 D \geq 2\sin A\sin C+2\sin B\sin D$;
(ii) $1 \geq \sin A\sin C+\sin B\sin D$.

1965★. *Proposed by Ji Chen, Ningbo University, China.*
Let P be a point in the interior of the triangle ABC, and let the lines AP, BP, CP intersect the opposite sides at D, E, F respectively.

(a) Prove or disprove that

$$PD \cdot PE \cdot PF \leq \frac{R^3}{8},$$

where R is the circumradius of $\triangle ABC$. Equality holds when ABC is equilateral and P is its centre.

(b) Prove or disprove that

$$PE \cdot PF + PF \cdot PD + PD \cdot PE \leq \frac{1}{4} \max\{a^2, b^2, c^2\},$$

where a, b, c are the sides of the triangle. Equality holds when ABC is equilateral and P is its centre, and also when P is the midpoint of the longest side of ABC.

1972. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Define a sequence $a_0, a_1, a_2, \ldots$ of nonnegative integers by: $a_0 = 0$ and

$$a_{2n} = 3a_n, \qquad a_{2n+1} = 3a_n + 1 \qquad \text{for } n = 0, 1, 2, \ldots.$$

(a) Characterize all nonnegative integers n so that there is exactly **one** pair (k, l) satisfying

$$k > l \qquad \text{and} \qquad a_k + a_l = n. \tag{1}$$

(b) For each n, let $f(n)$ be the number of pairs (k, l) satisfying (1). Find

$$\max_{n<3^{1972}} f(n).$$

1976. *Proposed by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*
If a, b and c are positive numbers, prove that

$$\frac{a(3a-b)}{c(a+b)} + \frac{b(3b-c)}{a(b+c)} + \frac{c(3c-a)}{b(c+a)} \leq \frac{a^3+b^3+c^3}{abc}.$$

1985. *Proposed by Murray S. Klamkin and Andy Liu, University of Alberta.*
Let $A_1A_2\ldots A_{2n}$ be a regular $2n$-gon, $n > 1$. Translate every even-numbered vertex $A_2, A_4, \ldots$, A_{2n} by an equal (nonzero) amount to get new vertices $A_2', A_4', \ldots, A_{2n}'$, and so that the new $2n$-gon $A_1A_2'A_3A_4'\ldots A_{2n-1}A_{2n}'$ is still convex. Prove that the perimeter of $A_1A_2'\ldots A_{2n-1}A_{2n}'$ is greater than the perimeter of $A_1A_2\ldots A_{2n}$.

1990. *Proposed by Leng Gangsong, Hunan Educational Institute, Changsha, China.*
Let r be the inradius of a tetrahedron $A_1A_2A_3A_4$, and let r_1, r_2, r_3, r_4 be the inradii of triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$ respectively. Prove that

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \leq \frac{2}{r^2},$$

with equality if the tetrahedron is regular.

1994. *Proposed by N. Kildonan, Winnipeg, Manitoba.*
This problem marks the one and only time that the number of a *Crux* problem is equal to the year in which it is published. In particular this is the *first* time that

$$\text{a problem number in an integer multiple of its publication year.} \qquad (1)$$

Assuming that *Crux* continues indefinitely to publish 10 problems per issue and 10 issues per year, will there be a *last* time (1) happens? If so, when will this occur?

2000. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
A 1000-element set is randomly chosen from $\{1, 2, \ldots, 2000\}$. Let p be the probability that the sum of the chosen numbers is divisible by 5. Is p greater than, smaller than, or equal to $1/5$?

2006. *Proposed by John Duncan, University of Arkansas, Fayetteville; Dan Velleman, Amherst College, Amherst, Massachusetts; and Stan Wagon, Macalester College, St. Paul, Minnesota.*
Suppose we are given $n \geq 3$ disks, of radii $a_1 \geq a_2 \geq \cdots \geq a_n$. We wish to place them in some order around an interior disk so that each given disk touches the interior disk and its two immediate neighbors. If the given disks are of widely different sizes (such as 100, 100, 100, 100, 1), we allow a disk to overlap other given disks that are not immediate neighbors. In what order should the given disks be arranged so as to maximize the radius of the interior disk?
[*Editor's note.* Readers may assume that for any ordering of the given disks the configuration of the problem exists and that the radius of the interior disk is unique, though, as the proposers point out, this requires a proof (which they supply).]

2009. *Proposed by Bill Sands, University of Calgary.*
Sarah got a good grade at school, so I gave her N two-dollar bills. Then, since Tim got a better grade, I gave him just enough five-dollar bills so that he got *more* money than Sarah. Finally, since Ursula got the best grade, I gave her just enough ten-dollar bills so that she got *more* money than Tim. What is the maximum amount of money that Ursula could have received? (This is a variation of problem 11 on the 1994 Alberta High School Mathematics Contest, First Part; see Skoliad Corner, this issue.)

2015. *Proposed by Shi-Chang Shi and Ji Chen, Ningbo University, China.*
Prove that

$$(\sin A + \sin B + \sin C)\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \geq \frac{27\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

2018. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
How many permutations $(x_1, \ldots, x_n)$ of $\{1, \ldots, n\}$ are there such that the cyclic sum

$$\sum_{i=1}^{n} |x_i - x_{i+1}|$$

(with $x_{n+1} = x_1$) is (a) a minimum, (b) a maximum?

2020. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Let a, b, c, d be **distinct** real numbers such that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4 \quad \text{and} \quad ac=bd.$$

Find the maximum value of

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}.$$

2022. *Proposed by K. R. S. Sastry, Dodballapur, India.*
Find the smallest integer of the form

$$\frac{A\star B}{B},$$

where A and B are three-digit positive integers and $A\star B$ denotes the six-digit integer formed by placing A and B side by side.

2023. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*
Let a, b, c, d, e be positive numbers with $abcde=1$.
(a) Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bcd}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea} +\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc}\geq\frac{10}{3}.$$

(b) Find a generalization!

2029★. *Proposed by Jun-hua Huang, The Middle School Attached To Hunan Normal University, Changsha, China.*
ABC is a triangle with area F and internal angle bisectors w_a, w_b, w_c. Prove or disprove that

$$w_bw_c+w_cw_a+w_aw_b\geq 3\sqrt{3}\,F.$$

2032. *Proposed by Tim Cross, Wolverley High School, Kidderminster, U. K.*
Prove that, for nonnegative real numbers x, y and z,

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\geq\sqrt{6\,(x+y+z)}.$$

When does equality hold?

2039★. *Proposed by Dong Zhou, Fudan University, Shang-hai, China, and Ji Chen, Ningbo University, China.*
Prove or disprove that

$$\frac{\sin A}{B}+\frac{\sin B}{C}+\frac{\sin C}{A}\geq\frac{9\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle. [Compare with *Crux* 1216 [1988: 120] and this issue!]

2044. *Proposed by Murray S. Klamkin, University of Alberta.*
Suppose that $n \geq m \geq 1$ and $x \geq y \geq 0$ are such that

$$x^{n+1} + y^{n+1} \leq x^m - y^m.$$

Prove that $x^n + y^n \leq 1$.

2048. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Find the least integer n so that, for every string of length n composed of the letters a, b, c, d, e, f, g, h, i, j, k (repititions allowed), one can find a nonempty block of (consecutive) letters in which no letter appears an odd number of times.

2049★. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*
Let a tetrahedron $ABCD$ with centroid G be inscribed in a sphere of radius R. The lines AG, BG, CG, DG meet the sphere again at A_1, B_1, C_1, D_1 respectively. The edges of the tetrahedron are denoted a, b, c, d, e, f. Prove or disprove that

$$\frac{4}{R} \leq \frac{1}{GA_1} + \frac{1}{GB_1} + \frac{1}{GC_1} + \frac{1}{GD_1} \leq \frac{4\sqrt{6}}{9}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}\right).$$

Equality holds if $ABCD$ is regular. (This inequality, if true, would be a three-dimensional version of problem 5 of the 1991 Vietnamese Olympiad; see [1994: 41].)

2053. *Proposed by Jisho Kotani, Akita, Japan.*
A figure consisting of two equal and externally tangent circles is inscribed in an ellipse. Find the eccentricity of the ellipse of minimum area.

2057★. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*
Let P be a point inside an equilateral triangle ABC, and let R_a, R_b, R_c and r_a, r_b, r_c denote the distances of P from the vertices and edges, respectively, of the triangle. Prove or disprove that

$$\left(1 + \frac{r_a}{R_a}\right)\left(1 + \frac{r_b}{R_b}\right)\left(1 + \frac{r_c}{R_c}\right) \geq \frac{27}{8}.$$

Equality holds if P is the centre of the triangle.

2064. *Proposed by Murray S. Klamkin, University of Alberta.*
Show that

$$3 \max\left\{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}, \frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right\} \geq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

for arbitrary positive real numbers a, b, c.

2073★. *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*
Let P be an interior point of an equilateral triangle $A_1A_2A_3$ with circumradius R, and let $R_1 = PA_1$, $R_2 = PA_2$, $R_3 = PA_3$. Prove or disprove that

$$R_1R_2R_3 \leq \frac{9}{8}R^3.$$

Equality holds if P is the midpoint of a side. [Compare this problem with *Crux* 1895 [1995: 204].]

2078★. *Proposed by Šefket Arslanagić, Berlin, Germany.*
Prove or disprove that

$$\sqrt{a-1}+\sqrt{b-1}+\sqrt{c-1} \le \sqrt{c\,(ab+1)}$$

for $a, b, c \ge 1$.

2084. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Prove that

$$\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}+\cos\frac{A}{2}\cos\frac{B}{2} \ge 1-2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

where A, B, C are the angles of a triangle.

2090. *Proposed by Peter Ivády, Budapest, Hungary.*
For $0 < x < \pi/2$ prove that

$$\left(\frac{\sin x}{x}\right)^2 < \frac{\pi^2 - x^2}{\pi^2 + x^2}.$$

2093★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let A, B, C be the angles (in radians) of a triangle. Prove or disprove that

$$(\sin A+\sin B+\sin C)\left(\frac{1}{\pi-A}+\frac{1}{\pi-B}+\frac{1}{\pi-C}\right) \le \frac{27\sqrt{3}}{4\pi}.$$

2095. *Proposed by Murray S. Klamkin, University of Alberta.*
Prove that

$$a^x(y-z)+a^y(z-x)+a^z(x-y) \ge 0$$

where $a > 0$ and $x > y > z$.

2099. *Proposed by Proof, Warszawa, Poland.*
The tetrahedron T is contained inside the tetrahedron W. Must the sum of the lengths of the edges of T be less than the sum of the lengths of the edges of W?

2100. *Proposed by Iliya Bluskov, student, Simon Fraser University, Burnaby, B. C.*
Find 364 five-element subsets $A_1, A_2, \ldots, A_{364}$ of a 17-element set such that $|A_i \cap A_j| \le 3$ for all $1 \le i < j \le 364$.

2101. *Proposed by Ji Chen, Ningbo University, China.*
Let a, b, c be the sides and A, B, C the angles of a triangle. Prove that for any $k \le 1$,

$$\sum \frac{a^k}{A} \ge \frac{3}{\pi}\sum a^k,$$

where the sums are cyclic.

2105. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Find all values of λ for which the inequality

$$2(x^3+y^3+z^3)+3(1+3\lambda)xyz \ge (1+\lambda)(x+y+z)(yz+zx+xy)$$

holds for all positive real numbers x, y, z.

2106. *Proposed by Yang Kechang, Yueyang University, Hunan, China.*
A quadrilateral has sides a, b, c, d (in that order) and area F. Prove that

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 4F.$$

When does equality hold?

2108. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*
Prove that

$$\frac{a+b+c}{3} \leq \frac{1}{4}\sqrt[3]{\frac{(b+c)^2(c+a)^2(a+b)^2}{abc}},$$

where $a, b, c > 0$. Equality holds if $a = b = c$.

2113. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*
Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n}(a_i + b_i)\right)\left(\sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i}\right)$$

for any positive numbers $a_1, \ldots, a_n$, $b_1, \ldots, b_n$.

2116. *Proposed by Yang Kechang, Yueyang University, Hunan, China.*
A triangle has sides a, b, c and area F. Prove that

$$a^3 b^4 c^5 \geq \frac{25\sqrt{5}(2F)^6}{27}.$$

When does equality hold?

2117. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
ABC is a triangle with $AB > AC$, and the bisector of $\sphericalangle A$ meets BC at D. Let P be an interior point of the side AC. Prove that $\sphericalangle BPD < \sphericalangle DPC$.

2128. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
$ABCD$ is a square. Let P and Q be interior points on the sides BC and CD respectively, and let E and F be the intersections of PQ with AB and AD respectively. Prove that

$$\pi \leq \sphericalangle PAQ + \sphericalangle ECF < \frac{5\pi}{4}.$$

2136. *Proposed by G. P. Henderson, Campbellcroft, Ontario.*
Let a, b, c be the lengths of the sides of a triangle. Given the values of $p = \sum a$ and $q = \sum ab$, prove that $r = abc$ can be estimated with an error of at most $r/26$.

2138. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
ABC is an acute angle triangle with circumcentre O. AO meets the circle BOC again at A', BO meets the circle COA again at B', and CO meets the circle AOB again at C'. Prove that $[A'B'C'] \geq 4\,[ABC]$, where $[XYZ]$ denotes the area of triangle XYZ.

2139. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*
Point P lies inside triangle ABC. Let D, E, F be the orthogonal projections from P onto the lines BC, CA, AB, respectively. Let O' and R' denote the circumcentre and circumradius of the triangle DEF, respectively. Prove that

$$[ABC] \geq 3\sqrt{3}R'\sqrt{R'^2 - (O'P)^2},$$

where [XYZ] denotes the area of triangle XYZ.

2145. *Proposed by Robert Geretschläger, Bundesrealgymnasium, Graz, Austria.*
Prove that $\prod_{k=1}^{n}(ak + b^{k-1}) \leq \prod_{k=1}^{n}(ak + b^{n-k})$ for all $a, b > 1$.

2146. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
ABC is a triangle with $AB > AC$, and the bisector of $\sphericalangle A$ meets BC at D. Let P be an interior point on the segment AD, and let Q and R be the points of intersection of BP and CP with sides AC and AB respectively. Prove that $PB - PC > RB - QC > 0$.

2153. *Proposed by Šefket Arslanagić, Berlin, Germany.*
Suppose that $a, b, c \in \mathbb{R}$. If, for all $x \in [-1, 1]$, $|ax^2 + bx + c| \leq 1$, prove that

$$|cx^2 + bx + a| \leq 2.$$

2163. *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*
Prove that if $n, m \in \mathbb{N}$ and $n \geq m^2 \geq 16$, then $2^n \geq n^m$.

2167. *Proposed by Šefket Arslanagić, Berlin, Germany.*
Prove, **without the aid the differential calculus**, the inequality, that in a right triangle

$$\frac{a^2(b+c) + b^2(a+c)}{abc} \geq 2 + \sqrt{2},$$

where a and b are the legs and c the hypotenuse of the triangle.

2172. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $x, y, z \geq 0$ with $x + y + z = 1$. For fixed real numbers a and b, determine the maximum $c = c(a, b)$ such that

$$a + bxyz \geq c(yz + zx + xy).$$

2173. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $n \geq 2$ and $x_1, \ldots, x_n > 0$ with $x_1 + \ldots + x_n = 1$. Consider the terms

$$l_n = \sum_{k=1}^{n} (1 + x_k) \sqrt{\frac{1 - x_k}{x_k}}$$

and

$$r_n = C_n \prod_{k=1}^{n} \frac{1 + x_k}{\sqrt{1 - x_k}}$$

where

$$C_n = (\sqrt{n-1})^{n+1} (\sqrt{n})^n / (n+1)^{n-1}.$$

1. Show $l_2 \leq r_2$.
2. Prove or disprove: $l_n \geq r_n$ for $n \geq 3$.

2176. *Proposed by Šefket Arslanagić, Berlin, Germany.*
Prove that

$$\sqrt[n]{\prod_{k=1}^{n} (a_k + b_k)} \geq \sqrt[n]{\prod_{k=1}^{n} a_k} + \sqrt[n]{\prod_{k=1}^{n} b_k}$$

where $a_1, a_2, \ldots, a_n > 0$ and $n \in \mathbb{N}$.

2178. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
If A, B, C are the angles of a triangle, prove that

$$\begin{aligned} \sin A \sin B \sin C &\leq 8(\sin^3 A \cos B \cos C + \sin^3 B \cos C \cos A + \sin^3 C \cos A \cos B) \\ &\leq 3\sqrt{3}\, (\cos^2 A + \cos^2 B + \cos^2 C). \end{aligned}$$

2180. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Prove that if $a > 0$, $x > y > z > 0$, $n \geq 0$ (natural), then

1. $a^x (yz)^n (y - z) + a^y (xz)^n (z - x) + a^z (xy)^n (x - y) \geq 0,$
2. $a^x \cosh x (y - z) + a^y \cosh y (z - x) + a^z \cosh z (x - y) \geq 0.$

2183. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*
Suppose that A, B, C are the angles of a triangle and that $k, l, m \geq 1$. Show that

$$\begin{aligned} 0 &< \sin^k A \cdot \sin^l B \cdot \sin^m C \\ &\leq k^k l^l m^m S^{\frac{S}{2}} \left[(Sk^2 + P)^{-\frac{k}{2}} \right] \left[(Sl^2 + P)^{-\frac{l}{2}} \right] \left[(Sm^2 + P)^{-\frac{m}{2}} \right], \end{aligned}$$

where $S = k + l + m$ and $P = klm$.

2188. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*
Suppose that a, b, c are the sides of a triangle with semi-perimeter s and area Δ. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{s}{\Delta}.$$

2190. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Determine the range of

$$\frac{\sin^2 A}{A} + \frac{\sin^2 B}{B} + \frac{\sin^2 C}{C}$$

where A, B, C are the angles of a triangle.

2191. *Proposed by Šefket Arslanagić, Berlin, Germany.*
Find all positive integers n, that satisfy the inequality

$$\frac{1}{3} < \sin\frac{\pi}{n} < \frac{1}{2\sqrt{2}}.$$

2192. *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*
Let $\{a_n\}$ be a sequence defined as follows:

$$a_{n+1} + a_{n-1} = \left(\frac{a_2}{a_1}\right) a_n, \qquad n \geq 1.$$

Show that if $\left|\frac{a_2}{a_1}\right| \geq 2$, then $\left|\frac{a_n}{a_1}\right| \geq n$.

2198. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*
Prove that, if a, b, c are the lengths of the sides of a triangle

$$(b-c)^2\left(\frac{2}{bc} - \frac{1}{a^2}\right) + (c-a)^2\left(\frac{2}{ca} - \frac{1}{b^2}\right) + (a-b)^2\left(\frac{2}{ab} - \frac{1}{c^2}\right) \geq 0,$$

with equality if and only if $a = b = c$.

2199. *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*
Find the maximum value of c for which $(x+y+z)^2 > cxz$ for all $0 \leq x < y < z$.

220A★. *Proposed by Ji Chen, Ningbo University, China.*
Let P be a point in the interior of the triangle ABC, and let $\alpha_1 = \sphericalangle PAB$, $\beta_1 = \sphericalangle PBC$, $\gamma_1 = \sphericalangle PCA$. Prove or disprove that $\sqrt[3]{\alpha_1\beta_1\gamma_1} \leq \pi/6$.

2202. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Suppose that $n \geq 3$. Let $A_1, \ldots, A_n$ be a convex n-gon (as usual with interior angles $A_1, \ldots, A_n$). Determine the greatest constant C_n such that

$$\sum_{k=1}^{n} \frac{1}{A_k} \geq C_n \sum_{k=1}^{n} \frac{1}{\pi - A_k}.$$

Determine when equality occurs.

2204. *Proposed by Šefket Arslanagić, Berlin, Germany.*
For triangle ABC such that $R(a+b) = c\sqrt{ab}$, prove that

$$r < \frac{3}{10}a.$$

Here, a, b, c, R, and r are the three sides, the circumradius and the inradius of $\triangle ABC$.

2206. *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*
Let a and b denote distinct positive real numbers.
(a) Show that if $0 < p < 1$, $p \neq \frac{1}{2}$, then

$$\frac{1}{2}(a^p b^{1-p} + a^{1-p} b^p) < 4p(1-p)\sqrt{ab} + [1 - 4p(1-p)]\frac{a+b}{2}.$$

(b) Use (a) to deduce Pólya's inequality:

$$\frac{a-b}{\ln a - \ln b} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right).$$

2213. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*
Suppose that the function $f(u)$ has a second derivative in the interval (a, b), and that $f(u) \geq 0$ for all $u \in (a, b)$. Prove that

1. $(y-z)f(x) + (z-x)f(y) + (x-y)f(z) > 0$ for all $x, y, z \in (a, b), z < y < x$ if and only if $f''(u) > 0$ for all $u \in (a, b)$;
2. $(y-z)f(x) + (z-x)f(y) + (x-y)f(z) = 0$ for all $x, y, z \in (a, b), z < y < x$ if and only if $f(u)$ is a linear function on (a, b).

2214. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $n \geq 2$ be a natural number. Show that there exists a constant $C = C(n)$ such that for all real $x_1, \ldots, x_n \geq 0$ we have

$$\sum_{k=1}^{n} \sqrt{x_k} \leq \sqrt{\prod_{k=1}^{n}(x_k + C)}.$$

Determine the minimum $C(n)$ for some values of n.
(For example, $C(2) = 1$.)

2232. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Find all solutions of the inequality:

$$n^2 + n - 5 < \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor < n^2 + 2n - 2, \quad (n \in \mathbb{N}).$$

2233. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let x, y, z be non-negative real numbers such that $x + y + z = 1$, and let p be a positive real number.
(a) If $0 < p \leq 1$, prove that

$$x^p + y^p + z^p \geq C_p[(xy)^p + (yz)^p + (zx)^p],$$

where

$$C_p = \begin{cases} 3^p & \text{if } p \leq \frac{\ln 2}{\ln 3 - \ln 2}, \\ 2^{p+1} & \text{if } p \geq \frac{\ln 2}{\ln 3 - \ln 2}. \end{cases}$$

(b)* Prove the same inequality for $p > 1$.

2236. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*
Let ABC be an arbitrary triangle and let P be an arbitrary point in the interior of the circumcircle of $\triangle ABC$. Let K, L, M, denote the feet of the perpendiculars from P to the lines AB, BC, CA, respectively. Prove that $[KLM] \le \dfrac{[ABC]}{4}$.
Note: $[XYZ]$ denotes the area of $\triangle XYZ$.

2240. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*
Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that $\dfrac{BD}{DC} \le \dfrac{BF}{FA} \le 1$ and $\dfrac{AE}{EC} \le \dfrac{AF}{FB}$. Prove that $[DEF] \le \dfrac{[ABC]}{4}$ with equality if and only if two of the three points D, E, F, (at least) are mid-points of the corresponding sides.
Note: $[XYZ]$ denotes the area of $\triangle XYZ$.

2256. *Proposed by Russell Euler and Jawad Sadek, Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, Missouri, USA.*
If $0 < y < x \le 1$, prove that $\dfrac{\ln(x) - \ln(y)}{x - y} > \ln\left(\dfrac{1}{y}\right)$.

2260. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*
Let n be a positive integer and $x > 0$. Prove that

$$(1+x)^{n+1} \ge \frac{(n+1)^{n+1}}{n^n} x.$$

2262. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Consider two triangles $\triangle ABC$ and $\triangle A'B'C'$ such that $\measuredangle A \ge 90^\circ$ and $\measuredangle A' \ge 90^\circ$ and whose sides satisfy $a > b \ge c$ and $a' > b' \ge c'$. Denote the altitudes to sides a and a' by h_a and h'_a.
Prove that (a) $\dfrac{1}{h_a h'_a} \ge \dfrac{1}{bb'} + \dfrac{1}{cc'}$, (b) $\dfrac{1}{h_a h'_a} \ge \dfrac{1}{bc'} + \dfrac{1}{b'c}$.

2275. *Proposed by M. Perisastry, Vizianagaram, Andhra Pradesh, India.*
Let $b > 0$ and $b^a \ge ba$ for all $a > 0$. Prove that $b = \mathrm{e}$.

2282. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*
A line, l, intersects the sides BC, CA, AB, of $\triangle ABC$ at D, E, F respectively such that D is the mid-point of EF. Determine the minimum value of $|EF|$ and express its length as elements of $\triangle ABC$.

2290. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*
For $x, y, z \ge 0$, prove that

$$[(x+y)(y+z)(z+x)]^2 \ge xyz(2x+y+z)(2y+z+x)(2z+x+y).$$

2296. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*
Show that

$$\sin^2 \frac{\pi x}{2} > \frac{2x^2}{1+x^2} \qquad \text{for } 0 < x < 1.$$

Hence or otherwise, deduce that

$$\pi < \frac{\sin \pi x}{x(1-x)} < 4 \qquad \text{for } 0 < x < 1.$$

2299. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $x, y, z > 0$ be real numbers such that $x + y + z = 1$. Show that

$$\prod_{\text{cyclic}} \left[\frac{(1-y)(1-x)}{x}\right]^{(1-y)(1-z)/x} \geq \frac{256}{81}.$$

Determine the cases of equality.

2300. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Suppose that ABC is a triangle with circumradius R. The circle passing through A and touching BC at its mid-point has radius R_1. Define R_2 and R_3 similarly.
Prove that

$$R_1^2 + R_2^2 + R_3^2 \geq \frac{27}{16} R^2.$$

2301. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Suppose that ABC is a triangle with sides a, b, c, that P is a point in the interior of $\triangle ABC$, and that AP meets the circle BPC again at A'. Define B' and C' similarly. Prove that the perimeter $\mathcal{P}$ of the hexagon $AB'CA'BC'$ satisfies

$$\mathcal{P} \geq 2\left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}\right).$$

2306. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*
(a) Give an elementary proof of the inequality

$$\left(\sin\frac{\pi x}{2}\right)^2 > \frac{2x^2}{1+x^2}; \qquad (0 < x < 1).$$

(b) Hence (or otherwise) show that

$$\tan \pi x \begin{cases} < \frac{\pi x(1-x)}{1-2x}; & (0 < x < \frac{1}{2}, \\ > \frac{\pi x(1-x)}{1-2x}; & (\frac{1}{2} < x < 1). \end{cases}$$

(c) Find the maximum value of $f(x) = \frac{\sin \pi x}{x(1-x)}$ on the interval $(0, 1)$.

2326★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove or disprove that if A, B and C are the angles of a triangle, then

$$\frac{2}{\pi} < \sum_{\text{cyclic}} \frac{(1-\sin\frac{A}{2})(1+2\sin\frac{A}{2})}{\pi - A} \leq \frac{9}{\pi}.$$

2340. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $\lambda > 0$ be a real number and a, b, c be the sides of a triangle. Prove that

$$\prod_{\text{cyclic}} \frac{s + \lambda a}{s - a} \geq (2\lambda + 3)^3.$$

[As usual s denotes the semiperimeter.]

2345. *Proposed by Vedula N. Murty, Andhra University, Visakhapatnam, India.*
Suppose that $x > 1$.
(a) Show that

$$\ln x > \frac{3(x^2-1)}{x^2+4x+1}.$$

(b) Show that

$$\frac{a-b}{\ln a - \ln b} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right),$$

where $a > 0$, $b > 0$ and $a \neq b$.

2349. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*
Suppose that $\triangle ABC$ has acute angles such that $A < B < C$. Prove that

$$\sin^2 B \sin\frac{A}{2}\sin\left(A+\frac{B}{2}\right) > \sin^2 A \sin\frac{B}{2}\sin\left(B+\frac{A}{2}\right).$$

2362. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*
Suppose that $a, b, c > 0$. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}.$$

2365. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*
Triangle DAC is equilateral. B is on the line DC so that $\sphericalangle BAC = 70°$. E is on the line AB so that $\sphericalangle ECA = 55°$. K is the mid-point of ED. Without the use of a computer, calculator or protractor, show that $60° > \sphericalangle AKC > 57.5°$.

2374. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Given triangle ABC with $\sphericalangle BAC > 60°$. Let M be the mid-point of BC. Let P be any point in the plane of $\triangle ABC$. Prove that $AP + BP + CP \geq 2\,AM$.

2382. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*
If $\triangle ABC$ has inradius r and circumradius R, show that

$$\cos^2\left(\frac{B-C}{2}\right) \geq \frac{2r}{R}.$$

2384. *Proposed by Paul Bracken, CRM, Université de Montréal, Québec.*
Prove that

$$2(3n-1)^n \geq (3n+1)^n \quad \text{for all } n \in \mathbb{N}.$$

2389. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*
Suppose that f is continuous on $\mathbb{R}^n$ and satisfies the condition that when any two of its variables are replaced by their arithmetic mean, the value of the function increases; for example:

$$f(a_1, a_2, a_3, \ldots, a_n) \leq f\left(\frac{a_1+a_3}{2}, a_2, \frac{a_1+a_3}{2}, a_4, \ldots, a_n\right).$$

Let $m = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$. Prove that

$$f(a_1, a_2, a_3, \ldots, a_n) \leq f(m, m, m, \ldots, m).$$

2392. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Suppose that x_i, y_i, $(1 \le i \le n)$ are positive real numbers. Let

$$A_n = \sum_{i=1}^{n} \frac{x_i y_i}{x_i + y_i}, \qquad B_n = \frac{\left(\sum\limits_{i=1}^{n} x_i\right)\left(\sum\limits_{i=1}^{n} y_i\right)}{\sum\limits_{i=1}^{n} (x_i + y_i)},$$

$$C_n = \frac{\left(\sum\limits_{i=1}^{n} x_i\right)^2 + \left(\sum\limits_{i=1}^{n} y_i\right)^2}{\sum\limits_{i=1}^{n} (x_i + y_i)}, \qquad D_n = \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{x_i + y_i}.$$

Prove that

1. $A_n \le C_n$,
2. $B_n \le D_n$,
3. $2A_n \le 2B_n \le C_n \le D_n$.

2393. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Suppose that a, b, c and d are positive real numbers. Prove that

1. $[(a+b)(b+c)(c+d)(d+a)]^{3/2} \ge 4abcd(a+b+c+d)^2$,
2. $[(a+b)(b+c)(c+d)(d+a)]^3 \ge 16(abcd)^2 \prod\limits_{\substack{a,b,c,d \\ \text{cyclic}}} (2a+b+c)$.

2394. *Proposed by Vedula N. Murty, Visakhapatnam, India.*
The inequality $a^a b^b \ge \left(\dfrac{a+b}{2}\right)^{a+b}$, where $a, b > 0$, is usally proved using Calculus. Give a proof without the aid of Calculus.

2400. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*
(a) Show that $1 + (\pi - 2)x < \dfrac{\cos(\pi x)}{1 - 2x} < 1 + 2x$ for $0 < x < 1/2$.
[Proposed by Bruce Shawyer, Editor-in-Chief.]
(b)★ Show that $\dfrac{\cos(\pi x)}{1 - 2x} < \dfrac{\pi}{2} - 2(\pi - 2)\left(x - \dfrac{1}{2}\right)^2$ for $0 < x < 1/2$.

2401. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*
In triangle ABC, CD is the altitude from C to AB. E and F are the mid-points of AB and CD respectively. P and Q are points on line segments BC and AC respectively, and are such that $PQ \parallel BA$. The projection of Q onto AB is R. PR and EF intersect at S. Prove that
(a) S is the mid-point of line segment PR,
(b) $\dfrac{1}{PR^2} \le \dfrac{1}{AB^2} + \dfrac{1}{CD^2}$.

2414. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, Dong Province, China, and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*
For $1 < x \le \mathrm{e} \le y$ or $\mathrm{e} \le x < y$, prove that $x^x y^{x^y} > x^{y^x} y^x$.

2422★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let A, B, C be the angles of an arbitrary triangle. Prove or disprove that

$$\frac{1}{A}+\frac{1}{B}+\frac{1}{C} \geq \frac{9\sqrt{3}}{2\pi(\sin A \sin B \sin C)^{1/3}}.$$

2423. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $x_1, x_2, \ldots, x_n > 0$ be real numbers such that $x_1+x_2+\ldots+x_n = 1$, where $n > 2$ is a natural number. Prove that

$$\prod_{k=1}^{n}\left(1+\frac{1}{x_k}\right) \geq \prod_{k=1}^{n}\left(\frac{n-x_k}{1-x_k}\right).$$

Determine the cases of equality.

2439. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that $ABCD$ is a square with side a. Let P and Q be points on sides BC and CD respectively, such that $\sphericalangle PAQ = 45^\circ$. Let E and F be the intersections of PQ with AB and AD respectively. Prove that $AE + AF \geq 2\sqrt{2}\,a$.

2443. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*
Without the use of any calculating device, find an explicit example of an integer, M, such that $\sin(M) > \sin(33) (\approx 0.99991)$. (Of course, M and 33 are in radians.)

2468. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
For $c > 0$, let $x, y, z > 0$ satisfy

$$xy + yz + zx + xyz = c. \qquad (1)$$

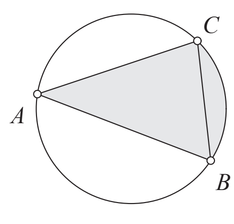
Determine the set of all $c > 0$ such that whenever (1) holds, then we have

$$x + y + z \geq xy + yz + zx.$$

2472. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*
If A, B, C are the angles of a triangle, prove that

$$\cos^2\left(\frac{A-B}{2}\right)\cos^2\left(\frac{B-C}{2}\right)\cos^2\left(\frac{C-A}{2}\right) \geq \left[8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]^3.$$

2477. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Given a non-degenerate $\triangle ABC$ with circumcircle Γ, let r_A be the inradius of the region bounded by BA, AC and arc(CB) (so that the region includes the triangle). Similarly, define r_B and r_C. Aus usual, r and R are the inradius and circumradius of $\triangle ABC$.



Prove that

(a) $\frac{64}{27}r^3 \leq r_A r_B r_C \leq \frac{32}{27}Rr^2$;

(b) $\frac{16}{3}r^2 \leq r_B r_C + r_C r_A + r_A r_B \leq \frac{8}{3}Rr$;

(c) $4r \leq r_A + r_B + r_C \leq \frac{4}{3}(R+r)$,

with equality occuring in all cases if and only if $\triangle ABC$ is equilateral.

2481. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that A, B, C are 2×2 commutative matrices. Prove that

$$\det((A+B+C)(A^3+B^3+C^3-3ABC)) \geq 0.$$

2482. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that p, q, r are complex numbers. Prove that

$$|p+q|+|q+r|+|r+p| \leq |p|+|q|+|r|+|p+q+r|.$$

2483. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that $0 \leq A, B, C$ and $A+B+C \leq \pi$. Show that

$$0 \leq A - \sin A - \sin B - \sin C + \sin(A+B) + \sin(A+C) \leq \pi.$$

There are, of course, similar inequalities with the angles permuted cyclically.

2497. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*
Given $\triangle ABC$ and a point D on AC, let $\sphericalangle ABD = \delta$ and $\sphericalangle DBC = \gamma$. Find all values of $\sphericalangle BAC$ for which $\frac{\delta}{\gamma} > \frac{AD}{DC}$.

2502. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
In $\triangle ABC$, the internal bisectors of $\sphericalangle BAC$, $\sphericalangle ABC$ and $\sphericalangle BCA$ meet BC, AC and AB at D, E and F respectively. Let p and q be the perimeters of $\triangle ABC$ and $\triangle DEF$ respectively. Prove that $p \geq 2q$, and that equality holds if and only if $\triangle ABC$ is equilateral.

2504. *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod_{\text{cyclic}} \cos(B-C)$.

2505. *Proposed by Hayo Ahlburg, Benidorm, Spain, and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Suppose that A, B and C are the angles of a triangle. Determine the best lower and upper bounds of $\prod_{\text{cyclic}} \sin(B-C)$.

2507. *Proposed by Ice B. Risteski, Skopje, Macedonia.*
Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n!+1, k!+1) > 1$.

2509. *Proposed by Ice B. Risteski, Skopje, Macedonia.*
Show that there are infinitely many pairs of distinct natural numbers, n and k such that $\gcd(n!-1, k!-1) > 1$.

2512. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
In $\triangle ABC$, the sides satisfy $a \geq b \geq c$. Let R and r be the circumradius and the inradius respectively. Prove that

$$bc \leq 6Rr \leq a^2,$$

with equality if and only if $a = b = c$.

2516. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
In isosceles $\triangle ABC$ (with $AB = AC$), let D and E be points on sides AB and AC respectively such that $AD < AE$. Suppose that BE and CD meet at P. Prove that $AE + EP < AD + DP$.

2522$^\bigstar$**.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Suppose that a, b and c are positive real numbers. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) \geq \frac{9}{1+abc}.$$

2523. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Prove that, if $t \geq 1$, then

$$\ln t \leq \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right).$$

Also, prove that, if $0 < t \leq 1$, then

$$\ln t \geq \frac{t-1}{2(t+1)}\left(1+\sqrt{\frac{2t^2+5t+2}{t}}\right).$$

2527. *Proposed by K. R. S. Sastry, Dodballapur, India.*
Let AD, BE and CF be concurrent cevians of $\triangle ABC$. Assume that:
(a) AD is a median; (b) BE bisects $\sphericalangle ABC$; (c) BE bisects AD.
Prove that $CF > BE$.

2529. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $G = \{A_1, A_2, \ldots, A_n\}$ be a set of points on a unit hemisphere. Let $\overbrace{A_iA_j}$ be the spherical distance between the points A_i and A_j. Suppose that $\overbrace{A_iA_j} \geq d$. Find $\max d$.

2531. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let F be a convex plane set and AB its diameter. The points A and B divide the perimeter of F into two parts, L_1 and L_2, say. Prove that

$$\frac{1}{\pi-1} < \frac{L_1}{L_2} < \pi - 1.$$

2532. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*
Suppose that a, b and c are positive real numbers satisfying $a^2+b^2+c^2 = 1$. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq 3 + \frac{2(a^3+b^3+c^3)}{abc}.$$

2536. *Proposed by Cristinel Mortici, Ovidius University of Constanta, Romania.*
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function such that for all positive integers n the following inequality holds:

$$\frac{|f(1)|}{1} + \frac{|f(2)|}{2} + \cdots + \frac{|f(n)|}{n} \leq 1.$$

Prove that there exists $c \in \mathbb{R}$ such that $f(c) = 0$ and $f(c+1) = 0$.

2539. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*
Let $ABCD$ be a convex quadrilateral with vertices oriented in the clockwise sense. Let X and Y be interior points on AD and BC, respectively. Suppose that P is a point between X and Y such that $\sphericalangle AXP = \sphericalangle BYP = \sphericalangle APB = \theta$ and $\sphericalangle CPD = \pi - \theta$ for some θ.
(a) Prove that $AD \cdot BC \geq 4PX \cdot PY$.
(b)$^\bigstar$ Find the case(s) of equality.

2542. *Proposed by Hassan Ali Shah Ali, Tehran, Iran.*
Suppose that k is a natural number and $\alpha_i \geq 0$, $i = 1, \ldots, n$, and $\alpha_{n+1} = \alpha_1$. Prove that

$$\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \alpha_i^{k-j} \alpha_{i+1}^{j-1} \geq \frac{k}{n^{k-2}} \left(\sum_{1 \leq i \leq n} \alpha_i \right)^{k-1} .$$

Determine the necessary and sufficient conditions for equality.

2551. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*
Suppose that a_k $(1 \leq k \leq n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise. Prove that

$$\prod_{j=1}^{n} \sum_{k=1}^{n} e_{j,k} a_k^2 \geq \prod_{j=1}^{n} \left(\sum_{k=1}^{n} d_{j,k} a_k \right)^2 .$$

2552. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*
Suppose that $a, b, c > 0$. If $x \geq \frac{a+b+c}{3\sqrt{3}} - 1$, prove that

$$\frac{(b+cx)^2}{a} + \frac{(c+ax)^2}{b} + \frac{(a+bx)^2}{c} \geq abc.$$

2554. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*
In triangle ABC, prove that at least one of the quantities

$$(a+b-c)\tan^2\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right),$$

$$(-a+b+c)\tan^2\left(\frac{B}{2}\right)\tan\left(\frac{C}{2}\right),$$

$$(a-b+c)\tan^2\left(\frac{C}{2}\right)\tan\left(\frac{A}{2}\right),$$

is greater than or equal to $\frac{2r}{3}$, where r is the radius of the incircle of $\triangle ABC$.

2555. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*
In any triangle ABC, show that

$$\sum_{\text{cyclic}} \frac{1}{\tan^3 \frac{A}{2} + \left(\tan \frac{B}{2} + \tan \frac{C}{2}\right)^3} < \frac{4\sqrt{3}}{3}.$$

2557. *Proposed by Gord Sinnamon, University of Western Ontario, London, Ontario, and Hans Heinig, McMaster University, Hamilton, Ontario.*
(a) Show that for all positive sequences $\{x_i\}$ and all integers $n > 0$,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \leq 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 x_k^{-1}.$$

(b)★ Does the above inequality remain true without the factor 2?
(c)★ What is the minimum constant c that can replace the factor 2 in the above inequality?

2571. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*
Suppose that a, b and c are the sides of a triangle. Prove that

$$\frac{1}{\sqrt{a}+\sqrt{b}-\sqrt{c}} + \frac{1}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{1}{\sqrt{c}+\sqrt{a}-\sqrt{b}} \geq \frac{3(\sqrt{a}+\sqrt{b}+\sqrt{c})}{a+b+c}.$$

2572. *Proposed by José Luis Díaz, Universitat Politècnica Catalunya, Terrassa, Spain.*
Let a, b, c be positive real numbers. Prove that

$$a^b b^c c^a \leq \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

2575. *Proposed by H. Fukagawa, Kani, Gifu, Japan.*
Suppose that $\triangle ABC$ has a right angle at C. The circle, centre A and radius AC meets the hypotenuse AB at D. In the region bounded by the arc DC and the line segments BC and BD, draw a square $EFGH$ of side y, where E lies on arc DC, F lies on DB and G and H lie on BC. Assume that BC is constant and that $AC = x$ is variable. Find $\max y$ and the corresponding value of x.

2580. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*
Suppose that a, b and c are positive real numbers. Prove that

$$\frac{b+c}{a^2+bc} + \frac{c+a}{b^2+ac} + \frac{a+b}{c^2+ab} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

2581. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*
Suppose that a, b and c are positive real numbers. Prove that

$$\frac{ab+c^2}{a+b} + \frac{bc+a^2}{b+c} + \frac{ca+b^2}{c+a} \geq a+b+c.$$

2585. *Proposed by Vedula N. Murty, Visakhapatnam, India.*
Prove that, for $0 < \theta < \pi/2$,

$$\tan\theta + \sin\theta > 2\theta.$$

2588. *Proposed by Niels Bejlegaard, Stavanger, Norway.*
Each positive whole integer a_k $(1 \le k \le n)$ is less than a given positive integer N. The least common multiple of any two of the numbers a_k is geater than N.

(a) Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k} < 2$.

(b)★ Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k} < \frac{6}{5}$.

(c)★ Find the smallest real number γ such that $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k} < \gamma$.

2590. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Spain.*
For $n = 1, 2, \ldots$, prove that $\displaystyle\prod_{k=1}^{n} \binom{n}{k}^2 \le \left(\frac{1}{n+1}\binom{2n}{n}\right)^n$.

2594. *Proposed by Nairi M. Sedrakyan, Yerevan, Armenia.*
Given a point M inside the triangle ABC, prove that

$$\min(MA, MB, MC) + MA + MB + MC < AB + BC + CA.$$

2596. *Proposed by Clark Kimberling, University of Evansville, Evansville, IN, USA.*
Write $r \ll s$ if there is an integer k satisfying $r < k < s$. Find, as a function of n $(n \ge 2)$ the least positive integer k satisfying

$$\frac{k}{n} \ll \frac{k}{n-1} \ll \frac{k}{n-2} \ll \cdots \ll \frac{k}{2} \ll k.$$

2597. *Proposed by Michael Lambrou, University of Crete, Crete, Greece.*
Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that

$$|\sphericalangle PBC - \sphericalangle PCB| \le \arcsin\left[2\sin\left(\frac{|\sphericalangle PAB - \sphericalangle PAC|}{2}\right)\right] - \left(\frac{|\sphericalangle PAB - \sphericalangle PAC|}{2}\right) \le |\sphericalangle PAB - \sphericalangle PAC|.$$

Show that the left inequality cannot be improved in the sense that there is a position Q of P on the ray AP giving an equality.

2603. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*
Suppose that A, B and C are the angles of a triangle. Prove that

$$\sin A + \sin B + \sin C \le \sqrt{\frac{15}{4} + \cos(A-B) + \cos(B-C) + \cos(C-A)}.$$

2604. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
(a) Determine the upper and lower bounds of

$$\frac{a}{a+b}+\frac{b}{b+c}-\frac{a}{a+c}$$

for all positive real numbers a, b and c.
(b)* Determine the upper and lower bounds (as functions of n) of

$$\sum_{j=1}^{n-1}\frac{x_j}{x_j+x_{j+1}}-\frac{x_1}{x_1+x_n}$$

for all positive real numbers $x_1, x_2, \ldots, x_n$.

2608***.** *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Suppose that $x, y, z \geq 0$ and $x^2+y^2+z^2=1$. Prove or disprove that

$$\text{(a)}\ \ 1 \leq \frac{x}{1-yz}+\frac{y}{1-zx}+\frac{z}{1-xy} \leq \frac{3\sqrt{3}}{2};$$

$$\text{(b)}\ \ 1 \leq \frac{x}{1+yz}+\frac{y}{1+zx}+\frac{z}{1+xy} \leq \sqrt{2}.$$

2615. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Suppose that $x_1, x_2, \ldots, x_n$ are non-negative numbers such that

$$\sum x_1^2+\sum (x_1x_2)^2=\frac{n(n+1)}{2},$$

where the sums here and subsequently are symmetric over the subscripts $1, 2, \ldots, n$.
(a) Determine the maximum of $\sum x_1$.
(b)* Prove or disprove that the minimum of $\sum x_1$ is $\sqrt{\frac{n(n+1)}{2}}$.

2623***.** *Proposed by Hassan Ali Shah Ali, Tehran, Iran.*
Suppose that $x_1, x_2, \ldots, x_n > 0$. Let $x_{n+1}=x_1$, $x_{n+2}=x_2$, etc.
For $k=0, 1, \ldots, n-1$, let

$$S_k=\sum_{j=1}^{n}\left(\frac{\sum_{i=0}^{k} x_{j+i}}{\sum_{i=0}^{k} x_{j+1+i}}\right).$$

Prove or disprove that $S_k \geq S_{k+1}$.

2625. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*
If R denotes the circumradius of triangle ABC, prove that

$$18R^3 \geq (a^2+b^2+c^2)R+\sqrt{3}abc.$$

2627. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $x_1, \ldots, x_n$ be positive real numbers and let $s_n=x_1+\cdots+x_n$ $(n \geq 2)$. Let $a_1, \ldots, a_n$ be non-negative real numbers. Determine the optimum constant $C(n)$ such that

$$\sum_{j=1}^{n}\frac{a_j(s_n-x_j)}{x_j} \geq C(n)\left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}.$$

2628. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*
Four points, X, Y, Z and W are taken inside or on triangle ABC. Prove that there exists a set of three of these points such that the area of the triangle formed by them is less than $\frac{3}{8}$ of the area of the given triangle.

2629. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
In triangle ABC, the symmedian point is denoted by S. Prove that

$$\frac{1}{3}(AS^2+BS^2+CS^2) \geq \frac{BC^2AS^2+CA^2BS^2+AB^2CS^2}{BC^2+CA^2+AB^2}.$$

2633. *Proposed by Mihály Bencze, Brasov, Romania.*
Prove that

$$\frac{n(n+1)}{2\mathrm{e}} < \sum_{k=1}^{n}(k!)^{\frac{1}{k}} < \frac{31}{20}+\frac{n(n+1)}{4}.$$

2635. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Consider triangle ABC, and three squares $BCDE$, $CAFG$ and $ABHI$ constructed on its sides, outside the triangle. Let XYZ be the triangle enclosed by the lines EF, DI and GH. Prove that $[XYZ] \leq (4-2\sqrt{3})\,[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

2637. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that ABC is an isosceles triangle with $AB = AC$. Let D be a point on side AB, and let E be a point on AC produced beyond C. The line DE meets BC at P. The incircle of $\triangle ADE$ touches DE at Q.
Prove that $BP \cdot PC \leq DQ \cdot QE$, and that equality holds if and only if $BD = CE$.

2641. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let H be a centrosymmetric convex hexagon, with area h, and let P be its minimal circumscribed parallelogram, with area p. Prove that

$$3p \leq 4h.$$

2645. *Proposed by Ho-joo Lee, student, Kwangwoon University, Kangwon-Do, South Korea.*
Suppose that a, b and c are positive real numbers. Prove that

$$\frac{2(a^3+b^3+c^3)}{abc}+\frac{9(a+b+c)^2}{a^2+b^2+c^2} \geq 33.$$

2650. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
In $\triangle ABC$, let a denote the side BC, and h_a, the corresponding altitude. Let r and R be the radii of the inscribed and circumscribed circles, respectively. Prove that $ra < h_aR$.

2651★**.** *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Let P be a non-exterior point of a regular n-dimensional simplex $A_0A_1A_2\ldots A_n$ of edge length e. If

$$F=\sum_{k=0}^{n}PA_k+\min_{0\leq k\leq n}PA_k, \quad F'=\sum_{k=0}^{n}PA_k+\max_{0\leq k\leq n}PA_k,$$

determine the maximum and minimum values of F and F'. (Professor Klamkin offers a prize of \$100 for the first correct solution received by the Editor-in-Chief.)

2652★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let d, e and f be the sides of the triangle determined by the three points at which the internal angle-bisectors of given $\triangle ABC$ meet the opposite sides. Prove that

$$d^2 + e^2 + f^2 \le \frac{s^2}{3},$$

where s is the semiperimeter of $\triangle ABC$.

2656★. *Proposed by Vedula N. Murty, Dover, PA, USA.*
For positive real numbers a, b and c, show that

$$\frac{(1-b)(1-bc)}{b(1+a)} + \frac{(1-c)(1-ca)}{c(1+b)} + \frac{(1-a)(1-ab)}{a(1+c)} \ge 0.$$

2662. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Suppose that $\triangle ABC$ is acute-angled, has inradius r and has area Δ. Prove that

$$\left(\sqrt{\cot A} + \sqrt{\cot B} + \sqrt{\cot C}\right)^2 \le \frac{\Delta}{r^2}.$$

2664. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*
Let a, b and c be positive real numbers such that $a + b + c = abc$. Prove that

$$a^5(bc-1) + b^5(ca-1) + c^5(ab-1) \ge 54\sqrt{3}.$$

2665. *Proposed by Aram Tangboondouangjit, Carnegie Mellon University, Pittsburgh, PA, USA.*
In $\triangle ABC$, we have $\sphericalangle ACB = 90^\circ$ and sides $AB = c$, $BC = a$ and $CA = b$. In $\triangle DEF$, we have $\sphericalangle EFD = 90^\circ$, $EF = (a+c)\sin\left(\frac{B}{2}\right)$ and $FD = (b+c)\sin\left(\frac{A}{2}\right)$. Show that $DE \ge c$.

2667. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
You are given a circle Γ and two points A and B outside of Γ such that the line through A and B does not intersect Γ. Let X be any point on Γ. Determine at which point X on Γ the sum $AX + XB$ attains its minimum value.

2668★. *Proposed by Vedula N. Murty, Dover, PA, USA.*
Suppose that $0 < r < q < 1$ and that $0 < m < \infty$. Show that

$$\begin{aligned}(1-q)(q+r-qr)\sqrt{1+m^2} + q(1-r)\sqrt{(q-2)^2+m^2q^2} \\ > (1-r)(q+r-qr)\sqrt{1+m^2} + r(1-q)\sqrt{(r-2)^2+m^2r^2}.\end{aligned}$$

2669★. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Let $A_1, A_2, \ldots, A_{2n}$, be any $2n$ points in $\mathbb{E}^m$. Determine the largest k_n such that

$$A_1A_2^2 + A_2A_3^2 + \cdots + A_{2n}A_1^2 \ge k_n\left(A_1A_{n+1}^2 + A_2A_{n+2}^2 + \cdots + A_nA_{2n}^2\right).$$

For $n = 2$, it is easily shown that $k_2 = 1$. That $k_3 = \frac{1}{2}$ is an Armenian Olympiad problem. (Professor Klamkin offers a prize of \$50 for the first correct solution received by the Editor-in-Chief.)

2672. *Proposed by Vedula N. Murty, Dover, PA, USA.*

(a) Suppose that $\alpha > 0$. Prove that $\displaystyle\sum_{k=1}^{n} k^\alpha < \frac{(n+1)^{\alpha+1}-1}{\alpha+1}$.

(b) Suppose that $-1 < \alpha < 0$. Prove that $\displaystyle\frac{(n+1)^{\alpha+1}-1}{\alpha+1} < \sum_{k=1}^{n} k^\alpha$.

[These two inequalities appear differently in "Analytic Inequalities" by Nicolas D. Kazarinoff, Holt Rinehart and Winston, p. 24. The term "-1" is missing from the numerators.]

2673. *Proposed by George Baloglou, SUNY Oswego, Oswego, NY, USA.*

Let $n \geq 2$ be an integer.

(a) Show that

$$(1 + a_1 \cdots a_n)^n \geq (a_1 \cdots a_n)\left(1 + a_1^{n-2}\right)\left(1 + a_2^{n-2}\right)\cdots\left(1 + a_n^{n-2}\right)$$

for all $a_1 \geq 1, a_2 \geq 1, \ldots, a_n \geq 1$, if and only if $n \leq 4$.

(b) Show that

$$\frac{1}{a_1\left(1 + a_2^{n-2}\right)} + \frac{1}{a_2\left(1 + a_3^{n-2}\right)} + \cdots + \frac{1}{a_n\left(1 + a_1^{n-2}\right)} \geq \frac{n}{1 + a_1 \cdots a_n}$$

for all $a_1 > 0, a_2 > 0, \ldots, a_n > 0$, if and only if $n \leq 3$.

(c) Show that

$$\frac{1}{a_1\left(1 + a_1^{n-2}\right)} + \frac{1}{a_2\left(1 + a_2^{n-2}\right)} + \cdots + \frac{1}{a_n\left(1 + a_n^{n-2}\right)} \geq \frac{n}{1 + a_1 \cdots a_n}$$

for all $a_1 > 0, a_2 > 0, \ldots, a_n > 0$, if and only if $n \leq 8$.

(d)* Show that

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)\left(\frac{1}{1 + a_1^{n-2}} + \frac{1}{1 + a_2^{n-2}} + \cdots + \frac{1}{1 + a_n^{n-2}}\right) \geq \frac{n^2}{1 + a_1 \cdots a_n}$$

for all $a_1 > 0, a_2 > 0, \ldots, a_n > 0$, if and only if $n \leq 5$.

2676. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let A, B and C be the angles of a triangle. Show that

$$(\sin A + \sin B + \sin C)^2 \leq 6\,(1 + \cos A \cos B \cos C).$$

When does equality occur?

2677. *Proposed by Péter Ivády, Budapest, Hungary.*

For $0 < x < \frac{\pi}{2}$, show that $\displaystyle\frac{\pi^2 - x^2}{\pi^2 + x^2} < \cos\left(\frac{x}{\sqrt{3}}\right)$.

2685. *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Let $\mathcal{C}$ be a bounded, closed and convex domain in the plane. Construct a parallelogram $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{A}(\mathcal{P}) \geq \frac{1}{2}\,\mathcal{A}(\mathcal{C})$, where $\mathcal{A}$ denotes area.

(b)* Prove that if, further, $\mathcal{C}$ is centrally symmetric, then one can construct a parallelogram $\mathcal{P}$ such that $\mathcal{A}(\mathcal{P}) \geq \frac{2}{\pi}\,\mathcal{A}(\mathcal{C})$.

2686*. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $\mathcal{C}$ be a bounded, closed and convex domain in space. Construct a parallelepiped $\mathcal{P}$ contained in $\mathcal{C}$ such that $\mathcal{V}(\mathcal{P}) \geq \frac{4}{9}\,\mathcal{V}(\mathcal{C})$, where $\mathcal{V}$ denotes volume.

2690. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let $\triangle ABC$ be such that $\sphericalangle A$ is the largest angle. Let r be the inradius and R the circumradius. Prove that

$$A \gtreqless 90^\circ \iff R + r \gtreqless \frac{b+c}{2}.$$

2693. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*
Given triangle ABC and a point P, the line through P parallel to BC, intersects AC, AB at Y_1, Z_1 respectively. Similarly, the parallel to CA intersects BC, AB at X_2, Z_2, and the parallel to AB intersects BC, AC at X_3, Y_3. Locate the point P for which the sum

$$Y_1P \cdot PZ_1 + Z_2P \cdot PX_2 + X_3P \cdot PY_3$$

of products of signed lengths is maximal.

2700. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*
Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \frac{k}{n+k}\binom{n}{k} < \sum_{k=1}^{n}\binom{n}{k}\log\left(\frac{n+k}{n}\right) < 2^{n-1}.$$

[Ed. "log" is, of course, the natural logarithm.]

2702. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let λ be an arbitrary real number. Show that

$$\left(\frac{s}{r}\right)^{2\lambda} s^2 \geq 3^{3\lambda+1}(s^2 - 8Rr - 2r^2),$$

where R, r and s are the circumradius, the inradius and the semi-perimeter of a triangle, respectively. Determine the cases of equality.

2704. *Proposed by Mihály Bencze, Brasov, Romania.*
Prove that

$$R - 2r \geq \frac{1}{12}\left(\sum_{\text{cyclic}} \sqrt{2(b^2+c^2) - a^2} - \frac{s^2 + r^2 + 4Rr}{R}\right) \geq 0,$$

where a, b and c are the sides of a triangle, and R, r and s are the circumradius, the inradius and the semi-perimeter of a triangle, respectively.

2707. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let ABC be a triangle and P a point in its plane. The feet of the perpendiculars from P to the lines BC, CA and AB are D, E and F respectively. Prove that

$$\frac{AB^2 + BC^2 + CA^2}{4} \leq AF^2 + BD^2 + CE^2,$$

and determine the cases of equality.

2709. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that

1. P is an interior point of $\triangle ABC$,
2. AP, BP and CP meet BC, CA and AB at D, E and F, respectively,
3. A' is a point on AD produced beyond D such that $DA' : AD = \kappa : 1$, where κ is a fixed positive number,
4. B' is a point on BE produced beyond E such that $EB' : BE = \kappa : 1$, and
5. C' is a point on CF produced beyond F such that $FC' : CF = \kappa : 1$.

Prove that $[A'B'C'] \leq \frac{(3\kappa+1)^2}{4}[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

2710. *Proposed by Jaroslav Švrček, Palacký University, Olomouc, Czech Republic.*
Determine the point P on the semicircle Γ, constructed externally over the side AB of the square $ABCD$, such that $AP^2 + CP^2$ is maximal.

2717. *Proposed by Mihály Bencze, Brasov, Romania.*
For any triangle ABC, prove that

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right).$$

2718. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $A_k \in M_m(\mathbb{R})$ with $A_iA_j = O_m$, $i,j \in \{1,2,\ldots,n\}$, with $i<j$ and $x_k \in \mathbb{R}^*$, $(k=1,2,\ldots,n)$. Prove that

$$\det\left(I_m + \sum_{k=1}^{n}(x_kA_k + x_k^2A_k^2)\right) \geq 0.$$

2723. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
For $1 \leq k \leq N$, let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n_1 + n_2 + \cdots + n_k = N$. Determine the minimum value of the sum

$$\sum_{j=1}^{k}\binom{n_j}{m} \quad \text{when} \quad \text{(a)}\ m=2; \quad \text{(b)}^\star\ m \geq 3.$$

2724★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let a, b, c be the sides of a triangle and h_a, h_b, h_c, respectively, the corresponding altitudes. Prove that the maximum range of validity of the inequality

$$\left(\frac{h_a^t + h_b^t + h_c^t}{3}\right)^{1/t} \leq \frac{\sqrt{3}}{2}\left(\frac{a^t + b^t + c^t}{3}\right)^{1/t},$$

where $t \neq 0$ is $\dfrac{-\ln 4}{\ln 4 - \ln 3} < t < \dfrac{\ln 4}{\ln 4 - \ln 3}$.

2729. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*
Let $Z(n)$ denote the number of trailing zeros of $n!$, where $n \in \mathbb{N}$.

(a) Prove that $\dfrac{Z(n)}{n} < \dfrac{1}{4}$.

(b)★ Prove or disprove that $\displaystyle\lim_{n\to\infty}\frac{Z(n)}{n} = \frac{1}{4}$.

2730. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*
Let $\mathrm{AM}(x_1, x_2, \ldots, x_n)$ and $\mathrm{GM}(x_1, x_2, \ldots, x_n)$ denote the arithmetic mean and the geometric mean of the real numbers $x_1, x_2, \ldots, x_n$, respectively. Given positive real numbers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$, prove that

(a) $\mathrm{GM}(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \geq \mathrm{GM}(a_1, a_2, \ldots, a_n) + \mathrm{GM}(b_1, b_2, \ldots, b_n)$.

For each real number $t \geq 0$, define $f(t) = \mathrm{GM}(t + b_1, t + b_2, \ldots, t + b_n) - t$.
(b) Prove that $f(t)$ is a monotonic increasing function of t, and that

$$\lim_{t\to\infty} f(t) = \mathrm{AM}(b_1, b_2, \ldots, b_n).$$

2732. *Proposed by Mihály Bencze, Brasov, Romania.*
Let ABC be a triangle with sides a, b, c, medians m_a, m_b, m_c, altitudes h_a, h_b, h_c, and area Δ. Prove that

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\,\Delta \max\left\{\frac{m_a}{h_a}, \frac{m_b}{h_b}, \frac{m_c}{h_c}\right\}.$$

2734. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Prove that

$$(bc)^{2n+3} + (ca)^{2n+3} + (ab)^{2n+3} \geq (abc)^{n+2}(a^n + b^n + c^n),$$

where a, b, c are non-negative reals, and n is a non-negative integer.

2738. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Let x, y and z be positive real numbers satisfying $x^2 + y^2 + z^2 = 1$. Prove that

$$\frac{x}{1 - x^2} + \frac{y}{1 - y^2} + \frac{z}{1 - z^2} \geq \frac{3\sqrt{3}}{2}.$$

2739. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Suppose that a, b and c are positive real numbers. Prove that

$$\frac{\sqrt{a+b+c} + \sqrt{a}}{b + c} + \frac{\sqrt{a+b+c} + \sqrt{b}}{c + a} + \frac{\sqrt{a+b+c} + \sqrt{c}}{a + b} \geq \frac{9 + 3\sqrt{3}}{2\sqrt{a+b+c}}.$$

2743. *Proposed by Péter Ivády, Budapest, Hungary.*
Show that, for $x, y \in \left(0, \frac{\pi}{2}\right)$,

$$\left(\frac{x}{\sin x} + \frac{y}{\sin y}\right) \cos\left(\frac{x}{2}\right) \cos\left(\frac{y}{2}\right) < 2.$$

2747. *Proposed by K. R. S. Sastry, Bangalore, India.*
Prove that the orthocentre of a triangle lies inside or on the incircle if and only if the inradius is a mean proportional to the two segments of an altitude, sectioned by the orthocentre.

2748. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ be non-negative real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $\sum_{k=1}^{n} a_k = 1$. Determine the least upper bound of $a_n \sum_{k=1}^{n}(n + 1 - k)a_k$.

2749. *Proposed by Christopher J. Bradley, Clifton College, Bristol, U. K.*
Suppose that P is an interior point of $\triangle ABC$. The line through P parallel to AB meets BC at L and CA at M'. The line through P parallel to BC meets CA at M and AB at N'. The line through P parallel to CA meets AB at N and BC at L'. Prove that

(a) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right)\left(\frac{BL'}{L'C}\right)\left(\frac{CM'}{M'A}\right)\left(\frac{AN'}{N'B}\right) = 1$;

(b) $\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right)\left(\frac{AN}{NB}\right) \le \frac{1}{8}$;

(c) $[LMN] = [L'M'N']$; [Note: $[XYZ]$ denotes the area of $\triangle XYZ$.]

(d) $[LMN] \le \frac{[ABC]}{3}$.

Locate the point P when equality holds in part (b) and (d).

2757★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let A, B and C be the angles of a triangle. Show that

$$\sum_{\text{cyclic}} \frac{1}{\tan\left(\frac{A}{2}\right) + 8\tan\left(\frac{\pi - A}{4}\right)^3} \le \frac{9\sqrt{3}}{11}.$$

2760. *Proposed by Michel Bataille, Rouen, France.*
Suppose that A, B, C are the angles of a triangle. Prove that

$$\begin{aligned} 8(\cos A + \cos B + \cos C) &\le 9 + \cos(A-B) + \cos(B-C) + \cos(C-A) \\ &\le \csc^2(A/2) + \csc^2(B/2) + \csc^2(C/2). \end{aligned}$$

2768. *Proposed by Mohammed Aassila, Strasbourg, France.*
Let $x_1, x_2, \ldots, x_n$ be n positive real numbers. Prove that

$$\frac{x_1}{\sqrt{x_1x_2 + x_2^2}} + \frac{x_2}{\sqrt{x_2x_3 + x_3^2}} + \cdots + \frac{x_n}{\sqrt{x_nx_1 + x_1^2}} \ge \frac{n}{\sqrt{2}}.$$

2769. *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*
In $\triangle ABC$, suppose that $\cos B - \cos C = \cos A - \cos B \ge 0$. Prove that

$$(b^2 + c^2)\cos A - (a^2 + b^2)\cos C \le (c^2 - a^2)\sec B.$$

2770. *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*
In $\triangle ABC$, suppose that $a \le b \le c$ and $\angle ABC \ne \frac{\pi}{2}$. Prove that

$$2 + \sec B \le \left(1 + \frac{b}{a}\right)\left(1 + \frac{b}{c}\right).$$

2774. *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*
Let x be a real number such that $0 < x \leq \frac{2}{9}\pi$. Prove that

$$(\sin x)^{\sin x} < \cos x .$$

(This is a generalization of Problem 10261 in the American Mathematical Monthly [1992 : 872, 1994 : 690]).

2775. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*
In $\triangle ABC$, let M be the mid-point of BC. Prove that

$$\cos\left(\frac{B-C}{2}\right) \geq \sin(\angle AMB) \geq 8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right).$$

2778. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that $z \neq 1$ is a complex number such that $z^n = 1$ $(n \geq 1)$. Prove that

$$|nz - (n+z)| \leq \frac{(n+1)(2n+1)}{6}\,|z-1|^2 .$$

2786★**.** *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Prove or disprove the inequality

$$3 \leq \frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx} \leq \frac{27}{8},$$

where $x+y+z = 1$ and $x, y, z \geq 0$.

2787★**.** *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Prove or disprove the inequality

$$\frac{27}{8} \leq \frac{1}{1-\left(\frac{x+y}{2}\right)^2} + \frac{1}{1-\left(\frac{y+z}{2}\right)^2} + \frac{1}{1-\left(\frac{z+x}{2}\right)^2} \leq \frac{11}{3},$$

where $x+y+z = 1$ and $x, y, z \geq 0$.

2791. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that $f : [0,1] \to (0,\infty)$ is a continuous function. Prove that if there exists $\alpha > 0$ such that, for $n \in \mathbb{N}$,

$$\int_0^1 x^{\alpha}(f(x))^n \,\mathrm{d}x \geq \frac{1}{(n+1)\alpha+1} \geq \int_0^1 (f(x))^{n+1}\,\mathrm{d}x,$$

then α is unique.

2792. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $A_k \in M_n(\mathbb{R})$ $(k = 1, 2, \dots, m \geq 2)$ for which

$$\sum_{1 \leq i < j \leq m} (A_i A_j + A_j A_i) = 0_n.$$

Prove that

$$\det \left(\sum_{k=1}^{m} (I_n + A_k)^2 - (m-2) I_n \right) \geq 0.$$

2794. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that $z_k \in \mathbb{C}^*$ $(k = 1, 2, \dots, n)$ such that

$$\begin{aligned} &|z_1 + z_2 + \cdots + z_n| + |z_2 + z_3 + \cdots + z_n| + \cdots + |z_{n-1} + z_n| + |z_n| \\ &\qquad = |z_1 + 2z_2 + \cdots + n z_n|. \end{aligned}$$

Prove that the z_k are collinear.

2795. *Proposed by Mihály Bencze, Brasov, Romania.*
A convex polygon with sides $a_1, a_2, \dots, a_n$, is inscribed in a circle of radius R. Prove that

$$\sum_{k=1}^{n} \sqrt{4R^2 - a_k^2} \leq 2nR \sin\left(\frac{(n-2)\pi}{n} \right).$$

2796★. *Proposed by Fernando Castro G., Matirín Estado Monagas, Vénézuéla.*
Let $\{p_n\}$ be the sequence of prime numbers. Prove that, for each $n \geq 2$, the set $I = \{1, 2, \dots, n\}$ can be partitioned into two sets A and B, where $A \cup B = I$, in such a way that

$$1 \leq \frac{\prod_{i \in A} p_i}{\prod_{j \in B} p_j} \leq 2.$$

2798★. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Prove or disprove the inequality

$$\sum_{j=1}^{n} \frac{1}{1 - \frac{P}{x_j}} \leq \frac{n}{1 - \left(\frac{1}{n}\right)^{n-1}},$$

where $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \geq 0$ $(j = 1, 2, \dots, n)$, and $P = \prod\limits_{j=1}^{n} x_j$.

2799★. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Prove or disprove the inequality

$$\sum_{\substack{i,j \in \{1,2,\dots,n\} \\ 1 \leq i < j \leq n}} \frac{1}{1 - x_i x_j} \leq \binom{n}{2} \frac{1}{1 - \frac{1}{n^2}},$$

where $\sum\limits_{j=1}^{n} x_j = 1$, $x_j \geq 0$.

2801. *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*
Suppose that $\triangle ABC$ is not obtuse. Denote (as usual) the sides by a, b, and c and the circumradius by R. Prove that

$$\left(\frac{2A}{\pi}\right)^{\frac{1}{a}}\left(\frac{2B}{\pi}\right)^{\frac{1}{b}}\left(\frac{2C}{\pi}\right)^{\frac{1}{c}} \leq \left(\frac{2}{3}\right)^{\frac{\sqrt{3}}{R}}.$$

When does equality hold?

2803. *Proposed by I. C. Draghicescu, Bucharest, Romania.*
Suppose that $x_1, x_2, \ldots, x_n$ $(n > 2)$ are real numbers such that the sum of any $n-1$ of them is greater than the remaining number. Let $s = \sum\limits_{k=1}^{n} x_k$. Prove that

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \geq \frac{s}{n-2}.$$

2806. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that $x, y, z > 0$, $\alpha \in \mathbb{R}$ and $x^\alpha + y^\alpha + z^\alpha = 1$. Prove that

a) $x^2 + y^2 + z^2 \geq x^{\alpha+2} + y^{\alpha+2} + z^{\alpha+2} + 2xyz(x^{\alpha-1} + y^{\alpha-1} + z^{\alpha-1})$,

b) $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} \geq x^{\alpha-2} + y^{\alpha-2} + z^{\alpha-2} + \dfrac{2\left(x^{\alpha+1} + y^{\alpha+1} + z^{\alpha+1}\right)}{xyz}$.

2807. *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*
In $\triangle ABC$, denote its area by $[ABC]$ (and its semi-perimeter by s). Show that

$$\min\left\{\frac{2s^4 - (a^4 + b^4 + c^4)}{[ABC]^2}\right\} = 38.$$

2810. *Proposed by I. C. Draghicescu, Bucharest, Romania.*
Suppose that a, b and $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ are positive real numbers. Let $s = \sum\limits_{k=1}^{n} x_k$. Prove that

$$\prod_{k=1}^{n}\left(a + \frac{b}{x_k}\right) \geq \left(a + \frac{nb}{s}\right)^n.$$

2811. *Proposed by Mihály Bencze, Brasov, Romania.*
Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy, for all real x,

$$f(x^3 + x) \leq x \leq f^3(x) + f(x).$$

2812. *Proposed by Mihály Bencze, Brasov, Romania.*
Determine all injective functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$(2a + b)f(ax + b) \geq af^2\left(\frac{1}{x}\right) + bf\left(\frac{1}{x}\right) + a$$

for all positive real x, where $a, b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ and $2a + b > 0$.

2814. *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*
Let a, b, and c be positive real numbers such that $a+b+c=abc$. Find the minimum value of

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}}.$$

2819. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $f:\mathbb{R}\to\mathbb{R}$ satisfy, for all real x and y, $f\left(\frac{2x+y}{3}\right)\geq f\left(\sqrt[3]{x^2y}\right)$. Prove that f is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$.

2821. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
In triangle $\triangle ABC$, let w_a, w_b, w_c be the lengths of the interior angle bisectors, and r the inradius. Prove that

$$\frac{1}{w_a^2}+\frac{1}{w_b^2}+\frac{1}{w_c^2}\leq\frac{1}{3r^2},$$

with equality if and only if $\triangle ABC$ is equilateral.

2829. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Given $\triangle ABC$ with sides a,b,c, prove that

$$\frac{3\left(a^4+b^4+c^4\right)}{(a^2+b^2+c^2)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2}\geq 2.$$

2831. *Proposed by Achilleas Pavlos Porfyriadis, Student, American College of Thessaloniki „Anatolia“, Thessaloniki, Greece.*
For a convex polygon, prove that it is impossible for two sides without a common vertex to be longer than the longest diagonal.

2833★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let a be a positive real number, and let $n\geq 2$ be integer. For each $k=1,2,\ldots,n$, let x_k be a non-negative real number, λ_k be a positive real number, and let $y_k=\lambda_k x_k+\frac{x_{k+1}}{\lambda_{k+1}}$. Here and elsewhere, indices greater than n are to be reduced modulo n.

(a) If $a>1$, prove that

$$n+\sum_{k=1}^{n}a^{y_k}\geq 2\sum_{k=1}^{n}a^{x_k}\qquad\text{and}\qquad 3n+\sum_{k=1}^{n}a^{y_k+y_{k+1}}\geq\sum_{k=1}^{n}(1+a^{x_k})^2.$$

(b) If $0<a<1$, prove that the opposite inequalities hold.

[The proposer has proofs for the cases $n=3$ and $n=4$.]

2835. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
For non-negative real numbers x and y, not both equal to 0, prove that

$$\frac{x^4+y^4}{(x+y)^4}+\frac{\sqrt{xy}}{x+y}\geq\frac{5}{8}.$$

2839. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*
Suppose that x, y, and z are real numbers. Prove that

$$(x^3+y^3+z^3)^2+3\,(xyz)^2 \geq 4\,(y^3z^3+z^3x^3+x^3y^3).$$

Determine the cases of equality.

2841. *Proposed by Mihály Bencze, Brasov, Romania.*
Prove the following inequalities:

$$\begin{aligned}
&\frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{3}{32n^2}-\frac{11}{128n^3}\right)\\
&\qquad\leq \left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}\\
&\qquad\leq \frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{3}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}\right).
\end{aligned}$$

2842. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

(a) $$\frac{\sum\limits_{k=1}^{n} x_k^n}{n\prod\limits_{k=1}^{n} x_k}+\frac{n\left(\prod\limits_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum\limits_{k=1}^{n} x_k} \geq 2,$$

(b) $$\frac{\sum\limits_{k=1}^{n} x_k^n}{\prod\limits_{k=1}^{n} x_k}+\frac{\left(\prod\limits_{k=1}^{n} x_k\right)^{\frac{1}{n}}}{\sum\limits_{k=1}^{n} x_k} \geq 1.$$

2843. *Proposed by Bektemirov Baurjan, student, Aktobe, Kazakstan.*
Suppose that $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=4+\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}$ for positive real x, y, z. Prove that

$$(1-x)(1-y)(1-z) \leq \frac{1}{64}.$$

2846. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
A regular simplex $S_n = A_1A_2A_3\ldots A_{n+1}$ is inscribed in the unit sphere Σ in $\mathbb{E}^n$. Let O be the origin in $\mathbb{E}^n$, $M \in \Sigma$, $u_k = \overrightarrow{OA_k}$ and $v = \overrightarrow{OM}$.
Find the maximum value of $\sum\limits_{k=1}^{n+1} |u_k \cdot v|$.

2852. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
In $\triangle ABC$, we have $AB < AC$. The internal bisector of $\sphericalangle BAC$ meets BC at D. Let P be an interior point of the line segment AD, and let E and F be the intersections of BP and CP with AC and AB, respectively. Prove that

$$\frac{PE}{PF} < \frac{AC}{AB}.$$

2859★. *Proposed by Mohammed Aassila, Université Louis Pasteur, Strasbourg, France.*
Prove that

$$\sum_{\text{cyclic}} \frac{ab}{c(c+a)} \geq \sum_{\text{cyclic}} \frac{a}{c+a},$$

where a, b, c represent the three sides of a triangle.

2860. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
In $\triangle ABC$ and $\triangle A'B'C'$, the lengths of the sides satisfy $a \geq b \geq c$ and $a' \geq b' \geq c'$. Let h_a and $h_{a'}$ denote the lengths of the altitudes to the opposite sides from A and A', respectively. Prove that

(a) $bb' + cc' \geq ah_{a'} + a'h_a$;

(b) $bc' + b'c \geq ah_{a'} + a'h_a$.

2863. *Proposed by Mihály Bencze, Brasov, Romania.*
Suppose that a, b, c are complex numbers such that $|a| = |b| = |c|$. Prove that

$$\left|\frac{ab}{a^2 - b^2}\right| + \left|\frac{bc}{b^2 - c^2}\right| + \left|\frac{ca}{c^2 - a^2}\right| \geq \sqrt{3}.$$

2864. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*
If a, b, c are the sides of an acute angled triangle, prove that

$$\sum_{\text{cyclic}} \sqrt{a^2 + b^2 - c^2}\sqrt{a^2 - b^2 + c^2} \leq ab + bc + ca.$$

2865. *Proposed by George Baloglou, SUNY Oswego, Oswego, NY.*
Suppose that D, E, F are the points at which the concurrent lines AD, BE, CF meet the sides of a given triangle ABC. Let p_1 and p_2 be the perimeters and δ_1 and δ_2 the areas of $\triangle ABC$ and $\triangle DEF$, respectively. Prove that

(a) $2p_2 \leq p_1$ if AD, BE, and CF are angle bisectors;

(b) $2p_2 \leq p_1$ if AD, BE, and CF are altitudes;

(c) $3p_2 \leq 2p_1$ for all D, E, F if and only if $\triangle ABC$ is equilateral;

(d) $4p_2 \leq p_1$ for all D, E, F and arbitrary $\triangle ABC$.

2869. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Given rectangle $ABCD$ with area S, let E and F be points on sides AB and AD, respectively, such that $[CEF] = \frac{1}{3}S$, where $[PQR]$ denotes the area of $\triangle PQR$. Prove that $\sphericalangle ECF \leq \frac{\pi}{6}$.

2871. *Proposed by Mihály Bencze, Brasov, Romania.*
In $\triangle ABC$, denote the sides by a, b, c, the symmedians by s_a, s_b, s_c, and the circumradius by R. Prove that

$$\frac{bc}{s_a} + \frac{ca}{s_b} + \frac{ab}{s_c} \leq 6R.$$

2874. *Proposed by Vedula N. Murty, Dover, PA, USA.*
Let a, b and c denote the side lengths BC, CA, and AB, respectively, of triangle ABC, and let s, r, and R denote the semi-perimeter, inradius, and circumradius of the triangle, respectively. Let $y = s/R$ and $x = r/R$. Show that

1. $\displaystyle\sum_{\text{cyclic}} \sin^2 A = 2 \iff y - x = 2 \iff \triangle ABC$ is right-angled;
2. $\displaystyle\sum_{\text{cyclic}} \sin^2 A > 2 \iff y - x > 2 \iff \triangle ABC$ is acute-angled;
3. $\displaystyle\sum_{\text{cyclic}} \sin^2 A < 2 \iff y - x < 2 \iff \triangle ABC$ is obtuse-angled.

2875. *Proposed by Michel Bataille, Rouen, France.*
Suppose that the incircle of $\triangle ABC$ is tangent to the sides BC, CA, AB, at D, E, F, respectively. Prove that

$$EF^2 + FD^2 + DE^2 \le \frac{s^2}{3},$$

where s is the semiperimeter of $\triangle ABC$.

2880. *Proposed by Mihály Bencze, Brasov, Romania.*
1. If $x, y, z > 1$, prove that

(a) $(\log_{yz} x^4yz)(\log_{zx} xy^4z)(\log_{xy} xyz^4) > 25$,
(b)★ $(\log_{yz} x^4yz)(\log_{zx} xy^4z)(\log_{xy} xyz^4) > 27$.

2.★ If $x_k > 1$ $(k = 1, 2, \ldots, n)$ and $\alpha \ge -1$, prove that

$$\prod_{k=1}^{n} \log_{b_k} b_k x_k^{\alpha+1} \ge \left(\frac{n+\alpha}{n-1}\right)^n,$$

where $b_k = x_1 \cdots x_{k-1} x_{k+1} \cdots x_n$.

2882. *Proposed by Mihály Bencze, Brasov, Romania.*
If $x \in (0, \frac{\pi}{2})$, $0 \le a \le b$, and $0 \le c \le 1$, prove that

$$\left(\frac{c + \cos x}{c+1}\right)^b < \left(\frac{\sin x}{x}\right)^a.$$

2883. *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzogovina.*
Suppose that $x, y, z \in [0, 1)$ and that $x + y + z = 1$. Prove that

$$\sqrt{\frac{xy}{z+xy}} + \sqrt{\frac{yz}{x+yz}} + \sqrt{\frac{zx}{y+zx}} \le \frac{3}{2}.$$

2884. *Proposed by Niels Bejlegaard, Copenhagen, Denmark.*
Suppose that a, b, c are the sides of a non-obtuse triangle. Give a geometric proof and hence, a geometric interpretation of the inequality

$$a + b + c \ge \sum_{\text{cyclic}} \sqrt{a^2 + b^2 - c^2}.$$

2886. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*
If a, b, c are positive real numbers such that $abc = 1$, prove that

$$ab^2 + bc^2 + ca^2 \geq ab + bc + ca.$$

2887. *Proposed by Vedula N. Murty, Dover, PA, USA.*
If a, b, c are the sides of $\triangle ABC$ in which at most one angle exceeds $\frac{\pi}{3}$, and if R is its circumradius, prove that

$$a^2 + b^2 + c^2 \leq 6R^2 \sum_{\text{cyclic}} \cos A.$$

2888★**.** *Proposed by Vedula N. Murty, Dover, PA, USA.*
Let a, b, c be the sides of $\triangle ABC$, in which at most one angle exceeds $\frac{\pi}{3}$. Give an algebraic proof of

$$8a^2b^2c^2 + \prod_{\text{cyclic}} (b^2 + c^2 - a^2) \leq 3abc \sum_{\text{cyclic}} a\,(b^2 + c^2 - a^2).$$

2889. *Proposed by Vedula N. Murty, Dover, PA, USA.*
Suppose that A, B, C are the angles of $\triangle ABC$, and that r and R are its inradius and circumradius, respectively. Show that

$$4\cos(A)\cos(B)\cos(C) \leq 2\left(\frac{r}{R}\right)^2.$$

2890. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Suppose that the polynomial $A(z) = z^n + \sum\limits_{k=0}^{n-1} a_k z^k$ can be factored into $A(z) = \prod\limits_{k=1}^{n} (z - z_k)$, where the z_k are positive real numbers. Prove that, for $k = 1, 2, \ldots, n-1$,

$$\left|\frac{a_{n-k}}{C(n,k)}\right|^{\frac{1}{k}} \geq \left|\frac{a_{n-k-1}}{C(n,k+1)}\right|^{\frac{1}{k+1}},$$

where $C(n,k)$ denotes the binomial coefficient $\binom{n}{k}$. When does equality occur?

2891. *Proposed by Vedula N. Murty, Dover, PA, USA, adapted by the editors.*
Two proofreaders, Chris and Pat, were asked to read a manuscript and find the errors. Let B be the number of errors which both Chris and Pat found, C the number of errors found only by Chris, and P the number found only by Pat; lastly, let N be the number of errors found by neither of them. Prove that

$$\sqrt{(B+P)(C+N)(B+C)(P+N)} \geq |BN - CP|.$$

2893. *Proposed by Vedula N. Murty, Dover, PA, USA.*
In [2001: 45–47], we find three proofs of the classical inequality

$$1 \leq \sum_{\text{cyclic}} \cos(A) \leq \frac{3}{2}.$$

In [2002: 86–87], we find Klamkin's illustrations of the Majorization (or Karamata) Inequality. Prove the above "classical inequality" using the Majorization Inequality.

2894. *Proposed by Vedula N. Murty, Dover, PA, USA.*
Suppose that $\triangle ABC$ is acute-angled. With the standard notation, prove that

$$4abc < (a^2+b^2+c^2)(a\cos A + b\cos B + c\cos C) \le \frac{9}{2}abc.$$

2895. *Proposed by Vedula N. Murty, Dover, PA, USA.*
Suppose that A and B are two events with probabilities $P(A)$ and $P(B)$ such that $0 < P(A) < 1$ and $0 < P(B) < 1$. Let

$$K = \frac{2\,[P(A\cap B) - P(A)P(B)]}{P(A)+P(B)-2P(A)P(B)}.$$

Show that $|K| < 1$, and interpret the value $K = 0$.

2899. *Proposed by Hiroshi Kotera, Nara City, Japan.*
Find the maximum area of a pentagon $ABCDE$ inscribed in a unit circle such that the diagonal AC is perpendicular to the diagonal BD.

2900★. *Proposed by Stanley Rabinowitz, Westford, MA, USA.*
Let I be the incentre of $\triangle ABC$, r_1 the inradius of $\triangle IAB$ and r_2 the inradius of $\triangle IAC$. Computer experiments using *Geometer's Sketchpad* suggest that $r_2 < \frac{5}{4}r_1$.

(a) Prove or disprove this conjecture.
(b) Can $\frac{5}{4}$ be replaced by a smaller constant?

2904. *Proposed by Mohammed Aassila, Strasbourg, France.*
Suppose that $x_1 > x_2 > \cdots > x_n$ are real numbers. Prove that

$$\sum_{k=1}^{n} x_k^2 - \sum_{1\le j<k\le n} \ln(x_j - x_k) \ge \frac{n(n-1)}{4}(1+2\ln 2) - \frac{1}{2}\sum_{k=1}^{n} k\ln k.$$

2906. *Proposed by Titu Zvonaru, Bucharest, Romania.*
Suppose that $k \in \mathbb{N}$. Find $\min\limits_{n\in\mathbb{N}}\left(\frac{2}{n}+\frac{n^2}{k}\right)$.

2911. *Proposed by Mihály Bencze, Brasov, Romania.*
(a) If $z, w \in \mathbb{C}$ and $|z| = 1$, prove that

$$(n-1)\sum_{k=1}^{n}|w+z^k| \ge \sum_{k=1}^{n-1}(n-k)|1-z^k|.$$

(b) If $x \in \mathbb{R}$, prove that

$$(n-1)\sum_{k=1}^{n}|\cos(kx)| \ge \sum_{k=1}^{n-1}(n-k)|\sin(kx)|.$$

2913. *Proposed by Mihály Bencze, Brasov, Romania.*
If $a, b, c > 1$ and $\alpha > 0$, prove that

$$a^{\sqrt{\alpha\log_a b}+\sqrt{\alpha\log_a c}} + b^{\sqrt{\alpha\log_b a}+\sqrt{\alpha\log_b c}} + c^{\sqrt{\alpha\log_c a}+\sqrt{\alpha\log_c b}} \le \sqrt{abc}\left(a^{\alpha-\frac{1}{2}}+b^{\alpha-\frac{1}{2}}+c^{\alpha-\frac{1}{2}}\right).$$

2916. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $S = A_1A_2A_3A_4$ be a tetrahedron and let M be the Steiner point; that is, the point M is such that $\sum_{j=1}^{4} A_jM$ is minimized. Assuming that M is an interior point of S, and denoting by A'_j the intersection of A_jM with the opposite face, prove that

$$\sum_{j=1}^{4} A_jM \geq 3\sum_{j=1}^{4} A'_jM.$$

2917★. *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzogovina.*
If $x_1, x_2, x_3, x_4, x_5 \geq 0$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 1$, prove or disprove that

$$\frac{x_1}{1+x_2} + \frac{x_2}{1+x_3} + \frac{x_3}{1+x_4} + \frac{x_4}{1+x_5} + \frac{x_5}{1+x_1} \geq \frac{5}{6}.$$

2918. *Proposed by Šefket Arslanagić and Faruk Zejnulahi, University of Sarajevo, Sarajevo, Bosnia and Herzogovina.*
Let $a_1, a_2, \ldots, a_{100}$ be real numbers satisfying:

$$\begin{aligned} a_1 \geq a_2 \geq \cdots \geq a_{100} \geq 0; \\ a_1^2 + a_2^2 \geq 200; \\ a_3^2 + a_4^2 + \cdots + a_{100}^2 \geq 200. \end{aligned}$$

What is the minimum value of $a_1 + a_2 + \cdots + a_{100}$?

2919★. *Proposed by Ross Cressman, Wilfrid Laurier University, Waterloo, ON.*
Let $n \in \mathbb{N}$ with $n > 1$, and let

$$T_n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; x_j > 0 \text{ for } j = 1, \ldots, n, \text{ and } \sum_{j=1}^{n} x_j = 1 \right\}.$$

Let $p, q, r \in T_n$ such that $\sum_{j=1}^{n} \sqrt{q_jr_j} < \sum_{j=1}^{n} \sqrt{p_jr_j}$. Prove or disprove:

(a) $\displaystyle\sum_{j=1}^{n} \sqrt{q_j(r_j + p_j)} < \sum_{j=1}^{n} \sqrt{p_j(r_j + p_j)}$,

(b) for all $\lambda \in [0, 1]$,

$$\sum_{j=1}^{n} \sqrt{q_j(\lambda r_j + (1-\lambda)p_j)} < \sum_{j=1}^{n} \sqrt{p_j(\lambda r_j + (1-\lambda)p_j)}.$$

[Proposer's remarks: (a) is the special case of (b) with $\lambda = \frac{1}{2}$. This question is connected with properties of the Shahshahani metric on T_n, a metric important for population genetics.]

2920. *Proposed by Simon Marshall, student, Onslow College, Wellington, New Zealand.*
Let a, b, and c be positive real numbers. Prove that

$$a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2) \geq 3(a^3b + b^3c + c^3a).$$

2923. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Suppose that $x, y \geq 0$ $(x, y \in \mathbb{R})$ and $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

2924. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*
Suppose that $x_1, \ldots, x_n$ $(n \geq 3)$ are positive real numbers satisfying

$$\frac{1}{1 + x_2^2 x_3 \cdots x_n} + \frac{1}{1 + x_1 x_3^2 \cdots x_n} + \cdots + \frac{1}{1 + x_1^2 x_2 \cdots x_{n-1}} \geq \alpha,$$

for some $\alpha > 0$. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1} \geq \frac{n\alpha}{n - \alpha} x_1 x_2 \cdots x_n.$$

2927★. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Suppose that a, b and c are positive real numbers. Prove that

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ca + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab + bc + ca)}{a + b + c}.$$

2928. *Proposed by Christopher J. Bradley, Bristol, UK.*
Suppose that ABC is an equilateral triangle and that P is a point in the plane of $\triangle ABC$. The perpendicular from P to BC meets AB at X, the perpendicular from P to CA meets BC at Y, and the perpendicular from P to AB meets CA at Z.

1. If P is in the interior of $\triangle ABC$, prove that $[XYZ] \leq [ABC]$.
2. If P lies on the circumcircle of ABC, prove that X, Y, and Z are collinear.

2930. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Suppose that a, b, and c are positive real numbers. Prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - 27\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)^{-2} \geq \frac{1}{3}\left[\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2\right].$$

2933. *Proposed by Titu Zvonaru, Bucharest, Romania.*
Prove, without the use of a calculator, that $\sin(40^\circ) < \sqrt{\frac{3}{7}}$.

2935. *Proposed by Titu Zvonaru, Bucharest, Romania.*
Suppose that a, b, and c are positive real numbers which satisfy $a^2 + b^2 + c^2 = 1$, and that $n > 1$ is a positive integer. Prove that

$$\frac{a}{1 - a^n} + \frac{b}{1 - b^n} + \frac{c}{1 - c^n} \geq \frac{(n+1)^{1+\frac{1}{n}}}{n}.$$

2937. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*
Suppose that $x_1, \ldots, x_n$ $(n \geq 2)$ are positive real numbers. Prove that

$$(x_1^2 + \cdots + x_n^2)\left(\frac{1}{x_1^2 + x_1 x_2} + \cdots + \frac{1}{x_n^2 + x_n x_1}\right) \geq \frac{n^2}{2}.$$

2938. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*
Suppose that $x_1, \ldots, x_n, \alpha$ are positive real numbers. Prove that

(a) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \geq \alpha + \sqrt[n]{x_1\cdots x_n}$;

(b) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \leq \alpha + \dfrac{x_1+\cdots+x_n}{n}$.

2946. *Proposed by Panos E. Tsaoussoglou, Athens, Greece.*
Let x, y, z be positive real numbers satisfying $x^2+y^2+z^2=1$. Prove that

(a) $\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right) - (x+y+z) \geq 2\sqrt{3}$,

(b) $\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right) + (x+y+z) \geq 4\sqrt{3}$.

2949★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let $n \geq 3$ be an *odd* natural number. Determine the smallest number $\mu = \mu(n)$ such that the entries of any row and of any column of the matrix

$$\begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,\mu} \\ 2 & a_{2,2} & \cdots & a_{2,\mu} \\ \vdots & \vdots & \ddots & \vdots \\ n & a_{n,2} & \cdots & a_{n,\mu} \end{pmatrix}$$

are distinct numbers from the set $\{1,2,\ldots,n-1,n\}$, and the numbers in each row sum to the same value.

2950★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let ABC be a triangle whose largest angle does not exceed $2\pi/3$. For $\lambda, \mu \in \mathbb{R}$, consider inequalities of the form

$$\cos\left(\frac{A}{2}\right)\cdot\cos\left(\frac{B}{2}\right)\cdot\cos\left(\frac{C}{2}\right) \geq \lambda + \mu\cdot\sin\left(\frac{A}{2}\right)\cdot\sin\left(\frac{B}{2}\right)\cdot\sin\left(\frac{C}{2}\right).$$

(a) Prove that $\lambda_{\max} \geq \dfrac{2\sqrt{3}-1}{8}$.

(b) Prove or disprove that

$$\lambda = \frac{2\sqrt{3}-1}{8} \quad \text{and} \quad \mu = 1+\sqrt{3}$$

yield the best inequality in the sense that λ cannot be increased. Determine also the cases of equality.

2953. *Proposed by Titu Zvonaru, Bucharest, Romania.*
Let m, n be positive integers with $n > 1$, and let a, b, c be positive real numbers satisfying $a^{m+1}+b^{m+1}+c^{m+1}=1$. Prove that

$$\frac{a}{1-ma^n}+\frac{b}{1-mb^n}+\frac{c}{1-mc^n} \geq \frac{(m+n)^{1+\frac{m}{n}}}{n}.$$

2955. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let n be a positive integer. For each positive integer k, let f_k be the k^{th} Fibonacci number; that is, $f_1 = 1$, $f_2 = 1$, and $f_{k+2} = f_{k+1} + f_k$ for all $k \geq 1$. Prove that

$$\left(\sum_{k=1}^{n} f_{k+1}^2\right)\left(\sum_{k=1}^{n} \frac{1}{f_{2k}}\right) \geq n^2.$$

2956. *Proposed by David Loeffler, student, Trinity College, Cambridge, UK.*
Let A, B, C be the angles of a triangle. Prove that

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) < 2$$

if and only if

$$\tan\left(\frac{A}{2}\right) + \tan\left(\frac{B}{2}\right) + \tan\left(\frac{C}{2}\right) < 2.$$

2959. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*
Given a non-isosceles triangle ABC, prove that there exists a unique inscribed equilateral triangle PQR of minimal area, with P, Q, R on BC, CA, and AB, respectively. Construct it by straightedge and compass.

2961. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let ABC and $A'B'C'$ be two right triangles with right angles at A and A'. If w_a and $w_{a'}$ are the interior angle bisectors of angles A and A', respectively, prove that $aw_a a'w_{a'} \geq bcb'c'$, with equality if and only if both ABC and $A'B'C'$ are isosceles.

2962. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let ABC and $A'B'C'$ be two triangles satisfying $a \geq b \geq c$ and $a' \geq b' \geq c'$. If $h_a, h_{a'}$ are the altitudes from the vertices A, A', respectively, to the opposite sides, prove that

$$\text{(i)}\ \ bb' + cc' \geq ah_{a'} + a'h_a, \qquad \text{(ii)}\ \ bc' + b'c \geq ah_{a'} + a'h_a.$$

Remark: Since this problem is identical to problem 2860, it is closed and no solutions will be accepted.

2963. *Proposed by Mihály Bencze, Brasov, Romania.*
Let ABC be any acute-angled triangle. Let r and R be the inradius and circumradius, respectively, and let s be the semiperimeter; that is, $s = \frac{1}{2}(a+b+c)$. Let m_a be the length of the median from A to BC, and let w_a be the length of the internal bisector of $\angle A$ from A to the side BC. We define m_b, m_c, w_b and w_c similarly. Prove that

$$\text{(a)}\ \ \frac{3s^2 - r^2 - 4Rr}{8sRr} \leq \sum_{\text{cyclic}} \frac{m_a}{aw_a} \leq \frac{s^2 - r^2 - 4Rr}{7sRr};$$

$$\text{(b)}\ \ \frac{3}{4} \leq \sum_{\text{cyclic}} \frac{m_a^2}{b^2 + c^2} \leq \frac{4R + r}{4R}.$$

2964. *Proposed by Joe Howard, Portales, NM, USA.*
(Inspired by Problem 80.D, Math. Gazette 80 (489) (1996) p. 606.)
Let $x \in \left(0, \frac{\pi}{2}\right)$. Show that:

$$\text{(a)}\quad \left[\frac{2+\cos x}{3}\right]\left[\frac{2(1-\cos x)}{x^2}\right] > \frac{1+\cos x}{2};$$

$$\text{(b)}\quad \frac{2+\cos x}{3} < \sqrt{\frac{1+\cos x}{2}} < \frac{2(1-\cos x)}{x^2}.$$

2967. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let

$$E_n = \sum_{i=1}^{n} \left(\sum_{j=0}^{n-1} a_i^j \right)^{-1}.$$

If $r = \sqrt[n]{a_1 a_2 \cdots a_n} \geq 1$, prove that $E_n \geq n \left(\sum_{j=0}^{n-1} r^j \right)^{-1}$ for:

(a) $n = 2$, (b) $n = 3$, (c)★ $n \geq 4$.

2968. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let

$$E_n = \frac{1+a_1a_2}{1+a_1} + \frac{1+a_2a_3}{1+a_2} + \cdots + \frac{1+a_na_1}{1+a_n}.$$

Let $r = \sqrt[n]{a_1 a_2 \cdots a_n} \geq 1$.

(a) Prove that $E_n \geq \frac{n(1+r^2)}{1+r}$ for $n = 3$ and $n = 4$.

(b)★ Prove or disprove that $E_n \geq \frac{n(1+r^2)}{1+r}$ for $n = 5$.

2969. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c, d, and r be positive real numbers such that $r = \sqrt[4]{abcd} \geq 1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq \frac{4}{(1+r)^2}.$$

2970. *Proposed by Titu Zvonaru, Bucharest, Romania.*
If m and n are positive integers such that $m \geq n$, and if $a, b, c > 0$, prove that

$$\frac{a^m}{b^m+c^m} + \frac{b^m}{c^m+a^m} + \frac{c^m}{a^m+b^m} \geq \frac{a^n}{b^n+c^n} + \frac{b^n}{c^n+a^n} + \frac{c^n}{a^n+b^n}.$$

2971. *Proposed by Michel Bataille, Rouen, France.*
For $a, b, c \in (0,1)$, find the least upper bound and the greatest lower bound of $a+b+c+abc$, subject to the constraint $ab+bc+ca = 1$.

2972. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
(a) Prove that if $0 \le \lambda \le 4$, then, for all positive real numbers x, y, z, t,

$$\begin{aligned}&(t^2+1)(x^3+y^3+z^3)+3(1-t^2)xyz\\ &\qquad \ge (1+\lambda t)(x^2y+y^2z+z^2x)+(1-\lambda t)(xy^2+yz^2+zx^2).\end{aligned}$$

(b) For $t=\frac{1}{4}$ and $\lambda = 4$, the above inequality becomes

$$17(x^3+y^3+z^3)+45xyz \ge 32(x^2y+y^2z+z^2x).$$

Find all positive values of δ such that the inequality

$$x^3+y^3+z^3+3\delta xyz \ge (1+\delta)(x^2y+y^2z+z^2x)$$

holds for all x, y, z which are: (i) positive real numbers: (ii) side lengths of a triangle.

2975. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Given an inscribed convex quadrilateral with sides of length m, n, p, q, taken in order around the quadrilateral, and diagonals of length d and d', prove that $\sqrt{mp+nq} \le \frac{1}{2}(d+d')$.

2976. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Let $a, b, c \in \mathbb{R}$. Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge (ab+bc+ca)^3.$$

2977. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers, let $r = \sqrt[n]{a_1a_2\cdots a_n}$, and let

$$E_n = \frac{1}{a_1(1+a_2)}+\frac{1}{a_2(1+a_3)}+\cdots+\frac{1}{a_n(1+a_1)}-\frac{n}{r(1+r)}.$$

(a) Prove that $E_n \ge 0$ for

(a_1) $n=3$;

(a_2) $n=4$ and $r \le 1$;

(a_3) $n=5$ and $\frac{1}{2} \le r \le 2$;

(a_4) $n=6$ and $r=1$.

(b)* Prove or disprove that $E_n \ge 0$ for

(b_1) $n=5$ and $r>0$;

(b_2) $n=6$ and $r \le 1$.

2983. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $a_1, a_2, \ldots, a_n < 1$ be non-negative real numbers satisfying

$$a = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \ge \frac{\sqrt{3}}{3}.$$

Prove that

$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+\cdots+\frac{a_n}{1-a_n^2} \ge \frac{na}{1-a^2}.$$

2988$^\bigstar$**.** *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove or disprove:

$$\text{(a)}\ \ xy^2 + yz^2 + zx^2 \geq \frac{1}{3}(xy + yz + zx);$$

$$\text{(b)}\ \ xy^2 + yz^2 + zx^2 \geq xy + yz + zx - \frac{2}{9}.$$

How do the right sides of (a) and (b) compare?

2989. *Proposed by Mihály Bencze, Brasov, Romania.*
Prove that if $0 < a < b < d < \pi$ and $a < c < d$ satisfy $a + d = b + c$, then

$$\frac{\cos(a-d) - \cos(b+c)}{\cos(b-c) - \cos(a+d)} < \frac{ad}{bc}.$$

2991. *Proposed by Mihály Bencze, Brasov, Romania.*
Let n be an integer, $n \geq 3$. For all $z_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, prove

$$(n-1)\left|\sum_{i=1}^{n} z_i^3 - 3\sum_{1\leq i<j<k\leq n} z_i z_j z_k\right| \leq \left|\sum_{i=1}^{n} z_i\right| \sum_{1\leq i<j\leq n}\left(|z_i - z_j|^2 + (n-3)|z_i + z_j|\right).$$

2992. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*
Let Q be a point interior to $\triangle ABC$. Let M, N, P be points on the sides BC, CA, AB, respectively, such that $MN \parallel AQ$, $NP \parallel BQ$, and $PM \parallel CQ$. Prove that

$$[MNP] \leq \frac{1}{3}[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

2993$^\bigstar$**.** *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove or disprove:

$$\text{(a)}\ \ \frac{x}{xy+1} + \frac{y}{yz+1} + \frac{z}{zx+1} \geq \frac{9}{10};$$

$$\text{(b)}\ \ \frac{x}{y^2+1} + \frac{y}{z^2+1} + \frac{z}{x^2+1} \geq \frac{9}{10}.$$

How do the left sides of (a) and (b) compare?

2994. *Proposed by Faruk Zejnulahi and Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Let a, b, c be non-negative real numbers satisfying $a + b + c = 3$. Show that

$$\text{(a)}\ \ \frac{a^2}{b+1} + \frac{b^2}{c+1} + \frac{c^2}{a+1} \geq \frac{3}{2};$$

$$\text{(b)}\ \ \frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1} \geq \frac{3}{2};$$

$$\text{(c)}\ \ \frac{a^2}{b^2+1} + \frac{b^2}{c^2+1} + \frac{c^2}{a^2+1} \geq \frac{3}{2};$$

$$\text{(d)}\ \ \frac{a}{b^2+1} + \frac{b}{c^2+1} + \frac{c}{a^2+1} \geq \frac{3}{2}.$$

2999. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let m, n be positive integers. Prove that

$$\left(\frac{m+1}{m}\sum_{k=1}^{n}\frac{k}{n^{m+2}}(n^m-k^m)\right)^m < \frac{1}{m+1}.$$

3000. *Proposed by Paul Dayao, Ateneo de Manila University, The Philippines.*
Let f be a continuous, non-negative, and twice-differentiable function on $[0,\infty)$. Suppose that $xf''(x)+f'(x)$ is non-zero and does not change sign on $[0,\infty)$. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers and c is their geometric mean, show that

$$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf(c),$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

3001. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*
Given $a, b, c, d, e > 0$ such that $a^2+b^2+c^2+d^2+e^2 \geq 1$, prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c} \geq \frac{\sqrt{5}}{3}.$$

3002. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*
Let $r, s \in \mathbb{R}$ with $0 < r < s$, and let $a, b, c \in (r, s)$. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq \frac{3}{2}+\frac{(r-s)^2}{2r(r+s)},$$

and determine when equality occurs.

3004. *Proposed by Mihály Bencze, Brasov, Romania.*
Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Prove that

$$\frac{(\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{a})^2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2} \leq \frac{4}{9}\left(\frac{R}{r}-2\right).$$

3005. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*
Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Let h_a, h_b, h_c be the lengths of the altitudes of $\triangle ABC$ issuing from A, B, C, respectively, and let w_a, w_b, w_c be the lengths of the interior angle bisectors of A, B, C, respectively. Prove that

$$\frac{h_a}{w_a}+\frac{h_b}{w_b}+\frac{h_c}{w_c} \geq 1+\frac{4r}{R}.$$

3007. *Proposed by Mihály Bencze, Brasov, Romania.*
Let ABC be a triangle, and let $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$ such that

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k > 0.$$

1. Prove that the segments AA_1, BB_1, CC_1 are the sides of a triangle.

Let T_k denote this triangle. Let R_k and r_k be the circumradius and inradius of T_k. Prove that:

2. $\mathcal{P}(T_k) < \mathcal{P}(ABC)$, where $\mathcal{P}(T)$ denotes the perimeter of triangle T;
3. $[T_k] = \dfrac{k^2+k+1}{(k+1)^2}[ABC]$, where $[T]$ denotes the area of triangle T;
4. $R_k \geq \dfrac{k\sqrt{k}\,\mathcal{P}(ABC)}{(k+1)(k^2+k+1)}$;
5. $r_k > \dfrac{k^2+k+1}{(k+1)^2}\,r$, where r is the inradius of $\triangle ABC$.

3009. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
With I the incentre of $\triangle ABC$, let the angle bisectors BI and CI meet the opposite sides at B' and C', respectively. Prove that $AB' \cdot AC'$ is greater than, equal to, or less than AI^2 according as $\sphericalangle A$ is greater than, equal to, or less than 90°.

3010. *Proposed by Mihály Bencze and Marian Dinca, Romania.*
Let ABC be a triangle inscribed in a circle Γ. Let $A_1, B_1, C_1 \in \Gamma$ such that

$$\frac{\sphericalangle A_1AB}{\sphericalangle CAB} = \frac{\sphericalangle B_1BC}{\sphericalangle ABC} = \frac{\sphericalangle C_1CA}{\sphericalangle BCA} = \lambda,$$

where $0 < \lambda < 1$. Let the inradius and semiperimeter of $\triangle ABC$ be denoted by r and s, respectively; let the inradius and semiperimeter of $\triangle A_1B_1C_1$ be denoted by r_1 and s_1, respectively. Prove that

1. $s_1 \geq s$;
2. $r_1 \geq r$;
3. $[A_1B_1C_1] \geq [ABC]$, where $[PQR]$ denotes the area of triangle PQR.

3012. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Triangles DBC, ECA, and FAB are constructed outwardly on $\triangle ABC$ such that $\sphericalangle DBC = \sphericalangle ECA = \sphericalangle FAB$ and $\sphericalangle DCB = \sphericalangle EAC = \sphericalangle FBA$. Prove that

$$AF + FB + BD + DC + CE + EA \geq AD + BE + CF.$$

When does equality hold?

3020. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $A_1A_2\cdots A_n$ be a regular polygon inscribed in the circle Γ, and let P be an interior point of Γ. The lines $PA_1, PA_2, \ldots, PA_n$ intersect Γ for the second time at $B_1, B_2, \ldots, B_n$, respectively.

(a) Prove that $\displaystyle\sum_{k=1}^{n}(PA_k)^2 \geq \sum_{k=1}^{n}(PB_k)^2$.

(b) Prove that $\displaystyle\sum_{k=1}^{n}PA_k \geq \sum_{k=1}^{n}PB_k$.

3021. *Proposed by Pierre Bornsztein, Maisons-Laffitte, France.*
Let E be a finite set of points in the plane, no three of which are collinear and no four of which are concyclic. If A and B are two distinct points of E, we say that the pair $\{A, B\}$ is *good* if there exists a closed disc in the plane which contains both A and B and which contains no other point of E. We denote by $f(E)$ the number of good pairs formed by the points of E.
Prove that if the cardinality of E is 1003, then $2003 \leq f(E) \leq 3003$.

3026. *Proposed by Michel Bataille, Rouen, France.*
Let $a > 0$. Prove that

$$\frac{a^2+1}{e^a} + \frac{3a^2-1}{3e^{3a}} + \frac{5a^2+1}{5e^{5a}} + \frac{7a^2-1}{7e^{7a}} + \cdots < \frac{\pi}{4}.$$

3028. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let $S_k = 1 + 2 + \cdots + k$. Prove the following

$$1 + \frac{(a_1 a_2^2)^{\frac{1}{S_2}}}{a_1 + 2a_2} + \frac{(a_1 a_2^2 a_3^3)^{\frac{1}{S_3}}}{a_1 + 2a_2 + 3a_3} + \cdots + \frac{(a_1 a_2^2 \cdots a_n^n)^{\frac{1}{S_n}}}{a_1 + 2a_2 + \cdots + na_n} \leq \frac{2n}{n+1}.$$

3029. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*
Let $a_1, a_2, \ldots, a_n$ be real numbers greater than -1, and let α be any positive real number. Prove that if $a_1 + a_2 + \cdots + a_n \leq \alpha n$, then

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1} \geq \frac{n}{\alpha+1}.$$

3030. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*
Show that, if $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$\frac{1}{a_1} + \frac{2}{(a_2)^{\frac{1}{2}}} + \frac{3}{(a_3)^{\frac{1}{3}}} + \cdots + \frac{n}{(a_n)^{\frac{1}{n}}} \geq \frac{S_n}{(a_1 a_2 \cdots a_n)^{\frac{1}{S_n}}}$$

where $S_n = 1 + 2 + \cdots + n$.

3032. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c be non-negative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-ca} \leq \frac{9}{2}.$$

3033. *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*
Let I be the incentre of $\triangle ABC$, and let R and r be its circumradius and inradius, respectively. Prove that

$$6r \leq AI + BI + CI \leq \sqrt{12(R^2 - Rr + r^2)}.$$

3034. *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*
Let a, b, c, x, y, z be positive real numbers. Prove that

$$\begin{aligned}&(bc + ca + ab)(yz + zx + xy)\\ &\qquad \geq bcyz + cazx + abxy + 2\sqrt{abcxyz(a+b+c)(x+y+z)},\end{aligned}$$

and determine when equality occurs.

3038. *Proposed by Virgil Nicula, Bucharest, Romania.*
Consider a triangle ABC in which $a = \max\{a, b, c\}$. Prove that the expressions

$$(a+b+c)\sqrt{2} - (\sqrt{a+b} + \sqrt{a-b}) \cdot (\sqrt{a+c} + \sqrt{a-c}) \quad \text{and} \quad b^2 + c^2 - a^2$$

have the same sign.

3039. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*
Let a, b be fixed non-zero real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f\left(x - \frac{b}{a}\right) + 2x \le \frac{a}{b}x^2 + 2\frac{b}{a} \le f\left(x + \frac{b}{a}\right) - 2x.$$

3040. *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*
Prove that, for any three distinct natural numbers a, b, c greater than 1,

$$\left(1 + \frac{1}{a}\right)\left(2 + \frac{1}{b}\right)\left(3 + \frac{1}{c}\right) \le \frac{91}{8}.$$

3042. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1x_2\cdots x_n = 1$. For $n \ge 3$ and
$0 < \lambda \le (2n-1)/(n-1)^2$, prove that

$$\frac{1}{\sqrt{1+\lambda x_1}} + \frac{1}{\sqrt{1+\lambda x_2}} + \cdots + \frac{1}{\sqrt{1+\lambda x_n}} \le \frac{n}{\sqrt{1+\lambda}}.$$

3043. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*
For any convex quadrilateral $ABCD$, prove that

$$\begin{aligned} 1 - \cos(A+B)\cos(A+C)\cos(A+D) \\ \le 2M \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{C+A}{2}\right), \end{aligned}$$

where $M = \max\{\sin A, \sin B, \sin C, \sin D\}$.

3045. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c be positive real numbers such that $abc \ge 1$. Prove that

$$\text{(a)}\;\; a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \ge 1; \qquad \text{(b)}\;\; a^{\frac{a}{b}} b^{\frac{b}{c}} c^{c} \ge 1.$$

The following problems have all been identified by the proposers to be dedicated to the lasting memory of Murray S. Klamkin.

KLAMKIN-01. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
(a) Let x and y be positive real numbers from the interval $\left[0, \frac{1}{2}\right]$. Prove that

$$2 \le \left(\frac{1-x}{1-y}\right)^{\frac{1}{4}} + \left(\frac{1-y}{1-x}\right)^{\frac{1}{4}} \le \frac{2}{(\sqrt{x}\sqrt{y} + \sqrt{1-x}\sqrt{1-y})^{\frac{1}{2}}}.$$

(b)$^\bigstar$ Is there a generalization of the above inequality to three or more numbers?

KLAMKIN-02. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
(a) Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$xyz\left(1 + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) \geq \frac{28}{27}.$$

(b)★ Prove or disprove the following generalization involving n positive real numbers $x_1, x_2, \ldots, x_n$ which sum to 1:

$$\left(\prod_{i=1}^{n} x_i\right)\left(1 + \sum_{i=1}^{n} \frac{1}{x_i^2}\right) \geq \frac{n^3 + 1}{n^n}.$$

KLAMKIN-03. *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*
If a, b, c are positive real numbers, prove that

$$\frac{(a+b+c)^2}{a^2+b^2+c^2} + \frac{1}{2}\left(\frac{a^3+b^3+c^3}{abc} - \frac{a^2+b^2+c^2}{ab+bc+ca}\right) \geq 4.$$

KLAMKIN-05. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let k and n be positive integers with $k < n$, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$. Prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \geq n(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_n a_{n+k})$$

(where the subscipts are taken modulo n) in the following cases:

(a) $n = 2k$; (b) $n = 4k$; (c)★ $2 < \dfrac{n}{k} < 4$.

KLAMKIN-06. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*
Let Γ be the circumcircle of $\triangle ABC$.
(a) Suppose that the median and the interior angle bisector from A intersect BC at M and N, respectively. Extend AM and AN to intersect Γ at M' and N', respectively. Prove that $MM' \geq NN'$.
(b)★ Suppose that P is a point in the interior of side BC and AP intersects Γ at P'. Find the location of P where PP' is maximal. Is this maximal P constructible by straightedge and compass?

KLAMKIN-07. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let a, b, c, d be real numbers such that $a > b \geq c > d > 0$. If $ad - bc > 0$, prove that

$$\prod_{k=1}^{n}\left(\frac{a^{\binom{n}{k}} - b^{\binom{n}{k}}}{c^{\binom{n}{k}} - d^{\binom{n}{k}}}\right)^k \geq \left(\frac{a^{\frac{2^n}{n+1}} - b^{\frac{2^n}{n+1}}}{c^{\frac{2^n}{n+1}} - d^{\frac{2^n}{n+1}}}\right)^{\binom{n+1}{2}}.$$

KLAMKIN-08. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let m and n be positive integers, and let $x_1, x_2, \ldots, x_m$ be positive real numbers. If λ is a real number, $\lambda \geq 1$, prove that

$$\left(\prod_{i=1}^{m} x_i\right)^{\frac{1}{m}} \leq \left(\frac{\lambda\left(\sum_{i=1}^{m} x_i\right)^n + (1-\lambda)\sum_{i=1}^{m} x_i^n}{\lambda m^n + (1-\lambda)m}\right)^{\frac{1}{n}} \leq \frac{1}{m}\sum_{i=1}^{m} x_i.$$

KLAMKIN-09. *Proposed by Phil McCartney, Northern Kentucky University, Highland Heights, KY, USA.*
For $0 < x < \pi/2$, prove or disprove that

$$\frac{\ln(1-\sin x)}{\ln(\cos x)} < \frac{2+x}{x}.$$

KLAMKIN-11. *Proposed by Mohammed Aassila, Strasbourg, France.*
Let P be an interior point of a triangle ABC, and let r_1, r_2, and r_3 be the inradii of the triangles APB, BPC, and CPA, respectively. Prove that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \geq \frac{6+4\sqrt{3}}{R},$$

where R is the circumradius of triangle ABC. When does equality hold?

KLAMKIN-12. *Proposed by Michel Bataille, Rouen, France.*
Let a, b, c be the sides of a spherical triangle. Show that

$$3\cos a\cos b\cos c \leq \cos^2 a+\cos^2 b+\cos^2 c \leq 1+2\cos a\cos b\cos c.$$

KLAMKIN-13. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $\mathcal{C}$ be a smooth closed convex curve in the plane. Theorems in analysis assure us that there is at least one circumscribing triangle $A_0B_0C_0$ to $\mathcal{C}$ having minimum perimeter. Prove that the excircles of $A_0B_0C_0$ are tangent to $\mathcal{C}$.

KLAMKIN-15. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*
A square $ABCD$ sits in the plane with corners A, B, C, D initially located at positions $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, respectively. The square is rotated counterclockwise through an angle θ $(0^\circ \leq \theta < 360^\circ)$ four times, with the centre of rotation at the points A, B, C, D in successive rotations. Suppose point A ends up on the x-axis or y-axis. Find all possible values of θ.

3051. *Proposed by Vedula N. Murty, Dover, PA, USA.*
Let $x, y, z \in [0,1)$ such that $x+y+z=1$. Prove that

(a) $$\sqrt{\frac{x}{x+yz}}+\sqrt{\frac{y}{y+zx}}+\sqrt{\frac{z}{z+xy}} \leq 3\sqrt{\frac{3}{2}};$$

(b) $$\frac{\sqrt{xyz}}{(1-x)(1-y)(1-z)} \leq \frac{3\sqrt{3}}{8}.$$

3052. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*
Let G be the centroid of $\triangle ABC$, and let A_1, B_1, C_1 be the mid-points of BC, CA, AB, respectively. If P is an arbitrary point in the plane of $\triangle ABC$, show that

$$PA+PB+PC+3PG \geq 2\,(PA_1+PB_1+PC_1).$$

3053. *Proposed by Avet A. Grigoryan and Hayk N. Sedrakyan, students, A. Shahinyan Physics and Mathematics School, Yerevan, Armenia.*
Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers whose sum is 1. Prove that

$$n-1 \leq \sqrt{\frac{1-a_1}{1+a_1}}+\sqrt{\frac{1-a_2}{1+a_2}}+\cdots+\sqrt{\frac{1-a_n}{1+a_n}} \leq n-2+\frac{2}{\sqrt{3}}.$$

3055. *Proposed by Michel Bataille, Rouen, France.*
Let the incircle of an acute-angled triangle ABC be tangent to BC, CA, AB at D, E, F, respectively. Let D_0 be the reflection of D through the incentre of $\triangle ABC$, and let D_1 and D_2 be the reflections of D across the diameters of the incircle through E and F. Define E_0, E_1, E_2 and F_0, F_1, F_2 analogously. Show that

$$\begin{aligned}&[D_0D_1D_2] + [E_0E_1E_2] + [F_0F_1F_2] \\ &\qquad = [DD_1D_2] = [EE_1E_2] = [FF_1F_2] \le \frac{1}{4}[ABC],\end{aligned}$$

where $[XYZ]$ denotes the area of $\triangle XYZ$.

3056. *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*
If $f(x)$ is a non-negative, continuous, concave function on the closed interval $[0,1]$ such that $f(0) = 1$, show that

$$2\int_0^1 x^2 f(x)\,\mathrm{d}x + \frac{1}{12} \le \left[\int_0^1 f(x)\,\mathrm{d}x\right]^2.$$

3057. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c be non-negative real numbers, and let $p \ge \frac{\ln 3}{\ln 2} - 1$. Prove that

$$\left(\frac{2a}{b+c}\right)^p + \left(\frac{2b}{c+a}\right)^p + \left(\frac{2c}{a+b}\right)^p \ge 3.$$

3058. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let A, B, C be the angles of a triangle. Prove that

(a) $\displaystyle \frac{1}{2-\cos A} + \frac{1}{2-\cos B} + \frac{1}{2-\cos C} \ge 2;$

(b) $\displaystyle \frac{1}{5-\cos A} + \frac{1}{5-\cos B} + \frac{1}{5-\cos C} \le \frac{2}{3}.$

3059. *Proposed by Gabriel Dospinescu, Onesti, Romania.*
Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 \le 1$. Prove that

$$ab + bc + cd + da + ac + bd \le 4\,abcd + \frac{5}{4}.$$

3061. *Proposed by Gabriel Dospinescu, Onesti, Romania.*
Find the smallest non-negative integer n for which there exists a non-constant function $f : \mathbb{Z} \to [0,\infty)$ such that for all integers x and y,

(a) $f(xy) = f(x)f(y)$, and

(b) $2\,f(x^2+y^2) - f(x) - f(y) \in \{0, 1, \ldots, n\}$.

For this value of n, find all the functions f which satisfy (a) and (b).

3062. *Proposed by Gabriel Dospinescu, Onesti, Romania.*
Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \ge \frac{3}{4}.$$

3065. *Proposed by Gabriel Dospinescu, Onesti, Romania.*
Let ABC be an acute-angled triangle, and let M be an interior point of the triangle. Prove that

$$\frac{1}{MA}+\frac{1}{MB}+\frac{1}{MC}\geq 2\left(\frac{\sin\sphericalangle AMB}{AB}+\frac{\sin\sphericalangle BMC}{BC}+\frac{\sin\sphericalangle CMA}{CA}\right).$$

3068. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{c+a}}+\sqrt{1+\frac{48c}{a+b}}\geq 15,$$

and determine when there is equality.

3070. *Proposed by Zhang Yun, High School attached to Xi An Jiao Tong University, Xi An City, Shan Xi, China.*
Let $x_1, x_2, \dots, x_n$ be positive real numbers such that

$$x_1+x_2+\cdots+x_n\geq x_1x_2\cdots x_n.$$

Prove that

$$(x_1x_2\cdots x_n)^{-1}\left(x_1^{n-1}+x_2^{n-1}+\cdots+x_n^{n-1}\right)\geq \sqrt[n-1]{n^{n-2}},$$

and determine when there is equality.

3071. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let $k>-1$ be a fixed real number. Let a, b, and c be non-negative real numbers such that $a+b+c=1$ and $ab+bc+ca>0$. Find

$$\min\left\{\frac{(1+ka)(1+kb)(1+kc)}{(1-a)(1-b)(1-c)}\right\}.$$

3072. *Proposed by Mohammed Aassila, Strasbourg, France.*
Find the smallest constant k such that, for any positive real numbers a, b, c, we have

$$abc\,(a^{125}+b^{125}+c^{125})^{16}\leq k\,(a^{2003}+b^{2003}+c^{2003}).$$

3073. *Proposed by Zhang Yun, High School attached to Xi An Jiao Tong University, Xi An City, Shan Xi, China.*
Let x, y, z be positive real numbers. Prove that

$$\frac{1}{x+y+z+1}-\frac{1}{(x+1)(y+1)(z+1)}\leq\frac{1}{8},$$

and determine when there is equality.

3074. *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.*
Let $f:\left[0,\frac{1}{2005}\right]\to\mathbb{R}$ be a function such that

$$f(x+y^2)\geq y+f(x),$$

for all real x and y with $x\in\left[0,\frac{1}{2005}\right]$ and $x+y^2\in\left[0,\frac{1}{2005}\right]$. Give an example of such a function, or show that no such function exists.

3076. *Proposed by Vedula N. Murty, Dover, PA, USA.*
If x, y, z are non-negative real numbers and a, b, c are arbitrary real numbers, prove that

$$(a(y+z)+b(z+x)+c(x+y))^2 \geq 4(xy+yz+zx)(ab+bc+ca).$$

(Note: If we impose the conditions that $x+y+z=1$ and that a, b, c are positive, then the above is equivalent to

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)} \leq a+b+c,$$

which is problem #8 of the 2001 Ukrainian Mathematical Olympiad, given in the December 2003 issue of *Crux with MAYHEM* [2003:498]. The solution of the Ukrainian problem appears on page 443.)

3077. *Proposed by Arkady Alt, San Jose, CA, USA.*
In $\triangle ABC$, we denote the sides BC, CA, AB as usual by a, b, c, respectively. Let h_a, h_b, h_c be the lengths of the altitudes to the sides a, b, c, respectively. Let d_a, d_b, d_c be the signed distances from the circumcentre of $\triangle ABC$ to the sides a, b, c, respectively. (The distance d_a, for example, is positive if and only if the circumcentre and vertex A lie on the same side of the line BC.) Prove that

$$\frac{h_a+h_b+h_c}{3} \leq d_a+d_b+d_c.$$

3078. *Proposed by D. J. Smeenk, Zaltbommel, The Netherlands.*
Let ABC be a triangle with $a > b$. Let D be the foot of the altitude from A to the line BC, let E be the mid-point of AC, and let CF be an external bisector of $\sphericalangle BCA$ with F on the line AB. Suppose that D, E, F are collinear.
(a) Determine the range of $\sphericalangle BCA$.
(b) Show that $c > b$.
(c) If $c^2 = ab$, determine the measures of the angles of $\triangle ABC$, and show that $\sin B = \cos^2 B$.

3079. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $x_1, x_2, \ldots, x_n$ be real numbers such that $x_1 \leq x_2 \leq \cdots \leq x_n$. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_y|\right)^4 \leq \frac{8(n-1)^2(n+1)(2n^2-3)}{15} \sum_{i,j=1}^{n} (x_i - x_j)^4.$$

3082. *Proposed by J. Walter Lynch, Athens, GA, USA.*
Suppose that four consecutive terms of a geometric sequence with common ratio r are the sides of a quadrilateral. What is the range of all possible values for r?

3084. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$\sum_{k=1}^{n} x_k = 0 \qquad \text{and} \qquad \sum_{k=1}^{n} x_k^4 = 1.$$

Prove that

$$\left(\sum_{k=1}^{n} k x_k\right)^4 \leq \frac{n^3(n^2-1)(3n^2-7)}{240}.$$

3086. *Proposed by Mihály Bencze, Brasov, Romania.*
If $a_k > 0$ for $k = 1, 2, \ldots, n$, prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq \frac{1}{n}\left(\sqrt[3]{\frac{a_1}{a_2}} + \sqrt[3]{\frac{a_2}{a_3}} + \cdots + \sqrt[3]{\frac{a_n}{a_1}}\right)^3 \geq n^2.$$

3087. *Proposed by Mihály Bencze, Brasov, Romania.*
Let ABC be a triangle with sides a, b, c opposite the angles A, B, C, respectively. If R is the circumradius and r the inradius of $\triangle ABC$, prove that:

(a) $\dfrac{3R}{r} \geq \dfrac{a+c}{b} + \dfrac{b+a}{c} + \dfrac{c+b}{a} \geq 6$;

(b) $\left(\dfrac{R}{r}\right)^3 \geq \left(\dfrac{a}{b} + \dfrac{b}{a}\right)\left(\dfrac{b}{c} + \dfrac{c}{b}\right)\left(\dfrac{a}{c} + \dfrac{c}{a}\right) \geq 8.$

(Both (a) and (b) are refinements of Euler's Inequality, $R \geq 2r$.)

3090. *Proposed by Arkady Alt, San Jose, CA, USA.*
Find all non-negative real solutions (x, y, z) to the following system of inequalities:

$$2x(3 - 4y) \geq z^2 + 1,$$
$$2y(3 - 4z) \geq x^2 + 1,$$
$$2z(3 - 4x) \geq y^2 + 1.$$

3091. *Proposed by Mihály Bencze and Marian Dinca, Romania.*
Let $A_1A_2 \ldots A_n$ be a convex polygon which has both an inscribed circle and a circumscribed circle. Let $B_1, B_2, \ldots, B_n$ denote the points of tangency of the incircle with sides A_1A_2, $A_2A_3, \ldots, A_nA_1$, respectively. Prove that

$$\frac{2sr}{R} \leq \sum_{k=1}^{n} B_kB_{k+1} \leq 2s\cos\left(\frac{\pi}{n}\right),$$

where R is the radius of the circumscribed circle, r is the radius of the inscribed circle, s is the semiperimeter of the polygon $A_1A_2 \ldots A_n$, and $B_{n+1} = B_1$.

3092. *Proposed by Vedula N. Murty, Dover, PA, USA.*
(a) Let a, b, and c be positive real numbers such that $a+b+c = abc$. Find the minimum value of $\sqrt{1+a^2} + \sqrt{1+b^2} + \sqrt{1+c^2}$.

[Compare with *Crux with MAYHEM* problem 2814 [2003 : 110; 2004 : 112].]

(b) Let a, b, and c be positive real numbers such that $a+b+c = 1$. Find the minimum value of

$$\frac{1}{\sqrt{abc}} + \sum_{\text{cyclic}} \sqrt{\frac{bc}{a}}.$$

3094. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers, where $n \geq 3$. Let $S = \sum_{k=1}^{n} x_k$ and $P = \prod_{k=1}^{n} (1 + x_k^2)$. Prove that

(a) $P \leq \max_{1 \leq k \leq n} \left\{ \left(1 + \frac{S^2}{k^2}\right)^k \right\}$;

(b) $P \leq \left(1 + \frac{S^2}{n^2}\right)^n \quad$ if $S > 2\sqrt{2}\,(n-1)$;

(c) $P \leq 1 + S^2 \quad$ if $S \leq 2\sqrt{2}$.

3095. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let a, b, c, p, and q be natural numbers. Using $\lfloor x \rfloor$ to denote the integer part of x, prove that

$$\min\left\{a, \left\lfloor \frac{c + pb}{q} \right\rfloor \right\} \leq \left\lfloor \frac{c + p(a+b)}{p+q} \right\rfloor.$$

3096. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let ABC be a triangle with sides a, b, c opposite the angles A, B, C, respectively. Prove that

$$\sum_{\text{cyclic}} \frac{bc}{b+c} \sin^2\left(\frac{A}{2}\right) \leq \frac{a+b+c}{8}.$$

3097. *Proposed by Mihály Bencze, Brasov, Romania.*
Let a and b be two positive real numbers such that $a < b$. Define $A(a,b) = \frac{a+b}{2}$ and $L(a,b) = \frac{b-a}{\ln b - \ln a}$. Prove that

$$L(a,b) < L\left(\frac{a+b}{2}, \sqrt{ab}\right) < \left(A(\sqrt{a}, \sqrt{b})\right)^2 < A(a,b).$$

3099. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n} \ln(1 + a_k) \leq \left(\ln\left(1 + \sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right)^n.$$

3105. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c, d be positive real numbers.
(a) Prove that the following inequality holds for $0 \leq x \leq (5 - \sqrt{17})/2$ and also for $x = 1$:

$$\sum_{\text{cyclic}} \frac{a}{a + (3-x)b + xc} \geq 1.$$

(b)$\star$ Prove the above inequality for $0 \leq x \leq 1$.

3109. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let ABC be a triangle in which angles B and C are both acute, and let a, b, c be the lengths of the sides opposite the vertices A, B, C, respectively. If h_a is the altitude from A to BC, prove that $\frac{1}{h_a^2} - \left(\frac{1}{b^2} + \frac{1}{c^2}\right)$ is positive, negative, or zero according as $\sphericalangle A$ is obtuse, acute, or right-angled.

3110. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let m_b be the length of the median to side b in $\triangle ABC$, and define m_c similarly. Prove that $4a^4 + 9b^2c^2 - 16m_b^2m_c^2$ is positive, negative, or zero according as angle A is acute, obtuse, or right-angled.

3111. *Proposed by Mihály Bencze, Brasov, Romania.*
Let a_k, b_k, and c_k be the length of the sides opposite the vertices A_k, B_k, and C_k, respectively, in triangle $A_kB_kC_k$, for $k = 1, 2, \ldots, n$. If r_k is the inradius of triangle $A_kB_kC_k$ and if R_k is its circumradius, prove that

$$\begin{aligned} 6\sqrt{3}\left(\prod_{k=1}^{n} r_k\right)^{\frac{1}{n}} &\leq \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} c_k\right)^{\frac{1}{n}} \\ &\leq 3\sqrt{3}\left(\prod_{k=1}^{n} R_k\right)^{\frac{1}{n}}. \end{aligned}$$

3113. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let ABC be a triangle and let a be the length of the side opposite the vertex A. If m_a is the length of the median from A to BC, and if R is the circumradius of $\triangle ABC$, prove that $m_a - R$ is positive, negative, or zero, according as $\sphericalangle A$ is acute, obtuse, or right-angled.

3114. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*
Let a, b, c be positive real numbers such that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2.$$

Prove that

$$\frac{1}{4a+1} + \frac{1}{4b+1} + \frac{1}{4c+1} \geq 1.$$

3115. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let a, b, c be the lengths of the sides opposite the vertices A, B, C, respectively, in triangle ABC. Prove that

$$\frac{\cos^8 A}{a} + \frac{\cos^8 B}{b} + \frac{\cos^8 C}{c} < \frac{a^2 + b^2 + c^2}{2abc}.$$

3116. *Proposed by Arkady Alt, San Jose, CA, USA.*
For arbitrary numbers a, b, c, prove that

$$\sum_{\text{cyclic}} a(b + c - a)^3 \leq 4abc(a + b + c).$$

3117. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*
Let a, b, c be the lengths of the sides and s the semi-perimeter of $\triangle ABC$. Prove that

$$\sum_{\text{cyclic}} (a + b)\sqrt{ab(s - a)(s - b)} \leq 3abc.$$

3119. *Proposed by Michel Bataille, Rouen, France.*
Let r and s denote the inradius and semi-perimeter, respectively, of triangle ABC. Show that

$$3\sqrt{3}\sqrt{\frac{r}{s}} \leq \sqrt{\tan\frac{A}{2}} + \sqrt{\tan\frac{B}{2}} + \sqrt{\tan\frac{C}{2}} \leq \sqrt{\frac{s}{r}}.$$

3121. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let n and r be positive integers. Show that

$$\left(\frac{1}{2^n}\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\left[1-\frac{1}{2^{nr}}\binom{n}{k}^r\right]\right)^r \leq \frac{r^r}{(r+1)^{r+1}}.$$

3122. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let $\triangle ABC$ and $\triangle A'B'C'$ have right angles at A and A', respectively, and let h_a and $h_{a'}$ denote the altitudes to the sides a and a', respectively. If $b \geq c$ and $b' \geq c'$, prove that

$$\sqrt{aa'} + 2\sqrt{h_a h_{a'}} \leq \sqrt{2}\left(\sqrt{bb'} + \sqrt{cc'}\right).$$

3123. *Proposed by Joe Howard, Portales, NM, USA.*
Let a, b, c be the sides of a triangle. Show that

$$\frac{abc(a+b+c)^2}{a^2+b^2+c^2} \geq 2abc + \prod_{\text{cyclic}}(b+c-a).$$

3124. *Proposed by Joe Howard, Portales, NM, USA.*
Let a, b, c be the sides of $\triangle ABC$ in which at most one angle exceeds $\pi/3$, and let r be its inradius. Show that

$$\frac{\sqrt{3}\,(abc)}{a^2+b^2+c^2} \geq 2r.$$

3125. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Let m_a, h_a, and w_a denote the lengths of the median, the altitude, and the internal angle bisector, respectively, to side a in $\triangle ABC$. Define m_b, m_c, h_b, h_c, w_b, and w_c similarly. Let R be the circumradius of $\triangle ABC$.

(a) Show that

$$\sum_{\text{cyclic}}\frac{b^2+c^2}{m_a} \leq 12R.$$

(b) Show that

$$\sum_{\text{cyclic}}\frac{b^2+c^2}{h_a} \geq 12R.$$

(c)★ Determine the range of

$$\frac{1}{R}\sum_{\text{cyclic}}\frac{b^2+c^2}{w_a}.$$

3127. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let H be the foot of the altitude from A to BC, where BC is the longest side of $\triangle ABC$. Let R, R_1, and R_2 be the circumradii of $\triangle ABC$, $\triangle ABH$, and $\triangle ACH$, respectively. Similarly, let r, r_1, r_2 be the inradii of these triangles. Prove that

(a) $R_1^2 + R_2^2 - R^2$ is positive, negative, or zero according as angle A is acute, obtuse, or right-angled.

(b) $r_1^2 + r_2^2 - r^2$ is positive, negative, or zero according as angle A is obtuse, acute, or right-angled.

3130. *Proposed by Michel Bataille, Rouen, France.*
Let A, B, C be the angles of a triangle. Show that

$$\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right)\left(\csc\frac{A}{2} + \csc\frac{B}{2} + \csc\frac{C}{2}\right) - \left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) \geq 6\sqrt{3}.$$

3132. *Proposed by Mihály Bencze, Brasov, Romania.*
Let $F(n)$ be the number of ones in the binary expression of the positive integer n. For example,

$$F(5) = F(101_{(2)}) = 2,$$
$$F(15) = F(1111_{(2)}) = 4.$$

Let $S_k = \sum\limits_{n=1}^{\infty} \frac{F^k(n)}{n(n+1)}$, where $F^k(n)$ is defined recursively by $F^1 = F$ and $F^k = F \circ F^{k-1}$ for $k \geq 2$.

(a) Prove that $S_1 = 2\ln 2$.

(b) Prove that $\frac{18}{5}\ln 2 - \frac{1}{15} \leq S_2 \leq 4\ln 2$.

(c) Prove that $\frac{218}{25}\ln 2 - \frac{7}{25} \leq S_3 \leq 11\ln 2$.

(d)★ Compute S_k.

3133. *Proposed by Mihály Bencze, Brasov, Romania.*
Let ABC be any triangle. Show that

$$\sum_{\text{cyclic}} \frac{1 + 2\sin A - \cos 2A}{8 + 3\cos\frac{A}{2}\cos\frac{B-C}{2} + \cos\frac{3A}{2}\cos\frac{3(B-C)}{2}} \leq 1.$$

3134. *Proposed by Mihály Bencze, Brasov, Romania.*
Let O be the circumcentre of $\triangle ABC$. Let D, E, and F be the mid-points of BC, CA, and AB, respectively; let K, M, and N be the mid-points of OA, OB, and OC, respectively. Denote the circumradius, inradius, and semiperimeter of $\triangle ABC$ by R, r, and s, respectively. Prove that

$$2(KD + ME + NF) \geq R + 3r + \frac{s^2 + r^2}{2R}.$$

3135. *Proposed by Marian Marinescu, Monbonnot, France.*
Let $\mathbb{R}^+$ be the set of non-negative real numbers. For all $a, b, c \in \mathbb{R}^+$, let $H(a,b,c)$ be the set of all functions $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$h(x) \geq h(h(ax)) + h(bx) + cx$$

for all $x \in \mathbb{R}^+$. Prove that $H(a,b,c)$ is non-empty if and only if $b \leq 1$ and $4ac \leq (1-b)^2$.

3140. *Proposed by Michel Bataille, Rouen, France.*
Let $a_1, a_2, \ldots, a_n$ be n distinct positive real numbers, where $n \geq 2$. For $i = 1, 2, \ldots, n$, let $p_i = \prod_{j \neq i} (a_j - a_i)$. Show that $\prod_{i=1}^{n} a_i^{\frac{1}{p_i}} < 1$.

3141. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let a, b, and c be the sides of a scalene triangle ABC. Prove that

$$\sum_{\text{cyclic}} \frac{(a+1)bc}{(\sqrt{a}-\sqrt{b})(\sqrt{a}-\sqrt{c})} < \frac{a^4+b^4+c^4}{abc}.$$

3142. *Proposed by Mihály Bencze, Brasov, Romania.*
If $x_k > 0$ for $k = 1, 2, \ldots, n$, prove that

$$\text{(a)} \quad \cos\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right) - \sin\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right) \geq \frac{1}{n}\sum_{k=1}^{n}\left(\cos\frac{1}{x_k} - \sin\frac{1}{x_k}\right);$$

$$\text{(b)} \quad \frac{\sum\limits_{k=1}^{n} \sin\frac{1}{x_k}}{\sum\limits_{k=1}^{n} \cos\frac{1}{x_k}} \geq \tan\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right).$$

3143. *Proposed by Mihály Bencze, Brasov, Romania.*
For $n \geq 1$ let $a_n = 1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}$. Prove that

$$\sum_{k=1}^{n} \frac{\sqrt[k]{k}}{a_k^2} < \frac{2n+1+(\ln n)^2}{n+1+\frac{1}{2}(\ln n)^2}.$$

3145★. *Proposed by Yuming Chen, Wilfrid Laurier University, Waterloo, ON.*
Let $f(x) = x - c^2 \tanh x$, where $c > 1$ is an arbitrary constant. It is not hard to show that $f(x)$ is decreasing on the interval $[-x_0, x_0]$, where $x_0 = \ln(c + \sqrt{c^2-1})$ is the positive root of the equation $\cosh x = c$. For each $x \in (-x_0, x_0)$, the horizontal line passing through $(x, f(x))$ intersects the graph of f at two other points with abscissas $x_1(x)$ and $x_2(x)$. Define a function $g : (-x_0, x_0) \to \mathbb{R}$ as follows:

$$g(x) = x + c^2 \tanh(x_1(x)) + c^2 \tanh(x_2(x)).$$

Prove or disprove that $g(x) > 0$ for all $x \in (0, x_0)$.

3146. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $p > 1$, and let $a, b, c, d \in [1/\sqrt{p}, \sqrt{p}]$. Prove that

$$\text{(a)} \quad \frac{p}{1+p} + \frac{2}{1+\sqrt{p}} \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}};$$

$$\text{(b)} \quad \frac{p}{1+p} + \frac{3}{1+\sqrt[3]{p}} \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \leq \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}.$$

3147. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania; and Gabriel Dospinescu, Paris, France.*
Let $n \geq 3$, and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n = 1$. For $n = 3$ and $n = 4$, prove that

$$\frac{1}{x_1^2 + x_1x_2} + \frac{1}{x_2^2 + x_2x_3} + \cdots + \frac{1}{x_n^2 + x_nx_1} \geq \frac{n}{2}.$$

3148. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $0 < m < 1$, and let $a, b, c \in [\sqrt{m}, 1/\sqrt{m}]$. Prove that

$$\frac{a^3 + b^3 + c^3 + 3(1+m)abc}{ab(a+b) + bc(b+c) + ca(c+a)} \geq 1 + \frac{m}{2}.$$

3149. *Proposed by David Martinez Ramirez, student, Universidad Nacional Autonoma de Mexico, Mexico.*
Let $P(z)$ be any non-constant complex monic polynomial. Show that there is a complex number w such that $|w| \leq 1$ and $|P(w)| \geq 1$.

3150. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*
Let a, b, c be the three sides of a triangle, and let h_a, h_b, h_c be the altitudes to the sides a, b, c, respectively. Prove that

$$\frac{h_a^2}{b^2 + c^2} \cdot \frac{h_b^2}{c^2 + a^2} \cdot \frac{h_c^2}{a^2 + b^2} \leq \left(\frac{3}{8}\right)^3.$$

3152. *Proposed by Michel Bataille, Rouen, France.*
Let $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ be real numbers such that $\sum\limits_{i=1}^{n} x_i = 0$ and $\sum\limits_{i=1}^{n} x_i^2 = 1$. Find the minimum and maximum of $\sum\limits_{i=1}^{n} |x_i|$.

3154. *Proposed by Challa K. S. N. M. Sankar, Andhrapradesh, India.*
(a) If $\beta > 1$ is a real constant, determine the number of possible real solutions of the equation

$$x - \beta \log_2 x = \beta - \beta \ln \beta.$$

(b) If $\alpha_1 < \alpha_2$ are two positive real solutions of the equation in (a), and if x_1 and x_2 are any two real numbers satisfying $\alpha_1 \leq x_1 < x_2 \leq \alpha_2$, prove that, for all λ such that $0 < \lambda < 1$,

$$\lambda \log_2 x_1 + (1-\lambda)\log_2 x_2 \geq \ln(\lambda x_1 + (1-\lambda)x_2).$$

Determine when equality occurs.

3159. *Proposed by Mihály Bencze, Brasov, Romania.*
Let n be a positive integer, and let γ be Euler's constant. Prove that

$$\gamma - \frac{1}{48n^3} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) < \gamma - \frac{1}{48(n+1)^3}.$$

3164. *Proposed by Mihály Bencze, Brasov, Romania.*
Let P be any point in the plane of $\triangle ABC$. Let D, E, and F denote the mid-points of BC, CA, and AB, respectively. If G is the centroid of $\triangle ABC$, prove that

$$0 \leq 3PG + PA + PB + PC - 2(PD + PE + PF) \leq \frac{1}{2}(AB + BC + CA).$$

3165. *Proposed by Mihály Bencze, Brasov, Romania.*
For any positive integer n, prove that there exists a polynomial $P(x)$, of degree at least $8n$, such that

$$\sum_{k=1}^{(2n+1)^2} |P(k)| < |P(0)|.$$

3166. *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*
Let P be an interior point of the triangle ABC. Denote by d_a, d_b, d_c the distances from P to the sides BC, CA, AB, respectively, and denote by D_A, D_B, D_C the distances from P to the vertices A, B, C, respectively. Further let P_A, P_B, and P_C denote the measures of $\sphericalangle BPC$, $\sphericalangle CPA$, and $\sphericalangle APB$, respectively. Prove that

$$d_a d_b \sin\left(\frac{P_A+P_B}{2}\right) + d_b d_c \sin\left(\frac{P_B+P_C}{2}\right) + d_c d_a \sin\left(\frac{P_C+P_A}{2}\right)$$
$$\leq \frac{1}{4}\left(D_B D_C \sin P_A + D_C D_A \sin P_B + D_A D_B \sin P_C\right).$$

3167. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let ABC be a non-obtuse triangle with circumradius R. If a, b, c are the lengths of the sides opposite angles A, B, C, respectively, prove that

$$a\cos^3 A + b\cos^3 B + c\cos^3 C \leq \frac{abc}{4\,R^2}.$$

3168. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let $x_1, x_2, \ldots, x_n$ be positive real numbers satisfying $\prod\limits_{i=1}^{n} x_i = 1$. Prove that

$$\sum_{i=1}^{n} x_i^n(1+x_i) \geq \frac{n}{2^{n-1}}\prod_{i=1}^{n}(1+x_i).$$

3170. *Proposed by Mihály Bencze, Brasov, Romania.*
Let a and b be real numbers satisfying $0 \leq a \leq \frac{1}{2} \leq b \leq 1$. Prove that

(a) $2\,(b-a) \leq \cos\pi a - \cos\pi b$;

(b) $(1-2a)\cos\pi b \leq (1-2b)\cos\pi a$.

3171. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*
Given a point P in the first quadrant, it is known that the line segment in the first quadrant joining the coordinate axes, passing through P, and having minimum length (Philo's line) is not constructible using straightedge and compass. However, the line which (together with the two axes) defines a triangle in the first quadrant with minimum perimeter is constructible. Give such a construction.

3179. *Proposed by Michel Bataille, Rouen, France.*
A transversal of $\triangle ABC$ makes angles α, β, and γ with the lines BC, CA, and AB, respectively. Express the minimum and maximum values of

$$(\cos\alpha\cos\beta\cos\gamma)^2 + (\sin\alpha\sin\beta\sin\gamma)^2$$

as functions of $p = \cos A\cos B\cos C$.

3182. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let a, b, and c be any positive real numbers, and let p be a real number such that $0 < p < 1$.

(a) Prove that

$$\frac{a}{(b+c)^p}+\frac{b}{(c+a)^p}+\frac{c}{(a+b)^p} \geq \frac{1}{2^p}\left(a^{1-p}+b^{1-p}+c^{1-p}\right).$$

(b) Prove that, if $p=1/3$, then

$$\frac{a}{(a+b)^p}+\frac{b}{(b+c)^p}+\frac{c}{(c+a)^p} \geq \frac{1}{2^p}\left(a^{1-p}+b^{1-p}+c^{1-p}\right).$$

(c)★ Prove or disprove

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \frac{1}{\sqrt{2}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

3183. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let ABC be a triangle with inradius r and circumradius R. If s is the semiperimeter of the triangle, prove that

$$\sqrt{3}\,s \leq r+4R.$$

3185. *Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.*
Let a_n denote the units digit of $(4n)^{(3n)^{(2n)^n}}$. Find all positive integers n such that $\sum\limits_{i=1}^{n} a_i \geq 4n$.

3186. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $f(x)$ be a function on an interval I which is convex for $x \geq a$ for some $a \in I$. Suppose that for all $x_1, x_2, \ldots, x_n \in I$ which satisfy $x_1+x_2+\cdots+x_n = na$, the following inequality holds:

$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \geq f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right).$$

Prove that this same inequality holds for all $x_1, x_2, \ldots, x_n \in I$ such that $x_1+x_2+\cdots+x_n \geq na$.

3188. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let x, y, z be positive real numbers. Prove that

$$\left(\frac{x}{y}+\frac{z}{\sqrt[3]{xyz}}\right)^2+\left(\frac{y}{z}+\frac{x}{\sqrt[3]{xyz}}\right)^2+\left(\frac{z}{x}+\frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12.$$

3194. *Proposed by Mihály Bencze, Brasov, Romania.*
Let n be any positive integer, and let $x_k, y_k \in \mathbb{R}$ for $k=1,2,\ldots,n$. Prove that

$$\min\left\{\sum_{k=1}^{n} x_k^2, \sum_{k=1}^{n} y_k^2\right\} \cdot \sum_{k=1}^{n}(x_k-y_k)^2 \geq \sum_{1\leq i<j\leq n}(x_iy_j-x_jy_i)^2.$$

3195. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
(a) Let n be a natural number, $n \geq 3$. Prove that there is a real number $q_n > 1$ such that for any real numbers $a_1, a_2, \ldots, a_n \in [1/q_n, q_n]$,

$$\frac{a_1}{a_2 + a_3} + \frac{a_2}{a_3 + a_4} + \cdot + \frac{a_n}{a_n + a_1} \geq \frac{n}{2}.$$

(b)$^\bigstar$ Does there exist a real number $q > 1$ such that the inequality in (a) holds for any natural number $n > 3$ and for any real numbers $a_1, a_2, \ldots, a_n \in [1/q, q]$?

3196. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\begin{aligned} &x_1^n + x_2^n + \cdots + x_n^n + n(n-1)x_1x_2\cdots x_n \\ &\qquad \geq x_1x_2\cdots x_n(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right). \end{aligned}$$

3197. *Proposed by Paul Deiermann, Southeast Missouri State University, Cape Girardeau, MO, USA.*
If AB is a fixed line segment, find the triangle ABC which has maximum area among those which satisfy $\sphericalangle AIO = \pi/2$, where I is the incentre of $\triangle ABC$ and O is its circumcentre.
What is this maximum area?

3208. *Proposed by Shaun White, student, Vincent Massey Secondary School, Windsor, ON.*
Find the largest integer k such that for all positive real numbers a, b, c, we have

$$(a^3 + 3)(b^3 + 6)(c^3 + 12) > k(a + b + c)^3.$$

3209. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let f be a convex function on an interval I. For $i = 1, 2, \ldots, n$, let $a_i \in I$. Define $a = \frac{1}{n}\sum\limits_{i=1}^{n} a_i$.
Prove that

$$\frac{n(n-2)}{2}f(a) + \sum_{i=1}^{n} f(a_i) \geq \frac{n}{2(n-1)}\sum_{i\neq j} f\left(a + \frac{a_i - a_j}{n}\right).$$

3210. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Prove that, for all real numbers $a_1, a_2, \ldots, a_n \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, we have

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \geq \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}},$$

where the subscripts are taken modulo n.

3212. *Proposed by José Luis Díaz-Barrero and Francisco Palacios Quiñonero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_k \geq 1$ for $1 \leq k \leq n$. Prove that

$$\prod_{k=1}^{n} a_k^{\left(\frac{2k}{n(n+1)}\right)^{1/2}} \leq \exp\left(\sqrt{\sum_{k=1}^{n} \ln^2 a_k}\right).$$

3214. *Proposed by Mihály Bencze, Brasov, Romania.*
Let ABC be an acute-angled triangle.

(a) Prove that $\dfrac{\tan A}{A} + \dfrac{\tan B}{B} + \dfrac{\tan C}{C} > \left(\dfrac{6}{\pi}\right)^2$.

(a) Prove that $A\cot A + B\cot B + C\cot C < \left(\dfrac{\pi}{2}\right)^2$.

(c)$^\bigstar$ Determine best constants $c_1 \geq \left(\dfrac{6}{\pi}\right)^2$ and $0 < c_2 < c_3 \leq \left(\dfrac{\pi}{2}\right)^2$ such that

$$\sum_{\text{cyclic}} \frac{\tan A}{A} \geq c_1 \qquad \text{and} \qquad c_2 \leq \sum_{\text{cyclic}} A\cot A \leq c_3.$$

3216. *Proposed by Mihály Bencze, Brasov, Romania.*
If a, b, c, and d are positive integers, prove that

$$45\left(\frac{1}{a+b+c+d+1} - \frac{1}{(a+1)(b+1)(c+1)(d+1)}\right)$$
$$\leq 4 + \sum_{\text{cyclic}}\left[\frac{1}{a+1} + \frac{1}{(a+1)(b+1)}\right].$$

3219. *Proposed by Dan Vetter, Regina, SK.*
A vulture with a university education, when approached by a car while dining on the road, will always fly off in a direction chosen to maximize the distance of closest approach of the car. Show that the ratio of the speed of the car to the speed of the bird is $\sec\theta$, where θ is the angle that the vulture's flight path makes with the road.

3221. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Let ABC be a triangle with sides $a \geq b \geq c$ opposite the angles A, B, C, respectively. Let AH be perpendicular to the side BC with H on BC. Set $m = BH$ and $n = CH$.
Prove that $a(bm + cn) - bc(b + c)$ is positive, negative, or zero according as $\sphericalangle A$ is obtuse, acute, or right-angled.

3222. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*
Given positive real numbers a, b, c such that $a + b + c = 1$, prove that

$$\frac{(1-a)(1-b)(1-c)}{(1-a^2)^2 + (1-b^2)^2 + (1-c^2)^2} \leq \frac{1}{8}.$$

3223. *Proposed by Achilleas Pavlos Porfyriadis, Student, American College of Thessaloniki „Anatolia“, Thessaloniki, Greece.*
Let a, b, c be positive real numbers which satisfy

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{abc}.$$

Prove that

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{3\sqrt{3}}{4}.$$

3226. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*
Let ABC be a triangle. Let $S = \sum\limits_{\text{cyclic}} \cos\frac{A}{2}$ and $P = \prod\limits_{\text{cyclic}} \cos\frac{A}{2}$. Prove that

(a) $\displaystyle\frac{S}{P} \leq 2\sqrt{3}\max\left\{\sec\frac{A}{2}, \sec\frac{B}{2}, \sec\frac{C}{2}\right\}$;

(b) $\displaystyle\frac{S}{P} \geq 4\max\left\{\sec^2\frac{B-C}{4}, \sec^2\frac{A-B}{4}, \sec^2\frac{C-A}{4}\right\}$.

3228. *Proposed by Mihály Bencze, Brasov, Romania.*
For $x \in (0, \frac{\pi}{2})$, prove that

$$\frac{(n+1)!}{2\prod\limits_{k=2}^{n}(k+\cos x)} \leq \left(\frac{x}{\sin x}\right)^{n-1} \leq \left(\frac{\pi}{2}\right)^{n-1} \cdot \frac{n!}{\prod\limits_{k=2}^{n}(k+\cos x)}.$$

3229. *Proposed by Mihály Bencze, Brasov, Romania.*
(a) Let x and y be positive real numbers, and let n be a positive integer. Prove that

$$(x+y)^n \sum_{k=0}^{n} \frac{1}{\binom{n}{k}x^{n-k}y^k} \geq n+1+2\sum_{i=1}^{n}\sum_{k=0}^{n-i} \frac{\binom{n}{k}}{\binom{n}{k+i}} \geq (n+1)^2.$$

(b)★ Let $x_1, x_2, \ldots, x_k$ be positive real numbers, and let n be a positive integer. Determine the minimum value of

$$(x_1+x_2+\cdots+x_k)^n \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\geq 0}} \frac{i_1!\,i_2!\cdots i_k!}{n!\,x_1^{i_1}\,x_2^{i_2}\cdots x_k^{i_k}}.$$

3230. *Proposed by Mihály Bencze, Brasov, Romania.*
Let a, x, and y be positive real numbers. Prove that

$$\begin{aligned}(x^{a+1}+x+y)(y^{a+1}+y+x)(x^{a+1}+(x^a+1)y)(y^{a+1}+(y^a+1)x)&\\ \geq (xy)^a(x+\sqrt{xy}+y)^4.&\end{aligned}$$

3232. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let P be a point in the interior of $\sphericalangle QOR$. Find the segment AB of minimum length which contains P with A on the ray OQ and B on the ray OR.

3236. *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*
Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq 3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

3239. *Proposed by Mihály Bencze, Brasov, Romania.*
Let n be a positive integer. If $\alpha = 1+\frac{1}{12(n+1)}$, prove that

$$\mathrm{e} < \left(\frac{(n+1)^{2n+1}}{(n!)^2}\right)^{\frac{1}{2n}} < \mathrm{e}^{\alpha}.$$

3241. *Proposed by Virgil Nicula, Bucharest, Romania.*
Let a, b, c be any real numbers such that $a^2 + b^2 + c^2 = 9$. Prove that

$$3 \cdot \min\{a, b, c\} \le 1 + abc.$$

3242. *Proposed by Virgil Nicula, Bucharest, Romania.*
Let $\mathcal{A} = \left\{z \in \mathbb{C}^* : \left|z + \frac{1}{z}\right| \le 2\right\}$. Let $n \ge 2$ be an integer. Prove that, if $\alpha^n \in \mathcal{A}$, then $\alpha \in \mathcal{A}$.

3246. *Proposed by Marian Tetiva, Bîrlad, Romania.*
Let a, b, c, d be any positive real numbers with $d = \min\{a, b, c, d\}$. Prove that

$$\begin{aligned} &a^4 + b^4 + c^4 + d^4 - 4abcd \\ &\qquad \ge 4d\left[(a-d)^3 + (b-d)^3 + (c-d)^3 - 3(a-d)(b-d)(c-d)\right]. \end{aligned}$$

3247. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let $a_1, a_2, \ldots, a_n$ be real numbers, each greater than 1. Prove that

$$\sum_{k=1}^{n} (1 + \log_{a_k}(a_{k+1}))^2 \ge 4n,$$

where $a_{n+1} = a_1$.

3248. *Proposed by Titu Zvonaru, Comăneşti, Romania, and Bogdan Ioniţă, Bucharest, Romania.*
If a, b, and c are positive real numbers, prove that

$$\frac{a^2(b+c-a)}{b+c} + \frac{b^2(c+a-b)}{c+a} + \frac{c^2(a+b-c)}{a+b} \le \frac{ab+bc+ca}{2}.$$

3249. *Proposed by Titu Zvonaru, Comăneşti, Romania, and Bogdan Ioniţă, Bucharest, Romania.*
Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$\frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ca} + \frac{(a+b)^2}{c^2+ab} \ge 6.$$

3251. *Proposed by Michel Bataille, Rouen, France.*
Let u_1, u_2, and u_3 be any real numbers. Prove that

$$\begin{aligned} &\frac{1}{6}\sum_{i=1}^{3}\left[\cos^2(u_i - u_{i+1}) + \cos^2(u_i + u_{i+1})\right] \\ &\qquad\qquad \ge (\cos u_1 \cos u_2 \cos u_3)^2 + (\sin u_1 \sin u_2 \sin u_3)^2, \end{aligned}$$

where the subscripts in the summation are taken modulo 3.

3253. *Proposed by Mihály Bencze, Brasov, Romania.*
Prove that

$$\log_{\mathrm{e}}(\mathrm{e}^\pi - 1)\log_{\mathrm{e}}(\mathrm{e}^\pi + 1) + \log_\pi(\pi^{\mathrm{e}} - 1)\log_\pi(\pi^{\mathrm{e}} + 1) < \mathrm{e}^2 + \pi^2.$$

3254. *Proposed by George Tsintsifas, Thessaloniki, Greece.*
Let $\mathcal{C}$ be a convex figure in the plane. A *diametrical chord* AB of $\mathcal{C}$ parallel to the direction vector $\overrightarrow{v}$ is a chord of $\mathcal{C}$ of maximal length parallel to the direction vector $\overrightarrow{v}$.
Prove that if every diametrical chord of $\mathcal{C}$ bisects the area enclosed by $\mathcal{C}$, then $\mathcal{C}$ must be centro-symmetric.

3260. *Proposed by Virgil Nicula, Bucharest, Romania.*
Let a, b be distinct positive real numbers such that $(a-1)(b-1) \geq 0$. Prove that

$$a^b + b^a \geq 1 + ab + (1-a)(1-b) \cdot \min\{1, ab\}.$$

3261. *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*
The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}, \qquad \text{for } n \geq 1; \\ L_0 &= 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}, \qquad \text{for } n \geq 1. \end{aligned}$$

Prove that

$$\sum_{n=1}^{\infty} \frac{\arctan\left(\frac{1}{L_{2n}}\right) \arctan\left(\frac{1}{L_{2n+2}}\right)}{\arctan\left(\frac{1}{F_{2n+1}}\right)} \leq \frac{4}{\pi} \arctan(\beta) \left(\arctan(\beta) + \frac{1}{3}\right),$$

where $\beta = \frac{1}{2}(\sqrt{5} - 1)$.

3263. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
The Fibonacci numbers F_n and Lucas numbers L_n are defined by the following recurrences:

$$\begin{aligned} F_0 &= 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1}, \qquad \text{for } n \geq 1; \\ L_0 &= 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+1} = L_n + L_{n-1}, \qquad \text{for } n \geq 1. \end{aligned}$$

Prove that for each positive integer n,

$$L_n L_{n+1} \leq 2 + \left(\sum_{k=1}^{n} L_k F_{2k}\right)^{\frac{1}{2}} \cdot \sum_{k=1}^{n} \frac{L_k^2}{\sqrt{F_k}}.$$

3265. *Proposed by Virgil Nicula, Bucharest, Romania.*
Let $ABCD$ be a trapezoid with $AB \parallel CD$ for which $AD = CD$ and $AC = BC$, and let E be the intersection of AC and BD. Let x, y, z denote the measures of angles ABC, BDC, AED, respectively. Show that $y \leq 30°$,

$$\tan y = \frac{2\tan x}{3 + \tan^2 x}, \quad \text{and} \quad \tan z = \frac{2\sin x + \sin 3x}{2\cos x + \cos 3x}.$$

3268. *Proposed by Bill Sands and John Wiest, University of Calgary, Calgary, AB.*
You are given an infinite sequence of cards $C_1, C_2, \ldots$, on each of which is written an infinite series of non-negative real numbers which sums to 1.

(a) Prove that there is a reordering $D_1, D_2, \ldots$ of the cards such that the series $\sum_{i=1}^{\infty} d_{ii}$ converges, where d_{ii} is the i^{th} term of the series on card D_i.

(b)* Is there necessarily a reordering such that $\sum_{i=1}^{\infty} d_{ii} \le 1$?

[Ed: Compare with problem 2620 [2002 : 127; 2005 : 319–326].]

3269. *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let n be a positive integer. Prove that

$$\exp\left(\frac{2^n}{n+1}\right)\sum_{k=1}^{n}\frac{k}{\exp\binom{n}{k}} \ge \binom{n+1}{2}.$$

3271. *Proposed by Virgil Nicula, Bucharest, Romania.*
Let a, b, and c be real numbers. Prove that $|a+b|+|b+c|+|c+a| \le 2$ if and only if $|a| \le 1$, $|b| \le 1$, $|c| \le 1$, and $|a+b+c| \le 1$.

3274. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, and c be non-negative real numbers. Prove that

$$\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2} \ge \frac{a+b+c}{3}.$$

3275. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let x, y, and z be non-negative real numbers satisfying $x+y+z=3$, and let $0 \le r \le 8$. Prove that

$$\frac{1}{xy^2+r}+\frac{1}{yz^2+r}+\frac{1}{zx^2+r} \ge \frac{3}{1+r}.$$

3281. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\left(\sum_{k=1}^{n} a_k^{\frac{n+1}{2}}\right)^n \le \prod_{k=1}^{n}\left(\sum_{k=1}^{n} a_k^k\right).$$

3287. *Proposed by Virgil Nicula, Bucharest, Romania.*
Let x, y, and z be positive real numbers satisfying $xy+yz+zx+xyz=4$. Prove that

(a) $(x+2)(y+2)+(y+2)(z+2)+(z+2)(x+2)=(x+2)(y+2)(z+2)$;

(b) there is a triangle whose sides have lengths

3290. *Proposed by Virgil Nicula, Bucharest, Romania.*
Let $ABCD$ be a trapezoid with $AD \parallel BC$. Denote the lengths of AD and BC by a and b, respectively. Let M be the mid-point of CD, and let P and Q be the mid-points of AM and BM, respectively. If N is the intersection of DP and CQ, prove that N belongs to the interior of $\triangle ABM$ if and only if $\frac{1}{3} < \frac{a}{b} < 3$.

3292. *Proposed by Mihály Bencze, Brasov, Romania.*
Let a, b, c, and d be arbitrary real numbers. Show that

$$11a^2 + 11b^2 + 221c^2 + 131d^2 + 22ab + 202cd + 48c + 6 \geq$$
$$98ac + 98bc + 38ad + 38bd + 12a + 12b + 12d.$$

3296. *Proposed by Michel Bataille, Rouen, France.*
Find greatest constant K such that

$$\frac{b^2c^2}{a^2(a-b)(a-c)} + \frac{c^2a^2}{b^2(b-c)(b-a)} + \frac{a^2b^2}{c^2(c-a)(c-b)} > K$$

for all distinct positive real numbers a, b, and c.

3297. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*
If A, B, and C are the angles of a triangle, prove that

$$\sin A + \sin B \sin C \leq \frac{1+\sqrt{5}}{2}.$$

When does equality hold?

3298. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA, USA.*
Let ABC be a triangle of area $\frac{1}{2}$ in which a is the length of the side opposite vertex A. Prove that

$$a^2 + \csc A \geq \sqrt{5}.$$

[Ed.: The proposer's only proof of this is by computer. He is hoping that some ***CRUX with MAYHEM*** reader will find a simpler solution.]

3299. *Proposed by Victor Oxman, Western Galilee College, Israel.*
Given positive real numbers a, b, and w_b, show that

(a) if a triangle ABC exists with $BC = a$, $CA = b$, and the length of the interior bisector of angle B equal to w_b, then it is unique up to isomorphism;

(b) for the existence of such a triangle in (a), it is necessary and sufficient that

$$b > \frac{2a|a-w_b|}{2a-w_b} \geq 0;$$

(c) if h_a is the length of the altitude to side BC in such a triangle in (a), we have $b > |a - w_b| + \frac{1}{2}h_a$.

3300. *Proposed by Arkady Alt, San Jose, CA, USA.*
Let a, b, and c be positive real numbers. For any positive integer n define

$$F_n = \left(\frac{3\left(a^n+b^n+c^n\right)}{a+b+c} - \sum_{\text{cyclic}} \frac{b^n+c^n}{b+c}\right).$$

(a) Prove that $F_n \geq 0$ for $n \leq 5$.

(b)* Prove or disprove that $F_n \geq 0$ for $n \geq 6$.

to be continued . . .

Inequalities proposed in
"Mathematical Reflections"

Last update: November 23, 2007

Please visit `http://reflections.awesomemath.org`

(An asterisk (★) after a number indicates that a problem was proposed without a solution.)

Juniors

J5. *Proposed by Cristinel Mortici, Valahia University of Târgovişte, Romania.*
Let x, y, z be positive real numbers such that $xyz = 1$. Show that the following inequality holds:

$$\frac{1}{(x+1)^2+y^2+1}+\frac{1}{(y+1)^2+z^2+1}+\frac{1}{(z+1)^2+x^2+1}\le\frac{1}{2}.$$

J8. *Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Romania.*
Let a, b be distinct real numbers such that

$$|a-1|+|b+1|=|a|+|b|=|a-1|+|b+1|.$$

Find the minimal possible value of $|a+b|$.

J14. *Proposed by Zdravko F. Starc, Vršac, Serbia and Montenegro.*
Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$a\left(b^2-\sqrt{b}\right)+b\left(c^2-\sqrt{c}\right)+c\left(a^2-\sqrt{a}\right)\ge 0.$$

J15. *Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Romania.*
Find the least positive number α with the following property: in every triangle, one can choose two sides of lengths a, b such that

$$1\le\frac{a}{b}<\alpha.$$

J17. *Proposed by Ivan Borsenco, student, Chişinău, Moldova.*
Let a, b, c be positive numbers. Prove the following inequality:

$$(ab+bc+ca)^3\le 3\,(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2).$$

J19. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Let a, b be real numbers such that $3(a+b)\ge 2|ab+1|$. Prove that

$$9(a^3+b^3)\ge|a^3b^3+1|.$$

J20. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Prove that:

(a) There are infinitely many quadrupels (a, b, c, d) of pairwise distinct positive integers such that $ab + cd = (a + b)(c + d)$.

(b) For any such quadruple, $\max(a,b,c,d)\ge\dfrac{4\sqrt{3}}{\sqrt{3}+1}(a+b+c+d)$.

J22. *Proposed by Liubomir Chiriac, Princeton University, USA.*
There are n 1's written on a board. At each step we can select two of the numbers on the board and replace them by $\sqrt[3]{\frac{a^2b^2}{a+b}}$. We keep applying this operation until there is only one number left. Prove that this number is not less than $\frac{1}{\sqrt[3]{n}}$.

J24. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Consider a triangle ABC and a point P in its interior. Denote by d_a, d_b, d_c the distances from P to the triangle's sides. Prove that

$$2\,S\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{R}\right) \geq d_a+d_b+d_c,$$

where S and R are the triangle's area and circumradius, respectively.

J26. *Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Romania.*
A line divides an equilateral triangle into two parts with the same perimeter and having areas S_1 and S_2, respectively. Prove that

$$\frac{7}{9} \leq \frac{S_1}{S_2} \leq \frac{9}{7}.$$

J30. *Proposed by Cezar Lupu, University of Bucharest, Romania.*
Let a, b, c be three nonnegative real numbers. Prove the inequality

$$\frac{a^3+abc}{b+c}+\frac{b^3+abc}{a+c}+\frac{c^3+abc}{a+b} \geq a^2+b^2+c^2.$$

J31. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Find the least perimeter of a right-angled triangle whose sides and altitude are integers.

J32. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Let a and b be real numbers such that

$$9a^2+8ab+7b^2 \leq 6.$$

Prove that $7a+5b+12ab \leq 9$.

J34. *Proposed by Magkos Athanasios, Kozani, Greece.*
Let ABC be a triangle and let I be its incenter. Prove that at least one of IA, IB, IC is greater than or equal to the diameter of the incircle of ABC.

J35. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Prove that among any four positive integers greater than or equal to 1 there are two, say a and b, such that

$$\frac{\sqrt{(a^2-1)(b^2-1)}+1}{ab} \geq \frac{\sqrt{3}}{2}.$$

J36. *Proposed by Iurie Boreico, Moldova.*
Let a, b, c, d be integers such that $\gcd(a, b, c, d) = 1$ and $ad - bc \neq 0$. Prove that the greatest possible value of $\gcd(ax + by, cx + dy)$ over all pairs (x, y) of relatively prime is $|ad - bc|$.

J37. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Let $a_1, a_2, \ldots, a_{2n+1}$ be distinct positive integers not exceeding $3n+1$. Prove that among them there are two such that

$$a_i - a_j = m, \text{ for all } m \in \{1, 2, \ldots, n\}.$$

J38. *Proposed by Cezar Lupu, University of Bucharest, Romania.*
Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{a^2+bc}{(a+b)(a+c)}+\frac{b^2+ca}{(b+a)(b+c)}+\frac{c^2+ab}{(c+a)(c+b)}.$$

J41. *Proposed by Daniel Campos Salas, Costa Rica.*
Let a, b, c be positive real numbers such that $a+b+c+1=4abc$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 3 \geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}.$$

J44. *Proposed by Mircea Lascu, Zalau, Romania.*
Consider a triangle ABC and let g_a, g_b, g_c and n_a, n_b, n_c be the Gergonne cevians and the Nagel cevians, respectively. Prove that

$$g_a+g_b+g_c+2\max(a,b,c) \geq n_a+n_b+n_c+2\min(a,b,c).$$

J47. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
In triangle ABC let m_a and l_a be the median and the angle bisector from vertex A, respectively. Prove that

$$0 \leq m_a^2 - l_a^2 \leq \frac{(b-c)^2}{2}.$$

J48. *Proposed by Ho Phu Thai, Da Nang, Vietnam.*
Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b(b+c)^2}+\frac{b}{c(c+a)^2}+\frac{c}{a(a+b)^2} \geq \frac{9}{4(ab+bc+ca)}.$$

J49. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Find the least k such that any k-element subset of $\{1, 2, \ldots, 10\}$ contains numbers whose sum is divisible by 11.

J51. *Proposed by Virgil Nicula and Cosmin Pohoata, Romania.*
Let a, b, c the sides of a triangle. Prove that

$$(a+b)(b+c)(c+a)+(-a+b+c)(a-b+c)(a+b-c) \geq 9abc.$$

J53. *Proposed by Cosmin Pohoata, Bucharest, Romania.*
Consider a triangle ABC. Let I be its incenter and let M, N, P be the midpoints of triangle's sides. Prove that

$$IM^2+IN^2+IP^2 \geq r(R+r),$$

where R and r are the circumradius and the inradius, respectively.

J56. *Proposed by Iurie Boreico, Harvard University.*
Two players, A and B, play the following game: player A divides an 9×9 square into strips of unit width and various lengths. After that player B picks an integer k, $1 \leq k \leq 9$, and removes all strips of length k. Find the largest area K that B can remove, regardless the way A divides the square into strips.

J57. *Proposed by Mircea Becheanu, Bucharest, Romania.*
Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Prove that

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq 16.$$

J60. *Proposed by Pham Huu Duc, Ballajura, Australia.*
Let a, b, c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab} \leq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

J62. *Proposed by Alex Anderson, New Trier High School, Winnetka, USA.*
Consider a right-angled triangle ABC with $\sphericalangle A = 90°$. Let $E \in AC$ and $F \in AB$ such that $\sphericalangle AEF = \sphericalangle ABC$ and $\sphericalangle AFE = \sphericalangle ACB$. Denote by E' and F' the projections of E and F onto BC, respectively. Prove that

$$E'E + EF + FF' \leq BC$$

and determine when equality holds.

J63. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Find the least n such that no matter how we color an $n \times n$ lattice point grid in two colors we can always find a parallelogram with all vertices to be monochromatic.

J64. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*
Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a+\sqrt[3]{4\left(b^3+c^3\right)}}+\frac{c+a}{b+\sqrt[3]{4\left(c^3+a^3\right)}}+\frac{a+b}{c+\sqrt[3]{4\left(a^3+b^3\right)}} \leq 2.$$

Seniors

S2. *Proposed by Ivan Borsenco, student, Chişinău, Moldova.*
Circles with radii r_1, r_2, r_3 are externally tangent to each other. Two other circles, with radii R, r, are tangent to all previous three circles. Prove that

$$Rr \geq \frac{r_1 r_2 r_3}{r_1+r_2+r_3}.$$

S6. *Proposed by Marian Tetiva, Bîrlad, Romania.*
Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

1. If $a \leq b \leq 1 \leq c$, then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1};$$

2. If $a \leq 1 \leq b \leq c$, then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}.$$

S7. *Proposed by Iurie Boreico and Marcel Teleucă, Chişinău, Moldova.*
Let $x_1, x_2, \ldots, x_n$ be real numbers greater than or equal to $\frac{1}{2}$. Prove that

$$\prod_{i=1}^{n}\left(1+\frac{2x_i}{3}\right)^{x_i} \geq \left(\frac{4}{3}\right)^n \sqrt[4]{(x_1+x_2)(x_2+x_3)\cdots(x_{n-1}+x_n)(x_n+x_1)}.$$

S8. *Proposed by Ivan Borsenco, student, Chişinău, Moldova.*
Let O, I, and r be the circumcenter, incenter, and inradius of a triangle ABC. Let M be a point inside the triangle, and let d_1, d_2, d_3, be the distances from M to the sides BC, AC, AB. Prove that if $d_1 \cdot d_2 \cdot d_3 \geq r^3$, then M lies inside the circle with center O and radius OI.

S9. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n}\left(\sum_{k=1}^{n} a_k^{T_k}\right) \geq \left(\sum_{k=1}^{n} a_k^{\frac{T_{n+1}}{3}}\right)^n,$$

where $T_k = \frac{k(k+1)}{2}$ is the k^{th} triangular number.

S10. *Proposed by Laurenţiu Panaitopol, University of Bucharest, Romania.*
Let $(a_n)_{n\geq 1}$ be a sequence of positive numbers such as $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Show that for all $n \geq 1$ we have $a_n \geq 2$.

S14. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*
Let a, b, c be the sides of a scalene triangle ABC and let S be its area. Prove that

$$\frac{2a+b+c}{a(a-b)(a-c)} + \frac{a+2b+c}{b(b-a)(b-c)} + \frac{a+b+2c}{c(c-a)(c-b)} < \frac{3\sqrt{3}}{4S}.$$

S16. *Proposed by Ivan Borsenco, student, Chişinău, Moldova.*
Let M_1 be a point inside triangle ABC and let M_2 be its isogonal conjugate. Let R and r denote the circumradius and the inradius of the triangle. Prove that

$$4R^2r^2 \geq (R^2 - OM_1^2)(R^2 - OM_2^2).$$

S17. *Proposed by Ivan Borsenco, student, Chişinău, Moldova.*
Let $m > n > 1$ be positive integers. A set of m real numbers is given. We are allowed to pick any n of them, say $a_1, a_2, \ldots, a_n$, and ask: is it true that $a_1 < a_2 < \cdots < a_n$? Determine k such that we can find the order of all m numbers asking at most k questions.

S20. *Proposed by Khoa Lu Nguyen, Massachusetts Institute of Technology, Cambridge, USA.*
Let ABC be an acute triangle and let P be a point in its interior. Prove that:

$$(AP + BP + CP)^2 \geq \sqrt{3}\,(PA \cdot BC + PB \cdot CA + PC \cdot AB).$$

S22. *Proposed by Iurie Boreico, Moldova.*
Let n and k be positive integers. Eve gives Adam k apples. However, she can first give him bitter apples, at most n. The procedure goes as follows: Eve gives Adam an apple at a time and Adam can either eat it (and find out whether it's sweet or not), or throw it away. Adam knows that the bitter apples come first, and the sweet last. Find, in terms of n, the least value of k for which Adam can be sure he eats more sweet apples than bitter.

S23. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Let a, b, c, d be positive real numbers. Prove that

$$3\,(a^2 - ab + b^2)(c^2 - cd + d^2) \geq 2\,(a^2c^2 - abcd + b^2d^2).$$

S24. *Proposed by Iurie Boreico, Moldova and Ivan Borsenco, University of Texas, Dallas, USA.*
Let ABC be an acute-angled triangle inscribed in a circle $\mathcal{C}$. Consider all equilateral triangles DEF with vertices on $\mathcal{C}$. The Simpson lines of D, E, F with respect to the triangle ABC form a triangle T. Find the greatest possible area of this triangle.

S25. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Prove that in any acute-angled triangle ABC,

$$\cos^3 A + \cos^3 B + \cos^3 C + \cos A \cos B \cos C \geq \frac{1}{2}.$$

S27. *Proposed by Pham Huu Duc, Australia.*
Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt[3]{\frac{a^2+bc}{b^2+c^2}} + \sqrt[3]{\frac{b^2+ca}{c^2+a^2}} + \sqrt[3]{\frac{c^2+ab}{a^2+b^2}} \geq \frac{9\sqrt[3]{abc}}{a+b+c}.$$

S28. *Proposed by Hung Quang Tran, Hanoi, Vietnam.*
Let M be a point in the plane of triangle ABC. Find the minimum of

$$MA^3 + MB^3 + MC^3 - \frac{3}{2}R \cdot MH^3,$$

where H is the orthocenter and R is the circumradius of the triangle ABC.

S29. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Prove that for any real numbers a, b, c the following inequality holds

$$3\,(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \geq a^3b^3 + b^3c^3 + c^3a^3.$$

S30. *Proposed by Pham Huu Duc, Australia.*
Prove that for all positive real numbers a, b, and c,

$$\frac{1}{a+b+c}\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{1}{ab+bc+ca} + \frac{1}{2(a^2+b^2+c^2)}.$$

S33. *Proposed by Cezar Lupu, University of Bucharest, Romania.*
Let a, b, c be nonnegative real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} + \frac{4(ab+bc+ca)}{(a+b)(b+c)(a+c)} \geq ab + bc + ca.$$

S35. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Let ABC be a triangle with the largest angle at A. On line AB consider the point D such that A lies between B and D and $AD = \frac{AB^3}{AC^2}$. Prove that

$$CD \leq \sqrt{3} \cdot \frac{BC^3}{AC^2}.$$

S37. *Proposed by Bogdan Enescu, “B. P. Hasdeu” National College, Romania.*
Let x, y, z be real numbers such that

$$\cos x + \cos y + \cos z = 0, \qquad \text{and} \qquad \cos 3x + \cos 3y + \cos 3z = 0.$$

Prove that

$$\cos 2x \cdot \cos 2y \cdot \cos 2z \leq 0.$$

S41. *Proposed by Pham Huu Duc, Ballajura, Australia.*
Prove that for any positive real numbers a, b and c,

$$\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}} \geq \sqrt{6\cdot\frac{a+b+c}{\sqrt[3]{abc}}}.$$

S42. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Prove that in any triangle there exist a pair (M_1, M_2) of isogonal conjugates such that $OM_1 \cdot OM_2 > OI^2$, where O and I are the circumcenter and the incenter, respectively.

S48. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Consider an equilateral triangle divided into 16 congruent equilateral triangles. Prove that no matter how we label these triangles with the numbers 1 through 16, there will be two adjacent triangles whose difference of the labels is at least 4.

S50. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Let $p \geq 5$ be a prime and let $q_1^{\beta_1} q_2^{\beta_2} \cdots q_n^{\beta_n}$ be the prime factorization of $(p-1)^p+1$. Prove that

$$\sum_{i=1}^{n} q_i\beta_i > p^2.$$

S54. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c be positive real numbers. Prove that

$$\frac{a^2-bc}{4a^2+4b^2+c^2}+\frac{b^2-ca}{4b^2+4c^2+a^2}+\frac{c^2-ab}{4c^2+4a^2+b^2} \geq 0$$

and find all equality cases.

S55. *Proposed by Iurie Boreico, Harvard University.*
Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of positive real numbers. Prove that there exist no more than $\frac{2^n}{\sqrt{n}}$ subsets of X, whose sum of elements is equal to 1.

S56. *Proposed by Tran Quang Hung, Ha Noi National University, Vietnam.*
Let G be the centroid of triangle ABC. Prove that

$$\sin\sphericalangle GBC+\sin\sphericalangle GCA+\sin\sphericalangle GAB \leq \frac{3}{2}.$$

S61. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Let ABC be a triangle. Prove that

$$\frac{1}{\sin\frac{A}{2}}+\frac{1}{\sin\frac{B}{2}}+\frac{1}{\sin\frac{C}{2}} \geq 4\sqrt{\frac{R}{r}},$$

where R and r are its circumradius and inradius, respectively.

S63. *Proposed by Pham Huu Duc, Ballajura, Australia.*
Let a, b, c be positive real numbers such that $ab+bc+ca \geq 3$. Prove that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \frac{3}{\sqrt{2}}.$$

Undergraduate

U6. *Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris.*
Find all positive integers a, b, c and all integers x, y, z satisfying the conditions:

a) $ax^2 + by^2 + cz^2 = abc + 2xyz - 1$

b) $ab + bc + ca \geq x^2 + y^2 + z^2$.

U9. *Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris.*
Let $||\cdot||$ be a norm on $\mathcal{M}_n(\mathbb{C})$ and let $A_1, A_2, \ldots, A_p$ be complex matrices of order n. Prove that for every $x > 0$ there exists $z \in \mathbb{C}$, with $|z| < x$, such that

$$\left\|(I_n - zA_1)^{-1} + (I_n - zA_2)^{-1} + \cdots + (I_n - zA_p)^{-1}\right\| \geq p.$$

U15. *Proposed by Cezar Lupu, University of Bucharest, and Tudorel Lupu, Decebal Highschool, Constanţa.*
Let $f : [a, b] \to \mathbb{R}$ be a continuous and convex function. Prove that

$$\int_a^b f(x)\,\mathrm{d}x \geq 2\int_{\frac{3a+b}{4}}^{\frac{3b+a}{4}} f(x)\,\mathrm{d}x \geq (b-a)f\left(\frac{a+b}{2}\right).$$

U26. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania.*
Let $f : [a, b] \to \mathbb{R}$ $(0 < a < b)$ be a continuous function on $[a, b]$ and diffentiable on (a, b). Prove that there is a $c \in (a, b)$ such that

$$\frac{2}{a-c} < f'(c) < \frac{2}{b-c}.$$

U30. *Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris.*
Let n be a positive integer. What is the largest cardinal of a subgroup G of $GL_n(\mathbb{Z})$ such that for any matrix $A \in G$, all elements of $A - I_n$ are even?

U31. *Proposed by Titu Andreescu, University of Texas, Dallas, USA.*
Find the minimum of the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{(x^2 - x + 1)^2}{x^6 - x^3 + 1}.$$

U31. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*
Let $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ be sequences of complex numbers. Prove that

$$\mathrm{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \leq \frac{1}{3n+2}\left(\sum_{k=0}^{n} |a_k|^2 + \frac{9n^2 + 6n + 2}{2}\sum_{k=0}^{n} |b_k|^2\right).$$

U42. *Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris.*
Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be points in a plane such that $B_iA_1 \cdot B_iA_2 \cdot \cdots \cdot B_iA_n \leq A_jB_i$ for all i and j. Prove that

$$\prod_{1\leq i<j\leq n} A_iA_j \cdot B_iB_j \leq n^{\frac{n}{2}}.$$

U44. *Proposed by Cezar Lupu, University of Bucharest, Romania.*
Let x, y be positive real numbers such that $x^y + y = y^x + x$. Prove that $x + y \leq 1 + xy$.

U47. *Proposed by Hung Quang Tran, Ha Noi National University, Vietnam.*
Let P be an arbitrary point inside equilateral triangle ABC. Find the minimum value of

$$\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}.$$

U48. *Proposed by Iurie Boreico, Moldova.*
Let n an integer greater than 1 and let $k \geq 1$ be a real number. For an n dimensional simplex $X_1X_2\dots X_{n+1}$ define its k-perimeter by

$$\sum_{1\leq i<j\leq n+1}|X_iX_j|^k.$$

Take now a regular simplex $A_1A_2\dots A_{n+1}$ and consider all simplexes $B_1B_2\dots B_{n+1}$ where B_i lies on the face $A_1\dots A_{i-1}A_{i+1}\dots A_n$. Find, in terms of the k-perimeter of $A_1A_2\dots A_{n+1}$, the minimal possible k-perimeter of $B_1B_2\dots B_{n+1}$.

U49. *Proposed by Cezar Lupu, Bucharest and Mihai Piticari, Campulung, Romania.*
Let $f:[0,1]\to[0,\infty)$ be an integrable function. Prove that

$$\int_0^1 f(x)\,\mathrm{d}x\cdot\int_0^1 x^3f(x)\,\mathrm{d}x \geq \int_0^1 xf(x)\,\mathrm{d}x\cdot\int_0^1 x^2f(x)\,\mathrm{d}x.$$

U56. *Proposed by Byron Schmuland, University of Alberta, Canada.*
Let x,y,z be positive real numbers. Prove that

$$\frac{3\sqrt{3}}{2}\leq\sqrt{x+y+z}\left(\frac{\sqrt{x}}{y+z}+\frac{\sqrt{y}}{z+x}+\frac{\sqrt{z}}{x+y}\right).$$

U59. *Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris.*
Let ϕ be Euler's totient function, where $\phi(1)=1$. Prove that for all positive integers n we have

$$1>\sum_{k=1}^{n}\frac{\phi(k)}{k}\ln\left(\frac{2^k}{2^k-1}\right)>1-\frac{1}{2^n}.$$

U62. *Proposed by Cezar Lupu, University of Bucharest, Romania.*
Let $x_1,x_2,\dots,x_n>0$ such that $x_1+x_2+\cdots+x_n=n$ and let $y_k=n-x_k$, $k=1,2,\dots,n$. Prove that

$$x_1^{x_1}\cdot x_2^{x_2}\cdots x_n^{x_n}\geq\left(\frac{y_1}{n-1}\right)^{y_1}\cdot\left(\frac{y_2}{n-1}\right)^{y_2}\cdots\left(\frac{y_n}{n-1}\right)^{y_n}.$$

Olympiad

O5. *Proposed by Gabriel Dospinescu, "Louis le Grand" College, Paris.*
Let p be a prime number of the form $4k+1$ such that $2^p\equiv 2\,(\mathrm{mod}\,p^2)$. Prove that there exists a prime number q, divisor of 2^p-1, such that $2^q>(6p)^p$.

O6. *Proposed by Vasile Cârtoaje, Ploieşti, Romania.*
Let x,y,z be nonnegative real numbers. Prove the inequality

$$x^4(y+z)+y^4(z+x)+z^4(x+y)\leq\frac{1}{12}(x+y+z)^5.$$

O8. *Proposed by Adrian Zahariuc, student, Bacău, Romania.*
Let a, b, c, x, y, z be real numbers and let $A = ax+by+cz$, $B = ay+bz+cx$ and $C = az+bx+cy$. Suppose that $|A-B| \geq 1$, $|B-C| \geq 1$ and $|C-A| \geq 1$. Find the smallest possible value of the product

$$(a^2+b^2+c^2)(x^2+y^2+z^2).$$

O11. *Proposed by Iurie Boreico and Ivan Borsenco, Chişinău, Moldava.*
Let a, b, c be distinct positive integers. Prove the following inequality:

$$\frac{a^2b+a^2c+b^2a+b^2c+c^2a+c^2b-6abc}{a^2+b^2+c^2-ab-bc-ac} \geq \frac{16abc}{(a+b+c)^2}.$$

O14. *Proposed by Ivan Borsenco, student, Chişinău, Moldova.*
The vertices of a planar graph G have degrees 3, 4, or 5 and vertices with the same degree are not connected. Suppose that the number of 5-sided faces is greater than the number of 3-sided faces. Denote by v the total number of vertices and by v_3 the number of vertices with degree 3. Prove that

$$v_3 \geq \frac{v+23}{4}.$$

O16. *Proposed by Iurie Boreico, student, Chişinău, Moldova.*
Let ABC be an acute-angled triangle. Let ω be the center of the nine point circle and let G be its centroid. Let $A', B', C', A'', B'', C''$ be the projections of ω and G on the corresponding sides. Prove that the perimeter of $A''B''C''$ is not less than the perimeter of $A'B'C'$.

O17. *Proposed by Bogdan Enescu, "B. P. Hasdeu" National College, Romania.*
Let α be a root of the polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_i \in [0,1]$, for $i = 0, 1, \ldots, n-1$. Prove that

$$\Re(\alpha) < \frac{1+\sqrt{5}}{2}.$$

O18. *Proposed by Nikolai Nikolov, Sofia, Bulgaria.*
Let x, y, z be real numbers such that $0 < y < x < 1$ and $0 < z < 1$. Prove that

$$(x^z - y^z)(1 - x^z y^z) > \frac{x-y}{1-xy}.$$

O19. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Let a, b, c be positive real numbers. Prove that:

a) $(a^3+b^3+c^3)^2 \geq (a^4+b^4+c^4)(ab+bc+ca)$,

b) $9(a^4+b^4+c^4)^2 \geq (a^5+b^5+c^5)(a+b+c)^3$.

O25. *Proposed by Darij Grinberg, Germany.*
For any triangle ABC, prove that

$$\cos\frac{A}{2}\cot\frac{A}{2} + \cos\frac{B}{2}\cot\frac{B}{2} + \cos\frac{C}{2}\cot\frac{C}{2} \geq \frac{\sqrt{3}}{2}\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right).$$

O27. *Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris.*
Let $a_1, a_2, \ldots, a_5$ be positive real numbers such that

$$a_1 a_2 \cdots a_5 = a_1(1+a_2) + a_2(1+a_3) + \cdots + a_5(1+a_1) + 2.$$

Find the minimal value of $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \dfrac{1}{a_4} + \dfrac{1}{a_5}$.

O30. *Proposed by Oleg Mushkarov, Bulgarian Academy of Sciences, Sofia.*
Prove that equation

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2} = \frac{n+1}{x_{n+1}^2}$$

has a solution in positive integers if and only if $n \geq 3$.

O32. *Proposed by Bin Zhao, University of Technology and Science, China.*
Let $a, b, c > 0$. Prove that

$$\sqrt{\frac{a^2}{4a^2+ab+4b^2}} + \sqrt{\frac{b^2}{4b^2+bc+4c^2}} + \sqrt{\frac{c^2}{4c^2+ca+4a^2}} \leq 1.$$

O33. *Proposed by Hung Quang Tran, Ha Noi National University, Vietnam.*
Let ABC be a triangle with circumcenter O and incenter I. Consider a point M lying on the small arc BC. Prove that

$$AM + 2OI \geq MB + MC \geq MA - 2OI.$$

O35. *Proposed by Iurie Boreico, Moldova.*
Let $0 < a < 1$. Find, with proof, the greatest real number b_0 such that if $b < b_0$ and $A_n \subset [0;1])_{n\in\mathbb{N}}$ are finite unions of disjoint segments with total length a, then there are two different $i, j \in \mathbb{N}$ such that $A_i \cap A_j$ is a union of segments with total length at least b. Generalize this result to numbers greater than 2: if $k \in \mathbb{N}$ find the least b_0 such that whenever $b < b_0$ and $(A_n \subset [0;1])_{n\in\mathbb{N}}$ are finite unions of disjoint segments with total length a, then there are k different $i_1, i_2, \ldots, i_k \in \mathbb{N}$ such that $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ is a union of segments with total length at least b.

O36. *Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris.*
Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers and let x_{ij} be the number of indices k such that $b_k \geq \max(a_i, a_j)$. Suppose that $x_{ij} > 0$ for any i and j. Prove that we can find an even permutation f and an odd permutation g such that

$$\sum_{i=1}^{n} \frac{x_{if(i)}}{x_{ig(i)}} \geq n.$$

O37. *Proposed by Vasile Cîrtoaje, University of Ploieşti, Romania.*
Let a, b, c, d be nonnegative numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that

$$\sqrt{2}(4 - ab - bc - cd - da) \geq (\sqrt{2}+1)(4 - a - b - c - d).$$

O39. *Proposed by Ho Phu Thai, Da Nang, Vietnam.*
Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+2bc}} + \frac{b}{\sqrt{b^2+2ca}} + \frac{c}{\sqrt{c^2+2ab}} \leq \frac{a+b+c}{\sqrt{ab+bc+ca}}.$$

O42. *Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris.*
Let $a_1, a_2, \ldots, a_5$ be positive real numbers such that

$$a_1 a_2 \cdots a_5 = a_1(1+a_2) + a_2(1+a_3) + \cdots + a_5(1+a_1) + 2.$$

Find the minimal value of

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5}.$$

O43. *Proposed by Vo Quoc Ba Can, Can Tho University, Vietnam.*
Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{b+c}{a}} + \sqrt{\frac{c+a}{b}} + \sqrt{\frac{a+b}{c}} \geq \sqrt{\frac{16(a+b+c)^3}{3(a+b)(b+c)(c+a)}}.$$

O48. *Proposed by Gabriel Dospinescu, Ecole Normale Superieure, Paris.*
Let $f \in \mathbb{Z}[X]$ be a monic irreducible polynomial of degree n whose zeros $x_1, x_2, \ldots, x_n$ are all real numbers. Let $S_k = x_1^{2k} + x_2^{2k} + \cdots + x_n^{2k}$. Prove that there exist a universal constant $c > 0$, such that

$$S_1 \cdot S_2 \cdot \cdots \cdot S_{n-1} \geq c \cdot \frac{\mathrm{e}^{2n}}{n^2}$$

holds for all n.

O49. *Proposed by Cezar Lupu, Romania and Darij Grinberg, Germany.*
Let A_1, B_1, C_1 be points on the sides BC, CA, AB of a triangle ABC. Lines AA_1, BB_1, CC_1 intersect again the circumcircle of triangle ABC at A_2, B_2, C_2, respectively. Prove that

$$\frac{AA_1}{A_1A_2} + \frac{BB_1}{B_1B_2} + \frac{CC_1}{C_1C_2} \geq \frac{3s^2}{r(4R+r)},$$

where s, r, R are the semiperimeter, inradius, and circumradius of triangle ABC, respectively.

O50. *Proposed by Iurie Boreico, Moldova and Ivan Borsenco, University of Texas, Dallas.*
Find the least k for which there exist integers $a_1, a_2, \ldots, a_k$, different from -1, such that numbers $x^2 + a_i y^2$, $x, y \in \mathbb{Z}$, $i = 1, 2, \ldots, k$, cover the set of prime numbers.

O56. *Proposed by Iurie Boreico, Harvard University.*
We have k hedgehogs in the upper-left unit square of a $m \times n$ grid. Each of them moves towards the lower-right unit square of the grid, by moving each minute either one unit to the right or one unit down. What is the least possible number of grid squares that are not visited by any of the hedgehogs?

O59. *Proposed by Ivan Borsenco, University of Texas, Dallas, USA.*
Let P_n and Q_n be the number of connected and disconnected unlabeled graphs in the graph with n vertices. Prove that

$$P_n - Q_n \geq 2\,(P_{n-1} - Q_{n-1}).$$

O61. *Proposed by Ciupan Andrei, Bucharest, Romania.*
Let a, b, c be positive numbers such that $4abc = a + b + c + 1$. Prove that

$$\frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + \frac{a^2+b^2}{c} \geq 2\,(ab+bc+ca).$$

to be continued . . .

Inequalities proposed in
"The American Mathematical Monthly"

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(An asterisk (★) after a number indicates that a problem was proposed without a solution.)

10354. *Proposed by Hassan Ali Shah Ali, Tehran, Iran.*
Determine the least natural number N such that, for all $n \geq N$, there exist natural numbers a, b with $n = \lfloor a\sqrt{2} + b\sqrt{3} \rfloor$.

10371. *Proposed by Emil Yankov Stoyanov, Antiem I Mathematical School, Vidin, Bulgaria.*
Let B' and C' be points on the sides AB and AC, respectively, of a given triangle ABC, and let P be a point on the segment $B'C'$. Determine the maximum value of

$$\frac{\min\{[BPB'], [CPC']\}}{[ABC]}$$

where $[F]$ denotes the area of F.

10374. *Proposed by David L. Bock, University of Maryland, College Park, MD.*
Given an integer N, characterize the smallest square in the plane containing N lattice points.

10383. *Proposed by Kevin Ford (student), University of Illinois, Urbana, IL.*
Let $B_1, B_2, \ldots, B_s$ denote subsets of a finite set B, and let $\lambda_i = \#(B_i)/\#(B)$ and $\lambda = \lambda_1 + \cdots + \lambda_s$. Show that, for every integer t satisfying $1 \leq t \leq \lambda$, there exist $r_1, r_2, \ldots, r_t$ with $r_1 < r_2 < \cdots < r_t$ and

$$\#(b_{r_1} \cap b_{r_2} \cap \cdots \cap B_{r_t}) \geq (\lambda - t + 1)\binom{s}{t}^{-1} \#(B).$$

10384. *Proposed by Franklin Kemp, East Texas State University, Commerce, TX.*
Suppose $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$. Define the correlation coefficient r in the usual way:

$$r = \frac{\sum_i (x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_i (x_i - \overline{x})^2 \cdot \sum_i (y_i - \overline{y})^2}}$$

where $\overline{x}$ and $\overline{y}$ are the average values of the x_i and y_i, respectively, and the sums run from 1 to n. Show that $r \geq 1/(n-1)$.

10391. *Proposed by Emre Alkan (student), Bosphorus University, İstanbul, Turkey, and the editors.*
If $a_1, a_2, \ldots, a_n$ are real numbers with $a_1 \geq a_2 \geq \cdots \geq a_n$, and if ϕ is a convex function defined on the closed interval $[a_n, a_1]$, then

$$\sum_{k=1}^{n} \phi(a_k) a_{k+1} \geq \sum_{k=1}^{n} \phi(a_{k+1} a_k$$

with the convention that $a_{n+1} = a_1$.

10392. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*
Determine the extreme values of

$$\frac{1}{1+x+u}+\frac{1}{1+y+v}+\frac{1}{1+z+w}$$

where $xyz=a^3$, $uvw=b^3$, and $x,y,z,u,v,w>0$.

10400. *Proposed by Itshak Borosh, Douglas Hensley, and Arthur M. Hobbs, Texas A&M University College, College Station, TX, and Anthony Evans, Write State University, Dayton, OH.*
Determine the set of all pairs (n,t) of integers with $0\le t\le n$ and

$$\sum_{k=0}^{t}\binom{n}{k}<\frac{n^t}{t!}.$$

10404. *Proposed by Behzad Djafari Rouhani, Shahid Beheshti University and Islamic Azad University, Tehran, Iran.*
Let $x_1,x_2,\ldots$ be a sequence of real numbers such that

$$|x_i-x_j|\ge|x_{i+1}-x_{j+1}|$$

for all positive integers i,j with $|i-j|\le 2$. Prove that $\langle x_n/n\rangle$ converges to a finite limit as $n\to\infty$.

10413. *Proposed by Mirel Mocanu, University of Craiova, Craiova, Romania.*
Four disjoint (except for boundary points) equilateral triangles of sides a,b,c and d, are enclosed ina regular hexagon of unit side.
(a) Prove that $3a+b+c+d\le 4\sqrt{3}$.
(b) When is $3a+b+c+d=4\sqrt{3}$?
(c)* Prove or disprove that $a+b+c+d\le 2\sqrt{3}$.

10417. *Proposed by Răzvan Satnoianu, A. S. E., Bucharest, Romania.*
Given the acute triangle ABC, let h_a, h_b, and h_c denote the altitudes and s the semiperimeter. Show that

$$\sqrt{3}\max\{h_a,h_b,h_c\}\ge s.$$

10419. *Proposed by Bill Correll, Jr. (student), Denison University, Granville, OH.*
Let k be an integer greater than or equal to 3. Let $S(k)$ be the set of nonnegative real numbers x for which

$$\left\lfloor\frac{x+k-2}{k}\right\rfloor\left\lfloor\frac{x+k-1}{k-1}\right\rfloor+\left\lfloor\frac{x}{k}\right\rfloor=\left\lfloor\frac{x+k-2}{k-1}\right\rfloor\left\lfloor\frac{x+k-1}{k}\right\rfloor+\left\lfloor\frac{x}{k-1}\right\rfloor.$$

(a) Determine the largest integer in $S(k)$.
(b) Show that $S(k)$ is the union of a finite number of intervals with the sum of the lengths of those intervals equal to $(k^2-3k+6)/2$.

10421. *Proposed by Gigel Militaru, University of Bucharest, Bucharest, Romania.*
Let n be an integer, $n\ge 3$, and let $z_1,\ldots,z_n$ and $t_1,\ldots,t_n$ be complex numbers. Prove that there exists an integer i, $1\le i\le n$ with

$$4|z_it_i|\le\sum_{j=1}^{n}|z_it_j+z_jt_i|.$$

10422. *Proposed by Adam Fieldsteel, Wesleyan University, Middletown, CT.*
Let $f : [0, 1] \to \mathbb{R}$ be a C^1 strictly increasing function with $f(1) = L$, where L is the length of the graph of f.
(a) Show that $\int_0^1 f(x)\,\mathrm{d}x \geq \pi/4$.
(b) Show that $\int_0^1 f(x)\,\mathrm{d}x = \pi/4$ only if the graph of f is a quarter circle.

10709. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.*
Let X be a standard normal random variable, and choose $y > 0$. Show that

$$\mathrm{e}^{-ay} < \frac{Pr(a \leq X \leq a+y)}{Pr(a-y \leq X \leq a)} < \mathrm{e}^{-ay+(1/2)ay^3}$$

when $a > 0$. Show that the reversed inequalities hold when $a < 0$.

10713. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*
Given a triangle with angles $A \geq B \geq C$, let a, b, and c be the lengths of the corresponding opposite sides, let r be the radius of the inscribed circle, and let R be the radius of the circumscribed circle. Show that A is acute if and only if

$$R + r < \frac{b+c}{2}.$$

10716. *Proposed by Michael L. Catalano-Johnson and Danial Loeb, Daniel Wagner Associates, Malvern, PA.*
What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?

10725. *Proposed by Vasile Mihai, Toronto, ON, Canada.*
Fix a positive integer n. Given a permutation α of $\{1, 2, \ldots, n\}$, let

$$f(\alpha) = \sum_{i=1}^{n} (\alpha(i) - \alpha(i+1))^2,$$

where $\alpha(n + 1) = \alpha(1)$. Find the extreme values of $f(\alpha)$ as α ranges over all permutations of $\{1, 2, \ldots, n\}$.

10730. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*
Fix an integer $n \geq 2$. Determine the largest constant $C(n)$ such that

$$\sum_{1 \leq i < j \leq n} (x_j - x_i)^2 \geq C(n) \cdot \min_{1 \leq i < n} (x_{i+1} - x_i)^2$$

for all real numbers $x_1 < x_2 < \cdots < x_n$.

10944. *Proposed by Marcin Mazur, University of Illinois, Urbana, IL.*
Prove that if a, b, c are positive real numbers such that $abc \geq 2^9$, then

$$\frac{1}{\sqrt{1+a}} + \frac{1}{\sqrt{1+b}} + \frac{1}{\sqrt{1+c}} \geq \frac{3}{\sqrt{1+\sqrt[3]{abc}}}.$$

11055. *Proposed by Razvan Satnoianu, City University, London, U. K.*
Let ABC be an acute triangle, with semiperimeter p and with inscribed and circumscribed circles of radius r and R, respectively.

a) Show that ABC has a median of length at most $p/\sqrt{3}$.

b) Show that ABC has a median of length at most $R+r$.

c) Show that ABC has an altitude of length at least $R+r$.

11069. *Proposed by Péter Ivády, Budapest, Hungary.*
Show that for $0 < x < 1$

$$\frac{1-x^2}{1+x^2}\left[1+x^3(1-x)^3\right] < \frac{\sin \pi x}{\pi x}.$$

11075. *Proposed by Götz Trenkler, University of Dortmund, Dortmund, Germany.*
Let a, b, and c be complex numbers. Show that

$$\left|\sqrt{a^2+b^2+c^2}\right| \le \max\{|a|+|b|, |b|+|c|, |a|+|c|\}.$$

to be continued ...

Inequalities proposed in
"The Mathematical Gazette"

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(An asterisk (★) after a number indicates that a problem was proposed without a solution.)

87.C. *Proposed by Nick Lord.*
Find the smallest value of α for which

$$\frac{1}{27} - xyz \leq \alpha \left[\frac{1}{3} - (xy + yz + zx)\right]$$

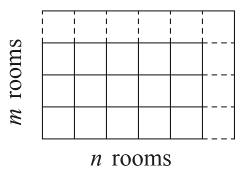
holds for all non-negative x, y, z satisfying $x + y + z = 1$.
(That $\alpha = \frac{7}{9}$ works in teh substance of BMO2 (1999) qn. 3.)

87.I. *Proposed by Michel Bataille.*
Let A, B, C and D be distinct points on a circle with radius r. Show that

$$AB^2 + BC^2 + CD^2 + DA^2 + AC^2 + BD^2 \leq 16r^2.$$

When does equality occur?

88.D. *Proposed by H. A. Shah Ali.*
Consider the $m \times n$ rectangular plan of rooms shown in the diagram: on each inner wall there could be a door. What is the minimum number of inner doors needed to allow entry into every room?



88.F. *Proposed by D. Mărghidanu.*
Let a, b, c, d be real numbers strictly between 0 and 1. Prove the inequality:

$$\left(\frac{a+b}{2}\right)^{\frac{(c+d)}{2}} + \left(\frac{b+c}{2}\right)^{\frac{(d+a)}{2}} + \left(\frac{c+d}{2}\right)^{\frac{(a+b)}{2}} + \left(\frac{d+a}{2}\right)^{\frac{(b+c)}{2}} > 2.$$

88.J. *Proposed by Péter Ivády.*
Show that, for $0 < x < \frac{\pi}{4}$ and $0 < y < \frac{\pi}{4}$, the following inequality holds:

$$cos(x - y) \leq \frac{4 \cos x \cos y}{(\cos x + \cos y)^2}.$$

to be continued ...

Inequalities proposed in
“Die $\sqrt{\textit{WURZEL}}$”

Last update: September 1, 2004

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ζ11 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Given the function

$$F(x) = \sin 3x \sin^3 x + \cos 3x \cos^3 x - \frac{3}{4} \cos 2x,$$

prove that

$$-\frac{1}{4} \leq F(x) \leq \frac{1}{4}$$

for all real x.

ζ13 *Proposed by Michael Möbius, Sulzbach, Germany*
Let a, b, c, d be real numbers satisfying $a^2 + b^2 \leq 1$ and $c^2 + d^2 \leq 1$. Prove that

$$\sqrt{(a+c)^2 + (b+d)^2} + \sqrt{(a-c)^2 + (b-d)^2} \leq 2\sqrt{2}.$$

When does equality hold?

ζ21 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*
Prove that for all real numbers x, y with $xy > 0$ the inequality

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy} + \frac{x+y}{2}$$

holds. When does exactly equality hold?

ζ23 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Let R, r be the circumradius and inradius, respectively, in a right-angled triangle with hypotenuse c and legs a, b. Find the maximum of the value $\frac{r}{R}$.

ζ37 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Let ABC a triangle with sides a, b, c and altitudes h_a, h_b, h_c. Prove the inequality

$$h_a^2 + h_b^2 + h_c^2 \leq \frac{3}{4}(a^2 + b^2 + c^2).$$

When does equality hold?

ζ38 *Proposed by Michael Heerdegen, Apolda, Germany*
Prove that

$$\sum_{i=0}^{n} \frac{(-1)^{n-i} \cdot 2^{i+1} \cdot \binom{n}{i}}{i+1} \geq 0$$

for all natural numbers n. When does equality hold?

ζ39 *Proposed by Zdravko F. Starc, Vršac, Yugoslavia*
Let a, b and c be positive real numbers. Prove that

$$\frac{a^5+b^5+a^2+b^2}{(a+b)(a^2+b^2)+1}+\frac{b^5+c^5+b^2+c^2}{(b+c)(b^2+c^2)+1}+\frac{c^5+a^5+c^2+a^2}{(c+a)(c^2+a^2)+1}<2\,(a^2+b^2+c^2).$$

η41 *Proposed by Hans Rudolf Moser, Bürglen, Switzerland*
The bottom face of a pyramid is a regular n-gon and its edges have all the same constant length s. Prove that the height of such a pyramid with maximum volume is independent of n.

ζ44 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Prove that

$$(y^3+x)(z^3+y)(x^3+z)\geq 125xyz,$$

where $x\geq 2$, $y\geq 2$, $z\geq 2$ are real numbers.

ζ47 *Proposed by Thomas Fischer, Jena, Germany*
For all integers $n\geq 2$ prove that

$$n^{2n}<(n-1)^{n-1}\cdot(n+1)^{n+1}.$$

ζ56 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*
Let n be a positive integer and $a_1,\ldots,a_n$ positive real numbers with $a_1+\cdots+a_n=1$. Prove that

$$\prod_{k=1}^{n}\left(n-2+\frac{1}{a_k}\right)\geq(2n-2)^n.$$

When does equality hold?

ζ58 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
For $n\in\mathbb{N}$ prove that

$$n^n\leq n!\cdot \mathrm{e}^{n-1}.$$

η44 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Prove that, in an isosceles triangle ABC with $AC=BC=a$, $AB=c$, and the angle-bisector $AD=w$, the inequalities

$$\frac{2ac}{a+c}>w>\frac{ac}{a+c}\sqrt{2}$$

hold.

η45 *Proposed by Prof. Walther Janous, Innsbruck, Austria*
Let x,y,z be nonnegative real numbers with $x+y+z=1$. Prove that

$$(1-x^2)^2+(1-y^2)^2+(1-z^2)^2\leq(1+x)(1+y)(1+z).$$

When does equality hold?

η48 *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany*
Let $f:[a,b]\to\mathbb{R}$ be a twice continuously differentiable and strictly convex function. Furthermore, $\int_a^b f(x)\,\mathrm{d}x=0$. Prove that

$$\frac{\left(f^2(b)-f^2(a)\right)^2}{4(f'(b)-f'(a))}<\int\limits_a^b f^3(x)\,\mathrm{d}x.$$

η49 *Proposed by Dr. Roland Mildner, Leipzig, Germany*
In a Cartesian coordinate system a circle K_1 (radius $2a$, centre $M_1(0,a)$) and a circle K_2 (radius a, centre $M_2(2a,0)$) are drawn with $a > 0$. Determine the smallest value of a such that the coordinates of the intersection points of K_1 and K_2 are integers.

η50 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Prove that

$$\sqrt{\frac{a+b}{c}}+\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}} \geq 3\sqrt{2},$$

where a, b, c are positive real numbers.

η51 *Proposed by Hans Rudolf Moser, Bürglen, Switzerland*
Given the linear system of equations in variables x, y, z with parameter p

$$px + y + z = p + 1$$

$$x + py + z = p$$

$$x + y + pz = p - 1.$$

For which values of p the solutions satisfy the inequalities $x < y < z$? When $x > y > z$ holds?

η52 *Proposed by Oleg Faynshteyn, Leipzig, Germany*
A sphere is inscribed in a (nonregular) tetrahedron with surface area A. Let ϵ be a plane parallel to one of the faces which touches the sphere. Determine the maximal area of a triangle that is formed by the intersection of ϵ and the tetrahedron.

η57 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers with $a_1 = 0$, $|a_2| = |a_1 + 1|$, $|a_3| = |a_2 + 1|$, $\ldots$, $|a_n| = |a_{n-1} + 1|$. Prove that, for each $n \in \mathbb{N}$,

$$\frac{a_1 + a_2 + \cdots a_n}{n} \geq -\frac{1}{2}.$$

ι7 *Proposed by Astrid Baumann, Friedberg, Germany*
Prove the inequality

$$(1 + x^n)(1 + x)^n \geq (1 + x^2)^n + 2^n x^n$$

for all $n \in \mathbb{N}$ and $x \geq 0$. In which cases equality holds?

κ47 *Proposed by Prof. Walther Janous, Innsbruck, Austria*
Prove that

$$y^4 + z^4 + 3 \geq y + z + 3 \cdot \frac{3yz+1}{4} \cdot \sqrt[3]{\frac{3yz+1}{4}}$$

for all real x, y.

λ31 *Proposed by Prof. Šefket Arslanagić, Sarajevo, Bosnia and Herzegovina*
Prove that $a^3 + b^3 + c^3 \geq 3abc$ for any $a, b, c \geq 0$.

λ32 *Proposed by after Mihály Bencze, Kronstadt, Romania*
Prove the inequality

$$\sum_{k=1}^{n} \frac{1}{k^3+1} < \frac{n}{2}\left(\frac{n+1}{n} - \sqrt[n]{\frac{2\,(n^2+n+1)}{3\,(n^2+n)}}\right).$$

λ34 *Proposed by Dr. Roland Mildner, Leipzig, Germany*
A buoy-similar solid consists of a circular cylinder (diameter d, height H), a circular cone (diameter d, height h) on the top of the cylinder and a half sphere (diameter d) on the bottom of the cylinder. How must the values of d, h and H be choosen to get the least surface area with given fixed volume V of the solid?

to be continued ...

Inequalities proposed in
"Elemente der Mathematik"

Last update: October 8, 2004

Please visit `http://www.birkhauser.ch/journals/1700/1700_tit.htm`

830. *Proposed by S. Gabler, Mannheim, BRD.*
Let $x_1, x_2, \dots, x_n$ $(n \geq 3)$ be positive real numbers satisfying $x_1 + x_2 + \cdots + x_n = 1$. Then

$$\binom{n}{2}^{-1} \sum_{1 \leq i < j \leq n} \frac{x_i}{1-x_i} \frac{x_j}{1-x_j} \geq \frac{1}{(n-1)^2}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n = \frac{1}{n}$. Prove this.

845. *Proposed by W. Janous, Innsbruck, A.*
For non-negative reals $x_1, \dots, x_n$ satisfying $x_1 + \cdots + x_n = s$ $(n \in \mathbb{N})$ prove that

$$n - \frac{ns}{s+n} \leq \sum_{i=1}^{n} \frac{1}{1+x_i} \leq n - \frac{s}{s+1}.$$

When does exactly equality occur?

846. *Proposed by P. Erdös.*
Let

$$n - 1 \leq k \leq \binom{n}{2}, \quad n, k \in \mathbb{N}.$$

Then always exist n points along the x-axis with $x_1 < x_2 < \cdots < x_n$, that determine exactly k different distances $x_i - x_j$ $(i > j)$. Prove this.

849. *Proposed by M. Bencze, Brasov, Romania.*
For natural numbers n prove that

$$\exp \frac{n(n-1)}{2} \leq 1^1 \cdot 2^2 \cdots n^n \leq \exp \frac{n(n-1)(2n+5)}{12}.$$

1084. *Proposed by Walther Janous, Innsbruck, A.*
Let a, b, λ be real numbers such that $\lambda > 0$ and $b - a \geq \pi/\sqrt{\lambda}$. The function $f : [a, b] \to \mathbb{R}$ is contigouos differentiable. Prove the existence of $t \in (a, b)$ such that $f'(t) < \lambda + f^2(t)$.

1091. *Proposed by Hansjürg Stocker, Wädenswil, CH; Jany Binz, Bolligen, CH.*
How many non-decreasing sequences of natural numbers with length $n \cdot p$ $(n, p \in \mathbb{N})$ exist which members a_k $(k, a_k \in \mathbb{N})$ satisfy the constraints

$$a_{pn} = a_{pn-1} = a_{pn-2} = \cdots = a_{pn-p+1} = n$$

and

$$a_{pi+1} \geq a_{pi} \geq a_{pi-1} \geq \cdots \geq a_{pi-p+1} \geq i \quad (i = n-1, n-2, \dots, 1)?$$

1094. *Proposed by R. Bil, Kiel, D.*
Prove that for all natural numbers n

$$\left(\frac{n+1}{n}\right)^{\left(\frac{\sqrt[4]{n}+\sqrt[4]{n+1}}{2}\right)^4} < \mathrm{e} < \left(\frac{n+1}{n}\right)^{\left(\frac{\sqrt[3]{n}+\sqrt[3]{n+1}}{2}\right)^3}$$

(e is as usual the Euler number.)

1126. *Proposed by Rolf Rose, Magglingen, CH.*
The sum of the surface areas of two solids with given shape is constant. Prove that the volumes of these solids are proportional to their surface areas if the sum of the volumes is a minimum. Furthermore, calculate this ratio of surface area to volume of two arbitrary solids with the same shape und determine this value if one solid is a cube and the other a regular tetrahedron.

1128. *Proposed by Wolfgang Moldenhauer, Erfurt, D.*
Let p be a polynomial of degree ≤ 3 and q a polynomial of degree ≤ 5 with

$$\begin{aligned} p(0) &= q(0), & p(1) &= q(1), \\ p'(0) &= q'(0), & p'(1) &= q'(1), \\ q''(0) &= 0, & q''(1) &= 0. \end{aligned}$$

Determine a constant $C > 0$ such that for all pairs (p, q) the inequality

$$\int_0^1 p(t)q(t)\,\mathrm{d}t \geq C \cdot \int_0^1 (p(t))^2\,\mathrm{d}t$$

holds.

1146. *Proposed by Šefket Arslanagić, Sarajevo, BIH.*
Prove or disprove: In each convex pentagon there are three diagonals from which one can construct a triangle.

1147. *Proposed by Zdravko F. Starc, Vršac, YU.*
Prove the following inequalities:

$$1^1 \cdot 2^2 \cdots n^n \leq 1! \cdot 2! \cdots n! \cdot \exp\left(\frac{n(n-1)}{2}\right), \tag{1}$$

$$f_1^{f_1} \cdot f_2^{f_2} \cdots f_n^{f_n} \leq f_1! \cdot f_2! \cdots f_n! \cdot \exp(f_{n+2} - n - 1). \tag{2}$$

Here f_n denotes the Fibonacci numbers: $f_1 = f_2 = 1, f_{n+2} = f_{n+1} + f_n$ for $n = 1, 2, 3, \ldots$.

1157. *Proposed by Roland Wyss, Flumenthal, CH.*
Given an ellipse with the equation $25x^2 + 9y^2 = 900$ and the points $O(0|0)$ and $C(1|0)$ on its minor axis. For which points P on the periphery is $\angle OPC$ a maximum?

1164. *Proposed by Jany C. Binz, Bolligen, CH.*
Three circles are inscribed in an isosceles triangle with base b, inradius ϱ and circumradius r: two of them with radius t touch each one of the legs, the base and the incircle; the other with radius u touches both legs and the incircle. Determine the smallest triangle such that ϱ is an integer multiple of t and b, ϱ, r, t, u are all integers.

1169. *Proposed by Péter Ivády, Budapest, H.*
Let $0 < x < \frac{\pi}{2}$. Prove the inequality

$$\left(\frac{2+\cos x}{3}\right)^3 < \left(\frac{\sin x}{x}\right)^2.$$

1174. *Proposed by Peter Hohler, Aarburg, CH.*
We consider sequences of $k > 2$ consecutive numbers:

$$n, n+1, n+2, \ldots, n+k-2, n+k-1.$$

Most of such sequences contain at least one number which is coprime to all other numbers of the sequence. Find the smallest sequence (that is, nk is minimum) with no number therein that is coprime to all other numbers of the sequence.

1190. *Proposed by Mihály Bencze, Sacele, RO.*
Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing differentiable function. Prove that

$$\sum_{l=1}^{n} \int_0^{nx_l-(x_1+x_2+\cdots+x_n)} f\left(x_l - \frac{t}{n-1}\right) \mathrm{d}t \geq 0,$$

where $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ are arbitrary real numbers.

1198. *Proposed by Götz Trenkler, Dortmund, D.*
Let a, b, c and d be complex numbers. Prove that

$$\sqrt{|ab+cd|} \leq \max\{|a|, |b|+|c|, |d|\}.$$

1200. *Proposed by Matthias Müller, Bad Saulgau, D.*
A "Ulam sequence" is defined recursively as follows: Two natural numbers u_1, u_2 are given with $u_1 < u_2$. For $n \geq 3$, let u_n be the smallest integer that is greater than u_{n-1} and that can be represented in the form $u_n = u_k + u_l$ with $0 < k < l < n$ exactly once. Let x_N be the number of terms of these Ulam sequence which are less than or equal to N.
Prove:

$$\lim \sup_{N\to\infty} \frac{x_N}{N} \leq \frac{1}{2}.$$

1201. *Proposed by Mihály Bencze, Sacele, RO.*
Let $1 \leq a < b$. Prove the following inequalities:

(a) $\left(\cos \frac{x}{\sqrt{a}}\right)^a < \left(\cos \frac{x}{\sqrt{b}}\right)^b$ for $0 < x < \frac{\pi}{2}$,

(b) $\left(\cos \frac{x}{\sqrt[3]{a}}\right)^a > \left(\cos \frac{x}{\sqrt[3]{b}}\right)^b$ for sufficient small positive x.

1205. *Proposed by Roland Wyss, Flumenthal, CH.*
The following problem is well known from the classroom: "Which rectangle with fixed perimeter has maximum area?". This will be generalized as follows: From a rectangular plate with sides ax and y $(a > 1)$, $m \geq 0$ squares of side x and $n \geq 0$ discs with diameter x should be cut. How x and y must be selected to maximize the area of the rest piece while the perimeter u remains constant? Prove also that a non-overlapping cutting of these $m + n$ pieces is actually possible.

1207. *Proposed by Šefket Arslanagić, Sarajevo, BIH.*
Prove that for positive numbers x, y, z the following inequality holds:

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x+y+z}{\sqrt[3]{xyz}}.$$

to be continued ...

Inequalities proposed in
"Mathscope"

Last update: October 19, 2006

Please visit `http://www.imo.org.yu/othercomp/Journ/mathscope.pdf`

(An asterisk (★) after a number indicates that a problem was proposed without a solution.)

153.1. *Proposed by Nguyễn Đông Yên.*
Prove that if $y \geq y^3 + x^2 + |x| + 1$, then $x^2 + y^2 \geq 1$. Find all pairs (x, y) such that the first inequality holds while equality in the second one attains.

153.2. *Proposed by Tạ Văn Tự.*
Given natural numbers m, n, and a real number $a > 1$, prove the inequality

$$a^{\frac{2n}{m}} - 1 \geq n(a^{\frac{n+1}{m}} - a^{\frac{n-1}{m}}).$$

153.3. *Proposed by Nguyễn Minh Đức.*
Prove that for each $0 < \epsilon < 1$, there exists a natural number n_0 such that the coefficients of the polynomial

$$(x + y)^n(x^2 - (2 - \epsilon)xy + y^2)$$

are all positive for each natural number $n \geq n_0$.

200.2. *Proposed by Trần Xuân Đáng.*
Let $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$, prove that

$$15(a^3 + b^3 + c^3 + ab + bc + ca) + 9abc \geq 7.$$

200.3. *Proposed by Đặng Hùng Thắng.*
Let a, b, c be integers such that the quadratic function $ax^2 + bx + c$ has two distinct zeros in the interval $(0, 1)$. Find the least value of a, b, and c.

200.5. *Proposed by Nguyễn Văn Mậu.*
Let $x, y, z, t \in [1, 2]$, find the smallest possible $p > 0$ such that the inequality

$$\frac{y+t}{x+z} + \frac{z+t}{t+x} \leq p\left(\frac{y+z}{x+y} + \frac{x+z}{y+t}\right)$$

holds.

200.6. *Proposed by Nguyễn Minh Hà.*
Let a, b, c be real positive numbers such that $a + b + c = \pi$, prove that

$$\sin a + \sin b + \sin c + \sin(a + b + c) \leq \sin(a + b) + \sin(b + c) + \sin(c + a).$$

208.1. *Proposed by Đặng Hùng Thắng.*
Let $a_1, a_2, \ldots, a_n$ be the odd numbers, none of which has a prime divisors greater than 5, prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < \frac{15}{8}.$$

213.1. *Proposed by Hồ Quang Vinh.*
Let a, b, c be positive real numbers such that $a+b+c=2r$, prove that

$$\frac{ab}{r-c}+\frac{bc}{r-a}+\frac{ca}{r-b}\geq 4r.$$

213.3. *Proposed by Nguyễn Minh Đức.*
Given three sequences of numbers $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$ such that x_0, y_0, z_0 are positive,

$$x_{n+1}=y_n+\frac{1}{z_n},\quad y_{n+1}=z_n+\frac{1}{x_n},\quad z_{n+1}=x_n+\frac{1}{y_n}\quad \text{for all } n\geq 0.$$

Prove that there exist positive numbers s and t such that $s\sqrt{n}\leq x_n\leq t\sqrt{n}$ for all $n\geq 1$.

216.3. *Proposed by Nguyễn Đễ.*
Prove that if $-1<a<1$ then

$$\sqrt[4]{1-a^2}+\sqrt[4]{1-a}+\sqrt[4]{1+a}<3.$$

216.5. *Proposed by Hoàng Đức Tân.*
Let P be any point interior to triangle ABC, let d_A, d_B, d_C be the distances of P to the vertices A, B, C respectively. Denote by p, q, r distances of P to the sides of the triangle. Prove that

$$d_A^2\sin^2 A+d_B^2\sin^2 B+d_C^2\sin^2 C\leq 3(p^2+q^2+r^2).$$

220.2. *Proposed by Phạm Ngọc Quang.*
Find triples of three non-negative integers (x, y, z) such that

$$3x^2+54=2y^2+4z^2,\quad 5x^2+74=3y^2+7z^2,$$

and $x+y+z$ is a minimum.

220.5. *Proposed by Phạm Hiến Bằng.*
In a triangle ABC, denote by l_a, l_b, l_c the internal angle bisectors, m_a, m_b, m_c the medians, and h_a, h_b, h_c the altitudes to the sides a, b, c of the triangle. Prove that

$$\frac{m_a}{l_b+h_b}+\frac{m_b}{l_c+h_c}+\frac{m_c}{l_a+h_a}\geq\frac{3}{2}.$$

221.1. *Proposed by Ngô Hân.*
Find the greatest possible natural number n such that 1995 is equal to the sum of n numbers $a_1, a_2, \ldots, a_n$, where a_i, $(i=1, 2, \ldots, n)$ are composite numbers.

221.5. *Proposed by Nguyễn Lê Dũng.*
Prove that if $a, b, c>0$ then

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\leq\frac{3(a^2+b^2+c^2)}{a+b+c}.$$

230.1. *Proposed by Trần Nam Dũng.*
Let $m\in\mathbb{N}$, $m\geq 2$, $p\in\mathbb{R}$, $0<p<1$. Let $a_1, a_2, \ldots, a_m$ be real positive numbers. Put $s=\sum\limits_{i=1}^{m}a_i$. Prove that

$$\sum_{i=1}^{m}\left(\frac{a_i}{s-a_i}\right)^p\geq\frac{1}{1-p}\left(\frac{1-p}{p}\right)^p,$$

with equality if and only if $a_1=a_2=\cdots=a_m$ and $m(1-p)=1$.

235.3. *Proposed by Đàm Văn Nhỉ.*
Find the maximum value of

$$\frac{a}{bcd+1}+\frac{b}{cda+1}+\frac{c}{dab+1}+\frac{d}{abc+1},$$

where $a, b, c, d \in [0, 1]$.

235.4. *Proposed by Trần Nam Dũng.*
Let M be any point in the plane of an equilateral triangle ABC. Denote by x, y, z the distances from P to the vertices and p, q, r the distances from M to the sides of the triangle. Prove that

$$p^2+q^2+r^2 \geq \frac{1}{4}(x^2+y^2+z^2),$$

and that this inequality characterizes all equilateral triangles in the sense that we can always choose a point M in the plane of a non-equilateral triangle such that the inequality is not true.

241.1. *Proposed by Nguyễn Khánh Trình, Trần Xuân Đáng.*
Prove that in any acute triangle ABC, we have the inequality

$$\sin A \sin B+\sin B \sin C+\sin C \sin A \leq (\cos A+\cos B+\cos C)^2.$$

241.2. *Proposed by Trần Nam Dũng.*
Given n real numbers $x_1, x_2, \ldots, x_n$ in the interval $[0, 1]$, prove that

$$\left[\frac{n}{2}\right] \geq x_1(1-x_2)+x_2(1-x_3)+\cdots+x_{n-1}(1-x_n)+x_n(1-x_1).$$

241.3. *Proposed by Trần Xuân Đáng.*
Prove that in any acute triangle ABC

$$\sin A \sin B+\sin B \sin C+\sin C \sin A \geq (1+\sqrt{2 \cos A \cos B \cos C})^2.$$

242.1. *Proposed by Phạm Hữu Hoài.*
Let α, β, γ real numbers such that $\alpha \leq \beta \leq \gamma$, $\alpha < \beta$. Let $a, b, c \in [\alpha, \beta]$ sucht that $a+b+c = \alpha+\beta+\gamma$. Prove that

$$a^2+b^2+c^2 \leq \alpha^2+\beta^2+\gamma^2.$$

242.2. *Proposed by Lê Văn Bảo.*
Let p and q be the perimeter and area of a rectangle, prove that

$$p \geq \frac{32q}{2q+p+2}.$$

242.3. *Proposed by Tô Xuân Hải.*
In triangle ABC with one angle exceeding $\frac{2}{3}\pi$, prove that

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2} \geq 4-\sqrt{3}.$$

243.2. *Proposed by Trần Nam Dũng.*
Given $2n$ real numbers $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$, suppose that $\sum\limits_{j=1}^{n} a_j \neq 0$ and $\sum\limits_{j=1}^{n} b_j \neq 0$. Prove the following inequality

$$\sum_{j=1}^{n} a_j b_j + \sqrt{\left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right)} \geq \frac{2}{n}\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} b_j\right),$$

with equaltiy if and only if

$$\frac{a_i}{\sum\limits_{j=1}^{n} a_j} + \frac{b_i}{\sum\limits_{j=1}^{n} b_j} = \frac{2}{n}, \quad i = 1, 2, \ldots, n.$$

243.5. *Proposed by Huỳnh Minh Việt.*
Given real numbers x, y, z such that $x^2 + y^2 + z^2 = k$, $k > 0$, prove the inequality

$$\frac{2}{k}xyz - \sqrt{2k} \leq x + y + z \leq \frac{2}{k}xyz + \sqrt{2k}.$$

244.1. *Proposed by Thái Viết Bảo.*
Given a triangle ABC, let D and E be points on the sides AB and AC, respectively. Points M, N are chosen on the line segment DE such that $DM = MN = NE$. Let BC intersect the rays AM and AN at P and Q, respectively. Prove that if $BP < PQ$, then $PQ < QC$.

244.2. *Proposed by Ngô Văn Thái.*
Prove that if $0 < a, b, c \leq 1$, then

$$\frac{1}{a+b+c} \geq \frac{1}{3} + (1-a)(1-b)(1-c).$$

244.3. *Proposed by Trần Chí Hòa.*
Given three positive real numbers x, y, z such that $xy + yz + zx + \frac{2}{a}xyz = a^2$, where a is a given positive number, find the maximum value of $c(a)$ such that the inequality

$$x + y + z \geq c(a)(xy + yz + zx)$$

holds.

248.1. *Proposed by Trần Văn Vương.*
Given three real numbers x, y, z such that $x \geq 4$, $y \geq 5$, $z \geq 6$ and $x^2 + y^2 + z^2 \geq 90$, prove that $x + y + z \geq 16$.

250.3. *Proposed by Nguyễn Khắc Minh.*
Consider the equation $f(x) = ax^2 + bx + c$ where $a < b$ and $f(x) \geq 0$ for all real x. Find the smallest possible value of

$$p = \frac{a+b+c}{b-a}.$$

250.5. *Proposed by Trần Nam Dũng.*
Prove that if $a, b, c > 0$ then

$$\frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2}\left(4 - \frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

250.6. *Proposed by Phạm Ngọc Quang.*
Given a positive integer m, show that there exist prime integers a, b such that the following conditions are simultaneously satisfied:

$$|a| \le m, \quad |b| \le m \quad \text{and} \quad 0 < a + b\sqrt{2} \le \frac{1+\sqrt{2}}{m+2}.$$

251.2. *Proposed by Nguyễn Thanh Hải.*
Given a cubic equation

$$x^3 - px^2 + qx - p = 0,$$

where $p, q \in \mathbb{R}^*$, prove that if the equation has only real roots, then the inequality

$$p \ge \left(\frac{1}{4} + \frac{\sqrt{2}}{8}\right)(q+3)$$

holds.

258.1. *Proposed by Đặng Hùng Thắng.*
Let a, b, c be positive integers such that

$$a^2 + b^2 = c^2(1 + ab),$$

prove that $a \ge c$ and $b \ge c$.

258.4. *Proposed by Đặng Kỳ Phong.*
Find all functions $f(x)$ that satisfy simultaneously the following conditions:

i) $f(x)$ is defined and continuous on $\mathbb{R}$;

ii) for each set of 1997 numbers $x_1, x_2, \dots, x_{1997}$ such that $x_1 < x_2 < \cdots < x_n$, the inequality

$$f(x_{999}) \ge \frac{1}{1996}\left(f(x_1) + f(x_2) + \cdots + f(x_{998}) + f(x_{1000}) + f(x_{1001}) + \cdots + f(x_{1997})\right).$$

holds.

259.2. *Proposed by Viên Ngọc Quang.*
Given four positive real numbers a, b, c and d such that the quartic equation $ax^4 - ax^3 + bx^2 - cx + d = 0$ has four roots in the interval $(0, \frac{1}{2})$, the roots not being necessarily distinct. Prove that

$$21a + 164c \ge 80b + 320d.$$

261.1. *Proposed by Hồ Quang Vinh.*
Given a triangle ABC, its internal angle bisectors BE and CF, and let M be any point on the line segment EF. Denote by S_A, S_B, and S_C the areas of triangles MBC, MCA, and MAB, respectively. Prove that

$$\frac{\sqrt{S_B} + \sqrt{S_C}}{\sqrt{S_A}} \le \sqrt{\frac{AC + AB}{BC}},$$

and determine when equality holds.

261.2. *Proposed by Editorial Board.*
Find the maximum value of the expression

$$A = 13\sqrt{x^2 - x^4} + 9\sqrt{x^2 + x^4} \qquad \text{for} \quad 0 \le x \le 1.$$

261.3. *Proposed by Editorial Board.*
The sequence (a_n), $n = 1, 2, 3, \ldots$, is defined by $a_1 > 0$, and $a_{n+1} = ca_n^2 + a_n$ for $n = 1, 2, 3, \ldots$, where c is a constant. Prove that

a) $a_n \ge \sqrt{c^{n-1} n^n a_1^{n+1}}$, and

b) $a_1 + a_2 + \cdots + a_n > n\left(na_1 - \dfrac{1}{c}\right)$ for $n \in \mathbb{N}$.

261.5. *Proposed by Vinh Competition.*
Prove that if $x, y, z > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ then the following inequality holds:

$$\left(1 - \frac{1}{1+x^2}\right)\left(1 - \frac{1}{1+y^2}\right)\left(1 - \frac{1}{1+z^2}\right) > \frac{1}{2}.$$

261.6. *Proposed by Đỗ Văn Đức.*
Given four real numbers x_1, x_2, x_3, x_4 such that $x_1+x_2+x_3+x_4 = 0$ and $|x_1|+|x_2|+|x_3|+|x_4| = 1$, find the maximum value of $\prod\limits_{1 \le i < j \le 4} (x_i - x_j)$.

262.1. *Proposed by Ngô Văn Hiệp.*
Let ABC an equilateral triangle of side length a. For each point M in the interior of the triangle, choose points D, E, F on the sides CA, AB, and BC, respectively, such that $DE = MA$, $EF = MB$, and $FD = MC$. Determine M such that $\triangle DEF$ has smallest possible area and calculate this area in terms of a.

264.1. *Proposed by Trần Duy Hinh.*
Prove that the sum of all squares of the divisors of a natural number n is less than $n^2\sqrt{n}$.

264.2. *Proposed by Hoàng Ngọc Cảnh.*
Given two polynomials

$$f(x) = x^4 - (1 + \mathrm{e}^x) + \mathrm{e}^2, \qquad g(x) = x^4 - 1,$$

prove that for distinct positive numbers a, b satisfying $a^b = b^a$, we have $f(a)f(b) < 0$ and $g(a)g(b) > 0$.

264.4. *Proposed by Nguyễn Minh Phương, Nguyễn Xuân Hùng.*
Let I be the incenter of triangle ABC. Rays AI, BI, and CI meet the circumcircle of triangle ABC again at X, Y, and Z, respectively. Prove that

a) $IX + IY + IZ \ge IA + IB + IC$, b) $\dfrac{1}{IX} + \dfrac{1}{IY} + \dfrac{1}{IZ} \ge \dfrac{3}{R}$.

265.2. *Proposed by Đàm Văn Nhỉ.*
Let AD, BE, and CF be the internal angle bisectors of triangle ABC. Prove that

$$p(DEF) \le \frac{1}{2}p(ABC),$$

where $p(XYZ)$ denotes the perimeter of triangle XYZ. When does equality hold?

266.1. *Proposed by Lê Quang Nẫm.*
Given real numbers $x, y, z \geq -1$ satisfying $x^3 + y^3 + z^3 \geq x^2 + y^2 + z^2$, prove that

$$x^5 + y^5 + z^5 \geq x^2 + y^2 + z^2.$$

266.4. *Proposed by Lưu Xuân Tình.*
Let x, y be real numbers in the interval $(0, 1)$ and $x + y = 1$, find the minimum of $x^x + y^y$.

267.1. *Proposed by Đỗ Thanh Hân.*
Let x, y, z be real numbers such that

$$x^2 + z^2 = 1,$$
$$y^2 + 2y(x + z) = 6.$$

Prove that $y(z - x) \leq 4$, and determine when equality holds.

267.2. *Proposed by Vũ Ngọc Minh, Phạm Gia Vĩnh Anh.*
Let a, b be real positive numbers, x, y, z be real numbers such that

$$x^2 + z^2 = b,$$
$$y^2 + (a - b)y(z + x) = 2ab^2.$$

Prove that $y(z - x) \leq (a + b)b$ with equality if and only if

$$x = \pm \frac{a\sqrt{b}}{\sqrt{a^2 + b^2}}, \quad z = \mp \frac{b\sqrt{b}}{\sqrt{a^2 + b^2}}, \quad z = \mp \sqrt{b(a^2 + b^2)}.$$

267.4. *Proposed by Trần Nam Dũng.*
In triangle ABC, denote by a, b, c the side lengths, and F the area. Prove that

$$F \leq \frac{1}{16}(3a^2 + 2b^2 + 2c^2),$$

and determine when equality holds. Can we find another set of the coefficients of a^2, b^2, and c^2 for which equality holds?

268.1. *Proposed by Đỗ Kim Sơn.*
In a triangle, denote by a, b, c the side lengths, and let r, R be the inradius and circumradius, respectively. Prove that

$$a(b + c - a)^2 + b(c + a - b)^2 + c(a + b - c)^2 \leq 6\sqrt{3}R^2(2R - r).$$

272.3. *Proposed by Hồ Quang Vinh.*
Let M and m be the greatest and smallest numbers in the set of positive numbers $a_1, a_2, \ldots, a_n$, $n \geq 2$. Prove that

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \leq n^2 + \frac{n(n-1)}{2}\left(\sqrt{\frac{M}{m}} - \sqrt{\frac{m}{M}}\right)^2.$$

274.1. *Proposed by Đào Mạnh Thắng.*
Let p be the semiperimeter and R the circumradius of triangle ABC. Furthermore, let D, E, F be the excenters. Prove that

$$DE^2 + EF^2 + FD^2 \geq 8\sqrt{3}pR,$$

and determine the equality case.

274.4. *Proposed by Nguyễn Hào Liễu.*
Prove the inequality for $x \in \mathbb{R}$:

$$\frac{1+2x\arctan x}{2+\ln(1+x^2)^2} \geq \frac{1+\mathrm{e}^{\frac{x}{2}}}{3+\mathrm{e}^x}.$$

275.1. *Proposed by Trần Hồng Sơn.*
Let x, y, z be real numbers in the interval $[-2, 2]$, prove the inequality

$$2(x^6+y^6+z^6)-(x^4y^2+y^4z^2+z^4x^2) \leq 192.$$

276.1. *Proposed by Vũ Đức Cảnh.*
Find the maximum value of the expression

$$f = \frac{a^3+b^3+c^3}{abc},$$

where a, b, c are real numbers lying in the interval $[1, 2]$.

276.2. *Proposed by Hồ Quang Vinh.*
Given a triangle ABC with sides $BC = a$, $CA = b$, and $AB = c$. Let R and r be the circumradius and inradius of the triangle, respectively. Prove that

$$\frac{a^3+b^3+c^3}{abc} \geq 4 - \frac{2r}{R}.$$

276.3. *Proposed by Phạm Hoàng Hà.*
Given a triangle ABC, let P be a point on the side BC, let H, K be the orthogonal projections of P onto AB, AC respectively. Points M, N are chosen on AB, AC such that $PM \parallel AC$ and $PN \parallel AB$. Compare the areas of triangles PHK and PMN.

279.5. *Proposed by Vũ Đức Sơn.*
Find all positive integers n such that $n < t_n$, where t_n is the number of positive divisors of n^2.

279.6. *Proposed by Trần Nam Dũng.*
Find the maximum value of the expression

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2},$$

where x, y, z are real numbers satisfying the condition $x + y + z = 1$.

279.7. *Proposed by Hoàng Hoa Trại.*
Given are three concentric circles with center O, and radii $r_1 = 1$, $r_2 = \sqrt{2}$, and $r_3 = \sqrt{5}$. Let A, B, C be three non-collinear points lying respectively on these circles and let F be the area of triangle ABC. Prove that $F \leq 3$, and determine the side lengths of triangle ABC.

281.2. *Proposed by Hồ Quang Vinh.*
In a triangle ABC, let $BC = a$, $CA = b$, $AB = c$ be the sides, r, r_a, r_b, and r_c be the inradius and exradii. Prove that

$$\frac{abc}{r} \geq \frac{a^3}{r_a}+\frac{b^3}{r_b}+\frac{c^3}{r_c}.$$

285.2. *Proposed by Vũ Đức Cảnh.*
Prove that if $x, y \in \mathbb{R}^*$ then

$$\frac{2x^2+3y^2}{2x^3+3y^3}+\frac{2y^2+3x^2}{2y^3+3x^3} \leq \frac{4}{x+y}.$$

285.4. *Proposed by Trần Tuấn Anh.*
Let a, b, c be non-negative real numbers, determine all real numbers x such that the following inequality holds:

$$\begin{aligned}[][a^2+b^2+(x-1)c^2][a^2+c^2+(x-1)b^2][b^2+c^2+(x-1)a^2]\\ \le (a^2+xbc)(b^2+xac)(c^2+xab).\end{aligned}$$

285.5. *Proposed by Trương Cao Dũng.*
Let O and I be the circumcenter and incenter of a triangle ABC. Rays AI, BI, and CI meet the circumcircle at D, E, and F, respectively. Let R_a, R_b, and R_c be the radii of the escribed circles of $\triangle ABC$, and let R_d, R_e, and R_f be the radii of the escribed circles of triangle DEF. Prove that

$$R_a+R_b+R_c \le R_d+R_e+R_f.$$

286.3. *Proposed by Vũ Đình Hòa.*
In a convex hexagon, the segment joining two of its vertices, dividing the hexagon into two quadrilaterals is called a *principal* diagonal. Prove that in every convex hexagon, in which the length of each side is equal to 1, there exists a principal diagonal with length not greater than 2 and there exists a principal diagonal with length greater than $\sqrt{3}$.

286.4. *Proposed by Đỗ Bá Chủ.*
Prove that in any acute or right triangle ABC the following inequality holds:

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}+\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \ge \frac{10\sqrt{3}}{9}.$$

286.5. *Proposed by Trần Tuấn Điệp.*
In triangle ABC, no angle exceeding $\frac{\pi}{2}$, and each angle is greater than $\frac{\pi}{4}$. Prove that

$$\cot A+\cot B+\cot C+3\cot A\cot B\cot C \le 4(2-\sqrt{2}).$$

289.2. *Proposed by Hồ Quang Vinh.*
Given a convex quadrilateral $ABCD$, let M and N be the midpoints of AD and BC, respectively, P be the point of intersection of AN and BM, and Q the intersection point of DN and CM. Prove that

$$\frac{PA}{PN}+\frac{PB}{PM}+\frac{QC}{QM}+\frac{QD}{QN} \ge 4,$$

and determine when equality holds.

290.1. *Proposed by Nguyễn Song Minh.*
Given $x, y, z, t \in \mathbb{R}$ and real polynomial

$$F(x,y,z,t)=9(x^2y^2+y^2z^2+z^2t^2+t^2x^2)+6xz(y^2+t^2)-4xyzt.$$

a) Prove that the polynomial can be factored into the product of two quadratic polynomials.
b) Find the minimum value of the polynomial F if $xy+zt=1$.

290.3. *Proposed by Đỗ Ánh.*
Consider a triangle ABC and its incircle. The internal angle bisector AD and median AM intersect the incircle again at P and Q, respectively. Compare the lengths of DP and MQ.

291.2. *Proposed by Đỗ Thanh Hân.*
Given three real numbers x, y, z that satisfy the conditions $0 < x < y \leq z \leq 1$ and $3x+2y+z \leq 4$. Find the maximum value of the expression $3x^3 + 2y^2 + z^2$.

291.5. *Proposed by Nguyễn Văn Thông.*
Find the maximum value of the expression

$$x^2(y-z) + y^2(z-y) + z^2(1-z),$$

where x, y, z are real numbers such that $0 \leq x \leq y \leq z \leq 1$.

291.6. *Proposed by Vũ Thành Long.*
Given an acute-angled triangle ABC with side lengths a, b, c. Let R, r denote its circumradius and inradius, respectively, and F its area. Prove the inequality

$$ab + bc + ca \geq 2R^2 + 2Rr + \frac{8}{\sqrt{3}}F.$$

292.1. *Proposed by Thái Nhật Phượng, Trần Hà.*
Let x, y, z be positive numbers such that $xyz = 1$, prove the inequality

$$\frac{x^2}{x+y+y^3z} + \frac{y^2}{y+z+z^3x} + \frac{z^2}{z+x+x^3y} \leq 1.$$

294.3. *Proposed by Vũ Trí Đức.*
If a, b, c are positive real numbers such that $ab + bc + ca = 1$, find the minimum value of the expression $w(a^2 + b^2) + c^2$, where w is a positive real number.

294.5. *Proposed by Trương Ngọc Đắc.*
Let x, y, z be positive real numbers such that $x = \max\{x, y, z\}$, find the minimum value of

$$\frac{x}{y} + \sqrt{1 + \frac{y}{z}} + \sqrt[3]{1 + \frac{z}{x}}.$$

294.6. *Proposed by Phạm Hoàng Hà.*
The sequence (a_n), $n = 1, 2, 3, \ldots$, is defined by $a_n = \frac{1}{n^2(n+2)\sqrt{n+1}}$ for $n = 1, 2, 3, \ldots$. Prove that

$$a_1 + a_2 + \cdots + a_n < \frac{1}{2\sqrt{2}} \qquad \text{for} \quad n = 1, 2, 3, \ldots.$$

294.7. *Proposed by Vũ Huy Hoàng.*
Given are a circle O of radius R, and an odd natural number n. Find the positions of n points $A_1, A_2, \ldots, A_n$ on the circle such that the sum $A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n + A_nA_1$ is a minimum.

295.1. *Proposed by Hoàng Văn Đắc.*
Let $a, b, c, d \in \mathbb{R}$ such that $a + b + c + d = 1$, prove that

$$(a+c)(b+d) + 2(ac+bd) \leq \frac{1}{2}.$$

295.4. *Proposed by Bùi Thế Hùng.*
Let A, B be respectively the greatest and smallest numbers from the set of n positive numbers $x_1, x_2, \ldots, x_n$, $n \geq 2$. Prove that

$$A < \frac{(x_1 + x_2 + \cdots + x_n)^2}{x_1 + 2x_2 + \cdots + nx_n} < 2B.$$

295.5. *Proposed by Trần Tuấn Anh.*
Prove that if $x, y, z > 0$ then

a) $(x+y+z)^3(y+z-x)(z+x-y)(x+y-z) \leq 27x^3y^3z^3$,

b) $(x^2+y^2+z^2)(y+z-x)(z+x-y)(x+y-z) \leq xyz(yz+zx+xy)$,

c) $(x+y+z)\left[2(yz+zx+xy)-(x^2+y^2+z^2)\right] \leq 9xyz$.

296.1. *Proposed by Thới Ngọc Anh.*
Prove that

$$\frac{1}{6} < \frac{3-\underbrace{\sqrt{6+\sqrt{6+\cdots+\sqrt{6}}}}_{n \text{ times}}}{3-\underbrace{\sqrt{6+\sqrt{6+\cdots+\sqrt{6}}}}_{(n-1) \text{ times}}} < \frac{5}{27},$$

where there are n radical signs in the expression of the numerator and $n-1$ ones in the expression of the denominator.

296.3. *Proposed by Nguyễn Văn Hiến.*
Let $k, n \in \mathbb{N}$ such that $k < n$. Prove that

$$\frac{(n+1)^{n+1}}{(k+1)^{k+1}(n-k+1)^{n-k+1}} < \frac{n!}{k!\,(n-k)!} < \frac{n^n}{k^k(n-k+1)^{n-k}}.$$

300.1. *Proposed by Vũ Trí Đức.*
Find the maximum and minimum values of the expression $x\sqrt{1+y}+y\sqrt{1+x}$, where x, y are non-negative real numbers such that $x+y=1$.

300.4. *Proposed by Võ Giang Giai, Mạnh Tú.*
Prove that if $a, b, c, d, e \geq 0$ then

$$\frac{a+b+c+d+e}{5} \geq \sqrt[5]{abcde}+\frac{q}{20},$$

where $q = (\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{d})^2+(\sqrt{d}-\sqrt{e})^2$.

301.2. *Proposed by Nguyễn Thế Bình.*
Find the smallest value of the expression

$$\frac{2}{ab}+\frac{1}{a^2+b^2}+\frac{a^4+b^4}{2}$$

where a, b are real positive numbers such that $a+b=1$.

301.3. *Proposed by Đỗ Anh.*
Suppose that a, b, c are side lengths of a triangle and $0 \leq t \leq 1$. Prove that

$$\sqrt{\frac{a}{b+c-ta}}+\sqrt{\frac{b}{c+a-tb}}+\sqrt{\frac{c}{a+b-tc}} \geq 2\sqrt{1+t}.$$

301.5. *Proposed by Trần Xuân Đáng.*
Find the maximum value of $3(a+b+c)-22abc$, where $a, b, c \in \mathbb{R}$ such that $a^2+b^2+c^2=1$.

301.7. *Proposed by Lê Hào.*
A convex quadrilateral $ABCD$ is inscribed in a circle with center O, radius R. Let CD intersect AB at E, a line through E meets the lines AD and BC at P, Q. Prove that

$$\frac{1}{EP} + \frac{1}{EQ} \le \frac{2EO}{EO^2 - R^2},$$

and determine when equality holds.

306.1. *Proposed by Phan Thị Mùi.*
Prove that if $x, y, z > 0$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ then

$$(x + y - z - 1)(y + z - x - 1)(z + x - y - 1) \le 8.$$

306.2. *Proposed by Trần Tuấn Anh.*
Given an integer $m \ge 4$, find the maximum and minimum values of the expression $ab^{m-1} + a^{m-1}b$, where a, b are real numbers such that $a + b = 1$ and $0 \le a, b \le \frac{m-2}{m}$.

309.1. *Proposed by Vũ Hoàng Hiệp.*
Given a positive integer n, find the smallest possible $t = t(n)$ such that for all real numbers $x_1, x_2, \ldots, x_n$ we have

$$\sum_{k=1}^{n} (x_1 + x_2 + \cdots + x_k)^2 \le t(x_1^2 + x_2^2 + \cdots + x_n^2).$$

309.2. *Proposed by Lê Xuân Sơn.*
Given a triangle ABC, prove that

$$\sin A \cos B + \sin B \cos C + \sin C \cos A \le \frac{3\sqrt{3}}{4}.$$

311.1. *Proposed by Nguyễn Xuân Hùng.*
The chord PQ of the circumcircle of a triangle ABC meets its incircle at M and N. Prove that $PQ \ge 2MN$.

319.1. *Proposed by Dương Châu Dinh.*
Prove the inequality

$$x^2y + y^2z + z^2x \le x^3 + y^3 + z^3 \le 1 + \frac{1}{2}(x^4 + y^4 + z^4),$$

where x, y, z are real non-negative numbers such that $x + y + z = 2$.

319.3. *Proposed by Trần Việt Anh.*
Suppose that AD, BE and CF are the altitudes of an acute triangle ABC. Let M, N, and P be the intersection points of AD and EF, BE and FD, CF and DE respectively. Denote the area of triangle XYZ by $F[XYZ]$. Prove that

$$\frac{1}{F[ABC]} \le \frac{F[MNP]}{F^2[DEF]} \le \frac{1}{8\cos A\cos B\cos C \cdot F[ABC]}.$$

320.1. *Proposed by Nguyễn Quang Long.*
Find the maximum value of the function $f = \sqrt{4x - x^3} + \sqrt{x + x^3}$ for $0 \le x \le 2$.

320.3. *Proposed by Hồ Quang Vinh.*
Let R and r be the circumradius and inradius of triangle ABC; the incircle touches the sides of the triangle at three points which form a triangle of perimeter p. Suppose that q is the perimeter of triangle ABC. Prove that $r/R \le p/q \le \frac{1}{2}$.

321.1. *Proposed by Lê Thanh Hải.*
Prove that for all positive numbers a, b, c, d

a) $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a} \ge \dfrac{a+b+c}{\sqrt[3]{abc}}$;

b) $\dfrac{a^2}{b^2} + \dfrac{b^2}{c^2} + \dfrac{c^2}{d^2} + \dfrac{d^2}{a^2} \ge \dfrac{a+b+c+d}{\sqrt[4]{abcd}}$.

322.2. *Proposed by Trần Tuấn Anh.*
Prove the inequality

$$\frac{1}{2}\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{x_i}\right) \ge n - 1 + \frac{n}{\sum\limits_{i=1}^{n} x_i},$$

where x_i $(i = 1, 2, \ldots, n)$ are positive real numbers such that $\sum\limits_{i=1}^{n} x_i^2 = n$, with n as an integer, $n > 1$.

323.2. *Proposed by Nguyễn Thế Phiệt.*
Prove that for an acute triangle ABC,

$$\cos A + \cos B + \cos C + \frac{1}{3}(\cos 3B + \cos 3C) \ge \frac{5}{6}.$$

324.1. *Proposed by Trần Nam Dũng.*
Find the greatest possible real number c such that we can always choose a real number x which satisfies the inequality

$$\sin(mx) + \sin(nx) \ge c$$

for each pair of positive integers m and n.

327.1. *Proposed by Hoàng Trọng Hảo.*
Let $ABCD$ be a bicentric quadrilateral (i.e., it has a circumcircle of radius R and an incircle of radius r). Prove that $R \ge r\sqrt{2}$.

328.4. *Proposed by Hàn Ngọc Đức.*
Find all real numbers a such that there exists a positive real number k and functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the inequality

$$\frac{f(x) + f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + k|x - y|^a,$$

for all real numbers x, y.

328.5. *Proposed by Vũ Hoàng Hiệp.*
In space, let $A_1, A_2, \ldots, A_n$ be n distinct points. Prove that

$$\text{a)}\ \sum_{i=1}^{n} \angle A_i A_{i+1} A_{i+2} \geq \pi,$$

$$\text{b)}\ \sum_{i=1}^{n} \angle A_i Q A_{i+1} \leq (n-1)\pi,$$

where A_{n+i} is equal to A_i and Q is an arbitrary point distinct from $A_1, A_2, \ldots, A_n$.

329.1. *Proposed by Hoàng Ngọc Minh.*
Find the maximum value of the expression

$$(a-b)^4 + (b-c)^4 + (c-a)^4,$$

for any real numbers $1 \leq a, b, c \leq 2$.

331.2. *Proposed by Bùi Đình Thân.*
Given positive reals a, b, c, x, y, z such that

$$a + b + c = 4 \quad \text{and} \quad ax + by + cz = xyz,$$

show that $x + y + z > 4$.

331.7. *Proposed by Trần Tuấn Anh.*
Find all positive integers $n \geq 3$ such that the following inequality holds for all real numbers $a_1, a_2, \ldots, a_n$ (assume $a_{n+1} = a_1$):

$$\sum_{1 \leq i < j \leq n} (a_i - a_j)^2 \leq \left(\sum_{i=1}^{n} |a_i - a_{i+1}| \right)^2.$$

332.2. *Proposed by Nguyên Khánh Nguyên.*
Suppose that ABC is an isosceles triangle with $AB = AC$. On the line perpendicular to AC at C, let point D such that points B, D are on different sides of AC. Let K be the intersection point of the line perpendicular to AB at B and the line passing through the midpoint M of CD, perpendicular to AD. Compare the lengths of KB and KD.

332.5. *Proposed by Phạm Xuân Trinh.*
Show that if $a \geq 0$ then

$$\sqrt{a} + \sqrt[3]{a} + \sqrt[6]{a} \leq a + 2.$$

332.8. *Proposed by Phùng Văn Sử.*
Prove that for any real numbers a, b, c

$$(a^2+3)(b^2+3)(c^2+3) \geq \frac{4}{27}(3ab + 3bc + 3ca + abc)^2.$$

332.10. *Proposed by Hoàng Ngọc Cảnh.*
Let $A_1A_2\dots A_n$ be a n-gon inscribed in the unit circle; let M be a point on the minor arc A_1A_n. Prove that

$$\text{a) } MA_1 + MA_3 + \cdots + MA_{n-2} + MA_n < \frac{n}{\sqrt{2}} \quad \text{for } n \text{ odd;}$$

$$\text{b) } MA_1 + MA_3 + \cdots + MA_{n-3} + MA_{n-1} \le \frac{n}{\sqrt{2}} \quad \text{for } n \text{ even.}$$

When does equality hold?

334.3. *Proposed by Nguyễn Duy Liên.*
Find the smallest possible odd natural number n such that n^2 can be expressed as the sum of an odd number of consecutive perfect squares.

334.4. *Proposed by Phạm Việt Hải.*
Find all positive numbers a, b, c, d such that

$$\frac{a^2}{b+c} + \frac{b^2}{c+d} + \frac{c^2}{d+a} + \frac{d^2}{a+b} = 1 \quad \text{and}$$
$$a^2 + b^2 + c^2 + d^2 \ge 1.$$

334.5. *Proposed by Đào Quốc Dũng.*
The incircle of triangle ABC (incenter I) touches the sides BC, CA, and AB respectively at D, E, F. The line through A perpendicular to IA intersects lines DE, DF at M, N, respectively; the line through B perpendicular to IB intersect EF, ED at P, Q, respectively; the line through C perpendicular to IC intersect lines FD, FE at S, T, respectively. Prove the inequality

$$MN + PQ + ST \ge AB + BC + CA.$$

335.1. *Proposed by Vũ Tiến Việt.*
Prove that for all triangles ABC

$$\cos A + \cos B + \cos C \le 1 + \frac{1}{6}\left(\cos^2\frac{A-B}{2} + \cos^2\frac{B-C}{2} + \cos^2\frac{C-A}{2}\right).$$

335.2. *Proposed by Phan Đức Tuấn.*
In triangle ABC, let $BC = a$, $CA = b$, $AB = c$ and F be its area. Suppose that M, N, and P are points on the sides BC, CA, and AB, respectively. Prove that

$$ab \cdot MN^2 + bc \cdot NP^2 + ca \cdot PM^2 \ge 4F^2.$$

336.2. *Proposed by Phạm Văn Thuận.*
Given two positive real numbers a, b such that $a^2 + b^2 = 1$, prove that

$$\frac{1}{a} + \frac{1}{b} \ge 2\sqrt{2} + \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2.$$

336.3. *Proposed by Nguyễn Hồng Thanh.*
Let P be an arbitrary point in the interior of triangle ABC. Let $BC = a$, $CA = b$, $AB = c$. Denote by u, v and w the distances of P to the lines BC, CA, AB, respectively. Determine P such that the product uvw is a maximum and calculate this maximum in terms of a, b, c.

336.5. *Proposed by Hoàng Minh Dũng.*
Prove that in any triangle ABC the following inequalities hold:

a) $\cos A + \cos B + \cos C + \cot A + \cot B + \cot C \geq \dfrac{3}{2} + \sqrt{3}$;

b) $\sqrt{3}\,(\cos A + \cos B + \cos C) + \cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} \geq \dfrac{9\sqrt{3}}{2}$.

337.1. *Proposed by Nguyễn Thị Loan.*
Given four real numbers a, b, c, d such that $4a^2 + b^2 = 2$ and $c + d = 4$, determine the maximum value of the expression $f = 2ac + bd + cd$.

337.3. *Proposed by Trần Tuấn Anh.*
Determine the maximum value of the expression $f = (x - y)(y - z)(z - x)(x + y + z)$, where x, y, z lie in the interval $[0, 1]$.

337.4. *Proposed by Hàn Ngọc Đức.*
Let n, $n \geq 2$, be a natural number, a, b be positive real numbers such that $a < b$. Suppose that $x_1, x_2, \ldots, x_n$ are n real numbers in the interval $[a, b]$. Find the maximum value of the sum

$$\sum_{1 \leq i < j \leq n} (x_i - x_j)^2.$$

337.5. *Proposed by Lê Hoài Bắc.*
A line through the incenter of a triangle ABC intersects sides AB and AC at M and N, respectively. Show that

$$\frac{MB \cdot NC}{MA \cdot NA} \leq \frac{BC^2}{4\,AB \cdot AC}.$$

338.1. *Proposed by Phạm Thịnh.*
Show that if a, b, c, d, p, q are positive real numbers with $p \geq q$ then the following inequality holds:

$$\frac{a}{pb + qc} + \frac{b}{pc + qd} + \frac{c}{pd + qa} + \frac{d}{pa + qb} \geq \frac{4}{p + q}.$$

Is the inequality still true if $p < q$?

338.3. *Proposed by Trần Việt Anh.*
Determine the smallest possible positive integer n such that there exists a polynomial $p(x)$ of degree n with integer coefficients satisfying the conditions
a) $p(0) = 1,\ p(1) = 1$;
b) $p(m)$ divided by 2003 leaves remainders 0 or 1 for all integers $m > 0$.

339.1. *Proposed by Ngô Văn Khương.*
Given five positive real numbers a, b, c, d, e such that $a^2 + b^2 + c^2 + d^2 + e^2 \leq 1$, prove that

$$\frac{1}{1 + ab} + \frac{1}{1 + bc} + \frac{1}{1 + cd} + \frac{1}{1 + de} + \frac{1}{1 + ea} \geq \frac{25}{6}.$$

339.3. *Proposed by Trần Hồng Sơn.*
Let I be the incenter of triangle ABC and let m_a, m_b, m_c be the lengths of the medians from vertices A, B and C, respectively. Prove that

$$\frac{IA^2}{m_a^2}+\frac{IB^2}{m_b^2}+\frac{IC^2}{m_c^2}\le\frac{3}{4}.$$

339.4. *Proposed by Quách Văn Giang.*
Given three positive real numbers a, b, c such that $ab+bc+ca=1$. Prove that the minimum value of the expression $x^2+ry^2+tz^2$ is $2m$, where m is the root of the cubic equation

$$2x^3+(r+s+1)x^2-rs=0$$

in the interval $(0,\sqrt{rs})$. Find all primes r, s such that $2m$ is rational.

339.7. *Proposed by Nguyễn Xuân Hùng.*
In the plane, given a circle with center O and radius r. Let P be a fixed point inside the circle such that $OP=d>0$. The chords AB and CD through P make a fixed angle α, $(0^\circ<\alpha\le 90^\circ)$. Find the maximum and minimum value of the sum $AB+CD$ when both AB and CD vary, and determine the position of the two chords.

340.1. *Proposed by Phạm Hoàng Hà.*
Find the maximum value of the expression

$$\frac{x+y}{1+z}+\frac{y+z}{1+x}+\frac{z+x}{1+y},$$

where x, y, z are real numbers in the interval $[\frac{1}{2},1]$.

340.3. *Proposed by Trần Tuấn Anh.*
Let a, b, c be the side lengths of a triangle, and F its area, prove that

$$F\le\frac{\sqrt{3}}{4}(abc)^{\frac{2}{3}},$$

and determine equality cases.

340.4. *Proposed by Hàn Ngọc Đức.*
Given non-negative integers n, k, $n>1$ and let $\{a_1,a_2,\dots,a_n\}$ be n real numbers, prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{a_ia_j}{\binom{k+2}{k+i+j}}\ge 0.$$

340.5. *Proposed by Trần Minh Hiền.*
Does there exist a function $f:\mathbb{R}^*\to\mathbb{R}^*$ such that

$$f^2(x)\ge f(x+y)(f(x)+y)$$

for all positive real numbers x, y?

341.5. *Proposed by Nguyễn Vũ Lươngg.*
Prove that if $x, y, z>0$ then

a) $\displaystyle\sqrt{\frac{x}{y+2z}}+\sqrt{\frac{y}{x+2z}}+2\sqrt{\frac{z}{x+y+z}}>2,$

b) $\displaystyle\sqrt[3]{\frac{x}{y+2z}}+\sqrt[3]{\frac{y}{x+2z}}+2\sqrt[3]{\frac{z}{x+y+z}}>2.$

342.1. *Proposed by Trần Văn Hinh.*
Let ABC be an isosceles triangle with $\angle ABC = \angle ACB = 36°$. Point N is chosen on the angle bisector of $\angle ABC$ such that $\angle BCN = 12°$. Compare the length of CN and CA.

342.3. *Proposed by Trần Tuấn Anh.*
Show that if $a \geq 0$, then

$$\sqrt{9+a} \geq \sqrt{a} + \frac{2\sqrt{2}}{\sqrt{1+a}}.$$

When does equality hold?

342.6. *Proposed by Nguyễn Trọng Quân.*
Let r, R be the inradius, circumradius of a triangle ABC, respectively. Prove that

$$\cos A \cos B \cos C \leq \left(\frac{r}{R\sqrt{2}}\right)^2.$$

342.8. *Proposed by Đỗ Thanh Sơn.*
Suppose that a, b, c, d are positive real numbers such that $(bc - ad)^2 = 3(ac + bd)^2$. Prove that

$$\sqrt{(a-c)^2 + (b-d)^2} \geq \frac{1}{2}\left(\sqrt{a^2+b^2} + \sqrt{c^2+d^2}\right).$$

343.3. *Proposed by Cao Xuân Nam.*
Let $a, b > 1$ be real numbers such that $a + b \leq 4$, find the minimum value of the expression

$$F = \frac{a^4}{(b-1)^3} + \frac{b^4}{(a-1)^3}.$$

343.9. *Proposed by Phan Tuấn Cộng.*
For a triangle ABC, find A, B, and C such that $\sin^2 A + \sin^2 B - \sin^2 C$ is a minimum.

344.1. *Proposed by Vũ Hữu Chín.*
Let ABC be a right isosceles triangle with hypotenuse BC. Let M be the midpoint of BC, G be a point chosen on the side AB such that $AG = \frac{1}{3}AB$, E be the foot of the perpendicular from M on CG. Let MG intersect AC at D, compare DE and BC.

344.4. *Proposed by Tạ Hoàng Thông.*
Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$, find the greatest possible constant λ such that

$$ab^2 + bc^2 + ca^2 \geq \lambda(ab + bc + ca)^2.$$

344.5. *Proposed by Hàn Ngọc Đức.*
Let X be any point on the side AB of the parallelogram $ABCD$. A line through X parallel to AD intersects AC at M and intersects BD at N; XD meets AC at P and XC cuts BD at Q. Prove that

$$\frac{MP}{AC} + \frac{NQ}{BD} \geq \frac{1}{3}.$$

When does equality hold?

344.8. *Proposed by Trần Nguyên An.*
Let $\{f_n(x)\}$, $(n = 0, 1, 2, \ldots)$ be a sequence of functions defined on $[0, 1]$ such that

$$f_0(x) = 0, \text{ and } f_{n+1}(x) = f_n(x) + \frac{1}{2}\left[x - (f_n(x))^2\right] \text{ for } n = 0, 1, 2, \ldots.$$

Prove that

$$\frac{nx}{2 + n\sqrt{x}} \leq f_n(x) \leq \sqrt{x}, \quad \text{for } n \in \mathbb{N}, x \in [0, 1].$$

344.11. *Proposed by Vietnam 1991.*
Let A, B, C be angles of a triangle, find the minimum of

$$(1 + \cos^2 A)(1 + \cos^2 B)(1 + \cos^2 C).$$

344.12. *Proposed by Vietnam 1991.*
Let $x_1, x_2, \ldots, x_n$ be real numbers in the interval $[-1; 1]$, and $x_1 + x_2 + \cdots + x_n = n - 3$, prove that

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 \leq n - 1.$$

345.1. *Proposed by Trần Tuấn Anh.*
Let x, y be real numbers in the interval $[0, \frac{1}{\sqrt{2}}]$, find the maximum of

$$p = \frac{x}{1 + y^2} + \frac{y}{1 + x^2}.$$

345.2. *Proposed by Cù Huy Toàn.*
Prove that

$$\frac{3\sqrt{3}}{4} \leq \frac{yz}{x(1 + yz)} + \frac{zx}{y(1 + zx)} + \frac{xy}{z(1 + xy)} \leq \frac{1}{4}(x + y + z),$$

where x, y, z are positive real numbers such that $x + y + z = xyz$.

345.3. *Proposed by Hoàng Hải Dương.*
Points E and D are chosen on the sides AB, AC of triangle ABC such that $AE/EB = CD/DA$. Let M be the intersection of BD and CE. Locate E and D such that the area of triangle BMC is a maximum, and determine the area in terms of triangle ABC.

346.1. *Proposed by Đỗ Bá Chủ.*
Determine, with proof, the minimum of

$$(x^2 + 1)\sqrt{x^2 + 1} - x\sqrt{x^4 + 2x^2 + 5} + (x - 1)^2.$$

346.4. *Proposed by Bùi Đình Thân.*
Given quadratic trinomials of the form $f(x) = ax^2 + bx + c$, where a, b, c are integers and $a > 0$, having two distinct roots in the interval $(0, 1)$. Find all such quadratic trinomials and determine the one with the smallest possible leading coefficient.

346.5. *Proposed by Phạm Kim Hùng.*
Prove that

$$xy + yz + zx \geq 8(x^2 + y^2 + z^2)(x^2y^2 + y^2z^2 + z^2x^2),$$

where x, y, z are non-negative numbers such that $x + y + z = 1$.

346.6. *Proposed by Lam Son, Thanh Hoa.*
Let x, y, z be real numbers greater than 2 such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, prove that

$$(x-2)(y-2)(z-2) \leq 1.$$

346.7. *Proposed by Huỳnh Duy Thuỷ.*
Given a polynomial $f(x) = mx^2 + (n-p)x + m + n + p$ with m, n, p being real numbers such that $(m+n)(m+n+p) \leq 0$, prove that

$$\frac{n^2+p^2}{2} \geq 2m(m+n+p) + np.$$

346.8. *Proposed by Vũ Thái Lộc.*
The incircle (I) of a triangle $A_1A_2A_3$ with radius r touches the sides A_2A_3, A_3A_1, A_1A_2 respectively at M_1, M_2, M_3. Let (I_i) be the cirlce touching the sides A_iA_j, A_iA_k and externally touching (I) $(i, j, k \in \{1, 2, 3\},\ i \neq j \neq k \neq i)$. Let K_1, K_2, K_3 be the points of tangency of (I_1) with A_1A_2, of (I_2) with A_2A_3, of (I_3) with A_3A_1 respectively. Let $A_iA_i = a_i,\ A_iK_i = b_i$, $(i = 1, 2, 3)$, prove that

$$\frac{1}{r}\sum_{i=1}^{3}(a_i + b_i) \geq 2 + \sqrt{3}.$$

When does equality hold?

347.3. *Proposed by Trần Hồng Sơn.*
The quadratic equation $ax^2 + bx + c = 0$ has two roots in the interval $[0, 2]$. Find the maximum of

$$f = \frac{8a^2 - 6ab + b^2}{4a^2 - 2ab + ac}.$$

348.2. *Proposed by Tạ Hoàng Thông.*
Find the greatest value of the expression

$$p = 3(xy + yz + zx) - xyz,$$

where x, y, z are positive real numbers such that

$$x^3 + y^3 + z^3 = 3.$$

348.5. *Proposed by Trương Ngọc Bắc.*
Given n positive real numbers $a_1, a_2, \ldots, a_n$ such that

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} i(i+1), \quad \text{for } k = 1, 2, 3, \ldots, n,$$

prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq \frac{n}{n+1}.$$

349.1. *Proposed by Thái Viết Thảo.*
Prove that in every triangle ABC with sides a, b, c and area F, the following inequalities hold

a) $(ab + bc + ca)\sqrt{\dfrac{abc}{a^3 + b^3 + c^3}} \geq 4F,$

b) $8R(R - 2r) \geq (a - b)^2 + (b - c)^2 + (c - a)^2.$

349.3. *Proposed by Phạm Văn Thuận.*
Let a, b, c, d be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 1,$$

prove that

$$\frac{1}{1-ab} + \frac{1}{1-bc} + \frac{1}{1-cd} + \frac{1}{1-ca} + \frac{1}{1-bd} + \frac{1}{1-da} \leq 8.$$

350.1. *Proposed by Nguyễn Tiến Lâm.*
Consider the sum of n terms

$$S_n = 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+n},$$

for $n \in \mathbb{N}$. Find the least rational number r such that $S_n < r$, for all $n \in \mathbb{N}$.

350.2. *Proposed by Phạm Hoàng Hà.*
Find the greatest and the least values of

$$\sqrt{2x+1} + \sqrt{3y+1} + \sqrt{4z+1},$$

where x, y, z are nonegative real numbers such that $x + y + z = 4$.

350.7. *Proposed by Trần Tuấn Anh.*
Find the greatest and least values of

$$f = a(b-c)^3 + b(c-a)^3 + c(a-b)^3,$$

where a, b, c are nonegative real numbers such that $a + b + c = 1$.

350.8. *Proposed by Trần Minh Hiền.*
Let I and G be the incenter and centroid of triangle ABC. Let r_A, r_B, r_C be the circumradii of triangles IBC, ICA, and IAB, respectively; let R_A, R_B, R_C be the circumradii of triangles GBC, GCA, and GAB. Prove that

$$r_A + r_B + r_C \geq R_A + R_B + R_C.$$

351.1. *Proposed by Mạc Đăng Nghị.*
Prove that for all real numbers x, y, z

$$(x+y+z)^8 + (y+z-x)^8 + (z+x-y)^8 + (x+y-z)^8 \leq 2188(x^8 + y^8 + z^8).$$

351.6. *Proposed by Phạm Văn Thuận.*
Prove that if $a, b, c, d \geq 0$ such that $a + b + c + d = 1$, then

$$(a^2+b^2+c^2)(b^2+c^2+d^2)(c^2+d^2+a^2)(d^2+a^2+b^2) \leq \frac{1}{64}.$$

351.7. *Proposed by Trần Việt Anh.*
Prove that

$$(2n+1)^{n+1} \leq (2n+1)!!\pi^n$$

for all $n \in \mathbb{N}$, where $(2n+1)!!$ denotes the product of odd positive integers from 1 to $2n+1$.

352.1. *Proposed by Đỗ Văn Ta.*
Let a, b, c be positive real numbers such that $abc \geq 1$, prove that

$$\frac{a}{\sqrt{b+\sqrt{ac}}}+\frac{b}{\sqrt{c+\sqrt{ab}}}+\frac{c}{\sqrt{a+\sqrt{bc}}} \geq \frac{3}{\sqrt{2}}.$$

352.2. *Proposed by Vũ Anh Nam.*
Let $ABCD$ be a convex function, let E and F be the midpoints of AD, BC respectively. Denote by M the intersection of AF and BE, N the intersection of CE and DF. Find the minimum of

$$\frac{MA}{MF}+\frac{MB}{ME}+\frac{NC}{NE}+\frac{ND}{NF}.$$

352.6. *Proposed by Trần Minh Hiền.*
In triangle ABC with $AB = c$, $BC = a$, $CA = b$, let h_a, h_b, and h_c be the altitudes from vertices A, B, and C respectively. Let s be the semiperimeter of triangle ABC. Point X is chosen on side BC such that the inradii of triangles ABX, and ACX are equal, and denote this radius r_A; r_B, and r_C are defined similarly. Prove that

$$2(r_A + r_B + r_C) + s \leq h_a + h_b + h_c.$$

to be continued . . .

Inequalities proposed in
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1.1. *Practice Set 1–3.*

(a) If $a, b, c \geq 0$ and $(1 + a)(1 + b)(1 + c) = 8,$ prove that $abc \leq 1.$

(b) If $a, b, c \geq 1$ prove that $4(abc + 1) \geq (1 + a)(1 + b)(1 + c).$

2.1. *Practice Set 4–3.*
If $a, a'; b, b';$ and c, c' are the lengths of the three pairs of opposite edges of an arbitrary tetrahedron, prove that

(i) there exists a triangle whose sides have lengths $a + a'$, $b + b'$, and $c + c'$;

(ii) the triangle in (i) is acute.

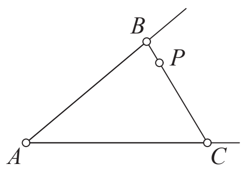
3.1. *Practice Set 5–1.*
A pack of 13 distinct cards is shuffled in some particular manner and then repeatedly in exactly the same manner. What is the maximum number of shuffles required for the cards to return to their original position?

4.1. *Practice Set 6–3.*
If $x, y, z \geq 0$, prove that

$$x^3 + y^3 + z^3 \geq y^2z + z^2x + x^2y$$

and determine when there is equality.

5.1. *The Eigth U.S.A. Mathematical Olympiad (May 1979), problem 4.*
Show how to construct a chord BPC of a given angle A, through a given point P within the angle A, such that $1/BP + 1/PC$ is a maximum.



6.1. *Eleventh Canadian Mathematics Olympiad (1979), problem 1.*
Given: (i) $a, b > 0$; (ii) a, A_1, A_2, b is an arithmetic progression; (iii) a, G_1, G_2, b is a geometric progression. Show that

$$A_1A_2 \geq G_1G_2.$$

6.2. *Eleventh Canadian Mathematics Olympiad (1979), problem 3.*
Let a, b, c, d, e be integers such that $1 \leq a < b < c < d < e$. Prove that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \leq \frac{15}{16},$$

where $[m, n]$ denotes the least common multiple of m and n (e. g. $[4, 6] = 12$).

6.3. *Eleventh Canadian Mathematics Olympiad (1979), problem 5.*
A walk consists of a sequence of steps of length 1 taken in directions north, south, east or west. A walk is *self-avoiding* if it never passes through the same point twice. Let $f(n)$ denote the number of n-step self-avoiding walks which begin at the origin. Compute $f(1)$, $f(2)$, $f(3)$, $f(4)$, and show that

$$2^n < f(n) \leq 4 \cdot 3^{n-1}.$$

7.1. *The XXI International Mathematical Olympiad, London 1979, problem 4.*
Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP + PR)/QR$ is a maximum.

8.1. *15th British Mathematical Olympiad (1979), problem 3.*
S is a set of distinct positive odd integers $\{a_i\}$, $i = 1, \ldots, n$. No two differences $|a_i - a_j|$ are equal, $1 \leq i < j \leq n$. Prove that

$$\sum_{i=1}^{n} a_i \geq \frac{1}{3} n(n^2 + 2).$$

8.2. *15th British Mathematical Olympiad (1979), problem 5.*
For n a positive integer, denote by $p(n)$ the number of ways of expressing n as the sum of one or more positive integers. Thus $p(4) = 5$, because there are 5 different sums, namely,

$$1+1+1+1, \qquad 1+1+2, \qquad 1+3, \qquad 2+2, \qquad 4.$$

Prove that, for $n > 1$,

$$p(n+1) - 2p(n) + p(n-1) \geq 0.$$

10.1. *Practice Set 8–3.*
Let n be a given natural number. Find nonnegative integers k and l so that their sum differs from n by a natural number and so that the following expression is as large as possible:

$$\frac{k}{k+l} + \frac{n-k}{n-(k+l)}.$$

12.1. *Practice Set 10–3.*
For $a \geq b \geq c \geq 0$, establish the inequality

$$b^m c + c^m a + a^m b \geq bc^m + ca^m + ab^m$$

(a) when m is a positive integer;
(b) find a proof valid for all real $m \geq 1$.

15.1. *"Jewish" Problems, J–1.*
Prove that

$$\left(\frac{\sin x}{x}\right)^3 \geq \cos x; \quad 0 < x \leq \frac{\pi}{2}.$$

15.2. *"Jewish" Problems, J–4.*
Let $ab = 4$, $c^2 + 4d^2 = 4$. Prove the inequality

$$(a-c)^2 + (b-d)^2 \geq 1.6.$$

15.3. *"Jewish" Problems, J–5.*
Let $ABCD$ be a tetrahedron with $DB \perp DC$ such that the perdendicular to the plane ABC coming through the vertex D intersects the plane of the triangle ABC at the orthocenter of this triangle. Prove that

$$(|AB| + |BC| + |AC|)^2 \leq 6\,(|AD|^2 + |BD|^2 + |CD|^2).$$

For which tetrahedra does the equality take place?

15.4. *"Jewish" Problems, J–6.*
What is more: $\sqrt[3]{60}$ or $2 + \sqrt[3]{7}$?

15.5. *"Jewish" Problems, J–7.*
Let $ABCD$ be a trapezoid with the bases AB and CD, and let K be a point in AB. Find a point M in CD such that the area of the quadrangle which is the intersection of the triangles AMB and CDK is maximal.

15.6. *"Jewish" Problems, J–8.*
Prove that $x \cos x < 0.71$ for all $x \in [0, \frac{\pi}{2}]$.

15.7. *"Other" Problems, O–3.*
Which is larger, $\sin(\cos x)$ or $\cos(\sin x)$?

15.8. *Practice Set 13–1.*
In n-dimensional Euclidean space $\mathbb{E}^n$, determine the least and greatest distances between the point $A = (a_1, a_2, \ldots, a_n)$ and the n-dimensional rectangular parallelepiped whose vertices are $(\pm\nu_1, \pm\nu_2, \ldots, \pm\nu_n)$ with $\nu_i > 0$.
(Some may find it helpful first to do the problem in $\mathbb{E}^3$ or even in $\mathbb{E}^2$.)

15.9. *Practice Set 13–3.*
(a) If $0 \leq x_i \leq a$, $i = 1, 2, \ldots, n$, determine the maximum value of

$$A \equiv \sum_{i=1}^{n} x_i - \sum_{1 \leq i < j \leq n} x_i x_j.$$

(b) If $0 \leq x_i \leq 1$, $i = 1, 2, \ldots, n$ and $x_{n+1} = x_1$, determine the maximum value of

$$B_n \equiv \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i x_{i+1}.$$

16.1. *The Ninth U.S.A. Mathematical Olympiad (May 1980), problem 2.*
Determine the maximum number of different three-term arithmetic progressions which can be chosen from a sequence of n real numbers $a_1 < a_2 < \cdots < a_n$.

16.2. *The Ninth U.S.A. Mathematical Olympiad (May 1980), problem 5.*
If $1 \geq a, b, c \geq 0$, prove that

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

16.3. *16th British Mathematical Olympiad (1980), problem 4.*
Find the set of real numbers a_0 for which the infinite sequence $\{a_n\}$ of real numbers defined by

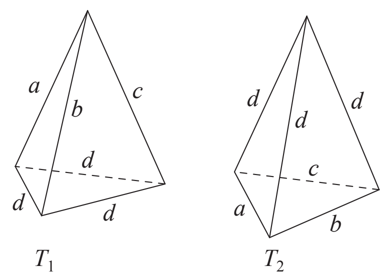
$$a_{n+1} = 2^n - 3a_n, \quad n = 0, 1, 2, \ldots$$

is strictly increasing, that is, $a_n < a_{n+1}$ for $n \geq 0$.

17.1. *Twelfth Canadian Mathematics Olympiad (1980), problem 2.*
The numbers from 1 to 50 are printed on cards. The cards are shuffled and then laid out face up in 5 rows of 10 cards each. The cards in each row are rearranged to make them increase from left to right. The cards in each column are then rearranged to make them increase from top to bottom. In the final arrangement, do the cards in the rows still increase from left to right?

17.2. *Twelfth Canadian Mathematics Olympiad (1980), problem 3.*
Among all triangles ABC having (i) a fixed angle A and (ii) an inscribed circle of fixed radius r, determine which triangle has the least perimeter.

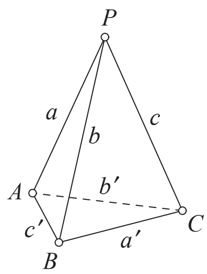
17.3. *Practice Set 14–1.*
Consider the tetrahedra T_1 and T_2 with edge lengths a, b, c, d, as shown in the figures. Under what conditions (on a, b, c, d) is the volume of T_1 greater than that of T_2?



17.4. *Practice Set 14–2.*
Determine the maximum volume of a tetrahedron if it has exactly k edges ($1 \leq k \leq 3$) of length greater than 1. For the case $k = 3$, it is also assumed that the three longest edges are not concurrent, since otherwise the volume can be arbitrarily large.

17.5. *Practice Set 14–3.*
If tetrahedron $PABC$ has edge lengths a, b, c, a', b', c' as shown in the figure, prove that

$$\frac{a'}{b+c} + \frac{b'}{c+a} > \frac{c'}{a+b}.$$



18.1. *Practice Set 15–3.*
Three circular arcs of fixed total length are constructed, each passing through two different vertices of a given triangle, so that they enclose the maximum area. Show that the three radii are equal.

19.1. *Ninth U.S.S.R. National Olympiad (1974), problem 2.*
Two players play the following game on a triangle ABC of unit area. The first player picks a point X on side BC, then the second player picks a point Y on CA, and finally the first player picks a point Z on AB. The first player wants triangle XYZ to have the largest possible area, while the second player wants it to have the smallest possible area. What is the largest area that the first player can be sure of getting?

19.2. *Ninth U.S.S.R. National Olympiad (1974), problem 3.*
The vertices of a convex 32-gon lie on the points of a square lattice whose squares have sides of unit length. Find the smallest perimeter such a figure can have.

19.3. *Ninth U.S.S.R. National Olympiad (1974), problem 8.*
Show that with the digits 1 and 2 one can form 2^{n+1} numbers, each having 2^n digits, and every two of which differ in at least 2^{n-1} places.

19.4. *Ninth U.S.S.R. National Olympiad (1974), problem 9.*
On a 7×7 square piece of graph paper, the centres of k of the 49 squares are chosen. No four of the chosen points are the vertices of a rectangle whose sides are parallel to those of the paper. What is the largest k for which this is possible?

19.5. *Ninth U.S.S.R. National Olympiad (1974), problem 11.*
A horizontal strip is given in the plane, bounded by straight lines, and n lines are drawn intersecting this strip. Every two of these lines intersect inside the strip and no three of them are concurrent. Consider all paths starting on the lower edge of the strip, passing along segments of the given lines, and ending on the upper edge of the strip, which have the following property: travelling along such a path, we are always going upward, and when we come to the point of intersection of two of the lines we must change over to the other line to continue following the path. Show that, among these paths,
(a) at least $\frac{1}{2}n$ of them have no point in common;
(b) there is some path consisting of at least n segments;
(c) there is some path passing along at most $\frac{1}{2}n+1$ of the lines;
(d) there is some path which passes along each of the n lines.

19.6. *Ninth U.S.S.R. National Olympiad (1974), problem 14.*
Prove that, for positive a, b, c, we have

$$a^3+b^3+c^3+3abc \geq bc(b+c)+ca(c+a)+ab(a+b).$$

19.7. *Ninth U.S.S.R. National Olympiad (1974), problem 16.*
Twenty teams are participating in the competition for the championships both of Europe and the world in a certain sport. Among them, there are k European teams (the results of their competitions for world champion count also towards the European championship). The tournament is conducted in round robin fashion. What is the largest value of k for which it is possible that the team getting the (strictly) largest number of points towards the European championship also gets the (strictly) smallest number of points towards the world championship, if the sport involved is
(a) hockey (0 for a loss, 1 for a tie, 2 for a win);
(b) volleyball (0 for a loss, 1 for a win, no ties).

19.8. *Ninth U.S.S.R. National Olympiad (1974), problem 17.*
Given real numbers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$, and positive numbers $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_n$, we form an $m\times n$ array in which the entry in the ith row ($i=1,2,\ldots,m$) and the jth column ($j=1,2,\ldots,n$) is

$$\frac{a_i+b_j}{p_i+q_j}.$$

Show that in such an array there is some entry which is no less than any other in the same row and no greater than an other in the same column
(a) when $m=2$ and $n=2$,
(b) for arbitrary m and n.

20.1. *"Jewish" Problems, J–11.*
Which is larger, $\sqrt[3]{413}$ or $6 + \sqrt[3]{3}$?

20.2. *"Jewish" Problems, J–17.*
Prove the inequality

$$\frac{1}{\sin^2 x} < \frac{1}{x^2} + 1 - \frac{4}{\pi^2} \qquad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

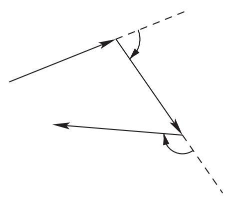
20.3. *"Jewish" Problems, J–19.*
Six points are given, one on each edge of a tetrahedron of volume 1, none of them being a vertex. Consider the four tetrahedra formed as follows: Choose one vertex of the original tetrahedron and let the remaining vertices be the three given points that lie on the three edges incident with the chosen vertex. Prove that at least one of these four tetrahedra has volume not exceeding $\frac{1}{8}$.

21.1. *Thirty-sixth Moscow Olympiad (1973), problem 5.*
A point is chosen on each side of a parallelogram in such a way that the area of the quadrilateral whose vertices are these four points is one-half the area of the parallelogram. Show that at least one of the diagonals of the quadrilateral is parallel to a side of the parallelogram.

21.2. *Thirty-sixth Moscow Olympiad (1973), problem 8.*
The faces of a cube are numbered $1, 2, \ldots, 6$ in such a way that the sum of the numbers on opposite faces is always 7. We have a chessboard of 50×50 squares, each square congruent to a face of the cube. The cube "rolls" from the lower left-hand corner of the chessboard to the upper right-hand corner. The "rolling" of the cube consists of a rotation about one of its edges so that one face rests on a square of the chessboard. The cube may roll only upward and to the right (never downward or to the left). On each square of the chessboard that was occupied during the trip is written the number of the face of the cube that rested there. Find the largest and the smallest sum that these numbers may have.

21.3. *Thirty-sixth Moscow Olympiad (1973), problem 9.*
On a piece of paper is an inkblot. For each point of the inkblot, we find the greatest and smallest distances from that point to the boundary of the inkblot. Of all the smallest distances we choose the maximum and of all the greatest distances we choose the minimum. If these two chosen numbers are equal, what shape can the inkblot have?

21.4. *Thirty-sixth Moscow Olympiad (1973), problem 10.*
A lion runs about the circular arena (radius 10 metres) of a circus tent. Moving along a broken line, he runs a total of 30 km. Show that the sum of the angles through which he turns (see figure) is not less than 2998 radians.



21.5. *Thirty-sixth Moscow Olympiad (1973), problem 12.*
On an infinite chessboard, a closed simple (i.e., non-self-intersecting) path is drawn, consisting of sides of squares of the chessboard. Inside the path are k black squares. What is the largest area that can be enclosed by the path?

21.6. *Thirty-sixth Moscow Olympiad (1973), problem 13.*
The following operation is performed on a 100-digit number: a block of 10 consecutively-placed digits is chosen and the first five are interchanged with the last five (the 1st with the 6th, the 2nd with the 7th, ..., the 5th with the 10th). Two 100-digit numbers which are obtained from each other by repeatedly performing this operation will be called *similar*. What is the largest number of 100-digit integers, each consisting of the digits 1 and 2, which can be chosen so that no two of the integers will be similar?

22.1. *Österreichisch-Polnischer Mathematik-Wettbewerb (1980), problem 1.*
Given three infinite arithmetic progressions of natural numbers such that each of the numbers 1, 2, 3, 4, 5, 6, 7, and 8 belongs to at least one of them, prove that the number 1980 also belongs to at least one of them.

22.2. *Österreichisch-Polnischer Mathematik-Wettbewerb (1980), problem 2.*
Let $\{x_n\}$ be a sequence of natural numbers such that

(a) $1 = x_1 < x_2 < x_3 < \cdots;$

(b) $x_{2n+1} \leq 2n \quad \text{for all } n.$

Prove that, for every natural number k, there exist terms x_r and x_s such that $x_r - x_s = k$.

22.3. *Österreichisch-Polnischer Mathematik-Wettbewerb (1980), problem 3.*
Prove that the sum of the six angles subtended at an interior point of a tetrahedron by its six edges is greater than 540°.

22.4. *Österreichisch-Polnischer Mathematik-Wettbewerb (1980), problem 6.*
Given a sequence $\{a_n\}$ of real numbers such that $|a_{k+m} - a_k - a_m| \leq 1$ for all positive integers k and m, prove that, for all positive integers p and q,

$$\left|\frac{a_p}{p} - \frac{a_q}{q}\right| < \frac{1}{p} + \frac{1}{q}.$$

22.5. *Österreichisch-Polnischer Mathematik-Wettbewerb (1980), problem 7.*
Find the greatest natural number n such that there exist natural numbers $x_1, x_2, \ldots, x_n, a_1, a_2, \ldots,$ a_{n-1} with $a_1 < a_2 < \cdots < a_{n-1}$ satisfying the following system of equations:

$$\begin{cases} x_1 x_2 \cdots x_n = 1980, \\ x_i + \frac{1980}{x_i} = a_i, \qquad i = 1, 2, \ldots, n-1. \end{cases}$$

22.6. *Österreichisch-Polnischer Mathematik-Wettbewerb (1980), problem 8.*
Let S be a set of 1980 points in the plane such that the distance between every pair of them is at least 1. Prove that S has a subset of 220 points such that the distance between every pair of them is at least $\sqrt{3}$.

22.7. *Competition in Mariehamn, Finland (Finland, Great Britain, Hungary, and Sweden), problem 2.*
The sequence $a_0, a_1, \ldots, a_n$ is defined by

$$a_0 = \frac{1}{2}, \qquad a_{k+1} = a_k + \frac{1}{n} a_k^2, \quad k = 0, 1, \ldots, n-1.$$

Prove that $1 - \frac{1}{n} < a_n < 1$.

22.8. *Romanian Mathematical Olympiad (1978), 9th class, problem 1.*
Determine the range of the function f defined for all real x by

$$f(x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1}.$$

22.9. *Romanian Mathematical Olympiad (1978), 9th class, problem 2.*

22.10. *Romanian Mathematical Olympiad (1978), 9th class, problem 3.*

22.11. *Romanian Mathematical Olympiad (1978), 10th class, problem 1.*

22.12. *Romanian Mathematical Olympiad (1978), 10th class, problem 4.*

to be continued ...

Inequalities proposed at
International Mathematical Olympiads

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2^{nd} IMO 1960, Sinaia, Romania. *Problem 2 (Proposed by Hungary).*
For what values of the variable x does the following inequality hold:

$$\frac{4x^2}{(1-\sqrt{1+2x})^2} < 2x+9?$$

2^{nd} IMO 1960, Sinaia, Romania. *Problem 6 (Proposed by Bulgaria).*
Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. Let V_1 be the volume of the cone and V_2 the volume of the cylinder.
(a) Prove that $V_1 \neq V_2$.
(b) Find the smallest number k for which $V_1 = kV_2$, for this case, construct the angle subtended by a diameter of the base of the cone at the vertex of the cone.

3^{rd} IMO 1961, Veszprem, Hungaria. *Problem 2 (Proposed by Poland).*
Let a, b, c be the sides of a triangle, and T its area. Prove:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}T.$$

In what case does equality hold?

3^{rd} IMO 1961, Veszprem, Hungaria. *Problem 4 (Proposed by G.D.R.).*
Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1},\ \frac{P_2P}{PQ_2},\ \frac{P_3P}{PQ_3}$$

at least one is ≤ 2 and at least one is ≥ 2.

3^{rd} IMO 1961, Veszprem, Hungaria. *Problem 5 (Proposed by Czechoslovakia).*
Construct triangle ABC if $AC = b$, $AB = c$ and $\angle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90^\circ$. Prove that a solution exists if and only if

$$b \tan\frac{\omega}{2} \leq c < b.$$

In what case does the equality hold?

4^{th} IMO 1962, České Budejovice, Czechoslovakia. *Problem 2 (Proposed by Hungary).*
Determine all real numbers x which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}.$$

5^{th} IMO 1963, Warsaw, Poland. *Problem 3 (Proposed by Hungary).*
In an n-gon all of whose interior angles are equal, the lengths of consecutive sides satisfy the relation $a_1 \geq a_2 \geq \cdots \geq a_n$. Prove that $a_1 = a_2 = \cdots = a_n$.

6^{th} IMO 1964, Moscow, U. S. S. R.. *Problem 2 (Proposed by Hungary).*
Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq 3abc.$$

6^{th} IMO 1964, Moscow, U. S. S. R.. *Problem 5 (Proposed by Romania).*
Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have.

7^{th} IMO 1965, Berlin, German Democratic Republic. *Problem 1 (Proposed by Yugoslavia).*
Determine all values x in the interval $0 \leq x \leq 2\pi$ which satisfy the inequality

$$2\cos x \leq |\sqrt{1+\sin 2x}-\sqrt{1-\sin 2x}| \leq \sqrt{2}.$$

7^{th} IMO 1965, Berlin, German Democratic Republic. *Problem 2 (Proposed by Poland).*
Consider the system of equations

$$a_{11}x_1+a_{12}x_2+a_{13}=0$$
$$a_{21}x_1+a_{22}x_2+a_{23}=0$$
$$a_{31}x_1+a_{32}x_2+a_{33}=0$$

with unknowns x_1, x_2, x_3. The coefficients satisfy the conditions:
(a) a_{11}, a_{22}, a_{33} are positive numbers;
(b) the remaining coefficients are negative numbers;
(c) in each equation, the sum of the coefficients is positive.
Prove that the given system has only the solution $x_1 = x_2 = x_3 = 0$.

7^{th} IMO 1965, Berlin, German Democratic Republic. *Problem 6 (Proposed by Poland).*
In a plane a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d. Prove that the number of diameters of the given set is at most n.

8^{th} IMO 1966, Sofia, Bulgaria. *Problem 3 (Proposed by Bulgaria).*
Prove: The sum of the distances of the vertices of a regular tetrahedron from the center of its circumscribed sphere is less than the sum of the distances of these vertices from any other point in space.

8^{th} IMO 1966, Sofia, Bulgaria. *Problem 6 (Proposed by Poland).*
In the interior of sides BC, CA, AB of triangle ABC, any points K, L, M, respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC.

9^{th} IMO 1967, Cetinje, Yugoslavia. *Problem 1 (Proposed by Czechoslovakia).*
Let $ABCD$ be a parallelogram with side lengths $AB = a$, $AD = 1$, and with $\angle BAD = \alpha$. If $\triangle ABD$ is acute, prove that the four circles of radius 1 with centers A, B, C, D cover the parallelogram if and only if

$$a \leq \cos\alpha + \sqrt{3}\sin\alpha.$$

9^{th} IMO 1967, Cetinje, Yugoslavia. *Problem 2 (Proposed by Poland).*
Prove that if one and only one edge of a tetrahedron is greater than 1, then its volume is $\leq \frac{1}{8}$.

9^{th} IMO 1967, Cetinje, Yugoslavia. *Problem 4 (Proposed by Italy).*
Let $A_0B_0C_0$ and $A_1B_1C_1$ be any two acute-angled triangles. Consider all triangles ABC that are similar to $\triangle A_1B_1C_1$ (so that vertices A_1, B_1, C_1 correspond to vertices A, B, C, respectively) and circumscribed about triangle $A_0B_0C_0$ (where A_0 lies on BC, B_0 on CA, and AC_0 on AB). Of all such possible triangles, determine the one with maximum area, and construct it.

10^{th} IMO 1968, Moscow, U. S. S. R.. *Problem 4 (Proposed by Poland).*
Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

11^{th} IMO 1969, Bucharest, Romania. *Problem 5 (Proposed by Mongolia).*
Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

11^{th} IMO 1969, Bucharest, Romania. *Problem 6 (Proposed by U.S.S.R.).*
Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$ with $x_1 > 0$, $x_2 > 0$, $x_1y_1 - z_1^2 > 0$, $x_2y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

12^{th} IMO 1970, Keszthely, Hungaria. *Problem 2 (Proposed by Romania).*
Let a, b and n be integers greater than 1, and let a and b be the bases of two number systems. A_{n-1} and A_n are numbers in the system with base a, and B_{n-1} and B_n are numbers in the system with base b; these are related as follows:

$$\begin{aligned}
&A_n = x_nx_{n-1}\cdots x_0, \quad A_{n-1} = x_{n-1}x_{n-2}\cdot x_0,\\
&B_n = x_nx_{n-1}\cdots x_0, \quad B_{n-1} = x_{n-1}x_{n-2}\cdot x_0,\\
&x_n \neq 0, \quad x_{n-1} \neq 0.
\end{aligned}$$

Prove that $\dfrac{A_{n-1}}{A_n} < \dfrac{B_{n-1}}{B_n}$ if and only if $a > b$.

12^{th} IMO 1970, Keszthely, Hungaria. *Problem 3 (Proposed by Sweden).*
The real numbers $a_0, a_1, \ldots, a_n, \ldots$ satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots.$$

The numbers $b_1, b_2, \ldots, b_n, \ldots$ are defined by

$$b_n = \sum_{k=1}^{n} \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}.$$

(a) Prove that $0 \leq b_n < 2$ for all n.
(b) Given c with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, \ldots$ with the above properties such that $b_n > c$ for large enough n.

12th IMO 1970, Keszthely, Hungaria. *Problem 5 (Proposed by Bulgaria).*
In the tetrahedron $ABCD$, angle BCD is a right angle. Suppose that the foot H of the perpendicular from d to the plane ABC is the intersection of the altitudes of $\triangle ABC$. Prove that

$$(AB + BC + CA)^2 \leq 6\,(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

12th IMO 1970, Keszthely, Hungaria. *Problem 6 (Proposed by U.S.S.R.).*
In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70 % of these triangles are acute-angled.

13th IMO 1971, Žilina, Czechoslovakia. *Problem 1 (Proposed by Hungary).*
Prove that the following assertion is true for $n = 3$ and $n = 5$, and that it is false for every other natural number $n > 2$: If $a_1, a_2, \ldots, a_n$ are arbitrary real numbers, then

$$\begin{aligned}&(a_1 - a_2)(a_1 - a_3)\cdots(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)\cdots(a_2 - a_n) + \cdots +\\ &(a_n - a_1)(a_n - a_2)\cdots(a_n - a_{n-1}) \geq 0.\end{aligned}$$

13th IMO 1971, Žilina, Czechoslovakia. *Problem 4 (Proposed by The Netherlands).*
All the faces of tetrahedron $ABCD$ are acute-angled triangles. We consider all closed polygonal paths of the form $XYZTX$ defined as follows: X is a point on edge AB distinct from A and B; similarly, Y, Z, T are interior points of edges BC, CD, DA, respectively. Prove that
(a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.
(b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then they are infinitely many shortest polygonal paths, their common length being $2AC\sin(\alpha/2)$, where $\alpha = \angle BAC + \angle CAD + \angle DAB$.

13th IMO 1971, Žilina, Czechoslovakia. *Problem 6 (Proposed by Sweden).*
Let $A = (a_{ij})$ $(i, j = 1, 2, \ldots, n)$ be a square matrix whose elements are non-negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is $\geq n$. Prove that the sum of all the elements of the matrix is $\geq n^2/2$.

14th IMO 1972, Torun, Poland. *Problem 4 (Proposed by The Netherlands).*
Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$\begin{aligned}&(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \leq 0, &&(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \leq 0,\\ &(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \leq 0, &&(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \leq 0,\\ &(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \leq 0,\end{aligned}$$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers.

14th IMO 1972, Torun, Poland. *Problem 5 (Proposed by Bulgaria).*
Let f and g be real-valued functions defined for all real values of x and y, and satisfying the equation

$$f(x + y) + f(x - y) = 2f(x)g(y)$$

for all x, y. Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all x, then $|g(y)| \leq 1$ for all y.

15$^{\text{th}}$ IMO 1973, Moscow, U. S. S. R.. *Problem 1 (Proposed by Czechoslovakia).*
Point O lies on line g; $\overrightarrow{OP_1}, \overrightarrow{OP_2}, \dots, \overrightarrow{OP_n}$ are unit vectors such that points $P_1, P_2, \dots, P_n$ all lie in a plane containing g and on one side of g. Prove that if n is odd,

$$|\overrightarrow{OP_1} + \overrightarrow{OP_2} + \cdots + \overrightarrow{OP_n}| \geq 1.$$

Here $|\overrightarrow{OM}|$ denotes the length of vector $\overrightarrow{OM}$.

15$^{\text{th}}$ IMO 1973, Moscow, U. S. S. R.. *Problem 3 (Proposed by Sweden).*
Let a and b be real numbers for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has at least one real solution. For all such pairs (a, b), find the minimum value of $a^2 + b^2$.

15$^{\text{th}}$ IMO 1973, Moscow, U. S. S. R.. *Problem 4 (Proposed by Yugoslavia).*
A soldier needs to check on the presence of mines in a region having the shape of an equilateral triangle. The radius of action of his detector is equal to half the altitude of the triangle. The soldier leaves from one vertex of the triangle. What path should he follow in order to travel the least possible distance and still accomplish his mission?

15$^{\text{th}}$ IMO 1973, Moscow, U. S. S. R.. *Problem 6 (Proposed by Sweden).*
Let $a_1, a_2, \dots, a_n$ be n positive numbers, and let q be a given real number such that $0 < q < 1$. Find n numbers $b_1, b_2, \dots, b_n$ for which

(a) $a_k < b_k$ for $k = 1, 2, \dots, n$,

(b) $q < \dfrac{b_{k+1}}{b_k} < \dfrac{1}{q}$ for $k = 1, 2, \dots, n-1$,

(c) $b_1 + b_2 + \cdots + b_n < \dfrac{1+q}{1-q}(a_1 + a_2 + \cdots + a_n)$.

16$^{\text{th}}$ IMO 1974, Erfurt, German Democratic Republic. *Problem 2 (Proposed by Finland).*
In the triangle ABC, prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if

$$\sin A \sin B \leq \sin^2 \frac{C}{2}.$$

16$^{\text{th}}$ IMO 1974, Erfurt, German Democratic Republic. *Problem 4 (Proposed by Bulgaria).*
Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following conditions:
(i) Each rectangle has as many white squares as black squares.
(ii) If a_i is the number of white squares in the i-th rectangle, then $a_1 < a_2 < \cdots < a_p$. Find the maximum value of p for which such a decomposition is possible. For this value of p, determine all possible sequences $a_1, a_2, \dots, a_p$.

16th IMO 1974, Erfurt, German Democratic Republic. *Problem 5 (Proposed by The Netherlands).*
Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

17th IMO 1975, Burgas, Bulgaria. *Problem 1 (Proposed by Czechoslovakia).*
Let x_i, y_i $(i = 1, 2, \ldots, n)$ be real numbers such that

$$x_1 \geq x_2 \geq \cdots \geq x_n \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_n.$$

Prove that, if $z_1, z_2, \ldots, z_n$ is any permutation of $y_1, y_2, \ldots, y_n$, then

$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2.$$

18th IMO 1976, Lienz, Austria. *Problem 1 (Proposed by Czechoslovakia).*
In a plane convex quadrilateral of area 32, the sum of the lengths of two opposite sides and one diagonal is 16. Determine all possible lengths of the other diagonal.

18th IMO 1976, Lienz, Austria. *Problem 3 (Proposed by The Netherlands).*
A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40 % of the box. Determine the possible dimensions of all such boxes.

18th IMO 1976, Lienz, Austria. *Problem 4 (Proposed by U.S.A.).*
Determine, with proof, the largest number which is the product of positive integers whose sum is 1976.

19th IMO 1977, Beograd, Yugoslavia. *Problem 2 (Proposed by Vietnam).*
In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

19th IMO 1977, Beograd, Yugoslavia. *Problem 4 (Proposed by Great Britain).*
Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta.$$

Prove that if $f(\theta) \geq 0$ for all real θ, then

$$a^2 + b^2 \leq 2 \quad \text{and} \quad A^2 + B^2 \leq 1.$$

19th IMO 1977, Beograd, Yugoslavia. *Problem 6 (Proposed by Bulgaria).*
Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n, then

$$f(n) = n \text{ for each } n.$$

20^{th} IMO 1978, Bucharest, Romania. *Problem 1 (Proposed by Cuba).*
m and n are natural numbers with $1 \le m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m+n$ has its least value.

20^{th} IMO 1978, Bucharest, Romania. *Problem 5 (Proposed by France).*
Let $\{a_k\}$ $(k = 1, 2, 3, \ldots, n, \ldots)$ be a sequence of distinct positive integers. Prove that for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \ge \sum_{k=1}^{n} \frac{1}{k}.$$

20^{th} IMO 1978, Bucharest, Romania. *Problem 6 (Proposed by The Netherlands).*
An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

21^{st} IMO 1979, London, United Kingdom. *Problem 4 (Proposed by U.S.A.).*
Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $(QP + PR)/QR$ is a maximum.

22^{nd} IMO 1981, Washington D.C., U.S.A. *Problem 1 (Proposed by Great Britain).*
P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.

22^{nd} IMO 1981, Washington D.C., U.S.A. *Problem 3 (Proposed by The Netherlands).*
Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying $m, n \in \{1, 2 \ldots, 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

23^{rd} IMO 1982, Budapest, Hungary. *Problem 3 (Proposed by U.S.S.R.).*
Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1, \text{ and for all } i \ge 0, \ x_{i+1} \le x_i.$$

(a) Prove that for every such sequence, there is an $n \ge 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \ge 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4 \quad \text{for all } n.$$

23rd IMO 1982, Budapest, Hungary. *Problem 6 (Proposed by Vietnam).*
Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments A_0A_1, A_1A_2, …, $A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $1/2$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not less than 198.

24th IMO 1983, Paris, France. *Problem 3 (Proposed by F.R.G.).*
Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$, where x, y and z are non-negative integers.

24th IMO 1983, Paris, France. *Problem 6 (Proposed by U.S.A.).*
Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality occurs.

25th IMO 1984, Prague, Czechoslovakia. *Problem 1 (Proposed by F.R.G.).*
Prove that

$$0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27},$$

where x, y and z are non-negative real numbers for which $x + y + z = 1$.

25th IMO 1984, Prague, Czechoslovakia. *Problem 5 (Proposed by Mongolia).*
Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices ($n > 3$), and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

26th IMO 1985, Joutsa, Finland. *Problem 3 (Proposed by The Netherlands).*
For any polynomial $P(x) = a_0 + a_1x + \cdots + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, \ldots$, let $Q_i(x) = (1+x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers such that $0 \leq i_1 < i_2 < \cdots < i_n$, then

$$w(Q_{i_1} + Q_{i_2} + \cdots + Q_{i_n}) \geq w(Q_{i_1}).$$

26th IMO 1985, Joutsa, Finland. *Problem 6 (Proposed by Sweden).*
For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting

$$x_{n+1} = x_n\left(x_n + \frac{1}{n}\right) \quad \text{for each } n \geq 1.$$

Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n.

27^{th} IMO 1986, Warsaw, Poland. *Problem 6 (Proposed by G.D.R.).*
One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white points and red points on L is not greater than 1? Justify your answer.

28^{th} IMO 1987, Havana, Cuba. *Problem 3 (Proposed by F.R.G.).*
Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the equation $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $|a_i| \leq k-1$ for all i and

$$|a_1x_1 + a_2x_2 + \cdots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

29^{th} IMO 1988, Canberra, Australia. *Problem 4 (Proposed by Ireland).*
Show that the set of real numbers x that satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

29^{th} IMO 1988, Canberra, Australia. *Problem 5 (Proposed by Greece).*
ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that

$$S \geq 2T.$$

30^{th} IMO 1989, Braunschweig, Germany. *Problem 2 (Proposed by Australia).*
In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that

(a) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$;
(b) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

30^{th} IMO 1989, Braunschweig, Germany. *Problem 3 (Proposed by The Netherlands).*
Let n and k be positive integers and let S be a set of n points in the plane such that

(a) no three points of S are collinear, and
(b) for every point P of S there are at least k points of S equidistant from P.

Prove that

$$k < \frac{1}{2} + \sqrt{2n}.$$

30^{th} IMO 1989, Braunschweig, Germany. *Problem 4 (Proposed by Iceland).*
Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

31st IMO 1990, Beijing, China. *Problem 2 (Proposed by C.S.F.R.).*
Let $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is *good* if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

32nd IMO 1991, Sigtuna, Sweden. *Problem 1 (Proposed by U.S.S.R.).*
Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C', respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

32nd IMO 1991, Sigtuna, Sweden. *Problem 3 (Proposed by China).*
Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers that are pairwise relatively prime.

32nd IMO 1991, Sigtuna, Sweden. *Problem 5 (Proposed by France).*
Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

32nd IMO 1991, Sigtuna, Sweden. *Problem 6 (Proposed by The Netherlands).*
An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be *bounded* if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$.
Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i-j|^a \geq 1$$

for every pair of distinct nonnegative integers i, j.

33rd IMO 1992, Moscow, Russia. *Problem 5 (Proposed by Italy).*
Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set A. (Note: the orthogonal projection of a point onto a plane is the foot of the perpendicular from the point to the plane.)

33rd IMO 1992, Moscow, Russia. *Problem 6 (Proposed by Great Britain).*
For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive square integers.
(a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.
(b) Find an integer n such that $S(n) = n^2 - 14$.
(c) Prove that there exist infinitely many positive integers n such that $S(n) = n^2 - 14$.

34th IMO 1993, Istanbul, Turkey. *Problem 4 (Proposed by Macedonia).*
For three points P, Q, R in the plane, we define $m(PQR)$ to be the minimum of the lengths of the altitudes of the triangle PQR (where $m(PQR) = 0$ when P, Q R are collinear).
Let A, B, C be given points in the plane. Prove that, for any point X in the plane,

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

35$^{\text{th}}$ IMO 1994, Hong Kong, H. K.. *Problem 1 (Proposed by France).*
Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j, $1 \leq i \leq j \leq m$, there exists k, $1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \frac{n+1}{2}.$$

36$^{\text{th}}$ IMO 1995, Toronto, Canada. *Problem 2 (Proposed by Russia).*
Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

36$^{\text{th}}$ IMO 1995, Toronto, Canada. *Problem 4 (Proposed by Poland).*
Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

(a) $x_0 = x_{1995}$;

(b) $x_{i-1} + \dfrac{2}{x_{i-1}} = 2x_i + \dfrac{1}{x_i}$ for each $i = 1, 2, \ldots, 1995$.

36$^{\text{th}}$ IMO 1995, Toronto, Canada. *Problem 5 (Proposed by New Zealand).*
Let $ABCDEF$ be a convex hexagon with

$$AB = BC = CD, \quad DE = EF = FA \quad \text{and} \quad \angle BCD = \angle EFA = 60°.$$

Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

37$^{\text{th}}$ IMO 1996, Mumbai, India. *Problem 4 (Proposed by Russia).*
The positive integers a and b are such that the numbers $15a+16b$ and $16a-15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares?

37$^{\text{th}}$ IMO 1996, Mumbai, India. *Problem 5 (Proposed by Armenia).*
Let $ABCDEF$ be a convex hexagon such that AB is parallel to ED, BC is parallel to FE, and CD is parallel to AF. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

38^{th} IMO 1997, Mar del Plata, Argentina. *Problem 1 (Proposed by Belarus).*
In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard).

For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along the edges of the squares.

Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let $f(m,n) = |S_1 - S_2|$.

(a) Calculate $f(m,n)$ for all positive integers m and n that are either both even or both odd.

(b) Prove that $f(m,n) \leq \frac{1}{2}\max\{m,n\}$ for all m and n.

(c) Show that there is no constant C such that $f(m,n) < C$ for all m and n.

38^{th} IMO 1997, Mar del Plata, Argentina. *Problem 3 (Proposed by Russia).*
Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and

$$|x_i| \leq \frac{n+1}{2} \quad \text{for } i = 1, 2, \ldots, n.$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

38^{th} IMO 1997, Mar del Plata, Argentina. *Problem 6 (Proposed by Lithuania).*
For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations that differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4; \quad 2+2; \quad 2+1+1; \quad 1+1+1+1.$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

39^{th} IMO 1998, Taipei, Taiwan. *Problem 2 (Proposed by India).*
In a competition, there are a contestants and b examiners, where $b \geq 3$ is an odd integer. Each examiner rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two examiners, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

39^{th} IMO 1998, Taipei, Taiwan. *Problem 6 (Proposed by Bulgaria).*
Consider all functions f from the set N of all positive integers into itself satisfying the equation

$$f(t^2 f(s)) = s(f(t))^2$$

for all s and t in N. Determine the least possible value of $f(1998)$.

40th IMO 1999, Bucharest, Romania. *Problem 2 (Proposed by Poland).*
Let n be a fixed integer, with $n \geq 2$.
(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \, (x_1 + x_2 + \cdots + x_n)^4$$

holds for all real numbers $x_1, \ldots, x_n \geq 0$.
(b) For this constant C, determine when equality holds.
(For a solution, see **BS00**, p. 23.)

40th IMO 1999, Bucharest, Romania. *Problem 3 (Proposed by Belarus).*
Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side.
N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.
Determine the smallest possible value of N.

41st IMO 2000, Taejon, Republic of Korea. *Problem 2 (Proposed by U.S.A.).*
Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

42nd IMO 2001, Washington D.C., U.S.A. *Problem 1 (Proposed by South Korea).*
Let ABC be an acute-angled triangle with circumradius O. Let P on BC be the foot of the altitude from A. Suppose that $\angle BCA \geq \angle ABC + 30°$. Prove that $\angle CAB + \angle COP < 90°$.

42nd IMO 2001, Washington D.C., U.S.A. *Problem 2 (Proposed by South Korea).*
Prove that

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1$$

for all positive real numbers a, b and c.

42nd IMO 2001, Washington D.C., U.S.A. *Problem 3 (Proposed by Germany).*
Twenty-one girls and twenty-one boys took part in a mathematical contest. Each contestant solved at most six problems. For each girl and each boy, at least one problem was solved by both of them. Prove that there was a problem that was solved by at least three girls and at least three boys.

43rd IMO 2002, Glasgow, United Kingdom. *Problem 4 (Proposed by Romania).*
Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$, where $1 = d_1 < d_2 < \cdots < d_k = n$. Define $D = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$.
(a) Prove that $D < n^2$.
(b) Determine all n for which D is a divisor of n^2.

43rd IMO 2002, Glasgow, United Kingdom. *Problem 6 (Proposed by Ukraine).*
Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \geq 3$. Denote their centres by $O_1, O_2, \ldots, O_n$, respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_i O_j} \leq \frac{(n-1)\pi}{4}.$$

44th IMO 2003, Tokyo, Japan. *Problem 5 (Proposed by Ireland).*
Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be real numbers with $x_1 \leq x_2 \leq \cdots \leq x_n$.
(a) Prove that

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

(b) Show that equality holds if and only if $x_1, x_2, \ldots, x_n$ is an arithmetic sequence.

45th IMO 2004, Athens, Greece. *Problem 4 (Proposed by South Korea).*
Let $n \geq 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

46th IMO 2005, Merida, Mexico. *Problem 3 (Proposed by South Korea).*
Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

46th IMO 2005, Merida, Mexico. *Problem 6 (Proposed by Romania).*
In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

47th IMO 2006, Ljubljana, Slovenia. *Problem 1 (Proposed by South Korea).*
Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies

$$\sphericalangle PBA + \sphericalangle PCA = \sphericalangle PBC + \sphericalangle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

47th IMO 2006, Ljubljana, Slovenia. *Problem 2 (Proposed by Serbia and Montenegro).*
Let P be a regular 2006-gon. A diagonal of P is called *good* if its endpoints divide the boundary of P into two parts, each composed of an odd number of sides of P. The sides of P are also called *good*.
Suppose P has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of P. Find the maximum number of isosceles triangles having two good sides that could appear in such a configuration.

47th IMO 2006, Ljubljana, Slovenia. *Problem 3 (Proposed by Ireland).*
Determine the least real number M such that the inequality

$$\left|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)\right| \leq M(a^2 + b^2 + c^2)^2$$

hold for all real numbers a, b and c.

47th IMO 2006, Ljubljana, Slovenia. *Problem 5 (Proposed by Romania).*
Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

47th IMO 2006, Ljubljana, Slovenia. *Problem 6 (Proposed by Serbia and Montenegro).*
Assign to each side b of a convex polygon P the maximum area of a triangle that has b as a side and is contained in P. Show that the sum of the areas assigned to the sides of P is at least twice the area of P.

48th IMO 2007, Hanoi, Vietnam. *Problem 1 (Proposed by New Zealand).*
Real numbers $a_1, a_2, \ldots, a_n$ are given. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$$

and let

$$d = \max\{d_i : 1 \le i \le n\}.$$

(a) Prove that, for any real numbers $x_1 \le x_2 \le \cdots \le x_n$,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \frac{d}{2}. \qquad (\star)$$

(b) Show that there are real numbers $x_1 \le x_2 \le \cdots \le x_n$ such that equality holds in $(\star)$.

48th IMO 2007, Hanoi, Vietnam. *Problem 3 (Proposed by Russia).*
In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its *size*.
Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged in two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

48th IMO 2007, Hanoi, Vietnam. *Problem 6 (Proposed by The Netherlands).*
Let n be a positive integer. Consider

$$S = \{(x, y, z) : x, y, z \in \{0, 1, \ldots, n\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

to be continued in 2008 . . .

Inequalities proposed in
"Five Hundred Mathematical Challenges"
by E. J. Barbeau, M. S. Klamkin, W. O. J. Moser

10, p. 2
Suppose that the center of gravity of a water jug is above the inside bottom of the jug, and that water is poured into the jug until the center of gravity of the combination of jug and water is as low as possible. Explain why the center of gravity of this "extreme" combination must lie at the surface of the water.

13, p. 2
Show that among any seven distinct positive integers not greater than 126, one can find two of them, say x and y, satisfying the inequalities $1 < \frac{y}{x} \le 2$.

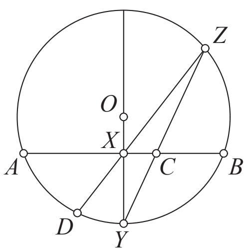
14, p. 2
Show that if 5 points are all in, or on, a square of side 1, then some pair of them will be no further than $\frac{\sqrt{2}}{2}$ apart.

15, p. 2
During an election campaign n different kinds of promises are made by the various political parties, $n > 0$. No two parties have exactly the same set of promises. While several parties may make the same promise, every pair of parties have at least one promise in common. Prove that there can be as many as 2^{n-1} parties, but no more.

19, p. 3
Give an elementary proof that

$$\sqrt{n}^{\sqrt{n+1}} > \sqrt{n+1}^{\sqrt{n}}, \quad n = 7, 8, 9, \ldots.$$

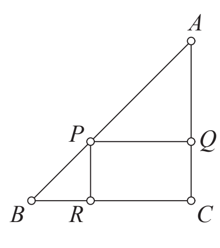
20, p. 3
If, in a circle with center O, OXY is perpendicular to chord AB, prove that $DX \le CY$ (see Figure). (P. Erdös and M. Klamkin)



28, p. 4
A boy lives in each of n houses on a straight line. At what point should the n boys meet so that the sum of the distances that they walk from their houses is as small as possible?

32, p. 4
Two points on a sphere of radius 1 are joined by an arc of length less than 2, lying inside the sphere. Prove that the arc must lie in some hemisphere of the given sphere. (USAMO 1974)

35, p. 4
Let ABC be the right-angled isosceles triangle whose equal sides have length 1. P is a point on the hypotenuse, and the feet of the perpendiculars from P to the other sides are Q and R. Consider the areas of the triangles APQ and PBR, and the area of the rectangle $QCRP$. Prove that regardless of how P is chosen, the largest of these three areas is at least $2/9$.



37, p. 4
A quadrilateral has one vertex on each side of a square of side-length 1. Show that the lengths a, b, c, and d of the sides of the quadrilateral satisfy the inequalities

$$2 \le a^2 + b^2 + c^2 + d^2 \le 4.$$

40, p. 5
Teams $T_1, T_2, \ldots, T_n$ take part in a tournament in which every team plays every other team just once. One point is awarded for each win, and it is assumed that there are no draws. Let $s_1, s_2, \ldots, s_n$ denote the (total) scores of $T_1, T_2, \ldots, T_n$ respectively. Show that, for $1 < k < n$,

$$s_1 + s_2 + \cdots + s_n \le nk - \frac{1}{2}k(k+1).$$

42, p. 5
In the following problem no "aids" such as tables, calculators, etc. should be used.
(a) Prove that the values of x for which $x = \frac{x^2+1}{198}$ lie between $\frac{1}{198}$ and $197.99494949\ldots$.
(b) Use the result of (a) to prove that $\sqrt{2} < 1.4142135642135642135 6\ldots$.
(c) Is it true that $\sqrt{2} < 1.41421356$?

58, p. 6
Let

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}.$$

Show that $2\sqrt{n+1} - 2 < s_n < 2\sqrt{n} - 1$.

59, p. 6
Show that for any quadrilateral inscribed in a circle of radius 1, the length of the shortest side is not more than $\sqrt{2}$.

65, p. 7
Let nine points be given in the interior of the unit square. Prove that there exists a triangle of area at most $\frac{1}{8}$ whose vertices are three of the nine points. (See also problem 14 or 43.)

67, p. 7
A triangle has sides of lengths a, b, c and respective altitudes of lengths h_a, h_b, h_c. If $a \ge b \ge c$ show that $a + h_a \ge b + h_b \ge c + h_c$.

75, p. 7
Given an $n \times n$ array of positive numbers

$$\begin{array}{cccc}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn},
\end{array}$$

let m_j denote the smallest number in the jth column, and m the largest of the m_j's. Let M_i denote the largest number in the ith row, and M the smallest of the M_i's. Prove that $m \le M$.

76, p. 7
What is the maximum number of terms in a geometric progression with common ratio greater than 1 whose entries all come from the set of integers between 100 and 1000 inclusive?

80, p. 8
Show that the integer N can be taken so large that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}$ is larger than 100.

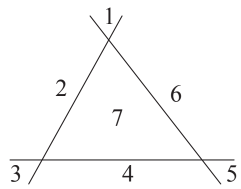
81, p. 8
Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be $2n$ positive real numbers. Show that either

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n$$

or

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq n.$$

83, p. 8
The Figure shows three lines dividing the plane into seven regions. Find the maximum number of regions into which the plane can be divided by n lines.



84, p. 8
In a certain town, the blocks are rectangular, with the streets (of zero width) running E–W, the avenues N–S. A man wishes to go from one corner to another m blocks east and n blocks north. The shortest path can be achieved in many ways. How many?

89, p. 9
Given n points in the plane, any listing (permutation) $p_1, p_2, \ldots, p_n$ of them determines the path, along straight segments, from p_1 to p_2, then from p_2 to p_3, …, ending with the segment from p_{n-1} to p_n. Show that the shortest such broken-line path does not cross itself.

93, p. 9
Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be any real numbers ≥ 1. Show that

$$(1 + a_1) \cdot (1 + a_2) \cdots (1 + a_n) \geq \frac{2^n}{n+1}(1 + a_1 + a_2 + \cdots + a_n).$$

97, p. 10
Let n be a fixed positive integer. For any choice of n real numbers satisfying $0 \leq x_i \leq 1$, $i = 1, 2, \ldots, n$, there corresponds the sum below. Let $S(n)$ denote the largest possible value of this sum. Find $S(n)$.

$$\begin{array}{rlllllll}
\displaystyle\sum_{1 \leq i < j \leq n} |x_i - x_j| = & |x_1 - x_2| & + |x_1 - x_3| & + |x_1 - x_4| & + \cdots & + |x_1 - x_{n-1}| & + |x_1 - x_n| \\
& & + |x_2 - x_3| & + |x_2 - x_4| & + \cdots & + |x_2 - x_{n-1}| & + |x_2 - x_n| \\
& & & + |x_3 - x_4| & + \cdots & + |x_3 - x_{n-1}| & + |x_3 - x_n| \\
& & & & + & & \\
& & & & & \ddots & \\
& & & & & + |x_{n-2} - x_{n-1}| & + |x_{n-2} - x_n| \\
& & & & & & + |x_{n-1} - x_n|
\end{array}$$

102, p. 10
Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person "A" phones person "B", "A" tells "B" everything he knows, while "B" tells "A" nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything?

105, p. 11
Show that every simple polyhedron has at least two faces with the same number of edges.

112, p. 11
Show that, for all positive real numbers p, q, r, s,

$$(p^2 + p + 1)(q^2 + q + 1)(r^2 + r + 1)(s^2 + s + 1) \geq 81pqrs.$$

117, p. 12
If a, b, c denote the lengths of the sides of a triangle show that

$$3\,(bc + ca + ab) \leq (a + b + c)^2 < 4\,(bc + ca + ab).$$

128, p. 13
Suppose the polynomial $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ can be factored into

$$(x + r_1)(x + r_2) \cdots (x + r_n),$$

where $r_1, r_2, \ldots, r_n$ are real numbers. Prove that $(n - 1)a_1^2 \geq 2na_2$.

129, p. 13
For each positive integer n, determine the smallest positive number $k(n)$ such that

$$k(n) + \sin\frac{A}{n}, \quad k(n) + \sin\frac{B}{n}, \quad k(n) + \sin\frac{C}{n}$$

are the sides of a triangle whenever A, B, C are the angles of a triangle.

130, p. 13
Prove that, for $n = 1, 2, 3, \ldots$,
(a) $(n + 1)^n \geq 2^n n!$;
(b) $(n + 1)^n(2n + 1)^n \geq 6^n(n!)^2$.

131, p. 13
Let z_1, z_2, z_3 be complex numbers satisfying:

$$(1)\ \ z_1z_2z_3 = 1,$$

$$(2)\ \ z_1 + z_2 + z_3 = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}.$$

Show that at least one of them is 1.

132, p. 13
Let m_a, m_b, m_c and w_a, w_b, w_c denote, respectively, the lengths of the medians and angle bisectors of a triangle. Prove that

$$\sqrt{m_a} + \sqrt{m_b} + \sqrt{m_c} \geq \sqrt{w_a} + \sqrt{w_b} + \sqrt{w_c}.$$

134, p. 13
If x, y, z are positive numbers, show that

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z}.$$

140, p. 14
Suppose that $0 \leq x_i \leq 1$ for $i = 1, 2, \ldots, n$. Prove that

$$2^{n-1}(1 + x_1x_2 \cdots x_n) \geq (1 + x_1)(1 + x_2) \cdots (1 + x_n),$$

with equality if and only if $n - 1$ of the x_i's are equal to 1.

141, p. 14
Sherwin Betlotz, the tricky gambler, will bet even money that you can't pick three cards from a 52-card deck without getting at least one of the twelve face cards. Would you bet with him?

146, p. 14
If $S = x_1 + x_2 + \cdots + x_n$, where $x_i > 0$ $(i = 1, 2, \ldots, n)$, prove that

$$\frac{S}{S - x_1} + \frac{S}{S - x_2} + \cdots + \frac{S}{S - x_n} \geq \frac{n^2}{n - 1},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

156, p. 15
Suppose that r is a nonnegative rational taken as an approximation to $\sqrt{2}$. Show that $\frac{r+2}{r+1}$ is always a better rational approximation.

159, p. 15
Prove that the sum of the areas of any three faces of a tetrahedron is greater than the area of the forth face.

160, p. 15
Let a, b, c be the lengths of the sides of a right-angled triangle, the hypotenuse having length c. Prove that $a + b \leq \sqrt{2}c$. When does equality hold?

161, p. 15
Determine all θ such that $0 \leq \theta \leq \frac{\pi}{2}$ and $\sin^5 \theta + \cos^5 \theta = 1$.

165, p. 15
If x is a positive real number not equal to unity and n is a positive integer, prove that

$$\frac{1 - x^{2n+1}}{1 - x} \geq (2n + 1)\, x^n.$$

169, p. 15
If a, b, c, d are positive real numbers, prove that

$$\frac{a^2 + b^2 + c^2}{a + b + c} + \frac{b^2 + c^2 + d^2}{b + c + d} + \frac{c^2 + d^2 + a^2}{c + d + a} + \frac{d^2 + a^2 + b^2}{d + a + b} \geq a + b + c + d$$

with equality only if $a = b = c = d$.

172, p. 16
Prove that, for real numbers x, y, z,

$$x^4(1 + y^4) + y^4(1 + z^4) + z^4(1 + x^4) \geq 6x^2y^2z^2.$$

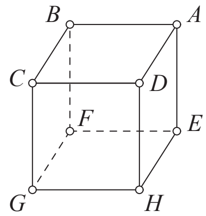
When is there equality?

175, p. 16
If $a_i \geq 1$ for $i = 1, 2, \ldots$, prove that, for each positive integer n,

$$n + a_1 a_2 \cdots a_n \geq 1 + a_1 + a_2 + \cdots + a_n$$

with equality if and only if no more than one of the a_i's is different from 1.

200, p. 18
If $ABCDEFGH$ is a cube, as shown in the Figure, determine the minimum perimeter of a triangle PQR whose vertices P, Q, R lie on the edges AB, CG, EH respectively.



206, p. 18
(a) In a triangle ABC, $AB = 2\,BC$. Prove that BC must be the shortest side. If the perimeter of the triangle is 24, prove that $4 < BC < 6$.
(b) If one side of a triangle is three times another and the perimeter is 24, find bounds for the length of the shortest side.

207, p. 18
Show that, if k is a nonnegative integer:

(a) $1^{2k} + 2^{2k} + 3^{2k} \geq 2 \cdot 7^k$;
(b) $1^{2k+1} + 2^{2k+1} + 3^{2k+1} \geq 6^{k+1}$.

When does equality occur?

209, p. 18
What is the smallest integer, which, when divided in turn by $2, 3, 4, \ldots, 10$ leaves remainders of $1, 2, 3, \ldots, 9$ respectively?

215, p. 19
Let $ABCD$ be a tetrahedron whose faces have equal areas. Suppose O is an interior point of $ABCD$ and L, M, N, P are the feet of the perpendiculars from O to the four faces. Prove that

$$OA + OB + OC + OD \geq 3\,(OL + OM + ON + OP).$$

219, p. 19
Sketch the graph of the inequality

$$|x^2 + y| \leq |y^2 + x|.$$

220, p. 19
Prove that the inequality

$$3a^4 - 4a^3 b + b^4 \geq 0$$

holds for all real numbers a and b.

224, p. 20
Prove or disprove the following statement. Given a line l and two points A and B not on l, the point P on l for which $\sphericalangle APB$ is largest must lie between the feet of the perpendiculars from A and B to l.

225, p. 20
Determine all triangles ABC for which

$$\cos A \cos B + \sin A \sin B \sin C = 1.$$

227, p. 20
Suppose that x, y, and z are nonnegative real numbers. Prove that

$$8\,(x^3 + y^3 + z^3)^2 \geq 9\,(x^2 + yz)(y^2 + zx)(z^2 + xy).$$

231, p. 20
If a, b, c are the lengths of the sides of a triangle, prove that

$$abc \geq (a + b - c)(b + c - a)(c + a - b).$$

232, p. 20
Prove that a longest chord of a centrally-symmetric region must pass through the center.

238, p. 21
Show that, for all real values of x (radians), $\cos(\sin x) > \sin(\cos x)$.

243, p. 21
If A, B, C denote the angles of a triangle, determine the maximum value of

$$\sin^2 A + \sin B \sin C \cos A.$$

250, p. 22
Given the equal sides of an isosceles triangle, what is the length of the third side which will provide the maximum area of the triangle?

253, p. 22
What is the smallest perfect square that ends with the four digits 9009?

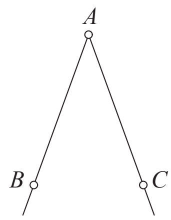
267, p. 23
(a) What is the area of the region in the Cartesian plane whose points (x, y) satisfy

$$|x| + |y| + |x + y| \leq 2?$$

(b) What is the volume of the region in space whose points (x, y, z) satisfy

$$|x| + |y| + |z| + |x + y + z| \leq 2?$$

269, p. 24
AB and AC are two roads with rough ground in between. (See Figure.) The distances AB and AC are both equal to p, while the distance BC is equal to q. A man at point B wishes to walk to C. On the road he walks with speed v, and on the rough ground his walking speed is w. Show that, if he wishes to take minimum time, he may do so by picking one of two particular routes. In fact, argue that he should go:
(a) by road through A if $2pw \leq qv$;
(b) along the straight path BC if $2pw \geq qv$.



271, p. 24
For positive integers n define

$$f(n) = 1^n + 2^{n-1} + 3^{n-2} + 4^{n-3} + \cdots + (n-2)^3 + (n-1)^2 + n.$$

What is the minimum value of $\frac{f(n+1)}{f(n)}$?

272, p. 24
Let a, b, c, d be natural numbers not less than 2. Write down, using parentheses, the various interpretations of

$$a^{b^{c^d}}.$$

For example, we might have $a^{((b^c)^d)} = a^{(b^{cd})}$ or $(a^b)^{(c^d)} = a^{b(c^d)}$. In general, these interpretations will not be equal to each other.

For what pairs of interpretaions does an inequality always hold? For pairs not necessarily satisfying an inequality in general, give numerical examples to illustrate particular instances of either inequality.

274, p. 24
There are $n!$ permutations $(s_1, s_2, \ldots, s_n)$ of $(1, 2, 3, \ldots, n)$. How many of them satisfy $s_k \geq k-2$ for $k = 1, 2, \ldots, n$?

275, p. 24
Prove that, for any quadrilateral with sides a, b, c, d, it is true that

$$a^2 + b^2 + c^2 > \frac{1}{3}d^2.$$

281, p. 25
Find the point which minimizes the sum of its distances from the vertices of a given convex quadrilateral.

301, p. 27
(a) Verify that

$$1 = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{20} + \frac{1}{41} + \frac{1}{110} + \frac{1}{1640}.$$

(b) Show that any representation of 1 as the sum of distinct reciprocals of numbers drawn from the arithmetic progression $\{2, 5, 8, 11, 14, 17, 20, \ldots\}$, such as is given in (a), must have at least eight terms.

303, p. 27
A pollster interviewed a certain number, N, of persons as to whether they used radio, television and/or newspapers as a source of news. He reported the following findings:

50 people used television as a source of news, either alone or in conjunction with other sources;

61 did not use radio as a source of news;

13 did not use newspapers as a source of news;

74 had at least two sources of news.

Find the maximum and minimum values of N consistent with this information.

Give examples of situations in which the maximum and in which the minimum values of N could occur.

305, p. 27
x, y, and z are real numbers such that

$$x + y + z = 5 \qquad \text{and}$$
$$xy + yz + zx = 3.$$

Determine the largest value that any one of the three numbers can be.

311, p. 28
Let

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

be a polynomial whose coefficients satisfy the conditions $0 \le a_i \le a_0$ $(i = 1, 2, \ldots, n)$. Let

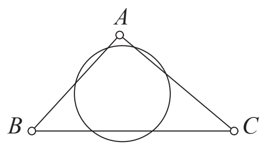
$$(f(x))^2 = b_0 + b_1x + \cdots + b_{n+1}x^{n+1} + \cdots + b_{2n}x^{2n}.$$

Prove that

$$b_{n+1} \le \frac{1}{2}(f(1))^2.$$

313, p. 28
Given a set of $(n+1)$ positive integers, none of which exceeds $2n$, show that at least one member of the set must divide another member of the set.

321, p. 29
ABC is a triangle whose angles satisfy $\sphericalangle A \ge \sphericalangle B \ge \sphericalangle C$. Circles are drawn such that each circle cuts each side of the triangle internally in two distinct points (see Figure).



(a) Show that the lower limit to the radii of such circles is the radius of the inscribed circle of the triangle ABC.
(b) Show that the upper limit to the radii of such circles is not necessarily equal to R, the radius of the circumscribed circle of triangle ABC. Find this upper limit in terms of R, A and B.

331, p. 31
Show that each of the following polynomials is nonnegative for all real values of the variables, but that neither can be written as a sum of squares of real polynomials:

(a) $x^2y^2 + y^2z^2 + z^2x^2 + w^4 - 4xyzw$;
(b) $x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$.

337, p. 31
Suppose u and v are two real numbers such that u, v and uv are the three roots of a cubic polynomial with rational coefficients. Show that at least one root is rational.

340, p. 31
Show without using a calculator that

$$7^{1/2} + 7^{1/3} + 7^{1/4} < 7 \qquad \text{and}$$
$$4^{1/2} + 4^{1/3} + 4^{1/4} > 4.$$

344, p. 31
Let u be an arbitrary but fixed number between 0 and 1, i.e., $0 < u < 1$. Form the sequence $u_1, u_2, u_3, \dots$ as follows:

$$u_1 = 1 + u$$
$$u_2 = \frac{1}{u_1} + u$$
$$u_3 = \frac{1}{u_2} + u$$

and so on, i.e., $u_n = \frac{1}{u_{n-1}} + u$ for $n = 2, 3, 4, \dots$. Does it ever happen that $u_n \le 1$?

350, p. 32
Let $P_1, P_2, \dots, P_m$ be m points on a line and $Q_1, Q_2, \dots, Q_n$ be n points on a distinct and parallel line. All segments P_iQ_j are drawn. What is the maximum number of points of intersection?

356, p. 32
Show that for any real numbers x, y, and any positive integer n,

(a) $0 \le [nx] - n[x] \le n - 1$,

(b) $[x] + [y] + (n-1)[x+y] \le [nx] + [ny]$.

($[z]$ denotes the greatest integer not exceeding z.)

358, p. 33
Let p be the perimeter and m the sum of the lengths of the three medians of any triangle. Prove that

$$\frac{3}{4}p < m < p.$$

359, p. 33
(a) Which is larger, $29\sqrt{14} + 4\sqrt{15}$ or 124?
(b) Which is larger, $759\sqrt{7} + 2\sqrt{254}$ or 2040?
(No calculators please.)

372, p. 34
Five gamblers A, B, C, D, E play together a game which terminates with one of them losing and then the loser pays to each of the other four as much as each has. Thus, if they start a game possessing $\alpha, \beta, \gamma, \delta, \epsilon$ dollars respectively, and say for example that B loses, then B gives A, C, D, E respectively $\alpha, \gamma, \delta, \epsilon$ dollars, after which A, B, C, D, E have $2\alpha, \beta - \alpha - \gamma - \delta - \epsilon, 2\gamma, 2\delta, 2\epsilon$ dollars respectively. They play five games: A loses the first game, B loses the second, C loses the third, D the fourth and E the fifth. After the final payment, made by E, they find that they are equally wealthy, i.e., each has the same integral number of dollars as the others. What is the smallest amount that each could have started with?

373, p. 35
Consider a square array of numbers consisting of m rows and m columns. Let a_{ij} be the number entered in the ith row and jth column. For each i, let r_i denote the sum of the numbers in the ith row, and c_i the sum of the numbers in the ith column. Show that there are distinct indices i and j for which $(r_i - c_i)(r_j - c_j) \le 0$.

374, p. 35
The function f has the property that

$$|f(a) - f(b)| \leq |a - b|^2$$

for any real numbers a and b. Show that f is a constant function.

375, p. 35
A rocket car accelerates from 0 kph to 240 kph in a test run of one kilometer. If the acceleration is not allowed to increase (but it may decrease) during the run, what is the longest time the run can take?

384, p. 36
A manufacturer had to ship 150 washing machines to a neighboring town. Upon inquiring he found that two types of trucks were available. One type was large and would carry 18 machines, the other type was smaller and would carry 13 machines. The cost of transporting a large truckload was \$35, that of a small one \$25. What is the most economical way of shipping the 150 machines?

388, p. 36
Let l and m be parallel lines and P a point between them. Find the triangle APB of smallest area, with A on l, B on m, and $\sphericalangle APB = 90°$.

394, p. 37
Show that if A, B, C are the angles of any triangle, then

$$3\left(\sin^2 A + \sin^2 B + \sin^2 C\right) - 2\left(\cos^3 A + \cos^3 B + \cos^3 C\right) \leq 6.$$

401, p. 38
It is intuitive that the smallest regular n-gon which can be inscribed in a given regular n-gon will have its vertices at the midpoints of the sides of the given n-gon. Give a proof!

402, p. 38
The real numbers x, y, z are such that

$$x^2 + (1 - x - y)^2 + (1 - y)^2 = y^2 + (1 - y - z)^2 + (1 - z)^2 = z^2 + (1 - z - x)^2 + (1 - x)^2.$$

Determine the minimum value of $x^2 + (1 - x - y)^2 + (1 - y)^2$.

405, p. 38
Determine the maximum value of

$$P = \frac{(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)}{(abc)^2},$$

where a, b, c are real and

$$\frac{b^2 + c^2 - a^2}{bc} + \frac{c^2 + a^2 - b^2}{ca} + \frac{a^2 + b^2 - c^2}{ab} = 2.$$

407, p. 39
If $S = a_1 + a_2 + \cdots + a_n$, where $a_1, a_2, \ldots, a_n$ are sides of a polygon, prove that

$$\frac{n+2}{S - a_k} \geq \sum_{i=1}^{n} \frac{1}{S - a_i} \qquad \text{for } k = 1, 2, \ldots, n.$$

409, p. 39
Determine the maximum area of a rectangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

410, p. 39
If w and z are complex numbers, prove that

$$2\,|w|\,|z|\,|w-z| \geq \{|w|+|z|\}|\,w|z|-z|w|\,|.$$

412, p. 39
If $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n > 0$, prove that any root r of the polynomial

$$P(z) \equiv a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

satisfies $|r| \leq 1$, i.e., all the roots lie inside or on the unit circle centered at the origin in the complex plane.

418, p. 40
For any simple closed curve there may exist more than one chord of maximum length. For example, in a circle all the diameters are chords of maximum length. In contrast, a proper ellipse has only one chord of maximum length (the major axis). Show that no two chords of maximum length of a given simple closed curve can be parallel.

423, p. 40
What is the least number of plane cuts required to cut a block $a \times b \times c$ into abc unit cubes, if piling is permitted? (L. Moser)

428, p. 41
What is the largest value of n, in terms of m, for which the following statement is true? If from among the first m natural numbers any n are selected, among the remaining $m-n$ at least one will be a divisor of another. (Student-Faculty Colloquium, Carleton College)

429, p. 41
Conjecture: If $f(t), g(t), h(t)$ are real-valued functions of a real variable, then there are numbers x, y, z such that $0 \leq x, y, z \leq 1$ and

$$|xyz - f(x) - g(y) - h(z)| \geq \frac{1}{3}.$$

Prove this conjecture. Show that if the number $\frac{1}{3}$ is replaced by a constant $c > \frac{1}{3}$, then the conjecture is false; i.e., the number $\frac{1}{3}$ in the conjecture is best possible.

431, p. 41
The digital expression $x_n x_{n-1} \ldots x_1 x_0$ is the representation of the number A to base a as well as that of B to base b, while the digital expression $x_{n-1} x_{n-2} \ldots x_1 x_0$ is the representation of C to base a and also of D to base b. Here, a, b, n are integers greater than one.
Show that $\frac{C}{A} < \frac{D}{B}$ if and only if $a > b$.

432, p. 41
Show that 5 or more great circles on a sphere, no 3 of which are concurrent, determine at least one spherical polygon having 5 or more sides. (L. Moser)

437, p. 41
A pack of 13 distinct cards is shuffled in some particular manner and then repeatedly in exactly the same manner. What is the maximum number of shuffles required for the cards to return to their original positions?

439, p. 41
If a, a' and b, b' and c, c' are the lengths of the three pairs of opposite edges of an arbitrary tetrahedron, prove that

(i) there exists a triangle whose sides have lengths $a + a'$, $b + b'$ and $c + c'$;

(ii) the triangle in (i) is acute.

440, p. 41
Determine the maximum value of

$$\sqrt[3]{4 - 3x + \sqrt{16 - 24x + 9x^2 - x^3}} + \sqrt[3]{4 - 3x - \sqrt{16 - 24x + 9x^2 - x^3}}$$

in the interval $-1 \leq x \leq 1$.

442, p. 42
If e and f are the lengths of the diagonals of a quadrilateral of area F, show that $e^2 + f^2 \geq 4F$, and determine when there is equality.

443, p. 42
Inside a cube of side 15 units there are 11000 given points. Prove that there is a sphere of unit radius within which there are at least 6 of the given points.

445, p. 42
Prove that if the top 26 cards of an ordinary shuffled deck contain more red cards than there are black cards in the bottom, then there are in the deck at least three consecutive cards of the same color. (L. Moser)

447, p. 42
If m and n are positive integers, show that

$$\frac{1}{\sqrt[n]{m}} + \frac{1}{\sqrt[m]{n}} > 1.$$

453, p. 43
Seventy-five coplanar points are given, no three collinear. Prove that, of all the triangles which can be drawn with these points as vertices, not more than seventy per cent are acute-angled.

454, p. 43
Let T_1 and T_2 be two acute-angled triangles with respective side lengths a_1, b_1, c_1 and a_2, b_2, c_2, areas Δ_1 and Δ_2, circumradii R_1 and R_2 and inradii r_1 and r_2. Show that, if $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, then $\Delta_1 \geq \Delta_2$ and $R_1 \geq R_2$, but it is not necessarily true that $r_1 \geq r_2$.

460, p. 43
Determine all real x, y, z such that

$$xa^2 + yb^2 + zc^2 \leq 0$$

whenever a, b, c are sides of a triangle.

462, p. 43
Determine the maximum value of

$$(\sin A_1)(\sin A_2)\cdots(\sin A_n)$$

if

$$(\tan A_1)(\tan A_2)\cdots(\tan A_n) = 1.$$

463, p. 43
Two triangles have sides (a_1, b_1, c_1), (a_2, b_2, c_2) and respective areas Δ_1, Δ_2. Establish the Newberg-Pedoe inequality

$$a_1^2\,(b_2^2 + c_2^2 - a_2^2) + b_1^2\,(c_2^2 + a_2^2 - b_2^2) + c_1^2\,(a_2^2 + b_2^2 - c_2^2) \geq 16\,\Delta_1\Delta_2,$$

and determine when there is equality.

465, p. 44
Let m and n be given positive numbers with $m \geq n$. Call a number x "good" (with respect to m and n) if:

$$m^2 + n^2 - a^2 - b^2 \geq (mn - ab)x \quad \text{for all } 0 \leq a \leq m,\ \ 0 \leq b \leq n.$$

Determine (in terms of m and n) the largest good number.

466, p. 44
Prove that, for any quadrilateral (simple or not, planar or not) of sides a, b, c, d

$$a^4 + b^4 + c^4 \geq \frac{d^4}{27}.$$

467, p. 44
Determine the maximum of x^2y, subject to constraints

$$x + y + \sqrt{2x^2 + 2xy + 3y^2} = k \ \ (\text{constant}), \quad x, y \geq 0.$$

468, p. 44
Prove

$$\frac{4^m}{2\sqrt{m}} < \binom{2m}{m} < \frac{4^m}{\sqrt{3m+1}}$$

for natural numbers $m > 1$.

484, p. 45
Find the rhombus of minimum area which can be inscribed (one vertex to a side) within a given parallelogram. (*Math. Gazette* 1904)

488, p. 45
Determine the largest real number k, such that

$$|z_2z_3 + z_3z_1 + z_1z_2| \geq k\,|z_1 + z_2 + z_3|$$

for all complex numbers z_1, z_2, z_3 with unit absolute value.

490, p. 45
If the roots of the equation

$$a_0x^n - na_1x^{n-1} + \frac{n(n-1)}{2}a_2x^{n-2} - \cdots + (-1)^n a_n = 0$$

are all positive, show that $a_r a_{n-r} > a_0 a_n$ for all values of r between 1 and $n-1$ inclusive, unless the roots are all equal. (A. Lodge, *Math. Gazette* 1896)

491, p. 46
Suppose $u \le 1 \le w$. Determine all values of v for which $u + vw \le v + wu \le w + uv$.

492, p. 46
Find the shortest distance between the plane $Ax + By + Cz = 1$ and the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

You can assume A, B, C are all positive and that the plane does not intersect the ellipsoid. (No calculus please.)

493, p. 46
One of the problems on the first William Lowell Putnam Mathematical Competition, was to find the length of the shortest chord that is normal to the parabola $y^2 = 2ax$ at one end. (Assume $a > 0$.) A calculus solution is straight forward. Give a completely "no calculus" solution.

495, p. 46
If P, Q, R are any three points inside or on a unit square, show that the smallest of the three distances determined by them is at most $2\sqrt{2-\sqrt{3}}$, i.e., show

$$\min(PQ, QR, RP) \le 2\sqrt{2-\sqrt{3}}.$$

Also determine when there is equality.

496, p. 46
Any 5 points inside or on a 2×1 rectangle determine 10 segments (joining the pairs of points). Show that the smallest of these 10 segments has a length at most $2\sqrt{2-\sqrt{3}}$. (Leo Moser)

Inequalities proposed in
"More Mathematical Morsels"
by R. Honsberger

1, p. 20
If $a, b, c,$ are nonnegative real numbers such that

$$(1+a)(1+b)(1+c) = 8,$$

prove that the product abc cannot exceed 1.

2, p. 26
Suppose S is a set of n *odd* positive integers $a_1 < a_2 < \cdots < a_n$ such that no two of the differences $|a_i - a_j|$ are the same. Prove, then, that the sum Σ of all the integers must be at least $n(n^2+2)/3$.

3, p. 33
For every integer $n > 1$, prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} > \frac{3n}{2n+1}.$$

4, p. 48
A and B play a game on a given triangle PQR as follows. First A chooses a point X on QR; then B takes his choice of Y on RP, and finally, A chooses Z on PQ. A's object is to make the inscribed $\triangle XYZ$ as large as possible (in area) while B is trying to make it as small as possible. What is the greatest area that A can be sure of getting?

5, p. 75
S is a set of 1980 points in the plane such that the distance between any two of them is at least 1. Prove that S must contain a subset T of 220 points such that the distance between each two of them is at least $\sqrt{3}$.

6, p. 86
S is a collection of disjoint intervals in the unit interval $[0, 1]$. If no two points of S are 1/10th of a unit apart, prove that the sum of the lengths of all the intervals in S cannot exceed 1/2.

7, p. 87
M is a set of $3n$ points in the plane such that the maximum distance between any two of the points is 1 unit. Prove that
(a) for any 4 points of M, the distance between some two of them is less than or at most $1/\sqrt{2}$,
(b) some circles of radius $\leq \sqrt{3}/2$ encloses the entire set M
(d) there is some pair of the $3n$ points of M whose distance apart is at most $4/(3\sqrt{n} - \sqrt{3})$.

8, p. 119
Suppose x and y vary over the nonnegative real numbers. If the value of

$$x + y + \sqrt{2x^2 + 2xy + 3y^2}$$

is always 4, prove that x^2y is always less than 4.

9, p. 125

In the plane, n circles of unit radius are drawn with different centers. Of course, overlapping circles partly cover each other's circumferences. A given circle could be so overlaid that any uncovered parts of its circumference are all quite small; that is, it might have no sizable uncovered arcs at all. However, this can't be true of every circle; prove that some circle must have a continuously uncovered arc which is at least $1/n$th of its circumference.

10, p. 147

The first n positive integers $(1, 2, 3, \ldots, n)$ are spotted around a circle in any order you wish and the positive differences $d_1, d_2, \ldots, d_n$ between consecutive pairs are determined. Prove that, no matter how the integers might be jumbled up around the circle, the sum of these n differences,

$$S = d_1 + d_2 + \cdots + d_n,$$

will always amount to at least $2n - 2$.

11, p. 149

Prove that a regular hexagon $H = ABCDEF$ of side 2 can be covered with 6 disks of unit radius, *but not by* 5.

12, p. 153

If 10 points are chosen in a circle C of diameter 5, prove that the distance between some pair of them is less than 2.

13, p. 163

If a, b, c, d are positive real numbers that add up to 1, prove that

$$S = \sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} < 6.$$

14, p. 187

Between what two integers does the sum S lie, where S is the unruly sum

$$S = \sum_{n=1}^{10^9} n^{-\frac{2}{3}} = 1 + \frac{1}{\sqrt[3]{2^2}} + \frac{1}{\sqrt[3]{3^2}} + \cdots + \frac{1}{\sqrt[3]{(10^9)^2}}?$$

15, p. 195

Prove that the positive root of

$$x(x+1)(x+2)\cdots(x+1981) = 1$$

is less than $1/1981!$.

16, p. 195

Let S be a collection of positive integers, not necessarily distinct, which contains the number 68. The average of the numbers in S is 56; however, if a 68 is removed, the average would drop to 55. What is the largest number that S can possibly contain?

17, p. 198

Prove that, among any seven real numbers $y_1, y_2, \ldots, y_7$, some two, y_i and y_j, are such that

$$0 \leq \frac{y_i - y_j}{1 + y_i y_j} \leq \frac{1}{\sqrt{3}}.$$

18, p. 199
A unit square is to be covered by 3 congruent circular disks.
(a) Show that there are disks of diameter less tahn the diagonal of the square that provide a covering.
(b) Determine the smallest possible diameter.

19, p. 203
Let $x_1, x_2, \ldots, x_n$, where $n \geq 2$, be positive numbers that add up to 1. Prove that

$$S = \frac{x_1}{1 + x_2 + x_3 + \cdots + x_n} + \frac{x_2}{1 + x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n}{1 + x_1 + x_2 + \cdots + x_{n-1}} \geq \frac{n}{2n-1}.$$

20, p. 205
If the positive real numbers $x_1, x_2, \ldots, x_{n+1}$ are such that

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_{n+1}} = 1,$$

prove that

$$x_1 x_2 \cdots x_{n+1} \geq n^{n+1}.$$

21, p. 244
If $0 \leq a, b, c, d \leq 1$, prove that

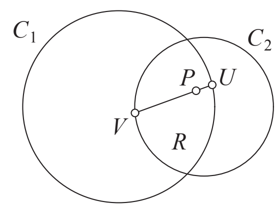
$$(1-a)(1-b)(1-c)(1-d) + a + b + c + d \geq 1.$$

22, p. 246
Determine an experiment in probability to justify the inequality

$$(1-p^m)^n + (1-q^n)^m > 1$$

for all positive integers m and n greater than 1 and all positive real numbers p and q such that $p + q \leq 1$.

23, p. 283
Let the fixed point P be taken anywhere inside the lens-shaped region of intersection R of two given circles C_1 and C_2. Let UV be a chord of R through P. Determine how to construct the chord which makes the product $PU \cdot PV$ a minimum.



Inequalities proposed in
“Old and New Inequalities”
by T. Andreescu, V. Cîrtoaje, G. Dospinescu, M. Lascu

1. *Kömal*
Prove the inequality

$$\sqrt{a^2+(1-b)^2}+\sqrt{b^2+(1-c)^2}+\sqrt{c^2+(1-a)^2}\geq\frac{3\sqrt{2}}{2}$$

holds for arbitrary real numbers a, b.

2. *Junior TST 2002, Romania*, [Dinu Şerbănescu]
If $a, b, c \in (0,1)$ prove that

$$\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}<1.$$

3. *Gazeta Matematică*, [Mircea Lascu]
Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{b+c}{\sqrt{a}}+\frac{c+a}{\sqrt{b}}+\frac{a+b}{\sqrt{c}}\geq\sqrt{a}+\sqrt{b}+\sqrt{c}+3.$$

4. *Tournament of the Towns, 1993*
If the equation $x^4+ax^3+2x^2+bx+1=0$ has at least one real root, then $a^2+b^2\geq 8$.

5.
Find the maximum value of the expression $x^3+y^3+z^3-3xyz$ where $x^2+y^2+z^2=1$ and x, y, z are real numbers.

6. *Ukraine, 2001*
Let a, b, c, x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$ax+by+cz+2\sqrt{(xy+yz+zx)(ab+bc+ca)}\leq a+b+c.$$

7. [Darij Grinberg]
If a, b, c are three positive real numbers, then

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\geq\frac{9}{4\,(a+b+c)}.$$

8. *Gazeta Matematică*, [Hojoo Lee]
Let $a, b, c\geq 0$. Prove that

$$\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4}\geq$$
$$\geq a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.$$

9. *JBMO 2002 Shortlist*
If a, b, c are positive real numbers such that $abc=2$, then

$$a^3+b^3+c^3\geq a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}.$$

When does equality hold?

10. *Gazeta Matematică*, [Ioan Tomescu]
Let $x, y, z > 0$. Prove that

$$\frac{xyz}{(1+3x)(x+8y)(y+9z)(z+6)} \leq \frac{1}{7^4}.$$

When do we have equality?

11. [Mihai Piticari, Dan Popescu]
Prove that

$$5\left(a^2+b^2+c^2\right) \leq 6\left(a^3+b^3+c^3\right)+1,$$

for all $a, b, c > 0$ with $a+b+c=1$.

12. [Mircea Lascu]
Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$, $n \geq 2$ and $a > 0$ such that

$$x_1+x_2+\cdots+x_n = a \qquad \text{and} \qquad x_1^2+x_2^2+\cdots+x_n^2 \leq \frac{a^2}{n-1}.$$

Prove that $x_i \in \left[0, \frac{2a}{n}\right]$, for all $i \in \{1, 2, \ldots, n\}$.

13. [Adrian Zahariuc]
Prove that for any $a, b, c \in (1, 2)$ the following inequality holds

$$\frac{b\sqrt{a}}{4b\sqrt{c}-c\sqrt{a}}+\frac{c\sqrt{b}}{4c\sqrt{a}-a\sqrt{b}}+\frac{a\sqrt{c}}{4a\sqrt{b}-b\sqrt{c}} \geq 1.$$

14.
For positive real numbers a, b, c such that $abc \leq 1$, prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a+b+c.$$

15. [Vasile Cirtoaje, Mircea Lascu]
Let a, b, c, x, y, z be positive real numbers such that $a+x \geq b+y \geq c+z$ and $a+b+c = x+y+z$. Prove that $ay+bx \geq ac+xz$.

16. *Junior TST 2003, Romania*, [Vasile Cirtoaje, Mircea Lascu]
Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$1+\frac{3}{a+b+c} \geq \frac{6}{ab+ac+bc}.$$

17. *JBMO 2002 Shortlist*
Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}.$$

18. *Russia 2004*
Prove that if $n > 3$ and $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{1+x_1+x_1x_2}+\frac{1}{1+x_2+x_2x_3}+\cdots+\frac{1}{1+x_n+x_nx_1} > 1.$$

19. [Marian Tetiva]
Let x, y, z be positive real numbers satisfying the condition

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

Prove that

(a) $xyz \leq \frac{1}{8}$;

(b) $xy + xz + yz \leq \frac{3}{4} \leq x^2 + y^2 + z^2$;

(c) $xy + xz + yz \leq \frac{1}{2} + 2xyz$.

20. *Gazeta Matematică*, [Marius Olteanu]
Let $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$ so that $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Prove that

$$|\cos x_1| + |\cos x_2| + |\cos x_3| + |\cos x_4| + |\cos x_5| \geq 1.$$

21. [Florina Cárlan, Marian Tetiva]
Prove that if $x, y, z > 0$ satisfy the condition $x + y + z = xyz$ then

$$xy + xz + yz \geq 3 + \sqrt{x^2+1} + \sqrt{y^2+1} + \sqrt{z^2+1}.$$

22. *JBMO, 2003*, [Laurenţiu Panaitopol]
Prove that

$$\frac{1+x^2}{1+y+z^2} + \frac{1+y^2}{1+z+x^2} + \frac{1+z^2}{1+x+y^2} \geq 2,$$

for any real numbers $x, y, z > -1$.

23.
Let $a, b, c > 0$ with $a + b + c = 1$. Show that

$$\frac{a^2+b}{b+c} + \frac{b^2+c}{c+a} + \frac{c^2+a}{a+b} \geq 2.$$

24. *Kvant, 1988*
Let $a, b, c \geq 0$ such that $a^4 + b^4 + c^4 \leq 2(a^2b^2 + b^2c^2 + c^2a^2)$. Prove that

$$a^2 + b^2 + c^2 \leq 2\,(ab + bc + ca).$$

25. *Vietnam, 1998*
Let $n \geq 2$ and $x_1, \ldots, x_n$ be positive real numbers satisfying

$$\frac{1}{x_1+1998} + \frac{1}{x_2+1998} + \cdots + \frac{1}{x_n+1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998.$$

26. [Marian Tetiva]
Consider positive real numbers x, y, z so that

$$x^2 + y^2 + z^2 = xyz.$$

Prove the following inequalities

a) $xyz \geq 27$;

b) $xy + xz + yz \geq 27$;

c) $x + y + z \geq 9$;

d) $xy + xz + yz \geq 2(x + y + z) + 9$.

27. *Russia, 2002*
Let x, y, z be positive real numbers with sum 3. Prove that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

28. *Gazeta Matematică*, [D. Olteanu]
Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{b+c} \cdot \frac{a}{2a+b+c} + \frac{b+c}{c+a} \cdot \frac{b}{2b+c+a} + \frac{c+a}{a+b} \cdot \frac{c}{2c+a+b} \geq \frac{3}{4}.$$

29. *India, 2002*
For any positive real numbers a, b, c show that the following inequality holds

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}.$$

30. *Proposed for the Balkan Mathematical Olympiad*
Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{b^2 - bc + c^2} + \frac{b^3}{c^2 - ac + a^2} + \frac{c^3}{a^2 - ab + b^2} \geq \frac{3(ab + bc + ca)}{a + b + c}.$$

31. [Adrian Zahariuc]
Consider the pairwise distinct integers $x_1, x_2, \ldots, x_n$, $n \geq 0$. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1x_2 + x_2x_3 + \cdots + x_nx_1 + 2n - 3.$$

32. *Crux Mathematicorum*, [Murray Klamkin]
Find the maximum value of the expression

$$x_1^2x_2 + x_2^2x_3 + \cdots + x_{n-1}^2x_n + x_n^2x_1$$

when $x_1, x_2, \ldots, x_{n-1}, x_n \geq 0$ add up to 1 and $n > 2$.

33. *IMO Shortlist, 1986*
Find the maximum value of the constant c such that for any $x_1, x_2, \ldots, x_n, \ldots > 0$ for which $x_{k+1} \geq x_1 + x_2 + \cdots + x_k$ for any k, the inequality

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \leq c\sqrt{x_1 + x_2 + \cdots + x_n}$$

also holds for any n.

34. *Russia, 2002*
Given are positive real numbers a, b, c and x, y, z, for which $a + x = b + y = c + z = 1$. Prove that

$$(abc + xyz)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) \geq 3.$$

35. *Gazeta Matematică*, [Viorel Vâjâitu, Alexandru Zaharescu]
Let a, b, c be positive real numbers. Prove that

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \leq \frac{1}{4}(a+b+c).$$

36.
Find the maximum value of the expression

$$a^3(b+c+d) + b^3(c+d+a) + c^3(d+a+b) + d^3(a+b+c),$$

where a, b, c, d are real numbers whose sum of squares is 1.

37. *Crux Mathematicorum 1654*, [Walther Janous]
Let x, y, z be positive real numbers. Prove that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} + \frac{y}{y+\sqrt{(y+z)(y+x)}} + \frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1.$$

38. *Iran, 1999*
Suppose that $a_1 < a_2 < \cdots < a_n$ are real numbers for some integer $n \geq 2$. Prove that

$$a_1a_2^4 + a_2a_3^4 + \cdots + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + \cdots + a_1a_n^4.$$

39. [Mircea Lascu]
Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 4\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

40.
Let $a_1, a_2, \ldots, a_n > 1$ be positive integers. Prove that at least one of the numbers $\sqrt[a_1]{a_2}$, $\sqrt[a_2]{a_3}, \ldots, \sqrt[a_{n-1}]{a_n}, \sqrt[a_n]{a_1}$ is less than or equal $\sqrt[3]{3}$.

41. [Mircea Lascu, Marian Tetiva]
Let x, y, z be positive real numbers which satisfy the condition

$$xy + xz + yz + 2xyz = 1.$$

Prove that the following inequalities hold

a) $xyz \leq \dfrac{1}{8}$;

b) $x + y + z \geq \dfrac{3}{2}$;

c) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 4(x+y+z)$;

d) $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 4(x+y+z) \geq \dfrac{(2z-1)^2}{z(2z+1)}$, where $z = \max\{x, y, z\}$.

42. [Manlio Marangelli]
Prove that for any positive real numbers x, y, z,

$$3\,(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq xyz(x + y + z)^3.$$

43. [Gabriel Dospinescu]
Prove that if a, b, c are real numbers such that $\max\{a, b, c\} - \min\{a, b, c\} \leq 1$, then

$$1 + a^3 + b^3 + c^3 + 6abc \geq 3a^2b + 3b^2c + 3c^2a.$$

44. [Gabriel Dospinescu]
Prove that for any positive real numbers a, b, c we have

$$27 + \left(2 + \frac{a^2}{bc}\right)\left(2 + \frac{b^2}{ca}\right)\left(2 + \frac{c^2}{ab}\right) \geq 6\,(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

45. *TST Singapore*
Let $a_0 = \frac{1}{2}$ and $a_{k+1} = a_k + \frac{a_k^2}{n}$. Prove that $1 - \frac{1}{n} < a_n < 1$.

46. [Călin Popa]
Let a, b, c be positive real numbers, with $a, b, c \in (0, 1)$ such that $ab + bc + ca = 1$. Prove that

$$\frac{a}{1 - a^2} + \frac{b}{1 - b^2} + \frac{c}{1 - c^2} \geq \frac{3}{4}\left(\frac{1 - a^2}{a} + \frac{1 - b^2}{b} + \frac{1 - c^2}{c}\right).$$

47. [Titu Andreescu, Gabriel Dospinescu]
Let $x, y, z \leq 1$ and $x + y + z = 1$. Prove that

$$\frac{1}{1 + x^2} + \frac{1}{1 + y^2} + \frac{1}{1 + z^2} \leq \frac{27}{10}.$$

48. [Gabriel Dospinescu]
Prove that if $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$, then

$$(1 - x)^2(1 - y)^2(1 - z)^2 \geq 2^{15}xyz(x + y)(y + z)(z + x).$$

49.
Let x, y, z be positive real numbers such that $xyz = x + y + z + 2$. Prove that

(1) $xy + yz + zx \geq 2\,(x + y + z)$;

(2) $\sqrt{x} + \sqrt{y} + \sqrt{z} \leq \frac{3}{2}\sqrt{xyz}$.

50. *IMO Shortlist, 1987*
Prove that if x, y, z are real numbers such that $x^2 + y^2 + z^2 = 2$, then

$$x + y + z \leq xyz + 2.$$

51. [Titu Andreescu, Gabriel Dospinescu]
Prove that for any $x_1, x_2, \ldots, x_n \in (0, 1)$ and for any permutation σ of the set $\{1, 2, \ldots, n\}$, we have the inequality

$$\sum_{i=1}^{n} \frac{1}{1 - x_i} \geq \left(1 + \frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{1 - x_i \cdot x_{\sigma(i)}}\right).$$

52. *Vojtech Jarnik*
Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\sum\limits_{i=1}^{n} \frac{1}{1+x_i} = 1$. Prove that

$$\sum_{i=1}^{n} \sqrt{x_i} \geq (n-1) \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}}.$$

53. *USAMO, 1999*, [Titu Andreescu]
Let $n > 3$ and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max\{a_1, a_2, \ldots, a_n\} \geq 2$.

54. [Vasile Cîrtoaje]
If a, b, c, d are positive real numbers, then

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0.$$

55. *France, 1996*
If x and y are positive real numbers, show that $x^y + y^x > 1$.

56. *MOSP, 2001*
Prove that if $a, b, c > 0$ have product 1, then

$$(a+b)(b+c)(c+a) \geq 4\,(a+b+c-1).$$

57.
Prove that for any $a, b, c > 0$,

$$(a^2+b^2+c^2)(a+b-c)(b+c-a)(c+a-b) \leq abc(ab+bc+ca).$$

58. *Kvant, 1988*, [D. P. Mavlo]
Let $a, b, c > 0$. Prove that

$$3 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\frac{(a+1)(b+1)(c+1)}{1+abc}.$$

59. [Gabriel Dospinescu]
Prove that for any positive real numbers $x_1, x_2, \ldots, x_n$ with product 1 we have the inequality

$$n^n \cdot \prod_{i=1}^{n} (x_i^n + 1) \geq \left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{x_i} \right)^n.$$

60. *Kvant, 1993*
Let $a, b, c, d > 0$ such that $a + b + c = 1$. Prove that

$$a^3 + b^3 + c^3 + abcd \geq \min\left\{ \frac{1}{4}, \frac{1}{9} + \frac{d}{27} \right\}.$$

61. *AMM*
Prove that for any real numbers a, b, c we have the inequality

$$\sum (1+a^2)^2(1+b^2)^2(a-c)^2(b-c)^2 \geq (1+a^2)(1+b^2)(1+c^2)(a-b)^2(b-c)^2(c-a)^2.$$

62. [Titu Andreescu, Mircea Lascu]
Let α, x, y, z be positive real numbers such that $xyz = 1$ and $\alpha \geq 1$. Prove that

$$\frac{x^\alpha}{y+z} + \frac{y^\alpha}{z+x} + \frac{z^\alpha}{x+y} \geq \frac{3}{2}.$$

63. *Korea, 2001*
Prove that for any real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that $x_1^2 + \cdots + x_n^2 = y_1^2 + \cdots + y_n^2 = 1$,

$$(x_1y_2 - x_2y_1)^2 \leq 2\left(1 - \sum_{k=1}^{n} x_k y_k\right).$$

64. *TST Romania*, [Laurenţiu Panaitopol]
Let $a_1, a_2, \ldots, a_n$ be pairwise distinct positive integers. Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3}(a_1 + a_2 + \cdots + a_n).$$

65. [Călin Popa]
Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{b\sqrt{c}}{a(\sqrt{3c} + \sqrt{ab})} + \frac{c\sqrt{a}}{b(\sqrt{3a} + \sqrt{bc})} + \frac{a\sqrt{b}}{c(\sqrt{3b} + \sqrt{ca})} \geq \frac{3\sqrt{3}}{4}.$$

66. [Titu Andreescu, Gabriel Dospinescu]
Let a, b, c, d be real numbers such that $(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) = 16$. Prove that

$$-3 \leq ab + bc + cd + da + ac + bd - abcd \leq 5.$$

67. *APMO, 2004*
Prove that

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca)$$

for any positive real numbers a, b, c.

68. [Vasile Cîrtoaje]
Prove that if $0 < x \leq y \leq z$ and $x + y + z = xyz + 2$, then

a) $(1 - xy)(1 - yz)(1 - xz) \geq 0$;

b) $x^2y \leq 1, \quad x^3y^2 \leq \dfrac{32}{27}$.

69. [Titu Andreescu]
Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that at least two of the inequalities

$$\frac{2}{a} + \frac{3}{b} + \frac{6}{c} \geq 6, \quad \frac{2}{b} + \frac{3}{c} + \frac{6}{a} \geq 6, \quad \frac{2}{c} + \frac{3}{a} + \frac{6}{b} \geq 6$$

are true.

70. [Gabriel Dospinescu, Marian Tetiva]
Let $x, y, z > 0$ such that $x + y + z = xyz$. Prove that

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3} - 10.$$

71. *Moldava TST, 2004*, [Marian Tetiva]
Prove that for any positive real numbers a, b, c,

$$\left| \frac{a^3 - b^3}{a + b} + \frac{b^3 - c^3}{b + c} + \frac{c^3 - a^3}{c + a} \right| \leq \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{4}.$$

72. *USAMO, 2004*, [Titu Andreescu]
Let a, b, c be positive real numbers. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

73. [Gabriel Dospinescu]
Let $n > 2$ and $x_1, x_2, \ldots, x_n > 0$ such that

$$\left(\sum_{k=1}^{n} x_k \right) \left(\sum_{k=1}^{n} \frac{1}{x_k} \right) = n^2 + 1.$$

Prove that

$$\left(\sum_{k=1}^{n} x_k^2 \right) \left(\sum_{k=1}^{n} \frac{1}{x_k^2} \right) > n^2 + 4 + \frac{2}{n(n-1)}.$$

74. [Gabriel Dospinescu, Mircea Lascu, Marian Tetiva]
Prove that for any positive real numbers a, b, c,

$$a^2 + b^2 + c^2 + 2abc + 3 \geq (1 + a)(1 + b)(1 + c).$$

75. *USAMO, 2003*, [Titu Andreescu, Zuming Feng]
Let a, b, c be positive real numbers. Prove that

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8.$$

76. *Austrian-Polish Competition, 1995*
Prove that for any positive real numbers x, y and any positive integers m, n,

$$\begin{aligned}(n-1)(m-1)(x^{m+n} + y^{m+n}) + (m+n-1)(x^m y^n + x^n y^m)\\ \geq mn(x^{m+n-1}y + y^{m+n-1}x).\end{aligned}$$

77. *Crux Mathematicorum 2023*, [Waldemar Pompe]
Let a, b, c, d, e be positive real numbers such that $abcde = 1$. Prove that

$$\begin{aligned}\frac{a + abc}{1 + ab + abcd} + \frac{b + bcd}{1 + bc + bcde} + \frac{c + cde}{1 + cd + cdea}\\ + \frac{d + dea}{1 + de + deab} + \frac{e + eab}{1 + ea + eabc} \geq \frac{10}{3}.\end{aligned}$$

78. *TST 2003, USA*, [Titu Andreescu]
Prove that for any $a, b, c \in \left(0, \frac{\pi}{2}\right)$ the following inequality holds

$$\frac{\sin a \cdot \sin(a-b) \cdot \sin(a-c)}{\sin(b+c)} + \frac{\sin b \cdot \sin(b-c) \cdot \sin(b-a)}{\sin(c+a)} + \frac{\sin c \cdot \sin(c-a) \cdot \sin(c-b)}{\sin(a+b)} \geq 0$$

79. *KMO Summer Program Test, 2001*
Prove that if a, b, c are positive real numbers, then

$$\sqrt{a^4+b^4+c^4} + \sqrt{a^2b^2+b^2c^2+c^2a^2} \geq \sqrt{a^3b+b^3c+c^3a} + \sqrt{ab^3+bc^3+ca^3}.$$

80. [Gabriel Dospinescu, Mircea Lascu]
For a given $n > 2$ find the smallest constant k_n with the property: if $a_1, \ldots, a_n > 0$ have product 1, then

$$\frac{a_1a_2}{(a_1^2+a_2)(a_2^2+a_1)} + \frac{a_2a_3}{(a_2^2+a_3)(a_3^2+a_2)} + \frac{a_na_1}{(a_n^2+a_1)(a_1^2+a_n)} \leq k_n.$$

81. *Kvant, 1989*, [Vasile Cîrtoaje]
For any real numbers a, b, c, x, y, z prove that the inequality holds

$$ax+by+cz+\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} \geq \frac{2}{3}(a+b+c)(x+y+z).$$

82. [Vasile Cîrtoaje]
Prove that the sides a, b, c of a triangle satisfy the inequality

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right) \geq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

83. *Crux Mathematicorum 2423*, [Walther Janous]
Let $n > 2$ and let $x_1, x_2, \ldots, x_n > 0$ add up to 1. Prove that

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right) \geq \prod_{i=1}^{n}\left(\frac{n-x_i}{1-x_i}\right).$$

84. *TST 1999, Romania*, [Vasile Cîrtoaje, Gheorghe Eckstein]
Consider positive real numbers $x_1, x_2, \ldots, x_n$ such that $x_1x_2\cdots x_n = 1$. Prove that

$$\frac{1}{n-1+x_1} + \frac{1}{n-1+x_2} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

85. *USAMO, 2001*, [Titu Andreescu]
Prove that for any nonnegative real numbers a, b, c such that $a^2+b^2+c^2+abc = 4$ we have
$0 \leq ab+bc+ca-abc \leq 2$.

86. *TST 2000, USA*, [Titu Andreescu]
Prove that for any positive real numbers a, b, c the following inequality holds

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \leq \max\{(\sqrt{a}-\sqrt{b})^2, (\sqrt{b}-\sqrt{c})^2, (\sqrt{c}-\sqrt{a})^2\}.$$

87. [Kiran Kedlaya]
Let a, b, c be positive real numbers. Prove that

$$\frac{a+\sqrt{ab}+\sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \frac{a+b}{2} \cdot \frac{a+b+c}{3}}.$$

88. *Vietnamese IMO Training Camp, 1995*
Find the greatest constant k such that for any positive integer n which is not a square,

$$|(1+\sqrt{n})\sin(\pi\sqrt{n})| > k.$$

89. *Vietnam, 2004*, [Dung Tran Nam]
Let $x, y, z > 0$ such that $(x+y+z)^2 = 32xyz$. Find the minimum and maximum of

$$\frac{x^4+y^4+z^4}{(x+y+z)^4}.$$

90. *Crux Mathematicorum 2393*, [George Tsintifas]
Prove that for any $a, b, c, d > 0$,

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4.$$

91. [Titu Andreescu, Gabriel Dospinescu]
Find the maximum value of the expression

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca},$$

where a, b, c are nonnegative real numbers which add up to 1 and n is some positive integer.

92.
Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{\sqrt[3]{abc}\,(1+\sqrt[3]{abc})}.$$

93. *Vietnam, 2002*, [Dung Tran Nam]
Prove that for any real numbers a, b, c such that $a^2+b^2+c^2 = 9$,

$$2\,(a+b+c) - abc \leq 10.$$

94. [Vasile Cîrtoaje]
Let a, b, c be positive real numbers. Prove that

$$\left(a+\frac{1}{b}-1\right)\left(b+\frac{1}{c}-1\right) + \left(b+\frac{1}{c}-1\right)\left(c+\frac{1}{a}-1\right)$$
$$\left(c+\frac{1}{a}-1\right)\left(a+\frac{1}{b}-1\right) \geq 3.$$

95. [Gabriel Dospinescu]
Let n be an integer greater than 2. Find the greatest real number m_n and the least real number M_n such that for any positive real numbers $x_1, x_2, \ldots, x_n$ (with $x_n = x_0, x_{n+1} = x_1$),

$$m_n \leq \sum_{i=1}^{n} \frac{x_i}{x_{i-1} + 2\,(n-1)x_i + x_{i+1}} \leq M_n.$$

96. *Gazeta Matematică*, [Vasile Cîrtoaje]
If x, y, z are positive real numbers, then

$$\frac{1}{x^2+xy+y^2} + \frac{1}{y^2+yz+z^2} + \frac{1}{z^2+zx+x^2} \geq \frac{9}{(x+y+z)^2}.$$

97. *Gazeta Matematică*, [Vasile Cîrtoaje]
For any $a, b, c, d > 0$ prove that

$$2\,(a^3+1)(b^3+1)(c^3+1)(d^3+1) \geq (1+abcd)(1+a^2)(1+b^2)(1+c^2)(1+d^2).$$

98. *Vietnam TST, 1996*
Prove that for any real numbers $a, b, c,$

$$(a+b)^4+(b+c)^4+(c+a)^4 \geq \frac{4}{7}(a^4+b^4+c^4).$$

99. *Bulgaria, 1997*
Prove that if a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \leq \frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}.$$

100. *Vietnam, 2001*, [Dung Tran Nam]
Find the minimum value of the expression $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}$ where a, b, c are positive real numbers such that $21ab+2bc+8ca \leq 12$.

101. [Titu Andreescu, Gabriel Dospinescu]
Prove that for any $x, y, z, a, b, c > 0$ such that $xy+yz+zx = 3$,

$$\frac{a}{b+c}(y+z)+\frac{b}{c+a}(z+x)+\frac{c}{a+b}(x+y) \geq 3.$$

102. *Japan, 1997*
Let a, b, c be positive real numbers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

103. [Vasile Cîrtoaje, Gabriel Dospinescu]
Prove that if $a_1, a_2, \ldots, a_n \geq 0$ then

$$a_1^n+a_2^n+\cdots+a_n^n-na_1a_2\cdots a_n \geq (n-1)\left(\frac{a_1+a_2+\cdots+a_{n-1}}{n-1}-a_n\right)^n$$

where a_n is the least among the numbers $a_1, a_2, \ldots, a_n$.

104. *Kvant*, [Turkevici]
Prove that for all positive real numbers $x, y, z, t,$

$$x^4+y^4+z^4+t^4+2xyzt \geq x^2y^2+y^2z^2+z^2t^2+t^2x^2+x^2z^2+y^2t^2.$$

105.
Prove that for any real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i,j=1}^{n} \frac{ij}{i+j-1}a_ia_j.$$

106. *TST Singapore*
Prove that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real numbers between 1001 and 2002, inclusively, such that $a_1^2+a_2^2+\cdots+a_n^2 = b_1^2+b_2^2+\cdots+b_n^2$, then we have the inequality

$$\frac{a_1^3}{b_1}+\frac{a_2^3}{b_2}+\cdots+\frac{a_n^3}{b_n} \leq \frac{17}{10}(a_1^2+a_2^2+\cdots+a_n^2).$$

107. [Titu Andreescu, Gabriel Dospinescu]
Prove that if a, b, c are positive real numbers which add up to 1, then

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq 8\,(a^2b^2+b^2c^2+c^2a^2)^2.$$

108. *Gazeta Matematică*, [Vasile Cîrtoaje]
If a, b, c, d are positive real numbers such that $abcd = 1$, then

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2} \geq 1.$$

109. *Gazeta Matematică*, [Vasile Cîrtoaje]
Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

110. *TST 2004, Romania*, [Gabriel Dospinescu]
Let $a_1, a_2, \ldots, a_n$ be real numbers and let S be a non-empty subset of $\{1, 2, \ldots, n\}$. Prove that

$$\left(\sum_{i \in S} a_i\right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + \cdots + a_j)^2.$$

111. [Dung Tran Nam]
Let $x_1, x_2, \ldots, x_{2004}$ be real numbers in the interval $[-1, 1]$ such that $x_1^3 + x_2^3 + \cdots + x_{2004}^3 = 0$. Find the maximal value of the $x_1 + x_2 + \cdots + x_{2004}$.

112. [Gabriel Dospinescu, Călin Popa]
Prove that if $n \geq 2$ and $a_1, a_2, \ldots, a_n$ are real numbers with product 1, then

$$a_1^2+a_2^2+\cdots+a_n^2-n \geq \frac{2n}{n-1}\cdot\sqrt[n]{n-1}\,(a_1+a_2+\cdots+a_n-n).$$

113. *Gazeta Matematică*, [Vasile Cîrtoaje]
If a, b, c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}} \leq 3.$$

114. *Iran, 1996*
Prove the following inequality for positive real numbers x, y, z

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

115.
Prove that for any x, y in the interval $[0, 1]$,

$$\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{(1-x)^2+(1-y)^2} \geq (1+\sqrt{5})(1-xy).$$

116. *Miklos Schweitzer Competition*, [Suranyi]
Prove that for any positive real numbers $a_1, a_2, \ldots, a_n$ the following inequality holds

$$(n-1)(a_1^n+a_2^n+\cdots+a_n^n)+na_1a_2\cdots a_n \geq (a_1+a_2+\cdots+a_n)(a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}).$$

117. *A generalization of Turkevici's inequality*
Prove that for any $x_1, x_2, \ldots, x_n > 0$ with product 1,

$$\sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq \sum_{i=1}^{n} x_i^2 - n.$$

118. [Gabriel Dospinescu]
Find the minimum value of the expression

$$\sum_{i=1}^{n} \sqrt{\frac{a_1 a_2 \cdots a_n}{1 - (n-1)a_i}}$$

where $a_1, a_2, \ldots, a_n < \frac{1}{n-1}$ add up to 1 and $n > 2$ is an integer.

119. [Vasile Cîrtoaje]
Let $a_1, a_2, \ldots, a_n < 1$ be nonnegative real numbers such that

$$a = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Prove that

$$\frac{a_1}{1 - a_1^2} + \frac{a_2}{1 - a_2^2} + \cdots + \frac{a_n}{1 - a_n^2} \geq \frac{na}{1 - a^2}.$$

120. [Vasile Cîrtoaje, Mircea Lascu]
Let a, b, c, x, y, z be positive real numbers such that

$$(a + b + c)(x + y + z) = (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

121. *Mathlinks Contest*, [Gabriel Dospinescu]
For a given $n > 2$, find the minimal value of the constant k_n, such that if $x_1, x_2, \ldots, x_n > 0$ have product 1, then

$$\frac{1}{\sqrt{1 + k_n x_1}} + \frac{1}{\sqrt{1 + k_n x_2}} + \cdots + \frac{1}{\sqrt{1 + k_n x_n}} \leq n - 1.$$

122. [Vasile Cîrtoaje, Gabriel Dospinescu]
For a given $n > 2$, find the maximal value of the constant k_n such that for any $x_1, x_2, \ldots, x_n > 0$ for which $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ we have the inequality

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) \geq k_n x_1 x_2 \cdots x_n.$$

Inequalities proposed in

"Olympiad Inequalities"

by Thomas J. Mildorf

The full document is available at
`http://www.artofproblemsolving.com/Resources/Papers/MildorfInequalities.pdf`

1.
Show that for positive reals a, b, c

$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2.$$

2.
Let a, b, c be positive reals such that $abc = 1$. Prove that

$$a + b + c \leq a^2 + b^2 + c^2.$$

3.
Let $P(x)$ be a polynomial with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$, then it holds for all $x > 0$.

4. *USAMO 78/1*
a, b, c, d, e are real numbers such that

$$a + b + c + d + e = 8$$
$$a^2 + b^2 + c^2 + d^2 + e^2 = 16.$$

What is the largest possible value of e?

5.
Show that for all positive reals a, b, c, d,

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

6. *USAMO 80/5*
Show that for all non-negative reals $a, b, c \leq 1$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

7. *USAMO 77/5*
If a, b, c, d, e are positive reals bounded by p and q with $0 < p \leq q$, prove that

$$(a+b+c+d+e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$$

and determine when equality holds.

8.
a, b, c are non-negative reals such that $a+b+c=1$. Prove that

$$a^3+b^3+c^3+6abc \geq \frac{1}{4}.$$

9. *USAMO 74/5*
If a, b, c, d are positive reals, then determine the possible values of

$$\frac{a}{a+b+d}+\frac{b}{b+c+a}+\frac{c}{b+c+d}+\frac{d}{a+c+d}.$$

10. *IMO 95/2*
a, b, c are positive reals with $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

11.
Let a, b, c be positive reals such that $abc=1$. Show that

$$\frac{2}{(a+1)^2+b^2+1}+\frac{2}{(b+1)^2+c^2+1}+\frac{2}{(c+1)^2+a^2+1} \leq 1.$$

12. *USAMO 98/3*
Let $a_0, \ldots, a_n$ be real numbers in the interval $(0, \frac{\pi}{2})$ such that

$$\tan\left(a_0-\frac{\pi}{4}\right)+\tan\left(a_1-\frac{\pi}{4}\right)+\cdots+\tan\left(a_n-\frac{\pi}{4}\right) \geq n-1.$$

Prove that

$$\tan(a_0)\tan(a_1)\cdots\tan(a_n) \geq n^{n+1}.$$

13.
Let a, b, c be positive reals. Prove that

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

with equality if and only if $a=b=c=1$.

14. *Romanian TST*
Let a, b, x, y, z be positive reals. Show that

$$\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by} \geq \frac{3}{a+b}.$$

15.
The numbers $x_1, x_2, \ldots, x_n$ obey $-1 \leq x_1, x_2, \ldots, x_n \leq 1$ and $x_1^3+x_2^3+\cdots+x_n^3=0$. Prove that

$$x_1+x_2+\cdots+x_n \leq \frac{n}{3}.$$

16. *Turkey*
Let $n \geq 2$ be an integer, and $x_1, x_2, \ldots, x_n$ positive reals such that $x_1^2+x_2^2+\cdots+x_n^2=1$. Determine the smallest possible value of

$$\frac{x_1^5}{x_2+x_3+\cdots+x_n}+\frac{x_2^5}{x_3+\cdots+x_n+x_1}+\cdots+\frac{x_n^5}{x_1+x_2+\cdots+x_{n-1}}.$$

17. *IMO Shortlist*
Find the minimum value of c such that for any n and any nonnegative reals $x_1, x_2, \ldots, x_n$ which satisfy $x_{i+1} \geq x_1 + x_2 + \cdots + x_i$ for $i = 1, \ldots, n-1$, we have

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \leq c\sqrt{x_1 + x_2 + \cdots + x_n}.$$

18. *Poland 95*
Let n be a positive integer. Compute the minimum value of the sum

$$x_1 + \frac{x_2^2}{2} + \frac{x_3^3}{3} + \cdots + \frac{x_n^n}{n}$$

where $x_1, x_2, \ldots, x_n$ are positive reals such that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n.$$

19.
Prove that for all positive reals a, b, c, d,

$$a^4b + b^4c + c^4d + d^4a \geq abcd(a + b + c + d).$$

20. *USAMO 01/3*
Let a, b, c be nonnegative reals such that $a^2 + b^2 + c^2 + abc = 4$. Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

21. *Vietnam 98*
Let $x_1, \ldots, x_n$ be positive reals such that

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998.$$

22. *Romania 99*
Show that for all positive reals $x_1, \ldots, x_n$ with $x_1 x_2 \cdots x_n = 1$, we have

$$\frac{1}{n - 1 + x_1} + \cdots + \frac{1}{n - 1 + x_n} \leq 1.$$

23. [Darij Grinberg]
Show that for all positive reals a, b, c,

$$\frac{\sqrt{b+c}}{a} + \frac{\sqrt{c+a}}{b} + \frac{\sqrt{a+b}}{c} \geq \frac{4(a+b+c)}{\sqrt{(a+b)(b+c)(c+a)}}.$$

24.
Show that for all positive numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1^3}{x_1^2 + x_1x_2 + x_2^2} + \frac{x_2^3}{x_2^2 + x_2x_3 + x_3^2} + \cdots + \frac{x_n^3}{x_n^2 + x_nx_1 + x_1^2} \geq \frac{x_1 + \cdots + x_n}{3}.$$

25.
Let a, b, c be positive reals such that $a + b \geq c$; $b + c \geq a$; and $c + a \geq b$, we have

$$2a^2(b+c) + 2b^2(c+a) + 2c^2(a+b) \geq a^3 + b^3 + c^3 + 9abc.$$

26.
Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}} + \frac{b}{\sqrt{2c^2+2a^2-b^2}} + \frac{c}{\sqrt{2a^2+2b^2-c^2}} \geq \sqrt{3}.$$

27. *IMO 99/2*
For $n \geq 2$ a fixed positive integer, find the smallest constant C such that for all nonnegative reals $x_1, \ldots, x_n$,

$$\sum_{1\leq i<j\leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^4.$$

28.
Show that for nonnegative reals a, b, c,

$$2a^6 + 2b^6 + 2c^6 + 16a^3b^3 + 16b^3c^3 + 16c^3a^3 \geq 9a^4(b^2+c^2) + 9b^4(c^2+a^2) + 9c^4(a^2+b^2).$$

29.
Let $0 \leq a, b, c \leq \frac{1}{2}$ be real numbers with $a+b+c=1$. Show that

$$a^3 + b^3 + c^3 + 4abc \leq \frac{9}{32}.$$

30. [Vasile Cîrtoaje]
Let $p \geq 2$ be a real number. Show that for all nonnegative reals a, b, c,

$$\sqrt[3]{\frac{a^3+pabc}{1+p}} + \sqrt[3]{\frac{b^3+pabc}{1+p}} + \sqrt[3]{\frac{c^3+pabc}{1+p}} \leq a+b+c.$$

31.
Let a, b, c be real numbers such that $abc = -1$. Show that

$$a^4 + b^4 + c^4 + 3(a+b+c) \geq \frac{a^2}{b} + \frac{a^2}{c} + \frac{b^2}{c} + \frac{b^2}{a} + \frac{c^2}{a} + \frac{c^2}{b}.$$

32. *MOP 2003*
Show that for all nonnegative reals a, b, c,

$$\begin{aligned} &a^4(b^2+c^2) + b^4(c^2+a^2) + c^4(a^2+b^2) + \\ &2abc(a^2b + a^2c + b^2c + b^2a + c^2a + c^2b - a^3 - b^3 - c^3 - 3abc) \\ &\qquad \geq 2a^3b^3 + 2b^3c^3 + 2c^3a^3. \end{aligned}$$

33. [Cezar Lupu]
Let a, b, c be positive reals such that $a+b+c+abc=4$. Prove that

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \frac{\sqrt{2}}{2}(a+b+c).$$

34. *Iran 1996*
Show that for all positive real numbers a, b, c,

$$(ab+bc+ca)\left(\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2}\right) \geq \frac{9}{4}.$$

35. *Japan 1997*
Show that for all positive reals a, b, c,

$$\frac{(a+b-c)^2}{(a+b)^2+c^2} + \frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} \geq \frac{3}{5}.$$

36. *MOP 02*
Let a, b, c be positive reals. Prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

37. [Mildorf]
Let $n \geq 2$ be an integer. Prove that for all reals $a_1, a_2, \ldots, a_n > 0$ and reals $p, k \geq 1$,

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{a_1^p + a_2^p + \cdots + a_n^p}\right)^k \geq \frac{a_1^k + a_2^k + \cdots + a_n^k}{a_1^{pk} + a_2^{pk} + \cdots + a_n^{pk}}$$

where inequality holds iff $p = 1$ or $k = 1$ or $a_1 = a_2 = \cdots = a_n$, flips if instead $0 < p < 1$, and flips (possibly again) if instead $0 < k < 1$.

38. [Vasile Cîrtoaje]
Show that for all real numbers a, b, c,

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3b + b^3c + c^3a).$$

39. [Anh-Cuong]
Show that for all nonnegative reals a, b, c,

$$a^3 + b^3 + c^3 + 3abc \geq ab\sqrt{2a^2 + 2b^2} + bc\sqrt{2b^2 + 2c^2} + ca\sqrt{2c^2 + 2a^2}.$$

40.
For $x \geq y \geq 1$, prove that

$$\frac{x}{\sqrt{x+y}} + \frac{y}{\sqrt{y+1}} + \frac{1}{\sqrt{x+1}} \geq \frac{y}{\sqrt{x+y}} + \frac{x}{\sqrt{x+1}} + \frac{1}{\sqrt{y+1}}.$$

41. [Vasile Cîrtoaje]
Show that for positive reals a, b, c,

$$\frac{1}{4a^2 - ab + 4b^2} + \frac{1}{4b^2 - bc + 4c^2} + \frac{1}{4c^2 - ca + 4a^2} \geq \frac{9}{7(a^2 + b^2 + c^2)}.$$

42. *USAMO 00/6*
Let $n \geq 2$ be an integer and $S = \{1, 2, \ldots, n\}$. Show that for all nonnegative reals $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$,

$$\sum_{i,j \in S} \min\{a_i a_j, b_i b_j\} \leq \sum_{i,j \in S} \min\{a_i b_j, a_j b_i\}.$$

43. [Gabriel Dospinescu]
For any $n > 2$ find the minimal value k_n such that for any positive reals $x_1, x_2, \ldots, x_n$ with product 1 we have

$$\sum_{i=1}^{n} \frac{1}{\sqrt{1 + k_n x_i}} \leq n - 1.$$

44. [Vasile Cîrtoaje]
For any $a, b, c, d > 0$ we have

$$2(a^3+1)(b^3+1)(c^3+1)(d^3+1) \geq (1+abcd)(1+a^2)(1+b^2)(1+c^2)(1+d^2).$$

45. [Vasile Cîrtoaje]
Prove that the sides a, b, c of any triangle obey

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1\right) \geq 2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

46. *Crux Mathematicorum*, [George Tsintifas]
Prove that for any $a, b, c, d > 0$ we have the inequality

$$(a+b)^3(b+c)^3(c+d)^3(d+a)^3 \geq 16a^2b^2c^2d^2(a+b+c+d)^4.$$

47. *Vietnam 2002*
Prove that for any reals x, y, z such that $x^2+y^2+z^2=9$,

$$2(x+y+z)-xyz \leq 10.$$

48. *MOP 2003*
For $n \geq 2$ a fixed positive integer, let $x_1, \ldots, x_n$ be positive reals such that

$$x_1+x_2+\cdots+x_n = \frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}.$$

Prove that

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+\cdots+\frac{1}{n-1+x_n} \leq 1.$$

49. *Taiwan 2002*
Show that for all positive $a, b, c, d \leq k$, we have

$$\frac{a^4+b^4+c^4+d^4}{(2k-a)^4+(2k-b)^4+(2k-c)^4+(2k-d)^4} \geq \frac{abcd}{(2k-a)(2k-b)(2k-c)(2k-d)}.$$

50. *IMO Shortlist 03/A6*, [Reid Barton]
Let $n \geq 2$ be a positive integer and $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ a sequence of $2n$ positive reals. Suppose $z_2, z_3, \ldots, z_{2n}$ is such that $z_{i+j}^2 \geq x_i y_j$ for all $i, j \in \{1, \ldots, n\}$. Let $M = \max\{z_2, z_3, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M+z_2+z_3+\cdots+z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1+\cdots+x_n}{n}\right)\left(\frac{y_1+\cdots+y_n}{n}\right).$$

51.
Show that for all positive reals a, b, c,

$$3a^2+3b^2+3c^2 \geq (a+b+c)^2.$$

52. *MOP 01?*
Show that for positive reals a, b, c,

$$\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(a+b)} \geq \frac{3}{4}.$$

53. *MOP 04*
Show that for all positive reals a, b, c,

$$\left(\frac{a+2b}{a+2c}\right)^3 + \left(\frac{b+2c}{b+2a}\right)^3 + \left(\frac{c+2a}{c+2b}\right)^3 \geq 3.$$

54. *MOP*
Show that if k is a positive integer and $a_1, a_2, \ldots, a_n$ are positive reals which sum to 1, then

$$\prod_{i=1}^{n} \frac{1-a_i^k}{a_i^k} \geq (n^k - 1)^n.$$

55.
Let $a_1, a_2, \ldots, a_n$ be nonnegative reals with a sum of 1. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n \leq \frac{1}{4}.$$

56. *Ukraine 01*
Let a, b, c, x, y, z be nonnegative reals such that $x + y + z = 1$. Show that

$$ax + by + cz + 2\sqrt{(ab + bc + ca)(xy + yz + zx)} \leq a + b + c.$$

57.
Let $n > 1$ be a positive integer and $a_1, a_2, \ldots, a_n$ positive reals such that $a_1a_2 \cdots a_n = 1$. Show that

$$\frac{1}{1+a_1} + \cdots + \frac{1}{1+a_n} \leq \frac{a_1 + \cdots + a_n + n}{4}.$$

58. [Aaron Pixton]
Let a, b, c be positive reals with product 1. Show that

$$5 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq (1+a)(1+b)(1+c).$$

59. [Valentin Vornicu]
Let a, b, c, x, y, z be arbitrary reals such that $a \geq b \geq c$ and either $x \geq y \geq z$ or $x \leq y \leq z$. Let $f : \mathbb{R} \to \mathbb{R}_0^+$ be either monotonic or convex, and let k be a positive integer. Prove that

$$f(x)(a-b)^k(a-c)^k + f(y)(b-c)^k(b-a)^k + f(z)(c-a)^k(c-b)^k \geq 0.$$

60. *IMO 01/2*
Let a, b, c be positive reals. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

61. [Vasile Cîrtoaje]
Show that for positive reals a, b, c,

$$\frac{a^3}{(2a^2+b^2)(2a^2+c^2)} + \frac{b^3}{(2b^2+c^2)(2b^2+a^2)} + \frac{c^3}{(2c^2+a^2)(2c^2+b^2)} \leq \frac{1}{a+b+c}.$$

62. *USAMO 04/5*
Let a, b, c be positive reals. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a+b+c)^3.$$

63. [Titu Andreescu]
Show that for all nonzero reals a, b, c,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a}{c}+\frac{c}{b}+\frac{b}{a}.$$

64. [Darij Grinberg]
Show that for positive reals a, b, c,

$$\frac{b^2+c^2-a^2}{a(b+c)}+\frac{c^2+a^2-b^2}{b(c+a)}+\frac{a^2+b^2-c^2}{c(a+b)} \geq \frac{3}{2}.$$

65. *IMO Shortlist 96*
Let a, b, c be positive reals with $abc = 1$. Show that

$$\frac{ab}{a^5+b^5+ab}+\frac{bc}{b^5+c^5+bc}+\frac{ca}{c^5+a^5+ca} \leq 1.$$

66.
Let a, b, c be positive reals such that $a+b+c = 1$. Prove that

$$\sqrt{ab+c}+\sqrt{bc+a}+\sqrt{ca+b} \geq 1+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}.$$

67. *IMO 00/2*
Positive reals a, b, c have product 1. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

68. *APMO 2005/2*
Let a, b, c be positive reals with $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(a^3+1)(b^3+1)}}+\frac{b^2}{\sqrt{(b^3+1)(c^3+1)}}+\frac{c^2}{\sqrt{(c^3+1)(a^3+1)}} \geq \frac{4}{3}.$$

69.
Show that for all positive reals a, b, c,

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \geq a+b+c.$$

70. *USAMO 97/5*
Prove that for all positive reals a, b, c,

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

71. *Moldova 1999*
Show that for all positive reals a, b, c,

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)} \geq \frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c}.$$

72. *Tuymaada 2000*
Prove that for all reals $0 < x_1, \ldots, x_n \leq \frac{1}{2}$,

$$\left(\frac{n}{x_1+\cdots+x_n}-1\right)^n \leq \prod_{i=1}^{n}\left(\frac{1}{x_i}-1\right).$$

73. *Mathlinks Lore*
Show that for all positive reals a, b, c, d with $abcd = 1$, and $k \geq 2$,

$$\frac{1}{(1+a)^k} + \frac{1}{(1+b)^k} + \frac{1}{(1+c)^k} + \frac{1}{(1+d)^k} \geq 2^{2-k}.$$

74. *Tiks*
Show that for all reals $a, b, c > 0$,

$$\frac{a^2}{(2a+b)(2a+c)} + \frac{b^2}{(2b+c)(2b+a)} + \frac{c^2}{(2c+a)(2c+b)} \leq \frac{1}{3}.$$

75. [Hyun Soo Kim]
Let a, b, c be positive reals with product not less than one. Prove that

$$\frac{1}{a+b^{2005}+c^{2005}} + \frac{1}{b+c^{2005}+a^{2005}} + \frac{1}{c+a^{2005}+b^{2005}} \leq 1.$$

76. *IMO 05/3*
Prove that for all positive a, b, c with product at least 1,

$$\frac{a^5-a^2}{a^5+b^2+c^2} + \frac{b^5-b^2}{b^5+c^2+a^2} + \frac{c^5-c^2}{c^5+a^2+b^2} \geq 0.$$

77. [Mildorf]
Let a, b, c, k be positive reals. Determine a simple, necessary and sufficient condition for the following inequality to hold:

$$(a+b+c)^k(a^kb^k + b^kc^k + c^ka^k) \leq (ab+bc+ca)^k(a^k+b^k+c^k).$$

78.
Let a, b, c be reals with $a+b+c = 1$ and $a, b, c \geq -\frac{3}{4}$. Prove that

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}.$$

79. [Mildorf]
Show that for all positive reals a, b, c,

$$\sqrt[3]{4a^3+4b^3} + \sqrt[3]{4b^3+4c^3} + \sqrt[3]{4c^3+4a^3} \leq \frac{4a^2}{a+b} + \frac{4b^2}{b+c} + \frac{4c^2}{c+a}.$$

80.
Let a, b, c, x, y, z be real numbers such that

$$(a+b+c)(x+y+z) = 3, \qquad (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$ax + by + cz \geq 0.$$

81. [Po-Ru Loh]
Let a, b, c be reals with $a, b, c > 1$ such that

$$\frac{1}{a^2-1} + \frac{1}{b^2-1} + \frac{1}{c^2-1} = 1.$$

Prove that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1.$$

82. [Weighao Wu]
Prove that

$$(\sin x)^{\sin x} < (\cos x)^{\cos x}$$

for all real numbers $0 < x < \frac{\pi}{4}$.

83. [Michael Rozenberg]
Show that for all positive reals a, b, c,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{3}{2} \cdot \frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

84. [Hungktn]
Prove that for all positive reals a, b, c,

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2.$$

85. *IMO 05/2*, [Mock]
Let a, b, c be positive reals. Show that

$$1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \leq \frac{3\sqrt{2}}{2}.$$

86. [Gabriel Dospinescu]
Let $n \geq 2$ be a positive integer. Show that for all positive reals $a_1, a_2, \ldots, a_n$ with $a_1 a_2 \cdots a_n = 1$,

$$\sqrt{\frac{a_1^2+1}{2}} + \cdots + \sqrt{\frac{a_n^2+1}{2}} \leq a_1 + \cdots + a_n.$$

87.
Let $n \geq 2$ be a positive integer, and let $k \geq \frac{n-1}{n}$ be a real number. Show that for all positive reals $a_1, a_2, \ldots, a_n$,

$$\left(\frac{(n-1)a_1}{a_2+\cdots+a_n}\right)^k + \left(\frac{(n-1)a_2}{a_3+\cdots+a_n+a_1}\right)^k + \cdots + \left(\frac{(n-1)a_n}{a_1+\cdots+a_{n-1}}\right)^k \geq n.$$

88.
Show that for reals x, y, z which are not all positive,

$$\frac{16}{9}(x^2-x+1)(y^2-y+1)(z^2-z+1) \geq (xyz)^2 - xyz + 1.$$

89. [Mildorf]
Let a, b, c be arbitrary reals such that $a \geq b \geq c$, and let x, y, z be nonnegative reals with $x + z \geq y$. Prove that

$$x^2(a-b)(a-c) + y^2(b-c)(b-a) + z^2(c-a)(c-b) \geq 0$$

and determine where equality holds.

90. *IMO 06/3*
Determine the least real number M such that for all reals a, b, c,

$$|a^3b + b^3c + c^3a - a^3c - b^3a - c^3b| \leq M \cdot (a^2+b^2+c^2)^2.$$

91. [Kiran Kedlaya]
Show that for all nonnegative $a_1, a_2, \ldots, a_n$,

$$\frac{a_1 + \sqrt{a_1a_2} + \cdots + \sqrt[n]{a_1 \cdots a_n}}{n} \le \sqrt[n]{a_1 \cdot \frac{a_1+a_2}{2} \cdots \frac{a_1+\cdots+a_n}{n}}.$$

92. [Vasile Cîrtoaje]
Prove that for all positive reals a, b, c such that $a+b+c=3$,

$$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} \ge \frac{3}{2}.$$

93. [Gabriel Dospinescu]
Prove that $\forall a, b, c, x, y, z \in \mathbb{R}^+ | xy+yz+zx = 3$,

$$\frac{a(y+z)}{b+c} + \frac{b(z+x)}{c+a} + \frac{c(x+y)}{a+b} \ge 3.$$

94. [Mildorf]
Let a, b, c be non-negative reals. Show that for all real k,

$$\sum_{\text{cyclic}} \frac{\max(a^k, b^k)(a-b)^2}{2} \ge \sum_{\text{cyclic}} a^k(a-b)(a-c) \ge \sum_{\text{cyclic}} \frac{\min(a^k, b^k)(a-b)^2}{2}$$

where $a, b, c \neq 0$ if $k \le 0$) and determine where equality holds for $k > 0$, $k = 0$, and $k < 0$ respectively.

95. [Vasile Cîrtoaje]
Let a, b, c, k be positive reals. Prove that

$$\frac{ab+(k-3)bc+ca}{(b-c)^2+kbc} + \frac{bc+(k-3)ca+ab}{(c-a)^2+kca} + \frac{ca+(k-3)ab+bc}{(a-b)^2+kab} \ge \frac{3(k-1)}{k}.$$

96. [Darij Grinberg and Vasile Cîrtoaje]
Show that for positive reals a, b, c, d,

$$\frac{1}{a^2+ab} + \frac{1}{b^2+bc} + \frac{1}{c^2+cd} + \frac{1}{d^2+da} \ge \frac{2}{\sqrt{abcd}}.$$

97. [Vasile Cîrtoaje; inspired by the next problem]
Show that for all positive reals a, b, c,

$$\frac{3a^2+ab}{(a+b)^2} + \frac{3b^2+bc}{(b+c)^2} + \frac{3c^2+ca}{(c+a)^2} \ge 3.$$

98. [Vasile Cîrtoaje; inspired by the next problem]
Show that for all positive reals a, b, c,

$$\frac{3a^2-2ab-b^2}{a^2+b^2} + \frac{3b^2-2bc-c^2}{b^2+c^2} + \frac{3c^2-2ca-a^2}{c^2+a^2} \ge 0.$$

99. [Mildorf]
Show that for all positive reals a, b, c,

$$\frac{3a^2-2ab-b^2}{3a^2+2ab+3b^2} + \frac{3b^2-2bc-c^2}{3b^2+2bc+3c^2} + \frac{3c^2-2ca-a^2}{3c^2+2ca+3a^2} \ge 0.$$

100. [Vasile Cîrtoaje]
Show that for real numbers a, b, c,

$$4\left(\sum_{\text{cyclic}} a^2b^2 - abc\sum_{\text{cyclic}} a\right)\left(\sum_{\text{cyclic}} a^4 - \sum_{\text{cyclic}} a^2b^2\right) \ge 3\left(\sum_{\text{cyclic}} a^3b - abc\sum_{\text{cyclic}} a\right)^2.$$