THE 1989 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let x_1, x_2, $\ldots$, x_n be positive real numbers, and let

$$S = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \cdots + \frac{S^n}{n!}\ .$$

Question 2

Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

Question 3

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1$, 2, and 3, suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Question 4

Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \leq a < b \leq n$. Show that there are at least

$$4m \cdot \frac{(m - \frac{n^2}{4})}{3n}$$

triples (a, b, c) such that (a, b), (a, c), and (b, c) belong to S.

Question 5

Determine all functions f from the reals to the reals for which

(1) $f(x)$ is strictly increasing,
(2) $f(x) + g(x) = 2x$ for all real x,

where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

THE 1990 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Given triagnle ABC, let D, E, F be the midpoints of BC, AC, AB respectively and let G be the centroid of the triangle.
For each value of $\angle BAC$, how many non-similar triangles are there in which $AEGF$ is a cyclic quadrilateral?

Question 2

Let a_1, a_2, $\ldots$, a_n be positive real numbers, and let S_k be the sum of the products of a_1, a_2, $\ldots$, a_n taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \cdots a_n$$

for $k = 1, 2, \ldots, n-1$.

Question 3

Consider all the triangles ABC which have a fixed base AB and whose altitude from C is a constant h. For which of these triangles is the product of its altitudes a maximum?

Question 4

A set of 1990 persons is divided into non-intersecting subsets in such a way that

1. No one in a subset knows all the others in the subset,

2. Among any three persons in a subset, there are always at least two who do not know each other, and

3. For any two persons in a subset who do not know each other, there is exactly one person in the same subset knowing both of them.

(a) Prove that within each subset, every person has the same number of acquaintances.

(b) Determine the maximum possible number of subsets.
Note: It is understood that if a person A knows person B, then person B will know person A; an acquaintance is someone who is known. Every person is assumed to know one's self.

Question 5

Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.

THE 1991 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let G be the centroid of triangle ABC and M be the midpoint of BC. Let X be on AB and Y on AC such that the points X, Y, and G are collinear and XY and BC are parallel. Suppose that XC and GB intersect at Q and YB and GC intersect at P. Show that triangle MPQ is similar to triangle ABC.

Question 2

Suppose there are 997 points given in a plane. If every two points are joined by a line segment with its midpoint coloured in red, show that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

Question 3

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1+b_1} + \frac{a_2^2}{a_2+b_2} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{a_1+a_2+\cdots+a_n}{2}\,.$$

Question 4

During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.

Question 5

Given are two tangent circles and a point P on their common tangent perpendicular to the lines joining their centres. Construct with ruler and compass all the circles that are tangent to these two circles and pass through the point P.

THE 1992 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

A triangle with sides a, b, and c is given. Denote by s the semiperimeter, that is $s = (a+b+c)/2$. Construct a triangle with sides $s-a$, $s-b$, and $s-c$. This process is repeated until a triangle can no longer be constructed with the side lengths given.
For which original triangles can this process be repeated indefinitely?

Question 2

In a circle C with centre O and radius r, let C_1, C_2 be two circles with centres O_1, O_2 and radii r_1, r_2 respectively, so that each circle C_i is internally tangent to C at A_i and so that C_1, C_2 are externally tangent to each other at A.
Prove that the three lines OA, O_1A_2, and O_2A_1 are concurrent.

Question 3

Let n be an integer such that $n > 3$. Suppose that we choose three numbers from the set $\{1, 2, \ldots, n\}$. Using each of these three numbers only once and using addition, multiplication, and parenthesis, let us form all possible combinations.

(a) Show that if we choose all three numbers greater than $n/2$, then the values of these combinations are all distinct.

(b) Let p be a prime number such that $p \leq \sqrt{n}$. Show that the number of ways of choosing three numbers so that the smallest one is p and the values of the combinations are not all distinct is precisely the number of positive divisors of $p-1$.

Question 4

Determine all pairs (h, s) of positive integers with the following property:

If one draws h horizontal lines and another s lines which satisfy

(i) they are not horizontal,

(ii) no two of them are parallel,

(iii) no three of the $h+s$ lines are concurrent,

then the number of regions formed by these $h+s$ lines is 1992.

Question 5

Find a sequence of maximal length consisting of non-zero integers in which the sum of any seven consecutive terms is positive and that of any eleven consecutive terms is negative.

THE 1993 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral such that all sides have equal length and angle ABC is 60 deg. Let l be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let M be the point of intersection of CE and AF.
Prove that $CA^2 = CM \times CE$.

Question 2

Find the total number of different integer values the function

$$f(x) = [x] + [2x] + [\frac{5x}{3}] + [3x] + [4x]$$

takes for real numbers x with $0 \leq x \leq 100$.

Question 3

Let

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad \text{and} \\ g(x) &= c_{n+1}x^{n+1} + c_n x^n + \cdots + c_0 \end{aligned}$$

be non-zero polynomials with real coefficients such that $g(x) = (x+r)f(x)$ for some real number r. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n + 1$.

Question 4

Determine all positive integers n for which the equation

$$x^n + (2+x)^n + (2-x)^n = 0$$

has an integer as a solution.

Question 5

Let P_1, P_2, $\ldots$, $P_{1993} = P_0$ be distinct points in the xy-plane with the following properties:

(i) both coordinates of P_i are integers, for $i = 1, 2, \ldots, 1993$;

(ii) there is no point other than P_i and P_{i+1} on the line segment joining P_i with P_{i+1} whose coordinates are both integers, for $i = 0, 1, \ldots, 1992$.

Prove that for some i, $0 \leq i \leq 1992$, there exists a point Q with coordinates (q_x, q_y) on the line segment joining P_i with P_{i+1} such that both $2q_x$ and $2q_y$ are odd integers.

THE 1994 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(i) For all $x, y \in \mathbb{R}$,
$$f(x) + f(y) + 1 \geq f(x + y) \geq f(x) + f(y),$$

(ii) For all $x \in [0, 1)$, $f(0) \geq f(x)$,

(iii) $-f(-1) = f(1) = 1$.

Find all such functions f.

Question 2

Given a nondegenerate triangle ABC, with circumcentre O, orthocentre H, and circumradius R, prove that $|OH| < 3R$.

Question 3

Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.

Question 4

Is there an infinite set of points in the plane such that no three points are collinear, and the distance between any two points is rational?

Question 5

You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

THE 1995 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Determine all sequences of real numbers a_1, a_2, ..., a_{1995} which satisfy:

$$2\sqrt{a_n - (n-1)} \geq a_{n+1} - (n-1), \text{ for } n = 1, 2, \ldots 1994,$$

and

$$2\sqrt{a_{1995} - 1994} \geq a_1 + 1.$$

Question 2

Let a_1, a_2, ..., a_n be a sequence of integers with values between 2 and 1995 such that:

(i) Any two of the a_i's are realtively prime,
(ii) Each a_i is either a prime or a product of primes.

Determine the smallest possible values of n to make sure that the sequence will contain a prime number.

Question 3

Let $PQRS$ be a cyclic quadrilateral such that the segments PQ and RS are not parallel. Consider the set of circles through P and Q, and the set of circles through R and S. Determine the set A of points of tangency of circles in these two sets.

Question 4

Let C be a circle with radius R and centre O, and S a fixed point in the interior of C. Let AA' and BB' be perpendicular chords through S. Consider the rectangles $SAMB$, $SBN'A'$, $SA'M'B'$, and $SB'NA$. Find the set of all points M, N', M', and N when A moves around the whole circle.

Question 5

Find the minimum positive integer k such that there exists a function f from the set $\mathbb{Z}$ of all integers to $\{1, 2, \ldots k\}$ with the property that $f(x) \neq f(y)$ whenever $|x - y| \in \{5, 7, 12\}$.

THE 1996 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

Question 2

Let m and n be positive integers such that $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n \ .$$

Question 3

Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1 be the incentre of the triangle $P_2P_3P_4$; I_2 be the incentre of the triangle $P_1P_3P_4$; I_3 be the incentre of the triangle $P_1P_2P_4$; I_4 be the incentre of the triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

Question 4

The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

1. All members of a group must be of the same sex; i.e. they are either all male or all female.
2. The difference in the size of any two groups is 0 or 1.
3. All groups have at least 1 member.
4. Each person must belong to one and only one group.

Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

Question 5

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \ ,$$

and determine when equality occurs.

THE 1997 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1 Given

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}} \ ,$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers (i.e. $k = n(n+1)/2$ for $n = 1, 2, \ldots, 1996$). Prove that $S > 1001$.

Question 2 Find an integer n, where $100 \le n \le 1997$, such that

$$\frac{2^n + 2}{n}$$

is also an integer.

Question 3 Let ABC be a triangle inscribed in a circle and let

$$l_a = \frac{m_a}{M_a} \ , \quad l_b = \frac{m_b}{M_b} \ , \quad l_c = \frac{m_c}{M_c} \ ,$$

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \ge 3,$$

and that equality holds iff ABC is an equilateral triangle.

Question 4 Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n ($n \ge 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.

(a) Prove that if this process is continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \ge 3$.

(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

Question 5 Suppose that n people A_1, A_2, …, A_n, ($n \ge 3$) are seated in a circle and that A_i has a_i objects such that

$$a_1 + a_2 + \cdots + a_n = nN,$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}. (Here A_{n+1} means A_1 and A_n means A_0.) How should this redistribution be performed so that the total number of objects transferred is minimum?

THE 1998 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let F be the set of all n-tuples $(A_1, \ldots, A_n)$ such that each A_i is a subset of $\{1, 2, \ldots, 1998\}$. Let $|A|$ denote the number of elements of the set A. Find

$$\sum_{(A_1,\ldots,A_n)\in F} |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

Question 2

Show that for any positive integers a and b, $(36a + b)(a + 36b)$ cannot be a power of 2.

Question 3

Let a, b, c be positive real numbers. Prove that

$$(1 + \frac{a}{b})(1 + \frac{b}{c})(1 + \frac{c}{a}) \geq 2 \cdot (1 + \frac{a+b+c}{\sqrt[3]{abc}})$$

Question 4

Let ABC be a triangle and D the foot of the altitude from A. Let E and F lie on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the segments BC and EF, respectively. Prove that AN is perpendicular to NM.

Question 5

Find the largest integer n such that n is divisible by all positive integers less than $\sqrt[3]{n}$.

THE 1999 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Find the smallest positive integer n with the following property: there does not exist an arithmetic progression of 1999 real numbers containing exactly n integers.

Question 2

Let $a_1, a_2, ...$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $i, j = 1, 2, ...$ Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

Question 3

Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Question 4

Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Question 5

Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n-1$ points in its interior and $n-1$ points in its exterior. Prove that the number of good circles has the same parity as n.

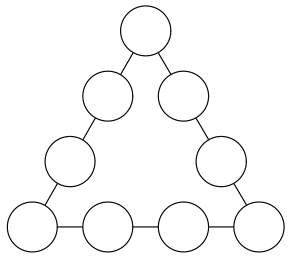
12th Asian Pacific Mathematics Olympiad

March 2000

Time allowed: 4 hours.
No calculators to be used.
Each question is worth 7 points.

1. Compute the sum $S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2}$ for $x_i = \frac{i}{101}$.

2. Given the following triangular arrangement of circles:



Each of the numbers 1, 2, …, 9 is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;

(ii) the sums of the squares of the four numbers on each side of the triangle are equal.

Find all ways in which this can be done.

3. Let *ABC* be a triangle. Let *M* and *N* be the points in which the median and the angle bisector, respectively, at *A* meet the side *BC*. Let *Q* and *P* be the points in which the perpendicular at *N* to *NA* meets *MA* and *BA*, respectively, and *O* the point in which the perpendicular at *P* to *BA* meets *AN* produced. Prove that *QO* is perpendicular to *BC*.

4. Let n, k be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k (n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k (n-k)^{n-k}}.$$

5. Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence 0, 1, …, n. A transposition of a_i with a_j is called *legal* if $a_i = 0$ for $i > 0$, and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after a number of legal transpositions it becomes $(1, 2, \ldots, n, 0)$. For which numbers n is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

END OF PAPER

THE 2001 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Problem 1.
For a positive integer n let $S(n)$ be the sum of digits in the decimal representation of n. Any positive integer obtained by removing several (at least one) digits from the right-hand end of the decimal representation of n is called a *stump* of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$.

Problem 2.
Find the largest positive integer N so that the number of integers in the set $\{1, 2, \ldots, N\}$ which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

Problem 3.
Let two equal regular n-gons S and T be located in the plane such that their intersection is a $2n$-gon ($n \geq 3$). The sides of the polygon S are coloured in red and the sides of T in blue.

Prove that the sum of the lengths of the blue sides of the polygon $S \cap T$ is equal to the sum of the lengths of its red sides.

Problem 4.
A point in the plane with a cartesian coordinate system is called a *mixed point* if one of its coordinates is rational and the other one is irrational. Find all polynomials with real coefficients such that their graphs do not contain any mixed point.

Problem 5.
Find the greatest integer n, such that there are $n + 4$ points A, B, C, D, $X_1, \ldots, X_n$ in the plane with $AB \neq CD$ that satisfy the following condition: for each $i = 1, 2, \ldots, n$ triangles ABX_i and CDX_i are equal.

THE 2002 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Problem 1.

Let $a_1, a_2, a_3, \ldots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \ .$$

Prove that

$$a_1! a_2! \ldots a_n! \geq (\lfloor A_n \rfloor !)^n \ ,$$

where $\lfloor A_n \rfloor$ is the greatest integer less than or equal to A_n, and $a! = 1 \times 2 \times \cdots \times a$ for $a \geq 1$ (and $0! = 1$). When does equality hold?

Problem 2.

Find all positive integers a and b such that

$$\frac{a^2 + b}{b^2 - a} \quad \text{and} \quad \frac{b^2 + a}{a^2 - b}$$

are both integers.

Problem 3.

Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R be the orthocentre of triangle ABP and S be the orthocentre of triangle ACQ. Let T be the point common to the segments BP and CQ. Find all possible values of $\angle CBP$ and $\angle BCQ$ such that triangle TRS is equilateral.

Problem 4.

Let x, y, z be positive numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Show that

$$\sqrt{x + yz} + \sqrt{y + zx} + \sqrt{z + xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

Problem 5.

Let $\mathbf{R}$ denote the set of all real numbers. Find all functions f from $\mathbf{R}$ to $\mathbf{R}$ satisfying:
(i) there are only finitely many s in $\mathbf{R}$ such that $f(s) = 0$, and
(ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all x, y in $\mathbf{R}$.

THE 2003 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Problem 1.

Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for $i = 1, 2, \ldots, 8$. Determine all possible values of f.

Problem 2.

Suppose $ABCD$ is a square piece of cardboard with side length a. On a plane are two parallel lines ℓ_1 and ℓ_2, which are also a units apart. The square $ABCD$ is placed on the plane so that sides AB and AD intersect ℓ_1 at E and F respectively. Also, sides CB and CD intersect ℓ_2 at G and H respectively. Let the perimeters of $\triangle AEF$ and $\triangle CGH$ be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

Problem 3.

Let $k \geq 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove:

(a) if $n = 2p_k$, then n does not divide $(n-k)!$;

(b) if $n > 2p_k$, then n divides $(n-k)!$.

Problem 4.

Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \geq 2$ be an integer. Show that

$$\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}\ .$$

Problem 5.

Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

THE 2004 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Problem 1.

Determine all finite nonempty sets S of positive integers satisfying

$$\frac{i+j}{(i,j)} \qquad \text{is an element of } S \text{ for all } i,j \text{ in } S,$$

where (i,j) is the greatest common divisor of i and j.

Problem 2.

Let O be the circumcentre and H the orthocentre of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

Problem 3.

Let a set S of 2004 points in the plane be given, no three of which are collinear. Let $\mathcal{L}$ denote the set of all lines (extended indefinitely in both directions) determined by pairs of points from the set. Show that it is possible to colour the points of S with at most two colours, such that for any points p, q of S, the number of lines in $\mathcal{L}$ which separate p from q is odd if and only if p and q have the same colour.

Note: A line ℓ *separates* two points p and q if p and q lie on opposite sides of ℓ with neither point on ℓ.

Problem 4.

For a real number x, let $\lfloor x \rfloor$ stand for the largest integer that is less than or equal to x. Prove that

$$\left\lfloor \frac{(n-1)!}{n(n+1)} \right\rfloor$$

is even for every positive integer n.

Problem 5.

Prove that

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca)$$

for all real numbers $a, b, c > 0$.

XVII Asian Pacific Mathematics Olympiad

Time allowed: 4 hours
Each problem is worth 7 points

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Problem 1. Prove that for every irrational real number a, there are irrational real numbers b and b' so that $a + b$ and ab' are both rational while ab and $a + b'$ are both irrational.

Problem 2. Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}.$$

Problem 3. Prove that there exists a triangle which can be cut into 2005 congruent triangles.

Problem 4. In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \leq i, j \leq n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \leq \frac{n}{2}$. During each subsequent time interval $[t, t+1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t. Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters? A house indexed by (i, j) is a *neighbor* of a house indexed by (k, ℓ) if $|i - k| + |j - \ell| = 1$.

Problem 5. In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio MN/BC in terms of R and r.

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Problem 1. Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n$ is an integer, there exists some i such that $\left| a_i - \frac{1}{2} \right| \geq f(n)$.

Problem 2. Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden mean $\tau = \frac{1+\sqrt{5}}{2}$. Here, an integral power of τ is of the form τ^i, where i is an integer (not necessarily positive).

Problem 3. Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.

Problem 4. Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.

Problem 5. In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.