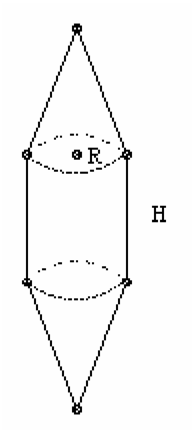
# 1st Putnam 1938

**Problem A1**
A solid in Euclidean 3-space extends from $z = -h/2$ to $z = +h/2$ and the area of the section $z = k$ is a polynomial in $k$ of degree at most 3. Show that the volume of the solid is $h(B + 4M + T)/6$, where $B$ is the area of the bottom ($z = -h/2$), $M$ is the area of the middle section ($z = 0$), and $T$ is the area of the top ($z = h/2$). Derive the formulae for the volumes of a cone and a sphere.

**Solution**
Let the polynomial be $az^3 + bz^2 + cz + d$. Then the volume is $\int_{-h/2}^{h/2} (az^3 + bz^2 + cz + d)\, dz = bh^3/12 + dh$. But $B + T = bh^2/2 + 2d$, $M = d$, so $h(B + 4M + T)/6 = bh^3/12 + dh$, which proves the formula. For a sphere radius $R$, we have $h = 2R$, $B + T = 0$ and $M = \pi R^2$, so the formula gives $4/3\ \pi R^3$, as usual. For a cone height $h$, base area $A$, we have $B = A$, $T = 0$, $M = A/4$, so the volume is $hA/3$, as usual.

**Problem A2**
A solid has a cylindrical middle with a conical cap at each end. The height of each cap equals the length of the middle. For a given surface area, what shape maximizes the volume?




**Solution**
Let the radius be $R$ and the height $H$. The area is $2\pi RH + 2\pi R\sqrt{R^2+H^2}$. The volume is $5/3\ \pi R^2 H$.

The area is fixed, so for some fixed $k$, we have $R\,( H + \sqrt{R^2+H^2} ) = k$. This gives $H = (k^2 - R^4)/(2kR)$. We must now choose $R$ to maximise $f(R) = R^2 H = R\,(k^2 - R^4)/2k$. Evidently the allowed range for $R$ is from $R = 0$ up to $\sqrt{k}$ (corresponding to $H = 0$). But $f(0) = 0$ and $f(\sqrt{k}) = 0$, so the maximum is at some interior point of the interval. Differentiating, we find it is at $R_{max} = (k^2/5)^{1/4}$. In terms of the area $A$, we have $A = 2\pi k$, so $R_{max} = (A/(\pi\, 2\sqrt{5}) )^{1/2}$.

**Problem A3**
A particle moves in the Euclidean plane. At time $t$ (taking all real values) its coordinates are $x = t^3 - t$ and $y = t^4 + t$. Show that its velocity has a maximum at $t = 0$, and that its path has an inflection at $t = 0$.

**Solution**
The speed squared is $(dx/dt)^2 + (dy/dt)^2 = 16t^6 + 9t^4 + 8t^3 - 6t^2 + 2$. Let this be $f(t)$. We have $f'(t) = 12t(8t^4 + 3t^2 + 2t - 1)$. So $f'(t) = 0$ at $t = 0$. Also for $t$ small (positive or negative), $8t^4 + 3t^2 + 2t - 1$ is close to $-1$ and hence negative, so $f'(t)$ is positive for $t$ just less than 0 and negative for $t$ just greater than 0. Hence $f(t)$ has a maximum at $t = 0$. Hence the speed does also.

The gradient $dy/dx = (4t^3 + 1)/(3t^2 - 1)$. Let this be $g(t)$. Then $g'(t) = 6t(2t^3 - 2t - 1)/(3t^2 - 1)^2$. Hence $g'(0) = 0$. Also $g'(t)$ is positive for $t$ just less than 0 and negative for $t$ just greater than 0, so it is a point of inflection.

**Problem A4**
A notch is cut in a cylindrical vertical tree trunk. The notch penetrates to the axis of the cylinder and is bounded by two half-planes. Each half-plane is bounded by a horizontal line passing through the axis of the cylinder. The angle between the two half-planes is $\theta$. Prove that the volume of the notch is minimized (for given tree and $\theta$) by taking the bounding planes at equal angles to the horizontal plane.

**Solution**
We find the volume of the notch above the horizontal plane. Suppose that the upper bounding half-plane is at an angle $\varphi$ to the horizontal. We may take the radius of the tree to be 1. A vertical section through the notch at a distance $x$ from its widest extent is a right-angled triangle with base $\sqrt{1 - x^2}$ and area $1/2\ (1 - x^2) \tan\varphi$. Hence the volume is $2/3 \tan\varphi$. So the total volume of the notch is $2/3 (\tan\varphi + \tan(\theta-\varphi) )$. So we have to find the angle $\varphi$ which minimises $(\tan\varphi + \tan(\theta - \varphi)$. Differentiating, or otherwise, we easily find that the minimum is at $\varphi/2$.

**Problem A5**
(1) Find $\lim_{x\to\inf} x^2/e^x$.
(2) Find $\lim_{k\to 0} 1/k \int_0^k\ (1 + \sin 2x)^{1/x}\, dx$.

**Solution**
(1) Let $f(x) = x^3e^{-x}$. Then $f'(x) = (3x^2 - x^3) e^{-x} < 0$ for $x > 3$. Hence $f(x) < f(3)$ for $x > 3$, so $x^2 e^{-x} < f(3)/x$ for $x > 3$. Hence $x^2 e^{-x}$ tends to zero.

(2) We use L'Hôpital's rule $\lim f(x)/g(x) = \lim f'(x)/g'(x)$. Applied to the expression given it gives $\lim (1 + \sin 2x)^{1/x}$. Write $(1 + \sin 2x)^{1/x} = \exp( 1/x \ln(1 + \sin 2x) )$. So apply the rule again to $1/x \ln( 1 + \sin 2x)$ to get $2 \cos 2x/(1 + \sin 2x)$ which tends to 2. Hence $(1 + \sin 2x)^{1/x}$ tends to $e^2$ and so does the original expression.

**Problem A6**
A swimmer is standing at a corner of a square swimming pool. She swims at a fixed speed and runs at a fixed speed (possibly different). No time is taken entering or leaving the pool. What path should she follow to reach the opposite corner of the pool in the shortest possible time?

**Solution**
Answer: let k be the running speed divided by the swimming speed. For $k > \sqrt{2}$, the unique solution is to run round the outside. For $k < \sqrt{2}$, the unique solution is to swim direct. For $k = \sqrt{2}$ there is no unique solution. Run along a side to X, swim to Y equidistant from the corner between X and Y, then run from Y. The time taken is independent of X.

We may take the side of the square to be 1, the swimming speed to be 1 and the running speed to be k. Let the square be ABCD. Suppose the start is at A and the finish at C. Possible routes are (1) run to X on AB, swim to Y on BC, run to C, (2) run to X on AD, swim to Y on CD, run to C, (3) run to X on AB, swim to Y on CD, run to C. We start by considering case (1). Take BX to be x, BY to be y. Then the time taken is $(2 - x - y)/k + \sqrt{x^2 + y^2}$. Note that this includes the extreme cases of running all the way ($x = y = 0$) and swimming all the way ($x = y = 1$).

Now $(x - y)^2 >= 0$, with equality iff $x = y$, so $(x + y)^2 <= 2(x^2 + y^2)$ and hence $(x + y) \le \sqrt{2} \sqrt{x^2 + y^2}$, with equality iff $x = y$. So if $k > \sqrt{2}$, then $(x + y) < k \sqrt{x^2 + y^2}$ and hence $2/k < (2 - x - y)/k + \sqrt{x^2 + y^2}$ unless $x = y = 0$ (when we have equality). Hence for $k > \sqrt{2}$, the unique solution is to run all the way.

If $k < \sqrt{2}$, then $(x + y) \le \sqrt{2} \sqrt{x^2 + y^2}$ implies $\sqrt{2} (\sqrt{2} - \sqrt{x^2 + y^2}) \le 2 - x - y$ and hence $k (\sqrt{2} - \sqrt{x^2 + y^2}) < 2 - x - y$ unless $x = y = 1$ (when we have equality). So $\sqrt{2} < (2 - x - y)/k + \sqrt{x^2 + y^2}$ unless $x = y = 1$. In other words, for $k < \sqrt{2}$ the unique solution is to swim all the way.

For $k = \sqrt{2}$ we have equality (in both the previous paragraphs) iff $x = y$. So any solution with $x = y$ is optimal in this case.

**Problem A7**
*Do either (1) or (2)*
(1) S is a thin spherical shell of constant thickness and density with total mass M and center O. P is a point outside S. Prove that the gravitational attraction of S at P is the same as the gravitational attraction of a point mass M at O.

(2) K is the surface $z = xy$ in Euclidean 3-space. Find all straight lines lying in S. Draw a diagram to illustrate them.

**Solution**
(1) Let Q be a point on S. The obvious coordinate is the angle $\theta$ = angle QOP. The density is $\rho = M/4\pi r^2$. By symmetry the attraction on P is towards O. Let the distance PO be d and the radius of the sphere be r. Let the gravitational constant be G. The component of the attraction towards O (per unit mass at P) is $G \int_0^\pi r\, d\theta\, 2 \pi r \sin \theta\, \rho\, (d - r \cos \theta) (d^2 + r^2 - 2dr \cos \theta)^{-3/2}$. Note that the factor $(d - r \cos \theta) (d^2 + r^2 - 2dr \cos \theta)^{-1/2}$ is needed to resolve the force in the direction PO. Writing $x = \cos \theta$, this becomes $2G \pi r^2\rho \int_{-1}^1 (d - r x)/(d^2 + r^2 - 2dr x)^{3/2} dx$.

This is not as bad as it looks. It is just the sum of a $(1 - x)^{-1/2}$ and a $(1 - x)^{-3/2}$ integral, both of which are straightforward. Moreover, when we come to substitute $x = \pm 1$, the factor $(d^2 + r^2 - 2dr x)^{1/2}$ becomes just $d - r$ or $d + r$. So we get after a little simplification $4 \pi r^2\rho/d^2 = MG/d^2$, which is the same result as if all the mass was concentrated at O.

(2) We can write a general line as $x = at + b$, $y = ct + d$, $z = et + f$, for some constants a, b, c, d, e, f and a parameter t which takes all real values. If this lies in $z = xy$, then $et + f = ac\, t^2 + (bc + ad) t + bd$ for all t. Hence a or c must be

zero. If a is 0, then z = by, so the line can be written as x = b, z = by. Similarly, if c = 0, then the line can be written as y = d, z = dx. Conversely, it is easy to see that these two families of lines lie in the surface.

**Problem B1**

*Do either (1) or (2)*

(1) Let A be matrix ($a_{ij}$), $1 \le i,j \le 4$. Let d = det(A), and let $A_{ij}$ be the cofactor of $a_{ij}$, that is, the determinant of the 3 x 3 matrix formed from A by deleting $a_{ij}$ and other elements in the same row and column. Let B be the 4 x 4 matrix ($A_{ij}$) and let D be det B. Prove $D = d^3$.

(2) Let P(x) be the quadratic $Ax^2 + Bx + C$. Suppose that P(x) = x has unequal real roots. Show that the roots are also roots of P(P(x)) = x. Find a quadratic equation for the other two roots of this equation. Hence solve $(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0$.

**Solution**

Answer: (2) The quadratic is $A^2x^2 + (AB + A)x + (AC + B + 1) = 0$. The quartic in y has roots 0, 1, 2, 2.

(1) We have $a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + a_{i4}A_{i4} = d$. But $a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} + a_{i4}A_{j4} = 0$ for i not equal to j (because it can be considered as an expansion of the determinant for the matrix derived from A by replacing row i by row j - the resulting matrix has two identical rows and hence zero determinant). So if we multiply the transpose of A by the matrix ($A_{ij}$) then we get d down the diagonal and zeros elsewhere. Hence $d D = d^4$, so $D = d^3$.

(2) It is obvious that if P(x) = x, then P(P(x)) = x.

P(P(x)) = x is $A(Ax^2 + Bx + C)^2 + B(Ax^2 + Bx + C) + C = x$. This is evidently a quartic and two of its roots are those of $Ax^2 + (B - 1)x + C = 0$. We could obtain the quadratic for the other two roots by multiplying out P(P(x)) - x and factorising it. But it is sufficient to obtain the coefficients of $x^4$, $x^3$ and $x^0$. This gives us the sum of the four roots as -2B/A and their product as $(AC + B + 1)C/A^3$. The sum and product of the two known roots are -B/A - 1/A and C/A. Hence the sum and product of the other two roots are -B/A + 1/A and $(AC + B + 1)/A^2$, so the roots are the roots of the quadratic $A^2x^2 + (AB + A)x + (AC + B + 1) = 0$.

$y^2 - 3y + 2 = 0$ has roots 1 and 2. So these values are also roots of $(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0$. The other two roots are also the roots of $x^2 + (-3 + 1)x + (2 - 3 + 1) = 0$. These are obviously 0 and 2.

**Problem B2**

Find all solutions of the differential equation zz'' - 2z'z' = 0 which pass through the point x=1, z=1.

**Solution**

Answer: z = 1/( A(x - 1) + 1).

We have z''/z' = 2 z'/z. Integrating, ln z' = 2 ln z + const, so $z' = - A/z^2$. Integrating again: 1/z = Ax + B. But z(1) = 1, so B = 1 - A.

**Problem B3**

A horizontal disk diameter 3 inches rotates once every 15 seconds. An insect starts at the southernmost point of the disk facing due north. Always facing due north, it crawls over the disk at 1 inch per second. Where does it again reach the edge of the disk?

**Solution**

Answer: at the northernmost point of the disk.

Take polar coordinates with r = 3/2, θ = 0 at the start. The equations of motion are dr/dt = - cos θ, r dθ/dt = 2rπ/15 + sin θ.

Differentiating the second equation: (dr/dt) (dθ/dt) + r $d^2θ/dt^2$ = (2π/15) dr/dt + (dθ/dt) cos θ. Substituting from the first equation, 2 (dr/dt) (dθ/dt) + r $d^2θ/dt^2$ = (2π/15) dr/dt. Multiplying through by r and integrating wrt t, we get $r^2$ dθ/dt = (π/15) $r^2$ + C, for some constant C. At t = 0, r = 3/2 and dθ/dt = 2π/15, so C = 3π/20. Thus $r^2$ dθ/dt = (π/15) $r^2$ + 3π/20. But the original equation gives $r^2$ dθ/dt = $2r^2π/15$ + r sin θ. Hence π $r^2$/15 + r sin θ = 3π/20 (**). Hence if r = ±3/2, we have sin θ = 0 and hence θ = 0 or π.

That is not quite enough to show that the insect reaches the edge again at $\theta = \pi$. But we can treat (**) as a quadratic in r and solve to get $r^2 = (\sin^2\theta + (\pi/5)^2)^{1/2} - \sin\theta$. This shows that r first decreases, but then increases again to $\pm 1$ at $\theta = \pi$. We can rule out -1 because $r^2$ is always positive, and r starts positive. So by continuity it must always remain positive. Thus the insect next reaches the edge at the northenmost point of the disk.

**Problem B4**
The parabola P has focus a distance m from the directrix. The chord AB is normal to P at A. What is the minimum length for AB?

**Solution**
Answer: $3\sqrt{3}$ m.

We may take the equation of P as $2my = x^2$. The gradient at the point A $(a, a^2/2m)$ is a/m, so the normal at (a, b) is $(y - a^2/2m) = -m/a\ (x - a)$. Substituting in $2my = x^2$, it meets P at (x, y) where $x^2 + 2m^2/a\ x - (2m^2 + a^2) = 0$, so the other point B has $x = -(2m^2/a + a)$.

Thus $AB^2 = (2a + 2m^2/a)^2 + 4m^2(1 + m^2/a^2)^2 = 4a^2(1 + m^2/a^2)^3$. Differentiating, we find the minimum is at $a^2 = 2m^2$ and is $AB^2 = 27m^2$.

**Problem B5**
Find the locus of the foot of the perpendicular from the center of a rectangular hyperbola to a tangent. Obtain its equation in polar coordinates and sketch it.

**Solution**
Answer: $r^2 = 2k^2 \sin 2\theta$. It is a figure of 8 with its axis along the line y = x and touching the x-axis and y-axis at the origin.

Take the hyperbola as $xy = k^2$. Then the tangent at $(a, k^2/a)$ is $(y - k^2/a) = - k^2/a^2\ (x - a)$. The perpendicular line through the origin is $y = a^2/k^2\ x$. They intersect at $x = 2k^2/(a\ (a^2/k^2 + k^2/a^2)\ )$, $y = 2a/(a^2/k^2 + k^2/a^2)$. So the polar coordinates r, θ satisfy $\tan\theta = y/x = a^2/k^2$, $r^2 = x^2 + y^2 = 4a^2(k^4/a^4 + 1)/(k^2/a^2 + a^2/k^2) = 4a^2(\cot^2\theta + 1)/(\tan\theta + \cot\theta)^2 = 4\ a^2 \cos^2\theta/(\sin^2\theta + \cos^2\theta)^2 = 4a^2\cos^2\theta = 4k^2\sin\theta\cos\theta = 2k^2 \sin 2\theta$. Thus the polar equation of the locus is $r^2 = 2k^2 \sin 2\theta$.

**Problem B6**
What is the shortest distance between the plane Ax + By + Cz + 1 = 0 and the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. You may find it convenient to use the notation $h = (A^2 + B^2 + C^2)^{-1/2}$, $m = (a^2A^2 + b^2B^2 + c^2C^2)^{1/2}$. What is the algebraic condition for the plane not to intersect the ellipsoid?

**Solution**
The tangent plane to the ellipsoid at (X, Y, Z) is $Xx/a^2 + Yy/b^2 + Zz/c^2 = 1$. It is parallel to Ax + By + Cz + 1 = 0 iff $X/a^2 = kA$, $Y/b^2 = kB$, $Z/c^2 = kC$ for some k. But $1 = X^2/a^2 + Y^2/b^2 + Z^2/c^2 = k^2(a^2A^2 + b^2B^2 + c^2C^2) = k^2m^2$, so $k = \pm 1/m$. There are two values corresponding to two parallel tangent planes (one on either side of the ellipse). The equation of the tangent plane is k(Ax + By + Cz) = 1.

The distance of the origin from the plane Ax + By + Cz + 1 = 0 is $1/(A^2 + B^2 + C^2)^{1/2} = h$. The distance of the origin from the tangent plane k(Ax + By + Cz) = 1 is $h/|k| = hm$. So if $m \geq 1$, the plane Ax + By + Cz + 1 = 0 lies between the two tangent planes and hence intersects the ellipse. So in this case the minimum distance is zero. If $m < 1$, then the distance between the plane Ax + By + Cz + 1 = 0 and the nearer tangent plane is h(1 - m) and that is the required shortest distance.

# 2nd Putnam 1939

**Problem A1**
Let C be the curve $y^2 = x^3$ (where x takes all non-negative real values). Let O be the origin, and A be the point where the gradient is 1. Find the length of the curve from O to A.

**Solution**
Ans: $8/27 (2\sqrt{2} - 1)$. Trivial integration.

**Problem A2**
Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. Prove that the gradient at B is 4 times the gradient at A.

**Solution**
Trivial. [Take the point as $(a,a^3)$. Write down the equation of the tangent. Write down its point of intersection with the curve: $(x^3 - a^3) = 3a^2(x - a)$. We know this has a repeated root x = a. The sum of the roots is zero, so the third root is x = - 2a. Finally, $3(-2a)^2 = 4$ times $3 a^2$.]

**Problem A3**
The roots of $x^3 + a x^2 + b x + c = 0$ are α, β and γ. Find the cubic whose roots are $\alpha^3$, $\beta^3$, $g^3$.

**Solution**
$x^3 + (a^3 - 3ab + 3c) x^2 + (b^3 - 3abc + 3c^2)x + c^3 = 0.$

A routine manipulation. Suppose the roots are α, β, γ. Then $\alpha + \beta + \gamma = - a$, $\alpha\beta + \beta\gamma + \gamma\alpha = b$, $\alpha\beta\gamma = -c$. So to get the coefficients of the desired polynomial we have to find the corresponding expressions in the cubes: $\alpha^3 + \beta^3 + \gamma^3$ etc. You obviously start with $(\alpha + \beta + \gamma)^3$ etc and then add additional terms to get the desired expressions.

**Problem A4**
Given 4 lines in Euclidean 3-space:

$L_1$: x = 1, y = 0;     $L_2$: y = 1, z = 0;     $L_3$: x = 0, z = 1;     $L_4$: x = y, y = -6z.

Find the equations of the two lines which both meet all of the $L_i$.

**Solution**
A routine computation. Assume the line meets $L_1$ at (1,0,a) and $L_2$ at (b,1,0). Then it is (x - 1) = t(x - b), y = t(y - 1), (z - a) = tz. So it can only cut $L_3$ if 1/b = 1 - a, and $L_4$ if 6a = 6ab - 1. This gives a quadratic for a, which we can solve to get a = -1/2 or 1/3. Hence the possible lines are (1,0,-1/2) + t(-1/3,1,1/2) and (1,0,1/3) + t(1/2,1,-1/3).

**Problem A5**
*Do either (1) or (2)*
(1) x and y are functions of t. Solve x' = x + y - 3, y' = -2x + 3y + 1, given that x(0) = y(0) = 0.

(2) A weightless rod is hinged at O so that it can rotate without friction in a vertical plane. A mass m is attached to the end of the rod A, which is balanced vertically above O. At time t = 0, the rod moves away from the vertical with negligible initial angular velocity. Prove that the mass first reaches the position under O at $t = \sqrt{(OA/g)} \ln (1 + \sqrt{2})$.

**Solution**
(1) Differentiate the first equation, use the second equation to eliminate y', and then the (undifferentiated) first equation to eliminate y, giving:
x'' - 4 x' + 5x = 10. Solving: $x = 2 + A e^{2t} \sin t + B e^{2t} \cos t$. But x(0) = 0, so B = -2. The first equation now gives $y = x' - x + 3 = 1 + (A + 2) e^{2t} \sin t + (A - 2) e^{2t} \cos t$. But y(0) = 1, so A = 1. The final solution is thus: $x = 2 + e^{2t} \sin t - 2 e^{2t} \cos t$; $y = 1 + 3 e^{2t} \sin t - e^{2t} \cos t$.

(2) Trivial, except for the integral, which is moderately hard, unless you happen to know it.

Let the angle the rod makes with the (upward) vertical be $\theta$. Conservation of energy gives immediately: $1/2\ OA^2(d\theta/dt)^2 = OA.g(1 - \cos\theta)$.
Now for the first time in this exam we come up against something that is not completely obvious. How do we do the integral?

You need the half-angle formulae, eg $(1 - \cos\theta) = 2\sin^2\theta/2$. Now if you can remember the integral for $1/\sin z$ (eg $\ln\sin z - \ln(1 + \cos z)$, or equivalently $\ln(\mathrm{cosec}\ z - \cot z)$, then you are home.

If not, use the half-angle formulae again: $\sin\theta/2 = 2\sin\theta/4\cos\theta/4$. Putting $c = \cos\theta/4$, we have to integrate $1/(c(1 - c^2))$. Expand using partial fractions and now the integral is just a sum of logs.

**Problem A6**
*Do either (1) or (2):*
(1) A circle radius r rolls around the inside of a circle radius 3r, so that a point on its circumference traces out a curvilinear triangle. Find the area inside this figure.

(2) A frictionless shell is fired from the ground with speed *v* at an unknown angle to the vertical. It hits a plane at a height *h*. Show that the gun must be sited within a radius $v/g\ (v^2 - 2gh)^{1/2}$ of the point directly below the point of impact.

**Solution**
(1) This is moderately difficult. It is not immediately obvious what coordinates to use (or at least, after meeting an integral I did not immediately recognize, I started worrying that there might be a better choice of coordinates), and it is not immediately obvious how to do the resulting integral.

Let C be the center of the large circle and let O be the initial point of contact between the two circles. Take O as the origin and OC as the x-axis, take the y-axis so that P the point of contact gets a positive y-coordinate just after rolling starts. The easiest parameter is to take angle OCP = $\theta$. Then it is not hard to see that $x/r = 3 - 2\cos\theta - \cos 2\theta$, $y/r = 2\sin\theta - \sin 2\theta$.

Evidently we need something like $\int y\, dx$. We need a little care on the limits of integration. Let A, B be the other vertices of the curvilinear triangle (A corresponding to $\theta = 2\pi/3$, B to $\theta = 4\pi/3$). Let X be the point where the curve AB cuts the x-axis and Y the point where the line AB cuts the x-axis. $\int_O^A$ gives the area under the curve OA, in other words the area OAX plus the area AXY. Then $\int_A^X$ gives minus the area AXY (because x is decreasing, so dx is negative). $\int_X^B$ gives minus area BXY (x increasing, but y negative), and $\int_B^O$ gives plus area BXY plus area OBX (x decreasing and y negative). So the entire integral $\int_{\theta=0}^{2\pi}$ gives the required area inside the curvilinear triangle OAB. In other words we need:
$2r^2 \int_{\theta=0}^{2\pi} (2\sin\theta - \sin 2\theta)(\sin\theta + \sin 2\theta)\, d\theta$.

This is the point at which we are likely to get stuck. Changing variable to $z = \cos\theta$ does not apparently help. The trick is to put things in terms of sin nx or cos nx. We may remember that $\sin^2 z = (1 - \cos 2z)/2$, and $\cos(w +/- z) = \cos w \cos z -/+ \sin w \sin z$, so that $\sin z \sin 2z = (\cos z - \cos 3z)/2$.

Expanding the integrand gives: $2\sin^2\theta + \sin\theta\sin 2\theta - \sin^2 2\theta$. Using the two formulae above transforms this to: $1 - \cos 2\theta + (\cos\theta - \cos 3\theta)/2 - 1/2 + 1/2 \cos 4\theta$. The cos terms all integrate to zero and the constant term 1/2 integrates to $\pi$, so the final answer is $2\pi r^2$.

(2) There is a slight trap here. We may be tempted to argue that the extreme case is where the shell reaches the plane with zero vertical velocity. The horizontal velocity does not change during the trajectory, so taking u as the horizontal velocity and w as the vertical, we can write down immediately that $w^2 = 2gh$ (energy) and hence the radius $r = w/g\ (v^2 - 2gh)^{1/2}$, which is unfortunately wrong. Notice that it is smaller than the answer sought. The reason is that impact at zero vertical velocity is not the extreme case. We can usually do better by having the shell peak before the plane, so that it hits it on the way down - the extra time to travel horizontally outweighs the loss of horizontal velocity.

So we have to write down the equations: $r = tu$, $h = tw - gt^2/2$. Squaring to eliminate u, w in favor of v, gives:
$g^2t^4/4 + (gh - v^2)t^2 + (h^2 + r^2) = 0$. (*)

For this to have a real root we require $(gh - v^2)^2 \geq g^2(h^2 + r^2)$ and hence $r \leq v/g\ (v^2 - 2gh)^{1/2}$.

We are not asked to prove that the plane can be hit from anywhere within this radius, so we could stop here. But $v^2 \geq 2gh > gh$, so (*) is a quadratic of the form $a z^2 - b z + c = 0$, with a, b, c positive. Hence if r satisfies the condition, it has two *positive* roots for z and (*) has two real (positive) roots for t. Having solved for t, we can then solve for the angle (or equivalently for u, w), which shows that we can hit the plane from anywhere inside the radius.

**Problem A7**
*Do either (1) or (2):*
(1) Let $C_a$ be the curve $(y - a^2)^2 = x^2(a^2 - x^2)$. Find the curve which touches all $C_a$ for $a > 0$. Sketch the solution and at least two of the $C_a$.

(2) Given that $(1 - hx)^{-1}(1 - kx)^{-1} = \sum_{i\geq0} a_i x^i$, prove that $(1 + hkx)(1 - hkx)^{-1}(1 - h^2x)^{-1}(1 - k^2x)^{-1} = \sum_{i\geq0} a_i^2 x^i$.

**Solution**
(1) It is not hard to see that $C_1$ is a figure 8 lying on its side with the double point at (1,0), vertical tangents at (1,1) and (-1,1) and horizontal tangents at $(1/\sqrt2, 1/2)$, $(1/\sqrt2, 3/2)$, $(-1/\sqrt2,1/2)$, $(-1/\sqrt2,3/2)$. $C_a$ is obtained from $C_1$ by transforming (x,y) to $(ax,a^2y)$ (so stretching by a factor a in the x-direction and a factor $a^2$ in the y-direction). So we get a sequence of ever-larger horizontal 8s centered ever-further up the y-axis. This suggests that the envelope is something like a parabola $y = k x^2$.

At this point it is distinctly helpful to know more classical differential geometry than today's undergraduate. The classical method for finding the envelope for a 1-parameter family of curves (which *usually* works) is to differentiate wrt the parameter. Thus we eliminate a from:

$$(y - a^2)^2 - x^2(a^2 - x^2) = 0, \text{ and } \quad 4a(y - a^2) + 2a\,x^2 = 0$$

giving $y = 3/4\ x^2$ (or the y-axis, which is presumably a spurious solution arising from the double points).

Alternatively, we might guess that that the envelope is a curve $y = k x^2$ for some k. The intersection of this with $C_a$ is given by:
$(k^2 + 1)x^4 - (2k + 1)\, a^2x^2 + a^4 = 0$, or $x^2 = (2k + 1)a^2/(2k^2 + 2) +/- a^2(4k - 3)^{1/2}/(2k^2 + 2)$.

For this to be a tangent we need double roots and hence $k = 3/4$. It is now easy to check that this parabola meets $C_a$ at $(2a/\sqrt5,3a^2/5)$, $(-2a/\sqrt5,3a^2/5)$ and to check that the gradients match.

(2) Fairly easy.

This is obviously not a general result (whereby we derive $\sum a_i^2x^i$ from $\sum a_ix^i$), so we need to evaluate the $a_i$. In fact, it is easily seen that $a_i = (h^{i+1} - k^{i+1})/(h - k)$. [For example, multiply the expansions of $(1 - hx)^{-1}$ and $(1 - kx)^{-1}$ to get $a_i = h^i + h^{i-1}k + ... + h\,k^{i-1} + k^i = (h^{i+1} - k^{i+1})/(h - k)$.]

Multiplying across to try to show that higher powers of $(1 - hkx)(1 - h^2x)(1 - k^2x) \sum a_i^2x^i$ have zero coefficients is a mistake (doable, but much algebra). It is better to evaluate $\sum a_i^2x^i$ directly. After substituting for $a_i$, we are going to get terms of the form $\sum z^i$ which evaluates immediately to $(1 - z)^{-1}$, giving the expression in the question in partial fraction form.

**Problem B1**
The points P(a,b) and Q(0,c) are on the curve $y/c = \cosh(x/c)$. The line through Q parallel to the normal at P cuts the x-axis at R. Prove that QR = b.

**Solution**
Trivial. [Let O be the origin. Then OR/OQ = sinh a/c, so $QR^2 = c^2(1 + \sinh^2 a/c)$, so QR = b.]

**Problem B2**
Evaluate $\int_1^3 (\,(x - 1)(3 - x)\,)^{-1/2}\,dx$ and $\int_1^{\text{inf}} (e^{x+1} + e^{3-x})^{-1}\,dx$.

**Solution**
(1) $\pi$. Trivial. [Write $(x - 1)(3 - x) = 1 - (x - 2)^2$. So we have a standard $\sin^{-1}z$ integral.]

(2) $\pi/(4 e^2)$. Trivial. [Multiply top and bottom by $e^x$. Change variable to $y = e^{x-1}$. We now have a standard $\tan^{-1}z$ integral.]

**Problem B3**
Given $a_n = (n^2 + 1) 3^n$, find a recurrence relation $a_n + p\, a_{n+1} + q\, a_{n+2} + r\, a_{n+3} = 0$. Hence evaluate $\sum_{n\geq 0} a_n x^n$

**Solution**
We can solve formally to get the recurrence relation, but it is quicker to get there informally. We look for a relation between $b_n = a_n$, $b_{n+1} = a_{n+1}/3$, $b_{n+2} = a_{n+2}/9$, $b_{n+3} = a_{n+3}/27$, because that takes care of the powers of 3. So, ignoring the $3^n$, we are looking at:

$n^2 + 1$
$n^2 + 2n + 2$
$n^2 + 4n + 5$
$n^2 + 6n + 10$

We try to get a linear combination of the first three which is constant. But that is easy: subtracting twice the second from the third gets rid of the n term, then adding the first gets rid of the $n^2$ term. So, $b_{n+2} - 2b_{n+1} + b_n = 2.3^n$. But $b_{n+3} - 2b_{n+2} + b_{n+1}$ has the *same* value, so subtracting:

$a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n = 0$, which is the required recurrence relation.

Let the power series sum to y. Then taking $y - 9x + 27x^2y - 27x^3y$ will give $a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n$ as the coefficient of $x^{n+3}$, so we need only worry about the early terms: $a_0 + (a_1 - 9a_0)x + (a_2 - 9a_1 + 27a_0)x^2 = (1 - 3x + 18x^2)$. Hence $y = (1 - 3x + 18x^2)/(1 - 9x + 27x^2 - 27x^3)$.

Using the ratio test, the original series evidently converges for $|x| < 1/3$, which may prompt us to notice that $1 - 9x + 27x^2 - 27x^3 = (1 - 3x)^3$.

That in turn may prompt us to try solving the problem backwards. We know that:

$1/(1 - z) = \sum z^n$; $1/(1 - z)^2 = \sum (n+1)z^n$; $1/(1 - z)^3 = \sum (n+1)(n+2)/2\, z^n$.

Hence $2/(1 - z)^3 - 3/(1 - z)^2 + 2/(1 - z) = \sum (n^2 + 1)z^n$. Replacing z by 3x gives $\sum a_n x^n = (1 - 3x + 18x^2)/(1 - 3x)^3$. Multiplying across by $(1 - 3x)^3$ now gives the required recurrence relation.

**Problem B4**
The *axis* of a parabola is its axis of symmetry and its *vertex* is its point of intersection with its axis. Find: the equation of the parabola which touches $y = 0$ at (1,0) and $x = 0$ at (0,2); the equation of its axis; and its vertex.

**Solution**
The general equation of the a parabola is: $(ax + by)^2 + cx + dy + e = 0$. Its intersection with $y = 0$ is given by $a^2x^2 + cx + e = 0$. This must have a double root, so $c = -2a^2$, $e = -a^2$. Considering the other tangent, we find: $d = -4b^2$, $e = 4b^2$. So (up to an irrelevant constant factor) we have: $a = 2, b = 1, c = -8, d = -4, e = 4$; or $a = 2, b = -1, c = -8, d = -4, e = 4$. But in the first case the equation can be written as $(2x + y - 2)^2 = 0$, which is a double line. It is debatable whether this qualifies as a parabola, but it would not normally be said to *touch* the points (1,0) and (0,2). So we are left with the parabola: $(2x - y)^2 - 8x - 4y + 4 = 0$.

We want to put this in the form $u = kv^2$. The line $x + 2y = 0$ is perpendicular to the line $2x - y = 0$, so we change variables to $X = 2x - y$, $Y = x + 2y$, giving: $X^2 - 16/5\, Y - 12/5\, X + 4 = 0$, or $16/5\, (Y - 4/5) = (X - 6/5)^2$, which is the equation of a parabola with vertex $X = 6/5$, $Y = 4/5$, axis $X = 6/5$. Changing back to the original coordinates, $x = (2X + Y)/5$, $y = (2Y - X)/5$, the vertex is (16/25, 2/25) and the axis is $10x - 5y = 6$.

**Problem B5**
*Do either (1) or (2):*

(1) Prove that $\int_1^k [x] f'(x)\,dx = [k] f(k) - \sum_1^{[k]} f(n)$, where $k > 1$, and $[z]$ denotes the greatest integer $\le z$. Find a similar expression for: $\int_1^k [x^2] f'(x)\,dx$.

(2) A particle moves freely in a straight line except for a resistive force proportional to its speed. Its speed falls from 1,000 ft/s to 900 ft/s over 1,200 ft. Find the time taken to the nearest 0.01 s. [No calculators or log tables allowed!]

**Solution**

(1) $[x]$ is constant over the interval $[i, i+1)$ for $i$ an integer, so we split the range of integration to get $\int_1^k = \int_{[k]}^k + \int_1^2 + \int_2^3 + \dots + \int_{[k]-1}^{[k]}$. We can write down each of these integrals, collect terms and get the result.

The same idea works the second integral, except that we divide at $\sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{[k^2]}$, giving the result: $[k^2] f(k) - ( f(1) + f(\sqrt{2}) + f(\sqrt{3}) + \dots + f(\sqrt{[k^2]}) )$.

(2) Easy. 1.26 s.

The equation of motion is $x'' = - k x'$. Integrating: $x' = k A e^{-kt}$. Integrating again, and putting $x(0) = 0$, $x = A(1 - e^{-kt})$. Suppose T is the required time. Then from the speed $e^{-kT} = 0.9$. $x(T) = 1200$, so $A = 12000$. The initial speed is 1000, so $k = 1/12$ and $T = -12 \ln 0.9$. The only slight snag is that in the exam calculators did not exist and log tables were not allowed. So we have to use the expansion $\ln(1 + x) = x - x^2/2 + x^3/3 - \dots$, or more usefully: $- \ln(1 - x) = x + x^2/2 + x^3/3 + \dots$, giving $T = 1.2 + 0.06 + 0.004 + 0.0003 + \dots$ or 1.26 s.

**Problem B6**

*Do either (1) or (2):*

(1) f is continuous on the closed interval $[a, b]$ and twice differentiable on the open interval $(a, b)$. Given $x_0 \in (a, b)$, prove that we can find $\xi \in (a, b)$ such that $( (f(x_0) - f(a))/(x_0 - a) - (f(b) - f(a))/(b - a) )/(x_0 - b) = f''(\xi)/2$.

(2) AB and CD are identical uniform rods, each with mass *m* and length 2*a*. They are placed a distance *b* apart, so that ABCD is a rectangle. Calculate the gravitational attraction between them. What is the limiting value as *a* tends to zero?

**Solution**

(1) We obviously have to use the mean value theorem. So we need to construct a suitable auxiliary function to apply it to. It is usually easiest to apply the MVT in cases where the function has equal values at the two ends of the interval. So we want to find some function g, such that g' has equal values at two different points. Let the value of the expression given, $( (f(x_0) - f(a))/(x_0 - a) - (f(b) - f(a))/(b - a) )/(x_0 - b)$, be $y_0$. Then we are looking for $g''(x)$ to be something like $1/2\, f''(x) - y_0$.

Let us start by rearranging the expression for $y_0$ to give $f(x_0) = f(a) + ( f(b) - f(a) )(x_0 - a)/(b - a) + y_0(x_0 - a)(x_0 - b)$. After a little experimentation we may try looking at:

$$g(x) = f(x) - f(a) - ( f(b) - f(a) )(x - a)/(b - a) - y_0(x - a)(x - b).$$

We notice that $g(a) = 0$, $g(b) = 0$, $g(x_0) = 0$, $g'(x) = f'(x) - ( f(b) - f(a) )/(b - a) - y_0(2x - a - b)$, $g''(x) = f''(x) - 2 y_0$. At this point we should realize that we are home, because we have to show that we can find $\xi$ such that $g''(\xi) = 0$. But the mean value theorem gives us a value in the interval $(a, x_0)$ at which g' is zero and another in $(x_0, b)$. Hence there must be a value between the two (and *a fortiori* in $(a, b)$ ) at which g'' is zero.

(2) Straightforward, apart from an awkward integral. Answer: $Gm^2( 1 - (1 + 4a^2/b^2)^{1/2} )/(2a^2)$, which tends to $Gm^2/b^2$ as the rods shorten to become point masses.

By symmetry the net force must be perpendicular to the rods, so we just calculate the perpendicular component. Take coordinates x along one rod and y along the other. Then we can write down immediately that the perpendicular component is:

$$Gm^2/(4a^2) \int (b^2 + (x - y)^2)^{-1}\, b\, (b^2 + (x - y)^2)^{-1/2}\, dx\, dy.$$

The integrand is of the form $(1 + z^2)^{-3/2}$. It helps to know that this integrates to $z/(1 + z^2)^{1/2}$ (as is easily checked). People used to mess around with trigonometric substitutions (eg tan θ reduces it to cos θ, which integrates immediately to sin θ, which one has to remember to think of as tan θ/sec θ). But it was always easier simply to learn a large number of integrals by rote. Nowadays, of course, one tends to look them up or use Mathematica/Maple.

So integrating first by y, we get $Gm^2/(4a^2) \int dx/b \{ (x - 2a)/b (1 + (x - 2a)^2/b^2)^{-1/2} - x/b (1 + x^2/b^2)^{-1/2} \}$. This is much less horrible than it looks, because we are just faced with integrating $z/(1 + z^2)^{1/2}$ which is obviously $(1 + z^2)^{1/2}$. So we get the expression above. For the limit, just expand the square root as a power series $1 + 2a^2/b^2 - a^4/(4b^4) + \dots$ .

**Problem B7**

*Do either (1) or (2):*

(1) Let $a_i = \sum_{n=0}^{\infty} x^{3n+i}/(3n+i)!$ Prove that $a_0^3 + a_1^3 + a_2^3 - 3 a_0a_1a_2 = 1$.

(2) Let O be the origin, λ a positive real number, C be the conic $ax^2 + by^2 + cx + dy + e = 0$, and $C_\lambda$ the conic $ax^2 + by^2 + \lambda cx + \lambda dy + \lambda^2 e = 0$. Given a point P and a non-zero real number k, define the transformation D(P,k) as follows. Take coordinates (x',y') with P as the origin. Then D(P,k) takes (x',y') to (kx',ky'). Show that D(O,λ) and D(A,-λ) both take C into $C_\lambda$, where A is the point $(-c\lambda/(a(1 + \lambda)), -d\lambda/(b(1 + \lambda)))$. Comment on the case λ = 1.

**Solution**

(1) Moderately hard, unless differentiating is a reflex, in which case it is easy.

The series are all absolutely convergent for all x, so we can carry out whatever operations we want. But it is not at all obvious what to do. Multiplying out the power series to get a complicated sum for the coefficient of $x^n$ is offputting. Much scope for algebraic error, and no guarantee that the eventual simplification will be obvious. So we look for some trick. The $a_i$ are closely related to the exponential series, eg $a_0 + a_1 + a_2 = e^x$, and the prevalence of 3 in the question may eventually suggest looking at ω, the cube root of 1. Indeed, $a_0 + \omega a_1 + \omega^2 a_2 = e^{\omega x}$. Since $\omega^2$ is also a root, we also have $a_0 + \omega^2 a_1 + \omega a_2 = e^{\omega 2x}$. Remembering that $1 + \omega + \omega^2 = 0$ might prompt us to multiply these three expressions together, but it helps to remember the product of the three left hand sides: $(a_0 + a_1 + a_2)(a_0 + \omega a_1 + \omega^2 a_2)(a_0 + \omega^2 a_1 + \omega a_2) = a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2$. Otherwise, you are faced with multiplying this out the hard way: after collecting terms, you get $(a_0^3 + a_1^3 + a_2^3)$, then 6 expressions of the type $a_0^2(1 + \omega + \omega^2)$, which are all zero, and $3a_0a_1a_2(\omega + \omega^2)$, which is $-3a_0a_1a_2$.

A more general approach, which is more likely to work, is to differentiate the expression $a_0^3 + a_1^3 + a_2^3 - 3 a_0a_1a_2$. Provided you notice that $a_0' = a_2$ etc, this gives the result almost immediately (the derivative is zero, so the expression must be constant, but its value for x = 0 is 1).

(2) Trivial. [D(O,λ) takes (x,y) to (λx,λy). So if (x,y) satisfies the equation for C, just check that (λx,λy) satisfies the equation for $C_\lambda$. Similarly, D(A,-λ) takes (x,y) to (-λx - λc/a, -λy - λd/b). Again, just check by substituting this into the equation for $C_\lambda$ and using the fact that (x,y) satisfies the equation for C. If λ = 1, then $C_\lambda$ = C, D(O,λ) is the identity transformation, and D(A,-λ) the central symmetry.]

## 3rd Putnam 1940

**Problem A1**
p(x) is a polynomial with integer coefficients. For some positive integer c, none of p(1), p(2), ... , p(c) are divisible by c. Prove that p(b) is not zero for any integer b.

**Solution**
Suppose p(b) = 0. Then p(x) = (x - b)q(x), where q is a polynomial with integer coefficients. Put b = cd + r, where $1 \le r \le c$ (note that this is different from the conventional $0 \le r < c$, but still possible because 1, 2, ... c are a complete set of residues mod c). Then p(r) = p(b - cd) = - cd q(r) which is divisible by c. Contradiction.

**Problem A2**
y = f(x) is continuous with continuous derivative. The arc PQ is concave to the chord PQ. X is a point on the arc PQ for which PX + XQ is a maximum. Prove that XP and XQ are equally inclined to the tangent at X.

**Solution**
Take s to be the arc-length PX and z to be PX + XQ. Suppose the tangent is WXY, so that angle WXP = $\theta$ and angle YXQ = $\varphi$. Then to first order if we vary X, the change $\delta z = \delta s (\cos \theta - \cos \varphi)$, which is can be made positive (by choosing the sign of $\delta s$ appropriately), thus contradicting the maximality of X, unless $\theta = \varphi$.

The official solution. If X is a point of S such that PX + XQ ≥ PY + YQ for all points Y of S, then we can easily show that S lies in the half-plane bounded by external angle bisector of PXQ. If X lies on PQ, then S must be a subset of the segment PQ and the result is trivial. So assume it does not. Now reflect Q in the bisector to get Q', with PXQ' a straight line. Then if Y is any point in the same half-plane as Q', we have YQ' < YQ and hence PY + YQ > PY + YQ' ≥ PQ' = PX + XQ' = PX + XQ, so Y is not in S.

This argument depends upon X achieving a global maximum. The original wording of the question, "a point ... for which ... is a maximum" (which on this point I quoted exactly), is somewhat ambiguous. Does it mean a local or a global maximum? If it means a local, then we have to take S to be a small arc for which the maximum is global. Then that arc lies on one side of the line. Since it also has a point in common (X) and is differentiable, the line must be the tangent to the arc at X.

This solution is somewhat harder and less obvious (unless you have seen it before), so I prefer the simpler solution.

**Problem A3**
$\alpha$ is a fixed real number. Find all functions f: R → R (where R is the reals) which are continuous, have a continuous derivative, and satisfy $\int_b^y f^\alpha(x)\,dx = \left( \int_b^y f(x)\,dx \right)^\alpha$ for all y and some b.

**Solution**
Straightforward to get the basic idea, but care is needed with the details. It is quite hard to get the answer exactly right (the official solution gives a spurious solution for the case (2) - overlooking the problem with the integration limits).

Answer:
(1) No solutions for $\alpha = 0$;
(2) No solutions for $\alpha < 0$;
(3) Any continuous f with continuous derivative is a solution for $\alpha = 1$;
(4) For $\alpha > 0$ and not of the form p/q, with p and q odd positive integers, $f = A e^{kx}$, where A is any real, and k is the positive real value of $\alpha^{1/(\alpha-1)}$;
(5) For $\alpha = p/q$, with p and q odd positive integers (but not both 1), $f = A e^{kx}$ or $A e^{-kx}$, where A is any real, and k is the positive real value of $\alpha^{1/(\alpha-1)}$.

Case (3) is obvious. Case (1) is almost obvious: if $\alpha = 0$, then the lhs varies with y ( = y - b), but the rhs does not ( = 1), so there are no solutions. Assume now that $\alpha$ is not 0 or 1.

Differentiate wrt y and put $g(y) = \int_b^y f(x)\,dx$. Then f = g' and we have $f^\alpha = \alpha g^{\alpha-1} f$. So f = k g, where $k = \alpha^{1/(\alpha-1)}$. Since f = g', we can integrate immediately to get $f(x) = A e^{kx}$ (*).

However, we have to consider how many real values k can have. If $\alpha > 0$, then k certainly has a positive value, but we can also take the corresponding negative value if $1/(\alpha - 1)$ involves an even root, in other words if $\alpha = p/q$ with p and q both odd. Finally, (*) is clearly necessary, but not necessarily sufficient, so we have to subsitute (*) back into the original equation. For $k > 0$, we find the solution works provided $b = -\infty$. If $k < 0$, then it works with $b = \infty$. This gives cases (4) and (5). [Note, however, that we can only allow A negative for $\alpha$ not of the form p/q with p and q odd, provided we are content for both sides of the original equation to have complex values (even though f is real valued).]

For $\alpha < 0$, all values of $\alpha^{1/(\alpha - 1)}$ are complex unless $\alpha = - p/q$ with p an even positive integer and q an odd positive integer. If $\alpha$ has that form, then we may take $k = - 1/|\alpha|^{1/(|\alpha|+1)}$. But now there is a problem with b. Taking $A = 1$ for simplicity, the $f^\alpha(x) = e^{|\alpha k|x}$, so the lhs = const $(e^{|\alpha k|y} - e^{|\alpha k|b})$ and the rhs = const $/(e^{- |k|y} - e^{- |k|b})^{|\alpha|}$. The constants are the same, but we need $b = -\infty$ to get rid of the b term on the lhs and $b = \infty$ to get rid of the term on the rhs. So there are no solutions in case (2).

**Problem A4**
p is a positive constant. Let R is the curve $y^2 = 4px$. Let S be the mirror image of R in the y-axis ($y^2 = - 4px$). R remains fixed and S rolls around it without slipping. O is the point of S initially at the origin. Find the equation for the locus of O as S rolls.

**Solution**
Answer: $x(x^2 + y^2) + 2p\, y^2 = 0$.

Take the point of contact as (X, Y), so $Y^2 = 4pX$. The tangent at to R at this point has gradient $2p/Y$, and hence has equation $(y - Y) = 2p/Y\, (x - X)$. The perpendicular to the tangent through the origin has equation $x = - 2p/Y\, y$. If (x, y) is their point of intersection, then (2x, 2y) is the point O (since S in its new position is the reflection of R in the tangent).

Solving for x, y: $y = Y^3/(2(Y^2 + 4p^2) )$, $x = - 2p/Y = - pY/(Y^2 + 4p^2)$. So the point O is $(-2pY^2/(Y^2 + 4p^2), Y^3/(Y^2 + 4p^2) )$ (*). Using $Y = -2py/x$, we get $x = -2py^2/(x^2 + y^2)$, or $x(x^2 + y^2) + 2p\, y^2 = 0$ (**). We have shown that the locus is given by (*) and that all points on (*) are on (**). However, we must check that (**) does not include additional points. Writing (**) as $y^2 = - x^3/(x + 2p)$, shows that for each value of x in the range (-2p, 0) there are exactly two possible values of y, and the only other point on (**) is the origin. Inspection shows that the same is true for (*), so the two expressions are equivalent.

**Problem A5**
Prove that the set of points satisfying $x^4 - x^2 = y^4 - y^2 = z^4 - z^2$ is the union of 4 straight lines and 6 ellipses.

**Solution**
$x^4 - x^2 = (x^2 - 1/2)^2 - 1/4$, so $x^4 - x^2 = y^4 - y^2$ is equivalent to $(x^2 - 1/2)^2 = (y^2 - 1/2)^2$ and hence to $x^2 - 1/2 = +/- (y^2 - 1/2)$, which is equivalent to $x = y$, or $x = -y$, or $x^2 + y^2 = 1$. Similarly, for $y^4 - y^2 = z^4 - z^2$. So the equation given is equivalent to: (1) $x = y$ and $y = z$, or (2) $x = y$ and $y = -z$, or (3) $x = -y$ and $y = z$, or (4) $x = -y$ and $y = -z$, or (5) $x = y$ and $y^2 + z^2 = 1$, or (6) $x = -y$ and $y^2 + z^2 = 1$, or (7) $x^2 + y^2 = 1$ and $y = z$, or (8) $x^2 + y^2 = 1$ and $y = -z$, or (9) $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

Clearly (1) - (4) are straight lines. (5) is the intersection of a plane and a cylinder, which is an ellipse. Similarly (6), (7) and (8). (9) is slightly harder to see. If one's visualization is good, then one can see that the intersection of two cylinders with the same radius and axes intersecting at right angles is two perpendicular ellipses with a common minor axis. Otherwise, subtracting the two equations we see that $x = z$ or $-z$ and the intersection is also given by the intersection of a cylinder with two planes.

**Problem A6**
p(x) is a polynomial with real coefficients and derivative $r(x) = p'(x)$. For some positive integers a, b, $r^a(x)$ divides $p^b(x)$. Prove that for some real numbers A and $\alpha$ and for some integer n, we have $p(x) = A(x - \alpha)^n$.

**Solution**
Write $p(x) = A \prod (x - \alpha_i)^{n_i}$.
Then $r(x) = p(x) \sum n_i/(x - \alpha_i) = (A \prod (x - \alpha_i)^{n_i - 1} )\, q(x)$, where no $\alpha_i$ is a root of q(x). This is easily seen, because q(x)

is a sum of terms, all but one of which has a factor $(x - \alpha_i)$. But $q(x)$ divides $p(x)^b$ which has no roots except the $\alpha_i$. Hence $q(x)$ must be a constant. But now the degree of $r(x)$ is wrong unless there is just one $\alpha_i$.

**Problem A7**
$a_i$ and $b_i$ are real, and $\sum_1^\infty a_i^2$ and $\sum_1^\infty b_i^2$ converge. Prove that $\sum_1^\infty (a_i - b_i)^p$ converges for $p \geq 2$.

**Solution**
Notice first that it is sufficient to prove the result for $p = 2$. For that is equivalent to the statement that $\sum |a_i - b_i|^2$ converges. Hence for sufficiently large i, $|a_i - b_i| < 1$, and hence $|a_i - b_i|^p \leq |a_i - b_i|^2$. So $\sum (a_i - b_i)^p$ is absolutely convergent and hence convergent.

$(a_i - b_i)^2 = a_i^2 - 2a_ib_i + b_i^2$. The only tricky part is the middle term. It may be positive, so we cannot simply argue that $0 \leq (a_i - b_i)^2 \leq a_i^2 + b_i^2$. However, it is true that $0 \leq (a_i - b_i)^2 = 2a_i^2 + 2b_i^2 - (a_i + b_i)^2 \leq 2a_i^2 + 2b_i^2$. That suffices, since $\sum a_i^2$ and $\sum b_i^2$ are absolutely convergent, hence also $\sum (2a_i^2 + 2b_i^2)$.

**Problem A8**
Show that the area of the triangle bounded by the lines $a_ix + b_iy + c_i = 0$ $(i = 1, 2, 3)$ is $\Delta^2/|2(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)|$, where $\Delta$ is the 3 x 3 determinant with columns $a_i$, $b_i$, $c_i$.

**Solution**
Fairly easy if you remember some formulae for determinants (which people did in those days). Of course, you can just slog through the expression (*) below in terms of $a_i$, $b_i$, $c_i$. That is doable, but completely mindless, the only required skill is doing elementary algebra fast without mistakes. I suppose I am fairly out of sympathy with the rather common Putnam style of problem where the basic idea is obvious, but you have to be skilful at evaluating integrals, determinants etc, often using tricks.

Take $A_i$ be the cofactor of $a_i$ in $\Delta$. Similarly, $B_i$ and $C_i$. [So, for example, $A_1 = b_2c_3 - b_3c_2$, $A_2 = b_3c_1 - b_1c_3$, $A_3 = b_1c_2 - b_2c_1$.]

The lines $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ intersect at $(A_1/C_1, B_1/C_1)$. Similarly, the other two points of intersection are $(A_2/C_2, B_2/C_2)$ and $(A_3/C_3, B_3/C_3)$.

The area of the triangle is therefore (the absolute value of) the determinant K with rows $A_1/C_1, B_1/C_1, 1; A_2/C_2, B_2/C_2, 1; A_3/C_3, B_3/C_3, 1$. (*) [For example, take a z-coordinate perpendicular to the plane, and take the cross product of the vectors along two sides.] But $K |C_1C_2C_3|$ is the determinant whose elements are the cofactors of the original determinant. This has value equal to $\Delta^2$. For example, on multiplying it by $\Delta$, we get $\Delta$ down the diagonal and zeros elsewhere, and hence $\Delta^3$.

**Problem B1**
A stone is thrown from the ground with speed v at an angle $\theta$ to the horizontal. There is no friction and the ground is flat. Find the total distance it travels before hitting the ground. Show that the distance is greatest when $\sin \theta \ln (\sec \theta + \tan \theta) = 1$.

**Solution**
Take coordinates (time and distance) zeroed on the peak of the trajectory. After time t, the stone travels a distance $x = v t \cos \theta$ horizontally and a distance $1/2\ g\ t^2$ vertically. So the trajectory is a parabola $2y = k x^2$, where $k = g/(v \cos \theta)^2$. The stone is on the ground at $t = +/- v/g \sin \theta$, at a horizontal distance +/- a from the peak, where $a = v^2/g \sin \theta \cos \theta$.

The length of the parabola $y = k/2\ x^2$ between $x = -a$ and $x = a$ is $2 \int_0^a (1 + k^2x^2)^{1/2} dx$.

To do the integral it helps to remember that $1 + \sinh^2 z = \cosh^2 z$. So substituting $kx = \sinh z$, will essentially give us the integral of $\cosh^2 z$. That is doable, using the analog of the double angle formulae. So setting $I = \int (1 + k^2x^2)^{1/2} dx$, and substituting $x = \sinh z$, we have $I = 1/k \int \cosh^2 z\ dz = 1/(2k) \int (\cosh 2z + 1) dz = 1/(4k) \sinh 2z + 1/(2k) z = 1/(2k) \sinh z \cosh z + 1/2\ z = x/2 (1 + k^2x^2)^{1/2} + 1/(2k) \sinh^{-1}(kx)$. We have $k = g/(v \cos \theta)^2$, and $a = v^2/g \sin \theta \cos \theta$, so $ka = \tan \theta$. So the required path length is $p(\theta) = a(1 + k^2a^2)^{1/2} + 1/k \sinh^{-1} a = v^2/g (\sin \theta + \cos^2\theta \sinh^{-1} \tan \theta)$.

To find the maximum, we differentiate, getting $p'(\theta) = \cos\theta - 2\cos\theta\sin\theta\sinh^{-1}\tan\theta + \cos^2\theta(1 + \tan^2\theta)^{-1/2}\sec^2\theta = 2\cos\theta(1 - \sin\theta\sinh^{-1}\tan\theta)$. In the range $[0, \pi/2]$, $\tan\theta$ is monotone increasing. $\sinh^{-1}z$ is strictly monotone increasing for positive z, so $\sinh^{-1}\tan\theta$ is strictly monotone increasing on $(0, \pi/2)$. Indeed it evidently tends to $\infty$ as $\theta$ tends to $\pi/2$. Hence $(1 - \sin\theta\sinh^{-1}\tan\theta)$ is strictly monotone decreasing on $(0, \pi/2)$ and crosses zero once. Cos $\theta$ is monotone decreasing and positive, so $p'(\theta)$ is strictly monotone decreasing and crosses zero once. Hence $p(\theta)$ has a single maximum on $[0, \pi/2]$, which is achieved for the value $\varphi$ in $(0, \pi/2)$ for which $(1 - \sin\varphi\sinh^{-1}\tan\varphi) = 0$. Rearranging, $\sinh(1/\sin\varphi) = \tan\varphi$. Squaring, adding 1, and taking the square root: $\cosh(1/\sin\varphi) = \sec\varphi$. Adding the last two equations: $e^{1/\sin\varphi} = \tan\varphi + \sec\varphi$, or $1/\sin\varphi = \ln(\tan\varphi + \sec\varphi)$, or $\sin\varphi\ln(\tan\varphi + \sec\varphi) = 1$.

**Problem B2**

$C_1$, $C_2$ are cylindrical surfaces with radii $r_1$, $r_2$ respectively. The axes of the two surfaces intersect at right angles and $r_1 > r_2$. Let S be the area of $C_1$ which is enclosed within $C_2$. Prove that $S = 8r_2^2A = 8r_1^2C - 8(r_1^2 - r_2^2)B$, where $A = \int_0^1 (1 - x^2)^{1/2}(1 - k^2x^2)^{-1/2}\,dx$, $B = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{-1/2}\,dx$, and $C = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{1/2}\,dx$, and $k = r_2/r_1$.

**Solution**

It is hard to see the point of the second half [ $8r_2^2A = 8r_1^2C - 8(r_1^2 - r_2^2)B$ ], which is trivial. Multiply top and bottom of the integrand in A by $(1 - x^2)^{1/2}$, so that we get $(1 - x^2)$ on the top. Then note that $k^2(1 - x^2) = 1 - k^2x^2 + (k^2 - 1)$.

For the first half, the part of $C_1$ enclosed inside $C_2$ comprises two bent ovals, one at each end. It is tempting to think that when rolled flat, each piece is an ellipse. If that were true, then the problem would be trivial - the semi-axes are $r_2$ and $r_1\sin^{-1}k$, so the total area would be $2\pi r_1r_2\sin^{-1}k$. But conics only remain conics when projected onto flat surfaces and here we are projecting onto a curved surface.

So we have to use integration. Take $C_1$ to be $x^2 + y^2 = r_1^2$, and $C_2$ to be $y^2 + z^2 = r_2^2$. We calculate the area of the quarter of the piece with $x > 0$ having $y, z > 0$. We may divide it into strips parallel to the z-axis. Take the angle $\theta$ to be the axial angle - the angle by which the strip has to be rotated about the z-axis from the $y = 0$ position (keeping in the surface $C_1$). [Points on the strip have the same x and y coordinates, but varying z-coordinates.] The strip has y-coordinate $r_1\sin\theta$ and hence length $(r_2^2 - r_1^2\sin^2\theta)^{1/2}$ and width $r_1 d\theta$. Thus the area of the quarter-oval is $\int_0^{\sin-1k} (r_2^2 - r_1^2\sin^2\theta)^{1/2} r_1\,d\theta$. Put $t = 1/k \sin\theta$, and the integral becomes $r_2^2 \int_0^1 (1 - t^2)^{1/2} (1 - k^2t^2)^{-1/2}\,dt$. Hence the complete oval is 4 times this, and the total area 8 times.

Comments. The original question was badly worded, because at first sight it appeared to be about volumes, not areas, which may have confused some. It is important to get a clear picture of the geometry - the easiest way to do this is to roll a piece of paper into a cylinder.

**Problem B3**

Let p be a positive real, let S be the parabola $y^2 = 4px$, and let P be a point with coordinates (a, b). Show that there are 1, 2 or 3 normals from P to S according as $4(2p - a)^2 + 27pb^2 >, = \text{ or } < 0$.

**Solution**

The general point on the parabola is $(pt^2, 2pt)$. The slope of the tangent is $2p/y = 1/t$, and so the slope of the normal is $-t$. Hence the equation of the normal is $(y - 2pt) = -t(x - pt^2)$. This passes through (a, b) iff $pt^3 + (2p - a)t - b = 0$ (*).

This is a cubic, so it has 1, 2 or 3 real roots. We have to decide which. If $(2p - a) > 0$, then $pt^3 + (2p - a)t$ is strictly increasing and takes all values in $(-\infty, \infty)$, so (*) has just one real root. The same is true if $(2p - a) = 0$, unless $b = 0$, in which case there are three coincident roots.

If $(2p - a) < 0$, then $pt^3 + (2p - a)t$ has a maximum and a minimum. Differentiating, we find that these are at $t = \pm\sqrt{-(2p - a)/(3p)}$, with values $\pm 2(2p - a)/3\sqrt{-(2p - a)/(3p)}$. So there are 1, 2 or 3 real roots according as $b^2 >, = \text{ or } < 4(2p - a)^2/9 \cdot -(2p - a)/3p$, which is the condition in the question. Note that if $(2p - a) \geq 0$, then this expression is certainly $> 0$, so the same rule applies.

**Problem B4**

Let S be the surface $ax^2 + by^2 + cz^2 = 1$ (a, b, c all non-zero), and let K be the sphere $x^2 + y^2 + z^2 = 1/a + 1/b + 1/c$ (known as the *director* sphere). Prove that if a point P lies on 3 mutually perpendicular planes, each of which is tangent to S, then P lies on K.

**Solution**
Let $f(x, y, z) = ax^2 + by^2 + cz^2$. Then the normal vector is the vector grad f, so the tangent plane at (u, v, w) is a.u.x + b.v.y + c.w.z = 1.

Note that this does not pass through the origin. The general plane not through the origin has equation $\mathbf{p}.(\mathbf{x} - \lambda\mathbf{p}) = 0$ or $\mathbf{p.x} = \lambda$, where $\mathbf{x}$ is the vector (x, y, z) representing a general point on the plane, $\mathbf{p}$ = (p, q, r) is a unit vector normal to the plane, and $\lambda > 0$ is the distance of the plane from the origin. If this is a tangent plane at some point of the quadric, then $a(p/\lambda a)^2 + b(q/\lambda b)^2 + c(r/\lambda c)^2 = 1$, or $p^2/a + q^2/b + r^2/c = \lambda^2$.

So suppose P is the point of intersection of three perpendicular tangent planes $p_ix + q_iy + r_iz = \lambda_i$, i = 1, 2, 3, where $\mathbf{p}_i$ are orthonormal vectors. The squared distance of P from the origin is $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = (p_1^2 + p_2^2 + p_3^2)/a + (q_1^2 + q_2^2 + q_3^2)/b + (r_1^2 + r_2^2 + r_3^2)/c = 1/a + 1/b + 1/c$, since $\mathbf{p}_i$ are orthonormal. [This is the key trick: if the rows of a matrix are orthonormal vectors, then the matrix is orthogonal and hence its *columns* are also orthonormal vectors.].

**Problem B5**
Find all rational triples (a, b, c) for which a, b, c are the roots of $x^3 + ax^2 + bx + c = 0$.

**Solution**
Answer: (0, 0, 0); (1, -1, -1), (1, -2, 0).

We require (1) a + b + c = -a, (2) ab + bc + ca = b, and (3) abc = -c.
From (3), either c = 0, or ab = -1. If c = 0, then (1) becomes b = -2a, and (2) becomes b(a - 1) = 0. Hence either a = b = 0, or a = 1, b = -2.
So assume $c \neq 0$, and ab = -1. (1) becomes c = - b - 2a. Substituting in (2), we get: -1 - (2a + b)(a + b) = b, so $-a^2 - 2a^4 + 3a^2 - 1 = -a$, or $2a^4 - 2a^2 - a + 1 = 0$. So a = 1, or $2a^3 + 2a^2 - 1 = 0$ (*). The first possibility gives a = 1, b = -1, c = -1. Suppose a = m/n is a root of (*) with m, n relatively prime integers. Then $2m^3 + 2m^2n - n^3 = 0$. So any prime factor of n must divide 2 and any prime factor of m must divide 1. Hence the only possibilities are a = 1, -1, 1/2, -1/2, and we easily check that these are not solutions. So (*) has no rational roots.

**Problem B6**
The n x n matrix $(m_{ij})$ is defined as $m_{ij} = a_ia_j$ for $i \neq j$, and $a_i^2 + k$ for i = j. Show that $\det(m_{ij})$ is divisible by $k^{n-1}$ and find its other factor.

**Solution**
Answer: $\det(m_{ij}) = k^{n-1}(k + \Sigma a_i^2)$.

Induction on n. Clearly true for n = 1.
Expanding by the first row we get $k.\ k^{n-2}(k + \Sigma_{i>1}a_i^2) + \det(m'_{ij})$, where $m'_{ij}$ is the same as $m_{ij}$ except that $m'_{11} = a_1^2$. Subtracting appropriate multiples of the first row from the others we zero all the elements outside the first row except those on the diagonal, which become k. Hence $\det(m'_{ij}) = k^{n-2}a_1^2$.

**Problem B7**
Given n > 8, let $a = \sqrt{n}$ and $b = \sqrt{(n+1)}$. Which is greater $a^b$ or $b^a$?

**Solution**
Answer: $a^b$ is greater.

$a^b = e^{b \ln a}$ and $b^a = e^{a \ln b}$. So we have to decide which of b ln a and a ln b is greater, or, equivalently, which of (ln a)/a and (ln b)/b is greater. The latter is clearly more promising. So set f(x) = (ln x)/x. Then $f'(x) = 1/x^2 - (\ln x)/x^2$ which is negative for x > e. Obviously b > a, so provided a > e, (ln a)/a > (ln b)/b and hence b ln a > a ln b and $a^b > b^a$. But $e^2 < 9$, so the result is certainly true for $n \geq 9$.

# 4th Putnam 1941

**Problem A1**
Prove that $(a - x)^6 - 3a(a - x)^5 + 5/2\ a^2(a - x)^4 - 1/2\ a^4(a - x)^2 < 0$ for $0 < x < a$.

**Solution**
Change variables to $t = 1 - x/a$ and the polynomial becomes $a^6(t^6 - 3t^5 + 5/2\ t^4 - 1/2\ t^2)$. This obviously has a factor $t^2$, and almost obviously $(t - 1)$. Dividing these out, we see that the resulting cubic has another factor $(t - 1)$. So we can write the original as $a^6t^2(t - 1)^2(t(t - 1) - 1/2)$, which evidently has the same sign as $t(t - 1) - 1/2$. But that is clearly negative for t between 0 and 1.

**Problem A2**
Define $f(x) = \int_0^x \sum_{i=0}^{n-1} (x - t)^i / i!\ dt$. Find the nth derivative $f^{(n)}(x)$.

**Solution**
Note that x appears both in the integrand and in the limits, so a little care is needed. Write $g_r(x, t) = \sum_{i=0}^{r-1} (x - t)^i / i!$ so that $f(x) = \int_0^x g_n(x, t) e^{nt}\ dt$. By definition $f'(x) = \lim_{\delta x \to 0} (\int_0^{x+\delta x} g_n(x+\delta x, t) e^{nt}\ dt - \int_0^x g_n(x, t) e^{nt}\ dt) / \delta x = e^{nx} g_n(x, x) + \int_0^x g_n'(x,t) e^{nt}\ dt$, where the ' denotes the partial derivative wrt the first variable. But $g_n(x, x) = 1$, and $g_r'(x, t) = g_{r-1}(x, t)$, so $f'(x) = e^{nx} + \int_0^x g_{n-1}(x, t) e^{nt}\ dt$. Hence by an easy induction $f^{(n)}(x) = e^{nx}( 1 + n + n^2 + ... + n^{n-1} )$.

**Problem A3**
A circle radius a rolls in the plane along the x-axis the envelope of a diameter is the curve C. Show that we can find a point on the circumference of a circle radius a/2, also rolling along the x-axis, which traces out the curve C.

**Solution**
Consider a circle radius a/2 with center initially at (0,a/2) rolling along the x-axis. After rolling through an angle $\theta$, the point initially at (0,a) is at $a/2 \sin \theta$, $a/2 \cos \theta$ relative to the center and hence at P $(a\theta/2 + (a/2) \sin \theta, a/2 + (a/2) \cos \theta)$. The tangent at P is $(y - (a/2)(1 + \cos \theta)/(x - (a/2)(\theta + \sin \theta) = (-\sin \theta)/(1 + \cos \theta)$, or $x \sin \theta + y(1 - \cos \theta) = (a/2)(\theta \sin \theta + 2 \cos \theta + 2)$.

If we put $\theta = 2\varphi$, then $\sin \theta = 2 \sin \varphi \cos \varphi$, $1 + \cos \theta = 2 \cos^2\varphi$, so the tangent at P has equation, $x \sin \varphi + y \cos \varphi = a(\varphi \sin \varphi + \cos \varphi)$.

Now consider the circle radius a with center initially at (0,a). When it has rolled through an angle $\varphi$, its center is at $(a\varphi, a)$, so the diameter which is initially horizontal lies on the line $(y-a)/(x-a\varphi) = -\tan \varphi$, or $x \sin \varphi + y \cos \varphi = a(\varphi \sin \varphi + \cos \varphi)$. In other words, the diameter is tangent to the point P of the curve traced out by the point on the circumference of the circle radius a/2. Hence the envelope of the diameter is that curve.

**Problem A4**
The real polynomial $x^3 + px^2 + qx + r$ has real roots $a \le b \le c$. Prove that f ' has a root in the interval $[b/2 + c/2, b/3 + 2c/3]$. What can we say about f if the root is at one of the endpoints?

**Solution**
$p(x) = (x - a)(x - b)(x - c)$, so $p'(x) = (x - a)(x - b) + (x - b)(x - c) + (x - a)(x - c)$.

We can write $p'(x) = (x - b)(x - c) + (x - a)(2x - b - c)$, so $p'(b/2 + c/2) = -1/4\ (c - b)^2 \le 0$, with equality iff $b = c$.

$p'(b/3 + 2c/3) = -2/9\ (c - b)^2 + (x - a)\ 1/3\ (c - b) = 1/3\ (c - b)\ (b - a) \ge 0$ with equality iff $a = b$ or $b = c$.

If $b = c$, then b is a repeated root and $p'(b) = 0$. If $a = b$, then $p'(b/3 + 2c/3) = 0$. Otherwise, $p'(x)$ is negative at $b/2 + c/2$ and positive at $b/3 + 2c/3$, so it has a zero in the interior of the interval.

**Problem A6**
f is defined for the non-negative reals and takes positive real values. The centroid of the area lying under the curve $y = f(x)$ between $x = 0$ and $x = a$ has x-coordinate $g(a)$. Prove that for some positive constant k, $f(x) = k\ g'(x)/(x - g(x))^2\ e^{\int 1/(t - g(t))\ dt}$.

**Problem A7**
*Do either (1) or (2):*

(1) *Do either (1) or (2):*

(1) Let A be the 3 x 3 matrix

$$\begin{matrix} 1+x^2-y^2-z^2 & 2(xy+z) & 2(zx-y) \\ 2(xy-z) & 1+y^2-z^2-x^2 & 2(yz+x) \\ 2(zx+y) & 2(yz-x) & 1+z^2-x^2-y^2 \end{matrix}$$

Show that det A = $(1 + x^2 + y^2 + z^2)^3$.

(2) A solid is formed by rotating about the x-axis the first quadrant of the ellipse $x^2/a^2 + y^2b^2 = 1$. Prove that this solid can rest in stable equilibrium on its vertex (corresponding to x = a, y = 0 on the ellipse) iff $a/b \le \sqrt{(8/5)}$.

**Solution**
(1) subtract z times row 2 from row 1 and add y times row 3 to row 1. After taking out the common factor $1+x^2+y^2+z^2$ from row 1 we get:

$$\begin{matrix} 1 & z & -y \\ 2(xy-z) & 1+y^2-z^2-x^2 & 2(yz+x) \\ 2(zx+y) & 2(yz-x) & 1+z^2-x^2-y^2 \end{matrix}$$

Subtract z times col 1 from col 2 and add y times col 1 to col 3. We get:

$$\begin{matrix} 1 & 0 & 0 \\ 2(xy-z) & 1+y^2+z^2-x^2-2xyz & 2x(1+y^2) \\ 2(zx+y) & -2x(1+z^2) & 1+z^2-x^2+y^2+2xyz \end{matrix}$$

Multiplying this out, we get $(1-x^2+y^2+z^2)^2 - 4x^2y^2z^2 + 4x^2(1+y^2+z^2+y^2z^2) = (1+x^2+y^2+z^2)^2$. Hence with the additional factor we took out, we get the result.

(2) We first have to find the position of the centre of mass on the axis. The moment about the y-axis of the solid is $\int_0^a \pi y^2 x\,dx = \pi b^2 \int_0^a (x - x^3/a^2)\,dx = \pi b^2a^2/4$. The volume is $\int_0^a \pi y^2\,dx = \pi b^2 \int_0^a (1 - x^2/a^2)\,dx = 2/3\,\pi b^2a$. Hence the centre of mass is a distance 3a/8 from the flat surface or 5a/8 from the point of contact.

Now suppose the point of contact is at (a cos t, b sin t). The tangent has gradient - b/a cot t, so the normal has gradient a/b tan t. So the equation of the normal is y - b sint = a/b tan t (x - a cos t). This meets the x-axis at $a(1 - b^2/a^2)$ cos t. For stability we want this to be closer to the origin than the centre of mass, in other words we want $(1 - b^2/a^2) \cos t < 3/8$. The point of contact is at cos t = 1, so we require $(1 - b^2/a^2) < 3/8$ or $b/a > \sqrt{(5/8)}$.

**Problem B1**
A particle moves in the plane so that its angular velocity about the point (1, 0) equals minus its angular velocity about the point (-1, 0). Show that its trajectory satisfies the differential equation $y' x(x^2 + y^2 - 1) = y(x^2 + y^2 + 1)$. Verify that this has as solutions the rectangular hyperbolae with center at the origin and passing through (±1, 0).

**Solution**
The angular momentum (for unit mass) about (1, 0) is (x - 1) dy/dt - y dx/dt. Hence the angular velocity is $( (x - 1)\,dy/dt - y\,dx/dt )/( (x - 1)^2 + y^2)$. Similarly, the angular velocity about (-1, 0) is $( (x + 1)\,dy/dt - y\,dx/dt )/( (x + 1)^2 + y^2)$. Hence the trajectory satisfies: $( (x - 1) y' - y)(x^2 + y^2 + 1 + 2x) + ( (x + 1) y' - y)(x^2 + y^2 + 1 - 2x) = 0$ or $2x y' (x^2 + y^2 + 1) - 4x y' - 2y(x^2 + y^2 + 1) = 0$, or $x y' (x^2 + y^2 - 1) = y (x^2 + y^2 + 1)$.

We see immediately that $xy = 0$ is a solution and almost immediately that $x^2 - y^2 = 1$ is a solution. Hence also all linear combinations $x^2 + k\,xy - y^2 = 1$. These are rectancular hyperbolae because the sum of the coefficients of $x^2$ and $y^2$ are zero.

**Problem B2**

Find:

(1) $\lim_{n\to\infty} \sum_{1\le i\le n} 1/\sqrt{(n^2 + i^2)}$;

(2) $\lim_{n\to\infty} \sum_{1\le i\le n} 1/\sqrt{(n^2 + i)}$;

(3) $\lim_{n\to\infty} \sum_{1\le i\le n^2} 1/\sqrt{(n^2 + i)}$;

**Solution**

(1) $1/\sqrt{(n^2 + i^2)} = (1/n) / \sqrt{(1 + (i/n)^2)}$. So the sum is just a Riemann sum for the integral $\int_0^1 dx/\sqrt{(1 + x^2)} = \sinh^{-1}1 = \ln(1 + \sqrt{2}) = 0.8814$.

(2) $1/\sqrt{(n^2 + i)} = (1/n) / \sqrt{(1 + i/n^2)}$. Each term is less than $1/n$, so the (finite) sum is less than 1. But each term is at least $(1/n) / \sqrt{(1 + 1/n)}$. So the sum is at least $1/\sqrt{(1 + 1/n)}$, which tends to 1. Hence the limit of the sum is 1.

(3) $1/\sqrt{(n^2 + i)} = n\,(1/n^2) / \sqrt{(1 + i/n^2)}$. Now $\sum (1/n^2) / \sqrt{(1 + i/n^2)}$ is just a Riemann sum for $\int_0^1 dx/\sqrt{(1 + x)} = 2\sqrt{(1 + x)}\,|_0^1 = 2(\sqrt{2} - 1)$. So the sum given tends to $2(\sqrt{2} - 1)\,n$, which diverges to infinity. [Or simpler, there are $n^2$ terms, each at least $1/\sqrt{(2n^2)} = 1/(n\sqrt{2})$, so the sum is at least $n/\sqrt{2}$ which diverges.]

**Problem B3**

Let y and z be any two linearly independent solutions of the differential equation $y'' + p(x)\,y' + q(x)\,y = 0$. Let $w = y\,z$. Find the differential equation satisfied by w.

**Solution**

We have $w = yz$, $w' = y'z + yz'$, $w'' = y''z + 2y'z' + yz''$. Hence $w'' + pw' + 2qw = 2y'z'$ (1). Now $(y'z')' = y''z' + y'z''$, $2py'z' = py'z' + py'z'$, $qw' = qyz' + qy'z$, so $(y'z')' + 2py'z' + qw' = 0$ (2).

Differentiating (1) we get: $w''' + p'w' + pw'' + 2q'w + 2qw' = 2(y'z')' = -4py'z' - 2qw'$ (using (2) ), $= -2p(w'' + pw' + 2qw) - 2qw'$. Rearranging, this gives: $w''' + 3p\,w'' + (p' + 2p^2 + 4q)\,w' + (4pq + 2q')w = 0$.

**Problem B4**

Given an ellipse center O, take two perpendicular diameters AOB and COD. Take the diameter A'OB' parallel to the tangents to the ellipse at A and B (this is said to be *conjugate* to the diameter AOB). Similarly, take C'OD' conjugate to COD. Prove that the rectangular hyperbola through A'B'C'D' passes through the foci of the ellipse.

**Solution**

Take the ellipse as $x^2/a^2 + y^2/b^2 = 1$ and the point A as $(a \cos t, b \sin t)$. Then AB has slope $b/a \tan t$, so CD has slope $-a/b \cot t$. The tangent at A has slope $-b/a \cot t$. Suppose C is $(a \cos u, b \sin u)$, then $b/a \tan u = -a/b \cot t$ and the tangent at C has slope $-b/a \cot u = b^3/a^3 \tan t$. Hence the line pair A'B', C'D' has equation $(y + x\, b/a \cot t)(y - x\, b^3/a^3 \tan t)$. Now we have the equations for two distinct conics through A'B'C'D': the original ellipse and the line pair A'B',C'D'. The equation of any other conic through these four points must be a linear combination of the equations of these two, in other words, $(x^2/a^2 + y^2/b^2 - 1) + k(y + x\, b/a \cot t)(y - x\, b^3/a^3 \tan t) = 0$ for some k.

The criterion for a rectangular hyperbola is that the coefficients of $x^2$ and $y^2$ should have sum zero, or that $1/a^2 + 1/b^2 + k - k\, b^4/a^4 = 0$. Hence $k = a^2/(b^4 - b^2a^2)$ and the equation of the rectangular hyperbola is $x^2 - y^2 + (b/a \tan t - a/b \cot t)\, xy = a^2 - b^2$. But the foci are at $(\pm(a^2 - b^2)^{1/2}, 0)$, so they lie on the rectangular hyperbola.

**Problem B5**

A wheel radius r is traveling along a road without slipping with angular velocity $\omega > \sqrt{(g/r)}$. A particle is thrown off the rim of the wheel. Show that it can reach a maximum height above the road of $(r\omega + g/\omega)^2/(2g)$. [Ignore air resistance.]

**Solution**

Suppose the pebble leaves the wheel from a point on the rim which is at an angle θ to the vertical. Its point of departure is a distance $r + r \cos \theta$ above the road. Its upward velocity is $r\,\omega \sin \theta$, so it ascends a further $(r\,\omega \sin \theta)^2/2g$. Thus the total height is $r + r^2\omega^2/2g + r \cos \theta - r^2\omega^2/2g \cos^2\theta = r + g/2\omega^2 + r^2\omega^2/2g - r^2\omega^2/2g\,(\cos \theta - g/r\omega^2)$

(*). We are given that $g/r\omega^2 < 1$, so (*) has a maximum when $\cos\theta = g/r\omega^2$ and the maximum value is $1/2g\ (r^2\omega^2 + 2gr + g^2/\omega^2) = (r\omega + g/\omega)^2/2g$.

**Problem B6**

f is a real valued function on [0, 1], continuous on (0, 1). Prove that $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x) f(y) f(z)\, dz\, dy\, dx = 1/6 \left( \int_{x=0}^{x=1} f(x)\, dx \right)^3$.

**Solution**

Let $S_{xyz}$ be the points in the cube for which $x \le y \le z$, let $S_{yxz}$ be the points for which $y \le x \le z$ and so on. Then the union of the six sets is the cube and the intersection of any two has measure zero. Also by changing the variables of integration we see that the integral of f(x) f(y) f(z) over each set is the same. Hence the integral over $S_{xzy}$ is 1/6 of the integral over the cube. But the integral over $S_{xzy}$ is just $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x) f(y) f(z)\, dz\, dy\, dx$ and the integral over the cube is $\left( \int_{x=0}^{x=1} f(x)\, dx \right)^3$.

**Problem B7**

*Do either (1) or (2):*

(1) f is a real-valued function defined on the reals with a continuous second derivative and satisfies $f(x + y) f(x - y) = f(x)^2 + f(y)^2 - 1$ for all x, y. Show that for some constant k we have $f''(x) = \pm k^2 f(x)$. Deduce that f(x) is one of $\pm\cos kx$, $\pm\cosh kx$.

(2) $a_i$ and $b_i$ are constants. Let A be the (n+1) x (n+1) matrix $A_{ij}$, defined as follows: $A_{i1} = 1$; $A_{1j} = x^{j-1}$ for $j \le n$; $A_{1\,(n+1)} = p(x)$; $A_{ij} = a_{i-1}^{j-1}$ for $i > 1, j \le n$; $A_{i\,(n+1)} = b_{i-1}$ for $i > 1$. We use the identity det A = 0 to define the polynomial p(x). Now given any polynomial f(x), replace $b_i$ by $f(b_i)$ and p(x) by q(x), so that det A = 0 now defines a polynomial q(x). Prove that f( p(x) ) is a multiple of &prodc; $(x - a_i)$ plus q(x).

**Solution**

(1) Putting y = 0 gives $f(0)^2 = 1$, so $f(0) = \pm 1$. Differentiating wrt y gives $f'(x+y) f(x-y) - f(x+y) f'(x-y) = 2 f(y) f'(y)$. Putting y = 0 gives $f'(0) = 0$. Differentiating wrt x gives $f''(x+y) f(x-y) = f(x+y) f''(x-y)$. Putting $x = y = z/2$ gives $f''(z) = h f(z)$, where $h = \pm f''(0)$. If h is positive, we may put $h = k^2$ and integrate, using $f(0) = \pm 1$, $f'(0) = 0$ to get $f(x) = \pm \cos kx$. If h is negative, we may put $h = - k^2$ and integrate to get $f(x) = \pm \cosh kx$.

(2)

# 5th Putnam 1942

**Problem A1**
ABCD is a square side 2a with vertices in that order. It rotates in the first quadrant with A remaining on the positive x-axis and B on the positive y-axis. Find the locus of its center.

**Solution**
Answer: the segment (a, a) to ($a\sqrt{2}$, $a\sqrt{2}$).

Let AB make an angle $\theta$ with the x-axis. Then we find that the coordinates of the center to be $x = y = a \cos \theta + a \sin \theta$. But $a \cos \theta + a \sin \theta = a \sqrt{2} \sin(\theta + \pi/4)$.

**Problem A2**
a and b are unequal reals. What is the remainder when the polynomial p(x) is divided $(x - a)^2(x - b)$.

**Solution**
Suppose the remainder is $cx^2 + dx + e$. We have $p(a) = ca^2 + da + e$, $p(b) = cb^2 + db + e$. Also, differentiating, we get $p'(a) = 2ca + d$. Solving, $c = p'(a)/(a - b) - p(a)/(a - b)^2 + p(b)/(a - b)^2$, $d = (2a\, p(a) - 2a\, p(b) - (a^2 - b^2)p'(a) )/(a - b)^2$, $e = p(a) - a^2(p(a) - p(b))/(a - b)^2 + ab\, p'(a) /(a - b)$.

**Problem A3**
Does $\sum_{n\ge 0} n!\, k^n/(n + 1)^n$ converge or diverge for k = 19/7?

**Solution**
The nth term divided by the n-1th term is $k\, n\, n^{n-1}/(n+1)^n = k/(1 + 1/n)^n$ which tends to k/e. But k/e < 1, so the series converges by the ratio test.

**Problem A4**
Let C be the family of conics $(2y + x)^2 = a(y + x)$. Find C', the family of conics which are orthogonal to C. At what angle do the curves of the two families meet at the origin?

**Solution**
For most points P in the plane we can find a unique conic in the family passing through the point. Thus we should be able to find the gradient of members of the family at (x, y) in a formula which is independent of a. We then use this to get a formula for the gradient of the orthogonal family and solve the resulting first-order differential equation to get the orthogonal family.

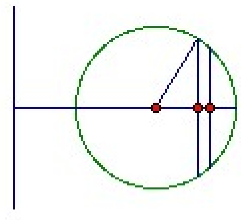
Thus we have $8y\, y' + 4x\, y' + 4y + 2x = ay' + a = (y' + 1)(2y + x)^2/(y + x)$. So $y'(2y+x)( 4(x+y) - (2y+x) ) = (2y+x)^2 - (x+2y)(x+y)$, or $y'(2y+x)(2y+3x) = -2x(2y+x)$, so $y' = -x/(2y+3x)$. Hence the orthogonal family satisfies $y' = (2y+3x)/x$. So $y'/x^2 - 2y/x^3 = 3/x^2$. Integrating $y = bx^2 - 3x$. These are all parabolas.

All members of both families pass through the origin. Changing coordinates to $X = x + 2y$, $Y = y - 2x$, the equation of a member of the first family becomes $X^2 = a(3X-Y)/5$ or $Y = - 5/a\, (X - 3a/10)^2 + 9a/20$. This has gradient 3 (in the new system) at the origin. In the old system the tangent is y = -x. The orthogonal set obviously has gradient -3 at the origin. If the angle between them is k, then $\tan k = (-1 +3)/(1+3) = 1/2$. So $k = \tan^{-1} 1/2$.

**Problem A5**
C is a circle radius a whose center lies a distance b from the coplanar line L. C is rotated through $\pi$ about L to form a solid whose center of gravity lies on its surface. Find b/a.

**Answer**
$(\pi + \sqrt{(\pi^2+2\pi-4)})/(2\pi-4)$ = about 2.9028

**Solution**
The solid is half a torus. We can divide it into a large number of thin disks. Each disk has variable thickness, with thickness proportional to the distance from L. So we must integrate to find the distance of the centroid of the disk from L. Take the density to be kd, where d is the distance from L.



Take x to be distance along the line perpendicular to x, and θ to be the angle between the radius vector and the x-axis. We have $x = a\cos\theta$, so $dx = -a\sin\theta\, d\theta$. The mass is $\int_0^\pi 2a \sin\theta\,(a\sin\theta\, d\theta)\, k(b + a\cos\theta) = 2a^2bk \int_0^\pi \sin^2\theta\, d\theta + 2a^3k \int_0^\pi \sin^2\theta\cos\theta\, d\theta = a^2bk\pi + 0$. So the mass times the centroid distance is $\int_0^\pi 2a^2k\sin^2\theta\,(a\cos\theta + b)^2\, d\theta = 2a^4k \int_0^\pi \sin^2\theta \cos^2\theta\, d\theta + 4a^3bk \int_0^\pi \sin^2\theta\cos\theta\, d\theta + 2a^2b^2k \int_0^\pi \sin^2\theta\, d\theta = \frac{1}{2}a^4k \int_0^\pi \sin^2 2\theta\, d\theta + 0 + a^2b^2k\pi = ka^2\pi(a^2/4 + b^2)$. So the centroid distance is $b + a^2/4b$. Thus we can regard the mass as uniformly spread over a semicircle radius $b + a^2/4b$.

We need another integration to find the distance of the mass of a semicircle radius r from its center. It is $(1/\pi r) \int_0^\pi r^2 \sin\theta\, d\theta = 2r/\pi$. Thus the cm of the half-torus is a distance $(2/\pi)(b + a^2/4b)$ from L. We want it to be a distance b-a from L so that it lies on the surface. Thus $(2/\pi)(b + a^2/4b) = b - a$, so $(2\pi-4)b^2 - 2\pi ab - a^2 = 0$. Hence $b/a = (\pi + \sqrt{(\pi^2+2\pi-4)})/(2\pi-4)$ = about 2.9028.

**Problem A6**
P is a plane and H is the half-space on one side of P. K is a fixed circle in P. C is a circle in P which cuts K at an angle α. Let C have center O and radius r. f(C) is the point in H on the normal to P through O and a distance r from O. Show that the locus of f(C) is a one-sheet hyperboloid and that it has two families of rulings in it.

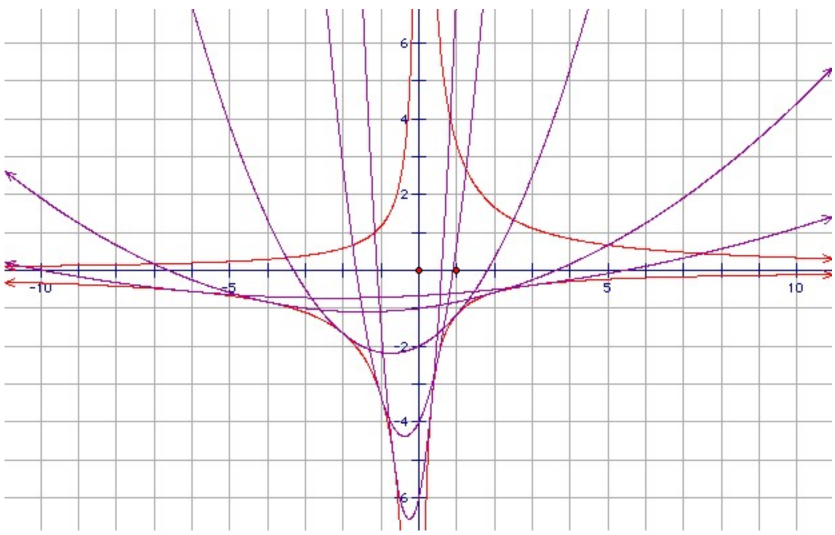
**Problem B1**
S is a solid square side 2a. It lies in the quadrant $x \ge 0$, $y \ge 0$, and it is free to move around provided a vertex remains on the x-axis and an adjacent vertex on the y-axis. P is a point of S. Show that the locus of P is part of a conic. For what P does the locus degenerate?

**Solution**
Let A be the vertex that moves along the x-axis and B the vertex that moves along the y-axis. Suppose that when AB is horizontal P has coordinates b, c. In the general configuration let be the angle BAO be θ. Then P has coordinates $x = (2a - b)\cos\theta + c\sin\theta$, $y = b\sin\theta + c\cos\theta$. Hence $cx - (2a - b)y = (b^2 + c^2 - 2ab)\sin\theta$, $bx - cy = (2ab - b^2 - c^2)\cos\theta$. Squaring and adding we eliminate θ to get: $(b^2 + c^2) x^2 - 4ac\, xy + (4a^2 + b^2 + c^2 - 4ab) y^2 = (b^2 + c^2 - 2ab)^2$, which is the equation of a conic. So the locus of P must form part of this conic.

The conic degenerates if $b^2 + c^2 = 2ab$. In this case, the equation becomes $2ab\, x^2 - 4ac\, xy + (4a^2 - 2ab)y^2 = 0$, or $bx^2 - 2c\, xy + (2a - b)y^2 = 0$, or $b^2x^2 - 2bc\, xy + c^2 y^2 = 0$, or $bx = cy$. So in this case the locus lies on a straight line. We may write the condition $b^2 + c^2 = 2ab$ as $(a - b)^2 + c^2 = a^2$, which shows that such P lie on the semicircle diameter AB.

**Problem B2**
Let $P_a$ be the parabola $y = a^3x^2/3 + a^2x/2 - 2a$. Find the locus of the vertices of $P_a$, and the envelope of $P_a$. Sketch the envelope and two $P_a$.

**Solution**
We can write the equation of $P_a$ as $(y + 35a/16) = (a^3/3)(x + 3/4a)^2$, so the vertex is $x = -3/4a$, $y = -35a/16$. The locus of the vertex is $xy = 105/64$.



The graph shows $P_3$, $P_2$, $P_1$, $P_{1/2}$, $P_{1/3}$ and the two hyperbolae yx = -7/6, yx = 10/3. It shows that for positive a, the parabolas touch the lower branches of the hyperbolae. For negative a they touch the upper branches.

That is not hard to verify. We claim that $P_a$ and xy = 10/3 touch at x = -2/a, y = -5a/3. The point obviously lies on xy = 10/3. We have $(a^3/3)(x + 3/4a)^2 = (1/3)(-2 + 3/4)^2$ a = 25a/48 = (-5a/3 + 35a/16), so the point also lies on $P_a$. The gradient of xy = 10/3 at the point is $-10/(3x^2) = -5a^2/6$. The gradient of $P_a$ at the point is $2a^3x/3 + a^2/2 = -5a^2/6$.

Similarly, we claim that $P_a$ and xy = -7/6 touch at x = 1/a, y = -7a/6. The point obviously lies on xy = -7/6. We have $(a^3/3)(x + 3/4a)^2 = (a/3)(1 + 3/4)^2$ = 49a/48 = (-7a/6 + 35a/16), so the point also lies on $P_a$. The gradient of xy = -7/6 at the point is $7/(6x^2) = 7a^2/6$. The gradient of $P_a$ at the point is $2a^3x/3 + a^2/2 = a^2(2/3 + 1/2) = 7a^2/6$.

It is less clear how you get the hyperbolas. One standard approach is to look for the singular points of the mapping $f(a,t) = (t, a^3t^2/3 + a^2t/2 - 2a)$. The matrix for the derivative is:

$$\begin{matrix} 1 & 0 \\ 2a^3t/3+a^2/2 & a^2t^2+at-2 \end{matrix}$$

which has zero determinant when at = 1 or -2, so xy = 7/6 or -10/3.

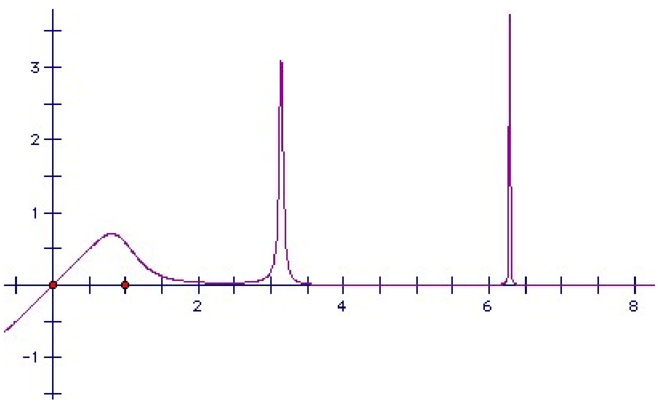
**Problem B3**
f(x, y) and g(x, y) satisfy the differential equation $f_1(x, y)\, g_2(x, y) - f_2(x, y)\, g_1(x, y) = 1$ (*). Taking r = f(x, y) and y as independent variables, and x = h(r, y), g(x, y) = k(r, y), show that $k_2(r, y) = h_1(r, y)$. Integrate and hence obtain a solution to (*). What other solutions does (*) have?

**Problem B4**
A particle moves in a circle through the origin under the influence of a force $a/r^k$ towards the origin (where r is its distance from the origin). Find k.

**Solution**
The equations of motion are $r(\theta')^2 - r'' = a/r^k$, $r^2\theta' = A$ (conservation of angular momentum).

If the particle moves in a circle as described, then we can write its orbit as r = B cos θ. Differentiating, $r' = -B\theta' \sin\theta = -AB/r^2 \sin\theta$. Differentiating again, $r'' = -AB\cos\theta\, A/r^2 + 2AB/r^3 \sin\theta\, r' = -A^2/r^3 - 2A^2B^2/r^5 \sin^2\theta = -A^2/r^3 - 2A^2B^2/r^5 (1 - r^2B^2) = A^2/r^3 - 2A^2B^2/r^5$. So substituting back in the equation of motion we get: $2A^2B^2/r^5 = a/r^k$. Hence k = 5.
Note that this is unphysical, since we require infinite velocity as we reach the origin.

**Problem B5**
Let $f(x) = x/(1 + x^6\sin^2x)$. Sketch the curve y = f(x) and show that $\int_0^\infty f(x)\,dx$ exists.




**Solution**
Obviously f(x) is positive for positive x. But it has an infinite number of spikes at x = nπ. The spike at nπ is height nπ, so we have to show that the integral is bounded above.

We have sin x > ½x near x = 0 (certainly for x < π/3). So $|\sin x| > 1/(n\pi)^k$ except possibly for $|x| < 2/(n\pi)^k$. Let $I_n$ be the interval centered on nπ width $4/(n\pi)^k$. For $x \in I_n$ we have f(x) < 2nπ, so the integral of f(x) over the interval is less than $8/(n\pi)^{k-1}$. The total integral over all such intervals is bounded provided that k > 2. Outside such intervals, $x^6 \sin^2x > ½\, x^6/x^{2k}$, so $f(x) < 2/x^{2k-5}$. Hence the interval of f(x) over 0 to ∞ excluding the intervals $I_n$ is bounded provided k > 5/2. By taking k = 2 1/4, for example, we get that the whole integral is bounded.

# 6th Putnam 1946

**Problem A1**
p(x) is a real polynomial of degree less than 3 and satisfies $|p(x)| \le 1$ for $x \in [-1, 1]$. Show that $|p'(x)| \le 4$ for $x \in [-1, 1]$.

**Solution**
Let $p(x) = ax^2 + bx + c$, so $p'(x) = 2ax + b$. It is evidently sufficient to show that $|2a + b|$ and $|2a - b| \le 4$. $p(0) = c$, $p(1) = a + b + c$, $p(-1) = a - b + c$, so $2a + b = 3/2\ p(1) + 1/2\ p(-1) + 2\ p(0)$. But $|p(1)|, |p(-1)|, |p(0)| \le 1$, so $|2a + b| \le 4$. Similarly, $2a - b = 1/2\ p(1) + 3/2\ p(-1) + 2\ p(0)$.

**Problem A2**
R is the reals. For functions $f, g : R \to R$ and $x \in R$ define $I(fg) = \int_1^x f(t) g(t)\, dt$. If a(x), b(x), c(x), d(x) are real polynomials, show that I(ac) I(bd) - I(ad) I(bc) is divisible by $(x - 1)^4$.

**Solution**
Let F(x) = I(ac) I(bd) - I(ad) I(bc). Then clearly F(1) = 0 (since all the integrals are over an empty range). Differentiating, we get, F' = acI(bd) + bdI(ac) - adI(bc) - bcI(ad). So F'(1) = 0. Differentiating again: F" = a'cI(bd) + ac'I(bd) + abcd + b'dI(ac) + bd'I(ac) + abcd - a'dI(bc) - ad'I(bc) - abcd - b'cI(ad) - bc'I(ad) - abcd = a'cI(bd) + ac'I(bd) + b'dI(ac) + bd'I(ac) - a'dI(bc) - ad'I(bc) - b'cI(ad) - bc'I(ad). So F"(1) = 1.

Differentiating again: F"' = a"cI(bd) + 2a'c'I(bd) + ac"I(bd) + a'bcd + abc'd + b"dI(ac) + 2b'd'I(ac) + bd"I(ac) + ab'cd + abcd' - a"dI(bc) - 2a'd'I(bc) - ad"I(bc) - a'bcd - abcd' - b"cI(ad) - 2b'c'I(ad) - bc"I(ad) - ab'cd - abc'd = a"cI(bd) + 2a'c'I(bd) + ac"I(bd) + b"dI(ac) + 2b'd'I(ac) + bd"I(ac) - a"dI(bc) - 2a'd'I(bc) - ad"I(bc) - b"cI(ad) - 2b'c'I(ad) - bc"I(ad). So F"'(1) = 1.

That is sufficient to prove the result. But notice that if we differentiate again, then just collecting the terms that do not involve I(fg) we get (after some cancellation) 2a'bc'd + 2ab'cd' - 2a'bcd' - 2ab'c'd, which is not, in general, zero for x = 1. So in general we do not have F(x) divisible by $(x - 1)^5$.

**Problem A3**
ABCD are the vertices of a square with A opposite C and side AB = s. The distances of a point P in space from A, B, C, D are a, b, c, d respectively. Show that $a^2 + c^2 = b^2 + d^2$, and that the perpendicular distance k of P from the plane ABCD is given by $8k^2 = 2(a^2 + b^2 + c^2 + d^2) - 4s^2 - (a^4 + b^4 + c^4 + d^4 - 2a^2c^2 - 2b^2d^2)/s^2$.

**Solution**
Let Q be the point of the plane ABCD closest to P. Let O be the center of the square ABCD. Let QO make an angle θ with AC. Then using the cosine rule we have: $AQ^2 = AO^2 + OQ^2 - 2AO\cdot OQ \cos\theta$, $CQ^2 = CO^2 + OQ^2 + 2CO.OQ \cos\theta$ (*). Adding: $AQ^2 + CQ^2 = 2AO^2 + 2QO^2$. But $a^2 = AQ^2 + k^2$ etc, so $a^2 + c^2 = 2k^2 + s^2 + 2QO^2$. Similarly, $b^2 + d^2 = 2k^2 + s^2 + 2QO^2$. We have established that $a^2 + c^2 = b^2 + d^2$.

We have also established that $8k^2 = 2(a^2 + b^2 + c^2 + d^2) - 4s^2 - 8QO^2$ (**). The angle φ between QO and BD is π/2 - θ. So cos φ = sin θ. Hence, going back to (*), $AQ^2 - CQ^2 = -4\ AO\cdot OQ \cos\theta$, and $BQ^2 - DQ^2 = \pm 4\ AO\cdot OQ \sin\theta$. But $AQ^2 - CQ^2 = a^2 - c^2$ etc. So $(a^2 - c^2)^2 + (b^2 - d^2)^2 = 16\ AO^2OQ^2 = 8s^2OQ^2$. Substituting in (**) gives the required result.

**Problem A4**
R is the reals. $f : R \to R$ has a continuous derivative, f(0) = 0, and |f '(x)| <= |f(x)| for all x. Show that f is constant.

**Solution**
Suppose f is not constant. Then take an interval [a, b] of length < 1/2 such that f(a) = 0, f(b) ≠ 0, and $|f(b)| \ge |f(x)|$ for $x \in [a, b]$. Now applying the mean value theorem to the interval gives an immediate contradiction.

**Problem A5**
Let T be a tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. What is the smallest possible volume for the tetrahedral volume bounded by T and the planes x = 0, y = 0, z = 0?

**Solution**
Answer: $\sqrt{3}\ abc/2$.

The normal at $(x_0, y_0, z_0)$ is $(xx_0/a^2, yy_0/b^2, zz_0/c^2)$. So the tangent plane is $x\, x_0/a^2 + y\, y_0/b^2 + z\, z_0/c^2 = 1$. This cuts the three axes at $x = a^2/x_0$, $y = b^2/y_0$, $z = c^2/z_0$. We can regard two of these lengths as defining the base of the tetrahedron, and the third as forming its height. Hence its volume is $a^2b^2c^2/(6x_0y_0z_0)$.

We wish to maximize $x_0y_0z_0$. That is equivalent to maximising $x_0^2/a^2\ y_0^2/b^2\ z_0^2/c^2$. But we know that the sum of these three numbers is 1, so their maximum product is 1/27 (achieved when they are all equal - the arithmetic/geometric mean result). Hence $x_0y_0z_0$ has maximum value $abc/(3\sqrt{3})$.

**Problem A6**
A particle moves in one dimension. Its distance x from the origin at time t is $at + bt^2 + ct^3$. Find an expression for the particle's acceleration in terms of a, b, c and its speed v.

**Solution**
Differentiating, $v = 3ct^2 + 2bt + a$ (*), and hence the acceleration $f = 6ct + 2b$. So $t = (f - 2b)/6c$. Substituting in (*) gives $v = (f - 2b)^2/12c + b(f - 2b)/3c + a$. Hence $12cv = f^2 - 4b^2 + 12ac$. So $f = 2\sqrt{(b^2 + 3cv - 3ac)}$.

**Problem B1**
Two circles $C_1$ and $C_2$ intersect at A and B. $C_1$ has radius 1. L denotes the arc AB of $C_2$ which lies inside $C_1$. L divides $C_1$ into two parts of equal area. Show L has length > 2.

**Solution**
Let $O_1$ be the center of $C_1$, and $O_2$ the center of $C_2$. Let the line $O_1O_2$ meet the arc AB of $C_2$ at P. If P lies between $O_1$ and $O_2$, then the tangent to $C_2$ at P divides $C_1$ into two unequal parts and the area $C_1 \cap C_2$ lies inside the smaller part. Contradiction. So $O_1$ must lie between P and $O_2$. But now the arc AP is greater than the segment AP, which is greater than $AO_1 = 1$. Hence L > 2.

**Problem B2**
$P_0$ is the parabola $y^2 = mx$, vertex K (0, 0). If A and B points on $P_0$ whose tangents are at right angles, let C be the centroid of the triangle ABK. Show that the locus of C is a parabola $P_1$. Repeat the process to define $P_n$. Find the equation of $P_n$.

**Solution**
The gradient at the point $(x_1 , y_1)$ is $1/2\ m/y_1$. So the tangents at $(x_1 , y_1)$, $(x_2 , y_2)$ are perpendicular iff $y_1y_2 = -m^2/4$. So we may take one point as $(x, y) = (t^2/m, t)$ and the other as $(1/16\ m^3/t^2, -1/4\ m^2/t)$. Hence the centroid is $(x, y)$, where $x = 1/3\ (t^2/m + 1/16\ m^3/t^2)$, $y = 1/3\ (t - 1/4\ m^2/t)$. But this lies on the parabola $y^2 = m/3\ (x - m/6)$. But since any value of t was possible and hence any value of y, any point of this parabola is a centroid from some pair of points A, B.
Repeating, we get that the equation for $P_n$ is $y^2 = m/3^n(x - m/6(1 + 1/3 + ... + 1/3^{n-1}) ) = m/3^n(x - m/4\ (1 - 1/3^n) )$.

**Problem B3**
The density of a solid sphere depends solely on the radial distance. The gravitational force at any point inside the sphere, a distance r from the center, is $kr^2$ (where k is a constant). Find the density (in terms of G, k and r), and the gravitational force at a point outside the sphere. [You may assume the usual results about the gravitational attraction of a spherical shell.]

**Solution**
A shell exerts no net attraction on a point inside and acts on a point outside as if all its mass was concentrated at its center. So let the density at a radial distance r be $\rho(r)$. Then $G/r^2 \int_0^r 4\pi t^2\rho(t)\, dt = kr^2$. Hence $\int_0^r t^2\rho(t)\, dt = k/(4G\pi)\ r^4$. Differentiating: $r^2\rho(r) = k/(G\pi)\ r^3$. So $\rho(r) = k/(G\pi)\ r$.

The force at the surface of the sphere (R) is $kR^2$. Hence the force at a distance r > R from the center of the sphere is $kR^4/r^2$. [We know the force is a simple inverse square law outside the sphere and it must be continuous at the surface.]

**Problem B4**

Define $a_n = 2(1 + 1/n)^{2n+1}/( (1 + 1/n)^n + (1 + 1/n)^{n+1})$. Prove that $a_n$ is strictly monotonic increasing.

**Solution**

$a_n = 2 (n+1)^{n+1}/(n^n (2n+1) )$. Let $f(n) = \ln a_n/2 = (n + 1) \ln(n + 1) - n \ln n - \ln(2n+1)$. Regard f as a function of a real variable and differentiate. $f'(x) = \ln(x + 1) - \ln x - 2/(2x + 1)$. Differentiate again: $f''(x) = 1/(x + 1) - 1/x + 4/(2x + 1)^2 = -1/( x(x+1)(2x+1)^2)$.

So $f''(x) < 0$ for all $x > 0$. Hence $f'(x)$ is decreasing. But we can write $f'(x) = \ln(1 + 1/x) - 2/(2x + 1)$ which tends to 0 as x tends to infinity. So $f'(x) > 0$ for all $x > 0$. Hence $f(x)$ is strictly increasing for all positive x. Hence $f(n+1) > f(n)$ and so $\exp f(n+1) > \exp f(n)$, which is the result we want.

**Problem B5**

Let m be the smallest integer greater than $(\sqrt{3} + 1)^{2n}$. Show that m is divisible by $2^{n+1}$.

**Solution**

This is a fairly well-known problem. $(\sqrt{3} - 1) < 1$ and (we will show) $(\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$ (*) is an integer. Hence it must be the required integer.

But it is easy to check that (*) is the solution of a recurrence relation, in fact the relation $u_n = 8u_{n-1} - 4u_{n-2}$, $u_0 = 2$, $u_1 = 8$. That establishes that (*) is an integer, and a trivial induction shows that (*) is divisible by $2^{n+1}$.

**Problem B6**

The particle P moves in the plane. At $t = 0$ it starts from the point A with velocity zero. It is next at rest at $t = T$, when its position is the point B. Its path from A to B is the arc of a circle center O. Prove that its acceleration at each point in the time interval $[0, T]$ is non-zero, and that at some point in the interval its acceleration is directly towards the center O.

**Solution**

This is almost trivial. Take polar coordinates with origin O. Let the radius of the circular arc be k. The particle's radial acceleration is $k (d\theta/dt)^2$ towards O. Its tangential acceleration is $k\, d^2\theta/dt^2$. Since the particle is not at rest between A and B, $(d\theta/dt)$ is non-zero, so the radial acceleration is non-zero (and hence its total acceleration).

$(d\theta/dt)$ is zero at A and at B, so its derivative (and hence the tangential acceleration) must be zero at some point between.

# 7th Putnam 1947

**Problem A1**
The sequence $a_n$ of real numbers satisfies $a_{n+1} = 1/(2 - a_n)$. Show that $\lim_{n\to\infty} a_n = 1$.

**Solution**
This is slightly messy. First, since $k = 1/(2 - k)$ implies $k = 1$, it is obvious that if the sequence tends to a limit, then the limit is 1.

Next, if $0 < a_n < 1$, then $1 < 2 - a_n < 2$, so $1/2 < a_{n+1} < 1$. So once the sequence gets into the interval (0, 1) it stays there. But $a_{n+1} - a_n = (a_n - 1)^2/(2 - a_n) > 0$ for $a_n < 2$ (*). So once the sequence gets into the interval (0, 1) it is monotonic increasing and bounded above by 1, and hence tends to a limit (which must be 1).

If $a_n < 0$, then $0 < a_{n+1} < 1$, so we are also home if a member of the sequence is negative. Similarly, $a_n > 2$ implies $a_{n+1} < 0$. If $a_n = 1$, then all following terms are 1 and so the limit is 1. So the only issue is if $1 < a_1 < 2$.

But then (*) shows that whilst the sequence remains in the interval (1, 2) it is monotonic increasing. It cannot tend to a limit, because that limit would have to be 1, which is impossible. So it cannot stay in the interval. We cannot have $a_m = 2$, because then we would not have $a_{m+1} = 1/(2 - a_m)$, so we must have $a_m > 2$ for some m and then we are home.

**Problem A2**
R is the reals. $f : R \to R$ is continuous and satisfies $f(r) = f(x) f(y)$ for all x, y, where $r = \sqrt{(x^2 + y^2)}$. Show that $f(x) = f(1)$ to the power of $x^2$.

**Solution**
Induction on n shows that $f(x \sqrt{n}) = f(x)^n$, and hence $f(n x) = f(x)$ to the power of $n^2$. In particular, taking $x = 1/n$, $f(1) = f(1/n)$ to the power of $n^2$. Hence, provided f(1) is non-zero, $f(1/n) = f(1)$ to the power of $1/n^2$. Hence $f(m/n) = f(1)$ to the power of $(m/n)^2$. So we have established that $f(x) = f(1)$ to the power of $x^2$ for all rational x. But f is continuous, so the relation holds for all x.
If $f(1) = 0$, then the same reasoning establishes that $f(x) = 0$ for all x.

**Problem A3**
ABC is a triangle and P an interior point. Show that we cannot find a piecewise linear path $K = K_1K_2 \dots K_n$ (where each $K_iK_{i+1}$ is a straight line segment) such that: (1) none of the $K_i$ do not lie on any of the lines AB, BC, CA, AP, BP, CP; (2) none of the points A, B, C, P lie on K; (3) K crosses each of AB, BC, CA, AP, BP, CP just once; (4) K does not cross itself.

**Solution**
Each time K crosses the boundary of a triangle it moves from the outside to the inside or vice versa. K has two endpoints, so we can find one of the three triangles ABP, BCP, CAP in which it does not start or finish. But that is impossible - on the first crossing it must go from outside to inside, on the second from inside to outside and on the third from outide to inside.

**Problem A4**
Take the x-axis as horizontal and the y-axis as vertical. A gun at the origin can fire at any angle into the first quadrant ($x, y \ge 0$) with a fixed muzzle velocity v. Assuming the only force on the pellet after firing is gravity (acceleration g), which points in the first quadrant can the gun hit?

**Solution**
Let the angle of the gun to the x-axis be $\theta$. Then the equations of motion are: $x = t v \cos \theta$, $y = t v \sin \theta - 1/2\ g t^2$. So the pellet moves along the parabola $y = x \tan \theta - x^2\ g/2v^2 \sec^2\theta$ (*).

We can view (*) as an equation for $\theta$ given x, y. Put $k = \tan \theta$, then the equation becomes $g/2v^2\ x^2\ k^2 - x\ k + y + gx^2/2v^2$. This has real roots iff $x^2 \ge 2g/v^2\ x^2y - g^2/v^4\ x^4$ and hence iff $y \le 2g^2/v^2 - x^2/v^4\ x^2$ (**). Since $x \ge 0$, we see directly from the quadratic that the sum and the product of the roots are both non-negative, so (**) is the condition for the equation to have at least one non-negative root in k and hence at least one root for $\theta$ in the range 0 to $\pi/2$. Thus the gun can hit points in the first quadrant under (or on) the parabola given by (**).

**Problem A5**
The sequences $a_n$, $b_n$, $c_n$ of positive reals satisfy: (1) $a_1 + b_1 + c_1 = 1$; (2) $a_{n+1} = a_n^2 + 2b_nc_n$, $b_{n+1} = b_n^2 + 2c_na_n$, $c_{n+1} = c_n^2 + 2a_nb_n$. Show that each of the sequences converges and find their limits.

**Solution**
$a_{n+1} + b_{n+1} + c_{n+1} = (a_n + b_n + c_n)^2$, so by a trivial induction $a_n + b_n + c_n = 1$. There appears to be symmetry between the three sequences, so we conjecture that each converges to 1/3.

Suppose $a_n \le b_n \le c_n$. We have $a_{n+1} = a_n^2 + 2b_nc_n \le a_nc_n + b_nc_n + c_nc_n = c_n(a_n + b_n + c_n) = c_n$. Similarly, $b_{n+1} = b_{n2} + 2a_nc_n \le b_nc_n + a_nc_n + c_n^2 = c^n$, and $c_{n+1} = c_n^2 + 2a_nb_n \le c_n^2 + a_nc_n + b_nc_n = c_n$. Hence the largest of $a_{n+1}$, $b_{n+1}$, $c_{n+1}$ is no bigger than the largest of $a_n$, $b_n$, $c_n$. An exactly similar argument works for the smallest. Hence the largest forms a monotonic decreasing sequence which is bounded below and the smallest forms a monotonic increasing sequence which is bounded above.

Let $b_n - a_n = h \ge 0$, $c_n - b_n = k \ge 0$. Then $a_{n+1} - b_{n+1} = (a_n - b_n)(a_n + b_n - 2c_n)$, so $|a_{n+1} - b_{n+1}| = h(h + 2k) \le (h + k)^2$. Similarly, $|b_{n+1} - c_{n+1}| = |b_n - c_n|\,|b_n + c_n - 2a_n| = k(2h + k) \le (h + k)^2$, and $|c_{n+1} - a_{n+1}| = |c_n - a_n|\,|c_n + a_n - 2b_n| = |(h + k)(k - h)| \le (h + k)^2$. So for n+1 the difference between the biggest and the smallest is the square of the difference for n. But $a_1$, $b_1$, $c_1$ are all positive and hence, by a trivial induction, $a_n$, $b_n$, $c_n$ are positive. Their sum is 1 so the difference between the biggest and smallest must be less than 1. Hence the difference tends to zero. Hence $a_n$, $b_n$, $c_n$ all tend to 1/3.

**Problem A6**
A is the matrix

a b c

d e f

g h i

det A = 0 and the cofactor of each element is its square (for example the cofactor of b is $fg - di = b^2$). Show that all elements of A are zero.

**Solution**
$a^2e^2 - b^2d^2 = (ei - fh)(ai - cg) - (fg - di)(ch - bi) = (ae - bd)\, i^2 + (cd - af)\, hi + (bf - ce)\, gi = (g^3 + h^3 + i^3)\, i = 0$, since $0 = \det A = g^3 + h^3 + i^3$. Hence $ae = \pm bd$. Similarly $cd = \pm af$, $bf = \pm ce$. Multiplying the three equations together we get abcdef = - abcdef unless at least one of the equations has a plus sign. In the first case, at least one of a, b, c, d, e, f is zero. In the second case, the element corresponding to the cofactor is zero - for example $ae = bd$ implies $i^2 = 0$ and hence $i = 0$. So either a member of the first two rows is zero, or a member of the last row is zero.

wlog we may assume a = 0. That implies b or d = 0 also. [Note that if, for example, i was the zero element, then we would have $ei = \pm fh$, by an argument similar to that above and hence f or h = 0). If b = 0, then since $a^3 + b^3 + c^3 = 0$, we have also c = 0. Similarly, if d = 0, then g = 0. So we now have a complete row or column zero. But now the square of any other element is a linear combination of elements in the that row or column and hence zero. Suppose, for example, a = b = c = 0. Then $g^2 = bf - ce = 0$, and similarly for the other five elements.

**Problem B1**
Let R be the reals. $f : [1, \infty) \to R$ is differentiable and satisfies $f'(x) = 1/(x^2 + f(x)^2)$ and $f(1) = 1$. Show that as $x \to \infty$, f(x) tends to a limit which is less than $1 + \pi/4$.

**Solution**
Clearly f '(x) is always positive. But f(1) = 1, so f(x) > 1 for all x. Hence $f'(x) < 1/(x^2 + 1)$ for all x. Hence $f(x) = 1 + \int_1^x f'(t)\, dt < 1 + 1 + \int_1^x 1/(1 + t^2)\, dt = 1 + (\tan^{-1}t)|_1^x = 1 + \tan^{-1}x - \pi/4 < 1 + \pi/2 - \pi/4 = 1 + \pi/4$. Since f '(x) is positive, f(x) is monotone increasing. It is bounded above by $1 + \pi/4$, so it must tend to a limit less than $1 + \pi/4$.

**Problem B2**
R is the reals. $f :(0, 1) \to R$ is differentiable and has a bounded derivative: $|f'(x)| <= k$. Prove that : $|\int_0^1 f(x)\, dx - \sum_1^n f(i/n)/n| \le k/n$.

**Solution**
The worst case for the difference between $1/n\ f(i/n)$ and $\int_{i/n-1/n}^{i/n} f(x)\,dx$ is if $f'(x) = k$ (or $-k$) for the entire range, in which case the difference is the area of a triangle base $1/n$ and height $k/n$. Hence the difference for the complete Riemann sum is at worst $k/(2n)$.

**Problem B3**
Let O be the origin (0, 0) and C the line segment $\{ (x, y) : x \in [1, 3], y = 1 \}$. Let K be the curve $\{ P :$ for some $Q \in C$, P lies on OQ and $PQ = 0.01 \}$. Let k be the length of the curve K. Is k greater or less than 2?

**Solution**
Answer: less.
If we use polar coordinates, then $r = \operatorname{cosec}\theta - .01$, so the length is $\int_{\pi/4}^{\arctan(1/3)} \sqrt{(\operatorname{cosec}\theta - .01)^2 + \operatorname{cosec}^2\theta\cot^2\theta}\ d\theta$. This is obviously horrendous.

The trick is that if we just remove the .01, then the integral gives the curve length for the line segment C, which is 2. But the presence of the -.01 obviously *reduces* the integrand at every point of the range, so the integral above must have value *less* than 2.

**Problem B4**
$p(z) \equiv z^2 + az + b$ has complex coefficients. $|p(z)| = 1$ on the unit circle $|z| = 1$. Show that $a = b = 0$.

**Solution**
In particular, $|p(1)| = |p(-1)| = 1$, so $1 + a + b$ and $1 - a + b$ lie on the unit circle. Hence their midpoint $1 + b$ lies in the unit disk. Similarly, $|p(i)| = |p(-i)| = 1$, so $-1 + ia + b$ and $-1 - ia + b$ lie on the unit circle and hence their midpoint $-1 + b$ lies in the unit disk. But $1 + b$ and $-1 + b$ are a distance 2 apart, so they must lie at either end of a diameter of the unit circle and hence $b = 0$. Now $1 + a$ and $1 - a$ lie on the unit circle, as does their midpoint 1. Hence they must coincide and so $a = 0$.

**Problem B5**
Let $p(x)$ be the polynomial $(x - a)(x - b)(x - c)(x - d)$. Assume $p(x) = 0$ has four distinct integral roots and that $p(x) = 4$ has an integral root k. Show that k is the mean of a, b, c, d.

**Solution**
a, b, c, d must be integers. $(k - a)(k - b)(k - c)(k - d) = 4$ and all of $(k - a)$, $(k - b)$, $(k - c)$, $(k - d)$ are integers. They all divide 4, so they must belong to $\{-4, -2, -1, 1, 2, 4\}$. They are all distinct, so at most two of them have absolute value 1. Hence none of them can have absolute value 4 - or their product would be at least 8. Hence they must be -2, -1, 1, 2. Hence their sum is 0, so $4k = a + b + c + d$.

**Problem B6**
P is a variable point in space. Q is a fixed point on the z-axis. The plane normal to PQ through P cuts the x-axis at R and the y-axis at S. Find the locus of P such that PR and PS are at right angles.

**Solution**
Answer: sphere centre Q, radius QO, where O is the origin, excluding the two circles formed by the intersection of the sphere with the y-z plane and the x-z plane.

Let Q be (0, 0, r). Let P be (a, b, c). If R is (x, 0, 0), then PR and PQ are perpendicular, so their dot product is 0, so $a(a - x) + b^2 + c(c - r) = 0$, hence $ax = (a^2 + b^2 + c^2 - cr)$. Similarly, if S is (0, y, 0), then $by = (a^2 + b^2 + c^2 - cr)$. We require PR and PS perpendicular so $a(a - x) + (b - y)b + c^2 = 0$, hence $ax + by = a^2 + b^2 + c^2$. So $a^2 + b^2 + c^2 - 2cr = 0$ and hence $a^2 + b^2 + (c - r)^2 = r^2$, which shows that P lies on the sphere centre Q radius QO.

Conversely, suppose P lies on the sphere. Then SP and SO are tangents to the sphere and hence equal. Similarly, RP = RO, so PRS and ORS are similar. Hence $\angle RPS = \angle ROS = 90^\circ$. However, if P lies in the z-x plane, then S is not a finite point and if P lies in the y-z plane, then R is not a finite point. So we must exclude points lying on these two circles of the sphere.

# 8th Putnam 1948

**Problem A1**
C is the complex numbers. f **:** C → R is defined by $f(z) = |z^3 - z + 2|$. What is the maximum value of f on the unit circle $|z| = 1$?

**Solution**
Answer: √13.
Put $z = e^{i\theta}$, and use $\cos 2\theta = 2 c^2 - 1$, $\cos 3\theta = 4 c^3 - 3c$, where $\cos \theta = c$, to get: $|f(z)|^2 = 6 - 2 \cos(3\theta - \theta) + 4 \cos^3\theta - 4 \cos \theta = 4(4c^3 - c^2 - 4c + 2)$. The cubic has stationary points where $12c^2 - 2c - 4 = 0$ or $c = 2/3$ or $-1/2$. So the maximum value is at $c = 1$ or $-1$ (the endpoints), or $2/3$ or $-1/2$ (the stationary points). Substituting in, we find that the maximum is actually at $c = -1/2$ with value 13.

**Problem A2**
K is a cone. s is a sphere radius r, and S is a sphere radius R. s is inside K touches it along all points of a circle. S is also inside K and touches it along all points of a circle. s and S also touch each other. What is the volume of the finite region between the two spheres and inside K?

**Solution**
A straight slog is fairly long and tiresome.
Slice off the top of a sphere radius r by a cut a distance d from the surface. A simple integration shows that the volume removed is $\pi d^2(3r - d)/3$.

Let h be the distance from the vertex of the cone along the axis to the nearest sphere. Similar triangles gives $(h+r)/r = (h+2r+R)/R$. Hence $h = 2r^2/(R-r)$. Let the plane through the circle of contact between the small sphere and the cone cut the cone's axis at a distance h+d from its vertex. Similarly, let the plane through the circle of contact of the other sphere cut the cone's axis a distance D from the point of contact between the two spheres. Let t be the distance from the cone's vertex to the circle of contact with the small sphere. Then $t^2 = h(h+2r)$. By similar triangles $(h+d)/t = t/(h+r)$, so $d = 2r^2/(R+r)$. Hence by similar triangles $D = 2rR/(R+r)$.

Let v be the volume in the cone between the vertex and the small sphere. We find this as the volume of a cone with circular base less the volume of a slice of sphere. The cone has height h+d. The square of the radius of the base is $(2rd-d^2)$. Hence $v = \pi/3 \; ( (h+d)(2rd-d^2) - d^2(3r-d) )$. This simplifies to $4r^5\pi/(3(R^2-r^2))$.

Hence V the volume of the corresponding region between the vertex and the large sphere (assuming the small sphere is temporarily removed) is $4r^2R^3\pi/(3(R^2-r^2))$. Hence the required volume is $V - v - 4\pi r^3/3 = 4\pi r^2R^2/3(R+r)$.

**Problem A3**
$a_n$ is a sequence of positive reals decreasing monotonically to zero. $b_n$ is defined by $b_n = a_n - 2a_{n+1} + a_{n+2}$ and all $b_n$ are non-negative. Prove that $b_1 + 2b_2 + 3b_3 + ... = a_1$.

**Solution**
We have that $b_1 + 2b_2 + 3b_3 + ... + n\, b_n = a_1 - (n+1)a_{n+1} + na_{n+2} = a_1 - (n+1)(a_{n+1} - a_{n+2}) - a_{n+2}$. Also we are given that all $b_i$ are non-negative. So S n $b_n$ is monotonic increasing and bounded above by $a_1$. So it converges to a limit $L \le a_1$.

$b_n = (a_n - a_{n+1}) - (a_{n+1} - a_{n+2})$. Hence $b_n + b_{n+1} + b_{n+2} + ... + b_{n+m} = (a_n - a_{n+1}) - (a_{n+m+1} - a_{n+m+2})$. But $a_m$ tends to zero, so $(a_{n+m+1} - a_{n+m+2})$ tends to zero as m tends to infinity. Hence $b_n + b_{n+1} + b_{n+2} + ...$ converges to $a_n - a_{n+1}$. Hence $(n+1)(b_{n+1} + b_{n+2} + b_{n+3} + ... )$ converges to $(n + 1)(a_{n+1} - a_{n+2})$. Hence $L \ge b_1 + 2b_2 + 3b_3 + ... + n\, b_n + (n+1)(b_{n+1} + b_{n+2} + b_{n+3} + ... ) = a_1 - (n+1)a_{n+1} + n\, a_{n+2} + (n+1)(a_{n+1} - a_{n+2}) = a_1 - a_{n+2}$. But $a_{n+2}$ tends to zero. Hence $L = a_1$.

**Problem A4**
Let D be a disk radius r. Given (x, y) □ D, and R > 0, let a(x, y, R) be the length of the arc of the circle center (x, y), radius R, which is outside D. Evaluate $\lim_{R\to 0} R^{-2} \int_D a(x, y, R)\, dx\, dy$.

**Solution**
Answer: $4\pi r$.
Let P be a point in the disk a distance x from its centre. Suppose that the circle centre P radius R cuts the disk

perimeter at A and B. Let angle APB be $2\theta$. We have $r^2 = x^2 + R^2 + 2xR \cos \theta$. Hence the length of arc outside the disk is $2R \cos^{-1}( (r^2-x^2-R^2)/2xR)$. This applies for $r-R <= x <= r$. For smaller x the small circle lies entirely in the disk. Thus we have to evaluate $1/R^2 \int 2\pi x\ 2R \cos^{-1}( (r^2-x^2-R^2)/2xR)\ dx$. Put $x = r - yR$, and we get $4\pi r \int_0^1 (1 - y R/r) \cos^{-1}( (2Rry - (y^2+1)R^2)/2R(r - yR) )\ dy$. As R tends to 0, $(1 - y R/r)$ tends to 1 and $(2Rry - (y^2+1)R^2)/2R(r - yR)$ tends to y, so we get $4\pi r \int_0^1 \cos^{-1}y\ dy = 4\pi r (y \cos^{-1}y - \sqrt{(1-y^2)} )|_0^1 = 4\pi r$.

**Problem A5**

Let $\alpha_1, \alpha_2, \dots , \alpha_n$ be the nth roots of unity. Find $\prod_{i<j} (\alpha_i - \alpha_j)^2$.

**Solution**

Let $\alpha_k = \exp(2pi(k-1)/n)$. So $\alpha_1 = 1$, and the other $\alpha_k$ are roots of $x^{n-1} + x^{n-2} + ... + x + 1 = 0$. Hence $(1 - \alpha_2), \dots , (1 - \alpha_n)$ are roots of $(1 - x)^{n-1} + ... + (1 - x) + 1 = 0$. The coefficient of $x^0$ is n and the coefficient of $x^{n-1}$ is $(-1)^{n-1}$. Hence $\prod_2^n(1 - \alpha_k) = n$.

Hence $n/\alpha_k = (\alpha_k - \alpha_1)(\alpha_k - \alpha_2) \dots (\alpha_k - \alpha_n)$, where the product excludes $(\alpha_k - \alpha_k)$. Hence $n^n/(\alpha_2 \dots \alpha_n) = \prod_{i \text{ not equ } j}(\alpha_i - \alpha_j) = (-1)^{n(n-1)/2} \prod_{i<j}(\alpha_i - \alpha_j)^2$. But $(\alpha_2 \dots \alpha_n) = (-1)^{n-1}$. Hence $\prod_{i<j}(\alpha_i - \alpha_j)^2 = (-1)^{(n-1)(n-2)/2} n^n$.

**Problem A6**

*Do either (1) or (2):*

(1) On each element ds of a closed plane curve there is a force 1/R ds, where R is the radius of curvature. The force is towards the center of curvature at each point. Show that the curve is in equilibrium.

(2) Prove that $x + 2/3\ x^3 + 2\cdot4/3\cdot5\ x^5 + \dots + 2\cdot4\cdot \dots .2n/(3\cdot5. \dots .2n+1)\ x^{2n+1} + \dots = (1 - x^2)^{-1/2} \sin^{-1}x$.

**Solution**

(2) Let $f(x) = x + 2/3\ x^3 + 8/15\ x^5 + \dots + 1/2\ n!\ n!/2n+1!\ (2x)^{2n+1} + \dots$ .
Hence $x f(x) = x^2 + 2/3\ x^4 + 8/15\ x^6 + \dots + 1/4\ n!\ n!/2n+1!\ (2x)^{2n+2} + \dots$ .
Also $f '(x) = 1 + 2x^2 + 8/3\ x^4 + \dots + n!\ n!/2n!\ (2x)^{2n} + \dots$ .
and $(1 - x^2) f '(x) = 1 + x^2 + 2/3\ x^4 + \dots + n-1!\ n-1!/2n!\ n/2\ (2x)^{2n} + \dots$
$= 1 + x^2 + 2/3\ x^4 + \dots + 1/4\ n-1!n-1!/2n-1!\ (2x)^{2n} + \dots$ .

Hence the derivative of $\sqrt{(1 - x^2)} f(x)$ is $1/\sqrt{(1 - x^2)} (- x f(x) + (1 - x^2) f '(x) ) = 1/\sqrt{(1 - x^2)}$. So $\sqrt{(1 - x^2)} f(x) = \text{const} + \sin^{-1}(x)$. Putting $x = 0$, we find that the constant is 0.

**Problem B1**

p(x) is a cubic polynomial with roots $\alpha, \beta, \gamma$ and p'(x) divides p(2x). Find the ratios $\alpha : \beta : \gamma$.

**Solution**

$p(x) = (x - \alpha)(x - \beta)(x - \gamma)$. Hence $p'(x) = 3x^2 - 2(\alpha+\beta+\gamma)x + \beta\gamma+\gamma\alpha+a\beta$. This must have roots $\alpha/2, \beta/2$. Hence $2(\alpha+\beta+\gamma)/3 = (\alpha+\beta)/2$ and $\alpha\beta/4 = (\beta\gamma+\gamma\alpha+\alpha\beta)/3$. This leads to $\alpha = (-2 + 2i/\sqrt{3})\gamma$, $\beta = (-2 - 2i/\sqrt{3})\gamma$.

**Probloem B2**

A circle radius r is tangent to the three coordinate planes ($x =0$, $y =0$, $z = 0$) in space. Find the locus of its center.

**Problem B3**

Show that $[\sqrt{n} + \sqrt{(n + 1)}] = [\sqrt{(4n + 2)}]$ for positive integers n.

**Solution**

We have $( \sqrt{n} + \sqrt{(n + 1)} )^2 = 2n + 1 + 2 \sqrt{(n^2 + n)}$. Now $n^2 < n^2 + n < n^2 + n + 1/4 = (n + 1/2)^2$, so $\sqrt{(4n+1)} < \sqrt{n} + \sqrt{(n+1)} < \sqrt{(4n+2)}$. Hence $[ \sqrt{(4n+1)} ] \le [ \sqrt{n} + \sqrt{(n+1)} ] \le [ \sqrt{(4n+2)} ]$. But a square must be congruent to 0 or 1 mod 4, so $[ \sqrt{(4n+2)} ] = [ \sqrt{(4n+1)} ]$. Hence result.

**Problem B4**

R is the reals. For what $\lambda$ can we find a continuous function $f : (0, 1) \to R$, not identically zero, such that $\int_0^1 \min(x, y) f(y)\ dy = \lambda f(x)$ for all x □ (0, 1)?

**Problem B5**
Find the area of the region { (x, y) **:** $|x + yt + t^2| \le 1$ for all t □ [0, 1] }.

**Solution**
We cannot have $|x| > 1$, for then the inequality is not satisfied at t = 0. We cannot have y > 1 for then $yt + t^2 > 2$ at t = 1 and hence $x + yt + t^2 > 1$. Similarly, we cannot have y < -3 for then $yt + t^2 < -2$ at t = 1 and hence $x + yt + t^2 < -1$.

We can write $x + yt + t^2 = x - y^2/4 + (t + y/2)^2$. So for $-3 \le y \le -2$, the maximum of $(t + y/2)^2$ occurs at t = 0 with value $y^2/4$, and its minimum is $(1 + y/2)^2$ at t = 1. Hence the minimum of $x + yt + t^2$ is 1 + x + y and the maximum is x. So the region is bounded by $x \le 1$ and $x + y \ge -2$.

For $-2 \le y \le -1$, the maximum of $(t + y/2)^2$ is $y^2/4$ at t = 0, but the minimum is 0 at t = -y/2. Hence the maximum of $x + yt + t^2$ is x and the minimum is $x - y^2/4$. So the region is bounded by $x \le 1$ and $x - y^2/4 \ge -1$.

For $-1 \le y \le 0$, the maximum of $(t + y/2)^2$ is $(1 + y/2)^2$ and the minimum is 0 at t = -y/2. Hence the maximum of $x + yt + t^2$ is 1 + x + y and the minimum is $x - y^2/4$. So the region is bounded by $x + y \le 0$ and $x - y^2/4 \ge -1$.

For $0 \le y \le 1$, the maximum of $(t + y/2)^2$ is $(1 + y/2)^2$ and the minimum is $y^2/4$. Hence the maximum of $x + yt + t^2$ is 1 + x + y and the minimum is x. So the region is bounded by $x + y \le 0$ and $x \ge -1$.

Thus the region is the rhombus with vertices at (1, -1), (1, -3), (-1, -1), (-1, 1), except that the vertex (-1, -1) is cut off by the curve $x = y^2/4 - 1$, which forms the boundary between (0, -2) and (-1, 0). The rhombus has area 4 (base 2 and height 2). The area between the parabola and the lines x = -1 and y = -2 is $\int_0^2 y^2/4 \, dy = 8/12$. The area between the line y = -2 -x and the lines x = -1 and y = -2 is 1/2, so the area cut off the rhombus by the parabola is 2/3 - 1/2 = 1/6. Hence the area required is 4 - 1/6 = 3 5/6.

**Problem B6**
*Do either (1) or (2):*

(1) Take the origin O of the complex plane to be the vertex of a cube, so that OA, OB, OC are edges of the cube. Let the feet of the perpendiculars from A, B, C to the complex plane be the complex numbers u, v, w. Show that $u^2 + v^2 + w^2 = 0$.

(2) Let $(a_{ij})$ be an n x n matrix. Suppose that for each i, $2 |a_{ii}| > \sum_1^n |a_{ij}|$. By considering the corresponding system of linear equations or otherwise, show that det $a_{ij} \ne 0$.

**Solution**
(2) If the det is non-zero, then we can find $x_i$not all zero so that $\sum x_i a_{ij} = 0$ for each j. Take k so that $|x_k| \ge |x_i|$ for all i. Then $|\sum_{i \text{ not } k} x_i a_{ik}| \le \sum_{i \text{ not } k} |x_i a_{ik}| \le |x_k| \sum_{i \text{ not } k} |a_{ik}| < |x_k| |a_{kk}|$, so we cannot have $\sum x_i a_{ik} = 0$. Contradiction.

# 9th Putnam 1949

**Problem A1**

*Do either (1) or (2)*

(1) Let L be the line through (0, -a, a) parallel to the x-axis, M the line through (a, 0, -a) parallel to the y-axis, and N the line through (-a, a, 0) parallel to the z-axis. Find the equation of S, the surface formed from the union of all lines K which intersect each of L, M and N.

(2) Let S be the surface $xy + yz + zx = 0$. Which planes cut S in circles? In parabolas?

**Problem A2**

Take points O, P, Q, R in space. Let the volume of the parallelepiped with edges OP, OQ, OR be V. Let V' be the volume of the parallelepiped which has O as one vertex and which has OP, OQ, OR as altitudes to three faces. Show that $V V' = OP^2OQ^2OR^2$. Generalize to n dimensions.

**Problem A3**

All the complex numbers $z_n$ are non-zero and $|z_m - z_n| > 1$ (for any $m \neq n$). Show that $\sum 1/z_n^3$ converges.

**Solution**

We show that it is absolutely convergent. Take a disk radius 1/2 centred on each $z_n$. The disks must be disjoint since $|z_m - z_n| > 1$ for distinct m, n. Now consider how many $z_n$ can lie in the annulus $N \le |z| < N+1$. If $z_n$ lies in the annulus, then at least half the corresponding disk must also, so if M is the number of $z_n$, then $M \pi/8 < \pi ( (N+1)^2 - N^2) = (2N+1) \pi$, and the sum of $1/|z_n|^3$ for these $z_n$ is at most $M/N^3 = 8(2N + 1)/N^3$. But $\sum 8(2N + 1)/N^3$ converges, since $\sum 1/N^2$ converges.

**Problem A4**

Take P inside the tetrahedron ABCD to minimize PA + PB + PC + PD. Show that ∠APB = ∠CPD and that the bisector of APB also bisects CPD.

**Problem A5**

Let $p(z) \equiv z^6 + 6z + 10$. How many roots lie in each quadrant of the complex plane?

**Solution**

There are no real roots, since p(z) has a minimum (for real value of z) at z = -1 of 5. Similarly there are no purely imaginary roots, since p(i y) has imaginary part 6iy (and z = 0 is not a root). Also a + ib is a root iff a - ib is a root, so either there is one root in each of the 1st and 4th quadrants and two in each of the 2nd and 3rd, or vice versa.

The argument principle states that the change in arg p(z) as we move (anti-clockwise) round a closed contour which passes through no zeros and contains k zeros is $2 \pi k$. Take a contour in the first quadrant from z = 0 to z = X, then around the arc |z| = X to iX, then back along the imaginary axis to z = 0. Moving along the real axis, arg p(z) does not change. Moving along the arc the change will be the same as that in arg $z^6$ for X sufficiently large and hence will be $3\pi$. As we move back down the imaginary axis, the imaginary part remains positive, so p(z) is confined to the upper half plane, and we end up at p(z) = 10, so the change must be $-\pi$. Thus the change for a complete circuit is $2\pi$, and so there is just one root in the first quadrant.

**Problem A6**

Show that $\prod_1^\infty (1 + 2 \cos(2z/3^n)/3 = (\sin z)/z$ for all complex z.

**Solution**

For z = 0, we may take (sin z)/z = 1 by continuity. In this case, each term on the lhs is 1, so the relation holds. Assume now that z is non-zero.

We have sin z = sin(2z/3 + z/3) = sin 2z/3 cos z/3 + cos 2z/3 sin z/3, and sin z/3 = sin(2z/3 - z/3) = sin 2z/3 cos z/3 - cos 2z/3 sin z/3. Hence sin z = (1 + cos 2z/3) sin z/3. Hence (sin z)/z = (sin z/3)/(z/3) (1 + cos 2z/3)/3. Iterating: $(\sin z)/z = (\sin z/3^N)/(z/3^N) \prod_1^N (1 + 2 \cos(2z/3^n) )/3$. But as N tends to infinity $z/3^N$ tends to zero and hence $(\sin z/3^N)/(z/3^N)$ tends to 1. Hence the result.

**Problem B1**
Show that for any rational $a/b$ □ $(0, 1)$, we have $|a/b - 1/\sqrt{2}| > 1/(4b^2)$.

**Solution**
If $|a/b - 1/\sqrt{2}| \le 1/(4b^2)$, then $|a^2/b^2 - 1/2| = |a/b - 1/\sqrt{2}|\ |a/b + 1/\sqrt{2}| < |a/b - 1/\sqrt{2}|\ (1 + 1) < 1/2b^2$. So $|2a^2 - b^2| < 1$. Hence $2\ a^2 = b^2$. But that is impossible since $\sqrt{2}$ is irrational. Note that the restriction to the interval $(0, 1)$ is unnecessary.

**Problem B2**
*Do either (1) or (2)*

(1) Prove that $\sum_2^\infty \cos(\ln \ln n) / \ln n$ diverges.

(2) Let $k, a, b, c$ be real numbers such that $a, k > 0$ and $b^2 < ac$. Show that $\int_U (k + ax^2 + 2bxy + cy^2)^{-2}\, dx\, dy = \pi/( k \sqrt{(ac - b^2)} )$, where $U$ is the entire plane.

**Solution**
(1) We have $\ln n^2 = 2 \ln n$, so for any $m$ in the range $n, n+1, n+2, \dots , n^2$ we have $1/\ln m > 1/2\ 1/\ln n$. Also $\ln \ln n^2 = \ln \ln n + \ln 2 < \ln \ln n + \pi/3$. So given any sufficiently large integer $N$, take $x > N$ so that $\ln \ln x$ = a multiple of $2\pi$, then we can find at least $N^2/2$ consecutive integers $n$ in the range $x$ to $x^2$such that (A) $1/\ln n > 1/(2 \ln N)$ and (B) $\ln \ln n$ lies between $2k\pi$ and $2k\pi + \pi/3$ (for some integer $k$) and hence so that $\cos \ln \ln n > 1/2$. Hence for each such integer $(\cos \ln \ln n)/\ln n > 1/(4 \ln N)$. So $\sum (\cos \ln \ln n)/\ln n$ over all these terms is at least $N^2/(8 \ln N)$ which can be made arbitrarily large. Hence the sequence does not converge.

(2) $ax^2 + 2bxy + cy^2 = h$ is the equation of an ellipse. If we rotate the axes to line up with its axes we get new coordinates $X, Y$ so that it becomes $AX^2 + BY^2 = h$ with $AB = ac - b^2$. Thus the integral is transformed to $\int_U (k + AX^2 + BY^2)^{-2}\, dX\, dY$. Take $u = X\sqrt{A}$, $v = Y\sqrt{B}$ and the integral becomes $1/\sqrt{(ac - b^2)} \int_U (k + u^2 + v^2)^{-2}\, du\, dv$. Now take $u = r \cos \theta$, $v = r \sin \theta$ and we get $1/\sqrt{(ac - b^2)} \int_U (k + u^2 + v^2)^{-2}\, du\, dv = 1/\sqrt{(ac - b^2)} \int_U (k + r^2)^{-2}\, r\, dr\, d\theta = 2\pi /\sqrt{(ac - b^2)} \int (k + r^2)^{-2}\, r\, dr = 2\pi /(k \sqrt{(ac - b^2)} ) \int (1 + s^2)^{-2}\, s\, ds = \pi /(k \sqrt{(ac - b^2)} ) (-1/(1 + s^2)|_0^{inf} = 2\pi /(k \sqrt{(ac - b^2)} )$.

**Problem B3**
$C$ is a closed plane curve. If $P, Q$ □ $C$, then $|PQ| < 1$. Show that we can find a disk radius $1/\sqrt{3}$ which contains $C$.

**Solution**
$C$ can be any set of points in the plane. The usual proof uses Helly's theorem. Given any three points of the set we can find a circle radius $1/\sqrt{3}$ which contains them. This is fairly obvious. Take the points to be $P, Q, R$ with $PQ$ the longest side of the triangle. Take $S$ on the same side of $PQ$ as $R$ so that $PQS$ is equilateral. Then $R$ must lie inside the bounding circle of $PQS$ (in fact it must lie inside the region bounded $PQ$, the arc of the circle centre $P$ from $S$ to $Q$, and the arc of the circle centre $Q$ from $S$ to $P$, and this region lies entirely inside the bounding circle of $PQS$).

Another way of expressing this is that the three circles radius $1/\sqrt{3}$ and centres $P, Q, R$ have a point in common. We can now apply Helly's theorem to the set of circles radius $1/\sqrt{3}$ and with centres at all the points of the set $C$. Every three of these circles have a point in common (proved above) and hence by Helly's theorem there is a point common to all of them. But this point can be taken as the centre of the required circle.

**Problem B4**
Let $(1 + x - \sqrt{(x^2 - 6x + 1)} )/4 = \sum_1^\infty a_n x^n$. Show that all $a_n$ are positive integers.

**Solution**
We have $1 + x - \sqrt{(x^2 - 6x + 1)} ) = 4 \sum a_n x^n$, and hence $1 + x + \sqrt{(x^2 - 6x + 1)} ) = 2x + 2 - 4 \sum a_n x^n$. Multiplying, $1 + 2x + x^2 - x^2 + 6x - 1 = 8x = 8(a_1 x + a_2 x^2 + \dots ) (1 + (1 - 2a_1)x - 2a_2 x^2 - 2a_3 x^3 - \dots )$. Hence:

$a_1 = 1$
$a_2 + a_1(1 - 2a_1) = 0$
$a_3 + a_2(1 - 2a_1) - 2a_1 a_2 = 0$
$a_4 + a_3(1 - 2a_1) - 2a_2 a_2 - 2a_1 a_3 = 0 \dots$
The nth equation has the form $a_n$ + combination of $a_1, \dots , a_{n-1}$ with integer coefficients, so by a simple induction, $a_n$ is integral.

**Problem B5**
$a_n$ is a sequence of positive reals. Show that $\limsup_{n\to\infty}( (a_1 + a_{n+1})/a_n)^n \ge e$.

**Solution**
It is sufficient to show that $(a_1 + a_{n+1})/a_n \ge 1 + 1/n$ for infinitely many n. For these n then form a sequence for which $( (a_1 + a_{n+1})/a_n )^n$ has a limit point (possibly infinity) not less than $\lim (1 + 1/n)^n = e$.

If that were false, then we would have $(a_1 + a_{n+1})/a_n < 1 + 1/n$ for all n not less than some N. In particular, taking n = N, N+1, N+2, ... we get $a_1/N+1 + a_{N+1}/N+1 < a_N/N$, $a_1/N+2 + a_{N+2}/N+2 < a_{N+1}/N+1$, $a_1/N+3 + a_{N+3}/N+3 < a_{N+2}/N+2$, ... . Hence $a_n/N > a_1(1/N+1 + 1/N+2 + ... + 1/N+M) + a_{N+M}/N+M > a_1(1/N+1 + 1/N+2 + ... + 1/N+M)$. But this holds for any M and $(1/N+1 + 1/N+2 + ... )$ diverges. Contradiction.

**Problem B6**
C is a closed convex curve. If P lies on C and $T_P$ is the tangent at P, then $T_P$ varies continuously with P. Let O be a point inside C. Given a point P on C, define f(P) to be the point where the perpendicular from O to $T_P$ intersects C. Given $P_1$, define the sequence $P_n$ by $P_{n+1} = f(P_n)$. Assume that f is continuous and that, for each P, C lies entirely on one side of $T_P$. Show that $P_n$ converges. Find $S = \{ P : P = \lim_{n\to\infty}P_n \text{ for some } P_1\}$.

## 10th Putnam 1950

**Problem A1**
a and b are positive reals and a > b. Let C be the plane curve $r = a - b \cos \theta$. For what values of b/a is C convex?

**Solution**
The curvature is $(x'y'' - x''y')/(x'^2 + y'^2)^{3/2}$. So for convexity we require $x'y'' \ge x''y'$. Putting $k = b/a$, $c = \cos \theta$, $s = \sin \theta$, we have $x/a = c - k c^2$, $x'/a = - s + 2kcs$, $x''/a = - c - 2ks^2 + 2kc^2$ and $y/a = s - ksc$, $y'/a = c - kc^2 + k s^2$, $y''/a = -s + 4ksc$. Thus the condition becomes after a little cancellation, $1 + 2k^2 - 3kc \ge 0$. c takes values in the range -1 to 1, so for the condition to be true for all points of the curve we require $1 + 2k^2 - 3k \ge 0$. But $1 + 2k^2 - 3k = (2k - 1)(k - 1)$. We are given that $k < 1$, so we must have $k < 1/2$.

For small k, the curve is approximately a circle centred on the origin. As k increases it develops a flattening near $x = a$, $y = 0$. For $k > 1/2$, this becomes a dimple in the surface, so that convexity is broken. For $k = 1$, the depression in the surface extends as far as the centre (the origin). For $k > 0$, the curve intersects itself at the origin so that it comprises two ovals one inside the other.

**Problem A2**
Does the series $\sum_2^\infty 1/\ln n!$ converge? Does the series $1/3 + 1/(3\ 3^{1/2}) + 1/(3\ 3^{1/2}\ 3^{1/3}) + ... + 1/(3\ 3^{1/2}\ 3^{1/3} ... 3^{1/n}) + ...$ converge?

**Solution**
(1) $\ln n! < n \ln n$, so $\sum_2^\infty 1/\ln n! > \sum_2^\infty 1/(n \ln n)$, which diverges. For $\int 1/(x \ln x) = \ln \ln x$.

(2) We have $1 + 1/2 + 1/3 + ... + 1/n$ is approx $\ln n$, and $1/3^{\ln n}$ is approx $1/n^{\ln 3}$. But $\ln 3 > 1$, so we expect this to converge. In fact $1 +1/2 + 1/3 + ... + 1/n > \int_1^{n+1} dx / x = \ln(n + 1) > \ln n$. So $1/3 + 1/(3\ 3^{1/2}) + 1/(3\ 3^{1/2}\ 3^{1/3}) + ... + 1/(3\ 3^{1/2}\ 3^{1/3} ... 3^{1/n}) + ... < 1/3^{\ln 1} + 1/3^{\ln 2} + 1/3^{\ln 3} + ... = 1/1 + 1/2^k + 1/3^k + ...$ , where $k = \ln 3$, which converges.

**Problem A3**
The sequence $a_n$ is defined by $a_0 = \alpha$, $a_1 = \beta$, $a_{n+1} = a_n + (a_{n-1} - a_n)/(2n)$. Find $\lim a_n$.

**Solution**
$a_{n+1} - a_n = -1/2n\ (a_n - a_{n-1})$, hence $a_{n+1} - a_n = (-1)^n/(2\cdot 4\cdot 6 ... .2n)\ (a_1 - a_0)$. Then $a_n = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + ... + (a_1 - a_0) + a_0$.
Hence the limit is $\beta + (\beta - \alpha)/2 - (\beta - \alpha)/2\cdot 4 + (\beta - \alpha)/2\cdot 4\cdot 6 - ....$ But $1/2 -1/2\cdot 4 + 1/2\cdot 4\cdot 6 - ... = 1 - 1/\sqrt{e}$, so the limit is $\alpha + (\beta - \alpha)/\sqrt{e}$.

**Problem A4**
*Do either (1) or (2)*

(1) P is a prism with triangular base. A is a vertex. The total area of the three faces containing A is 3k. Show that if the volume of P is maximized, then each of the three faces has area k and the two lateral faces are perpendicular to each other.

(2) Let $f(x) = x + x^3/(1\cdot 3) + x^5/(1\cdot 3\cdot 5) + x^7/(1\cdot 3\cdot 5\cdot 7) + ...$ , and $g(x) = 1 + x^2/2 + x^4/(2\cdot 4) + x^6/(2\cdot 4\cdot 6) + ....$ Show that $\int_0^x \exp( - t^2/2)\, dt = f(x)/g(x)$.

**Solution**
(1) Let the sides of the base at A be a, b with angle between them θ. Let the height be h. Then the volume is $V = 1/2\ abh \sin \theta$ and $3k = ah + bh + 1/2\ ab \sin \theta$. Now $V^2 = (1/2\ ab \sin \theta)\ (ah)\ (bh)\ 1/2 \sin \theta$. By the arithmetic geometric mean theorem we have that $(1/2\ ab \sin \theta)\ (ah)\ (bh) \le k^3$, with equality iff $ah = bh = 1/2\ ab \sin \theta$. So $V^2 \le k^3/2 \sin \theta$. Hence $V \le \sqrt{(k^3/2)}$ with equality iff $\theta = \pi/2$ and $ah = bh = 1/2\ ab$.

(2) $\exp(- t^2/2) = 1 - t^2/2 + t^4/2\cdot 4 - t^6 2\cdot 4\cdot 6 + ....$ We notice that this is similar to g(t). In fact $g(t) = \exp( t^2/2)$. So the required relation is $f(x) = \exp( x^2/2) \int_0^x \exp( -t^2/2)\, dt$. Differentiating gives $f '(x) = x f(x) + 1$. All that is just motivation.

We start from the series for f(x). By the ratio test it is absolutely convergent, so we may differentiate term by term to get $f'(x) = 1 + x^2/1 + x^4/1\cdot3 + x^6/1\cdot3\cdot5 + \dots = x f(x) + 1$. Multiplying this by $\exp(-x^2/2)$ we get $\exp(-x^2/2) f'(x) - x \exp(-x^2/2) f(x) = \exp(-x^2/2)$. Integrating and using $f(0) = 0$, we get $\exp(-x^2/2) f(x) = \int_0^x \exp(-t^2/2)\,dt$ or $f(x)/g(x) = \int_0^x \exp(-t^2/2)\,dt$.

**Problem A5**

Let N be the set of natural numbers {1, 2, 3, ... }. Let Z be the integers. Define $d : N \to Z$ by $d(1) = 0$, $d(p) = 1$ for p prime, and $d(mn) = m\,d(n) + n\,d(m)$ for any integers m, n. Determine d(n) in terms of the prime factors of n. Find all n such that $d(n) = n$. Define $d_1(m) = d(m)$ and $d_{n+1}(m) = d(d_n(m))$. Find $\lim_{n\to\infty} d_n(63)$.

**Solution**

$d(p^a) = a p^{a-1}$ by a trivial induction on a. Hence for $n = p^a q^b \dots$ we have $d(n) = n ( a/p + b/q + \dots )$ by a trivial induction on the number of primes.

Hence $d(n) = n$ iff $a/p + b/q + \dots = 1$. Multiplying through by all the prime denominators gives integral terms. All but the first are clearly divisible by p, so the first must be also. Hence $a/p = 1$ and b/q etc are zero. In other words, $n = p^p$ for some prime p.

We find $d_1(63) = 51$, $d_2(63) = 20$, $d_3(63) = 24$, $d_4(63) = 44$, $d_5(63) = 48$. Now suppose n is divisible by 16. Then $n = 2^k p^a q^b \dots$, where p, q, ... are all odd and $k >= 4$. Hence $d(n) = n( 4/2 + a/p + b/q + \dots )$. Now all of 2n, na/p, nb/q, ... are integral and multiples of 16. So d(n) is at least twice n and a multiple of 16. So if we have $d_k(m)$ divisible by 16, then $d_{h+k}(m) \ge 2^h$ which diverges. Hence $d_k(63)$ tends to infinity.

**Problem A6**

Let $f(x) = \sum_0^\infty a_n x^n$ and suppose that each $a_n = 0$ or 1. *Do either (1) or (2):*

(1) Show that if f(1/2) is rational, then f(x) has the form p(x)/q(x) for some integer polynomials p(x) and q(x).

(2) Show that if f(1/2) is not rational, then f(x) does not have the form p(x)/q(x) for any integer polynomials p(x) and q(x).

**Solution**

(1) f(1/2) is the binary expansion of some real in the closed interval [0, 1]. So if it is rational, then the binary expansion is periodic after some point. In other words, there are positive integers N and k such that $a_n = a_{n+k}$ for $n > N$. Hence $f(x) = a_0 + a_1x + \dots + a_Nx^N + (a_{N+1}x^{N+1} + \dots + a_{N+k-1}x^{N+k-1})(1 + x^k + x^{2k} + \dots ) = a_0 + a_1x + \dots + a_Nx^N + (a_{N+1}x^{N+1} + \dots + a_{N+k-1}x^{N+k-1})/(1 - x^k)$. Hence we can take $q(x) = 1 - x^k$ and $p(x) = (a_0 + a_1x + \dots + a_Nx^N)(1 - x^k) + (a_{N+1}x^{N+1} + \dots + a_{N+k-1}x^{N+k-1})$.
(2) is trivial. If $f(x) = p(x)/q(x)$, then $f(1/2) = p(1/2)/q(1/2)$ which is rational. The only slight complication is if $q(1/2) = 0$. But we can assume that p(x)/q(x) is in lowest terms, so if $q(1/2) = 0$, then p(x) is non-zero, but then $p(x) = f(x) q(x)$. We know that $f(1/2) \le 1/2 + 1/4 + 1/8 + \dots = 1$, so $f(1/2) q(1/2) = 0$. Hence $p(1/2) = 0$. Contradiction.

**Problem B1**

Given n, not necessarily distinct, points $P_1, P_2, \dots, P_n$ on a line. Find the point P on the line to minimize $\sum |PP_i|$.

**Solution**

Assume that the points are in the order given. For two points A and B, $|PA| + |PB| = AB$ for P on the segment AB and $|PA| + |PB| > AB$ for P outside it. Thus we minimise $|PP_1| + |PP_n|$ by placing P between $P_1$ and $P_n$. Similarly, we minimise $|PP_2| + |PP_{n-1}|$ by placing P between $P_2$ and $P_{n-1}$. And so on. But note that if we minimise $|PP_2| + |PP_{n-1}|$ then we also minimise $|PP_1| + |PP_n|$. Thus for n odd we must take P to be the central point, for n even we can take P to be any point on the segment between the two central points.

**Problem B2**

An ellipse with semi-axes a and b has perimeter length p(a, b). For b/a near 1, is $\pi(a + b)$ or $2\pi\sqrt{ab}$ the better approximation to p(a, b)?

**Solution**

Answer: $\pi(a + b)$.

Put $b^2 = a^2(1 - \varepsilon)$. The perimeter is $4 \int_0^{\pi/2} (a^2\sin^2 t + b^2\cos^2 t)^{1/2}\, dt = 4a \int (1 - \varepsilon \cos^2 t)^{1/2}\, dt = 4a \int ( 1 - 1/2\, \varepsilon \cos^2 t - 1/8\, \varepsilon^2 \cos^4 t + O(\varepsilon^4)\, dt = 2\pi a ( 1 - \varepsilon/4 - 3\varepsilon^2/64 + O(\varepsilon^4) )$.

$b = a(1 - \varepsilon)^{1/2} = a(1 - \varepsilon/2 - \varepsilon^2/8 + O(\varepsilon^3) )$. Hence $\pi (a + b) = 2\pi a (1 - \varepsilon/4 - 4\varepsilon^2/64 + O(\varepsilon^3) )$. Whilst $2\pi\sqrt{(ab)} = 2\pi a (1 - \varepsilon/2 - \varepsilon^2/8 + O(\varepsilon^3) )^{1/2} = 2\pi a (1 - \varepsilon/4 - 6\varepsilon^2/64 + O(\varepsilon^3) )$.

Hence the error in $\pi (a + b)$ is $\pi a \varepsilon^2/32 + O(\varepsilon^3)$, and the error in $2\pi\sqrt{(ab)}$ is $3\pi a \varepsilon^2/32 + O(\varepsilon^3)$, which is roughly three times as large for small $\varepsilon$.

**Problem B3**
Leap years have 366 days; other years have 365 days. Year $n > 0$ is a leap year iff (1) 4 divides n, but 100 does not divide n, or (2) 400 divides n. n is chosen at random from the natural numbers. Show that the probability that December 25 in year n is a Wednesday is not 1/7.

**Solution**
$365 = 52.7 + 1$, so in non-leap years Christmas day is a day later in the week than the previous year. In leap years it is two days later. Now consider a period of 400 years. There are 100 multiples of 4, 3 or which are multiples of 100 but not 400, so there are 97 leap years. Thus after 400 years the day has advanced $303\cdot 1 + 97\cdot 2 = 497$ days or exactly 41 weeks. In other words after 400 years Christmas is on the same day of the week. This cycle then repeats indefinitely. But 400 is not a multiple of 7, so each day cannot occur with exactly probability 1/7.

If we look in more detail at the 400 years we find that Su, Tu, Fr each occur 58 times, We and Th each occur 57 times, and Mo and Sa each occur 56 times (compared with the expected 57 1/7).

**Problem B4**
A long, light cylinder has elliptical cross-section with semi-axes $a > b$. It lies on the ground with its main axis horizontal and the major axes horizontal. A thin heavy wire of the same length as the cylinder is attached to the line along the top of the cylinder. [We could take the cylinder to be the surface $|x| \le L$, $y^2/a^2 + z^2/b^2 = 1$. Contact with the ground is along $|x| \le L$, $y = 0$, $z = -b$. The wire is along $|x| \le L$, $y = 0$, $z = b$.] For what values of $b/a$ is the cylinder in stable equilibrium?

**Solution**
Take the ellipse to be $x^2/a^2 + y^2b^2 = 1$. We need the normal to the ellipse at a point near $(0, -b)$ to meet the y-axis above $(0, b)$ (because then the heavy wire will be the correct side of the vertical through the point of contact to right the cylinder).

The gradient at $(a \cos t, b \sin t)$ is $-b/a \cot t$, so the normal has slope $a/b \tan t$, so its equation is $(y - b \sin t) = a/b \tan t\, (x - a \cos t)$. This meets the y-axis at $y = b \sin t - a^2/b \sin t$. For points near $(0, -b)$, $\sin t$ is $-(1 - \varepsilon)$ so we need $a^2/b - b > b$ or $a > b\sqrt{2}$.

**Problem B5**
*Do either (1) or (2):*
(1) Show that if $\sum(a_n + 2a_{n+1})$ converges, then so does $\sum a_n$.
(2) Let S be the surface $2xy = z^2$. The surface S and the variable plane P enclose a cone with volume $\pi a^3/3$, where a is a positive real constant. Find the equation of the envelope of P. What is the envelope in the case of a general cone?

**Solution**
(1) Suppose $a_n + 2a_{n+1}$ converges to 3k. We show that $a_n$converges to k.
Given any $\varepsilon > 0$, take N so that $a_n + 2a_{n+1}$ is within $\varepsilon$ of 3k for all $n \ge N$. Take a positive integer M such that $a_N$ is within $(2^M + 1)\varepsilon$ of k.

Then $a_{N+1}$ is within $( (2^M + 1)\varepsilon + \varepsilon)/2 = (2^{M-1} + 1)\varepsilon$ of $(3k - k)/2 = k$. By a trivial induction $a^{N+M}$ is within $2\varepsilon$ of k. Then $a^{N+M+1}$ is within $(2\varepsilon + \varepsilon)/2$, and hence within $2\varepsilon$, of k. So by a trivial induction, $a_n$ is within $2\varepsilon$ of k for all $n > N + M$.

(2) Answer: the 2-sheet hyperboloid $u^2 = v^2 + z^2 + a^2$.
The nature of the surface S becomes much clearer if we change coordinates to u, v, z with z unchanged and the x, y

axes rotated through $\pi/4$ to $u = (x + y)/\sqrt{2}$, $v = (-x + y)/\sqrt{2}$ (or inversely, $x = (u - v)/\sqrt{2}$, $y = (u + v)/\sqrt{2}$ ). Now S becomes $u^2 - v^2 = z^2$ or $v^2 + z^2 = u^2$, which is evidently a right circular cone with vertex the origin and axis the u-axis.

Evidently the plane $u = a$ forms a cone volume $1/3\ \pi a^3$ with the surface (the base is a circle radius a and the height is a). If the plane moves round to be almost tangent to the sides of the cone it should still cut off a cone with the same volume, so we might suspect that the envelope is a two sheet hyperboloid asymptotic to the cone and passing through the points $u = \pm a$, $v = z = 0$. This has equation $u^2 = v^2 + z^2 + a^2$. This is a surface of revolution formed by rotating the hyberbola $u^2 = v^2 + a^2$ about the u-axis, so it is sufficient to look at tangent planes that are parallel to (ie do not intersect) the z-axis. We need to show that these cut off cones volume $1/3\ \pi a^3$.

The tangent to the hyperbola at $u = a \cosh t$, $v = a \sinh t$ is $(v - a \sinh t) = \coth t\ (u - a \cosh t)$ or $v \sinh t - u \cosh t + a = 0$ (*). It cuts $v = u$ at $u = v = a/(\cosh t - \sinh t)$ and it cuts $v = -u$ at $u = -v = a/(\cosh t + \sinh t)$. So the corresponding plane cuts the hyperboloid in an ellipse with major axis length 2A, where 2A is the distance between these two points of intersection, which is $2a \cosh 2t$. At the centre of the ellipse (which we find as midway between the two points of intersection) we have $u = a \cosh t$, $v = a \sinh t$, so the extremities of minor axis have $z^2 = u^2 - v^2 = a^2$. Thus the minor axis has length 2a. Hence the area of the ellipse is $\pi a^2\sqrt{(2 \cosh 2t)}$. The distance of the origin from the plane is its distance from the line (*) which is $a/\sqrt{(2 \cosh 2t)}$. Hence the volume of the cone is 1/3 area of base x height = $1/3\ \pi a^3$, as we had guessed.
Finally, if we have a general cone (which does not have circular cross-section or right-angle), we can transform it into a right circular cone by an affine transformation. Such transformations preserve the ratios of distances and hence the ratios of volumes. They also preserve tangency, so the envelope in this more general case will still be a hyperboloid asymptotic to the cone.

**Problem B6**
(1) The convex polygon C' lies inside the polygon C. Is it true that the perimeter of C' is no longer than the perimeter of C?
(2) C is the convex polygon with shortest perimeter enclosing the polygon C'. Is it true that the perimeter of C is no longer than the perimeter of C' ?
(3) The closed convex surface S' lies inside the closed surface S. Is it true that area S' ≤ area S?
(4) S is the smallest convex surface containing the closed surface S'. Is it true that area S ≤ area S'?

**Solution**
(1) True. On each side of C' take a semi-infinite strip perpendicular to the side and extending outwards from C'. Since C' is convex, these strips are all disjoint. But the intersection of C with each strip is at least as long as the corresponding side of C', hence the total length of C that intersects the strips is at least as long as the perimeter of C'.

(2) True. Take C as the convex hull of C'. If is sufficient to prove that the perimeter of C is no longer than the perimeter of C'. Evidently the vertices of C form a subset of the vertices of C'. If C' includes a vertex P which is not a vertex of C, then removing it does not increase the perimeter of C' (by the triangle inequality) and leaves C unaffected. So we can assume that C and C' have the same vertices. If C' just has the vertices of C permuted cyclically (or in reverse order), then its perimeter is the same. If not, then there are vertices A, B which are adjacent in C' but not in C. We show that in this case we can permute the vertices of C' so as to shorten the perimeter. We can repeat this process, shortening the perimeter each time. But there are only finitely many permutations so we must eventually reach a perimeter with the same length as C.
Suppose C' has vertices in the order $X_1X_2 \ldots X_n$. wlog we may assume that $X_1$ and $X_2$ are not adjacent in C, so we can find $X_i$ on one side of $X_1X_2$ and $X_j$ on the other. Then one of the pairs $X_iX_{i+1}, X_{i+1}X_{i+2}, \ldots , X_{j-1}X_j$ must straddle $X_1X_2$. Suppose it is $X_kX_{k+1}$. Consider the quadrilateral $X_1X_kX_2X_{k+1}$. Suppose its diagonals $X_1X_2$ and $X_kX_{k+1}$ meet at O. Then $X_1X_2 + X_kX_{k+1} = X_1O + OX_2 + X_kO + OX_{k+1} = (X_1O + OX_k) + (X_2O + OX_{k+1}) > X_1X_k + X_2X_{k+1}$. So if we take C" to have vertices in the order $X_1X_kX_{k-1} \ldots X_3X_2X_{k+1}X_{k+2} \ldots X_n$, then C" has shorter perimeter.

(3) True. The same argument as (1) works. Take semi-infinite columns on each face of the inner polyhedron.

(4) False. Take the 4 vertices of a regular tetrahedron and its centre. We can find a surface S' with arbitrarily small area which contains these 5 points, but the convex hull S is the tetrahedron.

## 11th Putnam 1951

**Problem A1**
A is a skew-symmetric real 4 x 4 matrix. Show that det A ≥ 0.

**Solution**
Let A =

```
 0  a  b  c
-a  0  d  e
-b -d  0  f
-c -e -f  0
```

Then det A = $a^2f^2 + 2acdf - 2abef + b^2e^2 - 2bcde + c^2d^2$.
At this point it is helpful to notice that a only appears with f, b with e, and c with d. So putting X = af, Y = cd, Z = be, we have that det A = $X^2 + Y^2 + Z^2 + 2XY - 2XZ - 2YZ$. This easily factorizes as $(X + Y - Z)^2$.

**Problem A2**
k is a positive real and $P_1, P_2, \dots , P_n$ are points in the plane. What is the locus of P such that $\sum PP_i^2 = k$? State in geometric terms the conditions on k for such points P to exist.

**Solution**
Let $P_i$ have coordinates $(a_i, b_i)$. Take the origin at the centroid of the points, so that $\sum a_i = \sum b_i = 0$. Then if P has coordinates (x, y), we have $\sum PP_i^2 = \sum( (x - a_i)^2 + (y - b_i)^2 ) = n (x^2 + y^2) + \sum(a_i^2 + b_i^2)$. Let the origin be O, so that $\sum(a_i^2 + b_i^2) = \sum OP_i^2$. Then if $\sum OP_i^2 > k$, the locus is empty. Otherwise it is the circle with centre at the centroid of the n points and radius $\sqrt{( (k - \sum OP_i^2)/n )}$.

**Problem A3**
Find $\sum_0^\infty (-1)^n/(3n + 1)$.

**Solution**
We have $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$ (1). Put $\omega = e^{2\pi i/3}$, so that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 1$. Then $\omega^2 \ln(1 + \omega x) = x - \omega x^2/2 + \omega^2x^3/3 - x^4/4 + \dots$ (2) and $\omega \ln(1 + \omega^2x) = x - \omega^2x^2/2 + \omega x^3/3 - x^4/4 + \dots$ (3).

Adding (1), (2), (3) we get $3(x - x^4/4 + x^7/7 - \dots ) = \ln(1 + x) + \omega^2 \ln(1 + \omega x) + \omega \ln(1 + \omega^2x)$. Hence the required series sums to $1/3 (\ln 2 + \omega^2 \ln(1 + \omega) + \omega \ln(1 + \omega^2) )$.

If $k = \ln(1 + \omega)$, then $e^k = 1 + \omega = - \omega^2 = e^{i\pi/3}$, so $k = i\pi/3$. Hence $\omega^2 \ln(1 + \omega) = -(1 + i\sqrt{3})/2\ i\pi/3 = \pi/2\sqrt{3} - i\pi/6$. Similarly, $\omega \ln(1 + \omega^2) = -i\pi/3 (-1/2 + i\sqrt{3}/2) = \pi/2\sqrt{3} + i\pi/6$. Hence the series sums to $1/3 \ln 2 + \pi/3\sqrt{3}$.

**Problem A4**
Sketch the curve $y^4 - x^4 - 96y^2 + 100x^2 = 0$.

**Solution**
The graph is in three parts. It is symmetrical on reflection in the x-axis and on reflection in the y-axis. The central part is a figure of 8 with crossing point at the origin and touching the lines $y/x = \pm\sqrt{(100/96)}$. The top and bottom of the figure of 8 are at $(0, \pm\sqrt{96})$. The right hand part is a sort of dimpled hyperbola with asymptotes $y/x = \pm1$. It comes in along each asymptote and touches the line x = 8 at $y = \pm\sqrt{48}$, then it dimples back away from the origin to cut the x-axis at x = 10. The left hand part of the curve is just the reflection of this in the y-axis.

**Problem A5**
Show that a line in the plane with rational slope contains either no lattice points or an infinite number. Show that given any line L of rational slope we can find δ > 0, such that no lattice point is a distance k from L where 0 < k < δ.

**Solution**
The first part is obvious. If the line passes through (m, n) and has slope a/b (where m, n, a, b are all integers), then

it also passes through (m + kb, n + ka) for any integer a. Although not required, we note that there are lines with rational slope passing through no lattice points, for example, y = 1/2 and y = x + 1/2.

Suppose L is bx - ay + c = 0. The distance of L from (x, y) is $|bx - ay + c|/\sqrt{(a^2 + b^2)}$. If (x, y) is a lattice point then bx - ay is an integer. Hence |bx - ay + c| is either zero or at least 1 if c is an integer, or at least the distance of c from the nearest integer if c is not an integer. So we can find k > 0, so that |bx - ay + c| > k for all lattice points not on L. Hence the distance $|bx - ay + c|//\sqrt{(a^2 + b^2)}$ is at least $k/\sqrt{(a^2 + b^2)}$ for all lattice points not on L. So we can take δ to be $k/\sqrt{(a^2 + b^2)}$.

**Problem A6**
Let C be a parabola. Take points P, Q on C such that (1) PQ is perpendicular to the tangent at P, (2) the area enclosed by the parabola and PQ is as small as possible. What is the position of the chord PQ?

**Solution**
We may take the parabola as $y = x^2$ and P as $(a, a^2)$. The tangent at P has slope 2a, so the normal has slope -1/2a and equation $(y - a^2) + 1/2a\ (x - a) = 0$. This meets the parabola at $x^2 + x/2a - a^2 - 1/2 = 0$, so the other root is x = -1/2a - a and Q is $(-1/2a - a, (a + 1/2a)^2)$. The area under the curve between P and Q is $1/3\ (a^3 + (a + 1/2a)^3\ )$. The area under the chord PQ is $(2a + 1/2a)\ (a^2 + (a + 1/2a)^2)/2 = 2a^3 + 3a/2 + 1/2a + 1/16a^3$. Hence the area enclosed is $4/3\ a^3 + a + 1/4a + 1/48a^3$. We wish to minimise this. The gradient is $4a^2 + 1 -1/4a^2 - 1/16a^4 = 1/16a^4\ (4a^2 + 1)^2(2a - 1)(2a + 1)$. Hence there is a minimum at a = 1/2 (and another at a = -1/2). In the minimum position P is (1/2, 1/4) or (-1/2, 1/4), so the minimum positions are the intersection of the parabola with the perpendicular to the axis through the focus (0, 1/4).

**Problem A7**
Show that if $\sum a_n$ converges, then so does $\sum a_n/n$.

**Solution**
Put $s_n = a_1 + a_2 + ... + a_n$. Then $s_n - s_{n-1} = a_n$, and so $a_1/1 + a_2/2 + a_3/3 + ... + a_n/n = s_1(1/1 - 1/2) + s_2(1/2 - 1/3) + ... + s_n(1/n - 1/n+1) + s_n/n+1$. Now $s_1(1/1 - 1/2) + s_2(1/2 - 1/3) + ... + s_n(1/n - 1/n+1)+ ...$ is absolutely convergent, because the sequence $|s_n|$ is bounded, say by B. Hence $|s_1(1/1 - 1/2)| + |s_2(1/2 - 1/3)| + ... + |s_n(1/n - 1/n+1)| <= B( (1/1 - 1/2) + (1/2 - 1/3) + ... + (1/n - 1/n+1) ) = B(1 - 1/n+1) < B$. Obviously $s_n/n+1$ tends to zero, so $\sum a_n/n$ is absolutely convergent and hence convergent.

**Problem B1**
R is the reals. f, g **:** $R^2 \to R$ have continuous partial derivatives of all orders. What conditions must they satisfy for the differential equation f(x, y) dx + g(x, y) dy = 0 to have an integrating factor h(xy)?

**Solution**
If h(xy) is an integrating factor, then f(x, y) h(xy) dx + g(x, y) h(xy) dy is an exact form. In other words, ∂/∂y (f(x, y) h(xy) ) = ∂/∂x (g(x, y) h(xy). So h ∂f/∂y + x f h' = h ∂g/∂x + y g h'. Hence h'/h = (∂f/∂y - ∂g/∂x) / (yg - xf).

The lhs is a function of xy, so the rhs must be also. Equally if the rhs is a function of xy, then we can immediately integrate to get h(xy). Thus the required necessary and sufficient condition for the existence of an integrating factor is that (∂f/∂y - ∂g/∂x) / (yg - xf) should be a function of xy.

**Problem B2**
R is the reals. Find an example of functions f, g **:** R → R, which are differentiable, not identically zero, and satisfy (f/g)' = f '/g' .

**Solution**
Suppose we take f = gh. Then we require h' = h + g/g' h'. If we take g/g' = 2, then h' = -h. So this suggests $f(x) = e^{-x/2}$, $g(x) = e^{x/2}$. Checking, we see that that works.

**Problem B3**
Show that ln(1 + 1/x) > 1/(1 + x) for x > 0.

**Solution**
We have the well-known $e^y > 1 + y$ for all y not equal to 1. [Prove, for example, by differentiating.] So $\ln(1 + y) < y$. Put $y = -1/(1 + x)$, so $1 + y = x/(1 + x)$. Hence $1/(1 + x) < -\ln( x/(1+x) ) = \ln(1 + 1/x)$.

**Problem B4**
Can we find four distinct concentric circles all touching an ellipse?

**Solution**
Answer: yes, we can find four such circles for any ellipse except the circle.

We show first that we can find four such circles for any non-circular ellipse. Let the ellipse have centre O, major axis AB (length 2a) and minor axis CD (length 2b). Take a point P with $OP < (a - b)/2$. Then using the triangle inequality we have $PA > OA - OP = (a + b)/2$, $PB > OB - OP = (a + b)/2$. $PC < OC + OP = (a + b)/2$, $PD < OD + OP = (a + b)/2$. Put $k = (a + b)/2$. Then, if Q is a point on the ellipse, $(PQ - k)$ is positive at A and B and negative at C and D. Hence there must be at least four points Q on the ellipse at which it is stationary and hence at which PQ is normal to the ellipse. There is a point A' near A which is a maximum and hence has $PA' > k$. Similarly, there is a point B' near B with $PB' > k$. There is a point C' near C with $PC' < k$ and a point D' near D with $PD' < k$. Let us assume that P is in the quadrant AOC and not on either axis. The reflection D" of D' in the line AB lies at or near C' and hence has $PD" >= PC'$ (which is known to be a minimum). Put P lies closer to D" than to D'. Hence $PC' < PD'$. A similar argument shows that $PB' > PA'$. So we have $PB' > PA' > k > PD' > PC'$. Thus the circles centre P through A', B', C' and D' are all distinct and all touch the ellipse.

Note that we cannot find 4 circles in the case when the ellipse is circular. For given a point P and a circle E centre O, a circle centre P can only touch E on the line OP, but that implies there can only be two such circles.

**Problem B5**
T is a torus, center O. The plane P contains O and touches T. Prove that $P \cup T$ is two circles.

**Solution**
We may take the torus to be $( \sqrt{(x^2 + y^2)} - a )^2 + z^2 = b^2$, where $a > b$. Viewed in the x-z plane the torus appears as two circles each radius b and centred at $x = \pm a$, $z = 0$. The plane P appears as the line through the origin tangent to both circles and hence having equation $c z = b x$, where $c = \sqrt{(a^2 - b^2)}$. P also contains the y-axis which is perpendicular to the line $z = k x$. Hence the equation of the plane P is $z = k x$.

It is far from obvious that P meets T in two circles. If we draw the plane P we see that T contains the four points on the y-axis, $y = \pm a \pm b$, and also the points a distance c either side of the origin. The obvious way to put these onto two circles is to take circles radius c, centred at $(x, y, z) = (0, \pm b, 0)$. These can be written as $(x, y, z) = (c \cos t, \pm b + a \sin t, b \cos t)$. Hence $\sqrt{(x^2 + y^2)} = a \pm b \sin t$. So $( \sqrt{(x^2 + y^2)} - a )^2 = b^2\sin^2 t$ and $( \sqrt{(x^2 + y^2)} - a )^2 + z^2 = b^2\sin^2 t + b^2\cos^2 t = b^2$. Thus these circles do indeed lie on the torus.

We may write the equation of T as $2a \sqrt{(x^2 + y^2)} = x^2 + y^2 + z^2 + c^2$. Hence $4a^2(x^2 + y^2) = ( x^2 + y^2 + z^2 + c^2 )^2$. So $(x^2 + y^2 + z^2 - c^2)^2 = 4a^2x^2 + 4a^2y^2 - 4c^2(x^2 + y^2 + z^2) = 4b^2x^2 + 4b^2y^2 - 4c^2z^2$. Hence $(x^2 + y^2 + z^2 - c^2)^2 - (2by)^2 = 4b^2x^2 - 4b^2c^2$. So $(x^2 + (y + b)^2 + z^2 - a^2)(x^2 + (y - b)^2 + z^2 - a^2) = 4(bx - cz)(bx + cz)$.

So if a point lies on the plane P, which has equation $bx - cz = 0$ and on the torus T, then it also lies on one of the two spheres $(x^2 + (y + b)^2 + z^2 - a^2) = 0$ and $(x^2 + (y - b)^2 + z^2 - a^2) = 0$. But a plane always intersects a sphere in a circle (or not at all) so the intersection of P and T is at most two circles. We have already established that it contains two circles, so it cannot contain any other points.

**Problem B6**
The real polynomial $p(x) \equiv x^3 + ax^2 + bx + c$ has three real roots $\alpha < \beta < \gamma$. Show that $\sqrt{(a^2 - 3b)} < (\gamma - \alpha) \le 2 \sqrt{(a^2/3 - b)}$.

**Solution**
$a^2 - 3b = (\alpha+\beta+\gamma)^2 - (\alpha\beta + \beta\gamma + \gamma\alpha) = (\gamma - \alpha)^2 - (\gamma - \beta)(\beta - \alpha) < (\gamma - \alpha)^2$. Hence $\sqrt{(a^2 - 3b)} < (\gamma - \alpha)$.

Also $4(a^2 - 3b) - 3 (\gamma - \alpha)^2 = (\gamma - \alpha)^2 - 4(\gamma - \beta)(\beta - \alpha) = ( (\gamma - \beta) + (\beta - \alpha) )^2 - 4(\gamma - \beta)(\beta - \alpha) = ( (\gamma - \beta) - (\beta - \alpha) )^2 \ge 0$. Hence $(\gamma - \alpha) \le 2 \sqrt{(a^2/3 - b)}$.

**Problem B7**
In 4-space let S be the 3-sphere radius r: $w^2 + x^2 + y^2 + z^2 = r^2$. What is the 3-dimensional volume of S? What is the 4-dimensional volume of its interior?

**Solution**
Answer: $2\pi^2 r^3$, $1/2\ \pi^2 r^4$.

A hyperplane perpendicular to the x-axis will cut S in a sphere. If the hyperplane is at a distance k from the origin, then the radius of the sphere will be $\sqrt{(r^2 - k^2)}$. So the 4-dimensional volume = $\int_{-r}^{r} 4\pi/3\ (r^2 - x^2)^{3/2}\ dx = 4\pi\ r^4/3 \int_{-1}^{1} (1 - t^2)^{3/2}\ dt$.

Let $K = (1 - t^2)^{3/2}$ dt. We find that $A = t(1 - t^2)^{3/2}$ differentiates to $K - 3t^2(1 - t^2)^{1/2}$. $B = t(1 - t^2)^{1/2}$ differentiates to $(1 - t^2)^{1/2} - t^2(1 - t^2)^{-1/2}$. $C = \sin^{-1}t$ differentiates to $(1 - t^2)^{-1/2}$. Hence B + C differentiates to $2(1 - t^2)^{1/2}$, so 2A + 3(B + C) differentiates to 8K. So $\int_{-1}^{1} (1 - t^2)^{3/2}\ dt =$ ( terms involving powers of $(1 - t^2)^{1/2} + 3/8 \sin^{-1}t\ )|_{-1}^{1} = 3\pi/8$. Hence the 4-dimensional volume is $\pi^2 r^4/2$.

The volume of the 3-dimensional surface is the derivative wrt r of the 4-dimensional volume (because the 4-dimensional volume for $r + \delta r$ is the 4-dimensional volume for r + the 3-dimensional volume of the bounding sphere times $\delta r$). Hence the 3-dimensional volume is $2\pi^2 r^3$ .

# 12th Putnam 1952

**Problem A1**
p(x) is a polynomial with integral coefficients. The leading coefficient, the constant term, and p(1) are all odd. Show that p(x) has no rational roots.

**Solution**
Suppose there is a rational root p/q. Let $p(x) = \sum_0^n a_r x^r$. Then $\sum a_r p^r q^{n-r} = 0$. So p divides $a_0$ and hence is odd, and q divides $a_n$ and must also be odd. But now $\sum a_r p^r q^{n-r} = a_r = 1 \pmod 2$. Contradiction.

**Problem A2**
Show that the solutions of the differential equation $(9 - x^2)(y')^2 = 9 - y^2$ are conics touching the sides of a square.

**Solution**
We have $dy/dx = \pm \sqrt{(1 - (y/3)^2)} / \sqrt{(1 - (x/3)^2}$. Integrating, $\sin^{-1} y/3 = \pm \sin^{-1} x/3 + A$, where A is a constant. Hence $y/3 = \sin A \cos(\pm \sin^{-1} x/3) \pm x/3 \cos A = \sin A \sqrt{(1 - (x/3)^2)} \pm x/3 \cos A$. So $y = k(9 - x^2) \pm h x$, where $h^2 + k^2 = 1$. So $y^2 \pm 2hxy + x^2 = 9(1 - h^2)$. This is evidently a family of conics. Note that h is not restricted to being less than 1. By taking a suitable complex constant A we can get any value for h. But since negative values are possible, there is no benefit in retaining the ± sign. So we can write the family of conics simply as $y^2 + 2hxy + x^2 = 9(1 - h^2)$.

We might guess that the square is $x = \pm 3$, $y = \pm 3$. If we substitute $x = 3$ in the equation for the conic, we get $y^2 + 6hy + 9 = 9 - 9h^2$, or $(y - 3h)^2 = 0$. This has a repeated root $y = 3h$, which shows that the line $x = 3$ touches the conic. Similarly for the other four sides of the square.

**Problem A3**
Let the roots of the cubic $p(x) \equiv x^3 + ax^2 + bx + c$ be α, β, γ. Find all (a, b, c) so that $p(\alpha^2) = p(\beta^2) = p(\gamma^2) = 0$.

**Solution**
Answer: (0, 0, 0) roots 0, 0, 0
(-1, 0, 0) roots 0, 0, 1
(-2, 1, 0) roots 0, 1, 1
(0, -1, 0) roots 0, 1, -1
(1, 1, 0) roots 0, ω, $\omega^2$, where $\omega = e^{2\pi i/3}$
(-3, 3, -1) roots 1, 1, 1
(-1, -1, 1) roots 1, 1, -1
(1, -1, -1) roots 1, -1, -1
(0, 0, -1), roots 1, ω, $\omega^2$
(i, 1, -i), roots 1, -1, -i
(-i, 1, i), roots 1, -1, i
(1-ω, 1-ω, -ω, roots ω, ω, $\omega^2$
($1-\omega^2$, $1-\omega^2$, $-\omega^2$), roots ω, $\omega^2$, $\omega^2$
(1+ω, 1+ω, ω), roots ω, -ω, $\omega^2$
($1+\omega^2$, $1+\omega^2$, $\omega^2$), roots ω, $\omega^2$, $-\omega^2$
($(1-i\sqrt{7})/2$, $(-1-i\sqrt{7})/2$, -1), roots k, $k^2$, $k^4$, where $k = e^{2i\pi/7}$
($(1+i\sqrt{7})/2$, $(-1+i\sqrt{7})/2$, -1), roots $k^3$, $k^5$, $k^6$.

p(x) = 0 has only three roots, so the squares must all be found in { α, β, γ }. Let α be the root with the largest modulus, then $|\alpha|^2 > |\alpha|$ unless $|\alpha| \le 1$. Let γ be the root with the smallest non-zero modulus, then $|\gamma|^2 < |\beta|$ unless $|\gamma| \ge 1$. Hence all non-zero roots have modulus 1.
All roots zero is possible and gives (a, b, c) = (0, 0, 0).

Suppose β = γ = 0 and α is non-zero. Then $\alpha^2 = \alpha$, so α = 1. This gives (a, b, c) = (-1, 0, 0).

Suppose γ = 0, and α, b are non-zero. If $^{\alpha 2} = \alpha$, then α = 1. Then $\beta^2 = 1$ or β, so β = 1 or -1. This gives (a, b, c) = (-2, 1, 0) or (0, -1, 0). If $\alpha^2$ is not α (and $\beta^2$ not β), then we must have $\alpha^4 = \alpha$, so α and β are ω and $\omega^2$, where $\omega = e^{2i\pi/3}$. This gives (a, b, c) = (1, 1, 1).

The remaining possibility is that all roots are non-zero (and have modulus 1). There might be 3, 2, 1 or 0 roots equal to 1. The first case gives (-3, 3, -1). In the second case the third root must satisfy $\gamma^2 = 1$ or $\gamma$ and hence must be -1, giving (a, b, c) = (-1, -1, 1).

We consider the third case (just one root equal to 1). If both the roots not equal to 1 have square 1, then they must both be -1 and (a, b, c) = (1, -1, -1). If one has square 1 and one not. Then they must be -1 and $\pm i$. This gives (a, b, c) = (i, 1, i) or (-i, 1, -i). If neither have square 1, then we must have $\beta^2 = \gamma$, $\gamma^2 = \beta$, hence $\beta^3 = 1$ and $\beta$ and $\gamma$ are $\omega$ and $\omega^2$.

The final case is that no roots are 1. If we have $\alpha^2 = \beta$ and $\beta^2 = \alpha$, then $\alpha$ and $\beta$ are $\omega$ and $\omega^2$ and $\gamma^2 = \omega$ or $\omega^2$, so $\gamma = \pm\omega$ or $\pm\omega^2$. So the roots are $\omega, \omega, \omega^2$ or $\omega, \omega^2, \omega^2$ or $\omega, -\omega, \omega^2$ or $\omega, \omega^2, -\omega^2$. This gives (a, b, c) = $(1-\omega, 1-\omega, \omega)$ or $(1-\omega^2, 1-\omega^2, \omega^2)$ or $(1+\omega, 1+\omega, \omega)$ or $(1+\omega^2, 1+\omega^2, \omega^2)$.

The final possibility is $\alpha^2 = \beta$, $\beta^2 = \gamma$, $\gamma^2 = \alpha$. Hence $\alpha^7 = 1$. If we put $k = e^{i2\pi/7}$, then the roots could be $k, k^2, k^4$ or $k^6, k^5, k^3$. This corresponds to (a, b, c) = $((1-i\sqrt{7})/2, (-1-i\sqrt{7})/2, -1)$, $((1+i\sqrt{7})/2, (-1+i\sqrt{7})/2, -1)$.

**Problem A4**
A map represents the polar cap from latitudes -45° to 90°. The pole (latitude 90°) is at the center of the map and lines of latitude on the globe are represented as concentric circles with radii proportional to (90° - latitude). How are east-west distances exaggerated compared to north-south distances on the map at a latitude of -30°?

**Solution**
Answer: too high by a factor $4\pi/\sqrt{27} = 2.42$.

Let the globe have radius R. Then the distance from the pole to the circle of latitude at -30 is $2\pi R/3$. The circumference of the circle is $2\pi R (\sqrt{3})/2$. The ratio of circumference to distance is $(\sqrt{27})/2$. On the map the corresponding ratio is $2\pi$. Thus the map overstates by a factor $4\pi/\sqrt{27}$.

**Problem A5**
$a_i$ are reals $\neq 1$. Let $b_n = 1 - a_n$. Show that $a_1 + a_2b_1 + a_3b_1b_2 + a_4b_1b_2b_3 + ... + a_nb_1b_2 ... b_{n-1} = 1 - b_1b_2 ... b_n$.

**Solution**
Induction on n. Obvious for n = 1. Suppose true for n. Then $a_1 + a_2b_1 + a_3b_1b_2 + a_4b_1b_2b_3 + ... + a_{n+1}b_1b_2 ... b_n = 1 - b_1b_2 ... b_n + a_{n+1}b_1b_2 ... b_n = 1 - b_1b_2 ... b_n(1 - a_{n+1}) = 1 - b_1 ... b_{n+1}$, which is the result for n+1.

**Problem A6**
Prove that there are only finitely many cuboidal blocks with integer sides a x b x c, such that if the block is painted on the outside and then cut into unit cubes, exactly half the cubes have no face painted.

**Solution**
It is sufficient to show that abc = 2(a - 2)(b -2)(c - 2) has only finitely many solutions in integers $c \le b \le a$. If $c \le 4$, then $c \ge 2(c - 2)$, but $ab > (a - 2)(b - 2)$, so there are no solutions. If $c \ge 10$, then $c/(c - 2) \le 10/8$, so $D \le 125/64 < 2$, where for convenience we have written abc/( (a - 2)(b - 2)(c - 2) ) as D, so there are no solutions. Hence c = 5, 6, 7, 8, or 9.

If c = 5 and $b \ge 24$, then $D \le 5/3\ (24/22)^2 < 2$, so there are no solutions. Now a/(a - 2) is strictly monotonic decreasing, so there is at most one solution for given b, c. Hence there are only finitely many solutions for c = 5.

Similarly, for c = 6, we find that are only solutions for b < 16; for c = 7, we find b < 14; for c = 8, b < 12 and for c = 9, b < 12. Hence there are only finitely many solutions in each case.

**Problem A7**
Let O be the center of a circle C and $P_0$ a point on the circle. Take points $P_n$ on the circle such that angle $P_nOP_{n-1}$ = +1 for all integers n. Given that $\pi$ is irrational, show that given any two distinct points P, Q on C, the (shorter) arc PQ contains a point $P_n$.

**Solution**
All the points must be distinct, for if we had $P_n = P_m$ for $n < m$, then the integer $m - n$ would be a multiple of $2\pi$ and hence $\pi$ would be rational.

Choose N so that the arc length $PQ < 2\pi R/N$, where R is the radius of the circle. Divide the circle into N equal arcs length $2\pi R/N$. Then at least two of the points $P_1, P_2, \dots, P_{N+1}$ must lie in the same arc. Suppose the points are $P_n$ and $P_{n+m}$ (with m, n positive), so that $P_nP_{n+m}$ is an arc of length less than $2\pi R/N$. Hence the arcs $P_{n+m}P_{n+2m}$, $P_{n+2m}P_{n+3m}$ ... have the same length. So for some k, $P_{n+km}$ will lie in the arc PQ.

**Problem B1**
ABC is a triangle with, as usual, AB = c, CA = b. Find necessary and sufficient conditions for $b^2c^2/(2bc \cos A) = b^2 + c^2 - 2bc \cos A$.

**Solution**
Answer: BC = AB or CA.

We have that $a^2 = b^2 + c^2 - 2bc \cos A$. So the condition implies that $(b^2 + c^2 - a^2)/(2bc) = \cos A = bc/(2a^2)$. Hence, $b^2c^2 = a^2(b^2 + c^2 - a^2)$, so $(a^2 - b^2)(a^2 - c^2) = 0$. Hence a = b or c.

Conversely, if a = b, then the altitude from C meets AB at its midpoint and so $\cos A = \cos B = c/(2a) = bc/(2a^2)$. Hence $a^2 = bc/(2 \cos A)$, so $(b^2 + c^2 - 2bc \cos A) = b^2c^2/(2bc \cos A)$.

**Problem B2**
Find the surface comprising the curves which satisfy $dx/(yz) = dy/(zx) = dz/(xy)$ and which meet the circle $x = 0$, $y^2 + z^2 = 1$.

**Solution**
We have $x\,dx = y\,dy = z\,dz$. Integrating, $y^2 = x^2 + h$, $z^2 = x^2 + k$ (*). We are told that some point of the circle $x = 0$, $y^2 + z^2 = 1$ belongs to the curve, so h and k must be non-negative with sum 1. Hence the curve (*) lies in the surface $2x^2 + 1 = y^2 + z^2$, which is a one-sheet hyperboloid with the x-axis as an axis of symmetry. However, not all points of this surface lie on a curve (*). Clearly we require $|y| >= |x|$ and $|z| \ge |x|$. Equally, it is clear that this is a sufficient condition for we can then find h, k to satisfy (*).

**Problem B3**
Let A(x) = be the matrix

```
 0      a-x    b-x
-a-x     0     c-x
-b-x   -c-x     0
```

For which (a, b, c) does det A(x) = 0 have a repeated root in x?

**Solution**
Multiplying out, we find that $\det A(x) = -2x^3 + 2(bc - ac + ab)x$. This has roots 0 and $\pm\sqrt{(ab + bc - ac)}$. This has a repeated root iff $ac = ab + bc$.

**Problem B4**
The solid S consists of a circular cylinder radius r, height h, with a hemispherical cap at one end. S is placed with the center of the cap on the table and the axis of the cylinder vertical. For some k, equilibrium is stable if $r/h > k$, unstable if $r/h < k$ and neutral if $r/h = k$. Find k and show that if $r/h = k$, then the body is in equilibrium if any point of the cap is in contact with the table.

**Solution**
Let O be the centre of the base of the hemispherical cap. As the solid rolls, O is always a height r above the table. Consider a point P on the axis a distance d from O and inside the cap. If the solid is rolled to a position where the axis is at an angle $\theta$ to the vertical, then P ends up a vertical distance $d \cos \theta$ below O, so its height has increased. Similarly, a point on the axis a distance d above O would end up a vertical distance $d \cos \theta$ above O, so its height

would decrease. So equilibrium will be stable, neutral or unstable according as the centre of mass is below, at or above O.

We start by finding the position of the centre of mass of the cap. Take the x-axis along the axis with the origin at O. Assume the density is ρ. The moment about an axis perpendicular to the x-axis is $\int_0^r \pi(r^2 - x^2)\rho\, x\, dx = \pi \rho r^4/4$. The centre of mass of the cylindrical portion of the solid is obviously a distance h/2 from O, so we require $(\pi \rho r^2 h) h/2 = \pi \rho r^4/4$ or $r/h = \sqrt{2}$ get get the centre of mass at O. Thus $k = \sqrt{2}$.

If the centre of mass is at O, then the centre of mass is always at the same height above the table, so the solid is in equilibrium however far it is tilted from the vertical.

**Problem B5**
The sequence $a_n$ is monotonic and $\sum a_n$ converges. Show that $\sum n(a_n - a_{n+1})$ converges.

**Solution**
The sum of the first n terms is $(a_1 - a_2) + 2(a_2 - a_3) + ... + n(a_n - a_{n+1}) = a_1 + a_2 + a_3 + ... + a_n - n a_{n+1}$. We are given that $\sum a_n$ converges, so it is sufficient to show that the sequence $n a_{n+1}$ converges to zero.

But since $\sum a_n$ converges, $|a_{n+1} + a_{n+2} + ... + a_{2n}|$ is arbitrarily small for n sufficiently large. Since $a_n$ is monotonic, this implies that $n a_{n+1}$ is arbitrarily small for n sufficiently large.

**Problem B6**
A, B, C are points of a fixed ellipse E. Show that the area of ABC is a maximum iff the centroid of ABC is at the center of E.

**Solution**
Take the ellipse as $x^2/a^2 + y^2b^2 = 1$. Take the three points as A (a cos u, b sin u), B (a cos v, b sin v), C (a cos w, b sin w). If the area is a maximum then the tangent at A must be parallel to BC (otherwise we could keep B and C fixed and move A to increase the altitude and hence the area). So we have - b/a cot u = (b sin v - b sin w)/(a cos v - a cos w) and hence cos(u - v) = cos(u - w). Similarly, cos(v - u) = cos(v - w).

Hence 2u = v + w (mod 2π). Similarly 2v = u + w (mod 2π), so 3u = 3v (mod 2π). Hence u, v, w are (in some order and for some k) k, k + 2π/3, k - 2π/3. Hence the centroid is at x = a cos k + a cos(k+2π/3) + a cos(k-2π/3) = 0, y = b sin k + b sin(k+2π/3) + b sin(k-2π/3) = 0.

Conversely, if the centroid of A, B, C is at the centre, then cos u + cos v + cos w = 0, sin u + sin v + sin w = 0. Hence cos(u - v) = cos u (- cos u - cos w) + sin u (- sin u - sin w) = -1 - cos(u - w). Hence cos(u - v) + cos(u - w) = -1. Similarly, cos(v - u) + cos(v - w) = -1, hence cos(u - w) = cos(v - w). So cos w (cos u - cos v) = sin w (sin v - sin u), so -b/a cot w = (b sin v - b sin u)/(a cos v - a cos u), and hence the tangent at C is parallel to AB. Simiarly, the tangent at A is parallel to BC and the tangent at B is parallel to AC. Hence the area is a maximum.

**Problem B7**
Let R be the reals. Define $a_n$ by $a_1 = \alpha \in R$, $a_{n+1} = \cos a_n$. Show that $a_n$ converges to a limit independent of α.

**Solution**
$|\cos x| \le 1$, so $-1 \le a_2 \le 1$. Hence $0 < a_3 < 1$ and so $0 < a_n < 1$ for $n \ge 3$.

cos x is strictly monotonic decreasing over the range 0 to 1, so if $a_n < a_{n+1}$ then $a_{n+2} < a_{n+1}$. Now the gradient of cos x is greater than -1 throughout the interval (0, 1), so if $a_n < a_{n+1}$ then $\cos a_n - \cos a_{n+1} < a_{n+1} - a_n$. Hence $a_n < a_{n+2} < a_{n+1}$. Similarly, if $a_n > a_{n+1}$, then $a_n > a_{n+2} > a_{n+1}$. Thus either $a_{2n}$ is an increasing sequence bounded above and $a_{2n+1}$ is a decreasing sequence bounded below or vice versa. Hence both $a_{2n}$ and $a_{2n+1}$ converge.

By the mean value theorem $|a_{n+1} - a_{n+2}| = |\cos a_n - \cos a_{n+1}| = \sin k\, |a_n - a_{n+1}|$ for some k in (0, 1). But sin k < sin 1 < 0.9. So $a_{2n}$ and $a_{2n+1}$ must converge to the same limit. Suppose this limit is h. Then h = cos h. But cos x is strictly decreasing in (0, 1) and x is strictly increasing, so they have a single point of intersection in (0, 1). Thus h must be the unique point in (0, 1) at which h = cos h.

# 13th Putnam 1953

**Problem A1**
Show that $(2/3)\ n^{3/2} < \sum_1^n \sqrt{r} < (2/3)\ n^{3/2} + (1/2)\ \sqrt{n}$.

**Solution**
The gradient of $\sqrt{x}$ is falling from $x = 0$ to $x = n$, so $2/3\ n^{3/2} = \int_0^n \sqrt{x}\ dx < \sqrt{1} + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n}$. That gives the first inequality.

To get the second, we note that the chords joining $(r, \sqrt{r})$ to $(r+1, \sqrt{(r+1)}\ )$ all lie under the curve, so if we subtract the area of the little triangles from $\sum_1^n \sqrt{r}$ then we get something less than the integral. The triangles have area $1/2\ \sqrt{1} + 1/2\ (\sqrt{2} - \sqrt{1}) + 1/2\ (\sqrt{3} - \sqrt{2}) + \dots + 1/2\ (\sqrt{n} - \sqrt{(n-1)}\ ) = 1/2\ \sqrt{n}$. That gives the second inequality.

**Problem A2**
The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

**Solution**
Take any point A. It has 5 edges, so at least 3 of them must be the same color. wlog it is red. So we have B, C, D with AB, AC, AD all red. Now if any of the three edges BC, CD, DB is red, then that gives us a red triangle. But if they are all blue, then BCD is a blue triangle.

**Problem A3**
a, b, c are real, and the sum of any two is greater than the third. Show that $2(a + b + c)(a^2 + b^2 + c^2)/3 > a^3 + b^3 + c^3 + abc$.

**Solution**
wlog we may take $a \ge b \ge c$. c (and hence also a and b) must be positive. For if $c \le 0$, then $a \ge b + c$. Contradiction.

Multiplying out the required relation and cancelling, we must prove that $2(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) > a^3 + b^3 + c^3 + 3abc$. But we have $a^2(b + c) > a^3$, $b^2(c + a) > b^3$, $c^2(a + b) > c^3$. So it is sufficient to prove that $a^2b + a^2c + b^2a + b^2c + c^2a + c^2b > 3abc$. But $a^2b \ge abc$ and $ab^2 \ge abc$. Also $b^2c + bc^2 = bc(b + c) > abc$. Similarly, $ac(a + c) > abc$. So we have in fact proved the slightly stronger result with abc replaced by 4abc/3.

**Problem A4**
Using $\sin x = 2 \sin x/2 \cos x/2$ or otherwise, find $\int_0^{\pi/2} \ln \sin x\ dx$.

**Solution**
Let $I = \int_0^{\pi/2} \ln \sin x\ dx$. Making the suggested substitution, we get $I = \pi/2 \ln 2 + \int_0^{\pi/2} \ln \sin x/2 \cos x/2\ dx$. Putting $y = x/2$ we get $\int_0^{\pi/2} \ln \sin x/2 \cos x/2\ dx = 2 \int_0^{\pi/4} \ln \sin y + \ln \cos y\ dy$. But $\cos y = \sin(\pi/2 - y)$, so $\int_0^{\pi/4} \ln \cos y\ dy = \int_{\pi/4}^{\pi/2} \ln \sin y\ dy$. Hence $I = \pi/2 \ln 2 + 2I$, so $I = - \pi/2 \ln 2$.

**Problem A5**
S is a parabola with focus F and axis L. Three distinct normals to S pass through P. Show that the sum of the angles which these make with L less the angle which PF makes with L is a multiple of $\pi$.

**Solution**
We start by finding a formula for $\tan(A+B+C+D)$. Applying $\tan(A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$ twice and writing $a = \tan A$, $b = \tan B$, $c = \tan C$, $d = \text{tand } D$, we get $\tan(A+B+C+D) = (\sum a - \sum abc)/(1 - \sum ab + abcd)$.

Take the parabola as $y = x^2$. The gradient at $(k, k^2)$ is $2k$, so the normal has slope $-1/2k$. The normal has equation $2k^3 + (1 - 2y)k - x = 0$. For given x, y this is a cubic in k, so has 1 or real 3 roots. Their slopes satisfy the cubic $4x\ h^3 + 2(1 - 2y)\ h^2 + 1 = 0$ (*).

The focus is at $(0, 1/4)$, so the line joining $(x, y)$ has slope $(y - 1/4)/x$ and negative is $(1/4 - y)/x$. The quartic with this root and the 3 roots of (*) is $(h + (y - 1/4)/x\ )(4x\ h^3 + 2(1 - 2y)\ h^2 + 1)$ (**). So the angles the normals make with L less the angle which PF makes with L is a multiple of $\pi$ iff $(\sum a = \sum abc)$, where a, b, c, d are the slopes of the four lines and hence iff the coefficients of $h^3$ and h in (**) are the same. But the coefficient of $h^3$ is $2(1 - 2y) + 4x (y - 1/4)/x = 1$ and the coefficient of h is 1.

**Problem A6**

Show that $\sqrt{7}$, $\sqrt{7 - \sqrt{7}}$, $\sqrt{7 - \sqrt{7 + \sqrt{7}}}$, $\sqrt{7 - \sqrt{7 + \sqrt{7 - \sqrt{7}}}}$, ... converges and find its limit.

**Solution**

Answer: 2.

Let $x_n$ be the nth number in the sequence. We have $x_{n+2} = \sqrt{7 - \sqrt{7 + x_n}}$. Hence if $x_n < 2$, then $x_{n+2} > 2$, and if $x_n > 2$, then $x_{n+2} < 2$. So if the sequence converges, its limit must be 2.

If $x_n = 2 + \varepsilon$, with $0 < \varepsilon < 1$, then $9 + \varepsilon < 9 + \varepsilon + \varepsilon^2/36 = (3 + \varepsilon/6)^2$, so $x_{n+2} > \sqrt{7 - (3 + \varepsilon/6)} = \sqrt{4 - \varepsilon/6}$. But certainly $\varepsilon/6 < \varepsilon/3 - \varepsilon^2/144$, so $\sqrt{4 - \varepsilon/6} > 2 - \varepsilon/12$. Thus $x_{n+2}$ differs from 2 by less than $\varepsilon/12$. Similarly, if $x_n = 2 - \varepsilon$, with $0 < \varepsilon < 1$, then $9 - \varepsilon > 9 - \varepsilon/2 + \varepsilon^2/144 = (3 - \varepsilon/12)^2$, so $x_{n+2} < \sqrt{4 + \varepsilon/12} < 2 + \varepsilon/48$.

So we have established that if $|x_n - 2| < 1$, then $|x_{n+2} - 2| < 1/12\ |x_n - 2|$. But we certainly have $|x_1 - 2| < 1$, and $|x_2 - 2| < 1$, so $x_n$ converges to 2.

**Problem A7**

$p(x) \equiv x^3 + ax^2 + bx + c$ has three positive real roots. Find a necessary and sufficient condition on a, b, c for the roots to be cos A, cos B, cos C for some triangle ABC.

**Solution**

If $A + B + C = 180^o$, then $\cos A = - \cos(B + C) = \sin B \sin C - \cos B \cos C$. Squaring, we get $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$. So a necessary condition is $a^2 - 2b - 2c = 1$.

Conversely, suppose that this condition holds. Then if the roots are p, q, r, we have $p^2 + q^2 + r^2 + 2pqr = 1$ (*). We are given that the roots are all positive, so $2pqr > 0$, hence $p^2 < 1$ and so $p < 1$. Similarly for q and r. So we can find angles A, B, C greater than 0 and less than $90^o$ such that $p = \cos A$, $q = \cos B$, $r = \cos C$. Now we can rewrite (*) as $(1 - \cos^2 B)(1 - \cos^2 C) = \cos^2 A + 2 \cos A \cos B \cos C + \cos^2 B \cos^2 C = (\cos A + \cos B \cos C)^2$. But $(1 - \cos^2 B) = \sin^2 B$, $(1 - \cos C) = \sin^2 C$, so we have $\sin B \sin C = \pm(\cos A + \cos B \cos C)$. But we know that A, B, C are between 0 and $90^o$, so cos A, cos B, cos C, sin B, sin C are all positive. Hence we must use the + sign and we have $\cos A = \sin B \sin C - \cos B \cos C$, so $A + B + C = 180^o$. Hence the condition is also sufficient.

**Problem B1**

Does $\sum_1^\infty 1/n^{1 + 1/n}$ converge?

**Solution**

Answer: no.

$x < e^x$ for all $x \ge 0$, so $x^{1/x} < e < 3$. Hence $1/n^{1 + 1/n} > 1/3n$. But $\sum 1/n$ diverges.

**Problem B2**

p(x) is a real polynomial of degree n such that p(m) is integral for all integers m. Show that if k is a coefficient of p(x), then n! k is an integer.

**Solution**

Note that n! is best possible, because $1/n!\ x(x + 1) \dots (x + n - 1)$ is always integral for integral x (it is the binomial coefficient (x+n-1)Cn ).

We need a standard result from the calculus of differences. Let $\Delta f(x) = f(x + 1) - f(x)$. Then $p(x) = p(0) + \Delta_1 x + 1/2!\ \Delta_2\ x(x - 1) + 1/3!\ \Delta_3\ x(x - 1)(x - 2) + \dots + 1/n!\ \Delta_n\ x(x - 1) \dots (x - n + 1)$ (*), where $\Delta_m = \Delta^m p(0)$ (thus $\Delta_1 = p(1) - p(0)$, $\Delta_2 = p(2) - 2p(1) + p(0)$ etc).

Assume this is true. Then if p(x) is integral for all integral x, all the $\Delta_m$ must be integral. So the result above gives n! p(x) = an integral combination of integral polynomials. Hence all the coefficients of n! p(x) are integral.

To prove the result, notice first that if $f(x) = x(x - 1)(x - 2) \dots (x - m + 1)$, then $\Delta^r f(0) = m!$ for r = m and 0 otherwise. So if we call the rhs of (*) q(x) , then $\Delta^m q(0)$ is a sum of terms which are all zero except for $\Delta^m(1/m!$

$\Delta^m$ x(x - 1) ... (x - m + 1) )(0) = $\Delta_m$. Hence $\Delta^m$p(0) = $\Delta^m$q(0) for m = 1, 2, ... , n. Also p(0) = q(0), so by a simple induction p(m) = q(m) for m = 0, 1, 2, ... , n. But q(x) is a polynomial of degree at most n. If polyomials of degree at most n agree at n+1 points, then they must be identical. Hence p(x) = q(x).

**Problem B3**
k is real. Solve the differential equations $y' = z(y + z)^k$, $z' = y(y + z)^k$ subject to y(0) = 1, z(0) = 0.

**Solution**
Adding, $(y + z)' = (y + z)^{k+1}$. Integrating $(y + z)^k = 1/(1 - kx)$ (1).

Multiplying opposite sides of the two given equations together, we get $yy'(y + z)^k = zz' (y + z)^k$. Hence yy' = zz'. Integrating $y^2 - z^2 = 1$. Hence $(y + z)^k(y - z)^k = 1$, so $(y - z)^k = 1 - kx$ (2).

For k non-zero, we can immediately solve (1) and (2) to get $y = 1/2 ( (1 - kx)^{1/k} + 1/(1 - kx)^{1/k} )$, $z = 1/2 ( (1 - kx)^{1/k} - 1/(1 - kx)^{1/k} )$.

For k = 0, the original equations simplify to y' = z, z'= y. So y" = y. So y = A cosh x + B sinh x, z = A cosh x + B sinh x. Applying the initial conditions, y = cosh x, z = sinh x.

**Problem B4**
R is the reals. S is a surface in $R^3$ containing the point (1, 1, 1) such that the tangent plane at any point $P \in S$ cuts the axes at three points whose orthocenter is P. Find the equation of S.

**Solution**
Consider a plane cutting the axes at **a** = (a, 0, 0), **b** = (0, b, 0), **c** = (0, 0, c). If the orthocentre is at **p** = (x, y, z), then we have (**p** - **a**).(**b** - **c**) = (**p** - **b**).(**a** - **c**) = 0. But **a**.**b** = **b**.**c** = **c**.**a** = 0, so we have **p**.(**b** - **c**) = **p**.(**a** - **c**) = 0. In other words the line from the origin (0, 0, 0) to (x, y, z) is normal to the plane. So the surface satisfies the condition that all its normals pass through the origin and it passes through (1, 1, 1). This implies that it is the sphere $x^2 + y^2 + z^2 = 3$.

Note, however, that for points with one coordinate zero, the tangent plane will meet one axis at infinity, so we should arguable exclude all such points. That divides the sphere into 8 disconnected pieces. The piece containing (1, 1, 1) is that in the positive octant (x > 0, y > 0, z > 0).

**Problem B5**
The coefficients of the complex polynomial $z^4 + az^3 + bz^2 + cz + d$ satisfy $a^2d = c^2 \neq 0$. Show that the ratio of two of the roots equals the ratio of the other two.

**Solution**
We start with a straight slog. Let the roots be p, q, r, s. We have $a^2d - c^2 = (p + q + r + s)^2pqrs - (pqr + pqs + prs + qrs)^2$. The terms like $2p^2q^2rs$ all cancel, leaving $p^3qrs + pq^3rs + pqr^3s + pqrs^3 - p^2q^2r^2 - p^2q^2s^2 - p^2r^2s^2 - q^2r^2s^2$.

The trick now is to factorise this. We might suspect that pq - rs is a factor. But in that case pr - qs and ps - qr would presumably also be factors. (pq - rs)(pr - qs)(ps - qr) has degree 6, as required. It also has the correct number of terms (8). So we try multiplying it out and find that it is the same.

So (pq - rs)(pr - qs)(ps - qr) = 0. But that means that at least one factor must be zero, which gives the result. Note that the only reason for giving us that $a^2d$ is non-zero, is because that implies that none of the roots are zero (their product d is non-zero) and so having got pq = rs, we can divide to get p/r = s/q.

**Problem B6**
A and B are equidistant from O. Given k > OA, find the point P in the plane OAB such that OP = k and PA + PB is a minimum.

**Solution**
Let C be the circle centre O radius k. Take A' on the ray OA such that $OA \cdot OA' = k^2$ and B' on the ray OB such that $OB \cdot OB' = k^2$. If A'B' intersects the circle C, then the points of intersection give the positions where PA + PB is a

minimum. If not then the nearest point of C to A'B' (which is on the perpendicular bisector of AB) gives the minimum.

For given P on C, the triangles OAP and OPA' are similar, so PA = (AO/PO) PA'. Similarly, PB = (BO/PO) PB', so PA + PB = (AO/k) (PA' + PB'), so minimising PA + PB is equivalent to minimising PA' + PB'. If A'B' intersects C, then clearly the points of intersection minimise. If not, let Q be the point on C closest to A'B'. Let L be the tangent to the circle at that point (so that L is parallel to A'B'). Then QA + QB < RA + RB for other points R on L. Given another point P on C, take the perpendicular to L through P. If it intersects L at R, then PA + PB > RA + RB > QA + QB.

**Problem B7**
Show that we can express any irrational number $\alpha \in (0, 1)$ uniquely in the form $\sum_1^\infty (-1)^{n+1}\ 1/(a_1a_2 \dots a_n)$, where $a_i$ is a strictly monotonic increasing sequence of positive integers. Find $a_1$, $a_2$, $a_3$ for $\alpha = 1/\sqrt{2}$.

**Solution**
Answer: $a_1 = 1$, $a_2 = 3$, $a_3 = 8$.

Let $s_n$ be the sum of the first n terms. The terms alternate in sign and decrease in absolute value, so the odd terms of the sequence $s_n$ decrease and the even terms increase. Every odd term exceeds every even term, so the odds and the evens must each converge. But $s_n - s_{n+1} < 1/2^n$ which tends to zero, so they tend to a common limit.

Choose $a_n$ as follows. Take $a_1$ to be the smallest integer whose inverse exceeds $\alpha$. Having chosen $a_{2n-1}$, take $a_{2n}$ to be the largest integer such that $s_{2n} < \alpha$. Having chosen $a_{2n}$, take $a_{2n+1}$ to be the largest integer such that $s_{2n+1} > \alpha$.

We have to show that these choices are always possible, or, in other words, that they yield a strictly increasing sequence $a_n$. This depends on the relation $1/k - 1/k(k+1) = 1/(k+1)$ (*). For suppose we have just chosen $a_{2n-1}$. Then we know that $s_{2n-1} > \alpha$, but that if increased $a_{2n-1}$ by 1, then $s_{2n-1}$ would be $< \alpha$. Hence, using (*), taking $a_{2n} = a_{2n-1} + 1$ certainly gives $s_{2n} < \alpha$. On the other hand, if we take $a_{2n}$ to be sufficiently large, then $s_{2n}$ will be close to $s_{2n-1}$ and hence exceed $\alpha$ (note that $\alpha$ is irrational so it cannot equal any $s_m$). So the choice of $a_{2n}$ will exceed $a_{2n-1}$. A similar argument shows that $a_{2n+1}$ exceeds $a_{2n}$.

So we have established that we can find a sequence $a_n$ such that all the odd partial sums $s_n$ exceed $\alpha$ and all the even partial sums are less than $\alpha$. But we have also established that $s_n$ tends to a limit, so that limit must be $\alpha$. That establishes existence.

Suppose there is another expansion so that $\alpha = 1/a_1 - 1/a_1a_2 + \dots = 1/b_1 - 1/b_1b_2 + \dots$ . As above, we have that $1/(a_1 + 1) < \alpha < 1/a_1$ and also $1/(b_1 + 1) < \alpha < 1/b_1$. But since $a_1$ and $b_1$ are both integers that implies that $a_1 = b_1$. Suppose now that we have established that $a_i = b_i$ for $i \le n$. Then we have that $\beta = (-1)^n a_1 \dots a_n(\alpha - 1/a_1 + 1/a_1a_2 - \dots + (-1)^n/a_1 \dots a_n ) = 1/a_{n+1} - 1/a_{n+1}a_{n+2} + \dots$ . But we also have $\beta = 1/b_{n+1} - 1/b_{n+1}b_{n+2} + \dots$ . We now argue as before that $\beta$ lies between $1/(a_{n+1} + 1)$ and $1/a_{n+1}$ and also between $1/(b_{n+1} + 1)$ and $1/b_{n+1}$. Hence $a_{n+1} = b_{n+1}$. That establishes uniqueness.

Finally, consider $\alpha = 1/\sqrt{2}$. We have $1/2 < 1/\sqrt{2} < 1$, so $a_1 = 1$. We must pick $a_2$ as the largest integer so that $1 - 1/a_2 < 1/\sqrt{2}$, or $a_2 < 2 + \sqrt{2} = 3.4$. So $a_2 = 3$. We must pick $a_3$ as the largest integer so that $1 - 1/3 + 1/3a_3 > 1/\sqrt{2}$ or $2 + 1/x > 3/\sqrt{2}$ or $x < 3\sqrt{2} + 4 = 8.2$. So $a_3 = 8$.

## 14th Putnam 1954

**Problem A1**
Let N be the set $\{1, 2, \ldots, n\}$, where n is an odd integer. Let $f : N \times N \to N$ satisfy: (1) $f(r, s) = f(s, r)$ for all r, s; (2) $\{f(r, s) : s \in N\} = N$ for each r. Show that $\{f(r, r) : r \in N\} = N$.

**Solution**
We have a square array of numbers $a_{ij} = f(i, j)$. Each member of N occurs just once in each row (by (2) ), so it occurs n times in all. But the array is symmetric (by (1) ), so each member occurs an even number of times off the main diagonal. Hence it must occur an odd number of times, and hence at least once, on the main diagonal. But the main diagonal only has n entries, so if each of n numbers occurs at least once, then each must occur exactly once.

**Problem A2**
Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than $k = 1/\sqrt{2}$. Is this result true for a smaller k?

**Solution**
Let the square be ABCD and O its midpoint. Let the midpoints of AB, BC, CD, DA be P, Q, R, S respectively. Then ABCD is the union of the 4 smaller squares: APOS, BQOP, CROQ, DSOR, each with diameter k. At least one of the smaller squares must contain two points. So the two points are a distance apart $\leq k$. However, they cannot be a distance k apart, because the only pairs of points in a square realizing the diameter distance are opposite corners and they are not in the interior of ABCD. Hence the two points are a distance $< k$.
Take one point at O and the others on the main diagonals a distance $\varepsilon$ from the corners. Then shortest distance between two points is $k - \varepsilon$ which can be made arbitarily close to k.

**Problem A3**
Let S be the set of all curves satisfying $y' + a(x)\, y = b(x)$, where $a(x)$ and $b(x)$ are never zero. Show that if $C \in S$, then the tangent at the point $x = k$ on C passes through a point $P_k$ which is independent of C.

**Solution**
Let $C_h$ be the curve with $y(k) = h$. Then the gradient of $C_h$ at (k, h) is $b(k) - a(k)\, h$, so the tangent at (k, h) is $(y - h) = (b(k) - a(k)\, h)(x - k)$. This may be written as $(a(k)\, x - a(k)\, k - 1)\, h = (b(k)\, (x - k) - y)$. Evidently, this always passes through the point $(k + 1/a(k), b(k)/a(k))$ whatever the value of h.

**Problem A4**
A uniform rod length 2a is suspended in midair with one end resting against a smooth vertical wall at X and the other end attached by a string length 2b to a point on the wall above X. For what angles between the rod and the string is equilibrium possible?

**Solution**
Answer: always 0; additionally, if $b > a > b/2$, $\cos^{-1}((b^2 + 2a^2)/(3ab))$.

Let the other end of the rod be Y. Let the string be ZY (with Z on the wall) and M its midpoint. The line along which the rod's weight acts passes through the midpoint of the rod and hence also through M. So a necessary condition for equilibrium is that the normal force at X also passes through M, in other words that MX is normal to the wall.
By the cosine rule: $MX^2 = b^2 + 4a^2 - 4ab \cos\theta$ (where $\theta$ = angle XYZ). Similarly, $XZ^2 = 4b^2 + 4a^2 - 8ab \cos\theta$. But $MX^2 + XZ^2 = b^2$, so solving, $\cos\theta = (b^2 + 2a^2)/(3ab)$ (*).
Hence this is also a sufficient condition for equilibrium, because the geometry is now fixed. So we can solve for the tension in terms of the weight (resolving vertically) and then for the normal force (resolving horizontally). Hence the three forces have vector sum zero and the rod is in equilibrium.
Returning to (*), we know that $0 < a < b$. Put $x = a/b$, then $\cos\theta = (1 + 2x^2)/(3x)$. But it is easy to see that for $(1 + 2x^2)/(3x) < 1$, we require $x > 1/2$.

**Problem A5**
R is the reals. $f : (0, 1) \to R$ satisfies $\lim_{x\to 0} f(x) = 0$, and $f(x) - f(x/2) = o(x)$ as $x\to 0$. Show that $f(x) = o(x)$ as $x\to 0$.

**Solution**
We wish to show that given $\delta > 0$, $|f(x)| < x\,\delta$ for sufficiently small x. Certainly we can find $\varepsilon > 0$, such that for $x < \varepsilon$, $|f(x/2^r) - f(x/2^{r+1})| < x/2^{r+2}\,\delta$. Summing, $|f(x)| < x\,\delta\,(1/4 + 1/8 + ... ) + |f(x/2^n)| = x\,\delta/2 + |f(x/2^n)|$. But since $f(x/2^n)$ tends to zero, for any given x we can take n sufficiently large that $|f(x/2^n)| < x\,\delta/2$ and hence $|f(x)| < x\,\delta$ as required.

**Problem A6**
The real sequence $a_n$ satisfies $a_n = \sum_{n+1}^{\infty} a_k^2$. Show $\sum a_n$ does not converge unless all $a_n$ are zero.

**Solution**
Clearly $a_n \ge 0$. If any $a_n = 0$, then all subsequent $a_i$ must be zero, and, by a trivial induction, all previous $a_i$. So assume no $a_n = 0$.
Notice that $a_{n+1} = a_n^2 + a_n$. But we have just shown that $a_n^2 > 0$, so $a_{n+1} > a_n$.
If the sum converges, then we can take n sufficiently large that $a_{n+1} + a_{n+2} + a_{n+3} + ... < 1$. Then $a_n = a_{n+1}^2 + a_{n+2}^2 + a_{n+3}^2 + ... < a_n(a_{n+1} + a_{n+2} + a_{n+3} + ... ) < a_n$. Contradiction. So the sum does not converge.

**Problem A7**
Prove that the equation $m^2 + 3mn - 2n^2 = 122$ has no integral solutions.

**Solution**
If m, n is a solution, then $4m^2 + 12mn - 8n^2 = 488$, so $(2m + 3n)^2 - 17n^2 = 488$, so $(2m + 3n)^2 = 12 \pmod{17}$. But 12 is not a quadratic residue of 17 [check: $0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2 = 0, 1, 4, 9, 16, 8, 2, 15, 13 \pmod{17}$].

**Problem B1**
Show that for any positive integer r, we can find integers m, n such that $m^2 - n^2 = r^3$.

**Solution**
We notice that $m^2 - n^2 = (m + n)(m - n)$. This suggests taking $m + n = r^2$, $m - n = r$. This works: $m = r(r + 1)/2$, $n = r(r - 1)/2$.

**Problem B2**
Let $a_n = \sum_1^n (-1)^{i+1}/i$. Assume that $\lim_{n\to\infty} a_n = k$. Rearrange the terms by taking two positive terms, then one negative term, then another two positive terms, then another negative term and so on. Let $b_n$ be the sum of the first n terms of the rearranged series. Assume that $\lim_{n\to\infty} b_n = h$. Show that $b_{3n} = a_{4n} + a_{2n}/2$, and hence that $h \ne k$.

**Solution**
If we simply write out $a_{4n} = (1 - 1/2 + 1/3 - 1/4 + ... - 1/4n)$ and $a_{2n}/2 = (1/2 - 1/4 + 1/6 - 1/8 + ... - 1/4n)$ and add term by term we find that $a_{4n} + a_{2n}/2 = b_{3n}$.
Taking the limit, we have immediately that $h = 3k/2$. But clearly $k > 0$, since if we group the terms in pairs, each pair is positive. Hence $h \ne k$.

**Problem B3**
Let S be a finite collection of closed intervals on the real line such that any two have a point in common. Prove that the intersection of all the intervals is non-empty.

**Solution**
Let [A, B] be the interval with the largest left-hand endpoint, and let [a, b] be the interval with the smallest right-hand endpoint. Then since [A, B] and [a, b] overlap, we must have $A \le b$, so [A, b] is non-empty.

Now given any interval [x, y] in S, we have $x \le A$ and $y \ge b$, so $[A, b] \subseteq [x, y]$.

**Problem B4**
Let F be a point, and L and D lines, in the plane. Show how to construct the point of intersection (if any) between L and the parabola with focus F and directrix D.

**Solution**
If L and D are parallel and a distance d apart, then take the circle center F radius d. If it meets L, then the point(s) of intersection are the required points. If it does not, then there are none.

So assume L and D meet at O. If F lies on D, then the (degenerate) parabola is the line through F perpendicular to D. If it meets L, then the point of intersection is the required point. So assume F does not lie on D.

If L and D are perpendicular, then there is just one point. Take any point P (except O) on L and let the circle center P radius PO cut the line OF again at Q. Let the line through F parallel to PQ cut L at G. Then G is the required point (because an expansion center O takes the circle center P radius PO into a circle center G radius GO = GF).

So assume L and D are neither perpendicular nor parallel and meet at O. Take D' through O making the same angle with L as D. D and D' divide the plane into 4 sectors. If F lies in one of the two sectors containing L, then the required points exist. There are two required points unless F lies on D', when there is just one. [Any circle with center on L, touching D, must lie entirely within these two sectors, and hence F must also if the points exist.]

Take any point P on L (except O) and draw a circle center P touching D. Let the circle meet the line OF at Q. Let the line through F parallel to PQ cut L at G. Then G is the required point. If F does not lie on D', then there will be two possibilities for Q and hence two possibilities for G.

**Problem B5**
Let R be the reals. Let $f : (-1, 1) \to R$ be a function with a derivative at 0. Let $a_n$ be a sequence in (-1, 0) tending to zero and $b_n$ a sequence in (0, 1) tending to zero. Show that $\lim_{n\to\infty} (f(b_n) - f(a_n)/(b_n - a_n) = f'(0)$.

**Solution**
From the definition of derivative, we have that $( f(b_n) - f(0) )/b_n \to f'(0)$, and $( f(0) - f(a_n) )/-a_n \to f'(0)$. So $f(b_n) - f(0)$ lies within $\varepsilon/2$ of $b_n f'(0)$ for sufficiently large n, and $f(0) - f(a_n)$ lies within $\varepsilon/2$ of $-a_n f'(0)$ for sufficiently large n. Hence, adding, $f(b_n) - f(a_n)$ lies within $\varepsilon$ of $(b_n - a_n)f'(0)$ for sufficiently large n, which is what we need.

**Problem B6**
If x is a positive rational, show that we can find distinct positive integers $a_1, a_2, \ldots , a_n$ such that $x = \sum 1/a_i$.

**Solution**
Rather surprisingly, the simplest possible algorithm - the greedy algorithm - works: take $a_n$ to be the smallest integer not already chosen so that the sum does not exceed x. This terminates after a finite number of steps.

Note that the numbers involved can be quite large even in simple cases. For example: 4 = 1/1 + 1/2 + 1/3 + ... + 1/30 + 1/200 + 1/77706 + 1/16532869712 + 1/3230579689970657935732 + 1/36802906522516375115639735990520502954652700

The key idea is that the numerator of the difference gets smaller each time and hence the process must terminate.

First, we have to get the difference under 1. So take m such that $1 + 1/2 + \ldots + 1/m \le x < 1 + 1/2 + \ldots + 1/(m+1)$. This is possible since the harmonic series diverges. If we have equality, then we are home. If not then the difference $x - (1 + 1/2 + \ldots + 1/m) < 1/(m+1)$.

Let the difference $x - (1/a_1 + 1/a_2 + \ldots + 1/a_n)$ be $d_n = r_n/s_n$ (where $r_n$ and $s_n$ are integers).
So we may assume that we have picked $a_1, \ldots , a_n$ and that $d_n < 1/(a_n + 1)$. We now take $a_{n+1}$ so that $1/a_{n+1} \le d_n < 1/(a_{n+1} - 1)$. Now $d_{n+1} = d_n - 1/a_{n+1} = (r_n a_{n+1} - s_n)/(s_n a_{n+1})$. This expression may not be in lowest terms, but it is certainly *sufficient* to show that $(r_n a_{n+1} - s_n) < r_n$ or $d_n a_{n+1} - 1 < d_n$. But that is true since $d_n(a_{n+1} - 1) < 1$. Finally, we notice that $d_{n+1} < 1/(a_{n+1} + 1)$ is equivalent to $d_n - 1/a_{n+1} < 1/(a_{n+1} + 1)$, which is true since $1/(a_{n+1} - 1) - 1/a_{n+1} < 1/(a_{n+1} + 1)$, since $a_{n+1} > 1$.

**Problem B7**
Let $\alpha$ be a positive real. Let $a_n = \Sigma_1^n (\alpha/n + i/n)^n$. Show that $\lim a_n \in (e^\alpha, e^{\alpha+1})$.

**Solution**
The largest term is $(1 + \alpha/n)^n$ which tends to $e^\alpha$. The next largest is $(1 + (\alpha - 1)/n)^n$ which tends to $e^{\alpha - 1}$ and so on. All terms are positive, so the limit is at least $e^\alpha + e^{\alpha - 1} > e^\alpha$. Also the limit is at most $e^\alpha(1 + 1/e + 1/e^2 + \ldots ) < e^\alpha(1 + 1/2 + 1/4 + \ldots ) = 2e^\alpha < e^{\alpha + 1}$.

## 15th Putnam 1955

**Problem A1**
Prove that if a, b, c are integers and $a\sqrt{2} + b\sqrt{3} + c = 0$, then $a = b = c = 0$.

**Solution**

$\sqrt{3}$ is irrational. For if not, let $\sqrt{3} = r/s$ with r and s having no common factor. But now $3s^2 = r^2$, so 3 divides $r^2$ and hence r. Hence 3 divides $s^2$ and hence s. So r and s have a common factor 3. Contradiction. Similarly $\sqrt{2}$ is irrational.

If $a\sqrt{2} + b\sqrt{3} + c = 0$, then $2a^2 - 3b^2 - c^2 = 2bc\sqrt{3}$. But $\sqrt{3}$ is irrational, so b or c = 0. If b = 0, then $a\sqrt{2} + c = 0$. But $\sqrt{2}$ is irrational, so a = c = 0 also.

Hence c = 0. So $a\sqrt{2} + b\sqrt{3} = 0$. We may take a and b to be relatively prime (divide out any common factor). Squaring: $2a^2 = 3b^2$.

**Problem A2**
O is the center of a regular n-gon $P_1P_2 \dots P_n$ and X is a point outside the n-gon on the line $OP_1$. Show that $XP_1\, XP_2 \dots XP_n + OP_1^n = OX^n$.

**Solution**
Evidently it is sufficient to prove the result for the case $OP_1 = 1$. So represent $P_k$ by the complex number $\omega^k$, where $\omega = e^{i\,2\pi/n}$ an nth root of unity. Represent X by the real number r. Then we have to show that $|r - 1||r - \omega| \dots |r - \omega^{n-1}| = r^n - 1$, but that follows immediately because $r^n - 1 = (r - 1)(r - \omega) \dots (r - \omega^{n-1})$.

**Problem A3**
$a_n$ is a sequence of monotonically decreasing positive terms such that $\sum a_n$ converges. S is the set of all $\sum b_n$, where $b_n$ is a subsequence of $a_n$. Show that S is an interval iff $a_{n-1} \le \sum_n^\infty a_i$ for all n.

**Solution**
The condition is certainly necessary. For suppose $a_{n-1} = k_2 > k_3 = \sum_n^\infty a_i$. Let $k_1 = \sum_1^{n-2} a_i$. Then $k_1$ and $k_1 + k_2$ belong to S, but no number in the non-empty interval $(k_1 + k_3, k_1 + k_2)$ belongs to S, so S cannot be an interval.

Now assume that the condition holds. Let $k = \sum a_n$. We show that $S = [0, k]$. We get the endpoints by taking the subsequence to be the empty set or the whole sequence. So take $h \in (0, k)$. Define $b_n$ to be the earliest member of the sequence not so far chosen such that $\sum_1^n b_i \le h$. This is clearly possible since $a_m \to 0$. If at any point we get equality we are home.

So assume that the resulting $\{b_n\}$ is infinite. Clearly $\sum b_n \le h$. If $\{b_n\}$ is missing infinitely many members of $\{a_n\}$, then given any $\varepsilon > 0$, we can find $a_m < \varepsilon$ missing from the subsequence. But that means that for some n, we rejected $a_m$ because $b_1 + b_2 + \dots + b_n + a_m > h$. So $\sum b_i \ge b_1 + b_2 + \dots + b_n > h - \varepsilon$. Hence $\sum b_i = h$.

It remains to consider the case where only finitely many members are missing from the subsequence. Let $a_m$ be the largest such. Then for some n we have that $b_1 + b_2 + \dots + b_n + a_m > h$. We also have that $b_1 + b_2 + \dots + b_n + \sum_{m+1}^\infty a_i \le h$. But $a_m \le \sum_{m+1}^\infty a_i$. So we have a contradiction and this case cannot occur.

**Problem A4**
n vertices are taken on a circle and all possible chords are drawn. No three chords are concurrent (except at a vertex). How many points of intersection are there (excluding vertices)?

**Solution**
Answer: nC4.

Every 4 vertices correspond to a unique point of intersection (the intersection of the two diagonals defined by the 4 points).

[Actually, I got this the hard way. Take one of the vertices. Consider the diagonals from it. The diagonal to the next vertex but one has one vertex on one side and (n - 3) on the other. So it has 1(n - 3) points of intersection on it. The next diagonal has 2(n - 4) and so on up to (n - 3)1. We repeat for each vertex. That gives a total of n[1(n - 3) + 2(n - 4) + ... + (n - 3)1] points of intersection. But we count each diagonal twice and then each point of intersection twice. So the number of points of intersection is $n/4 \sum_1^{n-3} r(n - 2 - r) = n/4$ [ (n - 2) 1/2 (n - 3)(n - 2) - 1/6 (n - 3)(n - 2)(2n - 5) ] = 1/24 n(n - 2)(n - 3) [3(n - 2) - (2n - 5)] = nC4.
Of course, at this point, one wonders why the answer has such a nice form!]

**Problem A5**
Given a parabola, construct the focus (with ruler and compass).

**Solution**
You need to know two facts:
(1) Rays from infinity come to a focus at the focus. In other words, if a line L parallel to the axis of the parabola meets it at X and the focus is F, then L and XF are equally inclined to the normal at X;
(2) The line joining the midpoints of two parallel chords is parallel to the axis.

The construction is then fairly obvious. Take two parallel chords. Join their midpoints and extend to meet the parabola at X. Take a line M through X parallel to the chords. Then M is a tangent. The line perpendicular to M through X is the normal. Hence find the line XF (although not yet F). Repeat for another pair of parallel chords. The two lines interesect at F.

**Problem A6**
For what positive integers n does the polynomial $p(x) \equiv x^n + (2 + x)^n + (2 - x)^n$ have a rational root.

**Solution**
Answer: n = 1.

There are no roots at all if n is even. If n = 1, then -4 is the root. So suppose n is odd and at least 3.
We can use a similar argument to that showing that $\sqrt{2}$ is irrational. Suppose x = r/s with r and s having no common factor is a root. Then $r^n + (2s + r)^n + (2s - r)^n = 0$. The last two terms expand and add to give terms with even coefficients, so r must be even. So set r = 2t and we have $t^n + (s + t)^n + (s - t)^n = 0$. The same argument shows that t must be even. But now we have $t^n + 2(s^n + nC2\ s^{n-2} t^2 + ... ) = 0$. Since $n \ge 3$, s must be even also. Contradiction. So there are no rational roots for n odd $\ge 3$.

**Problem A7**
k is a real constant. y satisfies $y'' = (x^3 + kx) y$ with initial conditions y = 1, y' = 0 at x = 0. Show that the solutions of y = 0 are bounded above but not below.

**Solution**
For some $x_0$, $x^3 + kx > 0$ for all $x > x_0$. [For example, if $k \ge 0$, then we may take $x_0 = 0$. If $k < 0$, then we may take $x_0 = -k$.] Now suppose that a < b are two consecutive zeros greater than $x_0$. If y(x) > 0 on (a, b), then y" must be negative for at least part of the interval (a, b), but that is impossible, since $(x^3 + kx) y > 0$ for the whole interval. Similarly, if y(x) < 0 on (a, b), then y" must be positive for at least part of the interval (a, b), but that is impossible, since $(x^3 + kx) y < 0$ for the whole interval. So there is at most one root greater than $x_0$ and that provides a bound. [For completeness, we note that we require y not to be identically zero on a non-zero interval, but that is not possible because y(0) is non-zero.]

Suppose the second part is false. Then we can find $x_2$ such that for any $x < x_2$, y(x) is non-zero. There are two cases. Suppose first that y(x) > 0 for all $x < x_2$. We can take $x_3 < x_2$ such that $x^3 + kx < -2$ for all $x < x_3$. Hence, in particular, $y''(x) < 0$ for $x < x_3$. Take arbitary $a < b < x_3$, then as usual we can find $\xi \in (a, b)$ such that $y(a) = y(b) - (b - a) y'(b) + 1/2 (b - a)^2 y''(\xi) < y(b) - (b - a) y'(b)$. If y'(b) > 0, then for (b - a) sufficiently large y(a) < 0 (contradiction). So $y'(b) \ge 0$. So $y(a) \le y(b) + 1/2 (b - a)^2 y''(\xi)$. But now we use the stronger assumption that $\xi^3 + k\xi < -2$ and hence $y(a) < y(b)( 1 - (b - a)^2)$, which becomes negative for sufficiently large (b - a). Contradiction.

The remaining case is $y(x) < 0$ and $y''(x) > -2\,y(x)$ for all $x < x_3$. In particular, $y''(x) > 0$ for $x < x_3$. So arguing as before, $y'(b) \le 0$. As before $y(a) \ge -y(b)\,( (b - a)^2 - 1)$, which becomes positive for sufficiently large $(b - a)$. Contradiction.

**Problem B1**

The lines L and M are horizontal and intersect at O. A sphere rolls along supported by L and M. What is the locus of its center?

**Solution**

Let the angle between the two lines be $2\theta$. Evidently the center rolls above the angle bisector. Let its projection onto the angle bisector have length x and let it be a height y above the angle bisector. Suppose the sphere has radius r. Then $(x \sin \theta)^2 + y^2 = r^2$. So the locus is an ellipse with semi-axes r and $r/\sin \theta$.

If we assume the rolling is under gravity, then the sphere falls off when it reaches $x = r/\sin \theta$, so the locus is just the upper half of the ellipse. If we assume the sphere is somehow kept in contact with the lines, then it can roll back underneath and so the locus is the whole ellipse. That is the ambiguity.

The trap is that it can also roll the other way, over the other angle bisector, giving another ellipse (or upper half) with semi-axes r and $r/\cos \theta$.

**Problem B2**

Let R be the reals. $f : R \to R$ is twice differentiable, $f''$ is continuous and $f(0) = 0$. Define $g : R \to R$ by $g(x) = f(x)/x$ for $x \ne 0$, $g(0) = f'(0)$. Show that g is differentiable and that g' is continuous.

**Solution**

The only issue is $x = 0$, because elsewhere we have simply $g'(x) = f'(x)/x - f(x)/x^2$.

Given x, we can find $\xi$ such that $f(x) = f(0) + x f'(0) + 1/2\, x^2 f''(\xi) = x f'(0) + 1/2\, x^2 f''(\xi)$. Hence $\lim_{x\to 0} (g(x) - f'(0))/x = \lim_{x\to 0} 1/2\, f''(\xi) = 1/2\, f''(0)$. So $g'(0)$ exists.

Now $\lim_{x\to 0} g'(x) = \lim_{x\to 0} (x f'(x) - f(x))/x^2$. But $f'(x) = f'(0) + x f''(\zeta)$, and $f(x) = x f'(0) + 1/2\, x^2 f''(\xi)$ with $\xi, \zeta \in (0, x)$. So $\lim_{x\to 0} g'(x) = \lim_{x\to 0} f''(\zeta) - 1/2\, f''(\xi) = 1/2\, f''(0) = g'(0)$. So g' is continuous at 0.

**Problem B3**

Let S be a spherical cap with distance taken along great circles. Show that we cannot find a distance preserving map from S to the plane.

**Solution**

Let O be the center of the cap and C its perimeter. Then C is a circle center O. Its image must also be a circle with the same radius R, since the distance between each point of C and O is preserved. The circumference of the circle is also preserved. But the circumference is not equal to $2\pi R$.

**Problem B4**

Can we find n such that $\mu(n) = \mu(n + 1) = ... = \mu(n + 1000000) = 0$? [The Möbius function $\mu(r) = 0$ iff r has a square factor > 1.]

**Solution**

Answer: yes.

Let $p_n$ be the nth prime. We show how to find $a_n$ such that $p_1^2$ divides $a_n$, $p_2^2$ divides $a_n + 1$, $p_3^2$ divides $a_n + 2$, ... , $p_n^2$ divides $a_n + n - 1$.

Evidently we can take $a_1 = 4$. We now use induction. $a_n + k\, p_1^2 \dots p_n^2$ has the same property as $a_n$. So we need to pick k such that $a_n + k\, p_1^2 \dots p_n^2 = 0 \pmod{p_{n+1}^2}$. But that is always possible (by Euclid's algorithm, for example) since $p_{n+1}^2$ is relatively prime to $p_1^2 \dots p_n^2$.

**Problem B5**
n is a positive integer. An infinite sequence of 0s and 1s is such that it only contains n different blocks of n consecutive terms. Show that it is eventually periodic.

**Solution**
We use induction on n. It is obvious for n = 1. Suppose it is true for n. Let S be a sequence with just n + 1 different (n+1)-blocks. Consider the n-blocks formed by taking the first n elements of each (n+1)-block. These evidently form all the possible n-blocks in S. So if there are at most n of them, then we are home by induction.

But if there are n + 1 of them (all distinct), then the first n elements of each (n+1)-block determine the last element. So an n-block determines the entire sequence from that point on (because having fixed the (n + 1)th element, we move the n-block along one place and hence fix the (n + 2)nd element and so on). But there are only finitely many possible n-blocks, so there are two the same (in S). That gives a period, since the elements r places after the start of each block must be the same.

Note that the result is the best possible in the sense that we can construct a non-periodic sequence with just n + 1 different n-blocks. Take, for example, the n-blocks to be those with no or just one 1. Then if we take 1 followed by n + 1 zeros, followed by a 1, followed by n + 2 zeros, followed by a 1, followed by n + 3 zeros and so on, we have a non-periodic sequence involving only those n + 1 n-blocks.

**Problem B6**
Let N be the set of positive integers and $R^+$ the positive reals. $f : N \to R^+$ satisfies $f(n) \to 0$ as $n \to \infty$. Show that there are only finitely many solutions to $f(a) + f(b) + f(c) = 1$.

**Solution**
We show first that there are only finitely many solutions to $f(a) + f(b) = k$ ($> 0$). Take a to be $<$ b. Since $f(n) \to 0$ we can find N such that $f(n) < k/2$ for $n > N$. So a must be $\le N$. Now for each such $a_0$, we can find M such that $f(n) < k - f(a_0)$ for $n \ge M$, so there are only finitely many b such that $f(a_0) + f(b) = k$. Hence there are only finitely many solutions to $f(a) + f(b) = k$.

Now consider $f(a) + f(b) + f(c) = 1$. We can find N such that $f(n) < 1/3$ for $n > N$. So at least one of a, b, c must be $\le N$. But for each such a we then have only finitely many solutions to $f(b) + f(c) = 1 - f(a)$. Hence there are only finitely many solutions in total.

**Problem B7**
A three-dimensional solid acted on by four constant forces is in equilibrium. No two lines of force are in the same plane. Show that the four lines of force are rulings on a hyperboloid.

**Solution**
Let the lines be $L_1$, $L_2$, $L_3$, $L_4$. If the line L intersects each of $L_1$, $L_2$, $L_3$, then the moment of the first three forces about L is zero. Hence the moment of the fourth force about L is also zero and so L must also intersect $L_4$. This is sufficient to prove that $L_1$, $L_2$, $L_3$, $L_4$ lie in a ruled quadric.

In fact the union of all the lines which intersect $L_1$, $L_2$, $L_3$ is a ruled quadric. Let P be a point on $L_4$. For a point X and a line L not containing it we take XL to be the unique plane containing X and L. If L and L' are skew lines and X does not lie on either, then $XL \cap XL'$ is the unique line through X which meets L and L'. So $PL_1 \cap PL_2$ is the unique line through P which meets $L_1$ and $L_2$. Suppose it meets $L_1$ at Q. Now $QL_2 \cap QL_3$ meets $L_1$, $L_2$ and $L_3$ so it must also meet $L_4$. But there is a unique line through Q meeting $L_2$ and $L_4$, so $QL_2 \cap QL_3 = PL_1 \cap PL_2$. Hence P lies on the line $QL_2 \cap QL_3$ which is a line of the quadric. P was arbitrary, so we have shown that $L_4$ lies in the quadric.

## 16th Putnam 1956

**Problem A1**
$\alpha \neq 1$ is a positive real. Find $\lim_{x\to\infty} ( (\alpha^x - 1)/(\alpha x - x) )^{1/x}$.

**Solution**
$\lim_{x\to\infty} (1/x)^{1/x} = \lim_{y\to 0} y^y = 1$. If $\alpha > 1$, then $\lim 1/(\alpha - 1)^{1/x} = 1$, and $\lim (\alpha^x - 1)^{1/x} = \lim (\alpha^x)^{1/x} = \alpha$, so the whole expression tends to $\alpha$. If $\alpha < 1$, then $(\alpha^x - 1)/(\alpha - 1)$ tends to $1/(1 - \alpha)$, so the whole expression tends to 1.

**Problem A2**
Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

**Solution**
Let n have d digits. Take N = 12345678900 ... 0, ending in d+1 digits. One of N+1, N+2, ... , N+n must be a multiple of n and its first 10 digits are all different.

**Problem A3**
Find the trajectory of a particle which moves from rest in a vertical plane under (constant) gravity and a force k*v* perpendicular to its velocity *v*.

**Solution**
Take the x-axis horizontal and the y-axis vertically downwards. Then the equations of motion are: y'' = g - kx', x'' = ky'. If k = 0, then we have the usual constant acceleration downwards ($x = 0$, $y = 1/2\ gt^2$). So suppose $k \neq 0$.

By inspection, x = A (cos kt - 1) + B (sin kt - kt) (we have chosen the terms - A and -kBt to give x(0) = x'(0) = 0). Hence (integrating x'' = ky') we have y = - A sin kt + B(cos kt - 1). But y'(0) = 0, so A = 0. Finally, to satisfy y'' = g - kx', we $B = -g/k^2$. So $x = gt/k - g/k^2 \sin kt$, $y = g/k^2 (1 - \cos kt)$. This is the equation of a cycloid. Note that the particle is never able to fall more than a finite distance, it keeps being returned to its starting level.

**Problem A4**
Let p(x) be a real polynomial of degree n with leading coefficient 1 and all roots real. Let R be the reals and f **:** [a, b] → R be an n times differentiable function with at least n + 1 distinct zeros. Show that p(D) f(x) has at least one zero on [a, b], where D denotes d/dx.

**Solution**
The basic tool is Rolle's theorem which tells us that there is a zero of f '(x) between any two zeros of f(x). But we would like a zero of $(D - \alpha)f$ between any two zeros of f. For that, notice that the zeros of f(x) are also zeros of $e^{-\alpha x} f(x)$. So if f(x) has n + 1 zeros, then the derivative of $e^{-\alpha x} f(x)$ has at least n zeros. But the dervative of $e^{-\alpha x} f(x)$ is $e^{-\alpha x}$ times $(D - \alpha) f(x)$. $e^{-\alpha x}$ is never zero, so $(D - \alpha) f(x)$ has at least n zeros.

That (more than) proves the result for p(x) of degree 1. We now use induction on n, the degree of p(x). Suppose it is true for n. Let p(x) be a real polynomial with real roots and degree n + 1. Then $p(x) = q(x) (x - \alpha)$ for some $\alpha$ and q(x) of degree n. Let f be an n+1 times differentiable real function on [a, b] with at least n + 2 distinct zeros. Then we proved above that $(D - \alpha) f(x)$ has at least n + 1 distinct zeros. It is also at least n times differentiable. So, by the inductive hypothesis, $q(D) (D - \alpha) f(x)$ has at least one zero, which proves the result is true for n + 1.

**Problem A5**
Show that there are just (n-k+1)Ck subsets of {1, 2, ... , n} with k elements and not containing both i and i+1 for any i.

**Solution**
There is a bijection between subsets of {1, 2, ... , n} with k elements not containing both i and i+1 for any i and subsets of {1, 2, ... , n - k + 1} with k elements. Namely, associate $r_1 < r_2 < ... < r_k$ and $r_1, r_2 - 1, r_3 - 2, ... , r_k - k + 1$. But there are obviously just (n-k+1)Ck such subsets of {1, 2, ... , n - k + 1}.

**Problem A6**

Let R be the reals. Find $f : R \to R$ which preserves all rational distances but not all distances. Show that if $f : R^2 \to R^2$ preserves all rational distances then it preserves all distances.

**Solution**

Let $f(x) = x$ for x rational, $x + 1$ for x irrational. If $x - y$ is rational, then either both x and y are rational in which case $f(x) - f(y) = x - y$, or neither are in which case $f(x) - f(y) = (x + 1) - (y + 1) = x - y$. So f preserves rational distances. But f does not preserve all distances. For example $f(\sqrt{2}) - f(0) \neq \sqrt{2} - 0$.

Given points A and B in the plane a distance k apart, and any $\varepsilon > 0$, we can find a point C such that AC and BC are rational and $|AC - k| < \varepsilon/2$, $|BC| < \varepsilon/2$. Let $A' = f(A)$, $B' = f(B)$, $C' = f(C)$. Then $|A'C'| = |AC|$, $|B'C'| = |BC|$, and hence $|A'B'|$ lies within $\varepsilon$ of k. This is true for all $\varepsilon > 0$, so $|A'B'| = k$ and f preserves all distances.

**Problem A7**

Show that for any given positive integer n, the number of odd nCm with $0 \le m \le n$ is a power of 2.

**Solution**

(2n)C(2m+1) = 2n/(2m+1) (2n-1)C2m, which is even. There are n even factors in the numerator of 2nC2m and n even factors in its denominator. If we cancel 2 from each of these even factors, then we get nCm. So 2nC2m has the same parity as nCm. Hence there are the same number of odd binomial coefficients for 2n as for n.

(2n+1)C2m =2nC(2m-1) + 2nC2m, so (2n+1)C2m has the same parity as 2nC2m. Similarly, (2n+1)C(2m+1) = 2nC2m + 2nC(2m+1), so (2n+1)C(2m+1) has the same parity as 2nC2m. Hence there are twice as many odd binomial coefficients for 2n+1 as for 2n.
The required result now follows by a trivial induction.

**Problem B1**

The differential equation $a(x, y)\, dx + b(x, y)\, dy = 0$ is homogeneous and exact (meaning that $a(x, y)$ and $b(x, y)$ are homogeneous polynomials of the same degree and that $\partial a/\partial y = \partial b/\partial x$). Show that the solution $y = y(x)$ satisfies $x\, a(x, y) + y\, b(x, y) = c$, for some constant c.

**Solution**

$d(x a + y b) = (a\, dx + b\, dy) + x( \partial a/\partial x\, dx + \partial a/\partial y\, dy) + y( \partial b/\partial x\, dx + \partial b/\partial y\, dy)$.

Using exactness, this becomes: $(a\, dx + b\, dy) + x( \partial a/\partial x\, dx + \partial b/\partial x\, dy) + y( \partial a/\partial y\, dx + \partial b/\partial y\, dy)$. Homogeneity implies that for some integer k we have $x\, \partial a/\partial x + y\, \partial a/\partial y = k a$, and $x\, \partial b/\partial x + y\, \partial b/\partial y = k b$. So we get finally $d(x a + y b) = (a\, dx + b\, dy) + k (a\, dx + b\, dy)$. So if y is a solution, then $d(x a + y b) = 0$ and hence $x a + y b = c$ for some constant c.

**Problem B2**

Let P be the set of all subsets of the plane. $f : P \to P$ satisfies $f(X \cup Y) \supseteq f( f(X) ) \cup f(Y) \cup Y$ for all $X, Y \in P$ (*). Show that (1) $f(X) \supseteq X$, (2) $f( f(X) ) = f(X)$, (3) if $X \supseteq Y$, then $f(X) \supseteq f(Y)$, for all $X, Y \in P$. Show conversely that if $f : P \to P$ satisfies (1), (2), (3), then f satisfies (*).

**Solution**

Taking Y = X in (*) gives immediately that $f(X) = f(X \cup Y) \supseteq f( f(X) ) \cup f(X) \cup X \supseteq X$, which proves (1).
It also shows that $f(X) \supseteq f( f(X) )$. Now put $Y = f(X)$ in (*). Then $X \cup Y = f(X)$ (by (1) ), so (*) gives $f( f(X) ) \supseteq Y = f(X)$. So we have proved (2).

Finally, if $X \supseteq Y$, then $X \cup Y = X$, so $f(X) = f(X \cup Y) \supseteq f(Y)$, which proves (3).

Now assume (1), (2) and (3) and take any X, Y. Then $X \cup Y \supseteq Y$, so $f(X \cup Y) \supseteq f(Y)$ (using (3) ). But $f(Y) \supseteq Y$ (using (1) ), so also $f(X \cup Y) \supseteq Y$. Similarly, $f(X \cup Y) \supseteq f(X)$. But $f(X) = f( f(X) )$ (using (2) ), so $f(X \cup Y) \supseteq f( f(X) )$, which establishes (*).

**Problem B3**
ABCD is an arbitrary tetrahedron. The inscribed sphere touches ABC at S, ABD at R, ACD at Q and BCD at P. Show that the four sets of angles {ASB, BSC, CSA}, {ARB, BRD, DRA}, {AQC, CQD, DQA}, {BPC, CPD, DPB} are the same.

**Solution**
The key is to notice that the two angles in the sets subtended by the same side of the tetrahedron are the same. For example $\angle ASC = \angle AQC$. Let O be the centre of the sphere and r its radius. Then $AQ^2 + r^2 = AO^2 = AS^2 + r^2$, so AQ = AS. Similarly, CQ = CS. So the triangles AQC, ASC are similar.

Now the sum of the angles in each set is the same. But $\angle ASB$ in the first set equals $\angle ARB$ in the second set, so $\angle BSC + \angle CSA = \angle BRD + \angle DRA$. Similarly, $\angle CQD$ in the third set equals $\angle CPD$ in the fourth, so $\angle AQC + \angle DQA = \angle BPC + \angle DPB$. Adding these two equations and using $\angle BSC = \angle BPC$, $\angle CSA = \angle AQC$, $\angle BRD = \angle DPB$, $\angle DRA = \angle DQA$ gives $\angle CSA = \angle BRD$. In other words, the angles in the set subscribed by opposite sides of the tetrahedron are also the same.
That gives us all we need: $\angle ASC = \angle BRD = \angle AQC = \angle BPD$; $\angle ASB = \angle ARB = \angle CQD = \angle CPD$; $\angle BSC = \angle ARD = \angle AQD = \angle BPC$.

**Problem B4**
Show that for any triangle ABC, we have sin A cos C + A cos B > 0.

**Solution**
The result is obviously true unless B or C is obtuse. If C is obtuse, then |cos C| < cos B (because |cos C| = cos(A + B) < cos B). But sin A < A, so the result follows. So assume that B is obtuse.

We have A acute and hence A < tan A. So A cos A < sin A. But C is acute, so cos C is positive and A cos A cos C < sin A cos C. Now A sin A sin C is positive, so A cos(A + C) = A cos A cos C - A sin A sin C < A cos A cos C < sin A cos C. But cos(A + C) = -cos B, so - A cos B < sin A cos C.

**Problem B5**
Show that a graph with 2n points and $n^2 + 1$ edges necessarily contains a 3-cycle, but that we can find a graph with 2n points and $n^2$ edges without a 3-cycle.

**Solution**
Induction. For n = 2, the result is obviously true, because there is only one graph with 4 points and 5 edges and it certainly contains a triangle. Suppose the result is true for n. Consider a graph G with 2n + 2 points and $n^2 + 2n + 2$ edges. Take any two points P and Q joined by an edge. We now consider two cases. If there are less than 2n + 1 other edges to P and Q, then if we remove P and Q, we get a graph with 2n points and at least $n^2 + 1$ edges, which must contain a triangle, so G does also. If there are at least 2n + 1 other edges then at least one point must be joined to both P and Q, but that gives a triangle.

Let G and H be disjoint sets each with n points. Join every point of G to every point of H by an edge. Then the resulting graph has $n^2$ edges, but it does not contain any triangles, because two points of any triangle must belong to G or to H and in either case there is no edge connecting them.

**Problem B6**
The sequence $a_n$ is defined by $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. Show that any pair of values in the sequence are relatively prime and that $\sum 1/a_n = 1$.

**Solution**
We show by induction on k that $a_{n+k} = 1 \pmod{a_n}$. Obviously true for k = 1. Suppose it is true for k. Then for some m, $a_{n+k} = m a_n + 1$. Hence $a_{n+k+1} = a_{n+k}(m a_n + 1 - 1) + 1 = a_{n+k} m a_n + 1 = 1 \pmod{a_n}$. So the result is true for all k. Hence any pair of distinct $a_n$ are relatively prime.

We show by induction that $\sum_1^n 1/a_r = 1 - 1/(a_{n+1} - 1)$. For n = 2, this reduces to 1/2 = 1 - 1/(3 - 1), which is true. Suppose it is true for n. Then $\sum_1^{n+1} 1/a_r = 1 - k$, where $k = 1/(a_{n+1} - 1) - 1/a_{n+1} = 1/(a_{n+1}^2 - a_{n+1}) = 1/(a_{n+2} - 1)$, so it is true for n + 1. But $a_n \to \infty$, so $\sum 1/a_n = 1$.

**Problem B7**
$p(z)$ and $q(z)$ are complex polynomials with the same set of roots (but possibly different multiplicities). $p(z) + 1$ and $q(z) + 1$ also have the same set of roots. Show that $p(z) \equiv q(z)$.

**Solution**
This is much easier than it looks. We just have to consider $p - q$ and $p' - q'$. Suppose that $p$ has $A$ roots, of which $B$ are distinct, and that $p(z) + 1$ has $A$ roots, of which $C$ are distinct. Without loss of generality we may assume the degree of $q$ is at most $A$, so that $p - q$ has at most $A$ roots. Clearly the roots of $p(z)$ and $p(z) + 1$ do not overlap. The $B$ distinct roots of $p(z)$ and the $C$ distinct roots of $p(z) + 1$ must all be roots of $p - q$. So $B + C \le A$. On the other hand, $p'$ has at least $A - B$ roots in common with $p$ and at least $A - C$ in common with $p(z) + 1$, so it has at least $2A - (B + C)$ in total. But its degree is $A - 1$, so $(B + C) \ge A + 1$. Contradiction.

## 17th Putnam 1957

**Problem A1**
A surface S in 3-space is such that every normal intersects a fixed line L. Show that we can find a surface of revolution containing S.

**Solution**
Unfortunately, this is false.

It is not hard to paste together pieces of different surfaces of revolution. A simple example is as follows. Take a cylinder terminated at one end by a circle C. For part of the circumference of C continue the cylinder. Leave a gap either side and then on the rest join a portion of a cone (so that the surface is angled outwards away from the central axis).
It is possible, but hard, certainly too hard for a qu. 1, to prove that S is locally a surface of revolution. For details see Gleason et al.

**Problem A2**
k is a real number greater than 1. A uniform wire consists of the curve $y = e^x$ between $x = 0$ and $x = k$, and the horizontal line $y = e^k$ between $x = k - 1$ and $x = k$. The wire is suspended from $(k - 1, e^k)$ and a horizontal force applied at the other end, (0, 1) to keep it in equilibrium. Show that the force is directed towards increasing x.

**Solution**
If d is the density per unit length, then the total mass of wire is $d(1 + \int_0^k \sqrt{1 + e^{2x}}\, dx)$. The moment about the y-axis is $d(k - 1/2) + d \int_0^k x \sqrt{1 + e^{2x}}\, dx$. We require that the x-coordinate of the centre of mass exceeds k - 1. In other words, $k - 1/2 + \int_0^k x \sqrt{1 + e^{2x}}\, dx > (k - 1) ( 1 + \int_0^k \sqrt{1 + e^{2x}}\, dx )$.

Set $f(k) = \int_0^k x \sqrt{1 + e^{2x}}\, dx - (k - 1) \int_0^k \sqrt{1 + e^{2x}}\, dx$. Then we require $f(k) > -1/2$ for all $k > 1$. Clearly $f(0) = 0$. We try differentiating, and get $f'(k) = \sqrt{1 + e^{2k}} - \int_0^k \sqrt{1 + e^{2x}}dx$.

One possibility is to evaluate the integral explicitly (substitute $y = e^{2x}$, then $z^2 = 1 + y$, we end up with a messy but doable integral). But it is maybe easier to approximate. We have $1 + e^{2x} < 1 + e^{2x} + e^{-2x}/4 = (e^x + e^{-x}/2)^2$. So $\int_0^k \sqrt{1 + e^{2x}}dx < \int_0^k (e^x + e^{-x}/2)\, dx = (e^x - e^{-x}/2) |_0^k = e^k - e^{-k}/2 - 1/2 < e^k < \sqrt{1 + e^{2k}}$. So $f'(k) > 0$. Hence $f(k) > f(0) > 0 > -1/2$.

**Problem A3**
A and B are real numbers such that $\cos A \neq \cos B$. Show that for any integer $n > 1$, $|\cos nA \cos B - \cos A \cos nB| < (n^2 - 1) |\cos A - \cos B|$.

**Solution**
It is not clear how to use induction, expanding $\cos (n+1)A = \cos nA \cos A - \sin nA \sin A$ gives a profusion of sines and it is not clear how to get rid of them. So some other approach is needed.

An ingenious manipulation puts the expression entirely in terms of sines. Put $x = (A + B)/2$, $y = (A - B)/2$. Then $A = x + y$, $B = x - y$. Now $2 \cos nA \cos B = \cos(nA + B) + \cos(nA - B)$. Similarly, $2 \cos A \cos nB = \cos(nB + A) + \cos(nB - A)$. Switch to x and y and then go back to products of cosines. We have $\cos(nA + B) - \cos(nB + A) = \cos( (n+1)x + (n-1)y) - \cos( (n+1)x - (n-1)y ) = -2 \sin(n+1)x \sin(n-1)y$. Similarly, $\cos(nA - B) - \cos(nB - A) = - 2 \sin(n-1)x \sin(n+1)y$. Hence $|\cos nA \cos B - \cos A \cos nB| = |\sin(n+1)x \sin(n-1)y + \sin(n-1)x \sin(n+1)y|$. Obviously $|\cos A - \cos B| = |2 \sin x \sin y|$, so we have to prove that $|\sin(n+1)x \sin(n-1)y + \sin(n-1)x \sin(n+1)y| < 2(n^2 - 1) |\sin x \sin y|$.

If would evidently be sufficient to show that $|\sin mx| < m |\sin x|$. Unfortunately, that is not true. We have equality if $\sin x = 0$ or if $m = 1$. However, we can easily prove by induction that we have strict inequality in all other cases.

For $m = 2$, we have $\sin 2x = 2 \sin x \cos x$, which establishes the result since $|\cos x| < 1$ for $\sin x$ non-zero. Suppose it is true for m. Then $\sin(m+1)x = \sin mx \cos x + \sin x \cos mx$, so $|\sin(m+1)x| <= |\sin mx| |\cos x| + |\sin x| |\cos mx| <= |\sin mx| + |\sin x| < (m+1) |\sin x|$, so it is true for m+1.
Now if $\sin x = 0$, then A + B is a multiple of $2\pi$, so $\cos A = \cos B$. Similarly if $\sin y = 0$, then A - B is a multiple of $2\pi$, so $\cos A = \cos B$. But we are told that $\cos A \neq \cos B$, so $\sin x$ and $\sin y$ are both non-zero. Also we are given

that $n > 1$, so we have strict inequality on $|\sin(n+1)x| > (n+1) |\sin x|$ and $|\sin(n+1)y| > (n+1) |\sin y|$, whilst $|\sin(n-1)x| \geq (n-1) |\sin x|$ and $|\sin(n-1)y| \geq (n-1) |\sin y|$. Thus we get the required strict inequality.

**Problem A4**

p(z) is a polynomial of degree n with complex coefficients. Its roots (in the complex plane) can be covered by a disk radius r. Show that for any complex k, the roots of $n p(z) - k p'(z)$ can be covered by a disk radius $r + |k|$.

**Solution**

Let the roots of p(z) be $a_1, a_2, \dots , a_n$. Suppose they all lie in the disk centre c, radius r. Then $|c - a_n| \leq r$. Suppose that $|c - w| > r + |k|$. We show that w is not a root of $n p(z) - k p'(z)$. We have $|w - a_i| \geq |w - c| - |c - a_i| > r + |k| - r = |k|$. Now $p'(z)/p(z) = \sum 1/(z - a_i)$ (note that this is still true if we have repeated roots), so $|p'(w)/p(w)| < n/|k|$ and hence $|k p'(w)/p(w)| < n$. So $|n - k p'(w)/p(w)| > 0$. But $|p(w)| > 0$ (since w lies outside the disk containing all the roots of p(z) ), so $|n p(w) - k p'(w)| = |p(w)| |n - k p'(w)/p(w)| > 0$.

**Problem A5**

Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are a distance 1 apart.

**Solution**

Induction on n. Obviously true for $n \leq 3$. Suppose it is true for n. Take n+1 points. If no point is a distance 1 from more than 2 points, then we are done. So assume that A, B and C are all a distance 1 from P. wlog the largest of the three angles APB, APC, BPC is APB. It must be at most $60^o$, since $AB \leq 1$. So the ray PC lies between the rays PA and PB.

Now suppose there is another point D (apart from P) such that CD = 1. Then CD must intersect PA, because otherwise one of CP, CA, DP and DA would exceed 1. Similarly, it must intersect PB. But that is impossible. So there is no such point D. Hence if we remove C we lose only one realisation of the distance 1. But the remaining n points have at most n realisations, so the result is established.

**Problem A6**

Define $a_n$ by $a_1 = \ln \alpha$, $a_2 = \ln(\alpha - a_1)$, $a_{n+1} = a_n + \ln(\alpha - a_n)$. Show that $\lim_{n\to\infty} a_n = \alpha - 1$.

**Solution**

Unfortunately, this does not quite work as stated. If $\alpha$ is too small then $a_2 > \alpha$ and hence $a_3$ is undefined. The limit is just under 0.3442. So assume that $\alpha$ is sufficiently large that $a_2 < \alpha$.

For all non-zero x we have $e^x > 1 + x$. Hence for all positive x not equal to 1 we have $\ln x < x - 1$.

Suppose that $a_2 = \alpha - 1$. Then $a_n = \alpha - 1$ for all $n > 2$, which gives the result. So assume $a_2$ is not equal to $\alpha - 1$. Then for all $n \geq 2$ we have $\ln(\alpha - a_n) < \alpha - a_n - 1$, so $a_{n+1} = a_n + \ln(\alpha - a_n) < \alpha - 1$.

Also for n >= 3, we have that $\alpha - a_n > 1$, so $\ln(\alpha - a_n) > 0$, so $a_{n+1} > a_n$. Thus for $n \geq 3$, $a_n$ is a monotonic increasing sequence bounded above by $\alpha - 1$. So it must converge.

But $a_{n+1} - a_n = \ln(\alpha - a_n)$, so $\ln(\alpha - a_n)$ tends to 0 and hence $a_n$ tends to $\alpha - 1$.

**Problem A7**

Show that we can find a set of disjoint circles such that given any rational point on the x-axis, there is a circle touching the x-axis at that point. Show that we cannot find such a set for the irrational points.

**Solution**

Two disjoint circles touching the x-axis at A and B and each with radius r cannot have AB < 2r. Now suppose a circle radius $R \geq r$ touches at A and a disjoint circle radius r touches at B. The circle radius r touching at A does not extend outside the circle radius R, so the circle at B must also be disjoint from it. Hence we still have $AB \geq 2r$. Hence there can only be countably many disjoint circles radius r or more touching the x-axis. But a countable set of countable sets is still countable, so there can only be countably many disjoint circles touching the x-axis (any such circle has radius > 1/n for some n). There are uncountably many irrational points, so we cannot have disjoint circles touching at all the irrational points.

For the rational points take the circle touching at m/n (in lowest terms) to have radius $1/(3n^2)$. Now suppose the circles at m/n and a/b have centres P and Q. We have $PQ^2 = (m/n - a/b)^2 + (1/(3n^2) - 1/(3b^2))^2$. If they intersect, then $PQ \le (1/(3n^2) + 1/(3b^2))$ and hence $(m/n - a/b)^2 \le (1/(3n^2) + 1/(3b^2))^2 - (1/(3n^2) - 1/(3b^2))^2 = 4/9\ 1/(n^2b^2)$, so $(mb - an)^2 < 4/9$. But (mb - an) must be integral, so mb - an = 0 and hence m/n = a/b. So distinct circles do not intersect.

**Problem B1**
Let A be the 100 x 100 matrix with $a_{mn} = mn$. Show that the absolute value of each of the 100! products in the expansion of det A is congruent to 1 mod 101.

**Solution**
Each product is 100! 100! . But 101 is prime, so the numbers 1, 2, ... , 100 can be divided into pairs with the product of each pair being 1 mod 101.

**Problem B2**
The sequence $a_n$ is defined by its initial value $a_1$, and $a_{n+1} = a_n(2 - k\, a_n)$. For what real $a_1$ does the sequence converge to 1/k?

**Solution**
Answer: For $a_1$ strictly between 0 and 2/k.

Suppose $a_n$ has the opposite sign to k. Then $2 - k\, a_n$ is positive, and so $a_{n+1}$ also has the opposite sign to k, so we cannot get convergence. Similarly, if $a_n = 0$, then $a_{n+1} = 0$ and we cannot get convergence. So it is a necessary condition that $a_1$ should be non-zero and have the same sign as k.

If k is positive and $a_1 > 2/k$, then $a_2$ is negative and so the sequence does not converge to 1/k (as above). Similarly, if k is negative and $a_1 < 2/k$, then $a_2$ is positive and so we do not have convergence. Thus it is a necessary condition for $a_1$ to lie strictly between 0 and 2/k.

Suppose $a_n = 1/k + h$. Then $a_{n+1} = (1/k + h)(2 - (1 + kh)) = 1/k - kh^2$ (*). This is all we need. But to spell it out, consider first k positive. If $a_1$ lies strictly between 0 and 1/k, then h lies strictly between -1/k and 1/k. Now (*) shows that for $n \ge 2$, $a_n \le 1/k$ and is monotonic increasing. Hence it tends to a positive limit. If this limit is L, then referring to the original equation, L = L(2 - kL), so L = 1/k. Similarly for k negative.

**Problem B3**
$R^+$ is the positive reals, $f : [0, 1] \to R^+$ is monotonic decreasing. Show that $\int_0^1 f(x)\,dx \int_0^1 x\, f(x)^2\,dx \le \int_0^1 x\, f(x)\,dx \int_0^1 f(x)^2\,dx$.

**Solution**
$(f(x) - f(y))(y - x) \ge 0$. Also $f(x), f(y) \ge 0$, so $\int\int f(x) f(y) (f(x) - f(y))(y - x)\,dx\,dy \ge 0$. But $f(x) f(y) (f(x) - f(y))(y - x) = ( f(x)^2 y f(y) - f(x) y f(y)^2 ) + ( f(y)^2 x f(x) - f(y) x f(x)^2)$. So if we take the same limits of integration, 0 and 1, for both integrals, then we have $\int\int f(x) f(y) (f(x) - f(y))(y - x)\,dx\,dy = 2 \int f(x)^2 y f(y) - f(x) y f(y)^2\,dx\,dy = 2 \int f(x)^2\,dx \int y f(y)\,dy - 2 \int f(x)\,dx \int y f(y)^2\,dy$.

**Problem B4**
Show that the number of ways of representing n as an ordered sum of 1s and 2s equals the number of ways of representing n + 2 as an ordered sum of integers > 1. For example: 4 = 1 + 1 + 1 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 (5 ways) and 6 = 4 + 2 = 2 + 4 = 3 + 3 = 2 + 2 + 2 (5 ways).

**Solution**
We can establish a bijection as follows. Given a representation of n as an ordered sum of 1s and 2s, proceed as follows. Add a final 2 at the end. Now group together all summands up to and including the first 2, then all following summands up to and including the next 2, and so on. Add the members in each group. This gives us an ordered set of integers each at least 2 and summing to n+2.

Conversely, given a representation of n+2, write a summand m as m-2 1s followed by a 2. This gives an ordered sum of 1s and 2s. Finally remove the final 2. This gives us a representation of n. It is clear that these two operations are inverse and hence each is a bijection.

**Problem B5**
Let S be a set and P the set of all subsets of S. $f : P \to P$ is such that if $X \subseteq Y$, then $f(X) \subseteq f(Y)$. Show that for some K, $f(K) = K$.

**Solution**
Let K be the union of all subsets A of S such that $A \subseteq f(A)$. We show that $K \subseteq f(K)$. Take any x in K. Then x belongs to some A which satisfies $A \subseteq f(A)$. But $A \subseteq K$, so $f(A) \subseteq f(K)$. So x belongs to $A \subseteq f(A) \subseteq f(K)$, so x belongs to f(K).
Since $K \subseteq f(K)$, we have $f(K) \subseteq f( f(K) )$. Hence by definition $f(K) \subseteq K$.

**Problem B6**
y is the solution of the differential equation $(x^2 + 9) y'' + (x^2 + 4)y = 0$, $y(0) = 0$, $y'(0) = 1$. Show that $y(x) = 0$ for some $x \in [\sqrt{(63/53)}\, \pi, 3\pi/2]$.

**Solution**
We compare the equation given with (1) $z'' = -4/9\, z$, $z(0) = 0$, $z'(0) = 1$, and (2) $w'' = -53/63\, w$, $w(0) = 0$, $w'(0) = 1$.

For all $x > 0$ we have $(x^2 + 4)/(x^2 + 9) > 4/9$, so $y''z - yz'' = (4/9 - (x^2 + 4)/(x^2 + 9) )\, yz$. Integrating gives $y'z - yz'|_0^{3\pi/2} = \int (4/9 - (x^2 + 4)/(x^2 + 9) )\, yz\, dx$ (*). Now $y(0) = z(0) = 0$, $z = \sin 2x/3$, so $z(3\pi/2) = 0$, $z'(3\pi/2) = -2/3$, hence $y'z - yz'|_0^{3\pi/2} = 2/3\, y(3\pi/2)$. Now if y has no zeros on the open interval $(0, 3\pi/2)$, then $y(3\pi/2) \ge 0$ and the integrand $(4/9 - (x^2 + 4)/(x^2 + 9) )\, yz$ is negative for the entire interval. Hence lhs (*) $\ge 0$ and rhs (*) $< 0$. Contradiction. So y has at least on zero on $(0, 3\pi/2)$.

Turning to (2), we note that $\pi^2 < 10$, so $(3\pi/2)^2 < 22.5$. Hence on the interval $[0, 3\pi/2]$ we have $(x^2 + 4)/(x^2 + 9) < 26.5/31.5 = 53/63$. Now $w = \sin (\sqrt{(53/63)}\, x)$, so $w(x) > 0$ on the interval $(0, k)$, where $k = \sqrt{(63/53)}\, \pi$.

Now suppose that y has a root in $(0, k]$. Let h be the smallest such root, so that y(x) is positive on (0, h) and $y(h) = 0$, $y'(h) \le 0$. Integrate $w''y - w\, y'' = ( (x^2 + 4)/(x^2 + 9) - 53/63)$. The lhs is $w'y - w\, y' |_0^h = w(h)\, y'(h) \ge 0$. The rhs is $\int_0^h ( (x^2 + 4)/(x^2 + 9) - 53/63)\, y\, w\, dx$. But the integrand is negative on (0, h), so the rhs $< 0$. Contradiction. Hence y has no roots on $(0, k]$. So it must have a root on $(k, 3\pi/2)$.

**Problem B7**
Let P be a regular polygon and its interior. Show that for any $n > 1$, we can find a subset $S_n$ of the plane such that we cannot translate and rotate P to cover $S_n$ but we can translate and rotate P to cover any n points of $S_n$.

**Solution**
Let the radius of the inscribed circle in P be 1. Let P have m sides. We take $S_n$ to be a circle of radius slightly greater than 1. Clearly it cannot fit inside P. Place it so that its centre is at the centre of P. Suppose it cuts each side a distance $\tan^{-1}\theta$ either side of the midpoint, where $\theta$ is small. Then the arcs lying outside P subtend a total angle $2m\theta$ at the centre. Now rotate the circle about its centre. Let $\varphi$ measure the angle of rotation from the starting point. In order to keep a given point of the circle inside P, $\varphi$ must avoid intervals of total length $2m\theta$. So to keep any given n points of the circle inside P, $\varphi$ must avoid intervals of total length $2mn\theta$. The worst case is that these intervals are all disjoint, but provided $\theta < \pi/mn$ they cannot exhaust the available $2\pi$, so we can find angles $\varphi$ for which all n points are inside P.

# 18th Putnam 1958

**Problem A1**
Show that the real polynomial $\sum_0^n a_i x^i$ has at least one real root if $\sum a_i/(i + 1) = 0$.

**Solution**
Integrate from 0 to 1. We get zero. Hence the polynomial has at least one zero between 0 and 1.

**Problem A2**
A rough sphere radius R rests on top of a fixed rough sphere radius R. It is displaced slightly and starts to roll off. At what point does it lose contact?

**Solution**
Let the top of the fixed sphere be T and its centre O. Let X be the point of contact. Take angle XOT = $\theta$. Let the mass be M and the normal force between the two spheres N. Resolving radially, $MR(d<\theta/dt)^2 = Mg \cos \theta - N$ (1). Taking moments about an axis through X, $MgR \sin \theta = I\, d^2\theta/dt^2$ (2), where I is the moment of inertia of the sphere about X = the moment about a diameter + $MR^2 = 7/5\ MR^2$.

Integrating equation (2) gives $g(1 - \cos \theta) = 7/10\ R\ (d\theta/dt)^2$. Substituting in (1) gives $N = g \cos \theta - 10/7\ (1 - \cos \theta)$. Contact is lost when N becomes 0. In other words at the angle $\theta = \cos^{-1} 10/17$.

**Problem A3**
A sequence of numbers $\alpha_i \in [0, 1]$ is chosen at random. Show that the expected value of n, where $\sum_1^n \alpha_i > 1$, $\sum_1^{n-1} \alpha_i \le 1$ is e.

**Solution**
We can consider the possible values of $\alpha_1, \alpha_2, \dots , \alpha_n$ as points of the n-cube. The points corresponding to sum at most 1 are those in the corner at the origin, bounded by the hyperplane through (1, 0, ... , 0), (0, 1, 0, ... , 0) , ... , (0, ... , 0, 1). By an easy induction this has volume $1/(n-1)!$ . Let $p_n$ = the prob that the sum of the first n numbers is at most 1. We have just shown that $p_n = 1/(n-1)!$ .

Now the required expected value is $(1/1! - 1/2!)\ 2 + (1/2! - 1/3!)\ 3 + (1/3! - 1/4!)\ 4 + \dots = 2 + (3 - 2)/2! + (4 - 3)/3! + (5 - 4)/4! + \dots = e$.

**Problem A4**
$z_1, z_2, \dots , z_n$ are complex numbers with modulus $a > 0$. Let $f(n, m)$ denote the sum of all products of m of the numbers. For example, $f(3, 2) = z_1z_2 + z_2z_3 + z_3z_1$. Show that $|f(n, m)| /a^m = |f(n, n-m)| /a^{n-m}$.

**Solution**
It is obviously sufficient to take a = 1. Then if $z = e^{i\theta}$ we have $1/z = e^{-i\theta}$ = the complex conjugate of z. Now $f(n, m)/(z_1z_2 \dots z_n)$ = the sum of the inverses of the terms in $f(n, n-m)$ = the sum of the complex conjugates of the terms in $f(n, n-m)$ = complex conjugate of $f(n, n-m)$. But a number and its complex conjugate have the same modulus.

**Problem A5**
Let R be the reals. Show that there is at most one continuous function $f : [0, 1]^2 \to R$ satisfying $f(x, y) = 1 + \int_0^x\int_0^y f(s, t)\, dt\, ds$.

**Solution**
If there are two, then their difference $d(x, y)$ satisfies $d(x, y) = \int_0^x\int_0^y d(s, t)\, dt\, ds$ (*).

The domain is compact and d is continuous, so $|d|$ has some upper bound k. Now applying (*) gives $|d(x, y)| \le k\, xy$. Applying (*) again gives $|d(x, y)| \le k\, x^2/2\ y^2/2$. So by a simple induction $|d(x, y)| \le k\, x^n/n!\ y^n/n!$ . Hence $|d(x, y)| \le k/(n!\ n!)$ for all n, and so $d(x, y) = 0$.

**Problem A6**
Assume that the interest rate is r, so that capital of k becomes $k(1 + r)^n$ after n years. How much do we need to

invest to be able to withdraw 1 at the end of year 1, 4 at the end of year 2, 9 at the end of year 3, 16 at the end of year 4 and so on (in perpetuity)?

**Solution**
To meet the withdrawal at the end of year n we need to invest $n^2/(1+r)^n$. Thus the total investment required is $1/(1+r) + 4/(1+r)^2 + 9/(1+r)^3 + 16/(1+r)^4 + \dots$ . Now we know that $1 + x + x^2 + \dots = 1/(1 - x)$. Differentiating, we get: $x/(1 - x)^2 = x + 2x^2 + 3x^3 + \dots$ . Differentiating again: $x(1 + x)/(1 - x)^3 = 1^2x + 2^2x^2 + 3^2x^3 + \dots$ . Substituting $x = 1/(1+r)$ gives $(1 + r)(2 + r)/r^3$ as the required investment.

**Problem A7**
Show that we cannot place 10 unit squares in the plane so that no two have an interior point in common and one has a point in common with each of the others.

**Solution**
Note that we can place 9 squares - arrange them in a regular 3 x 3 array. Then the centre square touches 4 at the corners and the other 4 along the sides.

Consider first placing squares to touch a line. If neither square has a side in contact with the line, then it is easy to see that the two points of contact are more than 1 apart. If square U has a side in contact and square V does not, then together they occupy more than a length 1 of the line (from the far side of U to the point of contact of V).

So the only way to get three squares in contact with AB, a side of a fixed square S = ABCD, is to place one square so that its side is AB. One can then place a second square with its corner touching A and a third with its corner touching B.

So to get more than 8 = 4 x 2 squares touching S, we must get three squares to touch one side as above. Call them T2 in contact with AB, T1 touching at A and T3 touching at B. Now consider the side BC. T3 has a side BD. The most favourable case is that angle CBD = $90^o$ - T3 may be placed so that angle CBD is less than $90^o$. But certainly any square touching BC (apart from T3) cannot extend beyond the ray BD which is perpendicular to BC. But it is easy to see that this means we can get only one additional square to touch BC unless we place T4 with so that its side is BC and T5 touching C at a corner, which gives 2.

In particular, if 3 squares touch AB, then at most 2 (additional) squares touch each of BC and DA.

Hence at most 2 sides, which must be opposite, can have three squares touching. But in that case it is easy to see that the other two sides each have at most 1 additional square touching, so we get only 8 in total. Hence the only way to get 9 is to have 3 squares touching AB and 2 (additional) squares touching each of the other three sides. But those touching BC and DA must then be placed as indicated above and it is easy to see that we then get at most 1 (additional) square touching CD.

**Problem B1**
*Do both (1) and (2):*

(1) Given real numbers a, b, c, d with a > b, c, d, show how to construct a quadrilateral with sides a, b, c, d and the side length a parallel to that length b. What conditions must a, b, c, d satisfy?

(2) H is the foot of the altitude from A in the acute-angled triangle ABC. D is any point on the segment AH. BD meets AC at E, and CD meets AB at F. Show that ∠AHE = ∠AHF.

**Solution**
(1) A necessary and sufficient condition is that we should be able to form a triangle with sides c, d, (a - b). Hence we require $b + c + d \ge a$, $a + d \ge b + c$, and $a + c \ge b + d$. Construct two such triangles, one at each end of the segment a, then join the two tops to get the required quadrilateral.

(2) Take a line through A parallel to BC and project CF, HF, HE, BE to meet it in W, X, Y, Z. Then using similar triangles, we deduce that AY/CH = AZ/BC = BH·AW/CH (1/BC) = BH/CH AW/BC = BH/CH AX/BH = AX/CH, so AX = AY. But AH is perpendicular to XY, so the result follows.

**Problem B2**
Let n be a positive integer. Prove that $n(n + 1)(n + 2)(n + 3)$ cannot be a square or a cube.

**Solution**
$(n+1)(n+2) = n^2 + 3n + 2$ and $n(n+3) = n^2 + 3n$. So their product is $(n^2 + 3n + 1)^2 - 1$. Hence $n(n + 1)(n + 2)(n + 3)$ is 1 less than a square, so it cannot be a square.

One of n+1, n+2 must be odd. Suppose it is n+1. Then n+1 has no factor in common with $n(n + 2)(n + 3)$, so $n(n + 2)(n + 3) = n^3 + 5n^2 + 6n$ must be a cube. But $(n + 1)^3 = n^3 + 3n^2 + 3n + 1 < n^3 + 5n^2 + 6n < n^3 + 6n^2 + 12n + 8 = (n + 2)^3$, so $n^3 + 5n^2 + 6n$ cannot be a cube.

Similarly, suppose n + 2 is odd. Then it has no factor in common with $n(n + 1)(n + 3) = n^2 + 4n^2 + 3n$, so $n^2 + 4n^2 + 3n$ must be a cube. But for $n \ge 2$, $(n + 1)^3 < n^2 + 4n^2 + 3n < (n + 2)^3$, so $n^2 + 4n^2 + 3n$ cannot be a cube for $n \ge 2$. The case n = 1 is checked by inspection: 24 is not a cube.

**Problem B3**
In a tournament of n players, every pair of players plays once. There are no draws. Player i wins $w_i$ games. Prove that we can find three players i, j, k such that i beats j, j beats k and k beats i iff $\sum w_i^2 < (n - 1)n(2n - 1)/6$.

**Solution**
Suppose there are no such three players. Let A be the (or one of the) top-scoring players. We claim that A has n-1 wins. Suppose not, then he is beaten by some B. Now if every player beaten by A is also beaten by B, then B would have a higher score than A, so we must be able to find C who is beaten by A, but who beats B (who beats A). But we assumed there were no such triads. So A has n-1 wins. But we can now consider the remaining players. The same argument shows that the top scoring player amongst them beat n-2 of them. He did not beat A (who beat everyone), so his score is n-2. The argument repeats to show that the scores are n-1, n-2, ... , 1, 0. The sum of the squares is thus $(n-1)n(2n-1)/6$.

Conversely, it is clear that if the scores are n-1, n-2, ... , 1, 0 then there can be no triad. For the player with n-1 beats everyone, so he cannot be part of a triad. But now the player with n-2 beats everyone else, so he cannot either and so on.

Thus we have shown that there is a triad iff the scores are not n-1, n-2, ... , 1, 0. It remains to show that if the scores are not n-1, n-2, ... , 1, 0, then sum of the squares is less than $(n-1)n(2n-1)/6$. But if the scores are not n-1, n-2, ... , 1, 0, then two players must have the same score m. Changing the scores to m-1, m+1 (by changing the result of the match between the two players) increases the sum of the squares by 2. After a finite number of repetitions we must arrive at n-1, n-2, ... , 1, 0, so the starting sum must be strictly less.

**Problem B4**
Let S be a spherical shell radius 1. Find the average straight line distance between two points of S. [In other words S is the set of points (x, y, z) with $x^2 + y^2 + z^2 = 1$).

**Solution**
Answer: 4/3.

It is sufficient to fix one point and to find the average distance of the other points from it. Take the point as P and the centre as O. Now consider a general point Q on the surface. Let angle POQ = θ. The distance PQ is $2 \sin \theta/2$ and this is the same for all points in a band angular width dθ at the angle θ. The band has radius $\sin \theta$. Hence the average distance is $1/4\pi \int_0^\pi (2\pi \sin \theta) (2 \sin \theta/2)\, d\theta = \int 4 \sin^2\theta/2\, d(\sin \theta/2) = 4/3$.

**Problem B5**
S is an infinite set of points in the plane. The distance between any two points of S is integral. Prove that S is a subset of a straight line.

**Solution**
Suppose not. Take three non-collinear points A, B, C. Suppose AB = n. Any point P has $PA + AB \ge PB$, so $PB - PA \le n$. Similarly $PB - PA \ge -n$. But PB - PA is integral, so it must take one of the values -n, -(n-1), ... , 0, 1, ... , n. So P must lie on one of a finite number of hyperbolae $|PB - PA| = k$ (regarding the allowed line pair as a degenerate

hyperbola). Similarly it must lie on one of a finite number of hyperbolae |PA - PC| = h. But each pair of hyperbolae intersect in at most 4 points, so the number of points in addition to A, B, C is finite. Contradiction.

**Problem B6**

A particle of unit mass moves in a vertical plane under the influence of constant gravitational force g and a resistive force which is in the opposite direction to its velocity and with magnitude a function of its speed. The particle starts at time t = 0 and has coordinates (x, y) at time t. Given that x = x(t) and is not constant, show that $y(t) = - g\, x(t) \int_0^t ds/x'(s) + g \int_0^t x(s) / x'(s)\, ds + a\, x(t) + b$, where a and b are constants.

**Solution**

We can write the equations of motion as $x'' = - f(x',y')\, x'$, $y'' = - f(x', y')\, y' - g$. If $x'(t) = 0$ at some t, then a possible solution to the equations would be x = constant (and y found by integrating $y'' = - f(0, y')\, y' - g$). But solutions are unique, so this would be *the* solution. But we are told that x is not constant. So $x'(t)$ is never zero.

We do not know anything about f, so we resolve perpendicular to it to get $x'' \sin \theta - y'' \cos \theta = g \cos \theta$, or $x'' \tan \theta - y'' = g$. But $\tan \theta = dy/dx = y'/x'$, so $x''\, y'/x' - y'' = g$ (*).

Dividing by x' gives $x''\, y'/(x')^2 - y''/x' = g/x'$. Integrating, $y'/x' = A - g \int dt/x'$. So $y' = Ax' - g\, x' \int dt/x'$. Integrating again, $y(T) = A\, x + B - g \int_0^T x' \int_0^t ds/x'\, dt$. Now we can integrate the last integral by parts: $\int_0^T x' \int_0^t ds/x'\, dt = \int_{t=0}^T \int_0^t ds/x'\, dx = x(T) \int_0^T ds/x' - \int_0^T x/x'\, dt$. So we get finally the expression in the question.

**Problem B7**

R is the reals. $f : [a, b] \to R$ is continuous and $\int_a^b x^n f(x)\, dx = 0$ for all non-negative integers n. Show that f(x) = 0 for all x.

**Solution**

We deduce immediately that $\int_a^b x^n f(x)\, p(x)\, dx = 0$ for any polynomial p(x). We now need the Weierstrass approximation theorem: we can uniformly approximate any continuous function on a compact set by polynomials. In other words, we can find a polynomial p(x) such that $|f(x) - p(x)| < \varepsilon$ for all x in [a, b]. Also, since [a, b] is compact, f must be bounded, say by k, so $|f(x)| < k$ for all x in [a, b]. Now we have that $|\int_a^b f(x)^2\, dx| = |\int f(x)\, (f(x) - p(x))\, dx| \le \int |f(x)|\, |f(x) - p(x)|\, dx \le (b - a)\, k\, \varepsilon$, which can be made arbitrarily small by choosing ε sufficiently small. Hence $\int_a^b f(x)^2\, dx = 0$, so f(x) = 0 throughout the interval.

# 19th Putnam 1958

**A1.** Define f(i, j) as follows: f(i, 1) = f(1, j) = 1, f(i+1, j+1) = f(i, j+1) + f(i+1, j) + f(i, j). Let d(n) = f(1, n-1) + f(2, n-2) + ... + f(n-1, 1). Show that d(n + 2) = d(n) + 2 d(n + 1).

**A2.** Define $a_1 = 1$, $a_{n+1} = 1 + n/a_n$. Show that $\sqrt{n} \le a_n < 1 + \sqrt{n}$.

**A3.** Assuming that there is a unique function f(x) satisfying f(0) = 1, $f'(x) = f(x) + \int_0^1 f(t)\, dt$, find it.

**A4.** Find the general solution in real numbers a, b, c, d to the inequalities 2a > a + b > a + c > 2b > b + c > a + d > 2c > b + d > c + d > d + d. Find the smallest solution in positive integers.
**A5.** Let A = $(a_{ij})$ be the n x n matrix with $a_{ij} = 1$ if $i \ne j$, and $a_{ii} = 0$. Show that the number of non-zero terms in the expansion of det A is $n! \sum_o^n (-1)^i/i!$ .
**A6.** R is the reals. $\alpha \in [0, 1)$. $f : [0, 1] \to [0, \alpha]$ and $g : [0, 1] \to R$ are continuous. $\beta$ satisfies $\beta = \max (g(x) + f(x) \beta)$. When do we have $\beta = \max g(x)/(1 - f(x) )$?

**A7.** m, n are relatively prime positive integers with n even. Given any positive integer r, define f(r) to be the integer which minimizes |f(r)/r - m/n|. Show that $\lim_{k\to\infty} \sum_1^k |f(r)/r - m/n|\, r/k = 1/4$.
**B1.** Let $a_n = \sum_0^n 1/nCr$. Show that $a_n = 1 + a_{n-1} (n + 1)/(2n)$. Deduce that $\lim a_n = 2$.
**B2.** Let X be the set {1, 2, 3, ... , 2n}, take $Y \subseteq X$ with |Y| = n + 1. Show that we can find a, b $\in$ Y with a dividing b.
**B3.** Show that if X is a square side 1 and X = A $\cup$ B, then A or B has diameter at least $\sqrt{5}/2$. Show that we can find A and B both having diameter $\le \sqrt{5}/2$.

**B4.** Let R be the reals. $f : R \to R$ is three times differentiable. As $x \to \infty$, f(x) tends to a finite limit, and f '''(x) tends to zero. Show that f '(x) and f ''(x) also tend to zero.
**B5.** A sequence of points $P_n$ in the plane is defined by: $P_0$ is at the origin; $P_1$ is at (1, 0); and $P_{n-1}P_n$ is length 1/n and at an angle $\theta$ to the previous segment. Find the coordinates of lim $P_n$.
**B6.** A graph has n vertices {1, 2, ... , n} and a complete set of edges. Each edge is oriented, as either $i \to j$ or $j \to i$. Show that we can find a permutation of the vertices $a_i$ so that $a_1 \to a_2 \to a_3 \to ... \to a_n$.
**B7.** Let N = {1, 2, ... , n}. Given a permutation $f : N \to N$, define d(f) = no. of i such that f(i) > f(j) for all j > i. Find the mean of d(f) over all permutations f on N.

## 20th Putnam 1959

**Problem A1**
Prove that we can find a real polynomial $p(y)$ such that $p(x - 1/x) = x^n - 1/x^n$ (where n is a positive integer) iff n is odd.

**Solution**
Take n odd. For n = 1 the result is obvious. Now suppose the result is true for odd $n \le 2m - 1$. Expand $(x - 1/x)^{2m+1}$ by the binomial theorem. Since (2m+1)C0 = (2m+1)C(2m+1), (2m+1)C1 = (2m+1)C2m, (2m+1)C2 = (2m+1)C(2m-1), ... , (2m+1)Cm = (2m+1)C(m+1), we may group these pairs of terms together to get: $(x - 1/x)^{2m+1} = (x^{2m+1} - x^{-(2m+1)}) - (2m+1)C1\ (x^{2m-1} - x^{-(2m-1)}) + ... + (-1)^m\ (2m+1)Cm\ (x - 1/x)$. This gives the polynomial for n = 2m+1 in terms of the lower order polynomials.

Note that $x - 1/x$ has the same value 3/2 for $x = -1/2$ and $x = 2$. But $x^{2m} - 1/x^{2m}$ is negative for $x = -1/2$ and positive for $x = 2$, so we cannot express $x^{2m} - 1/x^{2m}$ as a function of $(x - 1/x)$, polynomial or otherwise.

**Problem A2**
Let $\omega^3 = 1$, $\omega \ne 1$. Show that $z_1$, $z_2$, $-\omega z_1 - \omega^2 z_2$ are the vertices of an equilateral triangle.

**Solution**
Let A be the point $z_1$, B the point $z_2$. Then $z_2 - z_1$ represents the vector from A to B. Now $-\omega^2$ has unit length and makes an angle $\pm\pi/3$ with the positive real axis, so multiplying $z_2 - z_1$ by it rotates AB through an angle $\pi/3$ (clockwise or counterclockwise). Adding the result to $z_2 - z_1$ gives a point C such that AC is at an angle $\pi/3$ to AB. In other words ABC is equilateral. It is easily checked that C is then $z_1 - \omega^2(z_2 - z_1) = -\omega z_1 - \omega^2 z_2$ (since $1 + \omega + \omega^2 = 0$).

**Problem A3**
Let C be the complex numbers. $f : C \to C$ satisfies $f(z) + z f(1 - z) = 1 + z$ for all z. Find f.

**Solution**
Putting $z = 1 - w$ we have $f(1 - w) + (1 - w) f(w) = 2 - w$. Hence $w f(1 - w) + (w - w^2) f(w) = 2w - w^2$. Hence $(1 + w - f(w)) + (w - w^2) f(w) = 2w - w^2$, giving $(w^2 - w + 1) f(w) = (w^2 - w + 1)$. So provided w is not $(1 \pm i \sqrt{3})/2$, we have $f(w) = 1$.

But at these two values f can be different. We can take one of them to be arbitrary. For example, take $f( (1 + i \sqrt{3})/2 ) = k$, any complex number. Then $f( (1 - i \sqrt{3})/2 ) = 1 + (1 - k)(1 - i \sqrt{3})/2$.

**Problem A4**
R is the reals. $f, g : [0, 1] \to R$ are arbitary functions. Show that we can find x, y such that $|xy - f(x) - g(y)| \ge 1/4$.

**Solution**
It is enough to consider the values at 0 and 1. If the pairs (0, 0), (0, 1), (1, 0) do not work for (x, y), then we have $|f(0) + g(0)| < 1/4$, $|f(0) + g(1)| < 1/4$, and $|f(1) + g(0)| < 1/4$. Hence $f(1) + g(1) \le f(1) + g(0) + f(0) + g(1) - (f(0) + g(0)) < 1/4 + 1/4 + 1/4$. So $1 - f(1) - g(1) > 1 - 3/4 = 1/4$. Hence the pair (1, 1) does work.

**Problem A5**
At a particular moment, A, T and B are in a vertical line, with A 50 feet above T, and T 100 feet above B. T flies in a horizontal line at a fixed speed. A flies at a fixed speed directly towards B, B flies at twice T's speed, also directly towards T. A and B reach T simultaneously. Find the distance traveled by each of A, B and T, and A's speed.

**Solution**
Answer: A travels $25(3 + \sqrt{73})/3 = 96.2$ ft, B travels $400/3$ ft = 133.3 ft, T $200/3$ ft = 66.7 ft, A's speed is $(3 + \sqrt{73})/8 = 1.443$ times T's speed.

Take the x-axis vertical and the y-axis horizontal, so that at t = 0, the target is at (0, 0) and the pursuer is at (a, 0). Assume the target (T) has speed v and the pursuer (A or B) has speed kv with k > 1. At time t, the target is at (0, vt). The equations of motion are: $(dx/dt)^2 + (dy/dt)^2 = kv$ and $y' = (y - vt)/x$. The first equation gives $kv/(dx/dt) = -\sqrt{1 + y'^2}$ (negative because x decreases with time).

Differentiating the second equation gives $xy'' = - v/(dx/dt)$. So $kxy'' = \sqrt{1 + y'^2}$. Integrating, $\ln(y' + \sqrt{1 + y'^2}) = \ln x + \text{const}$. At $x = a$, $y = 0$, so $(x/a)^{1/k} = y' + \sqrt{1 + y'^2}$. Hence $2y' = (x/a)^{1/k} - (x/a)^{-1/k}$. Integrating again, $2y = a/(1+1/k)\,(x/a)^{1+1/k} - a/(1-1/k)\,(x/a)^{1-1/k} - a/(1+1/k) + a/(1-1/k)$, since $y = 0$ at $x = a$. When pursuit ends, $x = 0$ and hence $y = ak/(k^2 - 1)$.

For B, $k = 2$, so $y = 2a/3 = 200/3$ ft. That is the distance travelled by T. B is travelling at twice the speed, so B travels a distance 400/3 ft. For A reaches T at the same y, hence $200/3 = 50\,k/(k^2 - 1)$, so $4k^2 - 3k - 4 = 0$, so $k = (3 + \sqrt{73})/8$.

**Problem A6**

Given any real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$, show that for $m, n > 1$ we can find m real n x n matrices $A_1, \ldots, A_m$ such that $\det A_i = \alpha_i$, and $\det(\sum A_i) = \beta$.

**Solution**

Start by setting $A_i$ to be the matrix with $1, 1, \ldots, 1, \alpha_i$ down the main diagonal and zeros elsewhere. Modify $A_1$ by changing the n, n-1 element to 1. Modify $A_m$ by changing the n-1, n element to $m(\alpha_1 + \ldots + \alpha_m) - \beta/m^{n-2}$. It is clear that this gives $\det A_i = \alpha_i$.

$A_1 + \ldots + A_m$ has $m, m, \ldots, m, (\alpha_1 + \ldots + \alpha_m)$ down the main diagonal. The only other non-zero elements are n, n-1, which is 1 and n-1, n, which is $m(\alpha_1 + \ldots + \alpha_m) - \beta/m^{n-2}$. Hence its determinant evaluates to $\beta$.

**Problem A7**

Let R be the reals. Let $f : [a, b] \to R$ have a continuous derivative, and suppose that if $f(x) = 0$, then $f'(x) \neq 0$. Show that we can find $g : [a, b] \to R$ with a continuous derivative, such that $f(x)g'(x) > f'(x)g(x)$ for all $x \in [a, b]$.

**Solution**

We note that the derivative of $f/g$ is $-(fg' - f'g)/g^2$ so if we take g so that f/g is decreasing, then we are almost home. Not quite, because of the difficulty that f may be zero. For example, if we take $g(x) = x f(x)$, then $fg' - f'g$ is zero whenever $f(x) = 0$.

f has only finitely many zeros, for otherwise the zeros would have a limit point c and then by continuity we would have $f(c) = f'(c) = 0$. So we may take a polynomial $p(x)$ such that $p(x) f'(x) = -1$ at each zero of f. Now consider $g(x) = x f(x) + k p(x)$. We have $f(x) g'(x) - f'(x) g(x) = f(x)^2 + k(f(x) p'(x) - f'(x) p(x))$. Now $f(x) p'(x) - f'(x) p(x) = 1$ at each zero of f, so we can find an open set K containing all these zeros such that $f(x) p'(x) - f'(x) p(x) > 1/2$ on K. Now $[a, b] - K$ is compact and contains no zeros of $f(x)^2$, which is continuous, so we can find $\varepsilon > 0$, so that $f(x)^2 > \varepsilon$ on $[a, b] - K$. But $[a, b]$ is compact, so $|f(x) p'(x) - f'(x) p(x)| <$ some M on $[a, b]$. Take $k < \varepsilon/M$. Then on $[a, b] - K$, $f(x)^2 > \varepsilon$ and $k|f(x) p'(x) - f'(x) p(x)| < \varepsilon$, so $f(x) g'(x) - f'(x) g(x) > 0$. On K, $f(x)^2$ is non-negative and $k(f(x) p'(x) - f'(x) p(x))$ is positive, so $f(x) g'(x) - f'(x) g(x) > 0$.

**Problem B1**

Join each of m points on the positive x-axis to each of n points on the positive y-axis. Assume that no three of the resulting segments are concurrent (except at an endpoint). How many points of intersection are there (excluding endpoints)?

**Solution**

Answer: $mn(m-1)(n-1)/4$.

Let us call the points on the y-axis $y_1 < y_2 < \ldots < y_n$ and the points on the x-axis $x_1 < x_2 < \ldots < x_m$. Let us join first $y_1$ to each $x_i$, then $y_2$ and so on. Joining $y_1$ to each of the $x_i$ creates no intersections. Joining $y_2$ to $x_1$ creates m-1 intersections. Joining it to $x_2$ creates m-2 intersections and so on. So, in all, joining $y_2$ gives $1 + 2 + \ldots + m-1 = m(m-1)/2$ intersections.

Similarly joining $y_3$ to $x_1$ gives $2(m-1)$ intersections. Joining it to $x_2$ gives $2(m-2)$ and so on. So, in all, $y_3$ gives $2 \times m(m-1)/2$. Similarly, $y_{r+1}$ gives $r \times m(m-1)/2$. So in total we get $m(m-1)/2\,(1 + 2 + \ldots + n-1) = mn(m-1)(n-1)/4$.

**Problem B2**
Show that any positive real can be expressed in infinitely many ways as a sum $\sum 1/(10\ a_n)$, where $a_1 < a_2 < a_3 < \dots$ are positive integers.

**Solution**
$\sum 1/n$ diverges, so $\sum 1/(10n)$ diverges. Let k be any positive real. Take N such that $1/(10N) < k$. Let M be any integer $> N$. We show how to find an expression $k = \sum 1/(10a_n)$ with $a_1 = M$. Take enough terms $a_2 = 1/(10(M+1))$, $a_3 = 1/(10(M+2))$, ... so that $\sum 1/(10a_n) < k$, but adding another term would give a sum $\geq k$. Let the difference $k - \sum 1/(10a_n)$ be $k_1 > 0$. Now take $M_1$ so that $1/(10\ M_1) < k_1$ and repeat. [In other words, take as many terms $1/(10\ M_1) + 1/(10(M_1+1)) + \dots$ as we can whilst keeping the sum $< k_1$.] Let $k_2$ be the new difference, and so on. This process gives a sum which converges to k. Each such sum is different because it has a different starting term.

**Problem B3**
Find a continuous function $f : [0, 1] \to [0, 1]$ such that given any $\beta \in [0, 1]$, we can find infinitely many $\alpha$ such that $f(\alpha) = \beta$.

**Solution**
We use an adapted space-filling curve. Let g(x) be a piecewise linear function with period 2: $g(x) = 0$ on [0, 1/3], 1 on [2/3, 4/3] and 0 on [5/3, 2] with linear connecting pieces. For t any real in the range $0 \leq t \leq 1$ let $f(t) = g(t)/2 + g(9t)/4 + g(81t)/8 + g(729t)/16 + \dots$ . The series is obviously uniformly convergent, so f is continuous.

Now suppose that in base 3 we have $t = 0.c_1c_2c_3 \dots$ with all $c_n = 0$ or 2. Then $9^n t$ = even integer + $0.c_{2n+1}c_{2n+2}\dots$ . Now $0.c_{2n+1}c_{2n+2}\dots$ lies in [0, 1/3] if $c_{2n+1} = 0$ and in [2/3, 1] if $c_{2n+1} = 2$, so $g(9^n t) = 0$ if $c_{2n+1} = 0$, 1 if $c_{2n+1} = 2$. Thus f(t) has the binary expansion $0.a_1a_2a_3 \dots$ , where $a_n = c_{2n-1}/2$. So given any s in [0, 1] we can find infinitely many t with $f(t) = s$, for odd n pick $c_n$ to give the correct binary expansion and for even n pick $c_n$ arbitrarily.

**Problem B4**
A is the 5 x 5 array:

```
11 17 25 19 16
24 10 13 15  3
12  5 14  2 18
23  4  1  8 22
 6 20  7 21  9
```

Pick 5 elements, one from each row and column, whose minimum is as large as possible (and prove it so).

**Solution**
Answer: 25, 23, 20, 18, 15.

We can only pick one of 16, 17, 19, 25; only one of 23, 24; only one of 18, 22; and only one of 20, 21. That only gives 4 elements, so the minimum cannot be larger than 15. The answer shows that 15 can be achieved.

**Problem B5**
$L_1$ is the line $\{ (t + 1, 2t - 4, -3t + 5) : t \text{ real} \}$ and $L_2$ is the line $\{ (4t - 12, -t + 8, t + 17) : t \text{ real} \}$. Find the smallest sphere touching $L_1$ and $L_2$.

**Solution**
The sphere has diameter PQ, where $P(s + 1, 2s - 4, -3s + 5)$ is on $L_1$, and $Q(4t - 12, -t + 8, t + 17)$ is on $L_2$. PQ is as short as possible. It can also be characterised as perpendicular to $L_1$ and $L_2$.

$PQ^2/2 = 7s^2 + st + 9t^2 + 25s - 52t + 457/2$. At the minimum the two partial derivatives must be zero, so $14s + t + 25 = 0$, $s + 18t - 52 = 0$. Solving $s = -2$, $t = 3$. This gives P (-3, -12, 11), Q(0, 5, 20), the radius (= PQ/2) as $1/2 \sqrt{251}$ and the centre as (-3/2, -7/2, 31/2).

**Problem B6**
$\alpha$ and $\beta$ are positive irrational numbers satisfying $1/\alpha + 1/\beta = 1$. Let $a_n = [n\ \alpha]$ and $b_n = [n\ \beta]$, for $n = 1, 2, 3, \dots$ . Show that the sequences $a_n$ and $b_n$ are disjoint and that every positive integer belongs to one or the other.

**Solution**
$\beta$ positive implies $\alpha > 1$, so $[\alpha (n+1)] > [\alpha n]$ and the sequence does not contain any integers twice. Similarly for $[\beta n]$.

If k appears in both sequences, then for some m, n we have: $k < n \alpha < k + 1$, $k < m \beta < k + 1$. Hence $k/\alpha < n < (k+1)/\alpha$, $k/\beta < m < (k+1)/\beta$. Adding gives $k < m+n < k + 1$. Contradiction. So an integer appears in at most one sequence.

Suppose k does not appear in the sequence $[n \alpha]$. Then for some n we have $n \alpha < k$, and $k + 1 < (n + 1) \alpha$. The first inequality implies that $n < k/\alpha = k - k/\beta$. Hence $(k - n) \beta > k$. The second inequality implies that $n + 1 > (k + 1)/\alpha = k + 1 - (k + 1)/\beta$. Hence $(k - n) \beta < k + 1$. So $[(k - n) \beta ] = k$.

**Problem B7**
Given any finite ordered tuple of real numbers X, define a real number [X], so that for all $x_i$, $\alpha$:
(1) [X] is unchanged if we permute the order of the numbers in the tuple X;

(2) $[(x_1 + \alpha, x_2 + \alpha, \dots , x_n + \alpha)] = [(x_1, x_2, \dots , x_n)] + \alpha$;

(3) $[(-x_1, -x_2, \dots , -x_n)] = - [(x_1, x_2, \dots , x_n)]$;

(4) for $y_1 = y_2 = \dots = y_n = [(x_1, x_2, \dots , x_n)]$, we have $[(y_1, y_2, \dots , y_n, x_{n+1})] = [(x_1, x_2, \dots , x_{n+1})]$.
Show that $[(x_1, x_2, \dots , x_n)] = (x_1 + x_2 + \dots + x_n)/n$.

**Solution**
It is convenient to write $[(x_1, \dots , x_n)]$ simply as $[x_1, \dots , x_n]$. We use induction on n. For $n = 1$, we have $[0] = - [0]$, so $[0] = 0$. Then (3) gives that $[x] = x$, which establishes the result for $n = 1$.

$[x, 0] = - [-x, 0] = - [0, -x] = - [x, 0] + x$, so $[x, 0] = x/2$. Hence $[x, y] = y + [x - y, 0] = y + (x - y)/2 = (x + y)/2$, which establishes the result for $n = 2$. Now suppose it is true for n.

We have that $[x, -x, 0, 0, \dots ,0] = - [-x, x, 0, 0, \dots , 0]$ by (3), $= - [x, -x, 0, 0, \dots ,0]$ by (1), so $[x, -x, 0, 0, \dots ,0] = 0$. Now suppose $x_1 + x_2 + \dots + x_{n+1} = 0$, then the n numbers $x_1, x_2, \dots , x_n$ and the n numbers $0, 0, \dots , 0, -x_{n+1}$ have the same sum, so by (4) we have for any z that $[x_1, \dots , x_n, z] = [0, 0, \dots , 0, -x_{n+1}, z]$. Take $z = x_{n+1}$, then $[0, \dots 0, -x_{n+1}, x_{n+1}] = 0$, so $[x_1, \dots , x_n, x_{n+1}] = 0$.

Finally, given arbitrary $x_1, \dots , x_{n+1}$, let k be their mean, then $x_1 - k, x_2 - k, \dots , x_{n+1} - k$ have zero sum, so $[x_1 - k, x_2 - k, \dots , x_{n+1} - k] = 0$. Hence $[x_1, \dots , x_{n+1}] = k$, as required.

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**Problem A1**
For n a positive integer find f(n), the number of pairs of positive integers (a, b) such that ab/(a + b) = n.

**Solution**
ab = n(a + b) implies a(b - n) = nb > 0, so a and b must both exceed n. Let b = n + d, then a = n(n+d)/d = n + $n^2$/d. This is a solution iff d divides $n^2$. So f(n) = the number of divisors of $n^2$.

**Problem A2**
Let S be the set consisting of a square with side 1 and its interior. Show that given any three points of S, we can find two whose distance apart is at most √6 - √2.

**Solution**
Suppose we can find three points of S so that their three distances exceed k = √6 - √2. If one of the points is not on the top side of S and another on the bottom side, we can expand S by a combination of translation and expansion in the direction perpendicular to those sides to get three points with larger minimum distance. Similarly, if one of the points is not on each of the other two sides of S, we can expand S by a combination of expansion and translation to get three points with larger minimum distance. So we end up with a point on each side. But there are only three points, so one point must be at a vertex of the square. The other points on the two sides meeting at the vertex are all a distance <= 1 < k from the vertex, so the other two points must be on the other two sides. Let the vertices be A, B, C, D (in order). Assume the vertex point is at A. Let X be the point on BC a distance 2 - √3 from B. Then AX = k, so the point on BC must lie on the segment XC. Similarly, let Y be the point on DC a distance 2 - √3 from D. Then AY = k, so the point on DC must lie on the segment YC. But XY = k, so the two points will be closer than k unless one is at X and the other at Y but in that case the distances AX, AY do not exceed k. So it is not possible to arrange for the three points to have all distances > k.

**Problem A3**
Let α, β, γ, δ, ε be arbitary reals. Show that $(1 - \alpha)\, e^{\alpha} + (1 - \beta)\, e^{\alpha+\beta} + (1 - \gamma)\, e^{\alpha+\beta+\gamma} + (1 - \delta)\, e^{\alpha+\beta+\gamma+\delta} + (1 - \varepsilon)\, e^{\alpha+\beta+\gamma+\delta+\varepsilon} \le k_4$, where $k_1 = e$, $k_2 = k_1^e$, $k_3 = k_2^e$, $k_4 = k_3^e$ (so $k_4$ is $10^k$ with k approx 1.66 million).

**Solution**
Differentiating shows that the maximum of $(1 - x)e^x$ is 1 and occurs at x = 0. Now consider $(1 - \alpha)\, e^{\alpha} + (1 - \beta)\, e^{\alpha+\beta} = e^{\alpha} ( 1 - \alpha + (1 - \beta)\, e^{\beta})$. For any given α, the maximum over β is 1, so we have to maximise $e^{\alpha}(2 - \alpha)$. Differentiating shows that this has maximum $k_1 = e$ at α = 1.

Now consider $(1 - \alpha)\, e^{\alpha} + (1 - \beta)\, e^{\alpha+\beta} + (1 - \gamma)\, e^{\alpha+\beta+\gamma} = e^{\alpha} ( 1 - \alpha + (1 - \beta)\, e^{\beta} + (1 - \gamma)\, e^{\beta+\gamma})$. We have just shown that the maximum of $(1 - \beta)\, e^{\beta} + (1 - \gamma)\, e^{\beta+\gamma}$ is $k_1$ at β = 1, γ = 0. So we have to maximise $(1 - \alpha + k_1)\, e^{\alpha}$. Differentiating shows that this is $k_2$ at $\alpha = k_1$.

Similarly, the maximum of $(1 - \alpha)\, e^{\alpha} + (1 - \beta)\, e^{\alpha+\beta} + (1 - \gamma)\, e^{\alpha+\beta+\gamma} + (1 - \delta)\, e^{\alpha+\beta+\gamma+\delta}$ is $k_3$ at $\alpha = k_2$, $\beta = k_1$, γ = 1, δ = 0. Finally, the maximum of the desired expression is $k_4$ at $\alpha = k_3$, $\beta = k_2$, $\gamma = k_1$, δ = 1, ε = 0.

**Problem A4**
Given two points P, Q on the same side of a line *l*, find the point X which minimises the sum of the distances from X to P, Q and *l*.

**Solution**
wlog take P to the right of Q and Q to be closer to the line than P. Take A on the line so that PA makes a 30° angle with the line and A is to the left of P. Similarly, take B on the line so that QB makes a 30° angle with the line and B is to the right of Q. There are three cases to consider.
(1) If the two segments PA and QB intersect, then their point of intersection X is the required point.
(2) If A is to the right of B, then join P to Q' the reflection of Q in the line. X is the point at which PQ' meets the line.
(3) If PA and QB do not meet, but A is to the left of B, then take X = Q.

We must now prove these statements. It is convenient to take the line *l* to be the x-axis. Take Q to be (a, b) and P to be (c, d). We assume that d > b > 0. Notice first that the sum of the three distances from the point X with coordinates (x, y) , which we will write as f(x, y), is a continuous and differentiable function of x and y. Also it tends to infinity as x or y tends to infinity. Hence the minimum must occur either (A) at a point where grad f = 0, or (B) on the boundary of the allowed region, in other words on y = 0, or (C) at a point where grad f does not exist.

We have $f(x, y) = y + \sqrt{(x - a)^2 + (y - b)^2} + \sqrt{(x - c)^2 + (y - d)^2}$. Hence grad f = ( (x - a)/QX + (x - c)/PX, 1 + (y - b)/QX + (y - d)/PX) = **r** + **q** + **p**, where **r** is the unit vector along the perpendicular line from the line *l* to X, **q** is the unit vector from Q to X, and **p** is the unit vector from P to X. By resolving along the line perpendicular to **r** we find that the angles between **q** and **r** and between **p** and **r** must be equal. Similarly, the other pairs of angles. Hence the angle between each pair of **p**, **q**, **r** is $120^o$. But it is clear that the only possible such point is the intersection H of the segments PA and QB (if it exists). H is thus the only candidate of type (A).

It is obvious that for points X on the line *l*, f(x, y) is minimised at K, the intersection of the line with PQ' (joining P to the reflection of Q), so that is the only candidate of type (B). Finally, the only points were the existence of grad f is doubtful are P and Q and indeed it is easy to see that grad f is not well defined there (for example, the limits as we approach parallel to the two axes are different). But clearly the distance sum is lower at Q than at P, so Q is the only candidate of type (C).

We now look separately at cases (1), (2) and (3). In case (1) all three candidates H, K and Q are available. We show first that H is better than Q. Let the perpendicular from Q to *l* meet *l* at C and the perpendicular from H to *l* meet *l* at D. Then angle HQC is $60^o$, so QC = HD + QH/2. By the cosine rule we have $PQ^2 = PH^2 + QH^2 + PH\cdot QH > (PH + QH/2)^2$. Hence PQ + QH/2 > PH + QH and so PQ + QC = PQ + QH/2 + HD > PH + QH + HD. Next we show that H is better than K. As before let HD be the perpendicular to *l*. It is sufficient to show that PH + QH + HD < PD + QD, since we know that PD + QD <= PK + QK. Now $PD^2 = PH^2 + HD^2 + PH\cdot HD > (PH + HD/2)^2$, so PD > PH + HD/2. Similarly, QD > QH + HD/2. Adding gives the required result. That shows that H minimises the distance sum for case (1).

In case (2) only candidates K and Q are available. Take QC and PE as the perpendiculars to *l*. Then $PQ^2 = CE^2 + (PE - QC)^2$, $PQ'^2 = CE^2 + (PE + QC)^2$. We wish to show that PQ' < PQ + QC, or $CE^2 + (PE + QC)^2 < CE^2 + (PE - QC)^2 + QC^2 + 2\ QC\ \sqrt{CE^2 + (PE - QC)^2}$, or $(4\ PE - QC)^2 < 4(CE^2 + PE^2 - 2\ PE\ QC + QC^2)$, or $4\ CE^2 > 12\ PE^2 - 3\ QC^2$. That is certainly true since we have $CE > (PE + QC)\ \sqrt{3}$.

Finally, in case (3) the only candidates are K and Q. So as in the previous paragraph we wish to show that $4\ CE^2 < 12\ PE^2 - 3\ QC^2$. Now we have $CE < \sqrt{3}\ (PE - QC)$ from which the inequality follows.

**Problem A5**
The real polynomial p(x) is such that for any real polynomial q(x), we have p(q(x)) = q(p(x)). Find p(x).

**Solution**
Take q(x) = x + k and set x = 0. Then we have p(k) = p(0) + k. This is true for all k, so p(x) must be x + c for some c. Now take $q(x) = x^2$. We get $x^2 + c = (x + c)^2 = x^2 + 2cx + c^2$. Hence c = 0 and p(x) = x.

**Problem A6**
A player throws a fair die (prob 1/6 for each of 1, 2, 3, 4, 5, 6 and each throw independent) repeatedly until his total score $\geq$ n. Let p(n) be the probability that his final score is n. Find lim p(n).

**Solution**
Answer: 2/7.

For i = 1, 2, 3, 4, 5, let $p_i(n)$ = prob that the final score is n+i if the player stops when his total score is at least n. We note that p(n) is also the probability that the player's total equals n on some throw if he throws repeatedly. Now we can see that $p_5(n) = 1/6\ p(n-1)$, because the only way to achieve a final score of n+5 without passing through n, n+1, n+2, n+3, n+4 is to reach n-1 and then throw a 6. Similarly, $p_4(n) = 1/6\ p(n-1) + 1/6\ p(n-2)$, because to reach n+4 without passing through n, n+1, n+2, n+3 you must either to through n-1, which requires reaching n-1 and then throwing a 5, or not, in which case you must reach n-2 and then throw a 6. Similarly, $p_3(n) = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3)$, $p_2 = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3) + 1/6\ p(n-4)$, and $p_1(n) = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3) + 1/6\ p(n-4) + 1/6\ p(n-5)$. Adding, we get: $1 = p(n) + p_1(n) + p_2(n) + p_3(n) + p_4(n) + p_5(n) = p(n) + 5/6\ p(n-1) + 4/6\ p(n-2) + 3/6\ p(n-3) + 2/6\ p(n-4) + 1/6\ p(n-5)$.

In the limit, $p(n-5) = p(n-4) = \dots = p(n)$. Hence we get: $p_1(n) = 5/6\ p(n)$, $p_2(n) = 4/6\ p(n)$, $p_3(n) = 3/6\ p(n)$, $p_4(n) = 2/6\ p(n)$, $p_5(n) = 1/6\ p(n)$. But they sum to 1, so $p(n) = 2/7$.

We have (as above) $p(n) = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3) + 1/6\ p(n-4) + 1/6\ p(n-5) + 1/6\ p(n-6)$ (*). Let $m(n) = \min\{ p(n-1), p(n-2), p(n-3), p(n-4), p(n-5), p(n-6) \}$. Then (*) establishes that $p(n) \geq m(n)$, and so $m(n+1) \geq m(n)$. Similarly, let $M(n) = \max\{ p(n-1), p(n-2), p(n-3), p(n-4), p(n-5), p(n-6) \}$. Then (*) shows that $p(n) \leq M(n)$, so $M(n+1) \leq M(n)$. Thus $m(n)$ is a monotonic increasing sequence and $M(n)$ is a monotonic decreasing sequence. But $m(n)$ is obviously bounded above by any $M(m)$, and $M(n)$ is bounded below by any $m(m)$. So both sequences converge. Suppose they converged to different limits. So $m(n)$ converges to $m$ and $M(n)$ converges to $M$ with $M - m > 36k > 0$. Take $n$ sufficiently large that $m(n) > m - 6k$. At least one of the terms on the rhs of (*) must equal $M(n)$ and the others are at least $m(n)$, so $p(n) \geq 5/6\ m(n) + 1/6\ M(n) > 5/6\ (m - 6k) + 1/6\ M > 5/6\ m - 5k + 1/6\ (m + 36k) = m+ k$. But that means that $m(n) > m+k$ for all sufficiently large n. Contradiction. Hence M and m are the same and p(n) must have the same limit.

**Problem A7**
Let f(n) be the smallest integer such that any permutation on n elements, repeated f(n) times, gives the identity. Show that $f(n) = p\ f(n - 1)$ if n is a power of p, and $f(n) = f(n - 1)$ if n is not a prime power.

**Solution**
A permutation on n elements can be written as a product of disjoint cycles of length <= n. So f(n) is the smallest number divisible by 2, 3, ... , n.

If n is not a power of a prime, then we can write $n = rs$, with r and s relatively prime and each greater than 1. Then $r, s < n - 1$, so r and s divide f(n-1) and hence n divides f(n-1). So $f(n) = f(n-1)$. If $n = p^{m+1}$, then $p^m$ divides f(n-1), but not $p^{m+1}$, so $f(n) = p\ f(n)$.

**Problem B1**
Find all pairs of unequal integers m, n such that $m^n = n^m$.

**Solution**
Answer: (2, 4), (4, 2), (-2, -4), (-4, -2).

Suppose first that m and n are both positive. Assume $m > n$. Then we can put $m = n + k$ with $k > 0$. Hence $(1 + k/n)^n = n^k$. But for $x > 1$ we have $1 + x < e^x$ (the derivative of $f(x) = e^x - x - 1$ is positive and $f(0) = 0$) and hence $(1 + k/n)^n < e^k$. So there are no solutions for $n > 2$. If $n = 1$, then $n^m = 1$ and hence $m = 1$, contradicting the fact that m and n are unequal. If $n = 2$, then m must be a power of 2. Suppose $m = 2^h$. Then we find h = 1 or 2. h = 1 is invalid (because m and n are unequal), so m = 4. There is also the corresponding solution with $m < n$.

If $n < 0$ and $m > 0$, then $n^m = 1/m^{-n}$. So m divides 1 and hence m = 1. But m must be even for to make $n^m$ positive. Contradiction. So there are no solutions of this type. If m and n are both negative, then -m, -n is a solution, so the only possibilities are (-2, -4) and (-4, -2) and it is readily checked that these are indeed solutions.

**Problem B2**
Let $f(m, n) = 3m+n+(m+n)^2$. Find $\sum_0^\infty \sum_0^\infty 2^{-f(m, n)}$.

**Solution**
Answer: 4/3.

The obvious approach is to sum over n and then over m (or vice versa). But it is not obvious how to sum $x^N$ where N runs through the squares. So we wonder just what values f(m, n) can take. A little experimentation suggests that it takes each even value just once.

In fact, it is clear that $r(r+1)$ is always even and that the difference between $r(r+1)$ and the next number in the sequence, $(r+1)(r+2)$ is $2r+2$, so any positive even number can be uniquely expressed as $r(r+1) + 2s$ with $0 \leq s \leq r$. But taking $m = s$, $n = r-s$, this is equivalent to the statement that any positive even number can be uniquely expressed as $3m+n+(m+n)^2$.

Hence the sum is just $1 + 1/4 + 1/4^2 + \dots = 4/3$.

**Problem B3**
Fluid flowing in the plane has the velocity $(y + 2x - 2x^3 - 2xy^2, - x)$ at $(x, y)$. Sketch the flow lines near the origin. What happens to an individual particle as $t \to \infty$ ?

**Solution**
The key insight is that the unit circle centre the origin is one possible trajectory. [The velocity is $(y + 2x - 2xr^2, - x) = (y, -x)$ on $r = 1$.] Hence fluid starting inside this circle remains inside it and fluid starting outside it remains outside. Differentiating $r^2 = x^2 + y^2$ gives $r\, dr/dt = 2x^2(1 - r^2)$. Hence if $r < 1$, then $dr/dt$ is positive, so points inside the unit circle move outwards towards it. Similarly, if $r > 1$, then $dr/dt$ is negative, so points outside the unit circle move inwards towards it.

To find the behaviour close to the origin, we drop the cubic terms. The equations are then linear and can be solved to get $x = (at + b)e^t$, $y = (-at + a - b)e^t$. This gives S shaped curves centred on the origin at a $45^o$ angle touching the line $y = -x$.

**Problem B4**
Show that if an (infinite) arithmetic progression of positive integers contains an nth power, then it contains infinitely many nth powers.

**Solution**
Let the difference between adjacent terms of the progression be d. Suppose that $N^n$ is an nth power in the progression. Then $(N+d)^n$ is a larger nth power in the progression (it is obvious from the binomial expansion that it has the form $N + kd$).

**Problem B5**
Define $a_n$ by $a_0 = 0$, $a_{n+1} = 1 + \sin(a_n - 1)$. Find $\lim (\sum_0^n a_i)/n$.

**Solution**
Answer: 1.

Note that $1 - a_{n+1} = \sin (1 - a_n)$. Put $c_n = 1 - a_n$. Note that $c_0$ belongs to the interval $(0, 1]$. Now for x in $(0, 1]$ we have that $\sin x < x$ and $\sin x$ is also in $(0 , 1]$. So it follows (by a trivial induction) that $1 = c_0 > c_1 > c_2 > ... > c_n > 0$. So $c_n$ is a monotonically decreasing sequence bounded below by 0. Hence it must tend to a limit $c \ge 0$. But c must satisfy $c = \sin c$. Hence $c = 0$. Hence $a_n$ converges to 1. Hence $(\sum_0^n a_i)/n$ converges to 1 also.

**Problem B6**
Let $2^{f(n)}$ be the highest power of 2 dividing n. Let $g(n) = f(1) + f(2) + ... + f(n)$. Prove that $\sum \exp( -g(n) )$ converges.

**Solution**
If n is odd, then $f(n) = 0$. If n is even, then $f(n) \ge 1$. Hence $g(n+2) \ge g(n) + 1$. So the series is dominated by $1 + (1/e + 1/e) + (1/e^2 + 1/e^2) + ... < 2/(1 - 1/e)$. So the sums are bounded above. All terms are positive, so it must converge.

**Problem B7**
Let R' be the non-negative reals. Let $f, g : R' \to R$ be continuous. Let $a : R' \to R$ be the solution of the differential equation: $a' + f a = g$, $a(0) = c$. Show that if $b : R' \to R$ satisfies $b' + f b \ge g$ for all x and $b(0) = c$, then $b(x) \ge a(x)$ for all x. Show that for sufficiently small x the solution of $y' + f y = y^2$, $y(0) = d$, is $y(x) = \max ( d e^{-h(x)} - \int_0^x e^{-(h(x)-h(t))} u(t)^2 dt )$, where the maximum is taken over all continuous $u(t)$, and $h(t) = \int_0^t (f(s) - 2u(s)) ds$.

**Solution**
Put $F(x) = \int_0^x f(t) dt$. Put $A(x) = a(x) e^{F(x)}$, $B(x) = b(x) e^{F(x)}$. Then $A' = (a' + f a) e^F = g e^F$, and $B' = (b' + f b) e^F \ge g e^F$, since $e^F > 0$. Hence $B' \ge A'$ for all x. But $B(0) = A(0)$, so $B(x) \ge A(x)$ for all x and hence $b(x) \ge a(x)$ for all x. Let u be any continuous function on R'. Put $U(x) = \int_0^x u(t) dt$. Put $h(x) = F(x) - 2 U(x) = \int_0^x f(t) - 2u(t) dt$. Then $h(0) = 0$. We have $(y - u)^2 \ge 0$, so $y^2 - 2uy \ge -u^2$. Now put $z = y e^h$, so that $z(0) = y(0) e^{h(0)} = d$.

$z' = (y' + h'y) e^h = (y' + (f - 2u)y) e^h = (y^2 - 2uy) e^h \ge -u^2 e^h$. Hence $z(x) = z(0) + \int_0^x z'(t) dt \ge d - \int_0^x u(t)^2 e^{h(t)} dt$.
Hence $y(x) = z(x) e^{-h(x)} \ge d e^{-h(x)} - \int_0^x e^{-(h(x) - h(t))} u(t)^2 dt$. So $z(x) \ge$ the maximum over all u.
If we put $u = y$ (the solution), then we get equality.

## 22nd Putnam 1961

**Problem A1**
The set of pairs of positive reals $(x, y)$ such that $x^y = y^x$ form the straight line $y = x$ and a curve. Find the point at which the curve cuts the line.

**Solution**
Put $y = kx$. Then we get $x = k^{1/(k-1)}$, $y = k^{k/(k-1)}$. This gives a curve which cuts $y = x$ at $x = \lim k^{1/(k-1)}$, where the limit is taken as k tends to 1. Put $k = 1 + 1/n$, then the limit is $(1 + 1/n)^n$, which is e. So the point of intersection is (e, e).

**Problem A2**
For which real numbers $\alpha$, $\beta$ can we find a constant k such that $x^\alpha y^\beta < k(x + y)$ for all positive x, y?

**Solution**
Answer: $\alpha, \beta \ge 0$ and $\alpha + b = 1$.

If $\alpha < 0$, then taking x sufficiently small and $y = 1$ gives $x^\alpha y^\beta > k(x + y)$ for any k. Similarly for $\beta < 0$. If $\alpha + b > 1$, then taking $x = y$ sufficiently large gives $x^\alpha y^\beta > k(x + y)$ for any k. On the other hand, if $\alpha + \beta < 1$, then taking $x = y$ sufficiently small gives $x^\alpha y^\beta > k(x + y)$ for any k. Suppose, on the other hand that $\alpha, b \ge 0$ and $\alpha + \beta = 1$. Take $x \ge y$. Then $x^\alpha y^\beta \le x^{\alpha+\beta} = x$, so the result holds with $k = 1$.

**Problem A3**
Find $\lim_{n\to\infty} \sum_1^N n/(N + i^2)$, where $N = n^2$.

**Solution**
As usual, we try integration. We can write the sum as $1/n \sum_1^N 1/(1 + (i/n)^2)$. This is a Riemann sum for the integral $\int_0^n 1/(1 + x^2)\, dx$ and hence tends to $\int_0^\infty 1/(1 + x^2)\, dx = \tan^{-1}x \,|_0^\infty = \pi/2$.

**Problem A4**
If $n = \prod p^r$ be the prime factorization of n, let $f(n) = (-1)^{\sum r}$ and let $F(n) = \sum_{d|n} f(d)$. Show that $F(n) = 0$ or 1. For which n is $F(n) = 1$?

**Solution**
Answer: $F(n) = 1$ for n square.

If some r is odd, then the factors of n can be grouped into pairs $p^a m$, $p^{r-a}m$ with $f(p^a m)$, $f(p^{r-a}m)$ having opposite signs. So in this case $F(n) = 0$.

If all r are even, so that n is a square, then we may use induction on the number N of prime factors. For $N = 1$, we have $n = p^{2m}$, so there are m+1 factors $p^{2r}$ with $f = 1$ and m factors $p(2r+1)$ with $f = -1$. So the result is true for $N = 1$. Suppose it is true for N. Take $n = p^{2m}P$, where P has m prime factors. Each divisor of n has the form $p^{2r}s$ or $p^{2r+1}s$, where s is a divisor of P. For the first type $f(p^{2r}s) = f(s)$, so summing over s gives 1. Then summing over r gives m+1. For the second type $f(p^{2r+1}s) = -f(s)$, so summing over s gives -1. Then summing over r gives -m. Hence $F(n) = 1$ and the result is true for N+1.

**Problem A5**
Let X be a set of n points. Let P be a set of subsets of X, such that if $A, B \in P$, then $X - A$, $A \cup B$, $A \cap B \in P$. What are the possible values for the number of elements of P?

**Solution**
Answer: $2, 4, 8, \dots , 2^n$.

Take $0 \le r < n$. Let $A_1 = \{1\}$, $A_2 = \{2\}$ , ... , $A_r = \{r\}$, $A_{r+1} = \{r+1, r+2, \dots , n\}$. Consider the collection of all unions of sets $A_i$. This collection has $2^{r+1}$ elements and satisfies the conditions. So this shows that we can achieve the values $2, 4, 8, \dots , 2^n$.

To prove the converse we use induction on n. It is obviously true for n = 1. Suppose it is true for all n < N. Let X have N points and P be a set of subsets of X satisfying the condition. If |P| = 2, then there is nothing to prove. Otherwise we can find a member Y of P other than the empty set and X which has no non-trivial subsets in P. Now consider the collection Q of subsets of X - Y which are in P. It is easy to check that (1) P is just the collection of all A and A ∩ Y, where A is in Q, so that |P| = 2|Q|, and (2) Q satisfies the conditions, so |Q| is a power of 2 by induction.

**Problem A6**
Consider polynomials in one variable over the finite field $F_2$ with 2 elements. Show that if n + 1 is not prime, then $1 + x + x^2 + ... + x^n$ is reducible. Can it be reducible if n + 1 is prime?

**Solution**
Let n+1 = ab, then $1 + x + x^2 + ... + x^n = (1 + x + x^2 + ... + x^{a-1})(1 + x^a + x^{2a} + ... + x^{ab-a})$. Note that this does not depend upon the field having two elements.

Yes. For example, $(1 + x + x^3)(1 + x^2 + x^3) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$.

**Problem A7**
S is a non-empty closed subset of the plane. The disk (a circle and its interior) D ⊇ S and if any disk D' ⊇ S, then D' ⊇ D. Show that if P belongs to the interior of D, then we can find two distinct points Q, R ∈ S such that P is the midpoint of QR.

**Solution**
The existence of D imposes strong restrictions on S. In fact, we will show that S must contain the entire perimeter C of D. For suppose it does not contain a point X on C. Then since S is closed, it does not contain a neighbourhood of X. So we can find a small circle C' centre X so that the disk D' enclosed by C' lies entirely outside S. Let C' meet C at A and B. Take a circle C" through A and B with centre on the same side of AB as the centre of C, but further away (so that it has a larger radius than C). Then C" contains all points of D - D' and hence all points of S. But it does not contain all points of D. Contradiction.

Finally, note that any point in the interior of D lies on the midpoint of some chord of C.

**Problem B1**
$a_n$ is a sequence of positive reals. $h = \lim (a_1 + a_2 + ... + a_n)/n$ and $k = \lim (1/a_1 + 1/a_2 + ... + 1/a_n)/n$ exist. Show that $h k \ge 1$.

**Solution**
Apply the arithmetic-geometric mean theorem to each sum. We get that $(a_1 + a_2 + ... + a_n)/n \ge (a_1a_2 ... a_n)^{1/n}$, $(1/a_1 + 1/a_2 + ... + 1/a_n)/n = 1/(a_1a_2 ... a_n)^{1/n}$. Hence their product is at least 1. Hence the product of the limits also.

**Problem B2**
Two points are selected independently and at random from a segment length β. What is the probability that they are at least a distance α (< β) apart?

**Solution**
Consider which points (x, y) of the square x = 0 to β, y = 0 to β have |x - y| ≥ α. Evidently the acceptable points lie in the two right-angled triangles: (0, α), (0, β), (β-α, β) and (α, 0), (β, 0), (β, β-α). These fit together to give a square side β-α, so the area of the acceptable points is $(β-α)^2$ out of a total area of $β^2$. Thus the probability is $(β-α)^2/(β)^2$.

**Problem B3**
A, B, C, D lie in a plane. No three are collinear and the four points do not lie on a circle. Show that one point lies inside the circle through the other three.

**Solution**
If the convex hull is a triangle, then the fourth point lies inside its circumcircle. So suppose ABCD is convex. One pair of opposite angles must have sum greater than 180° (otherwise the points would lie on a circle). Suppose they are A and C. Then we claim that C lies inside the circle through A, B, D. Take a point C' on the ray DC on the far

side of C from D such that $\angle CBC' = \angle A + \angle C - 180^o$. Then $\angle C' = 180^o - \angle A$. So C' lies on the circle. Hence C which lies on the segment C'D lies inside the circle.

**Problem B4**

Given $x_1, x_2, \dots, x_n \in [0, 1]$, let $s = \Sigma_{1\le i<j\le n} |x_i - x_j|$. Find f(n), the maximum value of s over all possible $\{x_i\}$.

**Solution**

Answer: $f(2m) = m^2$ (half the points 0, half 1); $f(2m+1) = m^2 + m$ (m+1 points 0, m points 1).

It remains to prove that we cannot do better than this. We prove that this is best possible by induction. It is obvious for n = 2, 3. Suppose it is true for n. Take n+2 points. Let A be the leftmost point and B the rightmost point. Then the new terms in the sum the n+2 points are AB and AC, CB for each existing C. But these 2n + 1 terms sum to at most n + 1 ($AC + CB = AB \le 1$). So $f(n+2) \le f(n) + n + 1$. For n = 2m this gives $f(2m+2) \le m^2 + 2m + 1 = (m + 1)^2$. For n = 2m + 1, it gives $f(2m+3) \le m^2 + m + 2m + 1 + 1 = (m+1)^2 + (m+1)$.

**Problem B5**

Let n be an integer greater than 2. Define the sequence $a_m$ by $a_1 = n$, $a_{m+1}$ = n to the power of $a_m$. *Either* show that $a_m < n!! \dots !$ (where the factorial is taken m times), *or* show that $a_m > n!! \dots !$ (where the factorial is taken m-1 times).

**Solution**

If it was true that $n^k < k!$. then the upper limit on $a_m$ would be almost trivial. Unfortunately, $n^k > k!$ for small k, which results in a few complications. Let $n_m = n! \dots !$ (m times). We will prove that $a_m < n_m$ by induction on m. For m = 1, the result just states that n < n!, which is obviously true. Suppose now that $n \ge 5$. Then $n! \ge n(n-1)3 > 2n^2$. So, a fortiori, $n_m > 2n^2$ for $m \ge 1$. It is easy to check that for n = 3, $a_2 = 27 < n_2 = 3!! = 720$, so the result is true for m = 2. Also $n_2 > 2n^2 = 18$. Similarly, for n = 4, $a_2 = 256 < n_2 = 4!! = 24!$ which establishes the result for m = 2. Also $n_2 > 2n^2 = 32$. So for all n we have a starting case m for which also $n_m > 2n^2$.

If $k \ge 2n^2$, then half the terms in k! are at least $n^2$, so their product is at least $n^{2\,k/2} = n^k$. In other words $k! > n^k$. So suppose that $a_m < n_m$, (*). Then since $n_m > 2n^2$, we may take (*) to the power of n to get $a_{m+1} = n^{lhs} < n^{rhs} < n_{m+1}$. Hence the result is true for all m.

The lower limit is harder because it is not true that $n^k > k!$ , so the result for m+1 does not follow in an immediate way from that for m.

Let $f_0(n) = n$, $f_1(n) = f_0(n)\, n!$, $f_2(n) = f_1(n)\, n!!$ , $f_3(n) = f_2(n)\, n!!!$, ... . We show that $f_{m+1}(n)$ < n to the power of $f_m(n)$. We use induction on m. For m = 0, this reads $n\, n! < n^n$, which is obviously true. Now assume it is true for m ***and all n***. Then we have $f_{m+1}(n) = n\, f_m(n!)$ < n (n! to the power of $f_{m-1}(n!)$ ) < (n n!) to the power of $f_{m-1}(n!)$ < $n^n$ to the power of $f_{m-1}(n!)$ = n to the power of $f_m(n)$.

It is now an easy induction that $f_m(n) < a_{m+1}$. For m = 1, this just states that $n\, n! < n^n$. Suppose it is true for m, then $f_{m+1}(n)$ < n to the power of $f_m(n)$ < n to the power of $a_{m+1} = a_{m+2}$. But obviously n! ... ! (! taken m times) < $f_m(n)$ and hence < $a_{m+1}$.

**Problem B6**

Let y be the solution of the differential equation $y'' = - (1 + \sqrt{x})\, y$ such that y(0) = 1, y'(0) = 0. Show that y has exactly one zero for $x \in (0, \pi/2)$ and find a positive lower bound for it.

**Solution**

Let z be the solution of z'' = -z with z(0) = 1, z'(0) = 0. Suppose y > 0 for $0 <= x < \pi/2$. Then since $y''z - yz'' = -(1 + \sqrt{x})\, yz + yz = - \sqrt{x}\, yz$, we have $\int_0\pi/2 - \sqrt{x}\, yz\, dx = \int_0^{\pi/2} (y''z - yz'')\, dx = (y'z - yz')\, |_0^{\pi/2} = y(\pi/2) \ge 0$. But $- \sqrt{x}\, yz < 0$ throughout the range, so $\int_0^{\pi/2} - \sqrt{x}\, yz\, dx < 0$. Contradiction. Hence y has at least one zero in $(0, \pi/2)$.

Let w be the solution of w'' = -3w with w(0) = 1, w'(0) = 0. Then $w = \cos (\sqrt{3})x$. Suppose y had a zero in $(0, \pi/(2\sqrt{3}))$. Then the same argument would show that w had a root in $(0, \pi/(2\sqrt{3})$ which is false. So $\pi/(2\sqrt{3})$is a positive lower bound for any zero of y.

Finally, suppose that y has more than one zero in $(0, \pi/2)$. Let the first be at $x = h$ and the second at $x = k$. Take $v(x)$ to be $A \cos( (\sqrt{3}) x + B)$. We select A and B so that $v(h) = 0$ and $v'(h) = y'(h)$. Now v cannot have another zero in $(h, k)$, so we can establish a contradiction by a similar argument to the above. We have $v'' = -3v$. Hence $y''v - yv'' = ( -(1 + \sqrt{x}) + 3) yv$ which is strictly positive on $(h, k)$. Hence $\int_h^k (y''v - yv'')\, dx > 0$. But $\int_h^k (y''v - yv'')\, dx = (y'v - yv') |_h^k = y'(k)v(k) < 0$. [y' must be positive since y is zero and $y(x) < 0$ for x just less than k, whilst we know that $v(k)$ is negative). Contradiction. Hence y has exactly one zero.

**Problem B7**

The sequence of non-negative reals satisfies $a_{n+m} \le a_n a_m$ for all m, n. Show that $\lim a_n^{1/n}$ exists.

**Solution**

$a_n \le a_1 a_{n-1} \le a_1^2 a_{n-2} \le ... \le a_1^n$. So $a_n^{1/n} \le a_1$. All $a_n$ are non-negative, so $a_n^{1/n} \ge 0$. Thus we have established that $\{a_n^{1/n}\}$ is bounded.

Fix n. Take any $N > n$. Then we may write $N = nq + r$, with $0 \le r < n$. We have $a_N \le a_n^q a_r$ , so $a_N^{1/N} \le a_n^s a_r^{1/N}$, where $s = q/N = (1 - r/N)/n$. But $(1 - r/N)$ tends to 1 as N tends to infinity, as does $a_r^{1/N}$. Hence $\limsup a_N^{1/N} \le a_n^{1/n}$. This is true for all n, so the sequence cannot have more than one limit point and hence converges.

# 23rd Putnam 1962

**Problem A1**
5 points lie in a plane, no 3 collinear. Show that 4 of the points form a convex quadrilateral.

**Solution**
If the convex hull has 4 or 5 vertices, then we are done. If not, then two of the points, say D and E, must lie inside the triangle ABC formed by the other three. DE must intersect just two of AB, BC and CA. Suppose that it does not intersect AB. Then the four points A, B, D, E form a convex quadrilateral.

**Problem A2**
Let R be the reals. Find all $f : K \to R$, where K is $[0, \infty)$ or a finite interval $[0, a)$, such that $(1/k \int_0^k f(x)\,dx)^2 = f(0) f(k)$ for all k in K.

**Solution**
Answer: for $[0, \infty)$ $f(x) = b/(cx + 1)^2$ with c non-negative. For $[0, a)$, $f(x) = b/(cx + 1)^2$ with $c >= -1/a$.

Put $y = \int_0^x f(t)\,dt$. Then the given equation becomes $y^2/x^2 = by'$, where $b = f(0)$. Integrating, $b/y = c + 1/x$, or $y = bx/(cx + 1)$. Differentiating gives $f(x) = y' = b/(cx + 1)^2$.

$f(-1/c)$ is undefined, so if K is the half-line, then c cannot be negative. If K is the finite interval $[0, a)$, then we require $c \ge -1/a$. Note that we can have equality because k must be strictly less than a. On the other hand, b can have any value.

**Problem A3**
ABC is a triangle and $k > 0$. Take A' on BC, B' on CA, C' on AB so that AB' = k B'C, CA' = k A'B, BC' = k C'A. Let the three points of intersection of AA', BB', CC' be P, Q, R. Show that the area PQR $(k^2 + k + 1)$ = area ABC $(k - 1)^2$.

**Solution**
There does not seem to be a geometric solution, so one is faced with a messy algebraic calculation. There are various ways of doing it. The worst is probably to use ordinary Cartesian coordinates.

Let us use points in the Argand diagram. Take A to be the origin and C to be the real point $1+k$. Take B to be $z(1+k)$. Then B' is k and A' is $zk+1$ (check: $( (zk+1) - (1+k) )/( z(1+k) - (zk+1) ) = (zk-k)/(z-1) = k$). We want to find R, the intersection of AA' and BB'. It lies on AA', so it is $r(zk+1)$ for some $0 < r < 1$. But it lies on BB', so it is $k + s(zk+z-k)$ for some $0 < s < 1$. After some manipulation, we get $r = (k^2+k)/(k^2+k+1)$.

Now take $k < 1$. Then area ARB = (AR/AA') area AA'B = r area AA'B (this is where we need $k < 1$, because if $k > 1$, then the geometry changes). But area AA'B = (A'B/CB) area ABC, so area ARB = $r/(1+k)$ area ABC = $k/(k^2+k+1)$ area ABC. Similarly, taking P as the intersection of BB' and CC' and Q as the intersection of AA' and CC', we have that area AQC = $k/(k^2+k+1)$ area ABC, and area CPB = $k/(k^2+k+1)$ area ABC. Adding we get (area ABC - area PQR), so area PQR/area ABC = $1 - 3k/(k^2+k+1) = (1 - k)^2/(k^2+k+1)$.

If $k > 1$, then we get the result with $1/k$ replacing k, but multiplying through by $k^2$ gives the same formula.

**Problem A4**
R is the reals. [a, b] is an interval with $b \ge a + 2$. $f : [a, b] \to R$ is twice differentiable and $|f(x)| \le 1$ and $|f''(x)| \le 1$. Show that $|f'(x)| \le 2$.

**Solution**
Take the interval to be [-1, 1]. Taylor's formula gives $f(1) = f(x) + (1 - x) f'(x) + (1 - x)^2 f''(h)/2$, $f(-1) = f(x) + (-1 - x) f'(x) + (-1 - x)^2 f''(k)/2$ for some h, k in the interval (note that we are expanding about x).

Subtracting, $f(1) - f(-1) = 2 f'(x) + (1 - x)^2 f''(h)/2 - (1 + x)^2 f''(k)/2$. Hence $2 |f'(x)| \le |f(1)| + |f(-1)| + (1 - x)^2 |f''(h)|/2 + (1 + x)^2 |f''(k)|/2 \le 2 + (1 - x)^2/2 + (1 + x)^2/2 = 3 + x^2 \le 4$. So $|f'(x)| \le 2$.

**Problem A5**

Find nC1 $1^2$ + nC2 $2^2$ + nC3 $3^2$ + ... + nCn $n^2$ (where nCr is the binomial coefficient).

**Solution**

Answer: $n(n+1)2^{n-2}$.

Differentiate $(1 + x)^n$, multiply by x and differentiate again. Set x = 1.

**Problem A6**

X is a subset of the rationals which is closed under addition and multiplication. $0 \notin X$. For any rational $x \neq 0$, just one of x, $-x \in X$. Show that X is the set of all positive rationals.

**Solution**

Either x or -x belongs to X. X is closed under multiplication, so the square $x^2 = (-x)^2$ belongs to X. In particular, 1 belongs to X. Hence by repeated addition all positive integers must belong to X. Suppose positive rational m/n does not belong to X. Then -m/n does, and hence by repeated addition -m. So m does not belong to X. Contradiction. So X contains all positive rationals. But if x is in X, -x is not, so X does not contain any negative rationals and hence X is just the set of all positive rationals.

**Problem B1**

Define $x^{(n)} = x(x - 1)(x - 2) \dots (x - n + 1)$ and $x^{(0)} = 1$. Show that $(x + y)^{(n)} = nC0\ x^{(0)}y^{(n)} + nC1\ x^{(1)}y^{(n-1)} + nC2\ x^{(2)}y^{(n-2)} + \dots + nCn\ x^{(n)}y^{(0)}$.

**Solution**

Induction on n. It is obviously true for n = 1. Suppose it is true for n. We now multiply the rhs by (x + y - n). We now write:

$$nC0\ x^{(0)}y^{(n)}(x + y - n) = (nC0\ x^{(0)}y^{(n)}\ (y-n)\ ) + (nC0\ x^{(0)}y^{(n)}\ x\ );$$
$$nC1\ x^{(1)}y^{(n-1)}\ (x + y - n) = (nC1\ x^{(1)}y^{(n-1)}\ (y-n+1)\ ) + (nC1\ x^{(1)}y^{(n-1)}\ (x-1)\ );$$
$$nC2\ x^{(2)}y^{(n-2)}\ (x + y - n) = (nC2\ x^{(2)}y^{(n-2)}\ (y-n+2)\ ) + (nC2\ x^{(2)}y^{(n-2)}\ (x-2)\ );$$
$$\dots$$
$$nCn\ x^{(n)}y^{(0)}\ (x + y - n) = (nCn\ x^{(n)}y^{(0)}\ y\ ) + (nCn\ x^{(n)}y^{(0)}\ (x-n)\ ).$$

Now the first term in the 1st equation gives $(n+1)C0\ x^{(0)}y^{(n+1)}$. The second term in the 1st equation and the first term in the 2nd equation give $(n+1)C1\ x^{(1)}y^{(n)}$. Then the second term in the 2nd equation and the first in the 3rd equation gives $(n+1)C2\ x^{(2)}y^{(n-1)}$ and so on. Finally the second term in the last equation gives $(n+1)C(n+1)\ x^{(n+1)}y^{(0)}$. So we have the result for n+1.

**Problem B2**

Let R be the reals, let N be the set of positive integers, and let $P = \{X : X \subseteq N\}$. Find $f : R \to P$ such that $f(a) \subset f(b)$ (and $f(a) \neq f(b)$ ) if $a < b$.

**Solution**

Let Q be the rationals. Define $g: Q \to N$ as follows. For m/n where m and n are positive integers without any common factor, let $g(m/n) = 2^m3^n$ and let $g(-m/n) = 2^m5^n$. Now define $f: R \to P$ by $f(x) = \{ g(r) : r <= x \text{ is rational} \}$. We claim that f has the required property.

Clearly g is injective. Now suppose x < y are reals. We can find a rational r such that x < r < y. So g(r) belongs to f(y) but not to f(x). But S belongs to f(x), then S = g(q) for some q in Q with q < x. So q < y and hence S belongs to f(y) also. So f(x) is a subset of f(y).

**Problem B3**

Show that a convex open set in the plane containing the point P, but not containing any ray from P, must be bounded. Is this true for any convex set in the plane?

**Solution**

Answer: no.

Let S be the convex set. Take any ray R from P. We can find X on R not in S. Since S is open we can take a small segment AB perpendicular to R through P with AP = PB which is entirely contained in S. Take A' on the line AX, the opposite side to A, and B' on BX the opposite side to B, with A'B' perpendicular to R. Then no point C' on the segment A'B' can be in S. For if it was, then we could project C'X to meet the segment AB in C and X would lie between two points of S and hence would have to be in S. But the angle A'PB' is greater than zero. So we have found an open interval about the direction PX (measured as a polar angle at P) such that S extends at most a distance AA' for directions within that interval.

The interval $[0, 2\pi]$ is compact and is covered by these open intervals, so a finite number of them cover it. Hence there is a finite distance d such that we can find a point not in S within a distance d of P in *any* direction. That means that no point Q a distance greater than d from P can be in S (otherwise all points on the segment QP would be in S and there would be no point not in S within a distance d in that direction). So S lies inside the disk radius d on P and hence is bounded.

Take the slab $0 < y < 1$ together with the point (0, 0). It is convex and unbounded but does not contain any ray through the origin.

**Problem B4**
A finite set of circles divides the plane into regions. Show that we can color the plane with two colors so that no two adjacent regions (with a common arc of non-zero length forming part of each region's boundary) have the same color.

**Solution**
Color points inside an odd number of circles blue and points inside an even number of circles red. Then we change color whenever we cross a circle.

**Problem B5**
Show that for $n > 1$, $(3n + 1)/(2n + 2) < \sum_1^n r^n/n^n < 2$.

**Solution**
Suppose $m \le n$, then expanding by the binomial theorem, $m^n = ( (m-1) + 1)^n = (m-1)^n + n (m-1)^{n-1}$ + further positive terms $> (m-1)^n + n (m-1)^{n-1} > 2 (m-1)^n$. Hence $n^n > (n-1)^n + (n-1)^n > (n-1)^n + (n-2)^n + (n-2)^n > \dots > (n-1)^n + (n-2)^n + \dots + 1^n$. So $2 n^n > \sum_1^n r^n$, which is one of the two required inequalities.

The other is slightly harder. We approximate the integral of $f(x) = x^n$ between 0 and 1. We have to do slightly better than the usual Riemann sums. We also take into account the small triangles. The curve has increasing gradient, so the area under the curve between $(r-1)/n$ and $r/n$ is less than the area under the chord joining $( (r-1)/n, f( (r-1)/n ) )$ and $(r/n, f(r/n) )$. Summing those areas gives: $(1/n) \sum_1^n (r/n)^n - (1/2n) \sum_1^n ( (r/n)^n - ( (r-1)/n )^n = (1/n) K - (1/2n) (n/n)^n$, where $K = \sum_1^n (r/n)^n$. The integral is just $1/(n+1)$, so we have: $1/(n+1) < K/n - 1/(2n)$, or $K > n/(n+1) + 1/2 = (3n+1)/(2n+2)$.

**Problem B6**
$f : [0, 2\pi) \to [-1, 1]$ satisfies $f(x) = \sum_0^n (a_j \sin jx + b_j \cos jx)$ for some real constants $a_j$, $b_j$. Also $|f(x)| = 1$ for just 2n distinct values in the interval. Show that $f(x) = \cos(nx + k)$ for some k.

**Solution**
We call f(x) a trigonometric sum of degree n. Let $x_1, x_2, \dots , x_{2n}$ be the points at which $f(x) = 1$ or $-1$. Two insights are needed. The first is that $f'(x_i) = 0$. For if the value was non-zero, then f(x) would lie outside [-1, 1] for points sufficiently close (and on the correct side). [Note that there is a minor complication at 0. If $f(0) = 1$ and $f'(0) < 0$, then the aberrant points would be just less than $2\pi$.]

The second is that if a trigonometric sum of degree n has 2n roots (counting multiplicities), then it is determined up to a multiplicative constant (and if it has more than 2n roots then it is identically zero). We will establish that later. But $1 - f(x)^2$ has the 2n roots $x_i$. Also, they all have multiplicity at least 2 since $f'(x_i) = 0$. So we have at least 4n roots (counting multiplicities). Using the familiar formulae $\cos(A + B)$ etc we can write $1 - f(x)^2$ in the same form as f(x) but with the sum from 0 to 2n.

Similarly, $f'(x)$ is a trigonometric sum of degree n with the 2n roots $x_i$. If we square it, each root is doubled, so $f'(x)^2$ is a trigonometric sum of degree 2n with the 4n roots $x_i$ (each counted twice). Hence we must have $f'(x)^2$ = multiple of $(1 - f(x)^2)$. Both are non-negative, so it must be a positive multiple. Hence we can take the square root to get $f'(x) = N\sqrt{(1 - f(x)^2)}$ for some fixed N. Solving, we get $f(x) = \cos(Nx + k)$. But for this to have just 2n values $\pm 1$ we must have $N = n$.

To prove the result about trigonometric polynomials put $z = e^{ix}$, then $f(x) = z^{-n} p(z)$, where p is a polynomial in z of degree 2n. Hence p has at most 2n roots and is determined by those roots up to a multiplicative constant. Note that if $f'(x) = 0$ and $f(x) = 0$, then $p(z) = 0$ and since $0 = f'(x)$ = combination of $p(z)$ and $p'(z)$, then $p'(z) = 0$ also, so double roots of f have corresponding roots of multiplicity at least 2 in p.

## 24th Putnam 1963

**Problem A1**
Dissect a regular 12-gon into a regular hexagon, 6 squares and 6 equilateral triangles. Let the regular 12-gon have vertices $P_1, P_2, \dots, P_{12}$ (in that order). Show that the diagonals $P_1P_9$, $P_{12}P_4$ and $P_2P_{11}$ are concurrent.

**Solution**
Take squares on the inside of alternate sides of the 12-gon. Each angle of the 12-gon is $150^o$, so the remaining angles after placing the squares are all $60^o$, so we have equilateral triangles on the inside of the remaining 6 sides of the 12-gon. Now the opposite sides of the squares form a hexagon. (As a check we may notice that if A, B, C, D are adjacent vertices of the 12-gon and we place squares ABPQ on AB and CDRS on CD, then S and P coincide and the equilateral triangle ABP and the two squares take $60^o + 90^o + 90^o$ out of the $360^o$ angle at P leaving $120^o$, which is the angle between sides of a regular hexagon.)

Let O be the centre of the 12-gon. $P_1P_9$ and $P_{12}P_4$ have the same length and one is obtained from the other by rotation about O through $90^o$. Thus they intersect on the perpendicular bisector of $P_{12}P_1$ a distance $P_{12}P_1/2$ from $P_{12}P_1$. $P_{11}P_2$ is parallel to $P_{12}P_1$ and its midpoint lies on the perpendicular bisector of $P_{12}P_1$. The angle $P_{11}P_2P_1$ is $180^o$ - angle $P_{12}P_1P_2 = 30^o$, so the distance between of the midpoint along the bisector is $P_{12}P_1 \sin 30^o = P_{12}P_1/2$. Hence the three lines intersect at this point.

**Problem A2**
The sequence $a_1, a_2, a_3, \dots$ of positive integers is strictly monotonic increasing, $a_2 = 2$ and $a_{mn} = a_m a_n$ for m, n relatively prime. Show that $a_n = n$.

**Solution**
The key observation is that if $a_N = N$ for N odd, then $a_{2N} = 2N$, and hence $a_{N+1} = N+1$, $a_{N+2} = N+2, \dots, a_{2N-1} = 2N-1$ (because $a_n$ is strictly increasing, so there are only N-1 holes for N-1 pegs. But we can now repeat using 2N-1 (provided 2N-1 > N, or N > 1) and the result is established for all n.

So we just need to get started with $a_3$. Suppose $a_3 = m$. Then $a_6 = 2m$. So $m+2 \le a_5 \le 2m-1$. Hence $a_{10} \le 4m-2$ and $a_9 \le 4m-3$. So $a_{18} \le 8m-6$ and hence $a_{15} \le 8m-9$. But $a_{15} = a_3a_5 \ge m^2+2m$, so $m^2+2m \le 8m-9$ or $(m-3)^2 \le 0$. Hence m = 3.

**Problem A3**
Let D be the differential operator d/dx and E the differential operator xD(xD - 1)(xD - 2) ... (xD - n). Find an expression of the form $y = \int_1^x g(t)\, dt$ for the solution of the differential equation Ey = f(x), with initial conditions $y(1) = y'(1) = \dots = y^{(n)}(1) = 0$, where f(x)is a continuous real-valued function on the reals.

**Solution**
The first step is to try to simplify Ey. Experimenting with small n, we soon suspect that $Ey = x^{n+1}y^{(n+1)}$ and that is easily proved by induction.

So we seek a solution to $x^{n+1}y^{(n+1)} = f(x)$, subject to $y(1) = y'(1) = \dots = y^{(n)}(1) = 0$. The trick is to use the Taylor expansion: $y(x) = y(1) + (x-1)y'(1) + (x-1)^2/2!\, y''(1) + \dots + (x-1)^n y^{(n)}(1) + \int_1^x g(t)\, dt$, where $g(t) = (x - t)^n y^{n+1}(t)/n! = (x - t)^n f(t)/(n!\, t^{n+1})$.

**Problem A4**
Show that for any sequence of positive reals, $a_n$, we have $\limsup_{n\to\infty} n( (a_{n+1} + 1)/a_n - 1) \ge 1$. Show that we can find a sequence where equality holds.

**Solution**
Suppose lim sup < 1. Then we cannot find an infinite subsequence with $n( (a_{n+1} + 1)/a_n - 1) \ge 1$, so we must have $n( (a_{n+1} + 1)/a_n - 1) < 1$ for all sufficiently large n. Suppose it is true for all $n \ge N$. Then $(a_{N+1} + 1)/a_N < (N + 1)/N$ and hence $a_N/N > (a_{N+1} + 1)/(N + 1) = a_{N+1}/(N+1) + 1/(N+1)$. But similarly, $a_{N+1}/(N+1) > a_{N+2}/(N+2) + 1/(N+2)$ and so on. Hence $a_N \ge 1/(N+1) + 1/(N+2) + \dots$, which is impossible since the rhs diverges. So we have established that lim sup $\ge 1$.

We can experiment with various series. $a_n = n$ gives 2. $a_n = n^2$ also gives 2. Higher powers give higher values. So we try looking at exponents between 1 and 2. $a_n = n^{1+k}$ gives 1+k. So by taking k arbitrarily small we can get close to 1, but we cannot reach it. It is often helpful to think of log n as $n^k$ with k infinitesimally small. So we try $a_n = n \log n$. That gives $n( (a_{n+1} + 1)/a_n - 1) = (1 + n \log (1 + 1/n) + \log(n + 1) )/\log n = (1 + n \log (1 + 1/n) + (\log(1 + 1/n) )/\log n + 1$. But $1 + n \log(1 + 1/n) + \log (1 + 1/n)$ is bounded and so $n( (a_{n+1} + 1)/a_n - 1)$ has limit 1 (and hence also lim sup = 1).

**Problem A5**
R is the reals. $f : [0, \pi] \to R$ is continuous and $\int_0^\pi f(x) \sin x\, dx = \int_0^\pi f(x) \cos x\, dx = 0$. Show that f is zero for at least two points in $(0, \pi)$. Hence or otherwise, show that the centroid of any bounded convex open region of the plane is the midpoint of at least three distinct chords of its boundary.

**Solution**
$\sin x > 0$ for all x in $(0, \pi)$, so if f did not change sign, then we could not have $\int_0^\pi f(x) \sin x\, dx = 0$. If there was only one sign change, at k say, then $\sin(x - k)$ would also have only one sign change and $f(x) \sin(x - k)$ would not change sign in the interval, so $\int_0^\pi f(x) \sin(x - k)\, dx = \cos k \int_0^\pi f(x) \sin x\, dx - \sin k \int_0^\pi f(x) \cos x\, dx$ would be non-zero. Contradiction. So f must have at least two sign changes and hence at least two zeros in the interval.

Take polar coordinates with the centroid as origin. Let the boundary be $f(\theta)$ for $0 \le \theta \le \pi$ and $-g(\theta)$ for $0 > -\theta > -\pi$.

Taking moments about the x-axis gives $\int_0^\pi f(\theta)\sin \theta\, d\theta = \int_0^\pi g(\theta) \sin \theta\, d\theta = \int_0^\pi h(\theta) \sin \theta\, d\theta$, where $h(\theta) = g(\pi - \theta)$. So $\int (f(\theta) - h(\theta) ) \sin \theta\, d\theta = 0$.

Similarly, taking moments about the y-axis gives $\int f(\theta) \cos \theta\, d\theta + \int g(\theta) \cos \theta\, d\theta = 0$ or $\int (f(\theta) - h(\theta) )\, d\theta = 0$. So the result proved in the first part gives that $f(\theta) = g(\pi-\theta)$ for at least two values in $(0, \pi)$. In other words, the centroid is the midpoint of at least two chords. But we could take the x-axis to be one of these chords and then repeat the result to get two more, giving three in total.

**Problem A6**
M is the midpoint of a chord PQ of an ellipse. A, B, C, D are four points on the ellipse such that AC and BD intersect at M. The lines AB and PQ meet at R, and the lines CD and PQ meet at S. Show that M is also the midpoint of RS.

**Solution**
The key is to generalise and think of the ellipse and the line pair as conics. Take M as the origin and PQ as the x-axis. The ellipse has equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. If P is (k, 0), then Q must be (-k, 0), so $ak^2 + dk + f = ak^2 - dk + f = 0$. Hence $d = 0$. The line AC must have equation $y = gx$ for some g and the line BD must have equation $y = hx$ for some h. So the line pair AC, BD is the conic with equation $(y - gx)(y - hx) = 0$. The ellipse and the line pair are two distinct conics through A, B, C, D, so any conic through those four points has equation $r(ax^2 + bxy + cy^2 + ey + f) + s(y - gx)(y - hx)$. This conic meets PQ (which is $y = 0$) at the points given by $r(ax^2 + f) + sgh x^2$. This has no x term, so the two points of intersection are equally spaced about M. In particular, this is true for the line pair AB, CD.

**Problem B1**
Find all integers n for which $x^2 - x + n$ divides $x^{13} + x + 90$.

**Solution**
Answer: n = 2.

If n is negative or zero, then the quadratic has two real roots. But we can easily check that the other polynomial has derivative everywhere positive and hence only one real root. So n must be positive.

If $x^2 - x + n$ divides $x^{13} + x + 90$, then $x^{13} + x + 90 = p(x) (x^2 - x + n)$, where p(x) is a polynomial with integer coefficients. Putting x = 0, we see that n must divide 90. Putting x = 1, we see that it must divide 92. Hence it must divide (92 - 90) - 2. So the only possibilities are 1 and 2. Suppose n = 1. Then putting x = 2, we have that 3 divides $2^{13} + 92$. But $2^{odd}$ is congruent to 2 mod 2, so $2^{13} + 92$ is congruent to 1 mod 3. So n cannot be 1.

To see that n = 2 is possible, we write explicitly: $(x^2 - x + 2)(x^{11} + x^{10} - x^9 - 3x^8 - x^7 + 5x^6 + 7x^5 - 3x^4 - 17x^3 - 11x^2 + 23x + 45) = x^{13} + x + 90$.

**Problem B2**
Is the set { $2^m3^n$**:** m, n are integers } dense in the positive reals?

**Solution**
Answer: yes.

It is easier to consider the set X = { m log 2 + n log 3: m, n integers }. We note first that there is no pair of integers m, n (except (0, 0) ) such that m log 2 + n log 3 = 0. For we would then have $2^m3^n = 1$. That is clearly impossible if m and n have the same sign. If m and n have opposite signs, then it would imply that $2^{|m|} = 3^{|n|}$, but a power of 2 is not divisible by 3.

If X has a least positive member k, then log 2 must be an integral multiple of k (for we can write log 2 = q k + r for some integer q and 0 <= r < k, but then r must be zero). Similarly log 3. So if log 2 = nk and log 3 = mk, then m log 2 - n log 3 = 0, contradicting what we just proved.

X contains the positive member log 2. But log 2 cannot be the least member, so it contains another member in (0, log 2). That cannot be the least member, so we can find another smaller positive member, and so on. So we get a countable infinity of members in (0, log 2). Hence for any ε > 0, there are infinitly many members in a subinterval of (0, log 2) of length ε. The difference between any two of these must also lie in X and will lie in the interval (0, ε). Now the multiples of this number all lie in X and form a grid of mesh less than ε, so at least one of them is within ε of any given real.

That establishes that X is dense in the reals. But the map $e^x$ maps the reals to the positive reals and is continuous, so the set { $2^m3^n$**:** m, n are integers } is dense in the positive reals.

**Problem B3**
R is the reals. Find all f **:** R → R which are twice differentiable and satisfy: $f(x)^2 - f(y)^2 = f(x + y) f(x - y)$.

**Solution**
Differentiate wrt x: 2 f(x) f '(x) = f '(x + y) f(x - y) + f(x + y) f '(x - y). Differentiate the result wrt y: 0 = f "(x + y) f(x - y) - f '(x + y) f '(x - y) + f '(x + y) f '(x - y) - f(x + y) f "(x - y). Put X = x + y, Y = x - y. Then we have f "(X) f(Y) = f(X) f "(Y).

If f(X) = 0 for all X, then we have a solution. So suppose for some $X_0$, $f(X_0)$ is non-zero and put k = f "$(X_0)/f(X_0)$, then f "(Y) = k f(Y). Now we consider separately k = 0, k ≤ 0 and k > 0. If k = 0, then integrating gives f(Y) = AY + B. But putting y = 0 in the original relation gives immediately that f(y) = 0, so B = 0 and we have the solution f(Y) = AY. This includes the solution f(Y) = 0 noticed earlier.

If k < 0, put k = - $a_2$. Then f(Y) = A sin aY + B cos aY. But f(0) = 0, so B = 0 and we have the solution f(Y) = A sin aY. If k > 0, put k = $a_2$. Then f(Y) = A sinh aY + B cosh aY. Again B = 0 and we have the solution f(Y) = A sinh aY.

It remains to check that these are solutions. It is obvious that f(Y) = AY is a solution. If f(Y) = A sin aY, then f(x + y)f(x - y) = $A^2$(sin ax cos ay + cos ax sin ay)(sin ax cos ay - cos ax sin ay) = $A^2(\sin^2 ax \cos^2 ay - \cos^2 ax \sin^2 ay)$ = $A^2(\sin^2 ax - \sin^2 ay) = f(x)^2 - f(y)^2$, as required. Similarly for the sinh expression.

**Problem B4**
Γ is a closed plane curve enclosing a convex region and having a continuously turning tangent. A, B, C are points of Γ such that ABC has the maximum possible perimeter p. Show that the normals to Γ at A, B, C are the angle bisectors of ABC. If A, B, C have this property, does ABC necessarily have perimeter p? What happens if Γ is a circle?

**Solution**
Let L be the external bisector of angle C (so that it is perpendicular to the internal bisector of angle ACB). Let B' be the reflection of B in the line L. If X is any point on the other side of L from ABC, then BX > B'X, so AX + BX >

AX + B'X. But AX + B'X ≥ AB' (with equality iff X lies on AB'), and AB' = AC + CB. So AX + BX > AC + CB. Thus the perimeter of AXB > p. Hence X cannot belong to the region. But if L does not coincide with the tangent at C then it will intersect the region in further points and so the region will contain points on both sides of L. So L must be the tangent.

No, this is not a sufficient condition. For example, take the curve Γ to be an equilateral triangle with rounded corners. Then take A, B, C to be the midpoints of the sides. In this case the perimeter is certainly not maximal.

If Γ is a circle, then the achieve the maximum by taking ABC to be equilateral.

**Problem B5**

The series $\sum a_n$ of non-negative terms converges and $a_i \le 100a_n$ for $i = n, n + 1, n + 2, \dots , 2n$. Show that $\lim_{n\to\infty} na_n = 0$.

**Solution**

We need to invert the inequality given. We are given a collection of $a_i$ which are less than a fixed $a_n$. We want to fix $a_n$ and find a collection of $a_j$ such that $a_n \le 100\ a_j$.

Evidently, $a_{2n} \le 100\ a_{2n-1}$, $a_{2n} \le 100\ a_{2n-2}$, $a_{2n} \le 100\ a_{2n-3}$, ... , $a_{2n} \le 100\ a_n$. Adding and multiplying by two, $2n\ a_{2n} \le 200\ (a_n + a_{n+1} + \dots + a_{2n-1})$. But $\sum a_n$ converges, so $(a_n + a_{n+1} + \dots + a_{2n-1}) < \varepsilon/200$ for all sufficiently large n, and hence $2n\ a_{2n} < \varepsilon$ for sufficiently large n. Similarly for $(2n+1)\ a^{2n+1}$. So $n\ a_n$ tends to zero.

**Problem B6**

Let $S = S_0$ be a set of points in space. Let $S_n = \{ P : P$ belongs to the closed segment AB, for some $A, B \in S_{n-1}\}$. Show that $S_2 = S_3$.

**Solution**

We have to show that $S_2$ is already the convex hull of S. We can define the convex hull of points $\mathbf{x_i}$ $i = 1, 2, \dots , n$ as the collection of all points $\sum \lambda_i\mathbf{x_i}$ with all $\lambda_i$ non-negative and $\sum \lambda_i = 1$. Given two such points $P = \sum \lambda_i\mathbf{x_i}$ and $Q = \sum \mu_i\mathbf{x_i}$, any point on the closed segment PQ can be written as $\lambda P + \mu Q$ with $\lambda$, $\mu$ non-negative and $\lambda + \mu = 1$, but $\lambda P + \mu Q = \sum (\lambda\ \lambda_i + \mu\ \mu_i)\ \mathbf{x_i}$ and $\sum (\lambda\ \lambda_i + \mu\ \mu_i) = 1$. So it follows immediately that all $S_n$ are subsets of the convex hull.

Now we claim that any point in the convex hull can be written as a sum $\sum \lambda_i\mathbf{x_i}$ with at most 4 non-zero terms. Suppose not. Then some point P requires at least $m \ge 5$ terms. wlog we may take these to be the first m terms, so that $P = \sum_1^m\lambda_i\mathbf{x_i}$. But now we can find $\mu_i$ not all zero so that $\sum_1^m \mu_i\mathbf{x_i} = 0$ (3 equations, one for each coordinate) and $\sum \mu_i = 0$ (1 equation). But now $P = \sum_1^m(\lambda_i + k\ \mu_i)\mathbf{x_i}$ for all k and we still have $\sum (\lambda_i + k\ \mu_i) = 1$. Take the smallest $\lambda_i/\mu_i$ and set k as its negative. Then $\lambda_i + k\ \mu_i = 0$ but the other terms $\lambda_j + k\ \mu_j$ remain non-negative, so we have expressed P as a sum of less than m terms. Contradiction.

So given any P in the convex hull we may write (wlog) $P = \lambda_1\mathbf{x_1} + \dots + \lambda_4\mathbf{x_4}$ for some non-negative $\lambda_i$ with sum 1. If two or three of the $\lambda_i$ are zero, then this shows that P is in $S_1$. So suppose $\lambda_1, \lambda_2, \lambda_3$ are all positive. Then $P = (\lambda_1 + \lambda_2)\ (\lambda_1\mathbf{x_1}/(\lambda_1 + \lambda_2) + \lambda_2\mathbf{x_2}/(\lambda_1 + \lambda_2)\ ) + (\lambda_3 + \lambda_4)\ (\lambda_3\mathbf{x_3}/(\lambda_3 + \lambda_4) + \lambda_4\mathbf{x_4}/(\lambda_3 + \lambda_4)\ )$, which shows that P is in $S_2$.

## 25th Putnam 1964

**Problem A1**
Let $A_1, A_2, A_3, A_4, A_5, A_6$ be distinct points in the plane. Let D be the longest distance between any pair, and d the shortest distance. Show that $D/d \ge \sqrt{3}$.

**Solution**
Let ABC be a triangle with sides $a = BC \ge b = CA \ge c = AB$. Suppose that angle $A \ge 2\pi/3$. It follows immediately from the cosine formula we have $a^2 = b^2 + c^2 - 2bc \cos A \ge b^2 + c^2 + bc \ge 3c^2$, and hence that $a/c \ge \sqrt{3}$. So it is sufficient to find a triangle with an angle of at least $2\pi/3$.

If the 6 points form a convex hexagon, then the six angles of the hexagon sum to $4\pi$ so at least one is at least $2\pi/3$. If not then one point is inside the convex hull of the others. Draw diagonals to triangulate the hull. Then the inside point must lie inside (or on) one of the triangles. But if P lies inside (or on) the triangle ABC, then at least one of the angles APB, BPC, CPA is at least $2\pi/3$.

**Problem A2**
$\alpha$ is a real number. Find all continuous real-valued functions $f :[0, 1] \to (0, \infty)$ such that $\int_0^1 f(x)\, dx = 1$, $\int_0^1 x\, f(x)\, dx = \alpha$, $\int_0^1 x^2 f(x)\, dx = \alpha^2$.

**Solution**
We have $\int_0^1 (\alpha - x)^2 f(x)\, dx = \alpha^2 - 2\alpha^2 + \alpha^2 = 0$. But the integrand is positive, except possibly for one point of the range, so the integral must also be positive. Contradiction. So there are no functions with this property.

**Problem A3**
The distinct points $x_n$ are dense in the interval (0, 1). $x_1, x_2, \dots , x_{n-1}$ divide (0, 1) into n sub-intervals, one of which must contain $x_n$. This part is divided by $x_n$ into two sub-intervals, lengths $a_n$ and $b_n$. Prove that $\sum a_n b_n(a_n + b_n) = 1/3$.

**Solution**
The trick is to notice that $3a_n b_n(a_n + b_n) = (a_n + b_n)^3 - a_n^3 - b_n^3$. The first n-1 points $x_1, \dots , x_{n-1}$ divide the interval into n sub-intervals. Let $c_n$ be the sum of the cubes of these sub-intervals. Then $3 \sum_1^{n-1} a_i b_i(a_i + b_i) = 1 - c_n$. So it is sufficient to prove that $c_n$ tends to zero.

Take $\varepsilon > 0$ and $< 1$. Since the points $x_n$ are dense, we can take N so that for $n > N$ all the subintervals are smaller than $\varepsilon$. Then $c_n < \varepsilon^2$ times the sum of the sub-interval lengths $= \varepsilon^2 < \varepsilon$.

**Problem A4**
The sequence of integers $u_n$ is bounded and satisfies $u_n = (u_{n-1} + u_{n-2} + u_{n-3}u_{n-4})/(u_{n-1}u_{n-2} + u_{n-3} + u_{n-4})$. Show that it is periodic for sufficiently large n.

**Solution**
This is almost trivial. The $u_n$ are bounded and integral, so there are only *finitely* many possible values. Hence there are only finitely many possible values for the 4-tuples $(u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4})$. So we must eventually get a repeat. Suppose the t-tuples are the same for n = N and n = N+M. Then the recurrence relation implies that they must also be the same for n = N+1 and n = N+M+1, and for n = N+2 and N+M+2 and so on. In other words the sequence is periodic (with period M) from this point onwards.

**Problem A5**
Find a constant k such that for any positive $a_i$, $\sum_1^\infty n/(a_1 + a_2 + \dots + a_n) \le k \sum_1^\infty 1/a_n$.

**Solution**
Let us write $s_n = a_1 + a_2 + \dots + a_n$ and $b_n = n/s_n$. If $\sum 1/a_n$ diverges, then there is nothing to prove. So assume it converges. Hence $a_n$ must diverge, so there are only finitely many values less than any given K, so we can arrange the terms in increasing order $c_1 \le c_2 \le c_3 \dots$ . Since all the terms are positive $\sum 1/a_n$ is absolutely convergent and equal to $\sum 1/c_n$. Also we have $s_n \ge c_1 + c_2 + \dots + c_n$. The trick is to notice that half the $c_i$ are at least $c_{n/2}$. It is convenient to consider separately n = 2m and n = 2m+1. So $s_{2m} \ge m\, c_m$. Hence $b_{2m} < 2/c_m$. Similarly, $s_{2m-1} > m\, c_m$, so $b_{2m-1} < 2/c_m$. Hence $\sum_1^{2m} b_i \le 4 \sum_1^m 1/c_i$, which gives the required result with k = 4.

**Problem A6**

S is a finite set of collinear points. Let k be the maximum distance between any two points of S. Given a pair of points of S a distance $d < k$ apart, we can find another pair of points of S also a distance d apart. Prove that if two pairs of points of S are distances a and b apart, then a/b is rational.

**Solution**

Since we are only interested in ratios, we may take $k = 1$ and the points to have coordinates $x_1 = 0, x_2 \dots , x_{n-1}, x_n = 1$. Let these points generate a vector space V of dimension m over Q.

If $m = 1$, then every $x_i$ is a rational multiple of $x_n$ and hence rational, so the ratio of any two distances is rational.

Suppose that $m > 1$. Take a basis $b_1, b_2, \dots , b_m$ for V as follows. Take $b_1 = 1$. Let $b_2 = x_t$, for some irrational $x_t$, then extend $\{b_1, b_2\}$ to a basis. Then each $x_i$ is a unique rational linear combination of the $b_j$. In particular, $x_1 = 0$, $x_t = b_2$ and $x_n= b_1$. Now define a linear map f from V to Q as follows. Let $f(b_2) = r\, b_2$, where r is rational and sufficiently large that $f(b_2) > 1$, and $f(b_i) = b_i/2$ for all other i.

Now suppose we take two distinct points in S, we can write them as $\sum r_ib_i$ and $\sum s_ib_i$, where all $r_i$ and $s_i$ are rational. Hence their images under f are $r\, r_2b_2 + \sum r_ib_i/2$ and $r\, s_2b_2 + \sum s_ib_i/2$, where the summations exclude $i = 2$. These must be unequal, otherwise we would have a linear combination of $b_i$ with rational coefficients which was zero.

So there is a unique point x in S with f(x) maximum and a unique point y in S with f(y) minimum. Then $f(x - y)$ is greater than f(d) where d is any other difference $x_i - x_j$. But every difference except $x_n - x_1$ and $x_1 - x_n$ occurs at least twice, so x and y must be the endpoints. Now $f(x_1) = 0$ and $f(x_n) = 1/2$, so $f(x - y) = 1/2$. But $f(x_t - x_1) = f(x_t) = f(b_2) > 1$. Contradiction.

**Problem B1**

$a_n$ are positive integers such that $\sum 1/a_n$ converges. $b_n$ is the number of $a_n$ which are <= n. Prove $\lim b_n/n = 0$.

**Solution**

If not, then for some $k > 0$ we can find arbitrarily large N such that $b_N > kN$. Then at least kN of the integers $a_1, \dots , a_N$ do not exceed N. Let S be the set of indices i from 1, 2, ... N with $a_i \le N$. Then since S has at least kN members, the sum over S of $1/a_i$ is at least k.
The idea is to take $N' > N$ with $b_{N'} > kN'$ and to ensure that half the indices i with $a_i \le N'$ were not included in S. Take S' to be these new indices. Then since S' does not overlap S, the sum over S and S' of $1/a_i$ is at least $k + k/2$. We then repeat always choosing the new set of indices not to overlap any of the previous sets. So at each stage we will add k/2 to the sum of $1/a_i$, which must therefore diverge, establishing a contradiction.

We can do this by taking the successor N' to N sufficiently big. In fact, it is sufficient to take $N'/2 > N/k$ because we know that all of the previous index sets are contained in $\{1, 2, \dots , N\}$, so to avoid them we do not have to drop more than $N < kN'/2$ indices, leaving at least $kN'/2$ new indices.

**Problem B2**

S is a finite set. A set P of subsets of S has the property that any two members of P have at least one element in common and that P cannot be extended (whilst keeping this property). Prove that P contains just half of the subsets of S.

**Solution**

For any subset X of S, P can contain at most one of X and S-X, so P cannot contain more than half the subsets.

Suppose P contains less than half the subsets. Then for some X, neither X nor S-X are in P. Hence there must be Y in P such that X and Y have no elements in common (otherwise we could extend P to include X). So Y is a subset of S-X. Similarly, P must contain a subset Z of X (otherwise we could extend P to include S-X). But that means Y and Z cannot overlap, contradicting the fact that as members of P they must have an element in common.

**Problem B3**

R is the reals. $f : R \to R$ is continuous and for any $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. Prove $\lim_{x\to\infty} f(x) = 0$.

**Solution**
Take $\varepsilon > 0$. Let $A_n = \{x : |f(nx)| \le \varepsilon\}$. Let $B_n$ be the intersection of all $A_m$ with $m \ge n$. The function f is continuous, so each $A_n$ is closed. Hence also each $B_n$. Now suppose we can show that an open interval (a, b) is contained in some $B_N$. That means that for any x in (a, b) and any $n > N$ we have $|f(nx)| < \varepsilon$. Hence $|f(x)| < \varepsilon$ for any x in (na, nb). But the intervals (na, nb) expand in size as they move to the right, so for n sufficiently large they overlap and we have that their union includes all sufficiently large x. In other words, for any sufficiently large x we have $|f(x)| < \varepsilon$. But $\varepsilon$ was arbitrary, so we have established that $\lim f(x) = 0$ as x tends to infinity.

It remains to show that we can find such an open interval (a, b). Now we are given that for any given x, $\lim f(nx) = 0$ (as n tends to infinity), so x belongs to some $B_n$ for n sufficiently large. In other words, the union of the $B_n$ is the entire line.

We now need the Baire category theorem which states that if a union of closed sets covers the line, then one of the sets contains an open interval. This is bookwork. [But straightforward. Assume none of the $B_n$ contain an open interval. Take a point not in $B_1$. Since $B_1$ is closed we may take a closed interval $C_1$about the point, not meeting $B_1$. Having chosen $C_n$, there must be a point in it not in $B_{n+1}$ (or $B_{n+1}$ would contain an open interval - the interior of $C_n$). Hence we may take $C_{n+1}$, a closed subinterval of $C_n$ which does not meet $B_{n+1}$. Now $C_n$ is a nested sequence of closed intervals, so there must be a point X in all the $C_n$ But $B_n$ cover the line, so X must be in some $B_n$. Contradiction.].

**Problem B4**
n great circles on the sphere are in general position (in other words at most two circles pass through any two points on the sphere). How many regions do they divide the sphere into?

**Solution**
Answer: $n^2 - n + 2$.

We use the well-known formula $E + 2 = V + F$, where E is the number of edges, V the number of vertices and F the number of faces. It is true for a sphere provided $V > 1$, so certainly for $n > 1$.
Each circle intersects every other circle in 2 vertices, so $V = n(n - 1)$. Each vertex has degree 4, so $E = 2V$. Hence $F = V + 2 = n^2 - n + 2$. It is easy to check that the formula also holds for $n = 1$.

**Problem B5**
Let $a_n$ be a strictly monotonic increasing sequence of positive integers. Let $b_n$ be the least common multiple of $a_1$, $a_2$, ... , $a_n$. Prove that $\sum 1/b_n$ converges.

**Solution**
We need a crude upper bound on the number of divisors for N. N is too crude. But we can do better by noticing that N has at most $\sqrt{N}$ divisors $\le \sqrt{N}$ and hence at most $2\sqrt{N}$ divisors in total (every divisor $d > \sqrt{N}$ has a matching divisor $N/d < \sqrt{N}$). Now we know that $b_n$ has at least n distinct divisors (namely $a_1, \dots , a_n$). Hence $b_n \ge n^2/4$. But we know that $\sum 1/n^2$ converges.

**Problem B6**
D is a disk. Show that we cannot find congruent sets A, B with $A \cap B = \varnothing$, $A \cup B = D$.

**Solution**
It is easy to find two congruent sets which overlap only at the centre O of the disk, so this suggests that the centre is the key.

Wlog we may assume O is in A and that the radius is 1. Let O' be the corresponding point in B. Let PQ be the diameter perpendicular to OO'. Then PO' and QO' are both greater than 1. But $OX \le 1$ for any point X in A, so $O'Y \le 1$ for any point Y in B. So P and Q must both be in A. Let P' and Q' be the corresponding points in B. Then $P'Q' = PQ = 2$, so P'Q' must be a diameter. But $O'P' = OP = 1$ and $O'Q' = OQ = 1$, so O' must be the midpoint of P'Q' and hence the centre of the disk. So $O' = O$. Contradiction.

## 26th Putnam 1965

**Problem A1**
The triangle ABC has an obtuse angle at B, and angle A is less than angle C. The external angle bisector at A meets the line BC at D, and the external angle bisector at B meets the line AC at E. Also, BA = AD = BE. Find angle A.

**Solution**
Answer: 12 degrees.

Let angle BAC = k. Then since BA = BE, angle BEA = k. Take B' on BA the opposite side of B to A. Then angle B'BE = 2k. Angle B'BC is bisected by BE, so angle CBE = 2k. Hence angle ACB = 3k. So angle DBA = 4k. But AD = BA, so angle BDA = 4k. But AD is the exterior bisector, so angle BAD = 90 - k/2. The angles in BAD must sum to 180 deg, so k = 12 deg.

**Problem A2**
Let k = [(n - 1)/2]. Prove that $\sum_0^k ( (n - 2r)/n )^2 (nCr)^2 = 1/n (2n-2)C(n-1)$ (where nCr is the binomial coefficient).

**Problem A3**
$\{a_r\}$ is an infinite sequence of real numbers. Let $b_n = 1/n \sum_1^n \exp(i\ a_r)$. Prove that $b_1, b_2, b_3, b_4, \ldots$ converges to k iff $b_1, b_4, b_9, b_{16}, \ldots$ converges to k.

**Solution**
If a sequence converges to k, then any subsequence also converges to k.

So suppose that the square terms converge to k. Let $c_r = \exp(i\ a_r)$. Take two consecutive squares $N = n^2$ and $N' = n^2 + 2n + 1$ and m between them. Then $|b_N - b_m| \le |b_N - 1/N \sum c_r| + |(1/N - 1/m) \sum c_r |$, where the sums are over m terms. But $|b_N - 1/N \sum c_r| \le 1/N \sum_{N+1}^m |c_r| = (m - N)/N \le 2n/n^2 = 2/n$. Also $|(1/N - 1/m) \sum c_r | \le (1/N - 1/m) \sum |c_r| < (1/n^2 - 1/(n+1)^2) (n+1)^2 = (2n+1)/n^2 < 3/n$. So the difference $|b_m - b_N|$ is arbitrarily small for n sufficiently large. Thus if we take m sufficiently large, then $|b_m - b_N| < \varepsilon$ and $|b_N - k| < \varepsilon$. So $|b_m - k| < 2\varepsilon$ and the sequence converges.

**Problem A4**
S and T and finite sets. U is a collection of ordered pairs (s, t) with $s \in S$ and $t \in T$. There is no element $s \in S$ such that all possible pairs $(s, t) \in U$. Every element $t \in T$ appears in at least one element of U. Prove that we can find distinct $s_1, s_2 \in S$ and distinct $t_1, t_2 \in T$ such that $(s_1, t_1), (s_2, t_2) \in U$, but $(s_1, t_2), (s_2, t_1) \notin U$.

**Solution**
Suppose that we cannot find such $s_i, t_i$. We will establish a contradiction.

Take t in T. Suppose that there are n distinct s in S such that (s, t) is in U. Suppose n > 0. Then take a specific s' such that (s', t) is in U. There must be some t' such that (s', t') is not in U. Now consider whether we have (s, t') in U. If (s, t) is not in U, then (s, t') cannot be in U (or we would have found $s_i, t_i$). But there are at most n-1 distinct s such that (s, t') is in U (the only candidates are the cases for which (s, t) is in U, and one of those, namely s', does not work).

Iterating, we must eventually get some x in T for which there is no s in S with (s, x) in U. Contradiction.

**Problem A5**
How many possible bijections f on {1, 2, ... , n} are there such that for each i = 2, 3, ... , n we can find j < n with f(i) - f(j) = ± 1?

**Solution**
Answer: $2^{n-1}$.

Consider the last element f(n). Suppose it is m, not 1 or n. Then the earlier elements fall into two non-empty sets A = {1, 2, ... , m-1} and B = {m+1, m+2, ... , n}. But the difference between an element of A and an element of B is at least 2. So if f(1) is in A, then the first time we get an element of B it has only elements of A preceding it. Contradiction. Similarly, if f(1) is in B.

So we conclude that the last element is always 1 or n. We can now prove the result by induction. Clearly given an arrangement for n we can derive one for n+1 by adding n+1 at the end. We can also derive one for n+1 by increasing each element by 1 and adding 1 at the end. Equally it is clear that all these are distinct and that there are no other arrangements for n+1 that end in 1 or n+1. So thee are twice as many arrangements for n+1 as for n.

**Problem A6**

$\alpha$ and $\beta$ are positive real numbers such that $1/\alpha + 1/\beta = 1$. Prove that the line $mx + ny = 1$ with m, n positive reals is tangent to the curve $x^\alpha + y^\alpha = 1$ in the first quadrant $(x, y \ge 0)$ iff $m^\beta + n^\beta = 1$.

**Solution**

Suppose $mx + ny = 1$ is tangent to the curve. Suppose it touches at (a, b). Differentiating, we see that the tangent at (a, b) is $a^{\alpha-1}x + b^{\beta-1} = 1$, so $m = a^{\alpha-1}$, $n = b^{\beta-1}$. Hence, using $\alpha\beta - \beta = \alpha$, we have that $m^\beta + n^\beta = a^\alpha + b^\alpha = 1$.

Conversely, suppose that $m^\beta + n^\beta = 1$. Take $a = m^{\beta/\alpha}$, $b = n^{\beta/\alpha}$. Then $a^\alpha + b^\alpha = 1$, so (a, b) lies on the curve in the first quadrant. Its tangent is $Mx + Ny = 1$, where $M = a^{\alpha-1}$, $N = b^{\beta-1}$. But $a = m^{\beta/\alpha}$ and $\beta/\alpha\ (\alpha - 1)= 1$, so M = m. Similarly, N = n. Thus we have established that $mx + ny = 1$ is tangent to the curve in the first quadrant as required.

**Problem B1**

X is the unit n-cube, $[0, 1]^n$. Let $k_n = \int_X \cos^2( \pi(x_1 + x_2 + ... + x_n)/(2n) )\ dx_1 ... dx_n$. What is $\lim_{n\to\infty} k_n$ ?

**Solution**

Let $y_i = 1 - x_i$ and change variables from $x_i$ to $y_i$. The sum $(x_1 + x_2 + ... + x_n)/(2n)$ becomes $1/2 - (y_1 + y_2 + ... + y_n)/(2n)$, so the integrand becomes $\sin^2( \pi(y_1 + y_2 + ... + y_n)/(2n) )$. So the integral is identical to the original except that cos has been changed to sin. Thus we can add it to the original to get $2k_n = \int_X dx_1\ dx_2 ... dx_n = 1$. So $k_n = 1/2$ for all n.

**Problem B2**

Every two players play each other once. The outcome of each game is a win for one of the players. Player n wins $a_n$ games and loses $b_n$ games. Prove that $\sum a_n^2 = \sum b_n^2$.

**Solution**

Suppose there are N players in total. Each player plays N-1 games, so $b_n = N - 1 - a_n$. Hence $\sum b_n^2 = \sum (N - 1)^2 - 2(N - 1) \sum a_n + \sum a_n^2 = N(N - 1)^2 - 2(N - 1) \sum a_n + \sum a_n^2$.

Each game is won by just one player, so $\sum a_n$ = no. of games = N(N - 1)/2. Hence $\sum b_n^2 = N(N - 1)^2 - 2(N - 1)N(N - 1)/2 + \sum a_n^2 = \sum a_n^2$.

**Problem B3**

Show that there are just three right angled triangles with integral side lengths a < b < c such that ab = 4(a + b + c).

**Solution**

Answer: 12, 16, 20; 10, 24, 26; 9, 40, 41.

We need the result that for some integral d, m, n we have $c = d(m^2 + n^2)$ and b = 2dmn, $a = d(m^2 - n^2)$ or $b = d(m^2 - n^2)$, a = 2dmn (*).

It follows that $4(a + b + c) =4d(2m^2 + 2mn)$, $ab = 2d^2mn(m^2 - n^2)$. Hence, 4 = dn(m - n). So we must have n = 1, 2 or 4 and (d, m, n) = (1, 5, 1), (2, 3, 1), (4, 2, 1), (1, 4, 2), (2, 3, 2) or (1, 5, 4), giving the three answers above.

It remains to prove (*). Let d be the gcd of a and b. It follows that d also divides c. Put a = dA, b = dB, c = dC, so that A, B, C are relatively prime in pairs. C cannot be even, for then A and B would both be odd and hence $A^2 + B^2$ would be congruent to 2 mod 4, which is impossible, since C is a square. So C must be odd and just one of A, B must be even. Assume A is odd. Then $A^2 = (C - B)(C + B)$. If an odd prime p divides A, then it must divide C - B or C + B. It cannot divide both, for then it would also divide B and C. So C - B and C + B must both be odd squares. Say $C + B = h^2$, $C - B = k^2$. Then A = hk, $B = (h^2 - k^2)/2$, $C = (h^2 + k^2)/2$, with h and k odd. Put m = (h + k)/2, n = (h - k)/2, then h = m + n, k = m - n and $A = m^2 - n^2$, B = 2mn, $C = m^2 + n^2$, so we have put a, b, c, in the form (*). On the other hand, it is obvious that if a, b, c have this form then they satisfy $a^2 + b^2 = c^2$.

**Problem B4**

Define $f_n$ **:**R → R by $f_n(x) = (nC0 + nC2\ x + nC4\ x^2 + ...) / (nC1 + nC3\ x + nC5\ x^2 + ...)$, where nCm is the binomial coefficient. Find a formula for $f_{n+1}(x)$ in terms of $f_n(x)$ and x, and determine $\lim_{n\to\infty} f_n(x)$ for all real x.

**Solution**

It is almost obvious that $f_{n+1} = (f_n + x)/(f_n + 1)$ (*). So if $f_n(x)$ tends to a limit k(x), then $k(x) = ( k(x) + x) /( k(x) + 1)$, and hence $k(x) = \sqrt{x}$.

Obviously, $f_n(0) = 1/n$, so $k(0) = 0$. We notice also that $f_n(x) = (\sqrt{x})\ N/D$, where $N = (1 + \sqrt{x})^n + (1 - \sqrt{x})^n$ and $D = (1 + \sqrt{x})^n - (1 - \sqrt{x})^n$.

Suppose $0 < x \le 1$. Then put $y = (1 - \sqrt{x})/(1 + \sqrt{x})$. We have $0 \le y < 1$ and $N/D = (1 + y^n)/(1 - y^n)$ which tends to 1, so in this case also $k(x) = \sqrt{x}$.

Similarly if $x > 1$, put $y = (\sqrt{x} - 1)/(1 + \sqrt{x})$ and we again get $k(x) = \sqrt{x}$.
It is clear from (*), that for $x < 0$, $f_n(x)$ does not tend to a limit.

**Problem B5**

Let S be a set with $n > 3$ elements. Prove that we can find a collection of $[n^2/4]$ 2-subsets of S such that for any three distinct elements A, B, C of the collection, $A \cup B \cup C$ has at least 4 elements.

**Solution**

This is just another form of finding a graph without triangles. If $n = 2m$, then take S to be the disjoint union of U and V, where U and V each have m elements. Now take K to be the collection all sets {u, v} with u in U and v in V. Evidently K has $m^2 = [n^2/4]$ members. Suppose we have three distinct elements A, B, C of K. $A \cup B \cup C$ cannot have less than three elements, for then two of A, B, C would be identical. So if it has less than 4 elements, then it has just three elements. But that means that for some a, b, c, we have A = {b, c}, B = {a, c}, C = {a, b}. Suppose a is in U. Then, considering B and C, c and b must be in V, but then A has two elements in V. Contradiction. Similarly if a is in V. So K has the required property.

Similarly, if $n = 2m+1$, take S to be the disjoint union of U and V, where U has m+1 elements and V has m elements. Then K, the set of all {u, v} with u in U and v in V has $m(m+1) = [n^2/4]$ elements and has the required property by the same argument.

**Problem B6**

Four distinct points $A_1, A_2, B_1, B_2$ have the property that any circle through $A_1$ and $A_2$ has at least one point in common with any circle through $B_1$ and $B_2$. Show that the four points are collinear or lie on a circle.

**Solution**

The trick is to take *concentric* circles. If $A_1A_2$ is not parallel to $B_1B_2$ then their perpendicular bisectors must intersect at some point O. Take circles centre O through $A_1$, $A_2$ and through $B_1$, $B_2$. These must either coincide, in which case the 4 points lie on a circle, or have no points in common.

In the case where $A_1A_2$ is parallel to $B_1B_2$. Assume they do not coincide (otherwise the 4 points would be collinear). Then take a point O on the perpendicular bisector of $A_1A_2$ on the opposite side of $A_1A_2$ to $B_1B_2$ and sufficiently distant from $A_1A_2$ that the circle through $A_1$, $A_2$ centre O only extends less than halfway towards the line $B_1B_2$. Similarly, take a circle through $B_1$, $B_2$ which extends less than halfway towards $A_1A_2$. Then these circles will not meet.

# 27th Putnam 1966

**Problem A1**
Let $f(n) = \sum_1^n [r/2]$. Show that $f(m + n) - f(m - n) = mn$ for $m > n > 0$.

**Solution**
It is a trivial induction to show that $f(2k) = k^2$, $f(2k+1) = k^2 + k$. So if m and n have the same parity, then $f(m+n) + f(m-n) = (m+n)^2/4 + (m-n)^2/4 = mn$. If m and n have opposite parity, then $f(m+n) + f(m-n) = (m+n-1)(m+n+1)/4 + (m-n-1)(m-n+1)/4 = mn$.

**Problem A2**
A triangle has sides a, b, c. The radius of the inscribed circle is r and $s = (a + b + c)/2$. Show that $1/(s - a)^2 + 1/(s - b)^2 + 1/(s - c)^2 \geq 1/r^2$.

**Solution**
Let $A = s-a$, $B = s-b$, $C = s-c$. Then $(A-B)^2 \geq 0$ with equality iff $a = b$. Hence $2/(AB) \leq 1/A^2 + 1/B^2$. Similarly for $2/(BC)$ and $2/(CA)$. Hence $1/(BC) + 1/(CA) + 1/(AB) \leq 1/A^2 + 1/B^2 + 1/C^2$ with equality iff the triangle is equilateral.

Now $1/(BC) + 1/(CA) + 1/(AB) = (A + B + C)/(ABC)$. But $A + B + C = s$. By Heron's theorem, $sABC = k^2$, where k is the area of the triangle. Also (dividing the triangle into three by connecting the incentre to each vertex, and considering the area of each part) $k = rs$. Hence $s/(ABC) = 1/r^2$, giving the required result.

**Problem A3**
Define the sequence $\{a_n\}$ by $a_1$ □ $(0, 1)$, and $a_{n+1} = a_n(1 - a_n)$. Show that $\lim_{n\to\infty} n\, a_n = 1$.

**Solution**
We show first that $a_n < 1/(n+1)$ for all $n > 1$. We have $a_1(1 - a_1) = 1/4 - (a_1 - 1/2)^2 < 1/3$, so it is true for $n = 2$. Also, the quadratic expression shows that $a_n(1 - a_n)$ is an increasing function of $a_n$ for $a_n < 1/2$. Hence if $a_n < 1/(n+1)$, then $a_{n+1} < (1 - 1/(n+1))/(n+1) = n/(n_2 + 2n + 1) < n/(n_2 + 2n) = 1/(n + 2)$. Hence, by induction it is true for all n.

Suppose that $1/2 > a_n > 1/(k + \sqrt{k})$. Then $a_{n+1} > (1/(k + \sqrt{k}))(1 - 1/(k + \sqrt{k}))$. We show that this is greater than $1/(k+1 + \sqrt{(k+1)})$ for k sufficiently large. We require $k^2 - 1 + (k + 1)\sqrt{k} + (k - 1)\sqrt{(k + 1)} + \sqrt{(k^2 + k)} > k^2 + k\sqrt{k} + k$. Obviously $\sqrt{(k^2 + k)} > k$, so it is sufficient to show that $(k - 1)\sqrt{(k + 1)} > (k - 1)\sqrt{k} + 1$. But this is almost obvious since $\sqrt{(k + 1)} > \sqrt{k} + 1/(3\sqrt{k})$, and $(k - 1) > 3\sqrt{k}$ for $k > 11$.

Now $a_1 > 0$, so $a_1 > 1/(k + \sqrt{k})$ for some k. Increasing k if necessary, we can take $k > 11$. It then follows by induction that $a_n > 1(n + k + \sqrt{(n + k)})$ for all n, where k is fixed. Hence $n\, a_n$ lies between $n/(n + k + \sqrt{(n + k)})$ and $n/(n + 1)$. But both $n/(n + k + \sqrt{(n + k)})$ and $n/(n + 1)$ converge to 1, so $n\, a_n$ does also.

**Problem A4**
Delete all the squares from the sequence 1, 2, 3, ... . Show that the nth number remaining is $n + m$, where m is the nearest integer to $\sqrt{n}$.

**Solution**
Any integer N in the sequence lies between $k^2$ and $(k + 1)^2$ for some k, and hence $N = k^2 + h$ for some k and some h such that $1 \leq h \leq 2k$.
Since there are just k squares smaller than N, N is the $(N - k)$th number in the sequence. But $(k - 1/2)^2 < k^2 - k + 1 \leq N - k \leq k^2 + k < (k + 1/2)^2$, so the nearest integer to the square root of $(N - k)$ is k and $N = n + m$, where $n = N - k$ and m is the nearest integer to the square root of n.

**Problem A5**
Let S be the set of continuous real-valued functions on the reals. $\varphi : S \to S$ is a linear map such that if f, g □ S and $f(x) = g(x)$ on an open interval $(a, b)$, then $\varphi f = \varphi g$ on $(a, b)$. Prove that for some h □ S, $(\varphi f)(x) = h(x)f(x)$ for all f and x.

**Solution**
Obviously if f = 0 on an open interval (a, b) then $\varphi f = 0$ on (a, b). But we need the stronger result that if $f(x_0) = 0$, then $\varphi f(x_0) = 0$.

So take any f and any $x_0$ such that $f(x_0) = 0$. Let $L(x) = f(x)$ for $x < x_0$ and 0 for $x \ge x_0$, and le $R(x) = 0$ for $x \le x_0$ and f(x) for $x > x_0$. Then f = L + R. Also $\varphi L(x) = 0$ for any x > 0. But $\varphi L$ is continuous, so $\varphi L(x_0) = 0$. Similarly, $\varphi R(x_0) = 0$. Hence $\varphi f(x_0) = 0$.

Let u be the function which has value 1 for all x. Let $h = \varphi u$. Then h is continuous. Also for any f and any $x_0$ the function $f - f(x_0)u$ is zero at $x_0$ and hence $\varphi f(x_0) = h(x_0)f(x_0)$.

**Problem A6**
Let $a_n = \sqrt{(1 + 2 \sqrt{(1 + 3 \sqrt{(1 + 4 \sqrt{(1 + 5 \sqrt{( ... + (n - 1) \sqrt{(1 + n)} ... )})})})})}$. Prove $\lim a_n = 3$.

**Solution**
Clearly $a_n < a_{n+1}$. Also, if we replace the final (1 + n) in $a_n$ by $(1 + n)^2$, then a simple induction shows that the resulting expression simplifies to 3. Hence $a_n < 3$. An increasing sequence which is bounded above must converge. So $a_n$ tends to a limit which is at most 3.

However, it is harder to show that the limit is 3. We need a new idea. Put $f(x) = \lim \sqrt{(1 + x \sqrt{(1 + (x+1) \sqrt{(1 + (x+2) \sqrt{(1 + (x+3) \sqrt{( ... + (x+n-1) \sqrt{(1 + x+n)} ... )})})})})}$. Then we may guess that f(x) = x + 1. The same idea as before shows that f(x) exists and is at most x + 1. Also, we have that $f(x)^2 = x f(x+1) + 1$ (*).

The trick is that a crude lower limit works, because we can use (*) to refine it repeatedly. Removing all the 1s and replacing (x+1), (x+2) ... by x gives that $f(x) \ge \lim \sqrt{( x \sqrt{( x \sqrt{( x \sqrt{( x ... )})})})} = x > 1/2 (x + 1)$. Now (*) gives $f(x)^2 \ge 1/2 (x^2 + 2x) + 1 > 1/2 (x + 1)^2$, so $f(x) > (x + 1)\sqrt{(1/2)}$. Using (*) again gives $f(x) > (x + 1) (1/2)^{1/4}$ and so on. Hence $f(x) \ge x + 1$ as required.

**Problem B1**
A convex polygon does not extend outside a square side 1. Prove that the sum of the squares of its sides is at most 4.

**Solution**
Form a right-angled triangle on each side of the polygon (and outside it), by taking the other two sides parallel to the sides of the square. The sum of the squares of the polygon's sides equals the sum of the squares of the non-hypoteneuse sides of the triangles. Because the polygon is convex, these triangle sides form 4 sets, one for each side of the square, and each set having lengths totalling less than 1 (the side of the square). So the sum of the squares in each set is less than 1 ($\sum x^2 < (\sum x)^2 = 1$).

**Problem B2**
Prove that at least one integer in any set of ten consecutive integers is relatively prime to the others in the set.

**Solution**
There are 5 odd numbers in the set. At most 2 are multiples of 3, at most 1 is a multiple of 5 and at most 1 is a multiple of 7. So there is at least one odd number, k, that is not divisible by 3, 5 or 7. Now if k has a common factor with another member in the set, then that factor must divide their difference, which is at most 9. But the common factor cannot be divisible by 2, 3, 5 or 7, so it must be 1.

**Problem B3**
$a_n$ is a sequence of positive reals such that $\sum 1/a_n$ converges. Let $s_n = \sum_1^n a_i$. Prove that $\sum n^2a_n/s_n^2$ converges.

**Solution**
Let $A = (\sum 1/a_n)^{1/2}$ and $B_N = \sum_1^N n^2a_n/s_n^2$.
Since all $a_n$ are positive, $s_{n-1} < s_n$ and hence $B_N < \sum_1^N n^2a_n/(s_ns_{n-1})$. But $a_n = s_n - s_{n-1}$, so for n > 1 we may write the summand as $n^2(1/s_{n-1} - 1/s_n)$. Hence $B_N < 1/a_1 + (4/s_1 - 4/s_2) + (9/s_2 - 9/s_3) + ... + (N^2/s_{N-1} - N^2/s_N) < 5/a_1 + 5/s_2 + 7/s_3 + 9/s_4 + ... + (2N-1)/s_{N-1} - N^2/s_N < 2/a_1 + 3(1/s_1 + 2/s_2 + 3/s_3 + ... + N/s_N)$.

Now $\sum_1^N n/s_n = \sum (1/\sqrt{a_n}) (n(\sqrt{a_n})/s_n) <= (\sum 1/a_n)^{1/2} (\sum n^2 a_n/s_n^2)^{1/2} < A B_N^{1/2}$. So we have $B_N < 2/a_1 + 3A B_N^{1/2}$. That implies that $B_N$ is bounded above. For example, we certainly have $B_N < (1 + 3/a_1 + 3A)^2$. But any increasing sequence which is bounded above must converge.

**Problem B4**

Given a set of (mn + 1) unequal positive integers, prove that we can either (1) find m + 1 integers $b_i$in the set such that $b_i$ does not divide $b_j$ for any unequal i, j, or (2) find n+1 integers $a_i$ in the set such that $a_i$ divides $a_{i+1}$ for i = 1, 2, ... , n.

**Solution**

Given any $a_1$ in the set, let $f(a_1)$ be the length of the longest sequence $a_1, a_2, \dots , a_k$ taken from the set such that $a_1|a_2, a_2|a_3, \dots , a_{k-1}|a_k$. If $f(a_1) > n$, then we are done, so assume that for all a in the set, $f(a) <= n$. Thus f can only have n possible values (1, 2, ... , n). There are mn+1 members in the set, so there must be at least m+1 members $b_1, b_2, \dots , b_{m+1}$ with the same value of f, say h. Now we cannot have any one of these members divide another, for if $b_i|b_j$, then we could extend the sequence length h for $b_j$ to give a sequence length h+1 for $b_i$ (in which each element divides the next).

**Problem B5**

Given n points in the plane, no three collinear, prove that we can label them $P_i$ so that $P_1P_2P_3 \dots P_n$ is a simple closed polygon (with no edge intersecting any other edge except at its endpoints).

**Solution**

Take arbitrary x, y axes. Take P to be the point with the smallest x-coordinate, or if there are two such, the one with the smaller y-coordinate. Take Q to be the point with the largest x-coordinate, or if there are two such the one with the smaller y-coordinate. Join P to Q by a path p along the lower part of the convex hull of the points (so that all other points are above the path).

Now order the remaining points not in the path according to the size of their x-coordinate (with the largest first). Continue the path from Q back to P by taking the remaining points in this order. We can never intersect the existing path p without going outside the convex hull and we cannot intersect the later part of the path because it has smaller x-coordinate.

**Problem B6**

$y = f(x)$ is a solution of $y'' + e^x y = 0$. Prove that f(x) is bounded.

**Solution**

We have $2 e^{-x} y' y'' + 2 y y' = 0$. Integrating from 0 to k gives $y(k)^2 = y(0)^2 - 2 \int_0^k e^{-x} y' y''\, dx$. Integrating by parts gives $2 \int_0^k e^{-x} y' y''\, dx = e^{-x} (y')^2|_0^k + \int_0^k (y')^2 e^{-x}\, dx = e^{-k}y'(k)^2 - y'(0)^2 + \int_0^k (y')^2 e^{-x}\, dx = A - y'(0)^2$, where $A > 0$.

Hence $y(k)^2 = y(0)^2 + y'(0)^2 - A < y(0)^2 + y'(0)^2$, which establishes that y is bounded.

## 28th Putnam 1967

**Problem A1**
We are given a positive integer n and real numbers $a_i$ such that $|\sum_1^n a_k \sin kx| \le |\sin x|$ for all real x. Prove $|\sum_1^n k\, a_k| \le 1$.

**Solution**Put $f(x) = \sum_1^n a_k \sin kx$. We note that $\sum_1^n k\, a_k = f'(0)$.

We also have $f'(0) = \lim ( f(x) - f(0) )/x = \lim f(x)/x = \lim f(x)/\sin x \lim (\sin x)/x = \lim f(x)/\sin x$. But $|f(x)| \le |\sin x|$, so $|f(x)/\sin x| \le 1$ and hence $|f'(0)| = |\lim f(x)/\sin x| \le 1$.

**Problem A2**
Let $u_n$ be the number of symmetric n x n matrices whose elements are all 0 or 1, with exactly one 1 in each row. Take $u_0 = 1$. Prove $u_{n+1} = u_n + n\, u_{n-1}$ and $\sum_0^\infty u_n x^n/n! = e^{f(x)}$, where $f(x) = x + (1/2) x^2$.

**Solution**
There is an obvious bijection between (1) n x n matrices satisfying the conditions and (2) (n+1) x (n+1) matrices satisfying the conditions which have 1 at the top left. [Just delete the first row and column to get from (2) to (1) ).

Similarly for any i = 2, 3, ... or n+1, there is an obvious bijection between (1) (n-1) x (n-1) matrices satisfying the conditions and (2) (n+1) x (n+1) matrices satisfying the conditions which have a 1 in row 1, col i (and hence also in row i, col 1). Just delete rows 1 and i and cols 1 and i to get from (2) to (1).
That establishes that $u_{n+1} = u_n + n\, u_{n-1}$. Also, we are given that $u_0 = 1$ and it is clear that $u_1 = 1$.

We have that $e^{f(x)} = 1 + (x + 1/2\, x^2) + (x + 1/2\, x^2)^2/2! + \dots$ which is clearly $1 + v_1x/1! + v_2x^2/2! + \dots$ for some $v_n$. Differentiating, we get that $(1 + x) (1 + v_1x/1! + v_2x^2/2! + \dots) = v_1 + v_2x/1! + v_3x^2/2! + \dots$ . Hence $v_1 = 1$, $v_2 = v_1 + 1$, $v_{n+1} = v_n + n\, v_{n-1}$. A trivial induction now shows that $v_n = u_n$.

**Problem A3**
Find the smallest positive integer n such that we can find a polynomial $nx^2 + ax + b$ with integer coefficients and two distinct roots in the interval (0, 1).

**Solution**
Answer: n = 5 with equation $5x^2 - 5x + 1$ with roots $(1 \pm \sqrt{(1/5)} )/2$.
The product of the roots lies in the interval (0, 1), so b must be 1, 2, 3, ... or n-1 (1). The larger root is $(-a + \sqrt{(a^2 - 4bn)} )/(2n)$. This must be less than 1, so $-a < b + n$ (2). The roots are real and distinct, so $a^2 > 4bn$ (3).

Putting (2) and (3) together we get: $(n + b - 1)^2 \ge a^2 \ge 4bn + 1$. So if b = 1, then $n \ge 5$. If b = 2, then $n \ge 6$. If b = 3, then $n \ge 8$. If b = 4, then $n \ge 10$. Thus there are no solutions for n < 5 (which requires b < 4 by (1) ). If n = 5, then the only possible solution is $5x^2 - 5x + 1$, which is easily verified to be a solution.

**Problem A4**
Let $1/2 < \alpha \in R$, the reals. Show that there is no function $f : [0, 1] \to R$ such that $f(x) = 1 + \alpha \int_x^1 f(t) f(t - x)\, dt$ for all $x \in [0, 1]$.

**Solution**
Suppose there is such a function. Let $K = \int_0^1 f(x)\, dx$. Then $K = 1 + \alpha \int_0^1\int_x^1 f(t) f(t - x)\, dt\, dx$.
Interchanging the order of integration gives $\int_0^1\int_x^1 f(t) f(t - x)\, dt\, dx = \int_0^1\int_0^t f(t) f(t - x)\, dx\, dt = \int_0^1 f(t) \int_0^t f(x)\, dx\, dt$ (*).
Put $g(x) = \int_0^x f(t)\, dt$. Then $g'(x) = f(x)$, so (*) gives $\int_0^1 g'(t) g(t)\, dt = 1/2\, g(1)^2 - 1/2\, g(0)^2$. But g(1) = K and g(0) = 0. Thus we have $K = 1 + \alpha/2\, K^2$, or rearranging $(K - 1/\alpha)^2 = -2/\alpha^2 (\alpha - 1/2)$. But that is impossible for $\alpha > 1/2$.

**Problem A5**
K is a convex, finite or infinite, region of the plane, whose boundary is a union of a finite number of straight line segments. Its area is at least $\pi/4$. Show that we can find points P, Q in K such that PQ = 1.

**Solution**
Suppose the result is false, so that the maximum distance is 2d < 1. Take a diameter of K (in other words two points

A, B for which AB = 2d). Let the x-axis lie along this diameter with the origin at its midpoint. Take the y-axis perpendicular to the x-axis as usual. Then K lies entirely between x = -d and x = +d. Let t(x) be the top boundary of K (above the x-axis) and -b(x) be the bottom boundary of K (below the x-axis).

The area A of K is the area under the curve t(x) plus the area between b(x) and the x-axis. In other words $A = \int_{-d}^{d} t(x)\,dx + \int_{-d}^{d} b(x)\,dx = \int_{-d}^{d} t(x)\,dx + \int_{-d}^{d} b(-x)\,dx = \int_{-d}^{d}( t(x) + b(-x) )\,dx$. Now $(t(x) + b(-x))^2 = D^2 - (2x)^2$, where D is the distance between the two points (x, t(x)) and (-x, b(-x)). Certainly, D < 1, so $\int_{-d}^{d}( t(x) + b(-x) )\,dx < \int_{-d}^{d} \sqrt{1 - 4x^2}\,dx$.
The indefinite integral is $1/2 \sqrt{1 - 4x^2} + (1/4) \sin^{-1}(2x)$, so integrated between -1/2 and 1/2 it gives $\pi/4$. The integrand is positive, so between -d and d it gives a smaller result. In other words we have established that if 2d < 1, then A < $\pi/4$. Contradiction.

**Problem A6**
$a_i$ and $b_i$ are reals such that $a_1b_2 \neq a_2b_1$. What is the maximum number of possible 4-tuples (sign $x_1$, sign $x_2$, sign $x_3$, sign $x_4$) for which all $x_i$ are non-zero and $x_i$ is a simultaneous solution of $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ and $b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0$. Find necessary and sufficient conditions on $a_i$ and $b_i$ for this maximum to be achieved.

**Solution**
Solving in terms of $x_3$, $x_4$ gives $x_1 = s_{23}/s_{12}\, x_3 + s_{24}/s_{12}\, x_4$, $x_2 = s_{31}/s_{12}\, x_3 + s_{41}/s_{12}\, x_4$, $x_3 = x_3$, $x_4 = x_4$, where $s_{ij} = (a_ib_j - a_jb_i)$. Plot the 4 lines $s_{23}/s_{12}\, x_3 + s_{24}/s_{12}\, x_4 = 0$, $x_2 = s_{31}/s_{12}\, x_3 + s_{41}/s_{12}\, x_4 = 0$, $x_3 = 0$, $x_4 = 0$ in the $x_3$, $x_4$ plane. We get 4 lines through the origin. Evidently $x_1$ changes sign if we cross the first, $x_2$ changes sign if we cross the second, $x_3$ changes sign if we cross the third and $x_4$ changes sign if we cross the fourth. So we have a different combination of signs in each sector, but the same combination throughout any given sector.

Thus the maximum is achieved when the four lines are distinct, giving 8 sectors (and hence 8 combinations). This requires that $s_{23}$ and $s_{24}$ are non-zero (otherwise the first line coincides with one of the last two) and that $s_{31}$ and $s_{41}$ are non-zero (otherwise the second line coincides with one of the last two). Finally, the first two lines must not coincide with each other. That requires that $s_{31}/s_{41}$ is not equal to $s_{23}/s_{24}$. After some slightly tiresome algebra that reduces to $s_{34}$ non-zero. So a necessary and sufficient condition to achieve 8 is that all $s_{ij}$ are non-zero.

**Problem B1**
A hexagon is inscribed in a circle radius 1. Alternate sides have length 1. Show that the midpoints of the other three sides form an equilateral triangle.

**Solution**
Each side length 1 forms an equilateral triangle with the centre. Let the other three sides subtend angles 2A, 2B, 2C at the centre. The midpoints of the sides subtending 2A and 2B are distances cos A and cos B from the centre and the line joining them subtends an angle A+B+60 at the centre. So, using the cosine formula, the square of the distance between them is $\cos^2A + \cos^2B - 2 \cos A \cos B \cos(A+B+60)$ (*). But A + B + C = 90, so cos(A+B+60) = cos(150-C) = $-\sqrt{3}/2 \cos C + 1/2 \sin C$. Hence we may write (*) as $(\cos^2A + \cos^2B + \cos^2C) + \sqrt{3} \cos A \cos B \cos C - \cos A \cos B \sin C - \cos^2C$. But we can write cos C as sin(A + B) = sin A cos B + cos A sin B and hence $\cos^2C$ as sin A cos B cos C + cos A sin B cos C. Thus (*) has the symmetrical form $(\cos^2A + \cos^2B + \cos^2C) + \sqrt{3} \cos A \cos B \cos C - (\cos A \cos B \sin C + \cos A \sin B \cos C + \sin A \cos B \cos C)$, which establishes that the triangle is equilateral.

**Problem B2**
A, B $\in$ [0, 1] and we have $ax^2 + bxy + cy^2 \equiv (A x + (1 - A )y)^2$, $(A x + (1 - A)y )(B x + (1 - B)y) \equiv dx^2 + exy + fy^2$. Show that at least one of a, b, c $\geq$ 4/9 and at least one of d, e, f $\geq$ 4/9.

**Solution**
For the first part, $a = A^2$, $b = 2 A (1 - A)$, $c = (1 - A)^2$. If a < 4/9, then A < 2/3. If c < 4/9 then A > 1/3. But b > 4/9 for 1/3 < A < 2/3.
For the second part, d = A B, f = (1 - A )(1 - B ), e = A + B - 2 AB.
Measure A along the x-axis and B along the y-axis. Consider the regions of the square for which each of d, e, f are $\geq$ 4/9. d $\geq$ 4/9 for points above the hyperbola y = 4/(9x) which passes through the points (4/9, 1), (2/3, 2/3), (1, 4/9). Similarly f $\geq$ 4/9 for points below the hyperbola y = 1 - 4/(9 - 9x), which passes through the points (0, 5/9), (1/3, 1/3), (5/9, 0). Finally, e > 4/9 for points lying between the two branches of the hyperbola y = (4 - x)/(9 - 18x). The bottom branch passes through (0, 4/9), (1/3, 1/3), (4/9, 0) and the top branch passes through (5/9, 1), (2/3, 2/3), (1,

5/9). Thus the bottom branch is entirely below the d = 4/9 hyperbola, except at the point of intersection (1/3, 1/3), and the top branch is entirely above the f = 4/9 hyperbola, except at the point of intersection (2/3, 2/3). This is easily checked by solving d = e = 4/9 and d = f = 4/9. Thus the three areas d ≥ 4/9, e ≥ 4/9, f ≥ 4/9 cover the unit square.

**Problem B3**
R is the reals. f, g are continuous functions R → R with period 1. Show that $\lim_{n\to\infty} \int_0^1 f(x)\, g(nx)\, dx = (\int_0^1 f(x)\, dx)(\int_0^1 g(x)\, dx)$.

**Solution**
The idea is to split the integration range into n equal parts. Thus we get $\int_0^1 f(x)\, g(nx)\, dx = \sum \int_{r/n}^{r/n+1/n} f(x)\, g(nx)\, dx$. For large n, f is roughly constant over the range, so we get $\approx f(r/n) \int_{r/n}^{r/n+1/n} g(nx)\, dx$. Changing the integration variable to t = nx, gives $\sum f(r/n)\, 1/n \int_0^1 g(t)\, dt$ since g is periodic. But $\lim \sum f(r/n)\, 1/n$ is just $\int_0^1 f(x)\, dx$, so we get the required $(\int_0^1 f(x)\, dx)(\int_0^1 g(x)\, dx)$.

It remains to look at the error involved in approximating f. The function f is continuous and [0, 1] is compact, so it must be uniformly continuous on [0, 1]. Thus we given any ε > 0, we can find N such that for n > N, we have |f(x) - f(r/n)| < ε on [r/n, r/n + 1/n] for each of r = 0, 1, 2, ... , n-1. So the error is at most $\sum \int_{r/n}^{r/n+1/n} |f(x) - f(r/n)|\, |g(nx)|\, dx < \sum \varepsilon\, 1/n \int_0^1 |g(t)|\, dt = \varepsilon \int_0^1 |g(t)|\, dt$, which can be made arbitrarily small.

**Problem B4**
We are given a sequence $a_1, a_2, \dots , a_n$. Each $a_i$ can take the values 0 or 1. Initially, all $a_i = 0$. We now successively carry out steps 1, 2, ... , n. At step m we change the value of $a_i$ for those i which are a multiple of m. Show that after step n, $a_i = 1$ iff i is a square. Devise a similar scheme give $a_i = 1$ iff i is twice a square.

**Solution**
$a_i$ is changed once for each divisor of i. If i is non-square then it has an even number of divisors (they come in pairs d, i/d), so $a_i$ ends as 0. If i is square it has an odd number of divisors and so $a_i$ ends as 1.
We change those that a multiple of 2, then those that are a multiple of 4, then those that are a multiple of 6 and so on. This only affects the even numbers and indeed $a_{2i}$ is changed once for each divisor of i, so the same argument as before shows that $a_{2i}$ ends as 1 iff i is a square. Note that $a_{2i+1}$ is never changed, so it remains 0.

**Problem B5**
The first n terms of the exansion of $(2 - 1)^{-n}$ are $2^{-n} ( 1 + n/1!\, (1/2) + n(n + 1)/2!\, (1/2)^2 + \dots + n(n + 1) \dots (2n -2)/(n - 1)!\, (1/2)^{n-1} )$. Show that they sum to 1/2.

**Solution**
Consider a random walk on a two dimensional lattice. The particle starts at (0, 0). At each step it moves up 1 step with probability 1/2 and right one step with probability 1/2. Suppose that we stop when the particle reaches x = n or y = n.

Suppose the particle first reaches x = n at y = m (0 ≤ m < n). Then its final move must be from (n-1, m) to (n, m). In order to reach (n-1, m) it must make n+m-1 moves, m of which must be upwards. So the probability is $(n+m-1Cm)(1/2)^{n+m}$. Thus the probability that it reaches x = n is $(n-1)C0\, (1/2)^n + nC1\, (1/2)^{n+1} + \dots + (2n-2)C(n-1)\, (1/2)^{2n-1}$. The probability that it reaches y = n is the same. Note that it cannot reach the point (n, n) (because it reaches (n, n-1) or (n-1, n) first and stops). So each series must total 1/2.

**Problem B6**
R is the reals. D is the closed unit disk $x^2 + y^2 = 1$ in $R^2$. The function f : D → R has partial derivatives $f_1(x, y)$ and $f_2(x, y)$ and all f(x, y) ∈ [-1, 1]. Show that there is a point (a, b) in the interior of D such that $f_1(a, b)^2 + f_2(a, b)^2 \le 16$.

**Solution**
Consider $f(x, y) + 2x^2 + 2y^2$. It is at least 1 on the entire boundary of D and at most 1 at the centre. So it is either constant at an interior point of D or has a minimum at an interior point. In either case, there is an interior point (a, b) at which its two partial derivatives are zero. So $f_1(a, b) = -4a$, $f_2(a, b) = -4b$ and $f_1^2 + f_2^2 = 16(a^2 + b^2) < 16$.

# 29th Putnam 1968

**Problem A1**
Prove that $\int_0^1 x^4(1 - x)^4/(1 + x^2)\,dx = 22/7 - \pi$.

**Solution**
Divide the denominator by the numerator to get: $(x^8 - 4x^7 + 6x^6 - 4x^5 + x^4) = (1 + x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - 4$. Now we can integrate to get $(x^7/7 - 2x^6/3 + x^5 - 4x^3/3 + 4x)|_0^1 - \int_0^1 4dx/(1 + x^2) = 22/7 - \pi$.

**Problem A2**
Given integers a, b, c, d such that $ad - bc \neq 0$, integers m, n and a real $\varepsilon > 0$, show that we can find rationals x, y, such that $0 < |ax + by - m| < \varepsilon$ and $0 < |cx + dy - n| < \varepsilon$.

**Solution**
Take k rational such that $0 < k < \varepsilon$. Now solve the simultaneous equations: $ax + by = m + k$, $cx + dy = n + k$. We get $x = (dm - bn + dk - bk)/(ad - bc)$, $y = (an - cm + ak - ck)/(ad - bc)$. Clearly x and y are rational and satisfy the required inequalities.

**Problem A3**
S is a finite set. P is the set of all subsets of S. Show that we can label the elements of P as $A_i$, such that $A_1 = \varnothing$ and for each n >= 1, either $A_{n-1} \subset A_n$ and $|A_n - A_{n-1}| = 1$, or $A_{n-1} \supset A_n$ and $|A_{n-1} - A_n| = 1$.

**Solution**
Let $N = 2^n$ and let $A_1, \dots , A_N$ be a solution for n with $A_N = \{n\}$. Now take $A_{N+m} = A_{2N+1-m}$ union $\{n+1\}$ for m = N+1, ... , 2N. This gives a solution for n+1.

**Problem A4**
Let $S_2$ be the 2-sphere $\{ (x, y, z) : x^2 + y^2 + z^2 = 1\}$. Show that for any n points on $S_2$, the sum of the squares of the $n(n - 1)/2$ distances between them (measured in space, not in $S_2$) is at most $n^2$.

**Solution**
Let the points be $(x_i, y_i, z_i)$ for i = 1, 2, ... n. The square of the distance between the pair i, j is $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = (x_i^2 + y_i^2 + z_i^2) + (x_j^2 + y_j^2 + z_j^2) - 2(x_ix_j + y_iy_j + z_iz_j) = 2 - 2 (x_ix_j + y_iy_j + z_iz_j)$. Hence the sum of the squares of the distances is $n(n-1) - 2 \sum (x_ix_j + y_iy_j + z_iz_j)$.

Now $(x_1 + x_2 + \dots + x_n)^2 = \sum x_i^2 + 2 \sum x_ix_j$. So $2 \sum (x_ix_j + y_iy_j + z_iz_j) = X^2 + Y^2 + Z^2 - \sum (x_i^2 + y_i^2 + z_i^2) = X^2 + Y^2 + Z^2 - n$, where $X = \sum x_i$, $Y = \Sigma y_i$, $Z = \sum z_i$. Hence the sum of the squares is $n^2 - (X^2 + Y^2 + Z^2)$, which is at most $n^2$.

**Problem A5**
Find the smallest possible $\alpha$ such that if $p(x) \equiv ax^2 + bx + c$ satisfies $|p(x)| \leq 1$ on [0, 1], then $|p'(0)| \leq \alpha$.

**Solution**
Answer: 8. Extreme case is $\pm(8x^2 - 8x + 1)$.
Note that $|p'(0)| = |b|$. So the question is how large we can make $|b|$ and still be able to find a and c such that p(x) lies between -1 and 1 on [0 ,1].

wlog $a > 0$. Write $p(x) = a(x + b/2a)^2 + c$. If $-b/2a \leq 0$, then p(x) has its minimum on [0, 1] at 0 and its maximum at 1, so we require $p(1) - p(0) \leq 2$ (we can then adjust c to get $|p(x)| \leq 1$ on the entire interval). But $p(1) - p(0) = a + b$, so $b \leq 2$ (and the extreme case is $p(x) = 2x - 1$).

If $-b/2a \geq 1$, then p(x) has its maximum at 0 and its minimum at 1 (on the interval). So we require $p(0) - p(1) < 2$ and hence $a + b \geq -2$. But $-b \geq 2a$, so $a \leq 2$ and $b \geq -4$ (and the extreme case is $2x^2 - 4x + 1$).
If $0 < -b/2a \leq 1/2$, then p(x) has its minimum at $-b/2a$ and its maximum at 1. So we require $a + b + b^2/4a \leq 2$ or $(b + 2a)^2/4a \leq 2$. But $0 < -b \leq a$, so $(b + 2a)^2 \geq a^2$ and hence $a^2/4a \leq 2$ or $a \leq 8$ and $0 < -b \leq 8$.
Finally, if $1/2 \leq -b/2a < 1$, the p(x) has its minimum at $-b/2a$ and its maximum at 0. So we require $b^2 /4a \leq 2$. So again $a^2 \leq b^2/4a \leq 2$. Hence $a \leq 8$. But $b^2 \leq 8a$, so $-b < 8$. Hence $-8 \leq b < 0$.

**Problem A6**
Find all finite polynomials whose coefficients are all ±1 and whose roots are all real.

**Solution**
Answer: $\pm(x + 1)$, $\pm(x - 1)$, $\pm(x^2 + x - 1)$, $\pm(x^2 - x - 1)$, $\pm(x^3 + x^2 - x - 1)$, $\pm(x^3 - x^2 - x + 1)$.

The linear and quadratic polynomials are easy to find (and it is easy to show that they are the only ones). Suppose the polynomial has degree $n \geq 3$. Let the roots be $k_i$. Then $\sum k_i^2 = (\sum k_i)^2 - 2 \sum k_i k_j = a^2 \pm 2b$, where a is the coefficient of $x^{n-1}$ and b is the coefficient of $x^{n-2}$. Hence the arithmetic mean of the squares of the roots is $(a^2 \pm 2b)/n$. But it is at least as big as the geometric mean which is 1 (because it is an even power of c, the constant term). So we cannot have $n > 3$. For $n = 3$, we must have b the opposite sign to the coefficient of $x^n$ and it is then easy to check that the only possibilities are those given above.

**Problem B1**
The random variables X, Y can each take a finite number of integer values. They are not necessarily independent. Express prob( min(X, Y) = k) in terms of $p_1$ = prob( X = k), $p_2$ = prob(Y = k) and $p_3$ = prob( max(X, Y) = k).

**Solution**
Let $q_1$ = prob(X = k, Y = k), $q_2$ = prob(X = k, Y > k), $q_3$ = prob(X > k, Y = k), $q_4$ = prob(X = k, Y < k), $q_5$ = prob(X < k, Y = k). These are all probabilities for disjoint events, so we can add them freely to get: prob( min(X, Y) = k) = $q_1 + q_2 + q_3$, $p_1 = q_1 + q_2 + q_4$, $p_2 = q_1 + q_3 + q_5$, $p_3 = q_1 + q_4 + q_5$. Hence prob( min(X, Y) = k) = $p_1 + p_2 - p_3$.

**Problem B2**
(G, *) is a finite group with n elements. K is a subset of G with more than n/2 elements. Prove that for every $g \in G$, we can find $h, k \in K$ such that $g = h * k$.

**Solution**
Take any g in G. Let H be the set of elements of the form $h^{-1}*g$ where h is in K. Since G is a group, H has the same number of elements as K. Hence it must overlap K (since between them they have more than n elements). So for some h, k in K we have $k = h^{-1}*g$, or $g = h*k$.

**Problem B3**
Given that a $60^o$ angle cannot be trisected with ruler and compass, prove that a $120^o/n$ angle cannot be trisected with ruler and compass for n = 1, 2, 3, ... .

**Solution**
This is a trap.

The obvious answer is that if we could trisect an angle $120^o/n$, that would give us an angle $40^o/n$ and hence $60^o/3$. But we are told we cannot trisect $60^o$.
This does not work for a slightly subtle reason. We are given the angle $120^o/n$. That ***in itself*** may help us to trisect $60^o$. For example, it certainly allows us to trisect $60^o$ if n = 6.

So we have to use the standard Galois theory approach. It is not hard to show that a straight-edge allows us to get rational distances and compasses allow us to get any finite number of square roots. So we can construct distances which are in an extension K of the rationals of degree $2^n$ for some n. Constructing an angle is equivalent to constructing its cosine.

So if we are given an angle A, then we can construct distances which are in any extension K of degree $2^n$over Q(cos A). Now if K is an extension of L of degree r and L is an extension of F of degree s, then K is an extension of degree rs of F. So suppose Q(cos A) has degree u and Q(cos(A/3 ) has degree v. If we can construct A/3 given A, then cos(A/3) must lie in an extension K of Q(cos A) of degree $2^n$. But K is also an extension of Q(cos(A/3) ) so v/u must be a power of 2.

Finally, it is well-known that Q(cos(360/m) ) has degree $\varphi(m)$ over Q, where $\varphi$ is Euler's function (so that $\varphi(m)$ is the number of 1, 2, ... m-1 relatively prime to m). So we have to show that $\varphi(9n)/\varphi(3n)$ is not a power of 2. But $\varphi(m) = m \prod(1 - 1/p)$, where the product is taken over all primes p dividing m. But 3n and 9n have the same prime divisors, so $\varphi(9n) = 3\,\varphi(3n)$, and 3 is not a power of 2.

**Problem B4**
R is the reals. $f : R \to R$ is continuous and $L = \int_{-\infty}^{\infty} f(x)\,dx$ exists. Show that $\int_{-\infty}^{\infty} f(x - 1/x)\,dx = L$.

**Solution**
Substitute $x = y - 1/y$. As y increases from $-\infty$ to 0, x increases (monotonically) from $-\infty$ to $+\infty$. Also $dx = (1 + 1/y^2)\,dy$, so we have $L = \int_{-\infty}^{0} f(y - 1/y)(1 + 1/y^2)\,dy = \int_{-\infty}^{0} f(y - 1/y)\,dy + \int_{-\infty}^{0} f(y - 1/y)\,1/y^2\,dy$.

In the last integral we may substitute $z = -1/y$ to get $\int_0^{\infty} f(z - 1/z)\,dz$ or, using y instead of z, $\int_0^{\infty} f(y - 1/y)\,dy$.

**Problem B5**
Let F be the field with p elements. Let S be the set of 2 x 2 matrices over F with trace 1 and determinant 0. Find |S|.

**Solution**
Answer: $p^2 + p$. Let the matrix be

a b
c d

If we take any a not 0 or 1, then d is fixed as 1 - a and is also not 0. Hence bc is fixed as a(1 - a) and is non-zero. So we can take any b not 0 and d is then fixed. Altogether that gives us (p - 2)(p - 1) possibilities.

If we take a as 0 or 1, the d is fixed as 1 - a and bc must be 0. That gives 2p-1 possibilities for bc, or 4p-2 possibilities in all.
So in total we have $(p - 2)(p - 1) + 4p - 2 = p^2 + p$.

**Problem B6**
A *compact* set of real numbers is closed and bounded. Show that we cannot find compact sets $A_1, A_2, A_3, \ldots$ such that (1) all elements of $A_n$ are rational and (2) given any compact set K whose members are all rationals, $K \subseteq$ some $A_n$.

**Solution**
Use a diagonalisation argument.

Suppose we can find such $A_n$. Let $S_n$ be the interval $[1/2^{2n}, 1/2^{2n-1}]$, and let $T_n$ be the set of rational points in $S_n$. Then the closure of $T_n$ is $S_n$, which contains irrational points, so there must be points of $T_n$ which are not in $A_n$. Let $x_n$ be one such. Now consider $\{x_n\}$. Its only limit point is 0, so K, the union of $\{x_n\}$ and $\{0\}$, is a compact set of rationals. But K is not contained in any of the $A_n$, because it has a member $x_n$ not in $A_n$ for each n. Contradiction.

# 30th Putnam 1969

**Problem A1**
$R^2$ represents the usual plane (x, y) with $-\infty < x, y < \infty$. p: $R^2 \to R$ is a polynomial with real coefficients. What are the possibilities for the image $p(R^2)$?

**Solution**
Answer: [k, k], [k, inf), (-inf, k], (-inf, inf), (k, inf), (-inf, k) for all real k.

The first four possibilities given in the answer are easily realised: p(x, y) = k gives [k, k]; p(x, y) = $x^2$ + k gives [k, inf); p(x, y) = k - $x^2$ gives (-inf, k]; p(x, y) = x gives (-inf, inf).
If p is not constant, then wlog there is a positive power of x. Let n be the highest such power. Fix y so that the p becomes a polynomial in x with a non-zero term in $x^n$ (this must be possible since only finitely many values of y can give a zero term). As x tends to inf, this term will dominate and tend to +inf or -inf. The domain $R^2$ is connected and p is continuous, so the image must be connected also. So certainly there are no other possibilities apart from those given in the Answer.

It is tempting to think that we cannot get (k, inf), but attempts to prove it fail. The trick is to use two square terms which cannot be zero simultaneously. For example, $(xy - 1)^2 + x^2 + k$. We can make x arbitrarily small and choose y to make the first term zero, but if we make x zero, then the first term is 1.

**Problem A2**
A is an n x n matrix with elements $a_{ij} = |i - j|$. Show that the determinant $|A| = (-1)^{n-1} (n - 1) 2^{n-2}$.

**Solution**
For i = 1, 2, 3, ... , n-2 subtract twice row i+1 from row i and add row i+2 to row i. For i < n-1, row i becomes all 0s except for a 2 in column i+1.
Now expand successively by the first, second, third ... rows to get $(-2)^{n-2}$ times the 2 x 2 determinant with first row n-2, 1 and second row n-1, 0. This 2 x 2 determinant has value -(n-1), so the $|A| = (-1)^{n-1} (n-1) 2^{n-2}$.

**Problem A3**
An n-gon (which is not self-intersecting) is triangulated using m interior vertices. In other words, there is a set of N triangles such that: (1) their union is the original n-gon; (2) the union of their vertices is the set consisting of the n vertices of the n-gon and the m interior vertices; (3) the intersection of any two distinct triangles in the set is either empty, a vertex of both triangles, or a side of both triangles. What is N?

**Solution**
Answer: N = n + 2m - 2.

We use the well-known relation F = E + 2 - V (*), where F is the number of faces, E the number of edges and V the number of vertices. In this case, F = N + 1, because there are N triangles and one n-gon (the exterior polygon). So 3N + 1.n gives each edge twice, in other words 2E = 3N + n. Clearly V = m + n. So (*) gives: N + 1 =3N/2 + n/2 + 2 - m - n. Hence N = n + 2m - 2.

**Problem A4**
Prove that $\int_0^1 x^x\,dx = 1 - 1/2^2 + 1/3^3 - 1/4^4 + \dots$ .

**Solution**
The rhs is a series, so this suggests that we should expand the integrand as a series and integrate term by term.
We have that $x^x = e^{x \ln x}$, so the obvious approach is to expand this as a series: $1 + (x \ln x) + (x \ln x)^2/2! + \dots$ . For this to work we need that $\int_0^1 (x \ln x)^n\,dx = n!\,(-1)^n/(n+1)^{n+1}$ (*).
The integral is easy to evaluate by parts. Each step reduces the exponent on the log term without affecting the exponent on the x term and the non-integral term always vanishes at both endpoints. In other words, we have $\int_0^1 x^n \ln^m x\,dx = -m/(n+1) \int_0^1 x^n \ln^{m-1} x\,dx$, which gives (*).

**Problem A5**
u(t) is a continuous function. x(t), y(t) is the solution of x' = -2y + u(t), y' = -2x + u(t) satisfying the initial condition

$x(0) = x_0$, $y(0) = y_0$. Show that if $x_0 \neq y_0$, then we do not have $x(t) = y(t) = 0$ for any t, but that given any $x_0 = y_0$ and any $T > 0$, we can always find some u(t) such that $x(T) = y(T) = 0$.

**Solution**
Subtracting, we have $z' = 2z$, where $z = x - y$. So $z(t) = Ae^{2t}$. If $x_0 \neq y_0$, then $A \neq 0$ and so $z(t) \neq 0$ for any t. Hence, in particular, we do not have $x(t) = y(t) = 0$ for any t.

If $x_0 = y_0 = k$, then $z(0) = 0$, so $A = 0$, so $x(t) = y(t)$ for all t. Take (for example) $u(t) = -(k/T)\ e^{-2t}$. Then we may solve $x' + 2x = u$ with $x(0) = k$ to get $x(t) = k(1 - t/T)\ e^{-2t}$. This has $x(T) = 0$ as required.

**Problem A6**
The sequence $a_1 + 2a_2, a_2 + 2a_3, a_3 + 2a_4, \dots$ converges. Prove that the sequence $a_1, a_2, a_3, \dots$ also converges.

**Solution**
Note that the result is not true for $a_i + a_2, a_2 + a_3, \dots$ or for $2a_1 + a_2, 2a_2 + a_3, \dots$ . In the first case, we could have 1, -1, 1, -1, 1, ... . In the second case, we could have 1, -2, 4, -8, 16, -32, ... .

Suppose $a_n + 2a_{n+1}$ converges to 3k. We show that $a_n$converges to k.
Given any $\varepsilon > 0$, take N so that $a_n + 2a_{n+1}$ is within $\varepsilon$ of 3k for all n >= N. Take a positive integer M such that $a_N$ is within $(2^M + 1)\varepsilon$ of k.
Then $a_{N+1}$ is within $( (2^M + 1)\varepsilon + \varepsilon)/2 = (2^{M-1} + 1)\varepsilon$ of $(3k - k)/2 = k$. By a trivial induction $a^{N+M}$ is within $2\varepsilon$ of k. Then $a^{N+M+1}$ is within $(2\varepsilon + \varepsilon)/2$, and hence within $2\varepsilon$, of k. So by a trivial induction, $a_n$ is within $2\varepsilon$ of k for all n > N + M.

**Problem B1**
The positive integer n is divisible by 24. Show that the sum of all the positive divisors of n - 1 (including 1 and n - 1) is also divisible by 24.

**Solution**
Let n = 24m. We show first that if d is a divisor of n - 1 = 24m - 1, then $d^2 - 1$ is divisible by 24. Clearly d is not a multiple of 3 (because n is), so 3 must divide $d^2 - 1$. Also d must be odd (like n - 1), so d - 1 and d + 1 are consecutive even numbers, so one must be a multiple of 4, and there product $d^2 - 1$ must be a multiple of 8.

Now 24m - 1 cannot be a square (because squares are congruent to 0 or 1 mod 4), so its divisors come in pairs d, (24m - 1)/d. But d + (24m - 1)/d is divisible by 24 (because $d^2 - 1$ and 24m are and no factor of 24 can divide d). Hence the sum of all the divisors of n - 1 is divisible by 24.

**Problem B2**
G is a finite group with identity 1. Show that we cannot find two proper subgroups A and B ($\neq \{1\}$ or G) such that $A \cup B = G$. Can we find three proper subgroups A, B, C such that $A \cup B \cup C = G$?

**Solution**
We have |A| divides |G| and hence $|A| \leq |G|/2$. Similarly, $|B| \leq |G|/2$. But 1 belongs to both A and B, so |A union B| < |G|.

For three subgroups, sometimes we can and sometimes we cannot. For example, if G is the group of order 4 defined by ba = ab, $a^2 = b^2 = 1$, then we can: A = {1, a}, B = {1, b}, C = {1, ab}. On the other hand, if G is a cyclic group of prime order, then it has no proper subgroups. [But these are far from the only exceptions. For example take G to be cyclic of order 15. Then any proper subgroup must have order 3 or 5. All subgroups contain 1, so any three subgroups contain at most 13 elements between them.]

**Problem B3**
The sequence $a_1, a_2, a_3, \dots$ satisfies $a_1a_2 = 1$, $a_2a_3 = 2$, $a_3a_4 = 3$, $a_4a_5 = 4$, ... . Also, $\lim_{n\to\infty} a_n/a_{n+1} = 1$. Prove that $a_1 = \sqrt{(2/\pi)}$.

**Solution**
Let $a_1 = 1/k$. Then we deduce successively that $a_2 = k$, $a_3 = 2/k$, $a_4 = (3/2)\ k$. By a trivial induction, $a_{2n} = (3.5.7 \dots 2n-1)/(2.4.6 \dots 2n-2)\ k$ and $a_{2n+1} = (2.4.6 \dots 2n)/(3.5.7 \dots 2n-1)\ 1/k$. Hence $a_{2n+1}/a_{2n+2} = (2/1)\ (2/3)\ (4/3)\ (4/5)\ (6/5) \dots$

$(2n/2n-1)\ 1/k^2$. We are given that this has limit 1, so (2/1) (2/3) (4/3) (4/5) (6/5) ... (2n/2n-1) has limit $k^2$.
So we need to establish that (2/1) (2/3) (4/3) (4/5) (6/5) (6/7) ... (2n/2n-1) (2n)/(2n+1) ... = $\pi/2$. This is the well-known Wallis product. It is usually established by proving the product representation for sin z, but that requires relatively advanced complex analysis, which is outside the Putnam syllabus, so we need a simpler approach. The following is rather unmotivated unless you have seen it before.

Let $I_n = \int_0\pi/2 \sin^n x\ dx$. Integrating by parts, we have that $I_n = -\int \sin^{n-1}x\ d(\cos x) = (n-1)\ I_{n-2} + (n-1)\ I_n$. Hence $I_n = (n-1)/n\ I_{n-2}$. But $I_0 = \pi/2$, $I_1 = 1$. So we find that $I_{2n} = (1/2)(3/4)(5/6) \dots (2n-1)/2n\ \pi/2$, $I_{2n+1} = (2/3)(4/5)(6/7) \dots (2n/2n+1)$.
Now $0 < \sin x < 1$ on $(0, \pi/2)$, so $I_{2n-1} < I_{2n} < I_{2n+1}$. Dividing by $I_{2n+1}$, we get $(2n+1)/2n > I_{2n}/I_{2n+1} > 1$. So $I_{2n}/I_{2n+1}$ tends to 1 as n tends to infinity, which estabishes the Wallis product.

**Problem B4**
$\Gamma$ is a plane curve of length 1. Show that we can find a closed rectangle area 1/4 which covers $\Gamma$.

**Solution**
Let the endpoints of the curve be A and B. Take lines p, q parallel to AB and on opposite sides of it so that the band between them just covers the curve. Similarly take lines r, s perpendicular to AB so that the band between them just covers the curve (and so that r is nearer to A than to B).

Reflect A in q to get $A_1$. Reflect $A_1$ in r to get $A_2$. Similarly, reflect B in p to get $B_1$, then $B_1$ in s to get $B_2$. Consider the line segment $A_2B_2$. By reflection we can derive from it a curve from A to B consisting of five straight line segments which intersects p, q, r and s. Moreover it is evidently the shortest curve with this property.

Let X be the distance between r and s, Y the distance between p and q, and Z the distance between A and B. We have $(X - 2Y)^2 + (X - Z)^2 + 2X(X - Z) \geq 0$ (*). Hence $(2X - Z)^2 + (2Y)^2 \geq 4XY$, which tells us that the square of the length $A_2B_2$ is greater than or equal to four times the area of the rectangle formed by the intersection of the two bands. Thus if the curve $\Gamma$ has length 1, then $A_2B_2$ must have length at least 1 and hence we have found a rectangle with area at most 1/4 which covers $\Gamma$. Note that (*) shows the area will be less than 1/4 unless Z = X (so that r goes through A and s through B) and X = 2Y.

**Problem B5**
The sequence $a_i$, i = 1, 2, 3, ... is strictly monotonic increasing and the sum of its inverses converges. Let f(x) = the largest i such that $a_i < x$. Prove that f(x)/x tends to 0 as x tends to infinity.

**Solution**
Suppose not. Then for some fixed k > 0, we can find arbitrarily large x such that f(x) > kx. So take a sequence $x_1 < x_2 < x_3 < \dots$ such that (1) $f(x_i) > kx_i$, (2) $kx_i/2$ gt; $f(x_{i-1})$. Then by (1) at least $kx_i$ members of the sequence $a_n$ are less than $x_i$. By (2) at most $kx_i/2$ are less than $x_{i-1}$. So at least $kx_i/2$ must lie between $x_{i-1}$ and $x_i$.

So $\sum 1/a_n$ has at least $kx_i/2$ terms between $1/x_i$ and $1/x_{i-1}$. These terms sum to at least k/2. All terms are positive, so the series diverges. Contradiction.

**Problem B6**
M is a 3 x 2 matrix, N is a 2 x 3 matrix. MN = $\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}$ Show that NM = $\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$

**Solution**
The key observation is that $(MN)^2 = 9\ MN$. [Of course, we expect this to be true since NM = 9 I, and it is easy to verify.]

It is also easy to check that MN has rank 2. The rank of NM must be at least as big as M(NM)N = 9 MN, so NM is non-singular. Now $(NM)^3 = N(MN)^2M = N(9\ MN)M = 9\ (NM)^2$. Multiplying by the inverse of NM twice gives that NM = 9 I.

# 31st Putnam 1970

**Problem A1**
$e^{bx}$ cos cx is expanded in a Taylor series $\sum a_n x^n$. b and c are positive reals. Show that either all $a_n$ are non-zero, or infinitely many $a_n$ are zero.

**Solution**
$e^{bx}$ cos cx = Re $e^{(b+ic)x}$, so $n!\ a_n = \text{Re}( (b + ic)^n )$. Let $b + ic = k e^{i\theta}$, then $n!\ a_n/k^n = \cos n\theta$. So $a_n = 0$ iff $n\theta = (2m+1)\pi/2$ for some integer m. So if there are any zeros then there are infinitely many.

**Problem A2**
$p(x, y) = a x^2 + b x y + c y^2$ is a homogeneous real polynomial of degree 2 such that $b^2 < 4ac$, and q(x, y) is a homogeneous real polynomial of degree 3. Show that we can find k > 0 such that p(x, y) = q(x, y) has no roots in the disk $x^2 + y^2 < k$ except (0, 0).

**Solution**
The disk is a strong hint that one should use polar coordinates, so put $x = r \cos\theta$, $y = r \sin\theta$. Then we get $r = p(\cos\theta, \sin\theta)/q(\cos\theta, \sin\theta)$. Now $|\cos\theta|$ and $|\sin\theta| \le 1$, so $|q(\cos\theta, \sin\theta)| \le h$, where h is the sum of the absolute values of the coefficients of q. So we are home if we can establish some inequality $|p(\cos\theta, \sin\theta)| \ge h'$, because we can then take $k = h'/h$.

Without loss of generality a > 0 and hence also c > 0. Now $p(\cos\theta, \sin\theta) = 1/2\ (a + c)(\cos^2\theta + \sin^2\theta) + 1/2\ (a - c)(\cos^2\theta - \sin^2\theta) + 1/2\ b \sin 2\theta = 1/2\ (a + c) + 1/2\ (a - c) \cos 2\theta + 1/2\ b \sin 2\theta$. Put $d = \sqrt{(b^2 + (a - c)^2)}$ and take $\varphi$ so that $\cos\varphi = (a - c)/d$, $\sin\varphi = b/d$. Then $2p(\cos\theta, \sin\theta) = (a + c) + d (\cos\varphi \cos 2\theta + \sin\varphi \sin 2\theta) = (a + c) + d \cos(\varphi - 2\theta) >= (a + c) - d$.

But $4ac > b^2$, so $(a + c)^2 > (a - c)^2 + b^2$, so $(a + c) > d$, and we are done.

**Problem A3**
A perfect square has *length* n if its last n digits (in base 10) are the same and non-zero. What is the longest possible length? What is the smallest square achieving this length?

**Solution**
Answer: 3, $38^2 = 1444$.

All squares end in the digit 0, 1, 4, 9, 6, or 5. A square ending in 11, 99, 66, 55 would be congruent to 3, 3, 2, 3 mod 4, but squares are congruent to 0 or 1 mod 4. So for length greater than 1 the square must end in 4. For example, $12^2 = 144$.
A square ending in 4444 would be congruent to 12 mod 16, but squares are congruent to 0, 1, 4 or 9 mod 16. So the maximum length is 2 or 3.

If $n^2$ ends in 4, then n must end in 2 or 8. $(100a + 10b + 2)^2 = 10000a^2 + 1000 (2ab) + 100 (4a + b^2) + 10 (4b) + 4$. So if this ends in 44, then b = 1 or 6. If b = 1, then $4a + b^2$ is odd, so the square cannot end in 444. If b = 6, then the square is 1000k + (4a + 38) 100 + 44. This will end in 444 if we take a = 4 or 9. Thus the smallest numbers ending in 2 whose square ends in 444 are 462 (square 213444) and 962 (square 925444).

$(100a + 10b + 8)^2 = 1000k + 100 (16a^2 + b^2) + 10 (16b + 6) + 4$. So if this ends in 44, then b = 3 or 8. $(100a + 38)^2 = 1000k + 100 (76a) + 1444$, so this ends in 444 if a = 0. This must be the smallest solution.

**Problem A4**
The real sequence $a_1, a_2, a_3, \ldots$ has the property that $\lim_{n\to\infty} (a_{n+2} - a_n) = 0$. Prove that $\lim_{n\to\infty} (a_{n+1} - a_n)/n = 0$.

**Solution**
Suppose we have any series $a_n$ satisfying the condition. Take a new series $b_n$ defined by $b_{2n+1} = 0$ and $b_{2n} = a_{2n}$. Then $b_n$ also satisfies the condition, but $(b_{2n+1} - b_{2n})/n = - a_{2n}/n$, so we must have the apparently stronger result that $a_{2n}/(2n)$ tends to zero (and similarly that $a_{2n+1}/(2n+1)$ tends to zero).

Thus we need to prove that if $c_n$ is any sequence such that $c_{n+1} - c_n$ tends to zero, then $c_n/n$ tends to zero.

Given any positive k, we have $|c_{n+1} - c_n| < k$ for all n > some N. Now for any M > N, we have $c_M = c_N + (c_{N+1} - c_N) + \dots + (c_M - c_{M-1})$. Hence $|c_M| \le |c_N| + (M-N)k$. If we take M sufficiently large that $|b_N|/M < k$, then $|c_M|/M < 2k$, which establishes that $c_n/n$ tends to zero.

**Problem A5**
Find the radius of the largest circle on an ellipsoid with semi-axes a > b > c.

**Solution**
Answer: b.

A circle lies in a plane, so we consider planes cutting the ellipsoid. The intersection of any plane P with the ellipsoid is an ellipse (assuming it cuts the ellipsoid in more than one point). That is fairly obvious. Substitute the linear equation for the plane into the quadratic equation for the ellipsoid and we get a quadratic equation for the projection of the intersection onto one of the coordinate planes. This must be a conic and, since bounded, an ellipse. Projecting back onto P shows that the intersection is an ellipse.

It is less obvious, but true, that parallel planes give similar ellipses. So to maximise the size we take the plane P through the centre of the ellipsoid. Suppose its intersection with the ellipsoid is a circle K. P cannot be normal to one of the semi-axes, because then its intersection is certainly an ellipse with unequal semi-axes. So P must meet the bc plane in a line. This line is a diameter of K. But it is also a diameter of an ellipse with semi-major axes b and c, so it has length at most 2b, so K has diameter is at most 2b. Similarly P meets the ab plane in a line, which has length at least 2b, so the diameter of K is at least 2b. So if K has radius b.

It remains to show that some planes do intersect the ellipsoid in a circle. Consider a plane through the semi-major axis b of the ellipsoid. The intersection K clearly has one diameter 2b. Moreover, the ellipse K is symmetrical about this diameter, so it must be one of its two semi-major axes. If we start with the plane also containing the semi-major axis c of the ellipsoid, then K has another diameter perpendicular to the first and length 2c. This must be the other semi-major axis of K. As we rotate the plane through a right-angle, the length of this diameter increases continuously to 2a. So at some angle it must be 2b. But an ellipse with equal semi-major axes is a circle.

**Problem A6**
x is chosen at random from the interval [0, a] (with the uniform distribution). y is chosen similarly from [0, b], and z from [0, c]. The three numbers are chosen independently, and $a \ge b \ge c$. Find the expected value of min(x, y, z).

**Solution**
Answer: $c/2 - c^2(1/(6a) + 1/(6b)) + c^3/(12ab)$.

Let $m(k) = \text{prob}(x \ge k \text{ and } y \ge k \text{ and } z \ge k)$. We have that $\text{prob}(x \ge k) = 1$ for $k < 0$, $1 - k/a$ for $0 \le k \le a$, 0 for $k > a$. Similarly for y and z. So m(k) is 1 for $k < 0$, and $(1 - k/a)(1 - k/b)(1 - k/c)$ for $0 \le k \le c$, and 0 for $k \ge a$.

Now the probability that the minimum lies between k and $k + \delta k$ is $m(k) - m(k + \delta k)$, so the probability density for min(x, y, z) to equal k is $-m'(k)$ which is: $(1/a + 1/b + 1/c) - 2k(1/(ab) + 1/(bc) + 1/(ca)) + 3k^2/(abc)$ for $0 \le k \le c$. Thus the expected value is the integral from 0 to c of $k(1/a + 1/b + 1/c) - 2k^2(1/(ab) + 1/(bc) + 1/(ca)) + 3k^3/(abc)$ which is $c^2(1/a + 1/b + 1/c)/2 - 2c^3(1/(ab) + 1/(bc) + 1/(ca))/3 + 3c^4/(4abc)$.

**Problem B1**
Let $f(n) = (n^2 + 1)(n^2 + 4)(n^2 + 9) \dots (n^2 + (2n)^2)$. Find $\lim_{n\to\infty} f(n)^{1/n}/n^4$.

**Solution**
Rearranging slightly, we may take $g(n) = (1 + (1/n)^2)(1 + (2/n)^2)(1 + (3/n)^2) \dots (1 + (2n/n)^2)$. We have to find $\lim g(n)^{1/n}$.

It is a mistake to approach this algebraically. It is not hard to show that the product of the pair of terms r and 2n+1-r is at least 4. There are n such pairs, so certainly $g(n)^{1/n} > 4$, but it is hard to get any further.

The key is to take logs. We then see immediately that $g(n)^{1/n}$ becomes a standard Riemann sum for $\int_0^2 \log(1 + x^2)\,dx$. So the limit is simply the integral.

x $\log(1 + x^2)$ differentiates to $\log(1 + x^2) + 2x^2/(1 + x^2)$. A little reflection then suggests using $2 \tan^{-1}x$, which differentiates to $2/(1 + x^2)$. So the complete integral is $x \log(1 + x^2) + 2 \tan^{-1}x - 2x$. Evaluating between 0 and 2 gives $2 \log 5 + 2 \tan^{-1}2 - 4 = k$, say. Then the original limit in the question is $e^k$. That is approximately 4.192.

**Problem B2**

A weather station measures the temperature T continuously. It is found that on any given day $T = p(t)$, where p is a polynomial of degree $\leq 3$, and t is the time. Show that we can find times $t_1 < t_2$, which are independent of p, such that the average temperature over the period 9am to 3pm is $( p(t_1) + p(t_2) / 2$. Show that $t_1$ = 10:16am, $t_2$ = 1:44pm.

**Solution**

Let t be time after 9am and T be 6 hrs after 9am, so that t = 0 represents 9am and t = T represents 3pm. We may write the polynomial p(t) as $at^3 + bt^2 + ct + d$ for some a, b, c, d. The average temperature is $1/T \int_0^T(at^3 + bt^2 + ct + d)\, dt = T^3 a/4 + T^2 b/3 + T c/2 + d$. We wish to find $t_1$, $t_2$ so that this equals $(t_1^3 + t_2^3) a/2 + (t_1^2 + t_2^2) b/2 + (t_1 + t_2) c/2 + d$ for all a, b, c, d.

The terms in d match. The term in c matches provided we take $t_1 + t_2 = T$. The term in b matches provided we take $t_1^2 + t_2^2 = 2T^2/3$. These two equations already determine $t_1$ and $t_2$. In fact, solving the quadratic, we get $t_1 = T(1 - 1/\sqrt{3})/2$ and $t_2 = T(1 + 1/\sqrt{3})/2$.

For the terms in a to match we need $t_1^3 + t_2^3 = T_3/2$. But we can easily check that the values above satisfy this relation also.

Finally note $t_1 = 6(1/2 - (\sqrt{3})/6 )$ hrs $= 3 - \sqrt{3} = 3 - 1.732 = 1.268$ hrs = 1 hr 16 min (approx), so $t_1$ represents about 10:16am and $t_2$ about 1 hr 16 min before 3pm or 1:44pm.

**Problem B3**

S is a closed subset of the real plane. Its projection onto the x-axis is bounded. Show that its projection onto the y-axis is closed.

**Solution**

Let Y be the projection onto the y-axis and X the projection on the x-axis. Let $\{ y_n\}$ be any Cauchy sequence in Y. Then $\{ y_n \}$ must converge to some real y. We have to show that y is in Y. For each n take any $x_n$ such that $(x_n, y_n)$ is in S. Then $\{ x_n \}$ lies in X, which is bounded, so $\{ x_n \}$ is bounded. But any bounded sequence has a convergent subsequence. So we can take a subsequence $\{ u_n \}$ of $\{ x_n \}$ which is convergent and therefore Cauchy. Let $\{ v_n \}$ be the corresponding subsequence of $\{ y_n \}$. Then $\{ v_n \}$ is also Cauchy and hence the sequence of points $P_n = (u_n, v_n)$ in S is Cauchy. But S is closed so P converges to some (u, v) in S. Hence $\{ v_n \}$ converges to v, which is in Y (it is the projection of (u, v) ). But since $\{ y_n \}$ converges to y, its subsequence $\{ v_n \}$ must also converge to y. Hence y = v and y is in Y.

**Problem B4**

A vehicle covers a mile (= 5280 ft) in less than a minute, starting and ending at rest and never exceeding 90 miles/hour. Show that its acceleration or deceleration exceeded 6.6 ft/sec$^2$.

**Solution**

Plot velocity (v in ft/sec ) against time ( t in sec ) .

The graph never gets above the line v = 132 ( ft/sec =90 mph).

If the vehicle's acceleration never exceeds 6.6 ft/sec$^2$, then in particular, its acceleration in the first 20 seconds does not exceed 6.6 ft/sec and hence the velocity curve never gets above the line $v = 6.6 t$, which cuts the line v = 132 at t = 20.

Similarly, if the vehicle's deceleration never exceeds 6.6 ft/sec$^2$, then its deceleration in the last 20 seconds never exceeds 6.6 ft/sec, so the velocity curve it never gets above the line $v = - 6.6 (t - 60)$, which represents constant deceleration at the maximum 6.6 ft/sec$^2$ from t = 40 until the finish at t = 60 when it is stationary. In fact the curve must lie strictly under this line, since the vehicle finishes in less than a minute.

Hence the area under the curve, which represents distance travelled, is less than the area of the quadrilateral bounded by $t = 0$, $v = 6.6t$, $v = 132$ and $v = -6.6(t - 60)$, which is 1/2 20. 132 + 20.132 + 1/2 20.132 = 40.132 = 5280 = 1 mile. But we know that the distance travelled was 1 mile. Hence either it accelerated at more than 6.6 ft/sec$^2$ at some point in the first 20 seconds, or it decelerated at more than 6.6 ft/sec$^2$ at some point in the last 20 seconds.

**Problem B5**

$k_n(x) = -n$ on $(-\infty, -n]$, $x$ on $[-n, n]$, and $n$ on $[n, \infty)$. Prove that the (real valued) function $f(x)$ is continuous iff all $k_n( f(x) )$ are continuous.

**Solution**

This is apparently almost trivial.

The $k_n$ are continuous, so if f is continuous, then so are all $k_nf$.

If f is not continuous, then there must be a point $x_0$ and an infinite sequence of points $x_m$ in the interval $(x_0 - 1, x_0 + 1)$ but not equal to $x_0$, such that $x_m$ tend to $x_0$, but $f(x_m)$ do not tend to $f(x_0)$. Take n sufficiently large that $(x_0 - 1, x_0 + 1)$ lies inside $(-n, n)$. Then $k_nf(x_i) = f(x_i)$, so $k_nf$ is not continuous at $x_0$.

**Problem B6**

The quadrilateral Q contains a circle which touches each side. It has side lengths a, b, c, d and area $\sqrt{(abcd)}$. Prove it is cyclic.

**Solution**

It helps a lot to know the following result about the area of quadrilaterals: the area of a quadrilateral side a, b, c, d is maximised by making it cyclic, in which case its area is $\sqrt{( (s - a)(s - b)(s - c)(s - d) )}$ (*), where $2s = a + b + c + d$.

If we assume this result, then the problem is fairly easy. Assume that the sides are in the order a, b, c, d, so that the side length a is opposite that length c. If Q contains a circle C which touches each side, then $a + c = b + d$. [The distances from a vertex or the quadrilateral to the two points of contact of its two sides are equal. So for some w, x, y, z we have $a = w + x$, $b = x + y$, $c = y + z$, $d = z + w$ and hence $a + c = b + d$.]

But this means that $s - a = (- a + c + b + d)/2 = (-a + c + a + c)/2 = c$. Similarly, $s - b = d$, $s - c = a$ and $s - d = b$, so the maximum possible area of the quadrilateral is $\sqrt{(abcd)}$, with equality iff it is cyclic. But we are given that the area is $\sqrt{(abcd)}$, so it must be cyclic.

To establish the result about quadrilaterals, drop the assumption that $a + c = b + d$, and let the angle between a and b be $\theta$ and the opposite angle be $\varphi$. Let the area be A. Then $2A = ab \sin \theta + cd \sin \varphi$, so $16A^2 = 4(a^2b^2 \sin^2\theta + c^2d^2 \sin^2\varphi ) + 8abcd \sin \theta \sin \varphi$ (*).

By the cosine rule we have $a^2 + b^2 - 2ab \cos \theta = c^2 + d^2 - 2cd \cos \varphi$, so $(a^2 + b^2 - c^2 - d^2)^2 = 4(a^2b^2 \cos^2\theta + c^2d^2 \cos^2\varphi ) - 8abcd \cos \theta \cos \varphi$. Adding to (*) gives $16A^2 = 4(a^2b^2 + c^2d^2) - (a^2 + b^2 - c^2 - d^2)^2 - 8abcd \cos(\theta + \varphi) \le 4(a^2b^2 + c^2d^2) - (a^2 + b^2 - c^2 - d^2)^2 + 8abcd$ (**), with equality iff $\cos(\theta + \varphi) = -1$, in other words, iff the quadrilateral is cyclic.

But we can easily check that $(-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d) = 4(a^2b^2 + c^2d^2) - (a^2 + b^2 - c^2 - d^2)^2 + 8abcd$.

# 32nd Putnam 1971

**Problem A1**
Given any 9 lattice points in space, show that we can find two which have a lattice point on the interior of the segment joining them.

**Solution**
We can divide the points into 8 categories according to the parity of each coordinate. There must be at least 2 points in the same category. The midpoint of the line joining them is then also a lattice point.

**Problem A2**
Find all possible polynomials f(x) such that f(0) = 0 and $f(x^2 + 1) = f(x)^2 + 1$.

**Solution**
Answer: f(x) = x.
f(0) = 0. $f(1) = f(0)^2 + 1 = 1$. $f(1^2 + 1) = f(1)^2 + 1 = 2$. Similarly, by an easy induction we can get an arbitrarily large number of integers n for which f(n) = n. So if f has degree m, we can find at least m+1 integers on which it agrees with the polynomial p(x) = x. Hence it is identically equal to p.

**Problem A3**
The vertices of a triangle are lattice points in the plane. Show that the diameter of its circumcircle does not exceed the product of its side lengths.

**Solution**
Let the side lengths be a, b, c and the circumradius R. Let θ be the angle opposite side a. Then the area of the triangle A = 1/2 bc sin θ. The side a subtends an angle 2θ at the centre of the circumcircle, so a = 2R sin θ. Hence 2A = abc/(2R). So we have to show that A ≥ 1/2.

This follows at once from the well-known Pick's theorem: the area of any (non-self-intersecting) polygon whose vertices are lattice points is v/2 + i - 1, where v is the number of lattice points on the perimeter and i is the number of lattice points inside the polygon (so since for a triangle v ≥ 3, and i ≥ 0, we have area at least 3/2 - 1 = 1/2).

**Problem A4**
k lies in the open interval (1, 2). Show that the polynomial formed by expanding $(x + y)^n(x^2 - k xy + y^2)$ has positive coefficients for sufficiently large n. Find the smallest such n for k = 1.998.

**Solution**
Answer: 1999.

The coefficient of $x^{n-r}y^{r+2}$ is ( nCr - k nCr+1 + nCr+2 ). We show that the worst case is r near n/2 and that even this is positive for sufficiently large n.
( nCr - k nCr+1 + nCr+2 ) = n!/(r+2! n-r!) ( (r+1)(r+2) - k(r+2)(n-r) + (n-r-1)(n-r) ). Let f(r) = ( (r+1)(r+2) - k(r+2)(n-r) + (n-r-1)(n-r) ). Then f '(r) = 2r+3 - kn +2kr + 2k - 2n + 2r + 1. So f '(r) = 0 iff (2k+4)r = (k+2)n - (2k+4) or r = n/2 - 1. Also it is clear that f '(r) is negative for smaller values and positive for larger values, so this represents a minimum. So if n = 2m the minimum occurs at r = m - 1 and is 2mCm-1 - k 2mCm + 2mCm+1. Now 2mCm-1 = 2mCm+1 = m/(m+1) 2mCm, so the minimum value is positive iff 2m/(m+1) > k which is certainly true for all sufficiently large m since k < 2.

Similarly if n = 2m+1, then the minimum occurs at r = m-1 or m. In either case, the minimum value is 2m+1Cm-1 - k 2m+1Cm + 2m+1Cm+1. But 2m+1Cm = 2m+1Cm+1 and 2m+1Cm-1 = m/(m+2) 2m+1Cm, so the minimum value is (m/(m+2) - (k-1) ) 2m+1Cm, which is certainly positive for all sufficiently large m since k-1 < 1.

We have 2m/(m+1) > 1.998 for m > 999, so the smallest even n with all coefficients positive is 2000. On the other hand, m/(m+2) > 0.998 for m > 998, so the smallest odd n with all coefficients positive is 1999. Thus the smallest n is 1999.

**Problem A5**
A player scores either A or B at each turn, where A and B are unequal positive integers. He notices that his cumulative score can take any positive integer value except for those in a finite set S, where $|S| = 35$, and $58 \in S$. Find A and B.

**Solution**
Answer: A, B = 8, 11.

Let us call a number *green* if it can be expressed as a non-negative multiple of A plus a non-negative multiple of B. Assume A > B. A and B must be coprime. Otherwise there would be infinitely many non-green numbers. The set of integers 0, A, 2A, ... , (B-1)A are all incongruent mod B (since A and B are coprime), so they form a complete set of residues mod B. Hence any integer ≥ (B-1)A can be expressed as the sum of a multiple of B and one of the members of the set and is thus green. In fact, the none of the numbers (B-1)A - B + 1, (B-1)A - B + 2, ... , (B-A)A - 1is a multiple of A and they are are incongruent to (B-1)A mod B, so they must equal one of the numbers 0, A, 2A, ... , (B-2)A plus a multiple of B. Thus any integer greater than (B-1)A - B is green. On the other hand, (B-1)A - B itself cannot because it is (B-1)A mod B, so it is incongruent to kA mod B for k < (B-1) (and it cannot be (B-1)A plus a multiple of B because it is too small).

So we need to look at the numbers ≤ AB - A - B. Exactly [A/B] of them are congruent to A mod B, but are not green. For if A = [A/B] B + r (with 0 < r < B), then they are r, r + B, r + 2B, ... , r + ([A/B] - 1)B. Similarly, [2A/B] are congruent to 2A mod B, but not green. So the total number of non-green numbers is: [A/B] + [2A/B] + ... + [(B-1)A/B] (*). But kA/B cannot be integral for k = 0, 1, ... , (B-1) since A and B are coprime, so [A/B] + [(B-1)A/B] = A-1, [2A/B] + [(B-2)A/B] = A-1, ... . If B - 1 is even, then this establishes that (*) is 1/2 (A-1)(B-1). If B - 1 is odd, then the central term is [A/2]. B is even, so A must be odd, so [A/2] = 1/2 (A - 1). So in this case also (*) is 1/2 (A-1)(B-1).

So we have $(A-1)(B-1) = 2\cdot 35 = 70 = 2\cdot 5\cdot 7$. So A = 11, B = 8, or A = 15, B = 6, or A = 36, B = 3. But the last two cases are ruled out because they do not have A, B coprime. So we have A = 11, B = 8. We can easily check that 58 is not green in this case.

**Problem A6**
$\alpha$ is a real number such that $1^\alpha, 2^\alpha, 3^\alpha, \dots$ are all integers. Show that $\alpha \ge 0$ and that $\alpha$ is an integer.

**Solution**
Note first that $\alpha$ is obviously not negative, because then $n^\alpha$ would be between -1 and 0 for sufficiently large n.

It is also fairly obvious that $\alpha$ cannot be rational (and non-integral). For if $2^{m/n} = A$, then $2^m = A^n$, so A must be a power of 2, but then m is a multiple of n. However, that line of approach gets us nowhere with irrational values. The hint is to use some sort of mean value theorem. The idea is that this gives us another expression for the difference between the value of f(x) at two points. We might then hope to show that the expression was non-integral, but the difference integral.

In principle, we have two choices for f(x): $x^\alpha$ or $n^x$. The latter does not seem promising because the derivative is $n^x \log n$. If x is allowed to range over some interval (as in the MVT) then we cannot be sure whether it is integral or non-integral. It is certainly not small.

On the other hand, the former is not immediately promising either. The derivative is $\alpha x^{\alpha-1}$ (*). But suppose $\alpha < 1$. Then $\alpha - 1 < 0$, so for sufficiently large x the derivative will be less than 1. It will still be positive, so it will be between 0 and 1 and hence non-integral. The MVT tells us that $f(n+1) - f(n) = f'(\xi)$ for some $\xi$ between n and n+1, so if n is suffiently large f(n+1) - f(n) is an integer, but $f'(\xi)$ is not. Contradiction. So we can rule out $0 < \alpha < 1$.

In principle, we could get a contradiction for larger $\alpha$ if we could use a larger derivative. We have $f^{(n)}(x) = \alpha(\alpha-1) \dots (\alpha-k+1) x^{\alpha-k}$, which will be between 0 and 1 provided that we choose k so that $k-1 < \alpha < k$ and take x sufficiently large.

So we need a MVT which involves $f^{(n)}(\xi)$ plus terms which are all integral. The usual generalisation is the Taylor series with remainder. That does not help, because the other terms are mainly non-integral - they involve the lower derivatives which all have a factor $\alpha$.

However, there is another generalisation. Define $\Delta f(x) = f(x+1) - f(x)$. We can iterate D, so that $D^2f(x) = \Delta f(x+1) - \Delta f(x) = f(x+2) - 2f(x+1) - f(x)$. We claim that $\Delta^2 f(x) = f^{(2)}(\xi)$ for some $\xi$ in $[x, x+2]$. The proof is almost immediate by applying the MVT to $\Delta f(x)$. [Let $g(x) = \Delta f(x)$. Then by the MVT $\Delta g(x) = g'(\zeta)$ for some $\zeta$ in $[x, x+1]$. But $g'(\zeta) = f'(\zeta+1) - f'(\zeta)$. By the MVT this is $f''(\xi)$ for some $\xi$ in $[\zeta, \zeta+1]$ and hence in $[x, x+2]$. ] By a simple induction, we can also show that $\Delta^n f(x) = f^{(n)}(\xi)$ for some $\xi$ in $[x, x+n]$.

This is the generalisation we need because $\Delta^k f(n)$ is an integral combination of values of f at integer points and hence integral. Whereas for suitable n and k we showed above that $f^{(k)}(n)$ is non-integral.

**Problem B1**

S is a set with a binary operation * such that (1) $a * a = a$ for all $a \in S$, and (2) $(a * b) * c = (b * c) * a$ for all $a, b, c \in S$. Show that * is associative and commutative.

**Solution**

$b*a = (b*a)(b*a)$ using (1)
$= ( (b*a)*b )*a$ using (2) twice
But $(b*a)*b = (b*b)*a$ using (2) twice
$= b*a$ using (1)
So $b*a = (b*a)*a$
$= (a*a)*b$ using (2) twice
$= a*b$
That shows that * is commutative. (2) now gives immediately that * is associative, because $(a*b)*c = (b*c)*a = a*(b*c)$.

**Problem B2**

Let X be the set of all reals except 0 and 1. Find all real valued functions f(x) on X which satisfy $f(x) + f(1 - 1/x) = 1 + x$ for all x in X.

**Solution**

The trick is that $x \to 1 - 1/x \to 1/(1-x) \to x$. Thus we have:
$f(x) + f(1 - 1/x) = 1 + x$ (1);
$f(1 - 1/x) + f(1/(1-x) ) = 2 - 1/x$ (2);
$f(1/(1-x) ) + f(x) = 1 + 1/(1-x)$ (3).
Now (1) - (2) + (3) gives $2 f(x) = x + 1/x + 1/(1-x)$ or $f(x) = (x^3 - x^2 - 1) / (2x^2 - 2x)$. It is easily checked that this does indeed satisfy the relation in the question.

**Problem B3**

Car A starts at time $t = 0$ and, traveling at a constant speed, completes 1 lap every hour. Car B starts at time $t = \alpha > 0$ and also completes 1 lap every hour, traveling at a constant speed. Let a(t) be the number of laps completed by A at time t, so that $a(t) = [t]$. Similarly, let b(t) be the number of laps completed by B at time t. Let $S = \{t \ge \alpha : a(t) = 2 b(t) \}$. Show that S is made up of intervals of total length 1.

**Solution**

Answer: $S = [ [\alpha]+\alpha, 2[\alpha]+1 ) \cup [ 2\alpha]+2, [\alpha]+\alpha+2 )$.

At all times in the interval $[ [\alpha]+\alpha, 2[\alpha]+1 )$ car A has completed $2[\alpha]$ laps and car B has completed $[\alpha]$ laps. At all times in the interval $[ 2\alpha]+2, [\alpha]+\alpha+2 )$ car A has completed $2[\alpha]+2$ laps and car B has completed $[\alpha]+1$ laps. The first interval has length $1-\{\alpha\}$ and the second interval has length $\{\alpha\}$, where $\{\alpha\}$ denotes the fractional part of $\alpha$.

During the interval $[ 2[\alpha]-1, 2[\alpha] )$ car A has completed $2[\alpha]-1$ laps and car B either $[\alpha]-2$ or $[\alpha]-1$ laps, so car A has completed more than twice as many laps as car B. Similarly, during the interval $[ 2[\alpha]-n, 2[\alpha]-n+1)$ car A has completed $2[\alpha]-n$ laps and car B $[\alpha]-n-1$ or $[\alpha]-n$ laps, so at all times before $2[\alpha]$ car A has completed more than twice as many laps as car B and these times do not form part of S.

During $[ 2\alpha]+1, [\alpha]+\alpha+1 )$ car A has completed $2[\alpha]+1$ laps and car B $[\alpha]$ laps, which is less than half the number. During $[ [\alpha]+\alpha+1, 2[\alpha]+2 )$ car A has completed $2[\alpha]+1$ laps and car B $[\alpha]+1$ laps, which is more than half the number. So no points in the gap between the two intervals of S belong in S.

Finally, during [ $[\alpha]+\alpha+2$, $2[\alpha]+3$ ) car A has completed $2[\alpha]+2$ laps and car B $[\alpha]+2$ laps, whilst during [ $2[\alpha]+n$, $2[\alpha]+n+1$ ) car A has completed $2[\alpha]+n$ laps and car B either $[\alpha]+n-1$ or $[\alpha]+n$ laps. So at all times after $[\alpha]+\alpha+2$ car A has completed less than twice the number of laps completed by car B and none of these times belong in S.

**Problem B4**

A and B are two points on a sphere. S(A, B, k) is defined to be the set {P : AP + BP = k}, where XY denotes the great-circle distance between points X and Y on the sphere. Determine all sets S(A, B, k) which are circles.

**Solution**

Answer: S(A, B, k) is a circle iff $k = \pi$ and A and B are not antipodal, in which case S(A, B, k) is the great circle perpendicular to the great circle through A and B and so that A and B are on the same side of and equidistant from the plane containing it.

Wlog we may take the sphere to have radius 1. Let the great-circle distance between A and B be d. So $0 \le d \le \pi$. If $d = 0$, then clearly any k (in the range $0 < k < \pi$) gives a circle. If $d = \pi$, then any point P on the sphere has PA + PB $= \pi$, so S is either empty or the whole sphere. So let us assume that $0 < d < \pi$.

Let C be the great circle through A and B. Let O be the centre. Let QR be the diameter with angle AOQ = angle BOR. Let B' be the reflection of B in QR. Then AB' is a diameter. Let C' be the great circle through Q and R perpendicular to C. For any point P on the sphere we have AP + PB' $= \pi$. If P lies on C', then by symmetry BP = B'P, so AP + PB $= \pi$. If P lies in the open hemisphere containing B, then BP < B'P, so AP + BP $< \pi$. Similarly, if P lies in the open hemisphere containing B', then BP > B'P, so AP + BP $> \pi$. Thus S(A, B, $\pi$) is the circle C'.

Let us now assume $0 < k < \pi$. If $k < d$, then S(A, B, k) is empty, since AP + BP cannot be less than AB. If $k = d$, then S(A, B, k) is the arc of C between A and B. So suppose $d < k < \pi$. Then there are two points on C' in S(A, B, k): Q' on the arc AQ and R' on the arc BR, with AQ' = BR'. Now S(A, B, k) must be symmetrical about the plane containing C', so if it is a circle C'', then it must be the circle diameter Q'R' and perpendicular to C'. It suffices to find a single point X on this circle with AX + BX > k. Let Y be the midpoint of the arc AB, and take X so that the great circle through X and Y is perpendicular to C'. Since we are assuming C'' is a circle we have YQ' = YX = YR'. Since the great circle through X and Y is perpendicular to C'', we must have AX > YX and BX > YX, so AX + BX > 2 YX = 2 YQ' = AQ' + BQ' = k, so Y is not in S(A, B, k) after all.

Finally consider $k > \pi$. Let A' be the antipodal point to B. Now for any point P on the sphere, AP $= \pi$ - B'P and BP $= \pi$ - A'P, so S(A, B, k) = S(A', B', $2\pi$-k).

**Problem B5**

A *hypocycloid* is the path traced out by a point on the circumference of a circle rolling around the inside circumference of a larger fixed circle. Show that the plots in the (x, y) plane of the solutions ( x(t), y(t) ) of the differential equations $x'' + y' + 6x = 0$, $y'' - x' + 6y = 0$ with initial conditions $x'(0) = y'(0) = 0$ are hypocycloids. Find the possible radii of the circles.

**Solution**

Answer: if we take the radius of the rolling circle to be R, then the radius of the fixed circle is 5/2 R or 5/3 R.

Take x, y coordinates and let the fixed circle have centre at the origin O and radius cR (with $c > 1$). Take the rolling circle to have centre C, radius R and to be initially touching the fixed circle at $x = cR$, $y = 0$. Take this point to be the point P on the rolling circle whose motion we track. After time t, let OC make an angle $\theta$ with the x-axis. Then the rolling circle will have rolled through an angle $c\theta$. We assume that the motion is uniform so that $\theta = bt$.

At time t, the coordinates of C will be $x = aR \cos bt$, $y = aR \sin bt$, where $a = c - 1$. CP makes an angle $c\theta - \theta = a\theta$ with the x-axis (in the opposite sense to $\theta$), so P has coordinates: $x = aR \cos bt + R \cos abt$, $y = aR \sin bt - R \sin abt$ (*).

However, we could also take the y-axis in the opposite direction, in which case the equations would become: $x = aR \cos bt + R \cos abt$, $y = -aR \sin bt + R \sin abt$ (**).

Turning to the differential equations given, take the first + i times the second and put $z = x + i y$. Then we get: $z'' - i z + 6z = 0$. This has general solution $z = A e^{3it} + B e^{-2it}$, or $x = A \cos 3t + B \cos 2t$, $y = A \sin 3t - B \sin 2t$. But we are told that $x'(0) = y'(0) = 0$, so $3A = 2B$. Hence $x = 2/3\ B \cos 3t + B \cos(2/3\ 3t)$, $y = 2/3\ B \sin 3t - B \sin(2/3\ 3t)$. Comparing with (*) we see that it represents a hypocycloid with $c = 5/3$.

Alternatively, we can cast the solution into the form (**) by writing it as: $x = 3/2\ A \cos 2t + A \cos(3/2\ 2t)$, $y = -3/2\ A \sin 2t + A \sin(3/2\ 2t)$. Comparing with (**) we see that this represents a hypocycloid with $c = 3/2 + 1 = 5/2$.

**Problem B6**

$|f(1)/1 + f(2)/2 + ... + f(n)/n - 2n/3| < 1$, where $f(n)$ is the largest odd divisor of n.

**Solution**

If n is odd, then $f(n)/n = 1$. If n is even, then $f(n) = f(n/2)$. If $n = 2m$, then $f(1)/1 + f(3)/3 + ... + f(2m-1)/(2m-1) = m$, and $f(2)/2 + f(4)/4 + ... + f(2m)/(2m) = 1/2\ ( f(1)/1 + f(2)/2 + ... + f(m)/m )$. So if we put $g(n) = f(1)/1 + f(2)/2 + ... + f(n)/n$, then we have the relations: $g(2n+1) = g(2n) + 1$ (*), and $g(2n) = n + g(n)/2$ (**). That allows us to establish by induction that $2n/3 < g(n) < 2n/3 + 5/6$ for n odd and $2n/3 < g(n) < 2n/3 + 1/2$ for n even.

For $g(1) = 1 = 2/3 + 1/3$, $g(2) = 3/2 = 2/3 \times 2 + 1/6$, so the result is true for $n = 1, 2$. Suppose it is true for $n < 2m$. Then $g(2m) = m + g(m)/2 > m + m/3 = 2/3\ (2m)$, and $g(2m) < m + m/3 + 1/2\ (5/6) < 2/3\ (2m) + 1/2$. Also $g(2m+1) = g(2m) + 1 > 2/3\ (2m) + 2/3 = 2/3\ (2m+1)$, and $g(2m+1) < 2/3\ (2m) + 1/2 + 1 = 2/3\ (2m+1) + 5/6$.

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**Problem A1**
Show that we cannot have 4 binomial coefficients nCm, nC(m+1), nC(m+2), nC(m+3) with $n, m > 0$ (and $m + 3 \le n$) in arithmetic progression.

**Solution**
Note that we can have three, for example: 7C1 = 7, 7C2 = 21, 7C3 = 35.
a, b, c are in arithmetic progression iff $2b = a + c$. We have nCm = (m+1)/(n - m) nC(m+1), so 2 nCm+1 = nCm + nC(m+2) iff $2 = (n - m - 1)/(m + 2) + (m + 1)/(n - m)$. Simplifying, this becomes: $m^2 - (n - 2)m + (n^2 - 5n + 2)/4 = 0$. For any given n, this is a quadratic in m, so it has at most two solutions. If we have 4 consecutive coefficients in arithmetic progression, then the two solutions must differ by 1. If they are $m_1$ and $m_2$, then $(m_1 - m_2) = 1$ implies $(m_1 - m_2)^2 = 1$ and hence $(m_1 + m_2)^2 - 4m_1m_2 = 1$. So $(n - 2)^2 - (n^2 - 5n + 2) = 1$, and hence $n = -1$. Hence for $n > 0$ we cannot have four coefficients in arithmetic progression.

Alternatively, because nCm = nC(n-m), m and m+1 must be centrally placed (in other words n = 2m+1). Otherwise we would have four distinct solutions to the quadratic, which is impossible. But (2m+1)C(m-1), (2m+1)Cm, (2m+1)C(m+1), (2m+1)C(m+2) cannot be a solution because both (2m+1)C(m-1) and (2m+1)C(m+2) are less than (2m+1)Cm.

**Problem A2**
Let S be a set with a binary operation * such that (1) a * (a * b) = b for all a, b $\in$ S, (2) (a * b) * b = a for all a, b $\in$ S. Show that * is commutative. Give an example for which S is not associative.

**Solution**
Consider (a*b)*( (a*b)*b). One can view it as c * (c * b), so that it is b by (1). Or one can consider first (a*b)*b, which is a by (2), so that the expression is (a*b)*a. Hence b = (a*b)*a. Multiplying on the right by a, we get b*a = ((a*b)*a)*a, which is (a*b) by (2). That proves * is commutative.

Take S to have three elements a, b, c. Let a*a = a, b*b = b, c*c = c, a*b = c, b*c = a, c*a = b and assume * is commutative. Then we easily check that the required conditions are met. But c = a*b = (a*a)*b, whereas a*(a*b) = a*c = b. So * is not always associative.

**Problem A3**
A sequence { $x_i$ } is said to have a Cesaro limit iff $\lim_{n\to\infty}(x_1 + x_2 + ... + x_n)/n$ exists. Find all (real-valued) functions f on the closed interval [0, 1] such that { $f(x_i)$ } has a Cesaro limit iff { $x_i$ } has a Cesaro limit.

**Solution**
Answer: the linear functions $f(x) = Ax + B$ where A is non-zero. [We need A non-zero, because otherwise *all* { $f(x_i)$ } would have a Cesaro limit.]

It is straightforward to show that these functions satisfy the condition.
Suppose $\{x_n\}$ has a Cesaro limit. Then we can find a limit k so that for any $\varepsilon > 0$ we have $|(x_1 + ... + x_n)/n - k| < \varepsilon/A$ for all sufficiently large n. Let $h = f(k)$, we wish to show that $|(f(x_1) + ... + f(x_n) )/n - h| < \varepsilon$ for all sufficiently large n. But $|(f(x_1) + ... + f(x_n) )/n - h| = | A(x_1 + ... + x_n)/n - Ak | < \varepsilon$.

Similarly, suppose { $f(x_n)$ } has a Cesaro limit. Then we can find a limit h so that for any $\varepsilon > 0$ we have $|(f(x_1) + ... + f(x_n) )/n - h| < A\varepsilon$ for all sufficiently large n. Take k so that $f(k) = h$. We wish to show that $|(x_1 + ... + x_n)/n - k| < \varepsilon$ for all sufficiently large n. But again this is obvious since we have $|(f(x_1) + ... + f(x_n) )/n - h| = | A(x_1 + ... + x_n)/n - Ak |$.

It is harder to show the converse - that any function satisfying the condition must be linear. Note first that it is not true for ordinary limits. If { $x_n$ } $\to$ k, then $f(x_n) \to f(k)$ for any continuous f (and it is not hard to show the converse). So the functions satisfying the corresponding condition for ordinary limits are just the homeomorphisms. The ordinary limit condition is stronger than Cesaro-summability - it is easy to show that if $x_n \to k$, then $(x_1 + x_2 + ... + x_n)/n \to k$ also. But it is easy to find sequences which do not tend to a limit but do have a Cesaro limit. For example, $x_n = (-1)^n$. So we need to use members of this wider class of sequences in order to prove the result.

Given any k in the open interval (0, 1), take a sequence $x_n$ of 0s and 1s whose average $s_n = (x_1 + ... + x_n)/n$ tends to k. Then the average $F_n = (f(x_1) + ... + f(x_n))/n$ tends to $(1 - k) f(0) + k f(1) = h$, since a fraction $(1 - k)$ of the n terms $f(x_i)$ are $f(0)$ and a fraction k are $f(1)$.

Now define a new sequence $y_n$ as follows. Take enough terms from the first sequence to bring $G_n = (g(y_1) + ... + g(y_n))/n$ within 1/2 of h. Then take enough terms k to bring $G_n$ within 1/4 of f(k). Then (starting where we left off before) take enough terms from the first sequence to bring $G_n$ within 1/8 of h, then enough ks to bring it within 1/16 of f(k) and so on. Thus we have a subsequence of the $G_n$s which tends to h and another which tends to f(k). But the average $(y_1 + ... + y_n)/n$ tends to k, so $G_n$ must tend to a single limit. Hence f(k) = h, or f(k) = a + bk, where a = f(0), b = f(1) - f(0).

**Problem A4**

Show that a circle inscribed in a square has a larger perimeter than any other ellipse inscribed in the square.

**Solution**

The first step is to show that if an ellipse touches all four sides of a square then its axes must lie along the diagonals of the square (in other words, it is symmetrically placed).

Let the ellipse have centre O, semi-major axis a and semi-minor axis b. Let the line A through O contain the ellipse's major axis and the line B through O contain its minor axis. Let S be the square in which the ellipse is inscribed. Let k be the similarity operation which leaves A invariant and expands by a factor a/b perpendicular to A. (So given a general point P, with X be the foot of the perpendicular from P to A, P goes to the point P' on the ray XP such that XP'/XP = a/b.)

k takes the ellipse into the circle centre O, radius a. It also takes parallel lines to parallel lines and touching curves to touching curves, so it takes the square S to a parallelogram whose sides all touch the circle. But that means the parallelogram must be a rhombus (with equal sides). But a line segment making an angle θ to the line A is taken by k to a line segment whose length is longer by a factor $\sqrt{1 + ((a/b)^2 - 1) \sin^2\theta}$, which is a strictly monotonically increasing function of θ. Since the equal sides of the square are taken to the equal sides of the rhombus, they must make equal angles to the line A. Hence they must be at 45° to it and hence the line A must lie along a diagonal of the square.

The second step is to show that the ellipse with the largest perimeter is the circle. Recall that the perimeter of an ellipse is not an elementary function (it is a complete elliptical integral of the second kind). So it is not clear how to attack the problem. An analytic approach could evidently get somewhat messy.
But to pursue that a little, the tangent at the point (X, Y) of the ellipse has equation $xX/a^2 + yY/b^2 = 1$. If this is at 45° (with X and Y both positive) then we have $Xb^2 = Ya^2$. The corresponding tangent at (X, -Y) is will evidently meet it on the x-axis at the point $(\sqrt{a^2 + b^2}, 0)$, so the side of the square has length $\sqrt{2a^2 + 2b^2}$.

The perimeter is $4a \int_0^{\pi/2} \sqrt{1 - c \cos^2\theta}\, d\theta$, where $c = (1 - (b/a)^2)$ (*). The easiest way to see this is to use the parametric equation of the ellipse x = a cos θ, y = b sin θ. Then if s represents arc length and ' differentiation wrt θ, we have $s' = \sqrt{(x')^2 + (y')^2}$ and (*) follows almost immediately.

Now $a/\sqrt{a^2 + b^2} = 1/\sqrt{2 - c}$, so we have to maximise $f(c) = 1/\sqrt{2 - c} \int_0^{\pi/2} \sqrt{1 - c \cos^2\theta}\, d\theta$ over the range [0, 1], with c = 0 corresponding to b = a or the circle.
Putting the factor $1/\sqrt{2 - c}$ inside the integral, the integrand becomes $I(c, \theta) = \sqrt{(1 - c \cos^2\theta) / (2 - c)}$. We note that at θ = π/4 this is always $\sqrt{1/2}$ irrespective of c. But for θ < π/4, the integrand is a decreasing function of c and for θ > π/4 it is an increasing function of c. So we cannot simply argue that the integrand is maximised by taking c = 0 and hence the integral also.

However, a slight elaboration of that argument does work. We can split the integral at π/4. Then for the range π/4 to π/2 we can make the substituion φ = π/2 - θ to get back to an integral over 0 to π/4 with integrand $J(c, \varphi) = \sqrt{(1 - c \sin^2\varphi) / (2 - c)}$. So now we have to maximise the integral over 0 to π/4 of I(c, θ) + J(c, θ). But now it is true that c = 0 maximises the integrand at every point of the range.

We just have to square twice. So $I(c, \theta) + J(c, \theta) \le \sqrt{2}$ is equivalent to $(1 - c \cos^2\theta) + (1 - c \sin^2\theta) + 2 \sqrt{(1 - c \cos^2\theta)(1 - c \sin^2\theta)} \le 2(2 - c)$, or $\sqrt{(1 - c \cos^2\theta)(1 - c \sin^2\theta)} \le 1 - c/2$. Squaring again, that is equivalent to 1 - c

$+ c \sin^2\theta \cos^2\theta \le 1 - c + c^2/4$ or $c^2 \sin 2\theta \le c^2$. But the last relation is certainly true with equality iff $c = 0$ or $\theta = \pi/4$. Hence for the integral we have equality iff $c = 0$.

**Problem A5**
Show that n does not divide $2^n - 1$ for $n > 1$.

**Solution**
Suppose that n does divide $2^n - 1$. Then n must be odd. Let p be the smallest prime dividing n. Then $2^{p-1} = 1$ (mod p). Let m be the smallest divisor of p - 1 such that $2^m = 1$ (mod p). Since m is smaller than p it must be coprime to n, so $n = qm + r$ with $0 < r < m$. Hence $2^r = 1$ (mod p). Contradiction.

**Problem A6**
f is an integrable real-valued function on the closed interval [0, 1] such that $\int_0^1 x^m f(x)\,dx = 0$ for $m = 0, 1, 2, \dots , n - 1$, and 1 for $m = n$. Show that $|f(x)| \ge 2^n(n + 1)$ on a set of positive measure.

**Solution**
The trick is to look at $(x - 1/2)^n f(x)$. Using the relations given in the question, we have immediately that $\int_0^1 (x - 1/2)^n f(x)\,dx = 1$. But we can also show that it must be small unless f(x) is large. For if $|f(x)| < 2^n(n + 1)$ except possibly on a set of zero measure, then $| \int | < 2^n(n + 1) \int_0^1 |x - 1/2|^n\,dx = 2^{n+1}(n + 1) \int_0^{1/2} x^n\,dx = 2^{n+1} x^{n+1}|_0^{1/2} = 1$. Contradiction.

**Problem B1**
Let $\sum_0^\infty x^n(x - 1)^{2n} / n! = \sum_0^\infty a_n x^n$. Show that no three consecutive $a_n$ are zero.

**Solution**
Let $p(x) = x(x - 1)^2$, and $f(x) = e^{p(x)}$. Then f(x) has the expansion given. Differentiating $f' = e^{p(x)}p'(x) = f\,p'$ (*). Now since p is a polynomial of degree 3, we have $p^{(n)} = 0$ for $n >= 4$. So differentiating (*) n times we get: $f^{(n+1)} = nC0\, f^{(n)} p' + nC1\, f^{(n-1)} p'' + nC2\, f^{(n-2)} p'''$. So if $f^{(n)}(0) = f^{(n-1)}(0) = f^{(n-2)}(0)$, then $f^{(m)}(0) = 0$ for all $m \ge n - 2$. In other words if three consecutive $a_n$ are zero then all subsequent $a_n$ are zero, and hence f(x) is a finite polynomial, which is impossible.

**Problem B2**
A particle moves in a straight line with monotonically decreasing acceleration. It starts from rest and has velocity v a distance d from the start. What is the maximum time it could have taken to travel the distance d?

**Solution**
Answer: 2d/v.
Plot velocity u(t) against time. We have $u(T) = v$. The area under the curve between $t = 0$ and $t = T$ is the distance d. But since the acceleration is monotonically decreasing, the curve is concave and hence the area under it is at least the area of the triangle formed by joining the origin to the point $t = T$, $u = v$ (other vertices $t = 0$, $u = 0$ and $t = T$, $u = 0$). Hence $d \ge 1/2\, vT$, so $T \le 2d/v$. This is achieved by a particle moving with constant acceleration.

**Problem B3**
A group has elements g, h satisfying: $ghg = hg^2h$, $g^3 = 1$, $h^n = 1$ for some odd n. Prove $h = 1$.

**Solution**
It is hard to get started on this. Or rather, it took me a long time to find the right way to get started. With hindsight, the correct approach is probably to work systematically through a set of increasingly complex expressions, trying to simplify them more than one way in order to get a new relation. It would not take long to reach $hg^2hg^2h$.
$gh^2 = (ghg)g^2h = (hg^2h)g^2h = (hg^2)(hg^2h) = hg^2(ghg) = h^2g$. From this point it is easy.

$gh^2g^2 = h^2$. Hence $gh^{2n}g^2 = h^{2n}$. Choose n so that $h^{2n} = h$. Then $ghg^2 = h$, or $gh = hg$. So the given relation implies $h = h^2$. Hence $h = 1$.

**Problem B4**
Show that for $n > 1$ we can find a polynomial p(a, b, c) with integer coefficients such that $p(x^n, x^{n+1}, x + x^{n+2}) \equiv x$.

**Solution**
By playing with small n, we soon find a general pattern:
$x = (x + x^{n+2})(1 - x^{n+1} + x^{2(n+1)} - ... + (-1)^{n-2}x^{(n-2)(n+1)} ) + (-1)^{n+1}(x^n)^n$.
The proof is immediate.

**Problem B5**
A, B, C and D are non-coplanar points. ∠ABC = ∠ADC and ∠BAD = ∠BCD. Show that AB = CD and BC = AD.

**Solution**
It is worth looking first at the coplanar case. If ABCD is convex, then opposite angles are equal, so it is a parallelogram and opposite sides are equal. In this case the result is true. But if ABDC is convex, then ∠ABC = ∠ADC implies that A, B, D, C lie on a circle and hence that ∠BAD = ∠BCD. In this case it is certainly not true that AB = CD or that BC = AD.

Applying the cosine formula to angles ABC and ADC gives: $(AB^2 + BC^2 - AC^2)/(2\ AB\ BC) = (AD^2 + CD^2 - AC^2)/(2\ AD\ CD)$. Hence $(AB\cdot CD - AD\cdot BC)(AB\cdot AD - BC\cdot CD) + AC^2(AB.BC - AD.CD) = 0$ (*). Similarly, applying it to the other two angles gives: $(BC^2 + CD^2 - BD^2)/(2\ BC\ CD) = (AB^2 + AD^2 - BD^2)/(2\ AB\cdot AD)$ and hence $(AD\cdot BC - AB\cdot CD)(AB\cdot BC - CD\cdot AD) + BD^2(BC\cdot CD - AB\cdot AD) = 0$ (**).

Put $x = (AB\cdot CD - AD\cdot BC)$, $y = (AB\cdot AD - BC\cdot CD)$, $z = (AB\cdot BC - AD\cdot CD)$. Then (*) and (**) become $xy + AC^2z = 0$, $xz + BD^2y = 0$. So if either of y or z is 0, then the other is also. But if y and z are 0, then AB/CD = BC/AD = AD/BC, so BC = AD and hence also AB = CD. On the other hand, if neither y nor z is 0, then we can deduce that $AC^2BD^2 = x^2$, and hence $AC\cdot BD = \pm x = \pm(AB\cdot CD - AD\cdot BC)$. Thus either $AC\cdot BD + AD\cdot BC = AB\cdot CD$ or $AC\cdot BD + AB\cdot CD = AD\cdot BC$.

In either case we can use Ptolemy's theorem which tells us that A, B, C, D must be (1) coplanar, and (2) concyclic (or collinear). But we are told that they are not coplanar. Hence BC = AD and AB = CD as required.

**Problem B6**
The polynomial p(x) has all coefficients 0 or 1, and p(0) = 1. Show that if the complex number z is a root, then $|z| \geq (\sqrt{5} - 1)/2$.

**Solution**
Let $h = (\sqrt{5} - 1)/2$. h is approx 0. 618, so certainly if $|z| \geq 1$, then $|z| > h$. So it is sufficient to show that if z is a root with $|z| < 1$, then $|z| \geq h$.

For such values of z, $1 + z + z^2 + ...$ converges, so we have $2 + z + z^2 + z^3 + ... - 2\ p(z) = \pm z \pm z^2 \pm z^3 \pm ...$ . Now $|rhs| \leq |z| + |z^2| + ... = |z|/(1 - |z| )$. If z is a root, then p(z) = 0, so $lhs = 2 + z + z^2 + ... = 2 + z/(1 - z)$ . If we could show that $|lhs| \geq (2 + |z|)/(1 + |z|)$, then we would have $|z|/(1 - |z|) \geq (2 + |z|)/(1 + |z|)$ and hence $|z|^2 + |z| - 1 \geq 0$. But $|z| \geq 0$, so $|z| \geq h$.

We require $|2 - z|\ (1 + |z|) \geq |1 - z|\ (2 + |z|)$. Squaring and using the polar form $z = r\ e^{i\theta}$, this is equivalent to $(4 - 4r \cos\theta + r^2)(1 + r)^2 \geq (1 - 2r\cos\theta + r^2)\ (2 + r)^2$, or after some simplification, $2r\ (1 - r^2)(1 + \cos\theta) \geq 0$, which is true.

## 34th Putnam 1973

**Problem A1**
ABC is a triangle. P, Q, R are points on the sides BC, CA, AB. Show that one of the triangles AQR, BRP, CPQ has area no greater than PQR. If BP ≤ PC, CQ ≤ QA, AR ≤ RB, show that the area of PQR is at least 1/4 of the area of ABC.

**Solution**
It is convenient to do the second part first. Let L be the midpoint of BC, M of AC and N of AB. Then LN is parallel to AC, so the ray PR meets the ray CA. Hence M is at least as close to PR as Q. So area PQR ≥ area PMR. Similarly, the ray MP meets the ray AB, so N is at least as close to PM as R. Hence area PMR ≥ area PMN. Finally MN is parallel to BC, so area PMN = area LMN. But area LMN = 1/4 area ABC, so area PQR ≥ 1/4 area ABC.

For the first part, there are two cases. Either each of AMN, BLN, CLM includes just one of P, Q, R, or one includes two of them. In the first case, we have the situation just considered (possibly with some relabeling of P, Q, R) , so area PQR ≥ 1/4 area ABC. If all three of AQR, BRP, CPQ had areas > area PQR, then in total their area plus that of PQR would exceed the area of ABC. Contradiction. Hence one of AQR, BRP, CPQ has area not greater than PQR.

In the second case, suppose AR <= AN and AQ <= AM. Let AP cut RM at X. Then AX ≤ AP/2, since RM cuts AP closer to A than MN, which bisects it (or possibly at the same point if R=N, Q=M). Hence AX ≤ PX. But area AQR/ area PQR = AX/PX, so area AQR ≤ area PQR.

**Problem A2**
$a_n = \pm 1/n$ and $a_{n+8} > 0$ iff $a_n > 0$. Show that if four of $a_1, a_2, \dots , a_8$ are positive, then $\sum a_n$ converges. Is the converse true?

**Solution**
Answer: yes.

Consider $a_n + a_{n+1} + \dots + a_{n+7}$. It is $( \pm(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7) \pm n(n+2)(n+3)(n+4)(n+5)(n+6)(n+7) \pm \dots \pm n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6) ) / ( n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7) )$. If there are an equal number of plus and minus signs, then there is no $n^7$ term in the numerator, so the expression is $O(1/n^2)$. On the other hand if there are an unequal number of plus and minus signs, then there is an $n^7$ term in the numerator and the expression is $O(1/n)$. But $\sum 1/n$ diverges, and $\sum 1/n^2$ converges.

Let $s_n$ be the sum of the first n terms. We have shown that with an unequal number of positive and negative signs $s_{8n}$ diverges and hence $s_n$ does not converge. With an equal number of signs, $s_{8n}$ converges. But each term tends to zero, so $|s_{8n+i} - s_{8n}|$ tends to zero for i = 1, ... 7. Hence $s_n$ converges.

**Problem A3**
n is a positive integer. Prove that $[\sqrt{4n + 1}\,] = [ \min (k + n/k) ]$, where the minimum is taken over all positive integers k.

**Solution**
We evaluate each side. It is probably easiest to tackle $[\sqrt{4n + 1}\,]$ first. Let $f(n) = [\sqrt{4n + 1}\,]$. If $n \le m^2 - 1$, then $4n + 1 \le 4m^2 - 3 < (2m)^2$, so $f(n) < 2m$. If $n \ge m^2$, then $4n + 1 > 4m^2$, so $f(n) \ge 2m$. If $n < m(m+1)$, then $4n + 1 < 4m^2 + 4m + 1 = (2m+1)^2$, so $f(n) < 2m+1$. If $n \ge m(m+1)$, then $4n+1 \ge (2m+1)^2$, so $f(n) \ge 2m+1$. Hence for $n = m^2, m^2 + 1, m^2 + 2, \dots , m^2 + m - 1$, we have $f(n) = 2m$. For $n = m^2 + m, m^2 + m + 1, m^2 + m + 2, \dots , m^2 + 2m$ we have $f(n) = 2m+1$.

Let g(n) be the minimum value of $[k + n/k]$, where k is a positive integer. Consider $h(x) = x + n/x$, with x real and positive. $h'(x) <0, =0, >0$ according as $x <\sqrt{n}, =\sqrt{n}, >\sqrt{n}$. So for positve real x, the smallest value of $x + n/x$ occurs at $x = \sqrt{n}$ and the smallest value with x restricted to be an integer must occur at $[\sqrt{n}]$ or $[\sqrt{n}] + 1$.

For $n = m^2, m^2 + 1, m^2 + 2, \dots , m^2 + m - 1$, we have $[\sqrt{n}] = m$. Also $0 < (m-1)/m < 1$, so $[m + n/m] = 2m$. Now $n > m^2 - 1$, so $n/(m+1) > m - 1$ and hence $[(m+1) + n/(m+1)] \ge [m+1 + m-1] = 2m$. So for these values of n we have $g(n) = 2m = f(n)$.

Now consider $n = m^2 + m, m^2 + m + 1, m^2 + m + 2, \dots , m^2 + 2m$. For these values of n also we have $[\sqrt{n}] = m$. For all except the last value of n, we have $[m + n/m] = 2m+1$. For the last we have $[m + n/m] = 2m+2$. But for all of them we have $[(m+1) + n/(m+1)] = 2m+1$. Hence for these values of n also $g(n) = 2m+1 = f(n)$.

**Problem A4**
How many real roots does $2^x = 1 + x^2$ have?

**Solution**
Answer: 3. $x = 0$, $x = 1$ and a value just over 4.

Clearly there are no roots for negative x, since for such x, $2^x < 1$, whereas $1 + x^2 > 1$. There are certainly roots at $x = 0$ and 1. Also $2^4 < 4^2 + 1$, whereas $2^5 > 5^2 + 1$, so there is a root between 4 and 5. We have to show that there are no other roots. Put $f(x) = 2^x - x^2 - 1$. Then $f''(x) = (\ln 2)^2 2^x - 2$. This is strictly increasing with a single zero. $f'(0) > 0$, so $f'(x)$ starts positive, decreases through zero to a minimum, then increases through zero. So it has just two zeros. Hence $f(x)$ has at most three zeros, which we have already found.

**Problem A5**
An object's equations of motion are: $x' = yz$, $y' = zx$, $z' = xy$. Its coordinates at time $t = 0$ are $(x_0, y_0, z_0)$. If two of these coordinates are zero, show that the object is stationary for all t. If $(x_0, y_0, z_0) = (1, 1, 0)$, show that at time t, $(x, y, z) = (\sec t, \sec t, \tan t)$. If $(x_0, y_0, z_0) = (1, 1, -1)$, show that at time t, $(x, y, z) = (1/(1 + t), 1/(1 + t), -1/(1 + t) )$. If two of the coordinates $x_0, y_0, z_0$ are non-zero, show that the object's distance from the origin $d \to \infty$ at some finite time in the past or future.

**Solution**
You are meant to assume the usual uniqueness theorems. So the only non-trivial part of the question is the last. For the first three parts, one just has to verify that the three solutions given satisfy the equations.

We have $xx' = yy' = zz' = xyz$. So integrating $y^2 = x^2 + A$, $z^2 = x^2 + B$. We may assume that $y_0$ and $z_0$ are non-zero and that $|x_0| \le |y_0|$ and $|z_0|$. So A and B are positive and $x' = ( (x^2 + A) (x^2 + B) )^{1/2}$. Suppose we take the positive square root so that x increases with time. Then $t = \int ( (x^2 + A) (x^2 + B) )^{1/2} dx$. But the integral converges as x tends to infinity, so x reaches infinity at a finite (future) time. If we had taken the negative root, then we conclude that x reaches infinity at a finite (past) time.

**Problem A6**
Show that there are no seven lines in the plane such that there are at least six points which lie on the intersection of just three lines and at least four points which lie on the intersection of just two lines.

**Solution**
From 7 lines we can choose just 21 pairs. An intersection of 3 lines accounts for 3 distinct pairs of lines, an intersection of 2 lines for 1 pair. Hence the configuration given would have at least $6 \times 3 + 4 \times 2 = 22$ distinct pairs of lines.

**Problem B1**
S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal.

**Solution**
Note that we need the collections to be both the same size and the same sum. Otherwise we could take $S = \{1, 1, 1, 1, 3\}$. If we remove a 1 we have: $3 = 1 + 1 + 1$, and if we remove the 3 we have: $1 + 1 = 1 + 1$.

The key is to notice that if we transform each element x of S to $ax + b$, then the new elements still satisfy the property given. In particular, by taking b sufficiently large we may make each element of S positive. Let $f(S)$ denote the sum of the elements of S. Let us say that a finite set has the property X if (1) all its elements are positive and integral, (2) it has the property given in the question, and (3) not all its elements are equal. So we have just established that if there is a set with the property given and not all elements equal, then there is a set with property X. We show that given such a set S we can find another such set T with $f(T) < f(S)$.

For each element x of S, f(S) - x is even, so all elements of S have the same parity. If they are all even, we may take T to be {x/2 for x in S}. If they are all odd, we may take T to be { (x + 1)/2 for x in S}. Either way, the transformed elements are all integral and positive and have the property given. Also unequal elements transform into unequal elements. So T has the property X. We always have $x/2 < x$, and we have $(x+1)/2 < x$ unless $x = 1$, in which case $(x+1)/2 = x$. The elements cannot all be 1 (because we are assuming they are not all equal). So at least one must be reduced by the transformation, and hence the $f(T) < f(S)$.

But this process could now be repeated indefinitely. That gives a contradiction, since positive integers cannot be indefinitely reduced and remain positive. Hence our assumption that all members were unequal must be wrong.

**Problem B2**

The real and imaginary parts of z are rational, and z has unit modulus. Show that $|z^{2n} - 1|$ is rational for any integer n.

**Solution**

We may put $z = \cos\theta + i \sin\theta$. Then $z^{2n} - 1 = (\cos 2n\theta - 1) + i \sin 2n\theta$, so $|z^{2n} - 1|^2 = 2 - 2\cos 2n\theta = 4\sin^2 n\theta$. Hence $|z^{2n} - 1| = 2|\sin n\theta|$. But $\sin n\theta$ is the imaginary part of $(\cos\theta + i\sin\theta)^n$. If we expand by the binomial theorem we get a series of terms $\cos^r\theta \sin^s\theta$ with integer coefficients. Each term is rational since $\cos\theta$ and $\sin\theta$ are rational. Hence $\sin n\theta$ is rational.

**Problem B3**

The prime p has the property that $n^2 - n + p$ is prime for all positive integers less than p. Show that there is exactly one integer triple (a, b, c) such that $b^2 - 4ac = 1 - 4p$, $0 < a \le c$, $-a \le b < a$.

**Solution**

Answer: (1, -1, p) is the only solution.

$b^2 = 4ac + 1 - 4p$, so $b^2$ is odd and hence b is odd (but not necessarily positive). Put $b = 2n - 1$. Then $4n^2 - 4n + 1 = 4ac + 1 - 4p$, so $n^2 - n + p = ac$. That is promising, because if n was in the required range we could conclude that a or c = 1 and, since $0 < a \le c$, that a = 1.
We are given that $n^2 - n + p$ is prime for n = 1, 2, ... , p-1. But it is obviously also prime for n = 0 and for n = -1, -2, ... , -(p-2) since $(-m)^2 - (-m) = m^2 + m = (m+1)^2 - (m+1)$ (so truth for m+1 implies truth for -m). So the required range is $-(p-2) \le n \le (p-1)$ or $(-2p+3) \le b \le (2p - 3)$.

$ac \ge a^2 \ge b^2$, so $b^2 - 4ac \le -3b^2$. If $|b| \ge p$, then $-3b^2 \le -3p^2 < 1 - 4p$ (since $(3p-1)(p-1) > 0$). But we know that $b^2 - 4ac = 1 - 4p$, so certainly $|b| < p$. But we know b is odd, so $|b| <= p - 2$. That is sufficient to guarantee that b is in the required range since $(p - 2) < (2p - 3)$.
So we can conclude that a = 1. But now b is odd and satisfies $-1 \le b < 1$, so b must be -1. Finally $b^2 - 4ac = 1 - 4p$ gives c = p.

**Problem B4**

f is defined on the closed interval [0, 1], f(0) = 0, and f has a continuous derivative with values in (0, 1]. By considering the inverse $f^{-1}$ or otherwise, show that $\left( \int_0^1 f(x)\,dx \right)^2 \ge \int_0^1 f(x)^3\,dx$. Give an example where we have equality.

**Solution**

We might suspect that there is nothing special about the upper limit 1. So put $g(y) = \left( \int_0^y f(x)\,dx \right)^2 - \int_0^y f(x)^3\,dx$. Then certainly g(0) = 0. So the obvious approach is to show that $g'(y) \ge 0$. Differentiating, we get $2 f(y) \int_0^y f(x)\,dx - f(y)^3$. We know that $f(y) \ge 0$ (because it is 0 at y = 0 and has non-negative derivative). So we need to show that $2 \int_0^y f(x)\,dx - f(y)^2 \ge 0$. Certainly it is 0 at y = 0, so we try differentiating again, hoping that the derivative will be non-negative. Differentiating gives $2 f(y) - 2 f(y) f'(y)$. But that is non-negative, because $f(y) \ge 0$ and $f'(y) \le 1$.

It is easy to see that f(x) = x gives equality. [Careful, f(x) = 0 is not an acceptable solution because the derivative is not strictly positive.]

[The suggestion about $f^{-1}$ was probably intended as follows. Let if y = f(x), let x = g(y). Then $\int f(x)\,dx = \int y\,dx/dy\,dy = \int y\,g'(y)\,dy$. Now think of the square as a double integral and use $g'(y) \ge 1$. But the straightforward solution above seems easier.]

**Problem B5**

If x is a solution of the quadratic $ax^2 + bx + c = 0$, show that, for any n, we can find polynomials p and q with rational coefficients such that $x = p(x^n, a, b, c) / q(x^n, a, b, c)$. Hence or otherwise find polynomials r, s with rational coefficients so that $x = r(x^3, x + 1/x) / s(x^3, x + 1/x)$.

**Solution**

The first part of this is somewhat confusing. Clearly we can find particular values of a, b, c for which we do not have $x = p(x^n, a, b, c) / q(x^n, a, b, c)$. For example, take $a = 1$, $b = 0$, $c = -2$. Then x is irrational, whereas any p and q would have rational values.

One can carry out a careful analysis of what values of a, b, c are allowed, but that does not seem to be what the question is after. It seems to want something much simpler.

An easy induction shows that $x^n = (x\, p_1(a, b, c) + p_2(a, b, c))/ p_3(a, b, c)$, where $p_1$, $p_2$ and $p_3$ are polynomials with integer coefficients. Just start with $x^2 = (- bx - c)/a$ and at each stage multiply through by x and substitute. Now $p_1$ cannot be identically zero because then both solutions to the original quadratic would have the same $|x|$, whereas it is easy to see that this is not true for most values of a, b, c. So we may divide by $p_1$ to get the required relation with $p(x^n, a, b, c) = x^n p_3(a, b, c) - p_2(a, b, c)$ and $q(x^n, a, ,b c) = p_1(a, b, c)$.

For the second part, it is simpler to start from scratch. $(x + 1/x)^2 = x^2 + 1/x^2 + 2$. So $x (x + 1/x)^2 = x^3 + 1/x + 2x$. Hence $x (x + 1/x)^2 - x = x^3 + (x + 1/x)$. That is essentially the relation we want: $x = (x^3 + y) / (y^2 - 1)$, where $y = x + 1/x$.

**Problem B6**

Show that $\sin^2x \sin 2x$ has two maxima in the interval $[0, 2\pi]$, at $\pi/3$ and $4\pi/3$. Let $f(x)$ = the absolute value of $\sin^2x \sin^34x \sin^38x \dots \sin^32^{n-1}x \sin 2^nx$. Show that $f(\pi/3) >= f(x)$. Let $g(x) = \sin^2x \sin^24x \sin^28x \dots \sin^22^nx$. Show that $g(x) \le 3^n/4^n$.

**Solution**

Let $h(x) = \sin^2x \sin 2x$. Then $h(x) = 2 \sin^3x \cos x$, and $h'(x) = 2 \sin^2x (3 - 4 \sin^2x)$. So $h'(x) = 0$ at $x = 0, \pi/3, 2\pi/3, 4\pi/3, 5\pi/3, 2\pi$ in the interval $[0, 2\pi]$. We easily see that $h(x)$ has maxima at $\pi/3$ and $4\pi/3$, minima at $2\pi/3$ and $5\pi/3$ and inflexions at $0, 2\pi$. At the inflexions $h(x) = 0$, at the other stationary values, $|h(x)| = (3\sqrt{3})/8$.

Notice that, for $n > 1$, $2^n\pi/3 = 2\pi/3$ or $4\pi/3 \pmod{2\pi}$. So at $x = 2^n\pi/3$, $|h(x)|$ attains its maximum value.

Let $f_n(x) = \sin^2x \sin^34x \sin^38x \dots \sin^32^{n-1}x \sin 2^nx$. Then $f_2(x) = h(x) h(2x)$ and in general $f_n(x) = f_{n-1}(x) h(2^{n-1}x)$, so that $f_n(x) = h(x) h(2x) h(4x) \dots h(2^{n-1}x)$. At $x = \pi/3$ each of $|h(2^ix)|$ is maximised separately and hence $|f_n(x)|$ is maximised.

We have in fact that $|f_n(x)| \le ( (\sqrt{3})/2 )^{3n}$. Hence also $|\sin x| |f_n(x)| |\sin^22^nx| \le ( (\sqrt{3})/2 )^{3n}$. But $|\sin x| |f_n(x)| |\sin^22^nx| = g(x)^{3/2}$, so $|g(x)| \le ( (\sqrt{3})/2 )^{2n} = (3/4)^n$, and hence $g(x) \le (3/4)^n$ (g is certainly non-negative).

## 35th Putnam 1974

**Problem A1**
S is a subset of {1, 2, 3, ... , 16} which does not contain three integers which are relatively prime in pairs. How many elements can S have?

**Solution**
Answer: 11

{2, 4, 6, 8, 10, 12, 14, 16, 3, 9, 15} has 11 elements. Any three integers from it include two which are multiples of 2 or two which are multiples of 3, so it does not contain three integers which are relatively prime in pairs.
On the other hand, any subset of 12 elements must include at least 3 members of {1, 2, 3, 5, 7, 11, 13} and those 3 will be relatively prime in pairs.

**Problem A2**
C is a vertical circle fixed to a horizontal line. P is a fixed point outside the circle and above the horizontal line. For a point Q on the circle, $f(Q) \in (0, \infty]$ is the time taken for a particle to slide down the straight line from P to Q (under the influence of gravity). What point Q minimizes f(Q)?

**Solution**
Answer: Let X be the lowest point of C. Q is the (other) point at which PX intersects C.

Let k be the angle which PQ makes to the horizontal. Then the acceleration of the particle along PQ is g sin k. Hence the time squared is proportional to PQ/sin k. So we minimise the time by minimising PQ/sin k. Let PQ meet the circle again at R and let PT be tangent to the circle. Then $PQ \cdot PR = PT^2$, a constant. So we minimise the time by maximising PR sin k. But PR sin k is just the vertical distance of R below P. That is clearly maximised by taking R = X.

**Problem A3**
Which odd primes p can be written in the form $m^2 + 16n^2$? In the form $4m^2 + 4mn + 5n^2$ ,where m and n may be negative? [You may assume that p can be written in the form $m^2 + n^2$ iff p = 1 (mod 4).]

**Solution**
Answer: p = 1 (mod 8); p = 5 (mod 8).

(A) $m^2 + 16n^2$.
We show first that $m^2 + 16n^2 = 1$ (mod 8). Clearly m is odd, otherwise $m^2 + 16n^2$ would be even. But odd squares are = 1 (mod 8) (because $(2N+1)^2 = 4N(N+1) + 1$ and N(N+1) is even). Hence $m^2 + 16n^2 = 1$ (mod 8).

Conversely, suppose p = 1 (mod 8). Take, $p = M^2 + N^2$. Without loss of generality, M is odd and N even. Hence $N^2 = 0$ (mod 8), so N is a multiple of 4.

(B) $4m^2 + 4mn + 5n^2$.
n must be odd (otherwise p would be even). But $4m^2 + 4mn + 5n^2 = (2m + n)^2 + (2n)^2$. The first term is congruent to 1 and the second to 4 mod 8. Hence p = 5 (mod 8).

Conversely, if $p = M^2 + N^2 = 5$ (mod 8), then wlog M is odd and N even. Take N = 2n. Since $M^2 = 1$ (mod 8), $N^2 = 4$ (mod 8) and hence n is odd. So M + n is even and we may set m = (M - n)/2. Now M = 2m + n, so $M^2 = 4m^2 + 4mn + n^2$ and $p = 4m^2 + 4mn + 5n^2$, as required.

**Problem A4**
Find $1/2^{n-1} \sum_1^{[n/2]} (n - 2i)$ nCi, where nCi is the binomial coefficient.

**Solution**
Answer: n( (n-1)C[n/2] - 1)/$2^{n-1}$

It suffices to show that $\sum_0^{[n/2]} (n - 2i)$ nCi = n ( (n-1)C[n/2] ).

We need two obvious facts: i nCi = n (n-1)C(i-1) (use factorials) and $\sum_0^{[n/2]}$ nCi = $2^{n-1}$ if n is odd, or $2^{n-1}$ + 1/2 nC[n/2] if n is even (use nCr = nC(n-r) ).

So consider first n odd. $\sum_0^{[n/2]}$ (n - 2i) nCi = n $2^{n-1}$ - 2 $\sum_0^{[n/2]}$ i nCi = n $2^{n-1}$ - 2n $\sum_1^{[n/2]}$ (n-1)C(i-1). Taking out the factor n we have $2^{n-1}$ - 2 $\sum_0^{[n/2]-1}$ (n-1)Ci = $2^{n-1}$ - ($2^{n-1}$ - (n-1)C[n/2] ) = (n-1)C[n/2], as required.

Similarly for n even, we have $\sum_0^{[n/2]}$ (n - 2i) nCi = n ($2^{n-1}$ + 1/2 nC[n/2] ) - 2n $\sum_0^{[n/2]-1}$ (n-1)Ci = n/2 ( nC[n/2] ) = n! / ( [n/2}! ([n/2]-1)! ) = n (n-1)C[n/2], as required.

**Problem A5**
The parabola y = $x^2$ rolls around the fixed parabola y = $-x^2$. Find the locus of its focus (initially at x = 0, y = 1/4).

**Solution**
Answer: horizontal line y = 1/4.

The tangent to the fixed parabola at (t, $-t^2$) has gradient -2t, so its equation is (y + $t^2$) = -2t(x - t). We wish to find the reflection of the focus of the fixed parabola in this line, because that is the locus of the rolling parabola when they touch at (t, $-t^2$). The line through the focus of the fixed parabola (0, -1/4) and its mirror image is perpendicular to the tangent and so has gradient 1/(2t). Hence its equation is (y + 1/4) = x/(2t). Hence it intersects the tangent at the point with y + $t^2$ = $-4t^2$(y + 1/4) + $2t^2$ or y(1 + $4t^2$) = 0, hence at the point (t/2, 0). So the mirror image is the point (t, 1/4). t can vary from $-\infty$ to $+\infty$, so the locus is the horizontal line through (0, 1/4).

**Problem A6**
Let f(n) be the degree of the lowest order polynomial p(x) with integer coefficients and leading coefficient 1, such that n divides p(m) for all integral m. Describe f(n). Evaluate f(1000000).

**Solution**
Answer: f(n) is smallest integer N such that n divides N! and f(1000000) = 25.

Put $p_N$(x) = x(x + 1)(x + 2) ... (x + N - 1). Note first that $p_N$(m) / N! = (m + N - 1)! / ( (m - 1)! N! ) which is a binomial coefficient and hence integral, so $p_N$(m) is certainly divisible by N! for all positive integers m. If m = 0, -1, ... , -(N - 1), then $p_N$(m) is zero, which can also be regarded as a multiple of N!. If m < -(N - 1), then we may put m = -m' - (N - 1) where m' is positive. Then $p_N$(m) = $(-1)^N$ $p_N$(m') which is divisible by N!. So certainly if we take N to be the smallest integer such that n divides N!, then we can find a polynomial with leading coefficient 1 (namely $p_N$(x) ) such that n divides p(m) for all integral m.

Now suppose p(x) is any polynomial with this property. We may write p(x) = $p_M$(x) + $a_1p_{M-1}$(x) + ... + $a_{M-1}p_1$(x) + $a_M$, where M is the degree of M and $a_i$ are integers (just choose successively, $a_1$ to match the coefficient of $x^{M-1}$, then $a_2$ to match the coefficient of $x^{M-2}$ and so on).

Taking x = n, we conclude that n must divide $a_M$, so there is another polynomial of the same degree, namely p(x) - $a_M$ with the same property. In other words, we may take $a_M$ = 0. Now take x = n - 1. Then $p_2$(x), $p_3$(x), ... are all divisible by n. So $a_{M-1}p_1$(n - 1) must also be divisible by n. But n and n - 1 are relatively prime, so $a_{M-1}$ must be divisible by n. Hence the polynomial $p_M$(x) + $a_1p_{M-1}$(x) + ... + $a_{M-2}p_2$(x) has the same property.

This argument continues on until we conclude that $p_M$(x) has the same property, but a little care is needed, because in general n and $p_i$(n - i ) are not relatively prime. However, their greatest common divisor is also the gcd of n and i! (just multiply out $p_i$(n - i) = (n - 1)(n - 2)...(n - i ) - all terms except the i! term have a factor n). But i! divides $p_i$(m) for all m (proved above), so the fact than n divides $a_{M-i}p_i$(n - i ) allows us to conclude that n divides $a_{M-i}p_i$(m) for all m and hence that we can drop the term.

So we have finally that $p_M$(x) also has the property. In particular, n must divide $p_M$(1) = M! so M must be at least as big as N, the smallest number with this property. That establishes that f(n) is just the smallest N such that n divides N!.

Finally we want f(1000000) = f($2^65^6$). Evidently 25 is the smallest N such that $5^6$ divides N! (25! includes 5 x 10 x 15 x 20 x 25 which is just the 6 powers of 5 needed). Certainly at least $2^{12}$ divides 25! (which includes 12 even numbers in the product: 2, 4, 6, ... , 24), so f(1000000) = 25.

**Problem B1**
P, Q, R, S, T are points on a circle radius 1. How should they be placed to maximise the sum of the perimeter and the five diagonals?

**Solution**
Answer: equally spaced (at the vertices of a regular pentagon).

We consider separately the questions of maximising the perimeter and the sum of the diagonals. We show first that if X lies on the arc AB, then we maximise AX + BX by taking X as the midpoint of the arc.

Let O be the centre of the circle and let the angle subtended at O by the arc by 2k. If the radius is R and angle AOX is x, then $AX + BX = 2R \sin(x/2) + 2R \sin(k - x/2) = 2R ( (1 - \cos k) \sin(x/2) + \sin k \cos(x/2) ) = 2R( 2 \sin^2(k/2) \sin(x/2) + 2 \sin(k/2) \cos(k/2) \cos(x/2) ) = 4R \sin(k/2) \cos(k/2 - x/2)$, which is uniquely maximised by taking $x = k$.

It follows immediately that equal spacing maximises the perimeter, for if the points are not equally spaced, then some side (say QR) is not equal to its counter-clockwise neighbouring side (say PQ). But then by moving Q to the midpoint of the arc PR we increase the perimeter, so the arrangement was not maximal.

But a similar (albeit slightly more complicated) argument shows that equal spacing also maximises the sum of the diagonals. Notice first that the relation $AX + BX = 4R \sin(k/2) \cos(k/2 - x/2)$ shows that the sum AX + BX is strictly decreasing as X moves away from the midpoint until it hits one of the points A, B. So if X is not at the midpoint then *any non-zero* move towards the midpoint increases the sum AX + BX.

Label the points so that the diagonals are PQ, QR, RS, ST, TP (and the order of the points around the circle is P, R, T, Q, S, moving counter-clockwise, say). Then if PQ = QR and QR = RS and RS = ST and ST = TP and TP = PQ, it follows that all the diagonals are equal and the points equally spaced. So if the points are not equally spaced then, without loss of generality, PQ does not equal QR. Now if no two points are coincident, it follows immediately that the arrangement is not maximal, because we can move Q towards the midpoint (of the arc PR on which it lies) and increase the sum.

However, if we allow coincident points, this argument fails, because if Q coincides with T (say) then it may be blocked by T from moving closer to the midpoint - if we move it beyond T, then QR ceases to be a diagonal and becomes a side.

Let us examine this case more closely. Since Q is blocked by T, we have that arc RT > arc PQ. Let QQ' be a diameter. Then P cannot coincide with Q' (since arc RT would then not exceed arc PQ). So if we move P closer to Q' we will increase QP + PT (= 2QP). If we move counter-clockwise around the circle from T we must reach P after Q' (otherwise arc PQ > arc Q'Q > arc RT, contradiction). So we can move P closer to Q' (and hence establish that the configuration is not maximal) unless it is blocked by R.

Now consider RS + ST. S is certainly not blocked from moving closer to the midpoint, because the two possible blocking points P and Q coincide with R and T. So if the configuration is to be maximal, then S must already be at the midpoint. Finally, consider QR + RS. The arc RT is greater than a semicircle so the midpoint must lie on it. P does not block because it lies on the other side. T does not block because it coincides with Q. So by moving R we can increase the sum, contradicting maximality.
Thus equal spacing maximises the perimeter and diagonal sum separately. So, a fortiori, it maximises their sum.

**Problem B2**
f(x) is a real valued function on the reals, and has a continuous derivative. $f'(x)^2 + f(x)^3 \to 0$ as $x \to \infty$. Show that f(x) and $f'(x) \to 0$ as $x \to \infty$.

**Solution**
The key to getting started is to notice that if $f' = 0$ for arbitrarily large values of x then the result is certainly true. Suppose $f'(x_n) = 0$ and $x_n \to \infty$. Then since $f'^2 + f^3 \to 0$, we have $f(x_n) \to 0$. But f is monotonic on the interval $[x_n, x_{n+1}\}$ since its derivative does not change sign, hence $f \to 0$. Hence also $f' \to 0$. So we may assume that for sufficiently large x, f ' does not change sign.

Now suppose f tends to a limit as $x \to \infty$. Then f ' must also tend to a limit. If that limit is non-zero, then f increases or decreases faster than some non-constant linear function for sufficiently large x and so cannot tend to a limit. Hence f ' must tend to zero. Hence f also.

So we may assume that either (1) for sufficiently large x, f is strictly monotonic increasing and tends to infinity, or (2) for sufficiently large x, f is strictly monotonic decreasing and tends to minus infinity.

The first case is impossible, because then $f^3$ and hence also $f^3 + f'^2$ would tend to infinity.

Showing that the second case is impossible needs a little more work. Suppose that for $x \ge X$, we have $f(x) < -1$ and $1/4 > f(x)^3 + f'(x)^2 > -1/4$. Then $1/2\ f(x)^3 < -1/2$, so $-1/2\ f(x)^3 > 1/2$. Hence $f'(x)^2 + 1/2\ f(x)^3 > -1/4 - 1/2\ f(x)^3 > 1/4 > 0$. So $f'(x)^2 > -1/2\ f(x)^3$. $f'(x)$ is negative, so $f'(x) < -1/2\ |f(x)|^{3/2}$ (*).

Now define g(x) to satisfy $g(X) = -1$, $g' = -1/2\ |g|^{3/2}$. Solving, we get $g(x) = (1 - (x - X)/4\ )^{-2}$ for x >= X. (*) shows that we must have $f(x) < g(x)$ for $x \ge X$. But $g(x) \to -\infty$ as $x \to 5X$, so f must be discontinuous on the interval (X, 6X). Contradiction.

**Problem B3**
Prove that $(\cos^{-1}(1/3)\ )/\pi$ is irrational.

**Solution**
Let $x = \cos^{-1}(1/3)$. If $x = m/n\ \pi$ for some integers m, n, then $\cos nx = \cos m\pi = \pm 1$. But we show that cos nx cannot be $\pm 1$. It follows that $x/\pi$ must be irrational as required.

As usual, we have $\cos nx = nC0\ c^n - nC2\ c^{n-2}s^2 + nC4\ c^{n-4}s^4 - \dots$ , where $c = \cos x$, $s = \sin x$. We may put $s^2 = 1 - c^2$ to get cos nx = a polynomial of degree n in c with integer coefficients. The coefficient of $c^n = nC0 + nC2 + nC4 + \dots = 2^{n-1}$. But $c = 1/3$, so $\cos nx = 2^{n-1}/3^n + k/3^{n-1} = (2^{n-1} + 3k)/3^n$ for some integer k. This must be in its lowest terms since $2^{n-1}$ is not divisible by 3. In particular, it cannot be $\pm 1$.

[A variant on this is to consider $\cos(2^n x)$. By a simple induction using $\cos 2y = 2\cos^2 y - 1$, we show that $\cos(2^n x) = a_n/b_n$, where $a_n$ is not a multiple of 3 and $b_n$is 3 to the power of $2^n$. It follows that as n runs through the natural numbers, all the values $\cos(2^n x)$ are distinct. But if $x/\pi$ was rational, there would only be finitely many distinct values.]

**Problem B4**
R is the reals. $f : R^2 \to R$ is such that $f_{x0} : R \to R$ defined by $f_{x0}(x) = f(x_0, x)$ is continuous for every $x_0$ and $g_{y0} : R \to R$ defined by $g_{y0}(x) = f(x, y_0)$ is continuous for every $y_0$. Show that there is a sequence of continuous functions $h_n : R^2 \to R$ which tend to f pointwise.

**Solution**
Let $M_n$ be the set of vertical lines at $1/2_n$ spacing, including the y-axis, ie all lines $y = m/2^n$ for integral m. Define $h_n$ to agree with f on $M_n$ and to be horizontally interpolated elsewhere, so if $x = m/2^n + x'$ , where $0 \le x' < 1/2^n$, then $h_n(x, y) = h_n(x, y) + (x'/2^n)(h_n(x + 1/2^n, y) - h_n(x, y)\ )$.

We show first that $h_n$ tends to f pointwise. Take any point P in $R^2$. f is continuous along the horizontal line L through P, so given any $\varepsilon > 0$, we may take $\delta > 0$ so that points on L within $\delta$ of P are taken by f to points within $\varepsilon$ of f(P). Take n sufficiently large so that the there are adjacent lines of $M_n$ flanking P and closer than $\delta$ to it. Let these lines cut L at Q and R. Then $h_n(P)$ lies between the values of f(Q) and f(R) and hence within $\varepsilon$ of f(P).

It remains to show that $h_n$ is continuous. This is slightly tricky to nail down rigorously.Take any point $P = (x_0, y_0)$ and any $\varepsilon > 0$. We need to show that $h_n$ takes poins close to P to values close to $h_n(P)$. Let L be the horizontal line through P. Assume that P does not lie on a line of $M_n$, so take H to be the first line to the left of P, and K the first to the right of P. Let H meet L at Q, and K meet L at R. Note that the separation of Q and R is non-zero and fixed, so the values f(Q) and f(R) may differ by a large amount. But we may take $\delta > 0$ so that points on H within $\delta$ of Q are taken by f to values within $\varepsilon/2$ of f(Q) and points on K within $\delta$ of R are taken by f to values within $\varepsilon/2$ of f(R). We want to show that for sufficiently small $\delta' > 0$, $h_n$ takes points inside the rectangle $|x - x_0| < \delta'$ , $|y - y_0| < \delta$ to values within $\varepsilon$ of $h_n(P)$.

It is now easiest to think geometrically. Let us retain the x-axis to represent values of x, but now use the y-axis to represent possible values of $h_n$. So at Q we have a vertical bar height ε centred on f(Q) and at R we have another vertical bar height ε centred on f(R). Joining the tops of these bars and joining the bottoms gives a parallelogram. A vertical line through this parallelogram at x gives the possible values of $h_n(x, y)$, where $|y - y_0| < \delta$ (because to get that value we must interpolate linearly between some point on the left-hand edge of the parallelogram and some point on the right-hand edge). Now suppose the top and bottom edges of the parallelogram make an angle θ with the y-axis. Then a thin vertical slice of the parallelogram centred on $y_0$ width δ ' will project onto a segment length ε + δ ' cot θ. By taking δ ' sufficiently small we can make this less than 2ε. In other words the spread of values of $h_n(x, y)$ for $|x - x_0| < \delta'$ and $|y - y_0| < \delta$ is less than 2ε, which is the statement that $h_n$ is continuous.

Finally, note that if P lies on a line of $M_n$, then we can deal separately with points to the left and right of P, using the same argument.

**Problem B5**
Let $f_n(x) = \sum_0^n x^i/i!$ . Show that $f_n(n) > e^n/2$. [Assume $e^x - f_n(x) = 1/n! \int_0^x (x - t)^n e^t\, dt$, and $\int_0^\infty t^n e^{-t}\, dt = n!$ ]

**Solution**
Using the two relations in the square brackets, we find almost immediately that it suffices to show that $\int_0^n t^n e^{-t}\, dt < \int_n^\infty t^n e^{-t}\, dt$.

We notice that the integrand (which we will call g(t) ) is positive for all real t and has a maximum at t = n (differentiate). It would certainly be sufficient to show that g(n + x) > g(n - x) for x in the interval (0, n). This turns out to be true. Taking logs, we need to show that n ln(n + x) - x > n ln(n - x) + x. Putting h(x) = n ln(n + x) - n ln(n - x) - 2x, we need to show that h(x) > 0. Differentiating, $h'(x) = 2x^2 /(n^2 - x^2) > 0$ for x in (0, n). But h(0) = 0, which gives the result.

**Problem B6**
Let S be a set with 1000 elements. Find a, b, c, the number of subsets R of S such that |R| = 0, 1, 2 (mod 3) respectively. Find a, b, c if |S| = 1001.

**Solution**
Answer: $[2^{1000}/3]$, $[2^{1000}/3]$, $[2^{1000}/3] + 1$; $[2^{1001}/3] + 1$, $[2^{1001}/3]$, $[2^{1001}/3] + 1$.

Let $f_0(n)$, $f_1(n)$, $f_2(n)$ be the number of subsets of a set with n elements whose size is = 0, 1, 2 (mod 3) respectively.

Let S be a set with n elements and X another element not in S, so that S' = S ∪ {X} has n+1 elements. Then the subsets of S' are just the subsets of S and the subsets of S with X adjoined. So $f_0(n+1) = f_0(n) + f_2(n)$. Similarly, $f_1(n+1) = f_1(n) + f_0(n)$ and $f_2(n+1) = f_2(n) + f_1(n)$.

These recurrence relations and the obvious initial values $f_0(1) = f_1(1) = 1$, $f_2(1) = 0$ are sufficient to determine $f_0$, $f_1$ and $f_2$. By looking at low values of n we quickly conjecture that $f_i(n) = [2^n/3]$ for n = 2i+2, 2i+3, 2i+4 (mod 6) and $[2^n/3] + 1$ for n = 2i-1, 2i, 2i+1 (mod 6). Proving that is an easy (but slightly monotonous) induction. The required answer then follows at once.

## 36th Putnam 1975

**Problem A1**
A triangular number is a positive integer of the form $n(n + 1)/2$. Show that m is a sum of two triangular numbers iff $4m + 1$ is a sum of two squares.

**Solution**
$4 ( a(a+1)/2 + b(b+1)/2 ) + 1 = 2a^2 + 2a + 2b^2 + 2b + 1 = (a - b)^2 + (a + b + 1)^2$.

If $A^2 + B^2 = 1 \pmod 4$, then one of A, B must be odd and the other even. Hence $(A + B - 1)$ and $(A - B - 1)$ are both even. Put $C = (A + B - 1)/2$, $D = (A - B - 1)/2$. Then $1/2\ C(C + 1) + 1/2\ D(D + 1) = 1/8 ( (A + B - 1)(A + B + 1) + (A - B - 1)(A - B + 1) ) = 1/8 ( (A + B)^2 - 1 + (A - B)^2 - 1) = 1/4 ( A^2 + B^2 - 1)$. So if $A^2 + B_2 = 4m + 1$, then $m = 1/2\ C(C + 1) + 1/2\ D(D + 1)$.

**Problem A2**
For what region of the real (a, b) plane, do both (possibly complex) roots of the polynomial $z^2 + az + b = 0$ satisfy $|z| < 1$?

**Solution**
Answer: the interior of the triangle with vertices $(\pm 2, 1)$, $(0, -1)$.

Clearly we require $|a| \le 2$, $|b| \le 1$. But that is not sufficient. For example, if $b = 0$, then we require $|a| \le 1$.
We consider first the region where $a \ge 0$. If $b > a^2/4$, then the roots are $z = a/2 \pm i \sqrt{(b - a^2/4)}$, so these satisfy $|z| < 1$ iff $(a/2)^2 + (b - a^2/4) < 1$ or $b < 1$. This gives the quasi-triangular region bounded by the two lines $a = 0$ and $b = 1$ on two sides and the curve $4b = a^2$ on the third side.

If $b < a^2/4$, then the roots are $z = a/2 \pm \sqrt{(a^2/4 - b)}$. These satisfy $|z| < 1$ iff $a/2 + \sqrt{(a^2/4 - b)} < 1$ which gives $a - b < 1$. This gives the quasi-triangular region bounded by the lines $a = 0$ and $a - b = 1$ on two sides and the curve $4b = a^2$ on the third. But since the line $a - b = 1$ touches the curve $b = a^2/4$ at $a = 2$, the two regions fit together to give the interior of triangle with vertices $(a, b) = (0, 1), (0, -1)$ and $(2, 1)$.

The region for $a \le 0$ is evidently the mirror image in the b-axis, so we get in total the (interior of the) triangular region with vertices (2, 1), (-2, 1) and (0, -1).

**Problem A3**
Let $0 < \alpha < \beta < \gamma \in R$, the reals. Let $K = \{ (x, y, z) : x^\beta + y^\beta + z^\beta = 1, \text{ and } x, y, z \ge 0\} \in R^3$. Define $f : K \to R$ by $f(x, y, z) = x^\alpha + y^\beta + z^\gamma$. At what points of K does f assume its maximum and minimum values?

**Solution**
Clearly for any (x, y, z) in K we have $x, y, z \le 1$. Hence $x^\alpha \ge x^\beta$. In fact $x^\alpha - x^\beta$ is zero at $x = 0$, increases to a maximum and then reduces to zero again at $x = 1$. Differentiating, we find the maximum is at $x = (\alpha/\beta)^{1/(\beta-\alpha)}$. Similarly $z^\gamma - z^\beta$ is non-negative, with a minimum at $z = (\beta/\gamma)^{1/(\gamma-\beta)}$.

To achieve a maximum for f(x, y, z), any non-zero value of z is unhelpful, whilst y is neutral, so the maximum is achieved at $x = (\alpha/\beta)^{1/(\beta-\alpha)}$, $y = (1 - (\alpha/\beta)^{\beta/(\beta-\alpha)})^{1\beta}$, $z = 0$. [The value of y is chosen so that $x^\beta + y^\beta = 1$.] Similarly, to achieve a minimum, we must take $x = 0$, $y = (1 - (\beta/g)^{\beta/(g-\beta)})^{1/\beta}$, $z = (\beta/\gamma)^{1/(\gamma-\beta)}$.

**Problem A4**
$m > 1$ is odd. Let $n = 2m$ and $\theta = e^{2\pi i/n}$. Find a finite set of integers $\{a_i\}$ such that $\sum a_i \theta^i = 1/(1 - \theta)$.

**Solution**
We have $\theta^m = -1$. Since m is odd we have $0 = (\theta^m + 1) = (\theta + 1)(\theta^{m-1} - \theta^{m-2} + \dots - \theta + 1)$. $\theta$ is not $-1$, so $\theta^{m-1} - \theta^{m-2} + \dots - \theta + 1 = 0$ (*).

Since m is odd, we may write (*) as: $1 - \theta(1 - \theta)(1 + \theta^2 + \theta^4 + \dots + \theta^{m-3} = 0$, or $1/(1 - \theta) = \theta + \theta^3 + \theta^5 + \dots + \theta^{m-2}$.

**Problem A5**
Let I be an interval and f(x) a continuous real-valued function on I. Let $y_1$ and $y_2$ be linearly independent solutions of y" = f(x) y, which take positive values on I. Show that from some positive constant k, k $\sqrt{(y_1 y_2)}$ is a solution of $y'' + 1/y^3 = f(x) y$.

**Solution**
Answer: k = $\sqrt{(2/d)}$ where d is the Wronksian.

The key is to show that $y_1y_2' - y_1'y_2$ is a constant. This expression is known as the Wronskian and it is well-known that it is non-zero iff $y_1$ and $y_2$ are linearly independent.
The derivative is $y_1y_2'' - y_1y_2'' = f\, y_1y_2 - f\, y_1y_2 = 0$, which shows that the expression is constant. Let its value be d.

Let us write $z = k \sqrt{(y_1y_2)}$, where k is a constant to be determined later. It is also convenient to put $c = k^2/2$, so that $z^2 = 2c\, y_1y_2$. Differentiating we get: $z z' = c(y_1y_2' + y_1'y_2)$ (*). Differentiating again: $z z'' + (z')^2 = c\, y_1y_2'' + c\, y_1''y_2 + 2c\, y_1'y_2' = 2c\, f\, y_1y_2 + 2c\, y_1'y_2' = f z^2 + 2c\, y_1'y_2'$.

Multiplying by $z^2$: $z^3z'' + (z z')^2 = f z^4 + 2z^2 c\, y_1'y_2' = f z^4 + 4 c^2 y_1y_2y_1'y_2'$. Using (*), $z^3 z'' + c^2(y_1y_2' + y_1'y_2)^2 = f z^4 + 4c^2 y_1y_2y_1'y_2'$. Hence $z^3z'' + c^2(y_1y_2' - y_1'y_2)^2 = f z^4$, or $z^3 z'' + c^2d^2 = f z^4$. So if we set c = 1/d, then $z'' + 1/z^3 = f z$.

**Problem A6**
Given three points in space forming an acute-angled triangle, show that we can find two further points such that no three of the five points are collinear and the line through any two is normal to the plane through the other three.

**Solution**
Let the points be A, B, C. Let the two additional points be D, E. There are three cases to consider:

(1) the plane ABC and the line DE; (2) the plane ADE and the line BC, and the two similar configurations; (3) the plane ABD and the line CE, and the five similar configurations.
Clearly (1) works provided the line DE is normal to ABC.

Now consider (2). Let the line DE intersect ABC at X. Let AX meet BC at Y. If BC is normal to ADE, then it is perpendicular to any line in ADE, so in particular it must be perpendicular to AY. In other words, AY must be an altitude of the triangle. But that is also sufficient. For we know that BC is perpendicular to DE (and if it is perpendicular to two non-parallel lines of the plane, then it must be normal to the plane). Similarly, X must lie on the other two altitudes. So this case works provided X is the orthocenter of ABC.

Take X as the origin. Let the vectors XA, XB, XC, XD, XE be **a**, **b**, **c**, **d**, k **d** respectively. A necessary and sufficient condition for AE to be normal to the plane BCD is that AE be perpendicular to BC and BD. We already know that it is perpendicular to BC (we have just shown that BC is normal to the plane ADE which contains it). So a necessary and sufficient condition is that AE be perpendicular to BD, or in vector language, (k **d** - **a**).(**b** - **d**) = 0, or k $\mathbf{d}^2$ = - **a**.**b**. This fixes k. Note that k is positive (so that D and E are on the same side of ABD, since **a**.**b** is negative). Now since X is the orthocenter of ABC, we have **a**.**b** = **b**.**c** = **c**.**a**. So with this choice of k we also have k $\mathbf{d}^2$ = - **b**.**c** = - **c**.**a**. It is easily checked that this is sufficient for the other five configurations also.

Finally, note that since ABC is acute, X lies strictly inside the triangle and hence **a.b** is non-zero, so k is non-zero and hence E is not collinear with any two of A, B, C. The only remaining point to check on non-collinearity is that D and E do not coincide. But then we would have AD perpendicular to BD, which is clearly false since AX is perpendicular to BX and D lies above X.

**Problem B1**
Let G be the group { (m, n) **:** m, n are integers } with the operation (a, b) + (c, d) = (a + c, b + d). Let H be the smallest subgroup containing (3, 8), (4, -1) and (5, 4). Let $H_{ab}$ be the smallest subgroup containing (0, a) and (1, b). Find a > 0 such that $H_{ab}$ = H.

**Solution**
Answer: 7.

$(3,8) = 3 (1,5) - (0,7)$, $(4,-1) = 4 (1,5) - 3 (0,7)$, $(5,4) = 5 (1,5) - 3 (0,7)$.
$(1,5) = (5,4) - (4,-1)$, $(0,7) = - 4 (3,8) - 7 (4,-1) + 8 (5,4)$.
This shows that $\{ (3,8), (4,-1), (5,4) \}$ and $\{ (1,5), (0,7) \}$ generate the same subgroups.

**Problem B2**
A *slab* is the set of points strictly between two parallel planes. Prove that a countable sequence of slabs, the sum of whose thicknesses converges, cannot fill space.

**Solution**
A slab thickness d intersects a sphere radius R in a volume less than $\pi R^2 d$. So the entire set of slabs fill less than $\pi R^2 D$ of the sphere, where D is the sum of their thicknesses. If we take $R > D$ this is less than the volume of the sphere, so the slabs cannot even fill the sphere.

**Problem B3**
Let n be a fixed positive integer. Let S be any finite collection of at least n positive reals (not necessarily all distinct). Let $f(S) = (\sum_{a \in S} a)^n$, and let $g(S)$ = the sum of all n-fold products of the elements of S (in other words, the nth symmetric function). Find $\sup_S g(S)/f(S)$.

**Solution**
Answer: 1/n!

For any n elements a, b, ... , w of S, the coefficient of $a^1 b^1 ... w^1$ in the multinomial expansion of f(s) is just $n!/(1!\ 1!\ ...\ 1!) = n!$. In other words, $f(S) = n!\ g(S)$ + other terms. But the other terms are all positive, so $f(S) > n!\ g(S)$. Hence $g(S)/f(S) < 1/n!$ . This establishes that 1/n! is an upper bound.

Take S to be m elements all 1. Then $f(S) = m^n$ and $g(S) = mCn$. Hence $g(S)/f(S) = (m/m) ( (m-1)/m) ( (m-2)/m) ... ( (m-n)/m) 1/n!$ . As m tends to infinity, each of the n terms $(m-r)/m$ tends to 1 and hence $g(S)/f(S)$ tends to 1/n! . Hence 1/n! is the least upper bound.

**Problem B4**
Does a circle have a subset which is topologically closed and which contains just one of each pair of diametrically opposite points?

**Solution**
Answer: no.

The map taking each point to the diametrically opposite point is a homeomorphism. [It is obviously (1,1) and its own inverse. So it is sufficient to prove it continuous. But that is almost obvious using an $\varepsilon$ $\delta$ argument.] Homeomorphisms take closed sets to closed sets. So if the subset was closed, then its complement would also be closed. So the circle would be the disjoint union of two closed sets and hence not connected. But it is connected.

**Problem B5**
Define $f_0(x) = e^x$, $f_{n+1}(x) = x\, f_n'(x)$. Show that $\sum_0^\infty f_n(1)/n! = e^e$.

**Solution**
The trick is to use the power series for $e^x$. Then we have immediately that $f_n(x) = \sum r^n x^r/r!$ Hence, $\sum f_n(x)/n! = 1 + x/1!\ (1 + 1/1! + 1^2/2! + ... ) + x^2/2!\ (1 + 2/1! + 2^2/2! + ... ) + ... = 1 + xe/1! + (xe)^2/2! + ... = e^{ex}$. [We assume the usual theorems about rearranging terms in absolutely convergent series.]

**Problem B6**
Let $h_n = \sum_1^n 1/r$. Show that $n - (n - 1)\, n^{-1/(n-1)} > h_n > n(n + 1)^{1/n} - n$ for $n > 2$.

**Solution**
Consider the numbers: $1 + 1/1, 1 + 1/2, 1 + 1/3, ... , 1 + 1/n$. Their arithmetic mean is $(n + h_n)/n$. The geometric mean is $(\prod (1 + 1/r) )^{1/n}$. But $\prod (1 + 1/r) = 2/1\ 3/2\ 4/3\ ...\ (n+1)/n = n + 1$. So the geometric mean is $(n + 1)^{1/n}$. The numbers are not all equal, so the arithmetic mean is strictly greater than the geometric mean. That gives the right inequality.

Similarly, consider the numbers: 1 - 1/2, 1 - 1/3, ... , 1 - 1/n. [One must be careful not to include 1 - 1/1, because that would make the geometric mean 0.] The arithmetic mean is $1 - (h_n - 1)/(n-1)$. The geometric mean is $n^{-1/(n-1)}$ (the terms telescope is a similar way). Rearranging gives the left inequality.

## 37th Putnam 1976

**Problem A1**
Given two rays OA and OB and a point P between them. Which point X on the ray OA has the property that if XP is extended to meet the ray OB at Y, then XP·PY has the smallest possible value.

**Solution**
Answer: take OX = OY.

Let M be the foot of the perpendicular from P to OA, and N be the foot of the perpendicular from P to OB. Obviously X must lie further from O than M - otherwise moving it closer to M would reduce both PX and PY. Similarly Y must lie further from O than N. Let $\varphi$ be the angle MPX and let $\theta$ be the angle MON. Then the angle NPY is $\theta-\varphi$. Hence $PX \cdot PY = PM \cdot PN/(\cos\varphi \cos \theta-\varphi)$. So we minimise PX.PY by maximising $(\cos\varphi \cos \theta-\varphi)$. But $(\cos\varphi \cos \theta-\varphi) = 1/2 (\cos(\varphi+\theta-\varphi) + \cos(2\varphi-\theta) ) = 1/2 \cos\theta + 1/2 \cos(2\varphi-\theta)$. This is obviously maximised by taking $\varphi = \theta-\varphi$. But that corresponds to angle PXM = angle PYN and hence OX = OY.

Alternatively, PX·PY may remind us of the elementary result that if XX'YY' is cyclic with diagonals XY, X'Y' meeting at P, then PX·PY = PX'·PY'. The problem is to find a suitable circle. With hindsight, it is fairly obvious. Take the circle which touches OA at X and OB at Y. Then any other line through P intersects the circle at X' and Y' before it intersects OA and OB (at X" and Y" say). Hence PX" > PX', PY" > PY' and PX'·PY' = PX·PY. However, I failed to find the right circle until after I had solved the problem trigonometrically.

**Problem A2**
Let a(x, y) be the polynomial $x^2y + xy^2$, and b(x, y) the polynomial $x^2 + xy + y^2$. Prove that we can find a polynomial $p_n(a, b)$ which is identically equal to $(x + y)^n + (-1)^n (x^n + y^n)$. For example, $p_4(a, b) = 2b^2$.

**Solution**
Let us write $E(n) = (x + y)^n + (-1)^n (x^n + y^n)$. We use induction on n. We have E(1) = 0, E(2) = 2b, E(3) = 3a. We find that $(x + y)^{n+3} = b (x + y)^{n+1} + a (x + y)^n$ and $x^{n+3} + y^{n+3} = b (x^{n+1} + y^{n+1}) - a (x^n + y^n)$. So it follows that E(n+3) = b E(n+1) + a E(n). That completes the induction.

**Problem A3**
Find all solutions to $p^n = q^m \pm 1$, where p and q are primes and $m, n \geq 2$.

**Solution**
Answer: $2^3 = 3^2 - 1$.

$p^n$ and $q^m$ have opposite parity, so one must be even. So without loss of generality we may take p = 2. Now $q^m \pm 1$ factors as $(q \pm 1)$ and a sum of m powers of q, each of which is odd. If m is odd, then the sum is odd, but that is impossible, since $2^n$ has no odd factors. So m is even. Take it to be 2M.
Consider first the case $2^n = q^{2M} - 1 = (q^M + 1)(q^M - 1)$. But $q^M + 1$ and $q^M - 1$ are successive even numbers, so one of them has an odd factor (which is impossible) unless then are 4 and 2. Thus the only solution of this type is $2^3 = 3^2 - 1$.
Finally consider $2^n = q^{2M} + 1$. $q^M$ is odd, so put $q^M = 2N + 1$, then $q^{2M} + 1 = 4N^2 + 4N + 2$, which is not divisible by 4. So n = 1 and q = 1, which is not a solution, since 1 is not prime.

**Problem A4**
Let $p(x) \equiv x^3 + ax^2 + bx - 1$, and $q(x) \equiv x^3 + cx^2 + dx + 1$ be polynomials with integer coefficients. Let $\alpha$ be a root of p(x) = 0. p(x) is irreducible over the rationals. $\alpha + 1$ is a root of q(x) = 0. Find an expression for another root of p(x) = 0 in terms of $\alpha$, but not involving a, b, c, or d.

**Solution**
Answer: the other roots are $-1/(\alpha+1)$ and $- (\alpha+1)/\alpha$.

Let the roots of p(x) be $\alpha, \beta, \gamma$. The polynomial q(x+1) has $\alpha$ as one of its roots. Suppose it is different from p(x). Then by subtracting we get either a quadratic or a linear equation which also has $\alpha$ as a root. It cannot be a linear equation, because then $\alpha$ would be rational and hence p(x) would not be irreducible over the rationals. If it was quadratic, then by multiplying it by a suitable rational factor rx + s and subtracting from p(x) we would get a linear factor ux + v which also had $\alpha$ as a root. If this factor is zero, then we have reduced p(x) over the rationals (as (rx + s) times the quadratic). If not, then $\alpha$ is a root of ux + v and hence rational, so that p(x) is still reducible. So we must have that p(x) and q(x+1) are identical. So the three roots of q(x) must be $\alpha+1, \beta+1, \gamma+1$. Hence we have the relations:

$\alpha\beta\gamma = 1$, $(\alpha+1)(\beta+1)(\gamma+1) = -1$ (derived from the constant terms of p(x) and q(x) respectively).

Hence $\alpha(\beta+\gamma)+(\beta+\gamma)+\alpha+1+\beta\gamma+\alpha\beta\gamma = -1$, so $(\beta+\gamma)(\alpha+1) = -(3+\alpha+1/\alpha)$, or $\beta+\gamma = -(3\alpha+\alpha^2+1) / (\alpha(\alpha+1))$. Hence $(\beta-\gamma)^2 = (\beta+\gamma)^2 - 4\beta\gamma = (\alpha^4+2\alpha^3+3\alpha^2+2\alpha+1)/(\alpha^2(\alpha+1)^2)$. So $\beta-\gamma = \pm(\alpha^2+\alpha+1)/(\alpha(\alpha+1))$ and hence $\beta = -1/(\alpha+1)$ or $-(\alpha+1)/\alpha$.

**Problem A5**

Let P be a convex polygon. Let Q be the interior of P and $S = P \cup Q$. Let p be the perimeter of P and A its area. Given any point (x, y) let d(x, y) be the distance from (x, y) to the nearest point of S. Find constants $\alpha, \beta, \gamma$ such that $\int_U e^{-d(x,y)}\, dx\, dy = \alpha + \beta p + \gamma A$, where U is the whole plane.

**Solution**

Answer: $A + p + 2\pi$.

For any point in S we have d(x, y) = 0. Hence the integral over S is just A. The locus of points a distance z from S is a set of segments parallel to the sides of P and displaced a distance z outwards, together with a set of arcs joining them. Each arc is centered on a vertex of P and has radius z. Together the arcs can be translated to form a complete circle radius z. Thus the set of points a distance z to $z + \delta z$ from S is a strip of area $p\,\delta z + 2\pi z\,\delta z$. Thus the integral outside S is just
$\int_0^\infty \exp(-z)(p + 2\pi z)\, dz$. This evaluates easily to $p + 2\pi$.

**Problem A6**

Let R be the real line. $f : R \to [-1, 1]$ is twice differentiable and $f(0)^2 + f'(0)^2 = 4$. Show that $f(x_0) + f''(x_0) = 0$ for some $x_0$.

**Solution**

Let $k(x) = f(x)^2 + f'(x)^2$. By the mean value theorem for some a in the interval (0, 2) we have $f'(a) = 1/2\,(f(2) - f(0))$. But $|f(x)| \le 1$ for all x, so $|f'(a)| \le 1$. Hence $k(a) \le 1 + 1 = 2$. Similarly, we can find b in the interval (-2, 0) with $k(b) \le 2$. We are given that k(0) = 4. Hence k(x) has a maximum at some interior point of (-2, 2). Let this point be c. Then certainly $k(c) \ge k(0) = 4$, $f(c)^2 \le 1$, so $|f'(c)| > 0$. We have $k'(c) = 0$. But $k'(c) = 2 f'(c)(f(c) + f''(c))$. We have just shown that $f'(c)$ is non-zero, so $f(c) + f''(c) = 0$.

**Problem B1**

Show that $\lim_{n\to\infty} 1/n \sum_1^n ( [2n/i] - 2[n/i] ) = \ln a - b$ for some positive integers a and b.

**Solution**

Answer: ln 4 - 1.

The expression inside the limit is a partial sum for $\int_0^1 [2/x] - [1/x]\, dx$ (taking the points x = 1/n, 2/n, ... , 1). So the limit is $\int_0^1 [2/x] - [1/x]\, dx$.
Evidently the integrand is non-zero on the intervals (1/(n+1), 1/(n + 1/2) ], for if x belongs to such an interval then $n + 1/2 <= x < n + 1$, so [1/x] = n, whilst $2n + 1 \le x \le 2n + 2$, so [2/x] = 2n+1. A similar argument shows it is zero on the complementary intervals (1/(n + 1/2), 1/n]. Thus the integral evaluates to (1/(1 + 1/2) - 1/2) + (1/(2 + 1/2) - 1/2 + ... = 2(1/3 - 1/4 + 1/5 - ... ) = 2(1 - 1/2 + 1/3 - 1/4 + ... ) - 1 = 2 ln 2 - 1 = ln 4 - 1.

**Problem B2**

G is a group generated by the two elements g, h, which satisfy $g^4 = 1$, $g^2 \ne 1$, $h^7 = 1$, $h \ne 1$, $ghg^{-1}h = 1$. The only subgroup containing g and h is G itself. Write down all elements of G which are squares.

**Solution**
Answer: $1, g^2, h, h^2, h^3, h^4, h^5, h^6$.

Obviously $g^{-1} = g^3$, $h^{-1} = h^6$. So we have $gh = h^6g$ (*). This allows us to write any element generated by g and h in the form $h^ng^m$ (with n = 0, 1, 2, ... , 6; m = 0, 1, 2, 3). Applying (*) we find that $(h^ng)^2 = g^2$, $(h^ng^2)^2 = h^{2n}$, $(h^ng^3)^2 = g^2$. Thus the only possible squares are $1, g^2, h^1$ [for example it is the square of $h^4$ ], $h^2, ... , h^6$. Moreover, these are all distinct. For if two powers of h were equal then we could deduce h = 1 (a contradiction). We are given that $g^2$ is not 1. If $g^2$ equalled a power of h, then the square of that power would be 1 and hence h would be 1 (a contradiction).

**Problem B3**
Let $0 < \alpha < 1/4$. Define the sequence $p_n$ by $p_0 = 1$, $p_1 = 1 - \alpha$, $p_{n+1} = p_n - \alpha\, p_{n-1}$. Show that if each of the events $A_1, A_2, ... , A_n$ has probability at least $1 - \alpha$, and $A_i$ and $A_j$ are independent for $| i - j | > 1$, then the probability of all $A_i$ occurring is at least $p_n$. You may assume all $p_n$ are positive.

**Solution**
This was a rare Putnam disaster - the question is wrong. Let $q_n$ be the probability that all of $A_1, ... , A_n$ occur. The idea was that one can more or less write down that $q_{n+1} \ge q_n - \alpha\, q_{n-1}$ (*). The problem is then to show that it follows that $q_n \ge p_n$ (despite the awkward minus sign in (*) ). Unfortunately, (*) requires that $A_{n+1}$ is independent of the event (all of $A_1, ... , A_{n-1}$ occur) and that does not follow from the fact that $A_{n+1}$ is independent of each of $A_1, ... , A_{n-1}$.

Let us assume first that the question was correctly worded. Let $E_n$ be the event that all of $A_1, A_2, ... , A_n$ occur. We write p(E) as the probability that event E occurs. We write ~E as the event (not E). We assume that $E_{n-1}$ and $A_{n+1}$ are independent. We may partition the event $E_n$ into the disjoint events $E_{n+1}$ and ($E_n$ & ~$A_{n+1}$). So $q_n = q_{n+1}$ + p($E_n$ & ~$A_{n+1}$). But $E_{n-1}$ implies $E_n$, so p($E_{n-1}$ & ~$A_{n+1}$) ≥ p($E_n$ & ~$A_{n+1}$). But $E_{n-1}$ and $A_{n+1}$ are assumed to be independent, so p($E_{n-1}$ & ~$A_{n+1}$) = $q_{n-1}$ p(~$A_{n+1}$) ≤ $\alpha\, q_{n-1}$. Hence p($E_n$ & ~$A_{n+1}$) ≤ $\alpha\, q_{n-1}$. Hence $q_{n+1} \ge q_n - \alpha\, q_{n-1}$. I have spelt that out at somewhat tedious length, but one should in fact be able to write the conclusion straight down.

Now let us assume that $q_{n+1} = q_n - \alpha\, q_{n-1} + \beta_{n+1}$, where $\beta_{n+1} \ge 0$. A simple induction establishes that $q_n = p_n + p_{n-2}\beta_1 + p_{n-3}\beta_2 + ... + p_0\beta_{n-1} + \beta_n$, so $q_n \ge p_n$ as required.

Finally, we need to show that the conclusion can be false if we assume no more than stated in the question. Take: p($A_1$ & $A_2$ & $A_3$ & $A_4$ & $A_5$) = p(~$A_1$ & $A_2$ & $A_3$ & $A_4$ & $A_5$) = p($A_1$ & ~$A_2$ & $A_3$ & $A_4$ & $A_5$) = p($A_1$ & $A_2$ & ~$A_3$ & $A_4$ & $A_5$) = p($A_1$ & $A_2$ & $A_3$ & ~$A_4$ & $A_5$) = p($A_1$ & $A_2$ & $A_3$ & $A_4$ & ~$A_5$) = 4/25, p(~$A_1$ & ~$A_2$ & ~$A_3$ & ~$A_4$ & ~$A_5$) = 1/25.

We can easily check that p($A_1$) = p($A_2$) = p($A_3$) = p($A_4$) = 20/25, so we may take $\alpha$ = 1/5. We can also check that the prob of any pair, such as p($A_1$ & $A_2$) = 16/25, so an independence assumption stronger than that in the question is satisfied. But p($A_1$ & $A_2$ & $A_3$ & $A_4$) = 4/25 ≤ $p_5$ (= 6/25 - 1/125).

**Problem B4**
Let an ellipse have center O and foci A and B. For a point P on the ellipse let d be the distance from O to the tangent at P. Show that $PA\cdot PB\cdot d^2$ is independent of the position of P.

**Solution**
Answer: $a^2b^2$.

Let the ellipse be $x^2/a^2 + y^2b^2 = 1$. Then the foci are at (± ae, 0) where the eccentricity e is given by $b^2 = a^2(1 - e^2)$. It is also a well-known property that PA + PB = 2a. Thus we may express the product 2 $PA\cdot PB$ as $(PA + PB)^2 - PA^2 - PB^2 = 4a^2 - (x + ae)^2 - y^2 - (x - ae)^2 - y^2 = 4a^2 - 2x^2 - 2y^2 - 2a^2e^2 = 2a^2 + 2b^2 - 2x^2 - 2y^2$.

The tangent at (x, y) is $x/b^2\, Y + x/a^2\, X = 1$, so it meets the axes at $(a^2/x, 0)$, $(0, b^2/y)$. These two points and the origin form a right-angled triangle with the origin a height d from the hypoteneuse. So we may calculate its area as $1/2\ a^2b^2/(xy)$ or as 1/2 d times hypoteneuse. Hence $d^2 = a^4b^4/( (xy)^2 (a^4x^2 + b^4/y^2) = a^4b^4/(a^4y^2 + b^4x^2)$. Using the fact that (x, y) lies on the ellipse, we have $a^4y^2 = a^4b^2 - a^2b^2x^2$, and $b^4x^2 = a^2b^4 - a^2b^2y^2$, so $a^4y^2 + b^4x^2 = a^2b^2 (a^2 + b^2 - x^2 - y^2) = a^2b^2 PA\cdot PB$. Hence $d^2 = a^2b^2/( PA\cdot PB )$.

**Problem B5**
Find $\sum_0^n (-1)^i$ nCi $( x - i )^n$, where nCi is the binomial coefficient.

**Solution**
Answer: n!

Given a polynomial p(x), define $\Delta p(x) = p(x) - p(x-1)$. If p(x) is of order n with leading coefficient a $x^n$, then $\Delta p(x)$ is of order n-1 with leading coefficient a n $x^{n-1}$. [Proof: let $p(x) = a x^n + b x^{n-1}$ + terms in $x^{n-2}$ and below. Then $\Delta p(x) = a(x^n - (x-1)^n) + b(x^{n-1} - (x-1)^{n-1})$ + terms in $x^{n-2}$ and below $= a(x^n - x^n + n x^{n-1}) + b(x^{n-1} - x^{n-1})$ + terms in $x^{n-2}$ and below = a n $x^{n-1}$ + lower terms.]
Hence $\Delta^n p(x) = a\, n!$ But the expression given is simply $\Delta^n p(x)$ with $p(x) = x^n$, so it evaluates to n!

That is the slick solution. The more plodding solution is to observe that the coefficient of $x^{n-m}$ is $(-1)^m$ nCm ( - nC1 $1^m$ + nC2 $2^m$ - nC3 $3^m$ + ... ). But the expression in parentheses is just the value at x = 1 of $(x\, d/dx)^m (1 - x)^n$ (*). For m < n, all terms in (*) will have a non-zero power of (1 - x) and hence will evaluate to zero. For m = n, the only term not evaluating to zero will be $x^n\, n!\, (-1)^n$ which gives $n!\, (-1)^n$. Hence the expression in the question has the coefficient of $x^r$ zero, except for the constant term, which is $(-1)^n$ nCn $n!\, (-1)^n = n!$ .

**Problem B6**
Let σ(n) be the sum of all positive divisors of n, including 1 and n. Show that if σ(n) = 2n + 1, then n is the square of an odd integer.

**Solution**
If n is odd and non-square, then the divisors can be arranged in pairs d, n/d. The components of each pair are odd, so their sum is even and hence σ(n) is even and cannot equal 2n+1. It is more difficult to show that n cannot be even.

We have $\sigma(n) = \prod(1 + p + p^2 + ... + p^k)$, where the product is taken over all primes p dividing n and k is the highest power of p dividing n. If p and k are odd, then the factor $(1 + p + ... + p^k)$ is even, and hence σ(n) is even. So if σ(n) = 2n+1, all k corresponding to odd primes must be even. In other words, we have $n = 2^a N^2$, for some odd integer N. Hence $\sigma(n) = (2^{a+1}-1)\sigma(N^2) = 2^{a+1}N^2 + 1$. So $N^2 = (2^{a+1}-1)(\sigma(N^2) - N^2) - 1$. Hence if q is any odd prime dividing $2^{a+1}-1$, then $N^2 = -1 \pmod q$. But -1 is a quadratic non-residue of primes of the form 4r+3. So every prime factor of $2^{a+1} - 1$ must be of the form 4r+1. But that is not possible if $a \geq 1$ (a product of numbers congruent to 1 mod 4 is congruent to 1, not -1). Hence a = 0 and n is an odd square.

## 38th Putnam 1977

**Problem A1**
Show that if four distinct points of the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear, then their average x-coordinate is some constant k. Find k.

**Solution**
Answer: - 7/8

Suppose the common line is $y = ax + b$, the the x-coordinates satisfy $2x^4 + 7x^3 + (3 - a)x - (5 + b) = 0$. This has at most 4 distinct roots. The arithmetic mean of the roots is 1/4 (-7/2) = -7/8.
The only other possibility is that the line is x = a, but that only meets the curve in one point.

**Problem A2**
Find all real solutions (a, b, c, d) to the equations a + b + c = d, 1/a + 1/b + 1/c = 1/d.

**Solution**
Answer: a, b arbitary; c = - a, d = b.

Take a, b arbitary. We then have: -c + d = a + b; cd = -ab. So -c and d are the roots of the quadratic $x^2 - (a + b)x + ab = 0$. Solving, the roots are a, b. So either c = -a, d = b, or c = -b, d = a.

**Problem A3**
R is the reals. f, g, h are functions R → R. f(x) = (h(x + 1) + h(x - 1) )/2, g(x) = (h(x + 4) + h(x - 4) )/2. Express h(x) in terms of f and g.

**Solution**
h(x) = g(x) - f(x - 3) + f(x - 1) + f(x + 1) - f(x + 3).

**Problem A4**
Find polynomials p(x) and q(x) with integer coefficients such that $p(x)/q(x) = \sum_0^\infty x^{2^n}/(1 - x^{2^{n+1}})$ for $x \in (0, 1)$.

**Solution**
It is an easy induction that the sum of the first n terms is:
$(x + x^2 + ... + x^{2^n-1})/(1 - x^{2^n})$.

But that may be written as $[ (1 - x^{2^n})/(1 - x) - 1 ] /(1 - x^{2^n}) = 1/(1 - x) - 1/(1 - x^{2^n})$. Since $x \in (0, 1)$, the second term tends to 1 as $n \to \infty$. So the result is 1/(1 - x) - 1 = x/(1 - x).

**Problem A5**
p is a prime and $m \ge n$ are non-negative integers. Show that (pm)C(pn) = mCn (mod p), where mCn is the binomial coefficient.

**Solution**
Let f(n) be the highest power of p dividing n. The multiples of p in (mp)! are mp, (m-1)p, ... , 2p, p. Hence f( (mp)! ) = $p^m$ f(m!). Similarly, f( (np)! ) = $p^n$ f(n!) and f( ((m-n)p)! ) = $p^{m-n}$ f( (m-n)! ). Hence f( mCn ) = f( (mp)C(np) ).

**Problem A6**
R is the reals. X is the square [0, 1] x [0, 1]. f : X → R is continuous. If $\int_Y f(x, y)\, dx\, dy = 0$ for all squares Y such that (1) $Y \subseteq X$, (2) Y has sides parallel to those of X, (3) at least one of Y's sides is contained in the boundary of X, is it true that f(x, y) = 0 for all x, y?

**Solution**
Answer: yes.

Given any square Z inside X with sides parallel to X, we can find squares $Y_1$, $Y_2$ satisfying the conditions in the question such that $Z = Y_1 \cap Y_2$. Hence $\int_Z f(x, y)\, dx\, dy = 0$ for all Z.

If $f(x, y) > 0$ for any interior point $(x, y)$ of X, then by continuity we can find a square Z such that $f(x, y) > 0$ on Z. Contradiction. Similarly for $f(x, y) < 0$ for an interior point of X. Hence $f(x, y) = 0$ on the interior of X and hence (by continuity) on the whole of X.

**Problem B1**
Find $\prod_2^\infty (n^3 - 1)/(n^3 + 1)$.

**Solution**
Answer: 2/3.

If we factor: $n^3 - 1 = (n - 1)(n^2 + n + 1)$, $n^3 + 1 = (n + 1)(n^2 - n + 1)$, then most of the terms cancel. Take the product up to $n = N$. Then the numerator is $1\cdot 2\cdot 3 \ldots (N - 1)\cdot 7\cdot 13\cdot 21\cdot 31 \ldots (N^2 + N + 1)$ and the denominator is $3\cdot 4\cdot 5 \ldots (N + 1)\cdot 3\cdot 7\cdot 13\cdot 21\cdot 31 \ldots (N^2 - N + 1)$. Hence the product up to $n = N$ is $2/(N(N + 1)) (N^2 + N + 1)/3 = 2/3 (N^2 + N + 1)/(N(N + 1))$, which tends to 2/3 as N tends to infinity.

**Problem B2**
P is a plane containing a convex quadrilateral ABCD. X is a point not in P. Find points A', B', C', D' on the lines XA, XB, XC, XD respectively so that A'B'C'D' is a parallelogram.

**Solution**
Let X' be the intersection of AC and BD. Take A', C' so that XA'X'C' is a parallelogram. Similarly take B', D' so that XB'X'D' is a parallelogram. Then both A'C' and B'D' have their midpoint at the midpoint of XX'. Hence A'B'C'D' is a parallelogram.

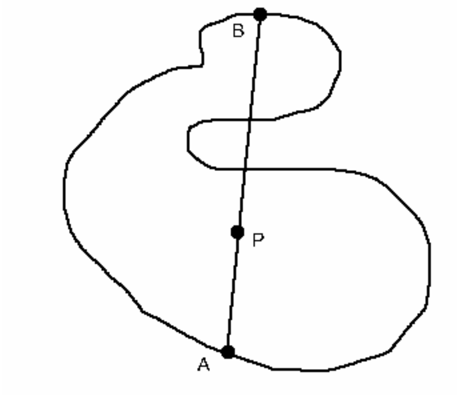
**Problem B3**
Let S be the set of all collections of 3 (not necessarily distinct) positive irrational numbers with sum 1. If $A = \{x, y, z\} \in S$ and $x > 1/2$, define $A' = \{2x - 1, 2y, 2z\}$. Does repeated application of this operation necessarily give a collection with all elements $< 1/2$?

**Solution**
Answer: no.

Write the three numbers in binary: $x = 0.x_1x_2x_3 \ldots$, $y = 0.y_1y_2y_3 \ldots$, $z = 0.z_1z_2z_3 \ldots$, where every $x_i$, $y_i$, $z_i$ is 0 or 1. Then after n operations (assuming a number $> 1/2$ at all stages) the three numbers are simply $x = 0.x_{n+1}x_{n+2}x_{n+3} \ldots$, $y = 0.y_{n+1}y_{n+2}y_{n+3} \ldots$, $z = 0.z_{n+1}z_{n+2}z_{n+3} \ldots$ . So we have to choose the $x_i$, $y_i$, $z_i$ so that (1) for each i, exactly one of $x_i$, $y_i$, $z_i$ is 1, (2) x, y, z are irrational. To achieve (2) we just have to ensure that there is no periodicity. So, for example, we could take: $x_i = 1$ for $i = 1, 4, 9, 16, \ldots$ ; $y_i = 1$ for $i = 2, 5, 10, 17, \ldots$; $z_i = 1$ if i is not a square or a square plus 1.
[If the triples are not required to be irrational, we have the even simpler solution: 1/7, 2/7, 4/7.]

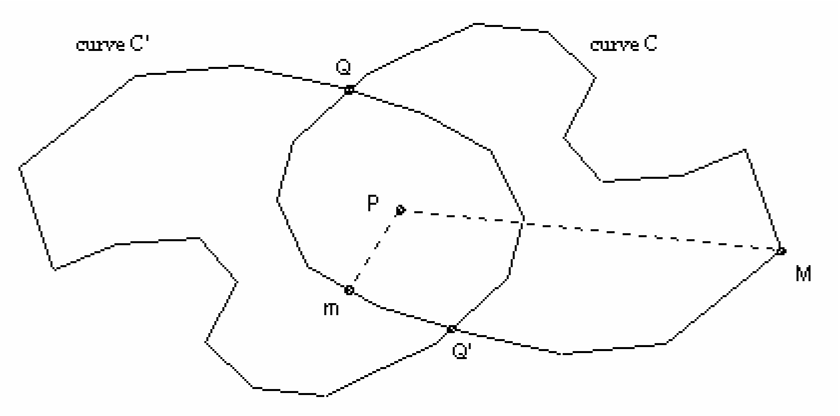
**Problem B4**
Let P be a point inside a continuous closed curve in the plane which does not intersect itself. Show that we can find two points on the curve whose midpoint is P.

**Solution**




Take an arbitrary chord AB through P. If P is the midpoint then we are done. So assume it is not. Let A' complete a circuit of the curve starting at A and returning to it. Let the chord through A' and P be A'B'. Let $f(A') = A'P/B'P$. Then f is a continuous function and $f(A_{start}) = 1/f(A_{finish})$. So for some point C on the curve f must assume the intermediate value 1, which means that P is the midpoint of this chord.

Let C be the curve and C' the curve obtained by rotating C through $180^o$ about P. Let m be a point on C closest to P, and M a point on C furthest from P. Then m must lie inside or on C', and M must lie outside or on C'. Hence C and C' must intersect. Take Q to be any common point. Then the point Q' obtained by rotating Q through $180^o$ must also lie on C and C'. Now P is the midpoint of QQ'.




**Problem B5**
$a_1, a_2, \dots , a_n$ are real and $b < (\sum a_i)^2/(n - 1) - \sum a_i^2$. Show that $b < 2a_ia_j$ for all distinct i, j.

**Solution**
It is sufficient to show that $(\sum a_i)^2/(n - 1) - \sum a_i^2 \le 2a_1a_2$. But this follows immediately from the Cauchy inequality for the two n-1 tuples:
$a_1 + a_2, a_3, a_4, \dots , a_n$; and $1, 1, \dots , 1$:

$(\sum a_i)^2 <= (n - 1)( (a_1 + a_2)^2 + a_3^2 + \dots + a_n^2) = (n - 1) \sum a_i^2 + (n - 1) 2a_1a_2$.

**Problem B6**
G is a group. H is a finite subgroup with n elements. For some element $g \in G$, $(gh)^3 = 1$ for all elements $h \in H$. Show that there are at most $3n^2$ distinct elements which can be written as a product of a finite number of elements of the coset Hg.

**Solution**
It is essential to notice first that $g^3 = 1$ (since $1 \in H$).

A little messing around should now convince us that we can simplify finite products of elements of the form hg. In fact, we show that they can always be written as (1) $h_1gh_2$, (2) $h_1g^2h_2$, or (3) $h_1g^2h_2g$.
Clearly hg is of the form (1), so it is sufficient to show that given an element k of form (1), (2) or (3), then k (hg) is also of one of these forms.

It is convenient to note that: $ghg = h^{-1}g^2h^{-1}$ (*) (post-multiply ghghgh = 1 successively by $h^{-1}$, $g^2$, $h^{-1}$); and $g^2hg^2 = h^{-1}gh^{-1}$ (**) (pre-multiply $gh^{-1}gh^{-1}gh^{-1} = 1$ successively by $g^2$, h, $g^2$).

So dealing with the three cases in turn: $(h_1gh_2)\, h_3g = h_1h_3^{-1}h_2^{-1}g^2h_3^{-1}h_2^{-1}g$, which is of form (2).

$(h_1g^2h_2)\, h_3g$ is obviously of form (3).

$(h_1g^2h_2g)\, h_3g = h_1g^2h_2h_3^{-1}g^2h_3^{-1} = h_1h_3h_2^{-1}gh_3h_2^{-1}h_3^{-1}$, which is of form (1).

# 39th Putnam 1978

**Problem A1**
Let S = {1, 4, 7, 10, 13, 16, ... , 100}. Let T be a subset of 20 elements of S. Show that we can find two distinct elements of T with sum 104.

**Solution**
Not best possible: we can make do with 19 elements. Note that the numbers have the form $3n + 1$ for $n = 0, 1, \dots, 33$. We seek $3n + 1$, $3m + 1$ so that $n + m = 34$. Evidently $n = 0$ and $n = 17$ do not help. The other 32 numbers form 16 pairs with the required sum. So if we take 19 numbers then we are sure to get two from the same pair.

**Problem A2**
Let A be the real n x n matrix $(a_{ij})$ where $a_{ij} = a$ for $i < j$, $b$ ($\neq a$) for $i > j$, and $c_i$ for $i = j$. Show that $\det A = (b\,p(a) - a\,p(b))/(b - a)$, where $p(x) = \prod (c_i - x)$.

**Solution**

$$\begin{vmatrix} c_1 & a & a & \dots & a \\ b & c_2 & a & \dots & a \\ b & b & c_3 & \dots & a \\ \dots & & & & \\ b & b & b & \dots & c_n \end{vmatrix} = \begin{vmatrix} a & a & a & \dots & a \\ b & c_2 & a & \dots & a \\ b & b & c_3 & \dots & a \\ \dots & & & & \\ b & b & b & \dots & c_n \end{vmatrix} + \begin{vmatrix} c_1-a & 0 & 0 & \dots & 0 \\ b & c_2 & a & \dots & a \\ b & b & c_3 & \dots & a \\ \dots & & & & \\ b & b & b & \dots & c_n \end{vmatrix}$$

To evaluate the first determinant on the right, we subtract the first column from each of the others. Then expanding by the top row we get a D, where D has zeros below the main diagonal and hence is just the product of the elements on its diagonal. In other words, the first determinant is just $a \prod_2^n (c_i - b)$.

If we expand the second determinant by the top row we get $(c_1 - a) \det A'$, where A' is the (n-1) x (n-1) matrix formed by deleting the first row and column of A. So we can use induction. The result is trivial for n = 1. So assume it is true for n - 1. Then for n we have $a \prod_2^n (c_i - b) + b/(b - a) \prod_1^n (c_i - a) - a/(b - a) (c_1 - a) \prod_2^n (c_i - b)$. Adding the first and third terms we get: $a (1 - (c_1 - a)/(b - a) ) \prod_2^n (c_i - b) = - a/(b - a) \prod_1^n (c_i - b)$. So the result is true for n.

**Problem A3**
Let $p(x) = 2(x^6 + 1) + 4(x^5 + x) + 3(x^4 + x^2) + 5x^3$. Let $a = \int_0^\infty x/p(x)\,dx$, $b = \int_0^\infty x^2/p(x)\,dx$, $c = \int_0^\infty x^3/p(x)\,dx$, $d = \int_0^\infty x^4/p(x)\,dx$. Which of a, b, c, d is the smallest?

**Solution**
Answer: b.

Let $a_1 = \int_0^1 x/p(x)\,dx$, $a_2 = \int_1^\infty x/p(x)\,dx$. Similarly, $b_1, b_2, c_1, c_2, d_1, d_2$. Also define $e_1 = \int_0^1 1/p(x)\,dx$. Using the subtitution $y = 1/x$ we find that $a_1 = c_2$, $a_2 = c_1$, $b_1 = b_2$, and $e_1 = d_2$. Hence, in particular, $a = a_1 + a_2 = c_2 + c_1 = c$.

But $x - 2x^2 + x^3 = x(x - 1)^2 > 0$ on (0, 1), so $a_1 - 2b_1 + c_1 > 0$. Hence $a = a_1 + a_2 > b_1 + b_2 = b$. Hence also $c > b$.

Similarly, $1 - 2x^2 + x^4 = (x^2 - 1)^2 > 0$ on (0, 1). So $e_1 - 2b_1 + d_1 > 0$. Hence $d = d_1 + d_2 > b_1 + b_2 = b$. So b is the smallest.

**Problem A4**
A binary operation (represented by multiplication) on S has the property that (ab)(cd) = ad for all a, b, c, d. Show that: (1) if ab = c, then cc = c; (2) if ab = c, then ad = cd for all d. Find a set S, and such a binary operation, which also satisfies: (A) a a = a for all a; (B) $ab = a \neq b$ for some a, b; (C) $ab \neq a$ for some a, b.

**Solution**
(1): (ab)(ab) = ab, so cc = c. (2): (ab)d = ( (ab)(ab) ) (dd) = (ab)(dd) = ad. Note that an exactly similar argument gives a(bc) = ac. So the operation is in fact associative and $a_1a_2\dots a_n = a_1a_n$.

In passing we note various possible special cases: (A) ab = k for all a, b (where k is fixed); (B) ab = a for all a, b; (C) ab = b for all a, b. The extra conditions are presumably designed to rule out these special cases.

So we need some operation which preserves something of the first element and something of the second. The easiest is to take S to consist of pairs (r, s) and to define the operation as: (r, s) (u, v) = (r, v). The simplest such example is S = { (0, 0), (0, 1), (1, 0), (1, 1) }. This obviously satisfies all the required conditions. Writing a = (0, 0), b = (0, 1), c = (1, 0), d = (1, 1), we have:

| | a | b | c | d |
|---|---|---|---|---|
| a | a | b | a | b |
| b | a | b | a | b |
| c | c | d | c | d |
| d | c | d | c | d |

**Problem A5**
Let $a_1, a_2, \dots , a_n$ be reals in the interval $(0, \pi)$ with arithmetic mean $\mu$. Show that $\prod (\sin a_i)/a_i \le ( (\sin \mu)/\mu )^n$.

**Solution**
Products are intractable, whereas sums are easy, so we take logs. Let $f(x) = \ln ( (\sin x)/x) = \ln \sin x - \ln x$. The required relation is now that $1/n \sum f(a_i) \le f(\mu)$. This is true if the curve is concave, in other words if $f''(x) \le 0$.

Differentiating, we have: $f'(x) = \cot x - 1/x$, $f''(x) = - \operatorname{cosec}^2 x + 1/x^2$. But $\sin x \le x$ on $[0, \pi]$, so $f''(x) \le 0$ on $[0, \pi]$.

**Problem A6**
Given n points in the plane, prove that less than $2n^{3/2}$ pairs of points are a distance 1 apart.

**Solution**
Label the points 1, 2, ... , n. Let $n_i$ be the number of points a distance 1 from point i. We wish to show that $\sum n_i < 4 n^{3/2}$. We can assume that all $n_i \ge 2$. [Keep removing points until those that are left have $n_i \ge 2$. If we exhaust the points before this happens then the number of pairs $\le n < 2n^{3/2}$. Otherwise, suppose we remove k points. Then the result follows from the result for n - k points since $k + 2(n - k)^{3/2} < 2n^{3/2}$.

The points a distance 1 from point i must all lie on $C_i$ the circle radius 1, center point i. Each pair of circles meets in 0, 1 or 2 points. So the total number of points of intersection of two circles is at most $n(n - 1)$ (where a point of intersection arises in more than one way we count one for each way it arises). Some of these points of intersection will be members of the original set of n points. Point i has $n_i$ circles through it, so it arises as a point of intersection in $1/2\ n_i(n_i - 1)$ ways. Thus all the points together give rise to $\sum 1/2\ n_i(n_i - 1)$ points of intersection. Hence $\sum n_i(n_i - 1) \le 2n(n - 1)$. Since $n_i \ge 2$, $n_i \le 2(n_i - 1)$, so we have that $\sum n_i^2 \le 2 \sum n_i(n_i - 1) \le 4n(n - 1) < 4n^2$. Now Cauchy's inequality gives that $n \sum n_i^2 \ge (\sum n_i)^2$. So $\sum n_i < 2 n^{3/2}$.

**Problem B1**
A convex octagon inscribed in a circle has 4 consecutive sides length 3 and the remaining sides length 2. Find its area.

**Solution**
Let the radius be R. The area is made up of four triangles sides 3, R, R and four triangles sides 2, R, R. The area is unchanged if we rearrange these triangles to form an octagon with sides alternately 2 and 3. But the new octagon is a square side $3 + 2\sqrt{2}$ with four corners lopped off, each sides 2, $\sqrt{2}$, $\sqrt{2}$. Hence its area is $(17 + 12\sqrt{2}) - 4 = 13 + 12\sqrt{2}$.

**Problem B2**
Find $\sum_1^\infty \sum_1^\infty 1/(i^2 j + 2ij + ij^2)$.

**Solution**
Answer: 7/4.

Let us fix i and sum over j. The term is $1/(i(i + 2)) (1/j - 1/(j + i + 2) )$. So, summing over j, all the terms cancel except for the first i + 2, giving: $1/(i(i + 2)) (1/1 + 1/2 + \dots + 1/(i + 2) )$. Also $1/(i(i + 2)) = 1/2 (1/i - 1/(i + 2) )$.

So summing over i (and multiplying by 2) we get:
$(1/1 - 1/3) (1 + 1/2 + 1/3) +$

(1/2 - 1/4) (1 + 1/2 + 1/3 + 1/4) +
(1/3 - 1/5) (1 + 1/2 + 1/3 + 1/4 + 1/5) +
(1/4 - 1/6) (1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6) + ...

Each of the minus terms partially cancels with the corresponding plus term two lines lower, so we get:
1/1(1 + 1/2 + 1/3) + 1/2(1 + 1/2 + 1/3 + 1/4) + 1/3(1/4 + 1/5) + 1/4(1/5 + 1/6) + ...
= 1 + 1/4 + 1/2 + (1/1 1/2 + 1/2 1/3 + 1/3 1/4 + ... ) + (1/1 1/3 + 1/2 1/4 + 1/3 1/5 + ... )
= 7/4 + ( (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4 ) + ... ) + 1/2 ( (1 - 1/3) + (1/2 - 1/4) + (1/3 - 1/5) + ... )
= 7/4 + 1 + 1/2 (1 + 1/2) = 7/2.
Hence the answer is 7/4.

**Problem B3**
The polynomials $p_n(x)$ are defined by $p_1(x) = 1 + x$, $p_2(x) = 1 + 2x$, $p_{2n+1}(x) = p_{2n}(x) + (n + 1)\, x\, p_{2n-1}(x)$, $p_{2n+2}(x) = p_{2n+1}(x) + (n + 1)\, x\, p_{2n}(x)$. Let $a_n$ be the largest real root of $p_n(x)$. Prove that $a_n$ is monotonic increasing and tends to zero.

**Solution**
It is fairly obvious that $a_n < 0$ and that the sequence is strictly monotonic increasing. The problem is proving it tends to zero.

Trivial inductions show that $p_n(0) = 1$ and $p_n(x) > 0$ for $x > 0$, so $a_n < 0$. Also we note that $a_1 = -1$, $a_2 = -1/2$. If $x > a_n$, then $p_n(x) > 0$ (otherwise there would be a root greater than $a_n$).

$p_{2n+1}(a_{2n}) = (n + 1)\, a_{2n}\, p_{2n-1}(a_{2n})$. By induction $p_{2n-1}(a_{2n}) > 0$, so $p_{2n+1}(a_{2n}) < 0$ and hence $a_{2n+1} > a_{2n}$. Similarly $a_{2n+2} > a_{2n+1}$. So, as claimed, $a_n$ is strictly monotonic increasing.

To show it tends to zero, it suffices to prove that: $a_{2n-1} > -1/(n-1)$ and $a_{2n} > -1/n$. This is true for n = 1. Suppose it is true for n. We have already established monotonicity, so $a_{2n+1} > -1/n$. Also $p_{2n-1}(-1/(n+1)) > 0$. But $p_{2n+2}(-1/(n+1)) = p_{2n+1}(-1/(n+1)) - p_{2n}(-1/(n+1)) = - p_{2n-1}(-1/(n+1)) < 0$, so $a_{2n+2} > -1/(n+1)$ and we have established the inductive hypothesis for n+1.

**Problem B4**
Show that we can find integers a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = abc + abd + acd + bcd$, and the smallest of a, b, c, d is arbitarily large.

**Solution**
If b, c, d are fixed, then the relation gives a quadratic in a whose two roots have sum bc + bd + cd. So if we have a solution a, b, c, d, then we can derive another solution a', b, c, d with a' = (bc + bc + cd) - a (*). If we take a < b < c < d, and all of a, b, c, d positive, then clearly a' > d.

So starting with any positive solution we may derive solutions with successively larger smallest members. In fact, starting with (1, 1, 1, 1), we get successively (1, 1, 1, 2), (1, 1, 2, 4), (1, 2, 4, 13), (2, 4, 13, 85). From this point on, the smallest member is increased each time.

**Problem B5**
Find the real polynomial p(x) of degree 4 with largest possible coefficient of $x^4$ such that $p( [-1, 1] ) \in [0, 1]$.

**Solution**
Let $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Then $a_4x^4 - a_3x^3 + a_2x^2 - a_1x + a_0$ also satisfies the conditions and hence also $a_4x^4 + a_2x^2 + a_0$. So we may take $a_3 = a_1 = 0$.

Now p(0), p(1/2), p(1) belong to [0, 1], so $0 \le a_0 \le 1$, $0 \le a_4/4 + a_2/2 + a_0 \le 1$, $0 \le a_4 + a_2 + a_0 \le 1$. Hence: $-2 \le -2a_0 \le 0$, $0 \le a_4 + 2a_2 + 4a_0 \le 4$, $-2 \le -2a_4 - 2a_2 - 2a_0 \le 0$. Adding: $-4 \le -a_4 < 4$. So the maximum value of $a_4$ is at most 4.

This can be achieved by $4(x^2 - 1/2)^2 = 4x^4 - 4x^2 + 1$.

**Problem B6**
$a_{ij}$ are reals in [0, 1]. Show that $( \sum_{i=1}^{n} \sum_{j=1}^{mi} a_{ij}/i )^2 \le 2m \sum_{i=1}^{n} \sum_{j=1}^{mi} a_{ij}$.

**Solution**

The question looks incredibly complicated, but is actually easy. We just use induction and the fact that each $a_{ij} \le 1$.

Put $b_i = \sum_1^{mi} a_{ij}$. Notice that since each $a_{ij} \le 1$, we have $b_i \le mi$. We use induction on n. For n = 1, the required result is: $b_1^2 \le 2mb_1$. But $b_1 \le m$ and $b_1 \ge 0$, so this is certainly true.

Now assume the result is true for n.

For n + 1, the lhs is $(b_1/1 + b_2/2 + ... + b_n/n)^2 + 2(b_1/1 + ... + b_n/n)\ b_{n+1}/(n+1) + b_{n+1}^2/(n+1)^2$, and the rhs is $2m(b_1 + ... + b_n) + 2mb_{n+1}$. Given the result for n, it is sufficient to show that: $2(b_1/1 + ... + b_n/n)\ b_{n+1}/(n+1) + b_{n+1}^2/(n+1)^2 \le 2mb_{n+1}$. Divding by $b_{n+1}$ and using $b_i/i$ le; m, we need: $2nm/(n+1) + m/(n+1) \le 2m$, which is clearly true.

## 40th Putnam 1979

**Problem A1**
Find the set of positive integers with sum 1979 and maximum possible product.

**Solution**
For $n > 4$, $2(n - 2) > n$, so the maximum product cannot include any integer greater than 4. Also, $2^3 < 3^2$, so it cannot include more than two 2s. Since $4 = 2\cdot 2$, it cannot include both a 2 and a 4. It obviously does not include a 1, since $n + 1 > n \times 1$. So the maximum product must be made up mainly of 3s, with either no, one or two 2s (or equivalently one 4). Hence for $1979 = 2 + 659\cdot 3$, the maximum product is $3^{659}2$.

**Problem A2**
R is the reals. For what real k can we find a continuous function $f : R \to R$ such that $f(f(x)) = k x^9$ for all x.

**Solution**
Evidently $k^{1/4} x^3$ works for any $k \ge 0$.

For $k \ne 0$, f must be (1, 1) [otherwise we would get $x^9 = y^9$ for $x \ne y$, contradiction.] But f is continuous, so it is strictly monotonic. If it is strictly monotonic increasing, then so is f(f(x)). But if it is strictly monotonic decreasing, then f(f(x)) is strictly monotonic *increasing*. So, either way, f(f(x)) is strictly monotonic increasing. If $k < 0$, then $k x^9$ is strictly monotonic decreasing, so we cannot have $k < 0$.

**Problem A3**
$a_n$ are defined by $a_1 = \alpha$, $a_2 = \beta$, $a_{n+2} = a_n a_{n+1}/(2a_n - a_{n+1})$. $\alpha$, $\beta$ are chosen so that $a_{n+1} \ne 2a_n$. For what $\alpha$, $\beta$ are infinitely many $a_n$ integral?

**Solution**
A trivial induction shows that $a_{n+2} = \alpha b/((n + 1)\alpha - n\beta) = \alpha\beta/(n(\alpha - \beta) + \alpha)$.

So we must have $\alpha = \beta$, otherwise the denominator grows without limit. But if $\alpha = \beta$, then all $a_n = \alpha$. So infinitely many $a_n$ are integral iff $\alpha = \beta =$ integer.

**Problem A4**
A and B are disjoint sets of n points in the plane. No three points of $A \cup B$ are collinear. Can we always label the points of A as $A_1, A_2, \dots , A_n$, and the points of B as $B_1, B_2, \dots , B_n$ so that no two of the n segments $A_iB_i$ intersect?

**Solution**
Answer: yes.

It is easy to waste a lot of time failing to find inductive arguments (can we find a line with the same number of A-points and B-points on one side of it etc).

The trick is to find another property which cannot be satisfied if two segments intersect. Take the labeling which minimises the total length of the segments (obviously exists, since there are only finitely many labelings). If $A_iB_i$ intersects $A_jB_j$ at X. Then $A_iB_j < A_iX + XB_j$, $A_jB_i < A_jX + XB_i$. So $A_iB_j + A_jB_i < A_iB_i + A_jB_j$. So the labeling was not minimal. Contradiction. Hence no pair of segments can intersect.

**Problem A5**
Show that we can find two distinct real roots $\alpha$, b of $x^3 - 10x^2 + 29x - 25$ such that we can find infinitely many positive integers n which can be written as $n = [r\alpha] = [s\beta]$ for some integers r, s.

**Solution**
The polynomial has value -25, -5, 1, -1, -5, -5, 5 at x = 0, 1, 2, 3, 4, 5, 6. So one root lies between 1 and 2, another between 2 and 3, and another between 5 and 6. In particular, all the roots are greater than 1, so if $\alpha$ is a root then all the values $[\alpha]$, $[2\alpha]$, $[3\alpha]$, $[4\alpha]$ ... are different. Hence just $[n/\alpha]$ of them lie in the range 1 to n. Similarly, if the other roots are $\beta$, $\gamma$, then $[n/\beta]$ values $[m\beta]$ lie in the range 1 to n, and $[n/\gamma]$ values $[m\gamma]$. So at least $[n/\alpha] + [n/\beta] + [n/\gamma] - n$ integers in the range 1 to n must have at least two of the representations $[m\alpha]$, $[m'\beta]$, $[m''\gamma]$. But $1/\alpha + 1/\beta + 1/\gamma >$

$1/2 + 1/3 + 1/6 = 1$. So as n tends to infinity, the number of integers <= n with at least two representations tends to infinity. In other words, infinitely many positive integers have at least two representations. But there are only finitely many possibilities, so we must be able to find two roots $\alpha$, $\beta$ such that infinitely many integers have the representations $[m\alpha]$ and $[m'\beta]$.
Interestingly, this argument does not tell us which two roots!

**Problem A6**
Given n reals $\alpha_i \in [0, 1]$ show that we can find $\beta \in [0, 1]$ such that $\sum 1/|\beta - \alpha_i| \le 8n \sum_1^n 1/(2i - 1)$.

**Solution**
The trick is to pick several candidates $\beta_j$ and then to take $\sum_j 1/|\beta_j - \alpha_i|$.

Divide [0, 1] into 2n equal intervals, each width 1/2n. Then at least n of them do not contain any $\alpha_i$. Take the $\beta_j$ to be the midpoints of n empty subintervals. Then certainly $|\beta_j - \alpha_i| \ge 1/4n$. If we fix j, then we cannot say more than that, so we get $\sum_i 1/|\beta_j - \alpha_i| \le n\ 4n$, which is not good enough.

But if we fix i, then we can say more: at most two $\beta_j$ can be at the minimum distance 1/4n, at most two more at 3/4n, at most two more at 5/4n and so on. We now sum over j. We do not know whether there are one or two $\beta_j$ at each stage (because if $\alpha_i$ is off-center, then fairly soon we run into the endpoint going one way, and so start getting just one $\beta_j$). But it is certainly conservative to assume that we have two at each stage and go on for n pairs. So summing over j gives: $\sum_j 1/|\beta_j - \alpha_i| < 8n \sum 1/(2i - 1)$. Now summing over all $a_i$ multiplies the result by n. At least one $b_j$ must give a result that is not above average, so we can find $\beta$ with the sum at most $8n \sum 1/(2i - 1)$.

**Problem B1**
Can we find a line normal to the curves y = cosh x and y = sinh x?

**Solution**
Answer: no.

The gradient of y = cosh x at x = a is sinh a, so the equation of the normal is sinh a (y - cosh a) + (x - a) = 0. Similarly, the normal to y = sinh x at x = b is cosh b (y - sinh b) + (x - b) = 0. For these two equations to be the same (so that the normals coincide) we require: sinh a = cosh b and a + sinh a cosh a = b + sinh b cosh b, or b - a = sinh a cosh a - sinh b cosh b = cosh a cosh b - sinh a sinh b = cosh (b - a). But that is impossible since cosh x > x for all x.

**Problem B2**
Given $0 < \alpha < \beta$, find $\lim_{\lambda\to 0} \left( \int_0^1 (\beta x + \alpha(1 - x) )^\lambda dx \right)^{1/\lambda}$.

**Solution**
Answer: $\beta^{\beta/(\beta - \alpha)}/( e\, \alpha^{\alpha/(\beta - \alpha)})$

Let $t = \beta x + \alpha(1 - x)$. Then the integral becomes $\int_\alpha^\beta t^\lambda\, dt/(\beta - \alpha) = 1/(1 + \lambda)\, (\beta^{\lambda+1} - a^{\lambda+1})/(\beta - \alpha)$. We evaluate the limits of $1/(1 + \lambda)^\lambda$ and $(\beta^{\lambda+1} - a^{\lambda+1})^\lambda/(\beta - \alpha)^\lambda$ separately. Put $k = 1/\lambda$. The first limit is $\lim 1/(1 + 1/k)^k = 1/e$. To evaluate the second, note that $\beta^x = e^{x \ln \beta} = (1 + x \ln \beta + O(x^2) )$, so the expression is $(1 + 1/k\, (\beta \ln \beta - \alpha \ln \alpha)/(\beta - \alpha) + O(1/k^2) )^k$, which tends to the limit $\exp( (\beta \ln \beta - \alpha \ln \alpha)/(\beta - \alpha) )$ as k tends to infinity, in other words, $\beta^{\beta/(\beta - \alpha)}/ \alpha^{\alpha/(\beta - \alpha)}$.

**Problem B3**
F is a finite field with n elements. n is odd. $x^2 + bx + c$ is an irreducible polynomial over F. For how many elements $d \in F$ is $x^2 + bx + c + d$ irreducible?

**Solution**
Since n is odd, $h \ne -h$ for any $h \in F$. So there are exactly (n + 1)/2 quadratic residues. Since n is odd, we may write b as 2k, and so we are interested in d for which $(x + k)^2 = k^2 - c - d$ is irreducible. In other words, d for which $k^2 - c - d$ is a quadratic non-residue. But $k^2 - c - d$ runs through all the elements of F, so there are just (n - 1)/2 values of d which give a quadratic non-residue.

**Problem B4**
Find a non-trivial solution of the differential equation $F(y) \equiv (3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 0$. $y = f(x)$ is the solution of $F(y) = 6(6x + 1)$ such that $f(0) = 1$, and $( f(-1) - 2)( f(1) - 6) = 1$. Find a relation of the form $( f(-2) - a)( f(2) - b) = c$.

**Solution**
Answer: $a = 6$, $b = 14$, $c = 1$.

The differential equation looks fairly horrible; there is no obvious systematic way of solving it. So we try various types of solutions. Trying simple polynomials leads to $x^2 + x$. Trying exponentials leads to $e^{3x}$. Obviously, a particular solution to $F(y) = 36x + 6$ is $y = 2$, so the general solution is $y = 2 + Ae^{3x} + Bx + Bx^2$. $f(0) = 1$ implies $A = -1$. $( f(-1) - 2)( f(1) - 6) = (- e^{-3} )( -4 - e^3 + 2B) = 1$, so $B = 2$.

$f(-2) = 2 - e^{-6} + 4$, $f(2) = 2 - e^6 + 12$, so we need $a = 6$, $b = 14$, $c = 1$ (and then the relation reduces to $( - e^{-6} )( - e^{6}) = 1$, as required).

**Problem B5**
A convex set S in the plane contains (0, 0) but no other lattice points. The intersections of S with each of the four quadrants have the same area. Show that the area of S is at most 4.

**Solution**
Let A, B, C, D be the points (1, 0), (0, 1), (-1, 0), (0, -1). These all lie outside S, so we may take support lines $S_A$, $S_B$, $S_C$, $S_D$ through each of them. If $S_A$ meets the x-axis in the interval [-1, 1], then we are home, because the part of S in the first quadrant lies inside a triangle area < 1. So assume each support line is either parallel to the other axis or meets it a distance more than 1 from the origin. Hence $S_A$ and $S_B$ meet at W in the first quadrant, $S_B$ and $S_C$ meet at X in the second quadrant, $S_C$ and $S_D$ meet at Y in the third quadrant, and $S_D$ and $S_A$ meet at Z in the fourth quadrant. S lies entirely inside WXYZ. At least one of the angles of this quadrilateral must be $\leq 90^o$. Suppose it is W.

Now we show that the area of the quadrilateral OAWB (where O is the origin) is at most 1. The required result then follows immediately. Area OAWB = area OAB + area WAB, and area OAB = 1/2. So we need to show that area WAB <= 1/2. But that is almost immediate. The locus of W given angle W is an arc through A, W, B. We maximise the area of the triangle by taking W the midpoint of the arc. But this triangle lies entirely inside the triangle AW'B with AW' = W'B and angle AW'B = $90^o$, which has area 1/2.

**Problem B6**
$z_i$ are complex numbers. Show that $|Re[ (z_1^2 + z_2^2 + ... + z_n^2)^{1/2} ]| \leq |Re\ z_1| + |Re\ z_2| + ... + |Re\ z_n|$.

**Solution**
Let $z_k = x_k + i\ y_k$, and let $(z_1^2 + z_2^2 + ... + z_n^2)^{1/2} = a + ib$.

Then, squaring, $a^2 - b^2 = \sum x_k^2 - \sum y_k^2$ (1), $ab = \sum x_k y_k$ (2). The Cauchy inequality gives that $|\sum x_k y_k| \leq ( \sum x_k^2 )^{1/2} ( \sum y_k^2 )^{1/2}$, so from (1) we have $|ab| \leq ( \sum x_k^2 )^{1/2} ( \sum y_k^2 )^{1/2}$ (3). If $|a| > ( \sum x_k^2 )^{1/2}$, then from (3) $|b| < ( \sum y_k^2 )^{1/2}$. But then $a^2 - b^2 > \sum x_k^2 - \sum y_k^2$, contradicting (1). So we must have that $|a| \leq ( \sum x_k^2 )^{1/2}$. The required result follows immediately since obviously $( \sum |x_k| )^2 \geq \sum x_k^2$.

# 41st Putnam 1980

**Problem A1**
Let $f(x) = x^2 + bx + c$. Let C be the curve $y = f(x)$ and let $P_i$ be the point $(i, f(i))$ on C. Let $A_i$ be the point of intersection of the tangents at $P_i$ and $P_{i+1}$. Find the polynomial of smallest degree passing through $A_1, A_2, \dots, A_9$.

**Solution**
$y = f(x) - 1/4$.

The answer suggests there ought to be a one-line solution, but I cannot see it.
The equation of the tangent at $x = i$ is $y - (i^2 + ib + c) = (2i + b)(x - i)$, or $y - (2i + b)x = c - i^2$. Solving, we find that $A_i$ is $(i + 1/2, i^2 + i + bi + b/2 + c)$. Clearly these do not lie on a straight line, so the degree is at least 2. But we see at once that *any* $A_i$ lies on the second degree $y = x^2 + bx + c - 1/4$.

**Problem A2**
Find f(m, n), the number of 4-tuples (a, b, c, d) of positive integers such that the lowest common multiple of any three integers in the 4-tuple is $3^m7^n$.

**Solution**
Answer: $(6m^2 + 3m + 1)(6n^2 + 3n + 1)$.

Each of a, b, c, d must be of the form $3^h7^k$ with $h \le m$, $k \le n$. We can consider h and k separately. At least two of a, b, c, d must have h = m. So there are three cases: (1) all of a, b, c, d have h = m (1 possibility); (2) three of a, b, c, d have h = m and the other has $0 \le h < m$ (3m possibilities); (3) two of a, b, c, d have h = m and the other two have $0 \le h < m$ ($6m^2$ possibilities). So, in all, there are $6m^2 + 3m + 1$ possibilities for h. Similarly, there are $6n^2 + 3n + 1$ possibilities for k. Each of the possibilities for h can be combined with each of the possibilities for k, so in all there are $(6m^2 + 3m + 1)(6n^2 + 3n + 1)$ possible 4-tuples.

**Problem A3**
Find $\int_0^{\pi/2} f(x)\,dx$, where $f(x) = 1/(1 + \tan^{\sqrt{2}}x)$.

**Solution**
Answer: $\pi/4$.

This involves a trick. $f(\pi/2 - x) = 1/(1 + \cot^{\sqrt{2}}x) = \tan^{\sqrt{2}}x/(1 + \tan^{\sqrt{2}}x) = 1 - f(x)$. Hence $\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_{\pi/4}^{\pi/2} f(x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_0^{\pi/4} f(\pi/2 - x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_0^{\pi/4} (1 - f(x))\,dx = \int_0^{\pi/4} dx = \pi/4$.
Note that the $\sqrt{2}$ is irrelevant - the argument works for any exponent.

**Problem A4**
Show that for any integers a, b, c, not all zero, and such that $|a|, |b|, |c| < 10^6$, we have $|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$. But show that we can find such a, b, c with $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

**Solution**
This needs a trick: $(a + b\sqrt{2} + c\sqrt{3})(a + b\sqrt{2} - c\sqrt{3})(a - b\sqrt{2} + c\sqrt{3})(a - b\sqrt{2} - c\sqrt{3}) = a^4 + 4b^4 + 9c^4 - 4a^2b^2 - 6c^2a^2 - 12b^2c^2$ which is an integer. Moreover, it is non-zero by a slight extension of the argument used to prove the irrationality of $\sqrt{2}$.

For suppose $a + b\sqrt{2} = c\sqrt{3}$ (*). Squaring: $2ab\sqrt{2} = 3c^2 - a^2 - 2b^2$. But $\sqrt{2}$ is irrational, so a or b is zero. But b cannot be zero since $\sqrt{3}$ is irrational, so a must be zero. Take b and c relatively prime (by dividing out any common factor if necessary). We have $2b^2 = 3c^2$. By the usual argument 2 divides both b and c. Contradiction. Hence (*) is impossible.

So the identity $(a + b\sqrt{2} + c\sqrt{3})(a + b\sqrt{2} - c\sqrt{3})(a - b\sqrt{2} + c\sqrt{3})(a - b\sqrt{2} - c\sqrt{3}) = a^4 + 4b^4 + 9c^4 - 4a^2b^2 - 6c^2a^2 - 12b^2c^2$ gives that $|a + b\sqrt{2} + c\sqrt{3}| \ge 1/|(a + b\sqrt{2} - c\sqrt{3})(a - b\sqrt{2} + c\sqrt{3})(a - b\sqrt{2} - c\sqrt{3})|$. But each of the three numbers in this product is at most $10^7$, so $|a + b\sqrt{2} + c\sqrt{3}| \ge 10^{-21}$.

For the other inequality we use a pigeon-hole argument. Take the $10^{18}$ numbers $a + b\sqrt{2} + c\sqrt{3}$ with $0 \le a, b, c < 10^6$. They all lie in the interval $[0, 5 \times 10^6]$, so if we divide this interval into $6 \times 10^{17}$ equal parts, then at least two numbers must lie in the same part. Subtracting them gives a number with absolute value at most $5/6\ 10^{-11}$.

**Problem A5**

Let p(x) be a polynomial with real coefficients of degree 1 or more. Show that there are only finitely many values $\alpha$ such that $\int_0^\alpha p(x) \sin x\, dx = 0$ and $\int_0^\alpha p(x) \cos x\, dx = 0$.

**Solution**

Note that either equation alone has infinitely many zeros.

Let $d(x) = p(x) - p''(x) + p^{(4)}(x) - \dots$ . Then we can easily check that $\int p(x) \sin x\, dx = -d(x) \cos x + d'(x) \sin x$, so the definite integral is $-d(\alpha) \cos \alpha + d'(\alpha) + d(0)$. Similarly, the second definite integral is $d(\alpha) \sin \alpha + d'(\alpha) \cos \alpha - d'(0)$. So any roots $\alpha$ of both equations also satisfy $d(\alpha) = d(0) \cos \alpha + d'(0) \sin \alpha$ (*). Let $k = |d(0)| + |d'(0)|$, then $|\text{rhs } (*)| \le k$. But d(x) has the same degree as p(x), so $|d(x)| > k$ for all $x >$ some k'. In other words all the roots of (*) must satisfy $|\alpha| \le k'$. By Rolle's theorem, if f(x) has only finitely many zeros, then so does $\int_0^x f(t)\, dt$. Both p(x) and sin x have only finitely many zeros in any finite range $|x| \le k'$, so $\int_0^\alpha p(x) \sin x\, dx$ has only finitely many zeros in the range $|\alpha| \le k'$. Hence result.

**Problem A6**

Let R be the reals and C the set of all functions $f : [0, 1] \to R$ with a continuous derivative and satisfying $f(0) = 0$, $f(1) = 1$. Find $\inf_C \int_0^1 | f'(x) - f(x) |\, dx$.

**Solution**

If it were not for the condition $f(0) = 0$, we could arrange for the infimum to be zero (which is obviously the lowest possible value), by taking $f(x) = e^{x-1}$. It is tempting to think that we can adjust this solution (and keep zero inf) by taking $f(x) = e^{x-1}$ for $[\varepsilon, 1]$ and kx for $[0, \varepsilon]$. However, this does not work because $\int_0^\varepsilon f'(x)\, dx = f(\varepsilon) = 1/e$.

We need the following (familiar) trick: $(f(x) e^{-x})' = (f'(x) - f(x))\, e^{-x}$. So $\int_0^1 | f'(x) - f(x) |\, dx = \int_0^1 | e^x (f(x) e^{-x})' |\, dx \ge \int_0^1 | (f(x) e^{-x})' |\, dx \ge \int_0^1 (f(x) e^{-x})'\, dx = f(1) e^{-1} - f(0) e^0 = 1/e$.

Finally, we just need to improve slightly our opening example. The integral of f(x) over $[0, \varepsilon]$ will be arbitrarily small for small $\varepsilon$. But derivative is discontinuous at $\varepsilon$. However, we can fix that by taking f(x) to be an arbitrary monotonic increasing function with the required values at 0 and $\varepsilon$ and with the required derivative at $\varepsilon$.

**Problem B1**

For which real k do we have $\cosh x \le \exp(k x^2)$ for all real x?

**Solution**

Answer: $k \ge 1/2$.

$\cosh x = 1 + x^2/2! + x^4/4! + \dots + x^{2n}/(2n)! + \dots$ and $\exp(x^2/2) = 1 + x^2/2 + x^4/8 + \dots + x^{2n}/(2^n n!)$. But $2^n n! = 2n.(2n - 2).(2n - 4) \dots 2 < (2n)!$ for $n > 1$, so $\cosh x < e^{x2/2}$ for all x. Hence the inequality holds for $k \ge 1/2$.

$\cosh x = 1 + x^2/2 + o(x)$, $\exp(k x^2) = 1 + kx^2 + o(x)$. So if $k < 1/2$, then $\cosh x > \exp(k x^2)$ for sufficiently small x. Thus the inequality does not hold for $k < 1/2$.

**Problem B2**

S is the region of space defined by $x, y, z \ge 0$, $x + y + z \le 11$, $2x + 4y + 3z \le 36$, $2x + 3z \le 24$. Find the number of vertices and edges of S. For which a, b is $ax + by + z \le 2a + 5b + 4$ for all points of S?

**Solution**

Answer: 7 vertices, 11 edges. Condn on a, b is $2/3 \le a \le 1$ and $b = 2 - a$.

The face of S in the xy plane has vertices (0,0,0), (11,0,0), (4,7,0) and (0,9,0). The face of S in the yz plane has vertices (0,0,0), (0,9,0), (0,3,8) and (0,0,8). The face of S in the zx plane has vertices (0,0,0), (11,0,0), (0,0,8) and (9,0,2). There are no other vertices. So 7 vertices in total. Note that (11,0,0), (4,7,0), (0,3,8) and (9,0,2) lie on $x + y + z = 11$; (0,3,8), (0,9,0) and (4,7,0) lie on $2x + 4y + 3z = 36$; and (0,0,8), (0,3,8) and (9,0,2) lie on $2x + 3z = 24$.

There are 6 faces (the six planes x=0, y=0, z=0, x + y + z = 11, 2x + 4y + 3z = 36 and 2x + 3z = 24). Hence the number of edges is 6 + 7 - 2 = 11. [Explicitly: (0,0,0) to (11,0,0), (0,9,0) and (0,0,8); (0,3,8) to (0,0,8), (0,9,0), (4,7,0) and (9,0,2); (0,9,0) to (4,7,0) to (11,0,0) to (9,0,2) to (0,0,8).]

The point (2,5,4) is the midpoint of the edge (0,3,8) to (4,7,0). If we fix a, b then 2a + 4b + 5 has some value k. The condition $ax + by + z \le k$ is then the condition that (x,y,z) lies on or to one side of the plane ax + by + z = k. But the point (2, 5, 4) lies on the plane, so we require that the plane is a support plane of S. Since (2, 5, 4) lies on an edge of S, that edge must lie in the plane. The two extreme positions are evidently x + y + z = 11 and 2x + 4y + 3z = 36, or equivalently 2/3 x + 4/3 y + z = 12, each of which contain a face of S including the edge.

So the acceptable a, b are $2/3 \le a < 1$, and b = 2 - a. [Check: this gives the condition $ax + (2 - a)y + z \le (14 - 3a)$. For the vertex (0,9,0) we have 18 - 9a < 14 - 3a, which is equivalent to $a \ge 2/3$. For the vertex (11,0,0) we have 11a < 14 - 3a, which is equivalent to $a \le 1$.]

**Problem B3**
Define $a_n$ by $a_0 = \alpha$, $a_{n+1} = 2a_n - n^2$. For which $\alpha$ are all $a_n$ positive?

**Solution**
Answer: $\alpha \ge 3$.

The trick is that we can solve the recurrence relation. A particular solution is obviously a polynomial with leading term $n^2$, so we soon find $a_n = n^2 + 2n + 3$. The general solution is then $a_n = n^2 + 2n + 3 + k2^n$. The initial condition $a_0 = \alpha$ gives that $k = \alpha - 3$. A necessary and sufficient condition for all $a_n$ to be positive is evidently $k \ge 0$, or $\alpha \ge 3$.

**Problem B4**
S is a finite set with subsets $S_1, S_2, \dots , S_{1066}$ each containing more than half the elements of S. Show that we can find $T \subseteq S$ with $|T| \le 10$, such that all $T \cap S_i$ are non-empty.

**Solution**
Let |S| = n. Consider the number of pairs (x, X) with $x \in S$ and X one of the 1066 subsets. There are > n/2 pairs (x, $X_0$) for a given $X_0$, and hence more than 533n in total. Hence there is an $x_1$ which is in at least 534 subsets. That leaves $\le 532$ subsets. Each of these contains > n/2 elements of S - $\{x_1\}$, so by the same argument we can find $x_2$ so that the number of subsets not containing either $x_1$ or $x_2$ is at most 265. Similarly, we can find $x_3$, so that at most 132 subsets do not intersect $\{x_1, x_2, x_3\}$. Continuing, the sequence goes 65, 32, 15, 7, 3, 1, 0 for ten elements.

**Problem B5**
$R^{0+}$ is the non-negative reals. For $\alpha \ge 0$, $C_\alpha$ is the set of continuous functions $f : [0, 1] \to R^{0+}$ such that: (1) f is convex [ $f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y)$ for $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$]; (2) f is increasing; (3) $f(1) - 2 f(2/3) + f(1/3) \ge \alpha ( f(2/3) - 2 f(1/3) + f(0) )$. For which $\alpha$ is $C_\alpha$ is closed under pointwise multiplication?

**Solution**
Answer: $\alpha \le 1$.

The function f(x) = x always belongs to $C_\alpha$ because it has f(1) - 2f(2/3) + f(1/3) = f(2/3) - 2f(1/3) + f(0) = 0. So if $C_\alpha$ is closed under pointwise multiplication, then $f(x) = x^2$ is also a member. But this requires $1 - 8/9 + 1/9 \ge \alpha (4/9 - 2/9 + 0)$ and hence $\alpha \le 1$.

Now if f and g are convex and increasing, then the pointwise product f*g is also convex. For if $x \le y$, we have $(f(y) - f(x))(g(y) - g(x)) \ge 0$ and hence $\lambda\mu(f(x)g(y) + f(y)g(x)) \le \lambda\mu(f(x)g(x) + f(y)g(y))$ (*).

But $f*g(\lambda x + \mu y) = f(\lambda x + \mu y)g(\lambda x + \mu y) \le (\lambda f(x) + \mu f(y) )(\lambda g(x) + \mu g(y) ) = \lambda^2 f(x)g(x) + \lambda\mu(f(x)g(y) + f(y)g(x) ) + \mu^2 f(y)g(y)$. So, using (*), we get $f*g(\lambda x + \mu y) \le (\lambda^2 + \lambda\mu)f(x)g(x) + (\lambda\mu + \mu^2)f(y)g(y) = \lambda\, f*g(x) + \mu\, f*g(y)$.

So it remains to show that if $\alpha \le 1$ and f, g are in $C_\alpha$, then f*g satisfies condition (3). This is not easy - one has to find the right collection of inequalities using the fact that f, g are increasing, convex and satisfy (3).

Since f(2/3) > f(1/3) and $g(1) - 2g(2/3) + g(1/3) \ge \alpha (g(2/3) - 2g(1/3) + g(0) )$, we have:

$f(2/3)\ (g(1) - 2g(2/3) + g(1/3)\ ) \ge \alpha\ f(1/3)\ (g(2/3) - 2g(1/3) + g(0)\ )$ **(1)**

Similarly, we have:

$(f(1) - 2f(2/3) + f(1/3)\ )\ g(2/3) \ge \alpha\ (f(2/3) - 2f(1/3) + f(0)\ )\ g(1/3)$ **(2)**

Since f and g are convex, we have $( f(1) - f(2/3) ) \ge ( f(2/3) - f(1/3) )$ and $( g(1) - g(2/3) ) \ge ( g(2/3) - g(1/3) )$. So $( f(1) - f(2/3) ) ( g(1) - g(2/3) \ge ( f(2/3) - f(1/3) ) ( g(2/3) - g(1/3) )$. The rhs is non-negative and $\alpha \le 1$, so $( f(2/3) - f(1/3) ) ( g(2/3) - g(1/3) ) \ge \alpha ( f(2/3) - f(1/3) ) ( g(2/3) - g(1/3) )$. Hence:

$( f(1) - f(2/3) ) ( g(1) - g(2/3) \ge \alpha ( f(2/3) - f(1/3) ) ( g(2/3) - g(1/3) )$ **(3)**

Similarly:

$( f(2/3) - f(1/3) ) ( g(2/3) - g(1/3) ) \ge \alpha ( f(1/3) - f(0) ) ( g(1/3) - g(0) )$ **(4)**

Adding **(1)**, **(2)**, **(3)** and **(4)** gives the required result:

$f(1)g(1) - 2f(2/3)g(2/3) + f(0)g(0) \ge \alpha ( f(2/3)g(2/3) - 2f(1/3)g(1/3) + f(0)g(0) )$.

**Problem B6**
An array of rationals f(n, i) where n and i are positive integers with i > n is defined by f(1, i) = 1/i, f(n+1, i) = (n+1)/i ( f(n, n) + f(n, n+1) + ... + f(n, i - 1) ). If p is prime, show that f(n, p) has denominator (when in lowest terms) not a multiple of p (for n > 1).

**Solution**
Generating functions are always a technique to bear in mind for sequences. They are strongly suggested here by the form of the recurrence relation. So let us put $f_n(x) = f(n, n)\ x^n + f(n, n+1)\ x^{n+1} + f(n, n+2)\ x^{n+2} + \dots$ . We can get the sum f(n, n) + f(n, n+1) + ... + f(n, i - 1) by multiplying by $(1 + x + x^2 + x^3 + \dots )$. For then the coefficient of $x^{i-1}$ is f(n, n) + f(n, n+1) + ... + f(n, i-1). We can easily get the factor (n+1) by multiplying by it. This gives us a series where the coefficient of $x^{i-1}$ is i f(n+1, i). In other words, it gives us $f'_{n+1}(x)$. So $f'_{n+1}(x) = (n+1)\ f_n(x)\ (1 + x + x^2 + \dots )$.

Let us examine the special case n = 1: $f'_2(x) = 2\ (x + x^2/2 + x^3/3 + \dots )(1 + x + x^2 + \dots )$. At this point we need to spot that $(1 + x + x^2 + \dots )$ is in fact the *derivative* of $(x + x^2/2 + x^3/3 + \dots )$. So integration gives simply $(x + x^2/2 + x^3/3 + \dots )^2$. In fact it is now obvious that we can carry out a simple induction to get $f_n(x) = (x + x^2/2 + x^3/3 + \dots )^n$.

This is all that we need. For the coefficient of $x^p$ is a sum of terms a/b, where a is an integer and b is a product of n integers taken from {1, 2, ..., p-n+1}. Hence we may add these terms to get a fraction with denominator $( (p-n+1)! )^n$. This has no factor p. A fortiori, when reduced to lowest terms.