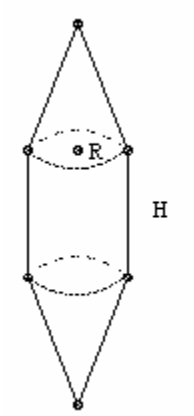
1st Putnam 1938

Problem A1
A solid in Euclidean 3-space extends from z = -h/2 to z = +h/2 and the area of the section z = k is a polynomial in k of degree at most 3. Show that the volume of the solid is h(B + 4M + T)/6, where B is the area of the bottom (z = -h/2), M is the area of the middle section (z = 0), and T is the area of the top (z = h/2). Derive the formulae for the volumes of a cone and a sphere.

Solution
Let the polynomial be $az^3 + bz^2 + cz + d$. Then the volume is $\int_{-h/2}^{h/2} (az^3 + bz^2 + cz + d)\, dz = bh^3/12 + dh$. But $B + T = bh^2/2 + 2d$, $M = d$, so $h(B + 4M + T)/6 = bh^3/12 + dh$, which proves the formula. For a sphere radius R, we have h = 2R, B + T = 0 and $M = \pi R^2$, so the formula gives $4/3\ \pi R^3$, as usual. For a cone height h, base area A, we have B = A, T = 0, M = A/4, so the volume is hA/3, as usual.

Problem A2
A solid has a cylindrical middle with a conical cap at each end. The height of each cap equals the length of the middle. For a given surface area, what shape maximizes the volume?



Solution
Let the radius be R and the height H. The area is $2\pi RH + 2\pi R\sqrt{(R^2+H^2)}$. The volume is $5/3\ \pi R^2 H$.

The area is fixed, so for some fixed k, we have $R\ (\ H + \sqrt{(R^2+H^2)}\ ) = k$. This gives $H = (k^2 - R^4)/(2kR)$. We must now choose R to maximise $f(R) = R^2H = R\ (k^2 - R^4)/2k$. Evidently the allowed range for R is from R = 0 up to $\sqrt{k}$ (corresponding to H = 0). But f(0) = 0 and $f(\sqrt{k}) = 0$, so the maximum is at some interior point of the interval. Differentiating, we find it is at $R_{max} = (k^2/5)^{1/4}$. In terms of the area A, we have $A = 2\pi k$, so $R_{max} = (A/(\pi\ 2\sqrt{5})\ )^{1/2}$.

Problem A3
A particle moves in the Euclidean plane. At time t (taking all real values) its coordinates are $x = t^3 - t$ and $y = t^4 + t$. Show that its velocity has a maximum at t = 0, and that its path has an inflection at t = 0.

Solution
The speed squared is $(dx/dt)^2 + (dy/dt)^2 = 16t^6 + 9t^4 + 8t^3 - 6t^2 + 2$. Let this be f(t). We have $f'(t) = 12t(8t^4 + 3t^2 + 2t - 1)$. So f '(t) = 0 at t = 0. Also for t small (positive or negative), $8t^4 + 3t^2 + 2t - 1$ is close to -1 and hence negative, so f '(t) is positive for t just less than 0 and negative for t just greater than 0. Hence f(t) has a maximum at t = 0. Hence the speed does also.

The gradient $dy/dx = (4t^3 + 1)/(3t^2 - 1)$. Let this be g(t). Then $g'(t) = 6t(2t^3 - 2t - 1)/(3t^2 - 1)^2$. Hence g'(0) = 0. Also g'(t) is positive for t just less than 0 and negative for t just greater than 0, so it is a point of inflection.

Problem A4
A notch is cut in a cylindrical vertical tree trunk. The notch penetrates to the axis of the cylinder and is bounded by two half-planes. Each half-plane is bounded by a horizontal line passing through the axis of the cylinder. The angle between the two half-planes is θ. Prove that the volume of the notch is minimized (for given tree and θ) by taking the bounding planes at equal angles to the horizontal plane.

Solution
We find the volume of the notch above the horizontal plane. Suppose that the upper bounding half-plane is at an angle φ to the horizontal. We may take the radius of the tree to be 1. A vertical section through the notch at a distance x from its widest extent is a right-angled triangle with base $\sqrt{(1 - x^2)}$ and area 1/2 $(1 - x^2)$ tan φ. Hence the volume is 2/3 tan φ. So the total volume of the notch is 2/3 (tan φ + tan(θ-φ)). So we have to find the angle φ which minimises (tan φ + tan(θ - φ). Differentiating, or otherwise, we easily find that the minimum is at φ/2.

Problem A5
(1) Find $\lim_{x->inf} x^2/e^x$.
(2) Find $\lim_{k->0} 1/k \int_0^k (1 + \sin 2x)^{1/x} dx$.

Solution
(1) Let $f(x) = x^3e^{-x}$. Then $f'(x) = (3x^2 - x^3) e^{-x} < 0$ for $x > 3$. Hence $f(x) < f(3)$ for $x > 3$, so $x^2 e^{-x} < f(3)/x$ for $x > 3$. Hence $x^2 e^{-x}$ tends to zero.

(2) We use L'Hôpital's rule lim f(x)/g(x) = lim f '(x)/g'(x). Applied to the expression given it gives lim $(1 + \sin 2x)^{1/x}$. Write $(1 + \sin 2x)^{1/x} = \exp(1/x \ln(1 + \sin 2x))$. So apply the rule again to 1/x ln(1 + sin 2x) to get 2 cos 2x/(1 + sin 2x) which tends to 2. Hence $(1 + \sin 2x)^{1/x}$ tends to e^2 and so does the original expression.

Problem A6
A swimmer is standing at a corner of a square swimming pool. She swims at a fixed speed and runs at a fixed speed (possibly different). No time is taken entering or leaving the pool. What path should she follow to reach the opposite corner of the pool in the shortest possible time?

Solution
Answer: let k be the running speed divided by the swimming speed. For $k > \sqrt{2}$, the unique solution is to run round the outside. For $k < \sqrt{2}$, the unique solution is to swim direct. For $k = \sqrt{2}$ there is no unique solution. Run along a side to X, swim to Y equidistant from the corner between X and Y, then run from Y. The time taken is independent of X.

We may take the side of the square to be 1, the swimming speed to be 1 and the running speed to be k. Let the square be ABCD. Suppose the start is at A and the finish at C. Possible routes are (1) run to X on AB, swim to Y on BC, run to C, (2) run to X on AD, swim to Y on CD, run to C, (3) run to X on AB, swim to Y on CD, run to C. We start by considering case (1). Take BX to be x, BY to be y. Then the time taken is $(2 - x - y)/k + \sqrt{(x^2 + y^2)}$. Note that this includes the extreme cases of running all the way (x = y = 0) and swimming all the way (x = y = 1).

Now $(x - y)^2 >= 0$, with equality iff x = y, so $(x + y)^2 <= 2(x^2 + y^2)$ and hence $(x + y) \le \sqrt{2}\ \sqrt{(x^2 + y^2)}$, with equality iff x = y. So if $k > \sqrt{2}$, then $(x + y) < k\ \sqrt{(x^2 + y^2)}$ and hence $2/k < (2 - x - y)/k + \sqrt{(x^2 + y^2)}$ unless x = y = 0 (when we have equality). Hence for $k > \sqrt{2}$, the unique solution is to run all the way.

If $k < \sqrt{2}$, then $(x + y) \le \sqrt{2}\ \sqrt{(x^2 + y^2)}$ implies $\sqrt{2}\ (\sqrt{2} - \sqrt{(x^2 + y^2)} \le 2 - x - y$ and hence $k\ (\sqrt{2} - \sqrt{(x^2 + y^2)} < 2 - x - y$ unless x = y = 1 (when we have equality). So $\sqrt{2} < (2 - x - y)/k + \sqrt{(x^2 + y^2)}$ unless x = y = 1. In other words, for $k < \sqrt{2}$ the unique solution is to swim all the way.

For $k = \sqrt{2}$ we have equality (in both the previous paragraphs) iff x = y. So any solution with x = y is optimal in this case.

Problem A7

Do either (1) or (2)

(1) S is a thin spherical shell of constant thickness and density with total mass M and center O. P is a point outside S. Prove that the gravitational attraction of S at P is the same as the gravitational attraction of a point mass M at O.

(2) K is the surface z = xy in Euclidean 3-space. Find all straight lines lying in S. Draw a diagram to illustrate them.

Solution

(1) Let Q be a point on S. The obvious coordinate is the angle θ = angle QOP. The density is $\rho = M/4\pi r^2$. By symmetry the attraction on P is towards O. Let the distance PO be d and the radius of the sphere be r. Let the gravitational constant be G. The component of the attraction towards O (per unit mass at P) is $G \int_0^{\pi} r\, d\theta\, 2\pi r \sin\theta\, \rho\, (d - r\cos\theta)\, (d^2 + r^2 - 2dr\cos\theta)^{-3/2}$. Note that the factor $(d - r\cos\theta)\, (d^2 + r^2 - 2dr\cos\theta)^{-1/2}$ is needed to resolve the force in the direction PO. Writing $x = \cos\theta$, this becomes $2G\pi r^2\rho \int_{-1}^{1} (d - rx)/(d^2 + r^2 - 2dr\, x)^{3/2}\, dx$.

This is not as bad as it looks. It is just the sum of a $(1 - x)^{-1/2}$ and a $(1 - x)^{-3/2}$ integral, both of which are straightforward. Moreover, when we come to substitute $x = \pm 1$, the factor $(d^2 + r^2 - 2dr\, x)^{1/2}$ becomes just d - r or d + r. So we get after a little simplification $4\pi r^2\rho/d^2 = MG/d^2$, which is the same result as if all the mass was concentrated at O.

(2) We can write a general line as x = at + b, y = ct + d, z = et + f, for some constants a, b, c, d, e, f and a parameter t which takes all real values. If this lies in z = xy, then $et + f = ac\, t^2 + (bc + ad)\, t + bd$ for all t. Hence a or c must be zero. If a is 0, then z = by, so the line can be written as x = b, z = by. Similarly, if c = 0, then the line can be written as y = d, z = dx. Conversely, it is easy to see that these two families of lines lie in the surface.

Problem B1

Do either (1) or (2)

(1) Let A be matrix (a_{ij}), $1 \le i,j \le 4$. Let d = det(A), and let A_{ij} be the cofactor of a_{ij}, that is, the determinant of the 3 x 3 matrix formed from A by deleting a_{ij} and other elements in the same row and column. Let B be the 4 x 4 matrix (A_{ij}) and let D be det B. Prove $D = d^3$.

(2) Let P(x) be the quadratic $Ax^2 + Bx + C$. Suppose that P(x) = x has unequal real roots. Show that the roots are also roots of P(P(x)) = x. Find a quadratic equation for the other two roots of this equation. Hence solve $(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0$.

Solution

Answer: (2) The quadratic is $A^2x^2 + (AB + A)x + (AC + B + 1) = 0$. The quartic in y has roots 0, 1, 2, 2.

(1) We have $a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + a_{i4}A_{i4} = d$. But $a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} + a_{i4}A_{j4} = 0$ for i not equal to j (because it can be considered as an expansion of the determinant for the matrix derived from A by replacing row i by row j - the resulting matrix has two identical rows and hence zero determinant). So if we multiply the transpose of A by the matrix (A_{ij}) then we get d down the diagonal and zeros elsewhere. Hence $d\, D = d^4$, so $D = d^3$.

(2) It is obvious that if P(x) = x, then P(P(x)) = x.

P(P(x)) = x is $A(Ax^2 + Bx + C)^2 + B(Ax^2 + Bx + C) + C = x$. This is evidently a quartic and two of its roots are those of $Ax^2 + (B - 1)x + C = 0$. We could obtain the quadratic for the other two roots

by multiplying out P(P(x)) - x and factorising it. But it is sufficient to obtain the coefficients of x^4, x^3 and x^0. This gives us the sum of the four roots as -2B/A and their product as $(AC + B + 1)C/A^3$. The sum and product of the two known roots are -B/A - 1/A and C/A. Hence the sum and product of the other two roots are -B/A + 1/A and $(AC + B + 1)/A^2$, so the roots are the roots of the quadratic $A^2x^2 + (AB + A)x + (AC + B + 1) = 0$.

$y^2 - 3y + 2 = 0$ has roots 1 and 2. So these values are also roots of $(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0$. The other two roots are also the roots of $x^2 + (-3 + 1)x + (2 - 3 + 1) = 0$. These are obviously 0 and 2.

Problem B2
Find all solutions of the differential equation zz" - 2z'z' = 0 which pass through the point x=1, z=1.

Solution
Answer: $z = 1/(A(x - 1) + 1)$.

We have $z''/z' = 2 z'/z$. Integrating, $\ln z' = 2 \ln z + \text{const}$, so $z' = - A/z^2$. Integrating again: $1/z = Ax + B$. But $z(1) = 1$, so $B = 1 - A$.

Problem B3
A horizontal disk diameter 3 inches rotates once every 15 seconds. An insect starts at the southernmost point of the disk facing due north. Always facing due north, it crawls over the disk at 1 inch per second. Where does it again reach the edge of the disk?

Solution
Answer: at the northernmost point of the disk.

Take polar coordinates with $r = 3/2$, $\theta = 0$ at the start. The equations of motion are $dr/dt = - \cos \theta$, $r\, d\theta/dt = 2r\pi/15 + \sin \theta$.

Differentiating the second equation: $(dr/dt)(d\theta/dt) + r\, d^2\theta/dt^2 = (2\pi/15)\, dr/dt + (d\theta/dt) \cos \theta$. Substituting from the first equation, $2 (dr/dt)(d\theta/dt) + r\, d^2\theta/dt^2 = (2\pi/15)\, dr/dt$. Multiplying through by r and integrating wrt t, we get $r^2\, d\theta/dt = (\pi/15)\, r^2 + C$, for some constant C. At t = 0, r = 3/2 and $d\theta/dt = 2\pi/15$, so $C = 3\pi/20$. Thus $r^2\, d\theta/dt = (\pi/15)\, r^2 + 3\pi/20$. But the original equation gives $r^2\, d\theta/dt = 2r^2\pi/15 + r \sin \theta$. Hence $\pi\, r^2/15 + r \sin \theta = 3\pi/20$ (**). Hence if $r = \pm 3/2$, we have $\sin \theta = 0$ and hence $\theta = 0$ or π.

That is not quite enough to show that the insect reaches the edge again at $\theta = \pi$. But we can treat (**) as a quadratic in r and solve to get $r^2 = (\sin^2\theta + (\pi/5)^2)^{1/2} - \sin \theta$. This shows that r first decreases, but then increases again to ± 1 at $\theta = \pi$. We can rule out -1 because r^2 is always positive, and r starts positive. So by continuity it must always remain positive. Thus the insect next reaches the edge at the northenmost point of the disk.

Problem B4
The parabola P has focus a distance m from the directrix. The chord AB is normal to P at A. What is the minimum length for AB?

Solution
Answer: $3\sqrt{3}$ m.

We may take the equation of P as $2my = x^2$. The gradient at the point A $(a, a^2/2m)$ is a/m, so the normal at (a, b) is $(y - a^2/2m) = -m/a\ (x - a)$. Substituting in $2my = x^2$, it meets P at (x, y) where $x^2 + 2m^2/a\ x - (2m^2 + a^2) = 0$, so the other point B has $x = -(2m^2/a + a)$.

Thus $AB^2 = (2a + 2m^2/a)^2 + 4m^2(1 + m^2/a^2)^2 = 4a^2(1 + m^2/a^2)^3$. Differentiating, we find the minimum is at $a^2 = 2m^2$ and is $AB^2 = 27m^2$.

Problem B5
Find the locus of the foot of the perpendicular from the center of a rectangular hyperbola to a tangent. Obtain its equation in polar coordinates and sketch it.

Solution
Answer: $r^2 = 2k^2 \sin 2\theta$. It is a figure of 8 with its axis along the line y = x and touching the x-axis and y-axis at the origin.

Take the hyperbola as $xy = k^2$. Then the tangent at $(a, k^2/a)$ is $(y - k^2/a) = -k^2/a^2\ (x - a)$. The perpendicular line through the origin is $y = a^2/k^2\ x$. They intersect at $x = 2k^2/(a\ (a^2/k^2 + k^2/a^2)\ )$, $y = 2a/(a^2/k^2 + k^2/a^2)$. So the polar coordinates r, θ satisfy $\tan\theta = y/x = a^2/k^2$, $r^2 = x^2 + y^2 = 4a^2(k^4/a^4 + 1)/(k^2/a^2 + a^2/k^2) = 4a^2(\cot^2\theta + 1)/(\tan\theta + \cot\theta)^2 = 4\ a^2\cos^2\theta/(\sin^2\theta + \cos^2\theta)^2 = 4a^2\cos^2\theta = 4k^2 \sin\theta \cos\theta = 2k^2 \sin 2\theta$. Thus the polar equation of the locus is $r^2 = 2k^2 \sin 2\theta$.

Problem B6
What is the shortest distance between the plane $Ax + By + Cz + 1 = 0$ and the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. You may find it convenient to use the notation $h = (A^2 + B^2 + C^2)^{-1/2}$, $m = (a^2A^2 + b^2B^2 + c^2C^2)^{1/2}$. What is the algebraic condition for the plane not to intersect the ellipsoid?

Solution
The tangent plane to the ellipsoid at (X, Y, Z) is $Xx/a^2 + Yy/b^2 + Zz/c^2 = 1$. It is parallel to $Ax + By + Cz + 1 = 0$ iff $X/a^2 = kA$, $Y/b^2 = kB$, $Z/c^2 = kC$ for some k. But $1 = X^2/a^2 + Y^2/b^2 + Z^2/c^2 = k^2(a^2A^2 + b^2B^2 + c^2C^2) = k^2m^2$, so $k = \pm 1/m$. There are two values corresponding to two parallel tangent planes (one on either side of the ellipse). The equation of the tangent plane is $k(Ax + By + Cz) = 1$.

The distance of the origin from the plane $Ax + By + Cz + 1 = 0$ is $1/(A^2 + B^2 + C^2)^{1/2} = h$. The distance of the origin from the tangent plane $k(Ax + By + Cz) = 1$ is $h/|k| = hm$. So if $m \geq 1$, the plane $Ax + By + Cz + 1 = 0$ lies between the two tangent planes and hence intersects the ellipse. So in this case the minimum distance is zero. If $m < 1$, then the distance between the plane $Ax + By + Cz + 1 = 0$ and the nearer tangent plane is $h(1 - m)$ and that is the required shortest distance.

2nd Putnam 1939

Problem A1
Let C be the curve $y^2 = x^3$ (where x takes all non-negative real values). Let O be the origin, and A be the point where the gradient is 1. Find the length of the curve from O to A.

Solution
Ans: 8/27 (2√2 - 1). Trivial integration.

Problem A2
Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. Prove that the gradient at B is 4 times the gradient at A.

Solution
Trivial. [Take the point as (a,a^3). Write down the equation of the tangent. Write down its point of intersection with the curve: $(x^3 - a^3) = 3a^2(x - a)$. We know this has a repeated root x = a. The sum of the roots is zero, so the third root is x = - 2a. Finally, $3(-2a)^2 = 4$ times $3\ a^2$.]

Problem A3
The roots of $x^3 + a\ x^2 + b\ x + c = 0$ are α, β and γ. Find the cubic whose roots are α^3, β^3, g^3.

Solution
$x^3 + (a^3 - 3ab + 3c)\ x^2 + (b^3 - 3abc + 3c^2)x + c^3 = 0$.

A routine manipulation. Suppose the roots are α, β, γ. Then $\alpha + \beta + \gamma = - a$, $\alpha\beta + \beta\gamma + \gamma\alpha = b$, $\alpha\beta\gamma = - c$. So to get the coefficients of the desired polynomial we have to find the corresponding expressions in the cubes: $\alpha^3 + \beta^3 + \gamma^3$ etc. You obviously start with $(\alpha + \beta + \gamma)^3$ etc and then add additional terms to get the desired expressions.

Problem A4
Given 4 lines in Euclidean 3-space:

L_1: x = 1, y = 0; L_2: y = 1, z = 0; L_3: x = 0, z = 1; L_4: x = y, y = -6z.

Find the equations of the two lines which both meet all of the L_i.

Solution
A routine computation. Assume the line meets L_1 at (1,0,a) and L_2 at (b,1,0). Then it is (x - 1) = t(x - b), y = t(y - 1), (z - a) = tz. So it can only cut L_3 if 1/b = 1 - a, and L_4 if 6a = 6ab - 1. This gives a quadratic for a, which we can solve to get a = -1/2 or 1/3. Hence the possible lines are (1,0,-1/2) + t(-1/3,1,1/2) and (1,0,1/3) + t(1/2,1,-1/3).

Problem A5
Do either (1) or (2)
(1) x and y are functions of t. Solve x' = x + y - 3, y' = -2x + 3y + 1, given that x(0) = y(0) = 0.

(2) A weightless rod is hinged at O so that it can rotate without friction in a vertical plane. A mass m is attached to the end of the rod A, which is balanced vertically above O. At time t = 0, the rod moves away from the vertical with negligible initial angular velocity. Prove that the mass first reaches the position under O at $t = \sqrt{(OA/g)}\ \ln(1 + \sqrt{2})$.

Solution
(1) Differentiate the first equation, use the second equation to eliminate y', and then the (undifferentiated) first equation to eliminate y, giving:
$x'' - 4x' + 5x = 10$. Solving: $x = 2 + A e^{2t} \sin t + B e^{2t} \cos t$. But $x(0) = 0$, so $B = -2$. The first equation now gives $y = x' - x + 3 = 1 + (A + 2) e^{2t} \sin t + (A - 2) e^{2t} \cos t$. But $y(0) = 1$, so $A = 1$. The final solution is thus: $x = 2 + e^{2t} \sin t - 2 e^{2t} \cos t$; $y = 1 + 3 e^{2t} \sin t - e^{2t} \cos t$.

(2) Trivial, except for the integral, which is moderately hard, unless you happen to know it.

Let the angle the rod makes with the (upward) vertical be θ. Conservation of energy gives immediately: $1/2\ OA^2(d\theta/dt)^2 = OA.g(1 - \cos\theta)$.
Now for the first time in this exam we come up against something that is not completely obvious. How do we do the integral?

You need the half-angle formulae, eg $(1 - \cos\theta) = 2\sin^2\theta/2$. Now if you can remember the integral for $1/\sin z$ (eg $\ln \sin z - \ln(1 + \cos z)$, or equivalently $\ln(\text{cosec } z - \cot z)$, then you are home.

If not, use the half-angle formulae again: $\sin\theta/2 = 2 \sin\theta/4 \cos\theta/4$. Putting $c = \cos\theta/4$, we have to integrate $1/(c(1 - c^2))$. Expand using partial fractions and now the integral is just a sum of logs.

Problem A6
Do either (1) or (2):
(1) A circle radius r rolls around the inside of a circle radius 3r, so that a point on its circumference traces out a curvilinear triangle. Find the area inside this figure.

(2) A frictionless shell is fired from the ground with speed *v* at an unknown angle to the vertical. It hits a plane at a height *h*. Show that the gun must be sited within a radius $v/g\ (v^2 - 2gh)^{1/2}$ of the point directly below the point of impact.

Solution
(1) This is moderately difficult. It is not immediately obvious what coordinates to use (or at least, after meeting an integral I did not immediately recognize, I started worrying that there might be a better choice of coordinates), and it is not immediately obvious how to do the resulting integral.

Let C be the center of the large circle and let O be the initial point of contact between the two circles. Take O as the origin and OC as the x-axis, take the y-axis so that P the point of contact gets a positive y-coordinate just after rolling starts. The easiest parameter is to take angle $OCP = \theta$. Then it is not hard to see that $x/r = 3 - 2\cos\theta - \cos 2\theta$, $y/r = 2\sin\theta - \sin 2\theta$.

Evidently we need something like $\int y\,dx$. We need a little care on the limits of integration. Let A, B be the other vertices of the curvilinear triangle (A corresponding to $\theta = 2\pi/3$, B to $\theta = 4\pi/3$). Let X be the point where the curve AB cuts the x-axis and Y the point where the line AB cuts the x-axis. $\int_O^A$ gives the area under the curve OA, in other words the area OAX plus the area AXY. Then $\int_A^X$ gives minus the area AXY (because x is decreasing, so dx is negative). $\int_X^B$ gives minus area BXY (x increasing, but y negative), and $\int_B^O$ gives plus area BXY plus area OBX (x decreasing and y negative). So the entire integral $\int_{\theta=0}^{2\pi}$ gives the required area inside the curvilinear triangle OAB. In other words we need:
$2r^2 \int_{\theta=0}^{2\pi} (2\sin\theta - \sin 2\theta)(\sin\theta + \sin 2\theta)\,d\theta$.

This is the point at which we are likely to get stuck. Changing variable to $z = \cos\theta$ does not apparently help. The trick is to put things in terms of $\sin nx$ or $\cos nx$. We may remember that $\sin^2 z$

= (1 - cos 2z)/2, and cos(w +/- z) = cos w cos z -/+ sin w sin z, so that sin z sin 2z = (cos z - cos 3z)/2.

Expanding the integrand gives: $2 \sin^2\theta + \sin\theta \sin 2\theta - \sin^2 2\theta$. Using the two formulae above transforms this to: $1 - \cos 2\theta + (\cos\theta - \cos 3\theta)/2 - 1/2 + 1/2 \cos 4\theta$. The cos terms all integrate to zero and the constant term 1/2 integrates to π, so the final answer is $2\pi r^2$.

(2) There is a slight trap here. We may be tempted to argue that the extreme case is where the shell reaches the plane with zero vertical velocity. The horizontal velocity does not change during the trajectory, so taking u as the horizontal velocity and w as the vertical, we can write down immediately that $w^2 = 2gh$ (energy) and hence the radius $r = w/g\ (v^2 - 2gh)^{1/2}$, which is unfortunately wrong. Notice that it is smaller than the answer sought. The reason is that impact at zero vertical velocity is not the extreme case. We can usually do better by having the shell peak before the plane, so that it hits it on the way down - the extra time to travel horizontally outweighs the loss of horizontal velocity.

So we have to write down the equations: $r = tu$, $h = tw - gt^2/2$. Squaring to eliminate u, w in favor of v, gives:

$$g^2t^4/4 + (gh - v^2)t^2 + (h^2 + r^2) = 0. \quad (*)$$

For this to have a real root we require $(gh - v^2)^2 \geq g^2(h^2 + r^2)$ and hence $r \leq v/g\ (v^2 - 2gh)^{1/2}$.

We are not asked to prove that the plane can be hit from anywhere within this radius, so we could stop here. But $v^2 \geq 2gh > gh$, so (*) is a quadratic of the form $a z^2 - b z + c = 0$, with a, b, c positive. Hence if r satisfies the condition, it has two *positive* roots for z and (*) has two real (positive) roots for t. Having solved for t, we can then solve for the angle (or equivalently for u, w), which shows that we can hit the plane from anywhere inside the radius.

Problem A7

Do either (1) or (2):

(1) Let C_a be the curve $(y - a^2)^2 = x^2(a^2 - x^2)$. Find the curve which touches all C_a for $a > 0$. Sketch the solution and at least two of the C_a.

(2) Given that $(1 - hx)^{-1}(1 - kx)^{-1} = \sum_{i\geq 0} a_i x^i$, prove that $(1 + hkx)(1 - hkx)^{-1}(1 - h^2x)^{-1}(1 - k^2x)^{-1} = \sum_{i\geq 0} a_i^2 x^i$.

Solution

(1) It is not hard to see that C_1 is a figure 8 lying on its side with the double point at (1,0), vertical tangents at (1,1) and (-1,1) and horizontal tangents at $(1/\sqrt{2}, 1/2)$, $(1/\sqrt{2}, 3/2)$, $(-1/\sqrt{2},1/2)$, $(-1/\sqrt{2},3/2)$. C_a is obtained from C_1 by transforming (x,y) to (ax,a^2y) (so stretching by a factor a in the x-direction and a factor a^2 in the y-direction). So we get a sequence of ever-larger horizontal 8s centered ever-further up the y-axis. This suggests that the envelope is something like a parabola $y = k x^2$.

At this point it is distinctly helpful to know more classical differential geometry than today's undergraduate. The classical method for finding the envelope for a 1-parameter family of curves (which *usually* works) is to differentiate wrt the parameter. Thus we eliminate a from:

$$(y - a^2)^2 - x^2(a^2 - x^2) = 0, \text{ and } \quad 4a(y - a^2) + 2a x^2 = 0$$

giving $y = 3/4\ x^2$ (or the y-axis, which is presumably a spurious solution arising from the double points).

Alternatively, we might guess that that the envelope is a curve $y = k x^2$ for some k. The intersection of this with C_a is given by:
$(k^2 + 1)x^4 - (2k + 1) a^2x^2 + a^4 = 0$, or $x^2 = (2k + 1)a^2/(2k^2 + 2) +/- a^2(4k - 3)^{1/2}/(2k^2 + 2)$.

For this to be a tangent we need double roots and hence k = 3/4. It is now easy to check that this parabola meets C_a at $(2a/\sqrt{5},3a^2/5)$, $(-2a/\sqrt{5},3a^2/5)$ and to check that the gradients match.

(2) Fairly easy.

This is obviously not a general result (whereby we derive $\sum a_i^2x^i$ from $\sum a_ix^i$), so we need to evaluate the a_i. In fact, it is easily seen that $a_i = (h^{i+1} - k^{i+1})/(h - k)$. [For example, multiply the expansions of $(1 - hx)^{-1}$ and $(1 - kx)^{-1}$ to get $a_i = h^i + h^{i-1}k + ... + h k^{i-1} + k^i = (h^{i+1} - k^{i+1})/(h - k)$.]

Multiplying across to try to show that higher powers of $(1 - hkx)(1 - h^2x)(1 - k^2x) \sum a_i^2x^i$ have zero coefficients is a mistake (doable, but much algebra). It is better to evaluate $\sum a_i^2x^i$ directly. After substituting for a_i, we are going to get terms of the form $\sum z^i$ which evaluates immediately to $(1 - z)^{-1}$, giving the expression in the question in partial fraction form.

Problem B1
The points P(a,b) and Q(0,c) are on the curve $y/c = \cosh (x/c)$. The line through Q parallel to the normal at P cuts the x-axis at R. Prove that QR = b.

Solution
Trivial. [Let O be the origin. Then $OR/OQ = \sinh a/c$, so $QR^2 = c^2(1 + \sinh^2 a/c)$, so QR = b.]

Problem B2
Evaluate $\int_1^3 ((x - 1)(3 - x))^{-1/2} dx$ and $\int_1^{\inf} (e^{x+1} + e^{3-x})^{-1} dx$.

Solution
(1) π. Trivial. [Write $(x - 1)(3 - x) = 1 - (x - 2)^2$. So we have a standard $\sin^{-1}z$ integral.]

(2) $\pi/(4 e^2)$. Trivial. [Multiply top and bottom by e^x. Change variable to $y = e^{x-1}$. We now have a standard $\tan^{-1}z$ integral.]

Problem B3
Given $a_n = (n^2 + 1) 3^n$, find a recurrence relation $a_n + p a_{n+1} + q a_{n+2} + r a_{n+3} = 0$. Hence evaluate $\sum_{n\geq 0} a_n x^n$

Solution
We can solve formally to get the recurrence relation, but it is quicker to get there informally. We look for a relation between $b_n = a_n$, $b_{n+1} = a_{n+1}/3$, $b_{n+2} = a_{n+2}/9$, $b_{n+3} = a_{n+3}/27$, because that takes care of the powers of 3. So, ignoring the 3^n, we are looking at:

$n^2 + 1$
$n^2 + 2n + 2$
$n^2 + 4n + 5$
$n^2 + 6n + 10$

We try to get a linear combination of the first three which is constant. But that is easy: subtracting twice the second from the third gets rid of the n term, then adding the first gets rid of the n^2 term. So, $b_{n+2} - 2b_{n+1} + b_n = 2.3^n$. But $b_{n+3} - 2b_{n+2} + b_{n+1}$ has the *same* value, so subtracting:

$a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n = 0$, which is the required recurrence relation.

Let the power series sum to y. Then taking $y - 9x + 27x^2y - 27x^3y$ will give $a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n$ as the coefficient of x^{n+3}, so we need only worry about the early terms: $a_0 + (a_1 - 9a_0)x + (a_2 - 9a_1 + 27a_0)x^2 = (1 - 3x + 18x^2)$. Hence $y = (1 - 3x + 18x^2)/(1 - 9x + 27x^2 - 27x^3)$.

Using the ratio test, the original series evidently converges for $|x| < 1/3$, which may prompt us to notice that $1 - 9x + 27x^2 - 27x^3 = (1 - 3x)^3$.

That in turn may prompt us to try solving the problem backwards. We know that:

$1/(1 - z) = \sum z^n$; $1/(1 - z)^2 = \sum (n+1)z^n$; $1/(1 - z)^3 = \sum (n+1)(n+2)/2\ z^n$.

Hence $2/(1 - z)^3 - 3/(1 - z)^2 + 2/(1 - z) = \sum (n^2 + 1)z^n$. Replacing z by 3x gives $\sum a_n x^n = (1 - 3x + 18x^2)/(1 - 3x)^3$. Multiplying across by $(1 - 3x)^3$ now gives the required recurrence relation.

Problem B4
The *axis* of a parabola is its axis of symmetry and its *vertex* is its point of intersection with its axis. Find: the equation of the parabola which touches y = 0 at (1,0) and x = 0 at (0,2); the equation of its axis; and its vertex.

Solution
The general equation of the a parabola is: $(ax + by)^2 + cx + dy + e = 0$. Its intersection with y = 0 is given by $a^2x^2 + cx + e = 0$. This must have a double root, so $c = -2a^2$, $e = -a^2$. Considering the other tangent, we find: $d = -4b^2$, $e = 4b^2$. So (up to an irrelevant constant factor) we have: a = 2, b = 1, c = -8, d = -4, e = 4; or a = 2, b = -1, c = -8, d = -4, e = 4. But in the first case the equation can be written as $(2x + y - 2)^2 = 0$, which is a double line. It is debatable whether this qualifies as a parabola, but it would not normally be said to *touch* the points (1,0) and (0,2). So we are left with the parabola: $(2x - y)^2 - 8x - 4y + 4 = 0$.

We want to put this in the form $u = kv^2$. The line x + 2y = 0 is perpendicular to the line 2x - y = 0, so we change variables to X = 2x - y, Y = x + 2y, giving: $X^2 - 16/5\ Y - 12/5\ X + 4 = 0$, or $16/5\ (Y - 4/5) = (X - 6/5)^2$, which is the equation of a parabola with vertex X = 6/5, Y = 4/5, axis X = 6/5. Changing back to the original coordinates, x = (2X + Y)/5, y = (2Y - X)/5, the vertex is (16/25, 2/25) and the axis is 10x - 5y = 6.

Problem B5
Do either (1) or (2):
(1) Prove that $\int_1^k [x]\ f'(x)\ dx = [k]\ f(k) - \sum_1^{[k]} f(n)$, where k > 1, and [z] denotes the greatest integer ≤ z. Find a similar expression for: $\int_1^k [x^2]\ f'(x)\ dx$.

(2) A particle moves freely in a straight line except for a resistive force proportional to its speed. Its speed falls from 1,000 ft/s to 900 ft/s over 1,200 ft. Find the time taken to the nearest 0.01 s. [No calculators or log tables allowed!]

Solution
(1) [x] is constant over the interval [i, i+1) for i an integer, so we split the range of integration to get $\int_1^k = \int_{[k]}^k + \int_1^2 + \int_2^3 + ... + \int_{[k]-1}^{[k]}$. We can write down each of these integrals, collect terms and get the result.

The same idea works the second integral, except that we divide at $\sqrt{2}, \sqrt{3}, \sqrt{4}, ..., \sqrt{[k^2]}$, giving the result: $[k^2]\ f(k) - (\ f(1) + f(\sqrt{2}) + f(\sqrt{3}) + ... + f(\sqrt{[k^2]})\ )$.

(2) Easy. 1.26 s.

The equation of motion is $x'' = - k x'$. Integrating: $x' = k A e^{-kt}$. Integrating again, and putting $x(0) = 0$, $x = A(1 - e^{-kt})$. Suppose T is the required time. Then from the speed $e^{-kT} = 0.9$. $x(T) = 1200$, so $A = 12000$. The initial speed is 1000, so $k = 1/12$ and $T = -12 \ln 0.9$. The only slight snag is that in the exam calculators did not exist and log tables were not allowed. So we have to use the expansion $\ln(1 + x) = x - x^2/2 + x^3/3 - ...$, or more usefully: $- \ln(1 - x) = x + x^2/2 + x^3/3 + ...$, giving $T = 1.2 + 0.06 + 0.004 + 0.0003 + ...$ or 1.26 s.

Problem B6

Do either (1) or (2):

(1) f is continuous on the closed interval [a, b] and twice differentiable on the open interval (a, b). Given $x_0 \in (a, b)$, prove that we can find $\xi \in (a, b)$ such that $((f(x_0) - f(a))/(x_0 - a) - (f(b) - f(a))/(b - a))/(x_0 - b) = f''(\xi)/2$.

(2) AB and CD are identical uniform rods, each with mass *m* and length 2*a*. They are placed a distance *b* apart, so that ABCD is a rectangle. Calculate the gravitational attraction between them. What is the limiting value as *a* tends to zero?

Solution

(1) We obviously have to use the mean value theorem. So we need to construct a suitable auxiliary function to apply it to. It is usually easiest to apply the MVT in cases where the function has equal values at the two ends of the interval. So we want to find some function g, such that g' has equal values at two different points. Let the value of the expression given, $((f(x_0) - f(a))/(x_0 - a) - (f(b) - f(a))/(b - a))/(x_0 - b)$, be y_0. Then we are looking for $g''(x)$ to be something like $1/2\ f''(x) - y_0$.

Let us start by rearranging the expression for y_0 to give $f(x_0) = f(a) + (f(b) - f(a))(x_0 - a)/(b - a) + y_0(x_0 - a)(x_0 - b)$. After a little experimentation we may try looking at:

$$g(x) = f(x) - f(a) - (f(b) - f(a))(x - a)/(b - a) - y_0(x - a)(x - b).$$

We notice that $g(a) = 0$, $g(b) = 0$, $g(x_0) = 0$, $g'(x) = f'(x) - (f(b) - f(a))/(b - a) - y_0(2x - a - b)$, $g''(x) = f''(x) - 2 y_0$. At this point we should realize that we are home, because we have to show that we can find ξ such that $g''(\xi) = 0$. But the mean value theorem gives us a value in the interval (a, x_0) at which g' is zero and another in (x_0, b). Hence there must be a value between the two (and *a fortiori* in (a, b)) at which g'' is zero.

(2) Straightforward, apart from an awkward integral. Answer: $Gm^2(1 - (1 + 4a^2/b^2)^{1/2})/(2a^2)$, which tends to Gm^2/b^2 as the rods shorten to become point masses.

By symmetry the net force must be perpendicular to the rods, so we just calculate the perpendicular component. Take coordinates x along one rod and y along the other. Then we can write down immediately that the perpendicular component is:

$$Gm^2/(4a^2) \int (b^2 + (x - y)^2)^{-1} b (b^2 + (x - y)^2)^{-1/2} dx\, dy.$$

The integrand is of the form $(1 + z^2)^{-3/2}$. It helps to know that this integrates to $z/(1 + z^2)^{1/2}$ (as is easily checked). People used to mess around with trigonometric substitutions (eg $\tan \theta$ reduces it to $\cos \theta$, which integrates immediately to $\sin \theta$, which one has to remember to think of as $\tan \theta/\sec \theta$). But it was always easier simply to learn a large number of integrals by rote. Nowadays, of course, one tends to look them up or use Mathematica/Maple.

So integrating first by y, we get $Gm^2/(4a^2) \int dx/b \{ (x - 2a)/b (1 + (x - 2a)^2/b^2)^{-1/2} - x/b (1 + x^2/b^2)^{-1/2} \}$. This is much less horrible than it looks, because we are just faced with integrating $z/(1 + z^2)^{1/2}$ which is obviously $(1 + z^2)^{1/2}$. So we get the expression above. For the limit, just expand the square root as a power series $1 + 2a^2/b^2 - a^4/(4b^4) + ...$.

Problem B7

Do either (1) or (2):

(1) Let $a_i = \sum_{n=0}^{\infty} x^{3n+i}/(3n+i)!$ Prove that $a_0^3 + a_1^3 + a_2^3 - 3 a_0a_1a_2 = 1$.

(2) Let O be the origin, λ a positive real number, C be the conic $ax^2 + by^2 + cx + dy + e = 0$, and C_λ the conic $ax^2 + by^2 + \lambda cx + \lambda dy + \lambda^2 e = 0$. Given a point P and a non-zero real number k, define the transformation D(P,k) as follows. Take coordinates (x',y') with P as the origin. Then D(P,k) takes (x',y') to (kx',ky'). Show that D(O,λ) and D(A,$-\lambda$) both take C into C_λ, where A is the point $(-c\lambda/(a(1 + \lambda)), -d\lambda/(b(1 + \lambda)))$. Comment on the case $\lambda = 1$.

Solution

(1) Moderately hard, unless differentiating is a reflex, in which case it is easy.

The series are all absolutely convergent for all x, so we can carry out whatever operations we want. But it is not at all obvious what to do. Multiplying out the power series to get a complicated sum for the coefficient of x^n is offputting. Much scope for algebraic error, and no guarantee that the eventual simplification will be obvious. So we look for some trick. The a_i are closely related to the exponential series, eg $a_0 + a_1 + a_2 = e^x$, and the prevalence of 3 in the question may eventually suggest looking at ω, the cube root of 1. Indeed, $a_0 + \omega a_1 + \omega^2 a_2 = e^{\omega x}$. Since ω^2 is also a root, we also have $a_0 + \omega^2 a_1 + \omega a_2 = e^{\omega 2x}$. Remembering that $1 + \omega + \omega^2 = 0$ might prompt us to multiply these three expressions together, but it helps to remember the product of the three left hand sides: $(a_0 + a_1 + a_2)(a_0 + \omega a_1 + \omega^2 a_2)(a_0 + \omega^2 a_1 + \omega a_2) = a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2$. Otherwise, you are faced with multiplying this out the hard way: after collecting terms, you get $(a_0^3 + a_1^3 + a_2^3)$, then 6 expressions of the type $a_0^2(1 + \omega + \omega^2)$, which are all zero, and $3a_0a_1a_2(\omega + \omega^2)$, which is $-3a_0a_1a_2$.

A more general approach, which is more likely to work, is to differentiate the expression $a_0^3 + a_1^3 + a_2^3 - 3 a_0a_1a_2$. Provided you notice that $a_0' = a_2$ etc, this gives the result almost immediately (the derivative is zero, so the expression must be constant, but its value for x = 0 is 1).

(2) Trivial. [D(O,λ) takes (x,y) to (λx,λy). So if (x,y) satisfies the equation for C, just check that (λx,λy) satisfies the equation for C_λ. Similarly, D(A,$-\lambda$) takes (x,y) to $(-\lambda x - \lambda c/a, -\lambda y - \lambda d/b)$. Again, just check by substituting this into the equation for C_λ and using the fact that (x,y) satisfies the equation for C. If $\lambda = 1$, then $C_\lambda = C$, D(O,λ) is the identity transformation, and D(A,$-\lambda$) the central symmetry.]

3rd Putnam 1940

Problem A1
p(x) is a polynomial with integer coefficients. For some positive integer c, none of p(1), p(2), ... , p(c) are divisible by c. Prove that p(b) is not zero for any integer b.

Solution
Suppose p(b) = 0. Then p(x) = (x - b)q(x), where q is a polynomial with integer coefficients. Put b = cd + r, where $1 \le r \le c$ (note that this is different from the conventional $0 \le r < c$, but still possible because 1, 2, ... c are a complete set of residues mod c). Then p(r) = p(b - cd) = - cd q(r) which is divisible by c. Contradiction.

Problem A2
y = f(x) is continuous with continuous derivative. The arc PQ is concave to the chord PQ. X is a point on the arc PQ for which PX + XQ is a maximum. Prove that XP and XQ are equally inclined to the tangent at X.

Solution
Take s to be the arc-length PX and z to be PX + XQ. Suppose the tangent is WXY, so that angle WXP = θ and angle YXQ = φ. Then to first order if we vary X, the change $\delta z = \delta s (\cos\theta - \cos\varphi)$, which is can be made positive (by choosing the sign of δs appropriately), thus contradicting the maximality of X, unless θ = φ.

The official solution. If X is a point of S such that PX + XQ ≥ PY + YQ for all points Y of S, then we can easily show that S lies in the half-plane bounded by external angle bisector of PXQ. If X lies on PQ, then S must be a subset of the segment PQ and the result is trivial. So assume it does not. Now reflect Q in the bisector to get Q', with PXQ' a straight line. Then if Y is any point in the same half-plane as Q', we have YQ' < YQ and hence PY + YQ > PY + YQ' ≥ PQ' = PX + XQ' = PX + XQ, so Y is not in S.

This argument depends upon X achieving a global maximum. The original wording of the question, "a point ... for which ... is a maximum" (which on this point I quoted exactly), is somewhat ambiguous. Does it mean a local or a global maximum? If it means a local, then we have to take S to be a small arc for which the maximum is global. Then that arc lies on one side of the line. Since it also has a point in common (X) and is differentiable, the line must be the tangent to the arc at X.

This solution is somewhat harder and less obvious (unless you have seen it before), so I prefer the simpler solution.

Problem A3
α is a fixed real number. Find all functions f: R → R (where R is the reals) which are continuous, have a continuous derivative, and satisfy $\int_b^y f^\alpha(x)\,dx = \left(\int_b^y f(x)\,dx\right)^\alpha$ for all y and some b.

Solution
Straightforward to get the basic idea, but care is needed with the details. It is quite hard to get the answer exactly right (the official solution gives a spurious solution for the case (2) - overlooking the problem with the integration limits).

Answer:
(1) No solutions for α = 0;
(2) No solutions for α < 0;
(3) Any continuous f with continuous derivative is a solution for α = 1;

(4) For $\alpha > 0$ and not of the form p/q, with p and q odd positive integers, $f = A e^{kx}$, where A is any real, and k is the positive real value of $\alpha^{1/(\alpha -1)}$;
(5) For $\alpha = p/q$, with p and q odd positive integers (but not both 1), $f = A e^{kx}$ or $A e^{-kx}$, where A is any real, and k is the positive real value of $\alpha^{1/(\alpha -1)}$.

Case (3) is obvious. Case (1) is almost obvious: if $\alpha = 0$, then the lhs varies with y (= y - b), but the rhs does not (= 1), so there are no solutions. Assume now that α is not 0 or 1.

Differentiate wrt y and put $g(y) = \int_b^y f(x)\, dx$. Then f = g' and we have $f^\alpha = \alpha\, g^{\alpha -1}\, f$. So f = k g, where $k = \alpha^{1/(\alpha -1)}$. Since f = g', we can integrate immediately to get $f(x) = A e^{kx}$ (*).

However, we have to consider how many real values k can have. If $\alpha > 0$, then k certainly has a positive value, but we can also take the corresponding negative value if $1/(\alpha - 1)$ involves an even root, in other words if $\alpha = p/q$ with p and q both odd. Finally, (*) is clearly necessary, but not necessarily sufficient, so we have to subsitute (*) back into the original equation. For $k > 0$, we find the solution works provided $b = -\infty$. If $k < 0$, then it works with $b = \infty$. This gives cases (4) and (5). [Note, however, that we can only allow A negative for α not of the form p/q with p and q odd, provided we are content for both sides of the original equation to have complex values (even though f is real valued).]

For $\alpha < 0$, all values of $\alpha^{1/(\alpha -1)}$ are complex unless $\alpha = -$ p/q with p an even positive integer and q an odd positive integer. If α has that form, then we may take $k = -1/|\alpha|^{1/(|\alpha|+1)}$. But now there is a problem with b. Taking A = 1 for simplicity, the $f^\alpha(x) = e^{|\alpha k|x}$, so the lhs = const $(e^{|\alpha k|y} - e^{|\alpha k|b})$ and the rhs = const $/(e^{-|k|y} - e^{-|k|b})^{|\alpha|}$. The constants are the same, but we need $b = -\infty$ to get rid of the b term on the lhs and $b = \infty$ to get rid of the term on the rhs. So there are no solutions in case (2).

Problem A4
p is a positive constant. Let R is the curve $y^2 = 4px$. Let S be the mirror image of R in the y-axis ($y^2 = - 4px$). R remains fixed and S rolls around it without slipping. O is the point of S initially at the origin. Find the equation for the locus of O as S rolls.

Solution
Answer: $x(x^2 + y^2) + 2p\, y^2 = 0$.

Take the point of contact as (X, Y), so $Y^2 = 4pX$. The tangent at to R at this point has gradient 2p/Y, and hence has equation (y - Y) = 2p/Y (x - X). The perpendicular to the tangent through the origin has equation x = - 2p/Y y. If (x, y) is their point of intersection, then (2x, 2y) is the point O (since S in its new position is the reflection of R in the tangent).

Solving for x, y: $y = Y^3/(2(Y^2 + 4p^2))$, $x = -2p/Y = -pY/(Y^2 + 4p^2)$. So the point O is $(-2pY^2/(Y^2 + 4p^2), Y^3/(Y^2 + 4p^2))$ (*). Using Y = -2py/x, we get $x = -2py^2/(x^2 + y^2)$, or $x(x^2 + y^2) + 2p\, y^2 = 0$ (**). We have shown that the locus is given by (*) and that all points on (*) are on (**). However, we must check that (**) does not include additional points. Writing (**) as $y^2 = - x^3/(x + 2p)$, shows that for each value of x in the range (-2p, 0) there are exactly two possible values of y, and the only other point on (**) is the origin. Inspection shows that the same is true for (*), so the two expressions are equivalent.

Problem A5
Prove that the set of points satisfying $x^4 - x^2 = y^4 - y^2 = z^4 - z^2$ is the union of 4 straight lines and 6 ellipses.

Solution
$x^4 - x^2 = (x^2 - 1/2)^2 - 1/4$, so $x^4 - x^2 = y^4 - y^2$ is equivalent to $(x^2 - 1/2)^2 = (y^2 - 1/2)^2$ and hence to $x^2 - 1/2 = +/- (y^2 - 1/2)$, which is equivalent to $x = y$, or $x = -y$, or $x^2 + y^2 = 1$. Similarly, for $y^4 - y^2 = z^4 - z^2$. So the equation given is equivalent to: (1) $x = y$ and $y = z$, or (2) $x = y$ and $y = -z$, or (3) $x = -y$ and $y = z$, or (4) $x = -y$ and $y = -z$, or (5) $x = y$ and $y^2 + z^2 = 1$, or (6) $x = -y$ and $y^2 + z^2 = 1$, or (7) $x^2 + y^2 = 1$ and $y = z$, or (8) $x^2 + y^2 = 1$ and $y = -z$, or (9) $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

Clearly (1) - (4) are straight lines. (5) is the intersection of a plane and a cylinder, which is an ellipse. Similarly (6), (7) and (8). (9) is slightly harder to see. If one's visualization is good, then one can see that the intersection of two cylinders with the same radius and axes intersecting at right angles is two perpendicular ellipses with a common minor axis. Otherwise, subtracting the two equations we see that $x = z$ or $-z$ and the intersection is also given by the intersection of a cylinder with two planes.

Problem A6
$p(x)$ is a polynomial with real coefficients and derivative $r(x) = p'(x)$. For some positive integers a, b, $r^a(x)$ divides $p^b(x)$. Prove that for some real numbers A and α and for some integer n, we have $p(x) = A(x - \alpha)^n$.

Solution
Write $p(x) = A \prod (x - \alpha_i)^n{}_i$.
Then $r(x) = p(x) \sum n_i/(x - \alpha_i) = (A \prod (x - \alpha_i)^n{}_i{}^{-1}) q(x)$, where no α_i is a root of $q(x)$. This is easily seen, because $q(x)$ is a sum of terms, all but one of which has a factor $(x - \alpha_i)$. But $q(x)$ divides $p(x)^b$ which has no roots except the α_i. Hence $q(x)$ must be a constant. But now the degree of $r(x)$ is wrong unless there is just one α_i.

Problem A7
a_i and b_i are real, and $\sum_1^\infty a_i^2$ and $\sum_1^\infty b_i^2$ converge. Prove that $\sum_1^\infty (a_i - b_i)^p$ converges for $p \geq 2$.

Solution
Notice first that it is sufficient to prove the result for $p = 2$. For that is equivalent to the statement that $\sum |a_i - b_i|^2$ converges. Hence for sufficiently large i, $|a_i - b_i| < 1$, and hence $|a_i - b_i|^p \leq |a_i - b_i|^2$. So $\sum (a_i - b_i)^p$ is absolutely convergent and hence convergent.

$(a_i - b_i)^2 = a_i^2 - 2a_ib_i + b_i^2$. The only tricky part is the middle term. It may be positive, so we cannot simply argue that $0 \leq (a_i - b_i)^2 \leq a_i^2 + b_i^2$. However, it is true that $0 \leq (a_i - b_i)^2 = 2a_i^2 + 2b_i^2 - (a_i + b_i)^2 \leq 2a_i^2 + 2b_i^2$. That suffices, since $\sum a_i^2$ and $\sum b_i^2$ are absolutely convergent, hence also $\sum (2a_i^2 + 2b_i^2)$.

Problem A8
Show that the area of the triangle bounded by the lines $a_ix + b_iy + c_i = 0$ $(i = 1, 2, 3)$ is $\Delta^2/|2(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)|$, where Δ is the 3 x 3 determinant with columns a_i, b_i, c_i.

Solution
Fairly easy if you remember some formulae for determinants (which people did in those days). Of course, you can just slog through the expression (*) below in terms of a_i, b_i, c_i. That is doable, but completely mindless, the only required skill is doing elementary algebra fast without mistakes. I suppose I am fairly out of sympathy with the rather common Putnam style of problem where the basic idea is obvious, but you have to be skilful at evaluating integrals, determinants etc, often using tricks.

Take A_i be the cofactor of a_i in Δ. Similarly, B_i and C_i. [So, for example, $A_1 = b_2c_3 - b_3c_2$, $A_2 = b_3c_1 - b_1c_3$, $A_3 = b_1c_2 - b_2c_1$.]

The lines $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ intersect at $(A_1/C_1, B_1/C_1)$. Similarly, the other two points of intersection are $(A_2/C_2, B_2/C_2)$ and $(A_3/C_3, B_3/C_3)$.

The area of the triangle is therefore (the absolute value of) the determinant K with rows A_1/C_1, B_1/C_1, 1; A_2/C_2, B_2/C_2, 1; A_3/C_3, B_3/C_3, 1. (*) [For example, take a z-coordinate perpendicular to the plane, and take the cross product of the vectors along two sides.] But $K\,|C_1C_2C_3|$ is the determinant whose elements are the cofactors of the original determinant. This has value equal to Δ^2. For example, on multiplying it by Δ, we get Δ down the diagonal and zeros elsewhere, and hence Δ^3.

Problem B1
A stone is thrown from the ground with speed v at an angle θ to the horizontal. There is no friction and the ground is flat. Find the total distance it travels before hitting the ground. Show that the distance is greatest when $\sin\theta \ln(\sec\theta + \tan\theta) = 1$.

Solution
Take coordinates (time and distance) zeroed on the peak of the trajectory. After time t, the stone travels a distance $x = v\,t\cos\theta$ horizontally and a distance $1/2\,g\,t^2$ vertically. So the trajectory is a parabola $2y = k\,x^2$, where $k = g/(v\cos\theta)^2$. The stone is on the ground at $t = +/- v/g \sin\theta$, at a horizontal distance +/- a from the peak, where $a = v^2/g \sin\theta\cos\theta$.

The length of the parabola $y = k/2\,x^2$ between x = -a and x = a is $2\int_0^a (1 + k^2x^2)^{1/2}\,dx$.

To do the integral it helps to remember that $1 + \sinh^2 z = \cosh^2 z$. So substituting $kx = \sinh z$, will essentially give us the integral of $\cosh^2 z$. That is doable, using the analog of the double angle formulae. So setting $I = \int (1 + k^2x^2)^{1/2}\,dx$, and substituting $x = \sinh z$, we have $I = 1/k \int \cosh^2 z\,dz = 1/(2k) \int (\cosh 2z + 1)\,dz = 1/(4k) \sinh 2z + 1/(2k)\,z = 1/(2k) \sinh z \cosh z + 1/2\,z = x/2\,(1 + k^2x^2)^{1/2} + 1/(2k) \sinh^{-1}(kx)$. We have $k = g/(v\cos\theta)^2$, and $a = v^2/g \sin\theta\cos\theta$, so $ka = \tan\theta$. So the required path length is $p(\theta) = a(1 + k^2a^2)^{1/2} + 1/k \sinh^{-1}a = v^2/g\,(\sin\theta + \cos^2\theta \sinh^{-1}\tan\theta)$.

To find the maximum, we differentiate, getting $p'(\theta) = \cos\theta - 2\cos\theta\sin\theta \sinh^{-1}\tan\theta + \cos^2\theta\,(1 + \tan^2\theta)^{-1/2} \sec^2\theta = 2\cos\theta(1 - \sin\theta \sinh^{-1}\tan\theta)$. In the range $[0, \pi/2]$, $\tan\theta$ is monotone increasing. $\sinh^{-1}z$ is strictly monotone increasing for positive z, so $\sinh^{-1}\tan\theta$ is strictly monotone increasing on $(0, \pi/2)$. Indeed it evidently tends to ∞ as θ tends to $\pi/2$. Hence $(1 - \sin\theta \sinh^{-1}\tan\theta)$ is strictly monotone decreasing on $(0, \pi/2)$ and crosses zero once. $\cos\theta$ is monotone decreasing and positive, so $p'(\theta)$ is strictly monotone decreasing and crosses zero once. Hence $p(\theta)$ has a single maximum on $[0, \pi/2]$, which is achieved for the value φ in $(0, \pi/2)$ for which $(1 - \sin\varphi \sinh^{-1}\tan\varphi) = 0$. Rearranging, $\sinh(1/\sin\varphi) = \tan\varphi$. Squaring, adding 1, and taking the square root: $\cosh(1/\sin\varphi) = \sec\varphi$. Adding the last two equations: $e^{1/\sin\varphi} = \tan\varphi + \sec\varphi$, or $1/\sin\varphi = \ln(\tan\varphi + \sec\varphi)$, or $\sin\varphi \ln(\tan\varphi + \sec\varphi) = 1$.

Problem B2
C_1, C_2 are cylindrical surfaces with radii r_1, r_2 respectively. The axes of the two surfaces intersect at right angles and $r_1 > r_2$. Let S be the area of C_1 which is enclosed within C_2. Prove that $S = 8r_2^2A = 8r_1^2C - 8(r_1^2 - r_2^2)B$, where $A = \int_0^1 (1 - x^2)^{1/2}(1 - k^2x^2)^{-1/2}\,dx$, $B = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{-1/2}\,dx$, and $C = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{1/2}\,dx$, and $k = r_2/r_1$.

Solution
It is hard to see the point of the second half [$8r_2^2A = 8r_1^2C - 8(r_1^2 - r_2^2)B$], which is trivial. Multiply

top and bottom of the integrand in A by $(1 - x^2)^{1/2}$, so that we get $(1 - x^2)$ on the top. Then note that $k^2(1 - x^2) = 1 - k^2x^2 + (k^2 - 1)$.

For the first half, the part of C_1 enclosed inside C_2 comprises two bent ovals, one at each end. It is tempting to think that when rolled flat, each piece is an ellipse. If that were true, then the problem would be trivial - the semi-axes are r_2 and $r_1\sin^{-1}k$, so the total area would be $2\pi\, r_1r_2\sin^{-1}k$. But conics only remain conics when projected onto flat surfaces and here we are projecting onto a curved surface.

So we have to use integration. Take C_1 to be $x^2 + y^2 = r_1^2$, and C_2 to be $y^2 + z^2 = r_2^2$. We calculate the area of the quarter of the piece with $x > 0$ having $y, z > 0$. We may divide it into strips parallel to the z-axis. Take the angle θ to be the axial angle - the angle by which the strip has to be rotated about the z-axis from the $y = 0$ position (keeping in the surface C_1). [Points on the strip have the same x and y coordinates, but varying z-coordinates.] The strip has y-coordinate $r_1\sin\theta$ and hence length $(r_2^2 - r_1^2\sin^2\theta)^{1/2}$ and width $r_1d\theta$. Thus the area of the quarter-oval is $\int_0^{\sin\text{-}1k} (r_2^2 - r_1^2\sin^2\theta)^{1/2}\, r_1\, d\theta$. Put $t = 1/k \sin\theta$, and the integral becomes $r_2^2 \int_0^1 (1 - t^2)^{1/2} (1 - k^2t^2)^{-1/2}\, dt$. Hence the complete oval is 4 times this, and the total area 8 times.

Comments. The original question was badly worded, because at first sight it appeared to be about volumes, not areas, which may have confused some. It is important to get a clear picture of the geometry - the easiest way to do this is to roll a piece of paper into a cylinder.

Problem B3
Let p be a positive real, let S be the parabola $y^2 = 4px$, and let P be a point with coordinates (a, b). Show that there are 1, 2 or 3 normals from P to S according as $4(2p - a)^2 + 27\, pb^2 >, =$ or < 0.

Solution
The general point on the parabola is $(pt^2, 2pt)$. The slope of the tangent is $2p/y = 1/t$, and so the slope of the normal is $-t$. Hence the equation of the normal is $(y - 2pt) = -t(x - pt^2)$. This passes through (a, b) iff $pt^3 + (2p - a)t - b = 0$ (*).

This is a cubic, so it has 1, 2 or 3 real roots. We have to decide which. If $(2p - a) > 0$, then $pt^3 + (2p - a)t$ is strictly increasing and takes all values in $(-\infty, \infty)$, so (*) has just one real root. The same is true if $(2p - a) = 0$, unless $b = 0$, in which case there are three coincident roots.

If $(2p - a) < 0$, then $pt^3 + (2p - a)t$ has a maximum and a minimum. Differentiating, we find that these are at $t = \pm\sqrt{(\ -(2p - a)/(3p)\ )}$, with values $\pm\, 2(2p - a)/3\ \sqrt{(\ -(2p - a)/(3p)\ )}$. So there are 1, 2 or 3 real roots according as $b^2 >, =$ or $< 4(2p - a)^2/9 - (2p - a)/3p$, which is the condition in the question. Note that if $(2p - a) \geq 0$, then this expression is certainly > 0, so the same rule applies.

Problem B4
Let S be the surface $ax^2 + by^2 + cz^2 = 1$ (a, b, c all non-zero), and let K be the sphere $x^2 + y^2 + z^2 = 1/a + 1/b + 1/c$ (known as the *director* sphere). Prove that if a point P lies on 3 mutually perpendicular planes, each of which is tangent to S, then P lies on K.

Solution
Let $f(x, y, z) = ax^2 + by^2 + cz^2$. Then the normal vector is the vector grad f, so the tangent plane at (u, v, w) is $a.u.x + b.v.y + c.w.z = 1$.

Note that this does not pass through the origin. The general plane not through the origin has equation $\mathbf{p}.(\mathbf{x} - \lambda\mathbf{p}) = 0$ or $\mathbf{p}.\mathbf{x} = \lambda$, where $\mathbf{x}$ is the vector (x, y, z) representing a general point on the plane, $\mathbf{p} = (p, q, r)$ is a unit vector normal to the plane, and $\lambda > 0$ is the distance of the plane from

the origin. If this is a tangent plane at some point of the quadric, then $a(p/\lambda a)^2 + b(q/\lambda b)^2 + c(r/\lambda c)^2 = 1$, or $p^2/a + q^2/b + r^2/c = \lambda^2$.

So suppose P is the point of intersection of three perpendicular tangent planes $p_ix + q_iy + r_iz = \lambda_i$, i = 1, 2, 3, where $\mathbf{p}_i$ are orthonormal vectors. The squared distance of P from the origin is $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = (p_1^2 + p_2^2 + p_3^2)/a + (q_1^2 + q_2^2 + q_3^2)/b + (r_1^2 + r_2^2 + r_3^2)/c = 1/a + 1/b + 1/c$, since $\mathbf{p}_i$ are orthonormal. [This is the key trick: if the rows of a matrix are orthonormal vectors, then the matrix is orthogonal and hence its *columns* are also orthonormal vectors.].

Problem B5
Find all rational triples (a, b, c) for which a, b, c are the roots of $x^3 + ax^2 + bx + c = 0$.

Solution
Answer: (0, 0, 0); (1, -1, -1), (1, -2, 0).

We require (1) a + b + c = -a, (2) ab + bc + ca = b, and (3) abc = -c.
From (3), either c = 0, or ab = -1. If c = 0, then (1) becomes b = -2a, and (2) becomes b(a - 1) = 0. Hence either a = b = 0, or a = 1, b = -2.
So assume $c \neq 0$, and ab = -1. (1) becomes c = - b - 2a. Substituting in (2), we get: $-1 - (2a + b)(a + b) = b$, so $-a^2 - 2a^4 + 3a^2 - 1 = -a$, or $2a^4 - 2a^2 - a + 1 = 0$. So a = 1, or $2a^3 + 2a^2 - 1 = 0$ (*). The first possibility gives a = 1, b = -1, c = -1. Suppose a = m/n is a root of (*) with m, n relatively prime integers. Then $2m^3 + 2m^2n - n^3 = 0$. So any prime factor of n must divide 2 and any prime factor of m must divide 1. Hence the only possibilities are a = 1, -1, 1/2, -1/2, and we easily check that these are not solutions. So (*) has no rational roots.

Problem B6
The n x n matrix (m_{ij}) is defined as $m_{ij} = a_ia_j$ for $i \neq j$, and $a_i^2 + k$ for i = j. Show that $\det(m_{ij})$ is divisible by k^{n-1} and find its other factor.

Solution
Answer: $\det(m_{ij}) = k^{n-1}(k + \Sigma a_i^2)$.

Induction on n. Clearly true for n = 1.
Expanding by the first row we get $k.\ k^{n-2}(k + \Sigma_{i>1}a_i^2) + \det(m'_{ij})$, where m'_{ij} is the same as m_{ij} except that $m'_{11} = a_1^2$. Subtracting appropriate multiples of the first row from the others we zero all the elements outside the first row except those on the diagonal, which become k. Hence $\det(m'_{ij}) = k^{n-2}a_1^2$.

Problem B7
Given n > 8, let $a = \sqrt{n}$ and $b = \sqrt{(n+1)}$. Which is greater a^b or b^a?

Solution
Answer: a^b is greater.

$a^b = e^{b \ln a}$ and $b^a = e^{a \ln b}$. So we have to decide which of b ln a and a ln b is greater, or, equivalently, which of (ln a)/a and (ln b)/b is greater. The latter is clearly more promising. So set f(x) = (ln x)/x. Then $f'(x) = 1/x^2 - (\ln x)/x^2$ which is negative for x > e. Obviously b > a, so provided a > e, (ln a)/a > (ln b)/b and hence b ln a > a ln b and $a^b > b^a$. But $e^2 < 9$, so the result is certainly true for $n \geq 9$.

4th Putnam 1941

Problem A1
Prove that $(a - x)^6 - 3a(a - x)^5 + 5/2\ a^2(a - x)^4 - 1/2\ a^4(a - x)^2 < 0$ for $0 < x < a$.

Solution
Change variables to $t = 1 - x/a$ and the polynomial becomes $a^6(t^6 - 3t^5 + 5/2\ t^4 - 1/2\ t^2)$. This obviously has a factor t^2, and almost obviously $(t - 1)$. Dividing these out, we see that the resulting cubic has another factor $(t - 1)$. So we can write the original as $a^6t^2(t - 1)^2(t(t - 1) - 1/2)$, which evidently has the same sign as $t(t - 1) - 1/2$. But that is clearly negative for t between 0 and 1.

Problem A2
Define $f(x) = \int_0^x \sum_{i=0}^{n-1} (x - t)^i / i!\ dt$. Find the nth derivative $f^{(n)}(x)$.

Solution
Note that x appears both in the integrand and in the limits, so a little care is needed. Write $g_r(x, t) = \sum_{i=0}^{r-1} (x - t)^i / i!$ so that $f(x) = \int_0^x g_n(x, t)\ e^{nt}\ dt$. By definition $f'(x) = \lim_{\delta x \to 0} (\int_0^{x+\delta x} g_n(x+\delta x, t)\ e^{nt}\ dt - \int_0^x g_n(x, t)\ e^{nt}\ dt) / \delta x = e^{nx}\ g_n(x, x) + \int_0^x g_n'(x,t)\ e^{nt}\ dt$, where the ' denotes the partial derivative wrt the first variable. But $g_n(x, x) = 1$, and $g_r'(x, t) = g_{r-1}(x, t)$, so $f'(x) = e^{nx} + \int_0^x g_{n-1}(x, t)\ e^{nt}\ dt$. Hence by an easy induction $f^{(n)}(x) = e^{nx}(1 + n + n^2 + ... + n^{n-1})$.

Problem A3
A circle radius a rolls in the plane along the x-axis the envelope of a diameter is the curve C. Show that we can find a point on the circumference of a circle radius a/2, also rolling along the x-axis, which traces out the curve C.

Solution
Consider a circle radius a/2 with center initially at (0,a/2) rolling along the x-axis. After rolling through an angle θ, the point initially at (0,a) is at a/2 sin θ, a/2 cos θ relative to the center and hence at P (aθ/2 + (a/2) sin θ, a/2 + (a/2) cos θ). The tangent at P is $(y - (a/2)(1 + \cos\theta)/(x - (a/2)(\theta + \sin\theta) = (-\sin\theta)/(1 + \cos\theta)$, or $x \sin\theta + y(1 - \cos\theta) = (a/2)(\theta \sin\theta + 2\cos\theta + 2)$.

If we put $\theta = 2\varphi$, then $\sin\theta = 2 \sin\varphi \cos\varphi$, $1 + \cos\theta = 2\cos^2\varphi$, so the tangent at P has equation, $x \sin\varphi + y \cos\varphi = a(\varphi \sin\varphi + \cos\varphi)$.

Now consider the circle radius a with center initially at (0,a). When it has rolled through an angle φ, its center is at (aφ, a), so the diameter which is initially horizontal lies on the line $(y-a)/(x-a\varphi) = -\tan\varphi$, or $x \sin\varphi + y \cos\varphi = a(\varphi \sin\varphi + \cos\varphi)$. In other words, the diameter is tangent to the point P of the curve traced out by the point on the circumference of the circle radius a/2. Hence the envelope of the diameter is that curve.

Problem A4
The real polynomial $x^3 + px^2 + qx + r$ has real roots $a \le b \le c$. Prove that f ' has a root in the interval $[b/2 + c/2, b/3 + 2c/3]$. What can we say about f if the root is at one of the endpoints?

Solution
$p(x) = (x - a)(x - b)(x - c)$, so $p'(x) = (x - a)(x - b) + (x - b)(x - c) + (x - a)(x - c)$.

We can write $p'(x) = (x - b)(x - c) + (x - a)(2x - b - c)$, so $p'(b/2 + c/2) = -1/4\ (c - b)^2 \le 0$, with equality iff b = c.

$p'(b/3 + 2c/3) = -2/9 (c - b)^2 + (x - a) 1/3 (c - b) = 1/3 (c - b) (b - a) \geq 0$ with equality iff a = b or b = c.

If b = c, then b is a repeated root and p'(b) = 0. If a = b, then p'(b/3 + 2c/3) = 0. Otherwise, p'(x) is negative at b/2 + c/2 and positive at b/3 + 2c/3, so it has a zero in the interior of the interval.

Problem A6
f is defined for the non-negative reals and takes positive real values. The centroid of the area lying under the curve y = f(x) between x = 0 and x = a has x-coordinate g(a). Prove that for some positive constant k, $f(x) = k\, g'(x)/(x - g(x))^2\, e^{\int 1/(t - g(t))\, dt}$.

Problem A7
Do either (1) or (2):

(1) *Do either (1) or (2):*
(1) Let A be the 3 x 3 matrix

$$\begin{matrix} 1+x^2-y^2-z^2 & 2(xy+z) & 2(zx-y) \\ 2(xy-z) & 1+y^2-z^2-x^2 & 2(yz+x) \\ 2(zx+y) & 2(yz-x) & 1+z^2-x^2-y^2 \end{matrix}$$

Show that $\det A = (1 + x^2 + y^2 + z^2)^3$.

(2) A solid is formed by rotating about the x-axis the first quadrant of the ellipse $x^2/a^2 + y^2b^2 = 1$. Prove that this solid can rest in stable equilibrium on its vertex (corresponding to x = a, y = 0 on the ellipse) iff $a/b \leq \sqrt{(8/5)}$.

Solution
(1) subtract z times row 2 from row 1 and add y times row 3 to row 1. After taking out the common factor $1+x^2+y^2+z^2$ from row 1 we get:

$$\begin{matrix} 1 & z & -y \\ 2(xy-z) & 1+y^2-z^2-x^2 & 2(yz+x) \\ 2(zx+y) & 2(yz-x) & 1+z^2-x^2-y^2 \end{matrix}$$

Subtract z times col 1 from col 2 and add y times col 1 to col 3. We get:

$$\begin{matrix} 1 & 0 & 0 \\ 2(xy-z) & 1+y^2+z^2-x^2-2xyz & 2x(1+y^2) \\ 2(zx+y) & -2x(1+z^2) & 1+z^2-x^2+y^2+2xyz \end{matrix}$$

Multiplying this out, we get $(1-x^2+y^2+z^2)^2 - 4x^2y^2z^2 + 4x^2(1+y^2+z^2+y^2z^2) = (1+x^2+y^2+z^2)^2$. Hence with the additional factor we took out, we get the result.

(2) We first have to find the position of the centre of mass on the axis. The moment about the y-axis of the solid is $\int_0^a \pi y^2\, x\, dx = \pi b^2 \int_0^a (x - x^3/a^2)\, dx = \pi b^2 a^2/4$. The volume is $\int_0^a \pi y^2\, dx = \pi b^2 \int_0^a (1 - x^2/a^2)\, dx = 2/3\, \pi b^2 a$. Hence the centre of mass is a distance 3a/8 from the flat surface or 5a/8 from the point of contact.

Now suppose the point of contact is at (a cos t, b sin t). The tangent has gradient - b/a cot t, so the normal has gradient a/b tan t. So the equation of the normal is y - b sint = a/b tan t (x - a cos t). This meets the x-axis at $a(1 - b^2/a^2) \cos t$. For stability we want this to be closer to the origin than the

centre of mass, in other words we want $(1 - b^2/a^2) \cos t < 3/8$. The point of contact is at $\cos t = 1$, so we require $(1 - b^2/a^2) < 3/8$ or $b/a > \sqrt{(5/8)}$.

Problem B1
A particle moves in the plane so that its angular velocity about the point (1, 0) equals minus its angular velocity about the point (-1, 0). Show that its trajectory satisfies the differential equation $y' x(x^2 + y^2 - 1) = y(x^2 + y^2 + 1)$. Verify that this has as solutions the rectangular hyperbolae with center at the origin and passing through (±1, 0).

Solution
The angular momentum (for unit mass) about (1, 0) is $(x - 1)\, dy/dt - y\, dx/dt$. Hence the angular velocity is $((x - 1)\, dy/dt - y\, dx/dt)/((x - 1)^2 + y^2)$. Similarly, the angular velocity about (-1, 0) is $((x + 1)\, dy/dt - y\, dx/dt)/((x + 1)^2 + y^2)$. Hence the trajectory satisfies: $((x - 1) y' - y)(x^2 + y^2 + 1 + 2x) + ((x + 1) y' - y)(x^2 + y^2 + 1 - 2x) = 0$ or $2x\, y' (x^2 + y^2 + 1) - 4x\, y' - 2y(x^2 + y^2 + 1) = 0$, or $x\, y' (x^2 + y^2 - 1) = y\, (x^2 + y^2 + 1)$.

We see immediately that $xy = 0$ is a solution and almost immediately that $x^2 - y^2 = 1$ is a solution. Hence also all linear combinations $x^2 + k\, xy - y^2 = 1$. These are rectancular hyperbolae because the sum of the coefficients of x^2 and y^2 are zero.

Problem B2
Find:
(1) $\lim_{n\to\infty} \sum_{1\le i\le n} 1/\sqrt{(n^2 + i^2)}$;
(2) $\lim_{n\to\infty} \sum_{1\le i\le n} 1/\sqrt{(n^2 + i)}$;
(3) $\lim_{n\to\infty} \sum_{1\le i\le n^2} 1/\sqrt{(n^2 + i)}$;

Solution
(1) $1/\sqrt{(n^2 + i^2)} = (1/n) / \sqrt{(1 + (i/n)^2)}$. So the sum is just a Riemann sum for the integral $\int_0^1 dx/\sqrt{(1 + x^2)} = \sinh^{-1} 1 = \ln(1 + \sqrt{2}) = 0.8814$.

(2) $1/\sqrt{(n^2 + i)} = (1/n) / \sqrt{(1 + i/n^2)}$. Each term is less than 1/n, so the (finite) sum is less than 1. But each term is at least $(1/n) / \sqrt{(1 + 1/n)}$. So the sum is at least $1/\sqrt{(1 + 1/n)}$, which tends to 1. Hence the limit of the sum is 1.

(3) $1/\sqrt{(n^2 + i)} = n\, (1/n^2) / \sqrt{(1 + i/n^2)}$. Now $\sum (1/n^2) / \sqrt{(1 + i/n^2)}$ is just a Riemann sum for $\int_0^1 dx/\sqrt{(1 + x)} = 2\sqrt{(1 + x)}\, |_0^1 = 2(\sqrt{2} - 1)$. So the sum given tends to $2(\sqrt{2} - 1)\, n$, which diverges to infinity. [Or simpler, there are n^2 terms, each at least $1/\sqrt{(2n^2)} = 1/(n\sqrt{2})$, so the sum is at least $n/\sqrt{2}$ which diverges.]

Problem B3
Let y and z be any two linearly independent solutions of the differential equation $y'' + p(x)\, y' + q(x)\, y = 0$. Let $w = y\, z$. Find the differential equation satisfied by w.

Solution
We have $w = yz$, $w' = y'z + yz'$, $w'' = y''z + 2y'z' + yz''$. Hence $w'' + pw' + 2qw = 2y'z'$ (1). Now $(y'z')' = y''z' + y'z''$, $2py'z' = py'z' + py'z'$, $qw' = qyz' + qy'z$, so $(y'z')' + 2py'z' + qw' = 0$ (2).

Differentiating (1) we get: $w''' + p'w' + pw'' + 2q'w + 2qw' = 2(y'z')' = -4py'z' - 2qw'$ (using (2)), $= -2p(w'' + pw' + 2qw) - 2qw'$. Rearranging, this gives: $w''' + 3p\, w'' + (p' + 2p^2 + 4q)\, w' + (4pq + 2q')w = 0$.

Problem B4
Given an ellipse center O, take two perpendicular diameters AOB and COD. Take the diameter A'OB' parallel to the tangents to the ellipse at A and B (this is said to be *conjugate* to the diameter AOB). Similarly, take C'OD' conjugate to COD. Prove that the rectangular hyperbola through A'B'C'D' passes through the foci of the ellipse.

Solution
Take the ellipse as $x^2/a^2 + y^2/b^2 = 1$ and the point A as (a cos t, b sin t). Then AB has slope b/a tan t, so CD has slope -a/b cot t. The tangent at A has slope -b/a cot t. Suppose C is (a cos u, b sin u), then b/a tan u = -a/b cot t and the tangent at C has slope -b/a cot u = b^3/a^3 tan t. Hence the line pair A'B', C'D' has equation (y + x b/a cot t)(y - x b^3/a^3 tan t). Now we have the equations for two distinct conics through A'B'C'D': the original ellipse and the line pair A'B',C'D'. The equation of any other conic through these four points must be a linear combination of the equations of these two, in other words, $(x^2/a^2 + y^2/b^2 - 1) + k(y + x\ b/a \cot t)(y - x\ b^3/a^3 \tan t) = 0$ for some k.

The criterion for a rectangular hyperbola is that the coefficients of x^2 and y^2 should have sum zero, or that $1/a^2 + 1/b^2 + k - k\ b^4/a^4 = 0$. Hence $k = a^2/(b^4 - b^2a^2)$ and the equation of the rectangular hyperbola is $x^2 - y^2 + (b/a \tan t - a/b \cot t)\ xy = a^2 - b^2$. But the foci are at $(\pm(a^2 - b^2)^{1/2}, 0)$, so they lie on the rectangular hyperbola.

Problem B5
A wheel radius r is traveling along a road without slipping with angular velocity $\omega > \sqrt{(g/r)}$. A particle is thrown off the rim of the wheel. Show that it can reach a maximum height above the road of $(r\omega + g/\omega)^2/(2g)$. [Ignore air resistance.]

Solution
Suppose the pebble leaves the wheel from a point on the rim which is at an angle θ to the vertical. Its point of departure is a distance r + r cos θ above the road. Its upward velocity is r ω sin θ, so it ascends a further $(r\ \omega \sin \theta)^2/2g$. Thus the total height is $r + r^2\omega^2/2g + r \cos \theta - r^2\omega^2/2g \cos^2\theta = r + g/2\omega^2 + r^2\omega^2/2g - r^2\omega^2/2g\ (\cos \theta - g/r\omega^2)$ (*). We are given that $g/r\omega^2 < 1$, so (*) has a maximum when $\cos \theta = g/r\omega^2$ and the maximum value is $1/2g\ (r^2\omega^2 + 2gr + g^2/\omega^2) = (r\omega + g/\omega)^2/2g$.

Problem B6
f is a real valued function on [0, 1], continuous on (0, 1). Prove that $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x)\ f(y)\ f(z)\ dz\ dy\ dx = 1/6\ (\int_{x=0}^{x=1} f(x)\ dx)^3$.

Solution
Let S_{xyz} be the points in the cube for which $x \le y \le z$, let S_{yxz} be the points for which $y \le x \le z$ and so on. Then the union of the six sets is the cube and the intersection of any two has measure zero. Also by changing the variables of integration we see that the integral of f(x) f(y) f(z) over each set is the same. Hence the integral over S_{xzy} is 1/6 of the integral over the cube. But the integral over S_{xzy} is just $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x)\ f(y)\ f(z)\ dz\ dy\ dx$ and the integral over the cube is $(\int_{x=0}^{x=1} f(x)\ dx)^3$.

Problem B7
Do either (1) or (2):

(1) f is a real-valued function defined on the reals with a continuous second derivative and satisfies $f(x + y)\ f(x - y) = f(x)^2 + f(y)^2 - 1$ for all x, y. Show that for some constant k we have $f''(x) = \pm k^2 f(x)$. Deduce that f(x) is one of ±cos kx, ±cosh kx.

(2) a_i and b_i are constants. Let A be the (n+1) x (n+1) matrix A_{ij}, defined as follows: $A_{i1} = 1$; $A_{1j} = x^{j-1}$ for $j \leq n$; $A_{1\,(n+1)} = p(x)$; $A_{ij} = a_{i-1}{}^{j-1}$ for $i > 1$, $j \leq n$; $A_{i\,(n+1)} = b_{i-1}$ for $i > 1$. We use the identity det A = 0 to define the polynomial p(x). Now given any polynomial f(x), replace b_i by $f(b_i)$ and p(x) by q(x), so that det A = 0 now defines a polynomial q(x). Prove that f(p(x)) is a multiple of $(x - a_i)$ plus q(x).

Solution

(1) Putting y = 0 gives $f(0)^2 = 1$, so $f(0) = \pm 1$. Differentiating wrt y gives $f'(x+y) f(x-y) - f(x+y) f'(x-y) = 2 f(y) f'(y)$. Putting y = 0 gives $f'(0) = 0$. Differentiating wrt x gives $f''(x+y) f(x-y) = f(x+y) f''(x-y)$. Putting x = y = z/2 gives $f''(z) = h f(z)$, where $h = \pm f''(0)$. If h is positive, we may put $h = k^2$ and integrate, using $f(0) = \pm 1$, $f'(0) = 0$ to get $f(x) = \pm \cos kx$. If h is negative, we may put $h = - k^2$ and integrate to get $f(x) = \pm \cosh kx$.

(2) -

5th Putnam 1942

Problem A1
ABCD is a square side 2a with vertices in that order. It rotates in the first quadrant with A remaining on the positive x-axis and B on the positive y-axis. Find the locus of its center.

Solution
Answer: the segment (a, a) to (a√2, a√2).

Let AB make an angle θ with the x-axis. Then we find that the coordinates of the center to be x = y = a cos θ + a sin θ. But a cos θ + a sin θ = a √2 sin(θ + π/4).

Problem A2
a and b are unequal reals. What is the remainder when the polynomial p(x) is divided $(x - a)^2(x - b)$.

Solution
Suppose the remainder is $cx^2 + dx + e$. We have $p(a) = ca^2 + da + e$, $p(b) = cb^2 + db + e$. Also, differentiating, we get $p'(a) = 2ca + d$. Solving, $c = p'(a)/(a - b) - p(a)/(a - b)^2 + p(b)/(a - b)^2$, $d = (2a\ p(a) - 2a\ p(b) - (a^2 - b^2)p'(a)\ )/(a - b)^2$, $e = p(a) - a^2(p(a) - p(b))/(a - b)^2 + ab\ p'(a)\ /(a - b)$.

Problem A3
Does $\sum_{n\geq 0} n!\ k^n/(n + 1)^n$ converge or diverge for k = 19/7?

Solution
The nth term divided by the n-1th term is $k\ n\ n^{n-1}/(n+1)^n = k/(1 + 1/n)^n$ which tends to k/e. But k/e < 1, so the series converges by the ratio test.

Problem A4
Let C be the family of conics $(2y + x)^2 = a(y + x)$. Find C', the family of conics which are orthogonal to C. At what angle do the curves of the two families meet at the origin?

Solution
For most points P in the plane we can find a unique conic in the family passing through the point. Thus we should be able to find the gradient of members of the family at (x, y) in a formula which is independent of a. We then use this to get a formula for the gradient of the orthogonal family and solve the resulting first-order differential equation to get the orthogonal family.

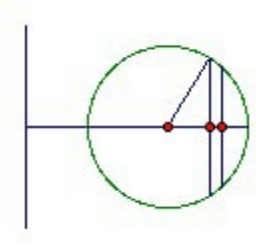
Thus we have $8y\ y' + 4x\ y' + 4y + 2x = ay' + a = (y' + 1)(2y + x)^2/(y + x)$. So $y'(2y+x)(\ 4(x+y) - (2y+x)\ ) = (2y+x)^2 - (x+2y)(x+y)$, or $y'(2y+x)(2y+3x) = -2x(2y+x)$, so $y' = -x/(2y+3x)$. Hence the orthogonal family satisfies $y' = (2y+3x)/x$. So $y'/x^2 - 2y/x^3 = 3/x^2$. Integrating $y = bx^2 - 3x$. These are all parabolas.

All members of both families pass through the origin. Changing coordinates to X = x + 2y, Y = y - 2x, the equation of a member of the first family becomes $X^2 = a(3X-Y)/5$ or $Y = - 5/a\ (X - 3a/10)^2 + 9a/20$. This has gradient 3 (in the new system) at the origin. In the old system the tangent is y = -x. The orthogonal set obviously has gradient -3 at the origin. If the angle between them is k, then tan k = (-1 +3)/(1+3) = 1/2. So $k = \tan^{-1}1/2$.

Problem A5
C is a circle radius a whose center lies a distance b from the coplanar line L. C is rotated through π about L to form a solid whose center of gravity lies on its surface. Find b/a.

Answer
$(\pi + \sqrt{(\pi^2+2\pi-4)})/(2\pi-4)$ = about 2.9028

Solution
The solid is half a torus. We can divide it into a large number of thin disks. Each disk has variable thickness, with thickness proportional to the distance from L. So we must integrate to find the distance of the centroid of the disk from L. Take the density to be kd, where d is the distance from L.



Take x to be distance along the line perpendicular to x, and θ to be the angle between the radius vector and the x-axis. We have $x = a \cos \theta$, so $dx = - a \sin \theta \, d\theta$. The mass is $\int_0^\pi 2a \sin \theta \, (a \sin \theta \, d\theta) \, k(b + a \cos \theta) = 2a^2bk \int_0^\pi \sin^2\theta \, d\theta + 2a^3k \int_0^\pi \sin^2\theta \cos \theta \, d\theta = a^2bk\pi + 0$. So the mass times the centroid distance is $\int_0^\pi 2a^2k \sin^2\theta \, (a \cos \theta + b)^2 \, d\theta = 2a^4k \int_0^\pi \sin^2\theta \cos^2\theta \, d\theta + 4a^3bk \int_0^\pi \sin^2\theta \cos \theta \, d\theta + 2a^2b^2k \int_0^\pi \sin^2\theta \, d\theta = ½a^4k \int_0^\pi \sin^2 2\theta \, d\theta + 0 + a^2b^2k\pi = ka^2\pi(a^2/4 + b^2)$. So the centroid distance is $b + a^2/4b$. Thus we can regard the mass as uniformly spread over a semicircle radius $b + a^2/4b$.

We need another integration to find the distance of the mass of a semicircle radius r from its center. It is $(1/\pi r) \int_0^\pi r^2 \sin \theta \, d\theta = 2r/\pi$. Thus the cm of the half-torus is a distance $(2/\pi)(b + a^2/4b)$ from L. We want it to be a distance b-a from L so that it lies on the surface. Thus $(2/\pi)(b + a^2/4b) = b - a$, so $(2\pi-4)b^2 - 2\pi ab - a^2 = 0$. Hence $b/a = (\pi + \sqrt{(\pi^2+2\pi-4)})/(2\pi-4)$ = about 2.9028.

Problem A6
P is a plane and H is the half-space on one side of P. K is a fixed circle in P. C is a circle in P which cuts K at an angle α. Let C have center O and radius r. f(C) is the point in H on the normal to P through O and a distance r from O. Show that the locus of f(C) is a one-sheet hyperboloid and that it has two families of rulings in it.

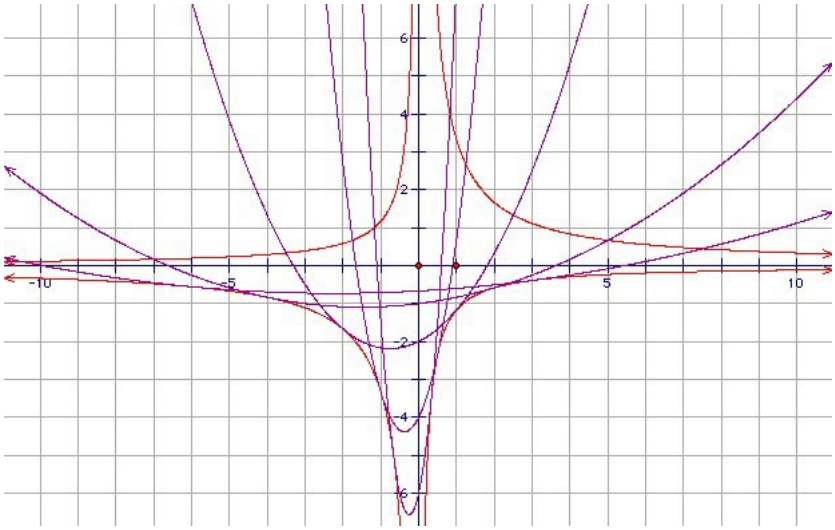
Problem B1
S is a solid square side 2a. It lies in the quadrant $x \geq 0$, $y \geq 0$, and it is free to move around provided a vertex remains on the x-axis and an adjacent vertex on the y-axis. P is a point of S. Show that the locus of P is part of a conic. For what P does the locus degenerate?

Solution
Let A be the vertex that moves along the x-axis and B the vertex that moves along the y-axis. Suppose that when AB is horizontal P has coordinates b, c. In the general configuration let be the angle BAO be θ. Then P has coordinates $x = (2a - b) \cos \theta + c \sin \theta$, $y = b \sin \theta + c \cos \theta$. Hence $cx - (2a - b)y = (b^2 + c^2 - 2ab) \sin \theta$, $bx - cy = (2ab - b^2 - c^2) \cos \theta$. Squaring and adding we eliminate θ to get: $(b^2 + c^2) x^2 - 4ac \, xy + (4a^2 + b^2 + c^2 - 4ab) y^2 = (b^2 + c^2 - 2ab)^2$, which is the equation of a conic. So the locus of P must form part of this conic.

The conic degenerates if $b^2 + c^2 = 2ab$. In this case, the equation becomes $2ab \, x^2 - 4ac \, xy + (4a^2 - 2ab)y^2 = 0$, or $bx^2 - 2c \, xy + (2a - b)y^2 = 0$, or $b^2x^2 - 2bc \, xy + c^2 y^2 = 0$, or $bx = cy$. So in this case the locus lies on a straight line. We may write the condition $b^2 + c^2 = 2ab$ as $(a - b)^2 + c^2 = a^2$, which shows that such P lie on the semicircle diameter AB.

Problem B2
Let P_a be the parabola $y = a^3x^2/3 + a^2x/2 - 2a$. Find the locus of the vertices of P_a, and the envelope of P_a. Sketch the envelope and two P_a.

Solution
We can write the equation of P_a as $(y + 35a/16) = (a^3/3)(x + 3/4a)^2$, so the vertex is $x = -3/4a$, $y = -35a/16$. The locus of the vertex is $xy = 105/64$.



The graph shows P_3, P_2, P_1, $P_{1/2}$, $P_{1/3}$ and the two hyperbolae $yx = -7/6$, $yx = 10/3$. It shows that for positive a, the parabolas touch the lower branches of the hyperbolae. For negative a they touch the upper branches.

That is not hard to verify. We claim that P_a and $xy = 10/3$ touch at $x = -2/a$, $y = -5a/3$. The point obviously lies on $xy = 10/3$. We have $(a^3/3)(x + 3/4a)^2 = (1/3)(-2 + 3/4)^2$ $a = 25a/48 = (-5a/3 + 35a/16)$, so the point also lies on P_a. The gradient of $xy = 10/3$ at the point is $-10/(3x^2) = -5a^2/6$. The gradient of P_a at the point is $2a^3x/3 + a^2/2 = -5a^2/6$.

Similarly, we claim that P_a and $xy = -7/6$ touch at $x = 1/a$, $y = -7a/6$. The point obviously lies on $xy = -7/6$. We have $(a^3/3)(x + 3/4a)^2 = (a/3)(1 + 3/4)^2 = 49a/48 = (-7a/6 + 35a/16)$, so the point also lies on P_a. The gradient of $xy = -7/6$ at the point is $7/(6x^2) = 7a^2/6$. The gradient of P_a at the point is $2a^3x/3 + a^2/2 = a^2(2/3 + 1/2) = 7a^2/6$.

It is less clear how you get the hyperbolas. One standard approach is to look for the singular points of the mapping $f(a,t) = (t, a^3t^2/3 + a^2t/2 - 2a)$. The matrix for the derivative is:

$$\begin{matrix} 1 & 0 \\ 2a^3t/3+a^2/2 & a^2t^2+at-2 \end{matrix}$$

which has zero determinant when $at = 1$ or -2, so $xy = 7/6$ or $-10/3$.

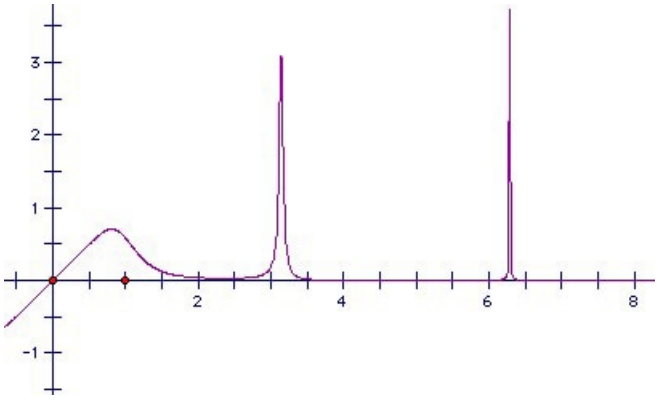
Problem B3
$f(x, y)$ and $g(x, y)$ satisfy the differential equation $f_1(x, y)\, g_2(x, y) - f_2(x, y)\, g_1(x, y) = 1$ (*). Taking $r = f(x, y)$ and y as independent variables, and $x = h(r, y)$, $g(x, y) = k(r, y)$, show that $k_2(r, y) = h_1(r, y)$. Integrate and hence obtain a solution to (*). What other solutions does (*) have?

Problem B4
A particle moves in a circle through the origin under the influence of a force a/r^k towards the origin (where r is its distance from the origin). Find k.

Solution
The equations of motion are $r\,(\theta')^2 - r'' = a/r^k$, $r^2\theta' = A$ (conservation of angular momentum).

If the particle moves in a circle as described, then we can write its orbit as $r = B \cos \theta$. Differentiating, $r' = - B \theta' \sin \theta = -AB/r^2 \sin \theta$. Differentiating again, $r'' = -AB \cos \theta A/r^2 + 2AB/r^3 \sin \theta r' = -A^2/r^3 - 2A^2B^2/r^5 \sin^2\theta = -A^2/r^3 - 2A^2B^2/r^5 (1 - r^2B^2) = A^2/r^3 - 2A^2B^2/r^5$. So substituting back in the equation of motion we get: $2A^2B^2/r^5 = a/r^k$. Hence $k = 5$.
Note that this is unphysical, since we require infinite velocity as we reach the origin.

Problem B5
Let $f(x) = x/(1 + x^6\sin^2 x)$. Sketch the curve $y = f(x)$ and show that $\int_0^\infty f(x)\,dx$ exists.



Solution
Obviously $f(x)$ is positive for positive x. But it has an infinite number of spikes at $x = n\pi$. The spike at $n\pi$ is height $n\pi$, so we have to show that the integral is bounded above.

We have $\sin x > \frac{1}{2}x$ near $x = 0$ (certainly for $x < \pi/3$). So $|\sin x| > 1/(n\pi)^k$ except possibly for $|x| < 2/(n\pi)^k$. Let I_n be the interval centered on $n\pi$ width $4/(n\pi)^k$. For $x \in I_n$ we have $f(x) < 2n\pi$, so the integral of $f(x)$ over the interval is less than $8/(n\pi)^{k-1}$. The total integral over all such intervals is bounded provided that $k > 2$. Outside such intervals, $x^6 \sin^2 x > \frac{1}{2} x^6/x^{2k}$, so $f(x) < 2/x^{2k-5}$. Hence the interval of $f(x)$ over 0 to ∞ excluding the intervals I_n is bounded provided $k > 5/2$. By taking $k = 2\ 1/4$, for example, we get that the whole integral is bounded.

6th Putnam 1946

Problem A1
p(x) is a real polynomial of degree less than 3 and satisfies $|p(x)| \le 1$ for $x \in [-1, 1]$. Show that $|p'(x)| \le 4$ for $x \in [-1, 1]$.

Solution
Let $p(x) = ax^2 + bx + c$, so $p'(x) = 2ax + b$. It is evidently sufficient to show that $|2a + b|$ and $|2a - b| \le 4$. $p(0) = c$, $p(1) = a + b + c$, $p(-1) = a - b + c$, so $2a + b = 3/2\ p(1) + 1/2\ p(-1) + 2\ p(0)$. But $|p(1)|$, $|p(-1)|$, $|p(0)| \le 1$, so $|2a + b| \le 4$. Similarly, $2a - b = 1/2\ p(1) + 3/2\ p(-1) + 2\ p(0)$.

Problem A2
R is the reals. For functions f, g **:** $R \to R$ and $x \in R$ define $I(fg) = \int_1^x f(t)\, g(t)\, dt$. If a(x), b(x), c(x), d(x) are real polynomials, show that I(ac) I(bd) - I(ad) I(bc) is divisible by $(x - 1)^4$.

Solution
Let F(x) = I(ac) I(bd) - I(ad) I(bc). Then clearly F(1) = 0 (since all the integrals are over an empty range). Differentiating, we get, F' = acI(bd) + bdI(ac) - adI(bc) - bcI(ad). So F'(1) = 0.

Differentiating again: F'' = a'cI(bd) + ac'I(bd) + abcd + b'dI(ac) + bd'I(ac) + abcd - a'dI(bc) - ad'I(bc) - abcd - b'cI(ad) - bc'I(ad) - abcd = a'cI(bd) + ac'I(bd) + b'dI(ac) + bd'I(ac) - a'dI(bc) - ad'I(bc) - b'cI(ad) - bc'I(ad). So F''(1) = 1.

Differentiating again: F''' = a''cI(bd) + 2a'c'I(bd) + ac''I(bd) + a'bcd + abc'd + b''dI(ac) + 2b'd'I(ac) + bd''I(ac) + ab'cd + abcd' - a''dI(bc) - 2a'd'I(bc) - ad''I(bc) - a'bcd - abcd' - b''cI(ad) - 2b'c'I(ad) - bc''I(ad) - ab'cd - abc'd = a''cI(bd) + 2a'c'I(bd) + ac''I(bd) + b''dI(ac) + 2b'd'I(ac) + bd''I(ac) - a''dI(bc) - 2a'd'I(bc) - ad''I(bc) - b''cI(ad) - 2b'c'I(ad) - bc''I(ad). So F'''(1) = 1.

That is sufficient to prove the result. But notice that if we differentiate again, then just collecting the terms that do not involve I(fg) we get (after some cancellation) 2a'bc'd + 2ab'cd' - 2a'bcd' - 2ab'c'd, which is not, in general, zero for x = 1. So in general we do not have F(x) divisible by $(x - 1)^5$.

Problem A3
ABCD are the vertices of a square with A opposite C and side AB = s. The distances of a point P in space from A, B, C, D are a, b, c, d respectively. Show that $a^2 + c^2 = b^2 + d^2$, and that the perpendicular distance k of P from the plane ABCD is given by $8k^2 = 2(a^2 + b^2 + c^2 + d^2) - 4s^2 - (a^4 + b^4 + c^4 + d^4 - 2a^2c^2 - 2b^2d^2)/s^2$.

Solution
Let Q be the point of the plane ABCD closest to P. Let O be the center of the square ABCD. Let QO make an angle θ with AC. Then using the cosine rule we have: $AQ^2 = AO^2 + OQ^2 - 2AO \cdot OQ \cos\theta$, $CQ^2 = CO^2 + OQ^2 + 2CO.OQ \cos\theta$ (*). Adding: $AQ^2 + CQ^2 = 2AO^2 + 2QO^2$. But $a^2 = AQ^2 + k^2$ etc, so $a^2 + c^2 = 2k^2 + s^2 + 2QO^2$. Similarly, $b^2 + d^2 = 2k^2 + s^2 + 2QO^2$. We have established that $a^2 + c^2 = b^2 + d^2$.

We have also established that $8k^2 = 2(a^2 + b^2 + c^2 + d^2) - 4s^2 - 8QO^2$ (**). The angle φ between QO and BD is $\pi/2 - \theta$. So $\cos\varphi = \sin\theta$. Hence, going back to (*), $AQ^2 - CQ^2 = -4\ AO \cdot OQ \cos\theta$, and $BQ^2 - DQ^2 = \pm 4\ AO \cdot OQ \sin\theta$. But $AQ^2 - CQ^2 = a^2 - c^2$ etc. So $(a^2 - c^2)^2 + (b^2 - d^2)^2 = 16\ AO^2OQ^2 = 8s^2OQ^2$. Substituting in (**) gives the required result.

Problem A4
R is the reals. f **:** R → R has a continuous derivative, f(0) = 0, and |f '(x)| <= |f(x)| for all x. Show that f is constant.

Solution
Suppose f is not constant. Then take an interval [a, b] of length < 1/2 such that f(a) = 0, f(b) ≠ 0, and |f(b)| ≥ |f(x)| for x ∈ [a, b]. Now applying the mean value theorem to the interval gives an immediate contradiction.

Problem A5
Let T be a tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. What is the smallest possible volume for the tetrahedral volume bounded by T and the planes x = 0, y = 0, z = 0?

Solution
Answer: $\sqrt{3}$ abc/2.

The normal at (x_0, y_0, z_0) is $(xx_0/a^2, yy_0/b^2, zz_0/c^2)$. So the tangent plane is $x\, x_0/a^2 + y\, y_0/b^2 + z\, z_0/c^2 = 1$. This cuts the three axes at $x = a^2/x_0$, $y = b^2/y_0$, $z = c^2/z_0$. We can regard two of these lengths as defining the base of the tetrahedron, and the third as forming its height. Hence its volume is $a^2b^2c^2/(6x_0y_0z_0)$.

We wish to maximize $x_0y_0z_0$. That is equivalent to maximising $x_0^2/a^2\ y_0^2/b^2\ z_0^2/c^2$. But we know that the sum of these three numbers is 1, so their maximum product is 1/27 (achieved when they are all equal - the arithmetic/geometric mean result). Hence $x_0y_0z_0$ has maximum value $abc/(3\sqrt{3})$.

Problem A6
A particle moves in one dimension. Its distance x from the origin at time t is $at + bt^2 + ct^3$. Find an expression for the particle's acceleration in terms of a, b, c and its speed v.

Solution
Differentiating, $v = 3ct^2 + 2bt + a$ (*), and hence the acceleration $f = 6ct + 2b$. So $t = (f - 2b)/6c$. Substituting in (*) gives $v = (f - 2b)^2/12c + b(f - 2b)/3c + a$. Hence $12cv = f^2 - 4b^2 + 12ac$. So $f = 2\sqrt{(b^2 + 3cv - 3ac)}$.

Problem B1
Two circles C_1 and C_2 intersect at A and B. C_1 has radius 1. L denotes the arc AB of C_2 which lies inside C_1. L divides C_1 into two parts of equal area. Show L has length > 2.

Solution
Let O_1 be the center of C_1, and O_2 the center of C_2. Let the line O_1O_2 meet the arc AB of C_2 at P. If P lies between O_1 and O_2, then the tangent to C_2 at P divides C_1 into two unequal parts and the area $C_1 \cap C_2$ lies inside the smaller part. Contradiction. So O_1 must lie between P and O_2. But now the arc AP is greater than the segment AP, which is greater than $AO_1 = 1$. Hence L > 2.

Problem B2
P_0 is the parabola $y^2 = mx$, vertex K (0, 0). If A and B points on P_0 whose tangents are at right angles, let C be the centroid of the triangle ABK. Show that the locus of C is a parabola P_1. Repeat the process to define P_n. Find the equation of P_n.

Solution
The gradient at the point (x_1 , y_1) is $1/2\ m/y_1$. So the tangents at (x_1 , y_1), (x_2 , y_2) are perpendicular iff $y_1y_2 = -m^2/4$. So we may take one point as $(x, y) = (t^2/m, t)$ and the other as $(1/16\ m^3/t^2, -1/4$

m^2/t). Hence the centroid is (x, y), where $x = 1/3\ (t^2/m + 1/16\ m^3/t^2)$, $y = 1/3\ (t - 1/4\ m^2/t)$. But this lies on the parabola $y^2 = m/3\ (x - m/6)$. But since any value of t was possible and hence any value of y, any point of this parabola is a centroid from some pair of points A, B.

Repeating, we get that the equation for P_n is $y^2 = m/3^n(x - m/6(1 + 1/3 + ... + 1/3^{n-1})) = m/3^n(x - m/4\ (1 - 1/3^n))$.

Problem B3
The density of a solid sphere depends solely on the radial distance. The gravitational force at any point inside the sphere, a distance r from the center, is kr^2 (where k is a constant). Find the density (in terms of G, k and r), and the gravitational force at a point outside the sphere. [You may assume the usual results about the gravitational attraction of a spherical shell.]

Solution
A shell exerts no net attraction on a point inside and acts on a point outside as if all its mass was concentrated at its center. So let the density at a radial distance r be $\rho(r)$. Then $G/r^2 \int_0^r 4\pi t^2\rho(t)\ dt = kr^2$. Hence $\int_0^r t^2\rho(t)\ dt = k/(4G\pi)\ r^4$. Differentiating: $r^2\rho(r) = k/(G\pi)\ r^3$. So $\rho(r) = k/(G\pi)\ r$.

The force at the surface of the sphere (R) is kR^2. Hence the force at a distance $r > R$ from the center of the sphere is kR^4/r^2. [We know the force is a simple inverse square law outside the sphere and it must be continuous at the surface.]

Problem B4
Define $a_n = 2(1 + 1/n)^{2n+1}/((1 + 1/n)^n + (1 + 1/n)^{n+1})$. Prove that a_n is strictly monotonic increasing.

Solution
$a_n = 2\ (n+1)^{n+1}/(n^n\ (2n+1))$. Let $f(n) = \ln a_n/2 = (n + 1) \ln(n + 1) - n \ln n - \ln(2n+1)$. Regard f as a function of a real variable and differentiate. $f'(x) = \ln(x + 1) - \ln x - 2/(2x + 1)$. Differentiate again: $f''(x) = 1/(x + 1) - 1/x + 4/(2x + 1)^2 = -1/(x(x+1)(2x+1)^2)$.

So $f''(x) < 0$ for all $x > 0$. Hence $f'(x)$ is decreasing. But we can write $f'(x) = \ln(1 + 1/x) - 2/(2x + 1)$ which tends to 0 as x tends to infinity. So $f'(x) > 0$ for all $x > 0$. Hence f(x) is strictly increasing for all positive x. Hence $f(n+1) > f(n)$ and so $\exp f(n+1) > \exp f(n)$, which is the result we want.

Problem B5
Let m be the smallest integer greater than $(\sqrt{3} + 1)^{2n}$. Show that m is divisible by 2^{n+1}.

Solution
This is a fairly well-known problem. $(\sqrt{3} - 1) < 1$ and (we will show) $(\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$ (*) is an integer. Hence it must be the required integer.

But it is easy to check that (*) is the solution of a recurrence relation, in fact the relation $u_n = 8u_{n-1} - 4u_{n-2}$, $u_0 = 2$, $u_1 = 8$. That establishes that (*) is an integer, and a trivial induction shows that (*) is divisible by 2^{n+1}.

Problem B6
The particle P moves in the plane. At t = 0 it starts from the point A with velocity zero. It is next at rest at t = T, when its position is the point B. Its path from A to B is the arc of a circle center O. Prove that its acceleration at each point in the time interval [0, T] is non-zero, and that at some point in the interval its acceleration is directly towards the center O.

Solution
Take polar coordinates with origin O. Let the radius of the circular arc be k. The particle's radial acceleration is $k\,(d\theta/dt)^2$ towards O. Its tangential acceleration is $k\,d^2\theta/dt^2$. Since the particle is not at rest between A and B, $(d\theta/dt)$ is non-zero, so the radial acceleration is non-zero (and hence its total acceleration).

$(d\theta/dt)$ is zero at A and at B, so its derivative (and hence the tangential acceleration) must be zero at some point between.

7th Putnam 1947

Problem A1
The sequence a_n of real numbers satisfies $a_{n+1} = 1/(2 - a_n)$. Show that $\lim_{n\to\infty} a_n = 1$.

Solution
This is slightly messy. First, since $k = 1/(2 - k)$ implies $k = 1$, it is obvious that if the sequence tends to a limit, then the limit is 1.

Next, if $0 < a_n < 1$, then $1 < 2 - a_n < 2$, so $1/2 < a_{n+1} < 1$. So once the sequence gets into the interval (0, 1) it stays there. But $a_{n+1} - a_n = (a_n - 1)^2/(2 - a_n) > 0$ for $a_n < 2$ (*). So once the sequence gets into the interval (0, 1) it is monotonic increasing and bounded above by 1, and hence tends to a limit (which must be 1).

If $a_n < 0$, then $0 < a_{n+1} < 1$, so we are also home if a member of the sequence is negative. Similarly, $a_n > 2$ implies $a_{n+1} < 0$. If $a_n = 1$, then all following terms are 1 and so the limit is 1. So the only issue is if $1 < a_1 < 2$.

But then (*) shows that whilst the sequence remains in the interval (1, 2) it is monotonic increasing. It cannot tend to a limit, because that limit would have to be 1, which is impossible. So it cannot stay in the interval. We cannot have $a_m = 2$, because then we would not have $a_{m+1} = 1/(2 - a_m)$, so we must have $a_m > 2$ for some m and then we are home.

Problem A2
R is the reals. f **:** R → R is continuous and satisfies $f(r) = f(x) f(y)$ for all x, y, where $r = \sqrt{(x^2 + y^2)}$. Show that f(x) = f(1) to the power of x^2.

Solution
Induction on n shows that $f(x \sqrt{n}) = f(x)^n$, and hence f(n x) = f(x) to the power of n^2. In particular, taking x = 1/n, f(1) = f(1/n) to the power of n^2. Hence, provided f(1) is non-zero, f(1/n) = f(1) to the power of $1/n^2$. Hence f(m/n) = f(1) to the power of $(m/n)^2$. So we have established that f(x) = f(1) to the power of x^2 for all rational x. But f is continuous, so the relation holds for all x.
If f(1) = 0, then the same reasoning establishes that f(x) = 0 for all x.

Problem A3
ABC is a triangle and P an interior point. Show that we cannot find a piecewise linear path $K = K_1K_2 \ldots K_n$ (where each K_iK_{i+1} is a straight line segment) such that: (1) none of the K_i do not lie on any of the lines AB, BC, CA, AP, BP, CP; (2) none of the points A, B, C, P lie on K; (3) K crosses each of AB, BC, CA, AP, BP, CP just once; (4) K does not cross itself.

Solution
Each time K crosses the boundary of a triangle it moves from the outside to the inside or vice versa. K has two endpoints, so we can find one of the three triangles ABP, BCP, CAP in which it does not start or finish. But that is impossible - on the first crossing it must go from outside to inside, on the second from inside to outside and on the third from outide to inside.

Problem A4
Take the x-axis as horizontal and the y-axis as vertical. A gun at the origin can fire at any angle into the first quadrant ($x, y \geq 0$) with a fixed muzzle velocity v. Assuming the only force on the pellet after firing is gravity (acceleration g), which points in the first quadrant can the gun hit?

Solution
Let the angle of the gun to the x-axis be θ. Then the equations of motion are: $x = t\ v \cos\theta$, $y = t\ v \sin\theta - 1/2\ g\ t^2$. So the pellet moves along the parabola $y = x \tan\theta - x^2\ g/2v^2 \sec^2\theta$ (*).

We can view (*) as an equation for θ given x, y. Put $k = \tan\theta$, then the equation becomes $g/2v^2\ x^2\ k^2 - x\ k + y + gx^2/2v^2$. This has real roots iff $x^2 \geq 2g/v^2\ x^2 y - g^2/v^4\ x^4$ and hence iff $y \leq 2g^2/v^2 - x^2/v^4\ x^2$ (**). Since $x \geq 0$, we see directly from the quadratic that the sum and the product of the roots are both non-negative, so (**) is the condition for the equation to have at least one non-negative root in k and hence at least one root for θ in the range 0 to π/2. Thus the gun can hit points in the first quadrant under (or on) the parabola given by (**).

Problem A5
The sequences a_n, b_n, c_n of positive reals satisfy: (1) $a_1 + b_1 + c_1 = 1$; (2) $a_{n+1} = a_n^2 + 2b_nc_n$, $b_{n+1} = b_n^2 + 2c_na_n$, $c_{n+1} = c_n^2 + 2a_nb_n$. Show that each of the sequences converges and find their limits.

Solution
$a_{n+1} + b_{n+1} + c_{n+1} = (a_n + b_n + c_n)^2$, so by a trivial induction $a_n + b_n + c_n = 1$. There appears to be symmetry between the three sequences, so we conjecture that each converges to 1/3.

Suppose $a_n \leq b_n \leq c_n$. We have $a_{n+1} = a_n^2 + 2b_nc_n \leq a_nc_n + b_nc_n + c_nc_n = c_n(a_n + b_n + c_n) = c_n$. Similarly, $b_{n+1} = b_{n2} + 2a_nc_n \leq b_nc_n + a_nc_n + c_n^2 = c^n$, and $c_{n+1} = c_n^2 + 2a_nb_n \leq c_n^2 + a_nc_n + b_nc_n = c_n$. Hence the largest of a_{n+1}, b_{n+1}, c_{n+1} is no bigger than the largest of a_n, b_n, c_n. An exactly similar argument works for the smallest. Hence the largest forms a monotonic decreasing sequence which is bounded below and the smallest forms a monotonic increasing sequence which is bounded above.

Let $b_n - a_n = h \geq 0$, $c_n - b_n = k \geq 0$. Then $a_{n+1} - b_{n+1} = (a_n - b_n)(a_n + b_n - 2c_n)$, so $|a_{n+1} - b_{n+1}| = h(h + 2k) \leq (h + k)^2$. Similarly, $|b_{n+1} - c_{n+1}| = |b_n - c_n|\ |b_n + c_n - 2a_n| = k(2h + k) \leq (h + k)^2$, and $|c_{n+1} - a_{n+1}| = |c_n - a_n|\ |c_n + a_n - 2b_n| = |(h + k)(k - h)| \leq (h + k)^2$. So for n+1 the difference between the biggest and the smallest is the square of the difference for n. But a_1, b_1, c_1 are all positive and hence, by a trivial induction, a_n, b_n, c_n are positive. Their sum is 1 so the difference between the biggest and smallest must be less than 1. Hence the difference tends to zero. Hence a_n, b_n, c_n all tend to 1/3.

Problem A6
A is the matrix

```
a  b  c
d  e  f
g  h  i
```

det A = 0 and the cofactor of each element is its square (for example the cofactor of b is fg - di = b^2). Show that all elements of A are zero.

Solution
$a^2e^2 - b^2d^2 = (ei - fh)(ai - cg) - (fg - di)(ch - bi) = (ae - bd)\ i^2 + (cd - af)\ hi + (bf - ce)\ gi = (g^3 + h^3 + i^3)\ i = 0$, since $0 = \det A = g^3 + h^3 + i^3$. Hence $ae = \pm bd$. Similarly $cd = \pm af$, $bf = \pm ce$. Multiplying the three equations together we get abcdef = - abcdef unless at least one of the equations has a plus sign. In the first case, at least one of a, b, c, d, e, f is zero. In the second case, the element corresponding to the cofactor is zero - for example ae = bd implies $i^2 = 0$ and hence i = 0. So either a member of the first two rows is zero, or a member of the last row is zero.

wlog we may assume a = 0. That implies b or d = 0 also. [Note that if, for example, i was the zero element, then we would have $ei = \pm fh$, by an argument similar to that above and hence f or h = 0). If

b = 0, then since $a^3 + b^3 + c^3 = 0$, we have also c = 0. Similarly, if d = 0, then g = 0. So we now have a complete row or column zero. But now the square of any other element is a linear combination of elements in the that row or column and hence zero. Suppose, for example, a = b = c = 0. Then g^2 = bf - ce = 0, and similarly for the other five elements.

Problem B1
Let R be the reals. f **:** [1, ∞) → R is differentiable and satisfies $f'(x) = 1/(x^2 + f(x)^2)$ and f(1) = 1. Show that as x → ∞, f(x) tends to a limit which is less than 1 + π/4.

Solution
Clearly f '(x) is always positive. But f(1) = 1, so f(x) > 1 for all x. Hence $f'(x) < 1/(x^2 + 1)$ for all x. Hence $f(x) = 1 + \int_1^x f'(t)\,dt < 1 + 1 + \int_1^x 1/(1 + t^2)\,dt = 1 + (\tan^{-1}t)|_1^x = 1 + \tan^{-1}x - \pi/4 < 1 + \pi/2 - \pi/4 = 1 + \pi/4$. Since f '(x) is positive, f(x) is monotone increasing. It is bounded above by 1 + π/4, so it must tend to a limit less than 1 + π/4.

Problem B2
R is the reals. f **:**(0, 1) → R is differentiable and has a bounded derivative: |f '(x)| <= k. Prove that : $|\int_0^1 f(x)\,dx - \sum_1^n f(i/n)/n| \le k/n$.

Solution
The worst case for the difference between 1/n f(i/n) and $\int_{i/n-1/n}^{i/n} f(x)\,dx$ is if f '(x) = k (or -k) for the entire range, in which case the difference is the area of a triangle base 1/n and height k/n. Hence the difference for the complete Riemann sum is at worst k/(2n).

Problem B3
Let O be the origin (0, 0) and C the line segment { (x, y) **:** x ∈ [1, 3], y = 1 }. Let K be the curve { P **:** for some Q ∈ C, P lies on OQ and PQ = 0.01 }. Let k be the length of the curve K. Is k greater or less than 2?

Solution
Answer: less.
If we use polar coordinates, then r = cosec θ - .01, so the length is $\int_{\pi/4}^{\arctan(1/3)} \sqrt{(\text{cosec}\,\theta - .01)^2 + \text{cosec}^2\theta\cot^2\theta}\,d\theta$. This is obviously horrendous.

The trick is that if we just remove the .01, then the integral gives the curve length for the line segment C, which is 2. But the presence of the -.01 obviously *reduces* the integrand at every point of the range, so the integral above must have value *less* than 2.

Problem B4
$p(z) \equiv z^2 + az + b$ has complex coefficients. |p(z)| = 1 on the unit circle |z| = 1. Show that a = b = 0.

Solution
In particular, |p(1)| = |p(-1)| = 1, so 1 + a + b and 1 - a + b lie on the unit circle. Hence their midpoint 1 + b lies in the unit disk. Similarly, |p(i)| = |p(-i)| = 1, so -1 + ia + b and -1 - ia + b lie on the unit circle and hence their midpoint -1 + b lies in the unit disk. But 1 + b and -1 + b are a distance 2 apart, so they must lie at either end of a diameter of the unit circle and hence b = 0. Now 1 + a and 1 - a lie on the unit circle, as does their midpoint 1. Hence they must coincide and so a = 0.

Problem B5
Let p(x) be the polynomial (x - a)(x - b)(x - c)(x - d). Assume p(x) = 0 has four distinct integral roots and that p(x) = 4 has an integral root k. Show that k is the mean of a, b, c, d.

Solution
a, b, c, d must be integers. (k - a)(k - b)(k - c)(k - d) = 4 and all of (k - a), (k - b), (k - c), (k - d) are integers. They all divide 4, so they must belong to {-4, -2, -1, 1, 2, 4}. They are all distinct, so at most two of them have absolute value 1. Hence none of them can have absolute value 4 - or their product would be at least 8. Hence they must be -2, -1, 1, 2. Hence their sum is 0, so 4k = a + b + c + d.

Problem B6
P is a variable point in space. Q is a fixed point on the z-axis. The plane normal to PQ through P cuts the x-axis at R and the y-axis at S. Find the locus of P such that PR and PS are at right angles.

Solution
Answer: sphere centre Q, radius QO, where O is the origin, excluding the two circles formed by the intersection of the sphere with the y-z plane and the x-z plane.

Let Q be (0, 0, r). Let P be (a, b, c). If R is (x, 0, 0), then PR and PQ are perpendicular, so their dot product is 0, so $a(a - x) + b^2 + c(c - r) = 0$, hence $ax = (a^2 + b^2 + c^2 - cr)$. Similarly, if S is (0, y, 0), then $by = (a^2 + b^2 + c^2 - cr)$. We require PR and PS perpendicular so $a(a - x) + (b - y)b + c^2 = 0$, hence $ax + by = a^2 + b^2 + c^2$. So $a^2 + b^2 + c^2 - 2cr = 0$ and hence $a^2 + b^2 + (c - r)^2 = r^2$, which shows that P lies on the sphere centre Q radius QO.

Conversely, suppose P lies on the sphere. Then SP and SO are tangents to the sphere and hence equal. Similarly, RP = RO, so PRS and ORS are similar. Hence $\angle RPS = \angle ROS = 90^o$. However, if P lies in the z-x plane, then S is not a finite point and if P lies in the y-z plane, then R is not a finite point. So we must exclude points lying on these two circles of the sphere.

8th Putnam 1948

Problem A1
C is the complex numbers. f **:** C → R is defined by f(z) = $|z^3 - z + 2|$. What is the maximum value of f on the unit circle $|z| = 1$?

Solution
Answer: √13.
Put $z = e^{i\theta}$, and use $\cos 2\theta = 2c^2 - 1$, $\cos 3\theta = 4c^3 - 3c$, where $\cos\theta = c$, to get: $|f(z)|^2 = 6 - 2\cos(3\theta - \theta) + 4\cos^3\theta - 4\cos\theta = 4(4c^3 - c^2 - 4c + 2)$. The cubic has stationary points where $12c^2 - 2c - 4 = 0$ or c = 2/3 or -1/2. So the maximum value is at c = 1 or -1 (the endpoints), or 2/3 or -1/2 (the stationary points). Substituting in, we find that the maximum is actually at c = -1/2 with value 13.

Problem A2
K is a cone. s is a sphere radius r, and S is a sphere radius R. s is inside K touches it along all points of a circle. S is also inside K and touches it along all points of a circle. s and S also touch each other. What is the volume of the finite region between the two spheres and inside K?

Solution
A straight slog is fairly long and tiresome.
Slice off the top of a sphere radius r by a cut a distance d from the surface. A simple integration shows that the volume removed is $\pi d^2(3r - d)/3$.

Let h be the distance from the vertex of the cone along the axis to the nearest sphere. Similar triangles gives (h+r)/r = (h+2r+R)/R. Hence $h = 2r^2/(R-r)$. Let the plane through the circle of contact between the small sphere and the cone cut the cone's axis at a distance h+d from its vertex. Similarly, let the plane through the circle of contact of the other sphere cut the cone's axis a distance D from the point of contact between the two spheres. Let t be the distance from the cone's vertex to the circle of contact with the small sphere. Then $t^2 = h(h+2r)$. By similar triangles (h+d)/t = t/(h+r), so $d = 2r^2/(R+r)$. Hence by similar triangles D = 2rR/(R+r).

Let v be the volume in the cone between the vertex and the small sphere. We find this as the volume of a cone with circular base less the volume of a slice of sphere. The cone has height h+d. The square of the radius of the base is $(2rd-d^2)$. Hence $v = \pi/3\ (\ (h+d)(2rd-d^2) - d^2(3r-d)\ )$. This simplifies to $4r^5\pi/(3(R^2-r^2))$.

Hence V the volume of the corresponding region between the vertex and the large sphere (assuming the small sphere is temporarily removed) is $4r^2R^3\pi/(3(R^2-r^2))$. Hence the required volume is $V - v - 4\pi r^3/3 = 4\pi r^2R^2/3(R+r)$.

Problem A3
a_n is a sequence of positive reals decreasing monotonically to zero. b_n is defined by $b_n = a_n - 2a_{n+1} + a_{n+2}$ and all b_n are non-negative. Prove that $b_1 + 2b_2 + 3b_3 + ... = a_1$.

Solution
We have that $b_1 + 2b_2 + 3b_3 + ... + n\,b_n = a_1 - (n+1)a_{n+1} + na_{n+2} = a_1 - (n+1)(a_{n+1} - a_{n+2}) - a_{n+2}$. Also we are given that all b_i are non-negative. So S n b_n is monotonic increasing and bounded above by a_1. So it converges to a limit $L \le a_1$.

$b_n = (a_n - a_{n+1}) - (a_{n+1} - a_{n+2})$. Hence $b_n + b_{n+1} + b_{n+2} + ... + b_{n+m} = (a_n - a_{n+1}) - (a_{n+m+1} - a_{n+m+2})$. But a_m tends to zero, so $(a_{n+m+1} - a_{n+m+2})$ tends to zero as m tends to infinity. Hence $b_n + b_{n+1} + b_{n+2} + ...$ converges to $a_n - a_{n+1}$. Hence $(n+1)(b_{n+1} + b_{n+2} + b_{n+3} + ...\ )$ converges to $(n + 1)(a_{n+1} - a_{n+2})$. Hence L

$\geq b_1 + 2b_2 + 3b_3 + ... + n\ b_n + (n+1)(b_{n+1} + b_{n+2} + b_{n+3} + ...) = a_1 - (n+1)a_{n+1} + n\ a_{n+2} + (n+1)(a_{n+1} - a_{n+2}) = a_1 - a_{n+2}$. But a_{n+2} tends to zero. Hence $L = a_1$.

Problem A4
Let D be a disk radius r. Given (x, y) □ D, and R > 0, let a(x, y, R) be the length of the arc of the circle center (x, y), radius R, which is outside D. Evaluate $\lim_{R\to 0} R^{-2} \int_D a(x, y, R)\ dx\ dy$.

Solution
Answer: 4πr.
Let P be a point in the disk a distance x from its centre. Suppose that the circle centre P radius R cuts the disk perimeter at A and B. Let angle APB be 2θ. We have $r^2 = x^2 + R^2 + 2xR \cos\theta$. Hence the length of arc outside the disk is $2R \cos^{-1}((r^2-x^2-R^2)/2xR)$. This applies for r-R <= x <= r. For smaller x the small circle lies entirely in the disk. Thus we have to evaluate $1/R^2 \int 2\pi\ x\ 2R \cos^{-1}((r^2-x^2-R^2)/2xR)\ dx$. Put x = r - yR, and we get $4\pi r \int_0^1 (1 - y\ R/r) \cos^{-1}((2Rry - (y^2+1)R^2)/2R(r - yR))\ dy$. As R tends to 0, (1 - y R/r) tends to 1 and $(2Rry - (y^2+1)R^2)/2R(r - yR)$ tends to y, so we get $4\pi r \int_0^1 \cos^{-1}y\ dy = 4\pi r\ (y \cos^{-1}y - \sqrt{(1-y^2)}\ )|_0^1 = 4\pi r$.

Problem A5
Let $\alpha_1, \alpha_2, ... , \alpha_n$ be the nth roots of unity. Find $\prod_{i<j} (\alpha_i - \alpha_j)^2$.

Solution
Let $\alpha_k = \exp(2pi(k-1)/n)$. So $\alpha_1 = 1$, and the other α_k are roots of $x^{n-1} + x^{n-2} + ... + x + 1 = 0$. Hence $(1 - \alpha_2), ... , (1 - \alpha_n)$ are roots of $(1 - x)^{n-1} + ... + (1 - x) + 1 = 0$. The coefficient of x^0 is n and the coefficient of x^{n-1} is $(-1)^{n-1}$. Hence $\prod_2^n(1 - \alpha_k) = n$.

Hence $n/\alpha_k = (\alpha_k - \alpha_1)(\alpha_k - \alpha_2) ... (\alpha_k - \alpha_n)$, where the product excludes $(\alpha_k - \alpha_k)$. Hence $n^n/(\alpha_2...\alpha_n) = \prod_{i \text{ not equ } j}(\alpha_i - \alpha_j) = (-1)^{n(n-1)/2} \prod_{i<j}(\alpha_i - \alpha_j)^2$. But $(\alpha_2...\alpha_n) = (-1)^{n-1}$. Hence $\prod_{i<j}(\alpha_i - \alpha_j)^2 = (-1)^{(n-1)(n-2)/2} n^n$.

Problem A6
Do either (1) or (2):

(1) On each element ds of a closed plane curve there is a force 1/R ds, where R is the radius of curvature. The force is towards the center of curvature at each point. Show that the curve is in equilibrium.

(2) Prove that $x + 2/3\ x^3 + 2\cdot4/3\cdot5\ x^5 + ... + 2\cdot4\cdot\ ... \ .2n/(3\cdot5.\ ... \ .2n+1)\ x^{2n+1} + ... = (1 - x^2)^{-1/2} \sin^{-1}x$.

Solution
(2) Let $f(x) = x + 2/3\ x^3 + 8/15\ x^5 + ... + 1/2\ n!\ n!/2n+1!\ (2x)^{2n+1} + ...$.
Hence $x\ f(x) = x^2 + 2/3\ x^4 + 8/15\ x^6 + ... + 1/4\ n!\ n!/2n+1!\ (2x)^{2n+2} + ...$.
Also $f'(x) = 1 + 2x^2 + 8/3\ x^4 + ... + n!\ n!/2n!\ (2x)^{2n} + ...$.
and $(1 - x^2)\ f'(x) = 1 + x^2 + 2/3\ x^4 + ... + n\text{-}1!\ n\text{-}1!/2n!\ n/2\ (2x)^{2n} + ...$
$= 1 + x^2 + 2/3\ x^4 + ... + 1/4\ n\text{-}1!n\text{-}1!/2n\text{-}1!\ (2x)^{2n} + ...$.

Hence the derivative of $\sqrt{(1 - x^2)}\ f(x)$ is $1/\sqrt{(1 - x^2)}\ (- x\ f(x) + (1 - x^2)\ f'(x)\ ) = 1/\sqrt{(1 - x^2)}$. So $\sqrt{(1 - x^2)}\ f(x) = \text{const} + \sin^{-1}(x)$. Putting x = 0, we find that the constant is 0.

Problem B1
p(x) is a cubic polynomial with roots α, β, γ and p'(x) divides p(2x). Find the ratios α : β : γ.

Solution
$p(x) = (x - \alpha)(x - \beta)(x - \gamma)$. Hence $p'(x) = 3x^2 - 2(\alpha+\beta+\gamma)x + \beta\gamma+\gamma\alpha+a\beta$. This must have roots α/2,

β/2. Hence 2(α+β+γ)/3 = (α+β)/2 and αβ/4 = (βγ+γα+αβ)/3. This leads to α = (-2 + 2i/√3)γ, β = (-2 - 2i/√3)γ.

Probloem B2
A circle radius r is tangent to the three coordinate planes (x =0, y =0, z = 0) in space. Find the locus of its center.

Problem B3
Show that [√n + √(n + 1)] = [√(4n + 2)] for positive integers n.

Solution
We have (√n + √(n + 1))^2 = 2n + 1 + 2 √(n^2 + n). Now n^2 < n^2 + n < n^2 + n + 1/4 = (n + 1/2)^2, so √(4n+1) < √n + √(n+1) < √(4n+2). Hence [√(4n+1)] ≤ [√n + √(n+1)] ≤ [√(4n+2)]. But a square must be congruent to 0 or 1 mod 4, so [√(4n+2)] = [√(4n+1)]. Hence result.

Problem B4
R is the reals. For what λ can we find a continuous function f **:** (0, 1) → R, not identically zero, such that $\int_0^1$ min(x, y) f(y) dy = λ f(x) for all x □ (0, 1)?

Problem B5
Find the area of the region { (x, y) **:** |x + yt + t^2| ≤ 1 for all t □ [0, 1] }.

Solution
We cannot have |x| > 1, for then the inequality is not satisfied at t = 0. We cannot have y > 1 for then yt + t^2 > 2 at t = 1 and hence x + yt + t^2 > 1. Similarly, we cannot have y < -3 for then yt + t^2 < -2 at t = 1 and hence x + yt + t^2 < -1.

We can write x + yt + t^2 = x - y^2/4 + (t + y/2)^2. So for -3 ≤ y ≤ -2, the maximum of (t + y/2)^2 occurs at t = 0 with value y^2/4, and its minimum is (1 + y/2)^2 at t = 1. Hence the minimum of x + yt + t^2 is 1 + x + y and the maximum is x. So the region is bounded by x ≤ 1 and x + y ≥ -2.

For -2 ≤ y ≤ -1, the maximum of (t + y/2)^2 is y^2/4 at t = 0, but the minimum is 0 at t = -y/2. Hence the maximum of x + yt + t^2 is x and the minimum is x - y^2/4. So the region is bounded by x ≤ 1 and x - y^2/4 ≥ -1.

For -1 ≤ y ≤ 0, the maximum of (t + y/2)^2 is (1 + y/2)^2 and the minimum is 0 at t = -y/2. Hence the maximum of x + yt + t^2 is 1 + x + y and the minimum is x - y^2/4. So the region is bounded by x + y ≤ 0 and x - y^2/4 ≥ -1.

For 0 ≤ y ≤ 1, the maximum of (t + y/2)^2 is (1 + y/2)^2 and the minimum is y^2/4. Hence the maximum of x + yt + t^2 is 1 + x + y and the minimum is x. So the region is bounded by x + y ≤ 0 and x ≥ -1.

Thus the region is the rhombus with vertices at (1, -1), (1, -3), (-1, -1), (-1, 1), except that the vertex (-1, -1) is cut off by the curve x = y^2/4 - 1, which forms the boundary between (0, -2) and (-1, 0). The rhombus has area 4 (base 2 and height 2). The area between the parabola and the lines x = -1 and y = -2 is $\int_0^2$ y^2/4 dy = 8/12. The area between the line y = -2 -x and the lines x = -1 and y = -2 is 1/2, so the area cut off the rhombus by the parabola is 2/3 - 1/2 = 1/6. Hence the area required is 4 - 1/6 = 3 5/6.

Problem B6
Do either (1) or (2):

(1) Take the origin O of the complex plane to be the vertex of a cube, so that OA, OB, OC are edges of the cube. Let the feet of the perpendiculars from A, B, C to the complex plane be the complex numbers u, v, w. Show that $u^2 + v^2 + w^2 = 0$.

(2) Let (a_{ij}) be an n x n matrix. Suppose that for each i, $2 |a_{ii}| > \sum_1^n |a_{ij}|$. By considering the corresponding system of linear equations or otherwise, show that det $a_{ij} \neq 0$.

Solution

(2) If the det is non-zero, then we can find $x_i$not all zero so that $\sum x_i a_{ij} = 0$ for each j. Take k so that $|x_k| \geq |x_i|$ for all i. Then $|\sum_{i \text{ not } k} x_i a_{ik}| \leq \sum_{i \text{ not } k} |x_i a_{ik}| \leq |x_k| \sum_{i \text{ not } k} |a_{ik}| < |x_k| |a_{kk}|$, so we cannot have $\sum x_i a_{ik} = 0$. Contradiction.

9th Putnam 1949

Problem A1
Do either (1) or (2)
(1) Let L be the line through (0, -a, a) parallel to the x-axis, M the line through (a, 0, -a) parallel to the y-axis, and N the line through (-a, a, 0) parallel to the z-axis. Find the equation of S, the surface formed from the union of all lines K which intersect each of L, M and N.
(2) Let S be the surface xy + yz + zx = 0. Which planes cut S in circles? In parabolas?

Problem A2
Take points O, P, Q, R in space. Let the volume of the parallelepiped with edges OP, OQ, OR be V. Let V' be the volume of the parallelepiped which has O as one vertex and which has OP, OQ, OR as altitudes to three faces. Show that $V\,V' = OP^2 OQ^2 OR^2$. Generalize to n dimensions.

Problem A3
All the complex numbers z_n are non-zero and $|z_m - z_n| > 1$ (for any $m \neq n$). Show that $\sum 1/z_n^3$ converges.

Solution
We show that it is absolutely convergent. Take a disk radius 1/2 centred on each z_n. The disks must be disjoint since $|z_m - z_n| > 1$ for distinct m, n. Now consider how many z_n can lie in the annulus $N \leq |z| < N+1$. If z_n lies in the annulus, then at least half the corresponding disk must also, so if M is the number of z_n, then $M\,\pi/8 < \pi\,((N+1)^2 - N^2) = (2N+1)\,\pi$, and the sum of $1/|z_n|^3$ for these z_n is at most $M/N^3 = 8(2N + 1)/N^3$. But $\sum 8(2N + 1)/N^3$ converges, since $\sum 1/N^2$ converges.

Problem A4
Take P inside the tetrahedron ABCD to minimize PA + PB + PC + PD. Show that □APB = □CPD and that the bisector of APB also bisects CPD.

Problem A5
Let $p(z) \equiv z^6 + 6z + 10$. How many roots lie in each quadrant of the complex plane?

Solution
There are no real roots, since p(z) has a minimum (for real value of z) at z = -1 of 5. Similarly there are no purely imaginary roots, since p(i y) has imaginary part 6iy (and z = 0 is not a root). Also a + ib is a root iff a - ib is a root, so either there is one root in each of the 1st and 4th quadrants and two in each of the 2nd and 3rd, or vice versa.

The argument principle states that the change in arg p(z) as we move (anti-clockwise) round a closed contour which passes through no zeros and contains k zeros is $2\,\pi\,k$. Take a contour in the first quadrant from z = 0 to z = X, then around the arc |z| = X to iX, then back along the imaginary axis to z = 0. Moving along the real axis, arg p(z) does not change. Moving along the arc the change will be the same as that in arg z^6 for X sufficiently large and hence will be 3π. As we move back down the imaginary axis, the imaginary part remains positive, so p(z) is confined to the upper half plane, and we end up at p(z) = 10, so the change must be $-\pi$. Thus the change for a complete circuit is 2π, and so there is just one root in the first quadrant.

Problem A6
Show that $\prod_1^{\infty} (1 + 2\cos(2z/3^n)/3 = (\sin z)/z$ for all complex z.

Solution
For z = 0, we may take (sin z)/z = 1 by continuity. In this case, each term on the lhs is 1, so the relation holds. Assume now that z is non-zero.

We have sin z = sin(2z/3 + z/3) = sin 2z/3 cos z/3 + cos 2z/3 sin z/3, and sin z/3 = sin(2z/3 - z/3) = sin 2z/3 cos z/3 - cos 2z/3 sin z/3. Hence sin z = (1 + cos 2z/3) sin z/3. Hence (sin z)/z = (sin z/3)/(z/3) (1 + cos 2z/3)/3. Iterating: (sin z)/z = (sin $z/3^N$)/($z/3^N$) $\prod_1^N$ (1 + 2 cos($2z/3^n$))/3. But as N tends to infinity $z/3^N$ tends to zero and hence (sin $z/3^N$)/($z/3^N$) tends to 1. Hence the result.

Problem B1
Show that for any rational a/b □ (0, 1), we have |a/b - 1/√2| > $1/(4b^2)$.

Solution
If |a/b - 1/√2| ≤ $1/(4b^2)$, then |a^2/b^2 - 1/2| = |a/b - 1/√2| |a/b + 1/√2| < |a/b - 1/√2| (1 + 1) < $1/2b^2$. So |$2a^2$ - b^2| < 1. Hence 2 a^2 = b^2. But that is impossible since √2 is irrational. Note that the restriction to the interval (0, 1) is unnecessary.

Problem B2
Do either (1) or (2)

(1) Prove that $\sum_2^\infty$ cos(ln ln n) / ln n diverges.

(2) Let k, a, b, c be real numbers such that a, k > 0 and b^2 < ac. Show that $\int_U$ (k + ax^2 + 2bxy + cy^2)$^{-2}$ dx dy = π/(k √(ac - b^2)), where U is the entire plane.

Solution
(1) We have ln n^2 = 2 ln n, so for any m in the range n, n+1, n+2, ... , n^2 we have 1/ln m > 1/2 1/ln n. Also ln ln n^2 = ln ln n + ln 2 < ln ln n + π/3. So given any sufficiently large integer N, take x > N so that ln ln x = a multiple of 2 π, then we can find at least $N^2/2$ consecutive integers n in the range x to x^2such that (A) 1/ln n > 1/(2 ln N) and (B) ln ln n lies between 2k π and 2k π + π/3 (for some integer k) and hence so that cos ln ln n > 1/2. Hence for each such integer (cos ln ln n)/ln n > 1/(4 ln N). So ∑ (cos ln ln n)/ln n over all these terms is at least $N^2/(8 \ln N)$ which can be made arbitrarily large. Hence the sequence does not converge.

(2) ax^2 + 2bxy + cy^2 = h is the equation of an ellipse. If we rotate the axes to line up with its axes we get new coordinates X, Y so that it becomes AX^2 + BY^2 = h with AB = ac - b^2. Thus the integral is transformed to $\int_U$ (k + AX^2 + BY^2)$^{-2}$ dX dY. Take u = X√A, v = Y√B and the integral becomes 1/√(ac - b^2) $\int_U$ (k + u^2 + v^2)$^{-2}$ du dv. Now take u = r cos θ, v = r sin θ and we get 1/√(ac - b^2) $\int_U$ (k + u^2 + v^2)$^{-2}$ du dv = 1/√(ac - b^2) $\int_U$ (k + r^2)$^{-2}$ r dr dθ = 2 π /√(ac - b^2) ∫ (k + r^2)$^{-2}$ r dr = 2 π /(k √(ac - b^2)) ∫ (1 + s^2)$^{-2}$ s ds = π /(k √(ac - b^2)) (-1/(1 + s^2))$|_0^{inf}$ = 2 π /(k √(ac - b^2)).

Problem B3
C is a closed plane curve. If P, Q □ C, then |PQ| < 1. Show that we can find a disk radius 1/√3 which contains C.

Solution
C can be any set of points in the plane. The usual proof uses Helly's theorem. Given any three points of the set we can find a circle radius 1/√3 which contains them. This is fairly obvious. Take the points to be P, Q, R with PQ the longest side of the triangle. Take S on the same side of PQ as R so that PQS is equilateral. Then R must lie inside the bounding circle of PQS (in fact it must lie inside the region bounded PQ, the arc of the circle centre P from S to Q, and the arc of the circle centre Q from S to P, and this region lies entirely inside the bounding circle of PQS).

Another way of expressing this is that the three circles radius $1/\sqrt{3}$ and centres P, Q, R have a point in common. We can now apply Helly's theorem to the set of circles radius $1/\sqrt{3}$ and with centres at all the points of the set C. Every three of these circles have a point in common (proved above) and hence by Helly's theorem there is a point common to all of them. But this point can be taken as the centre of the required circle.

Problem B4
Let $(1 + x - \sqrt{(x^2 - 6x + 1)})/4 = \sum_1^\infty a_n x^n$. Show that all a_n are positive integers.

Solution
We have $1 + x - \sqrt{(x^2 - 6x + 1)} = 4 \sum a_n x^n$, and hence $1 + x + \sqrt{(x^2 - 6x + 1)} = 2x + 2 - 4 \sum a_n x^n$. Multiplying, $1 + 2x + x^2 - x^2 + 6x - 1 = 8x = 8(a_1 x + a_2 x^2 + \ldots)(1 + (1 - 2a_1)x - 2a_2 x^2 - 2a_3 x^3 - \ldots)$. Hence:

$a_1 = 1$
$a_2 + a_1(1 - 2a_1) = 0$
$a_3 + a_2(1 - 2a_1) - 2a_1 a_2 = 0$
$a_4 + a_3(1 - 2a_1) - 2a_2 a_2 - 2a_1 a_3 = 0$ ...
The nth equation has the form a_n + combination of $a_1, \ldots, a_{n-1}$ with integer coefficients, so by a simple induction, a_n is integral.

Problem B5
a_n is a sequence of positive reals. Show that $\limsup_{n\to\infty} ((a_1 + a_{n+1})/a_n)^n \geq e$.

Solution
It is sufficient to show that $(a_1 + a_{n+1})/a_n \geq 1 + 1/n$ for infinitely many n. For these n then form a sequence for which $((a_1 + a_{n+1})/a_n)^n$ has a limit point (possibly infinity) not less than $\lim (1 + 1/n)^n = e$.

If that were false, then we would have $(a_1 + a_{n+1})/a_n < 1 + 1/n$ for all n not less than some N. In particular, taking n = N, N+1, N+2, ... we get $a_1/N+1 + a_{N+1}/N+1 < a_N/N$, $a_1/N+2 + a_{N+2}/N+2 < a_{N+1}/N+1$, $a_1/N+3 + a_{N+3}/N+3 < a_{N+2}/N+2$, ... Hence $a_n/N > a_1(1/N+1 + 1/N+2 + \ldots + 1/N+M) + a_{N+M}/N+M > a_1(1/N+1 + 1/N+2 + \ldots + 1/N+M)$. But this holds for any M and $(1/N+1 + 1/N+2 + \ldots)$ diverges. Contradiction.

Problem B6
C is a closed convex curve. If P lies on C and T_P is the tangent at P, then T_P varies continuously with P. Let O be a point inside C. Given a point P on C, define f(P) to be the point where the perpendicular from O to T_P intersects C. Given P_1, define the sequence P_n by $P_{n+1} = f(P_n)$. Assume that f is continuous and that, for each P, C lies entirely on one side of T_P. Show that P_n converges. Find $S = \{ P : P = \lim_{n\to\infty} P_n$ for some $P_1 \}$.

10th Putnam 1950

Problem A1
a and b are positive reals and a > b. Let C be the plane curve r = a - b cos θ. For what values of b/a is C convex?

Solution
The curvature is $(x'y'' - x''y')/(x'^2 + y'^2)^{3/2}$. So for convexity we require $x'y'' \geq x''y'$. Putting k = b/a, c = cos θ, s = sin θ, we have $x/a = c - k c^2$, $x'/a = - s + 2kcs$, $x''/a = - c - 2ks^2 + 2kc^2$ and $y/a = s - ksc$, $y'/a = c - kc^2 + k s^2$, $y''/a = -s + 4ksc$. Thus the condition becomes after a little cancellation, $1 + 2k^2 - 3kc \geq 0$. c takes values in the range -1 to 1, so for the condition to be true for all points of the curve we require $1 + 2k^2 - 3k \geq 0$. But $1 + 2k^2 - 3k = (2k - 1)(k - 1)$. We are given that k < 1, so we must have k < 1/2.

For small k, the curve is approximately a circle centred on the origin. As k increases it develops a flattening near x = a, y = 0. For k > 1/2, this becomes a dimple in the surface, so that convexity is broken. For k = 1, the depression in the surface extends as far as the centre (the origin). For k > 0, the curve intersects itself at the origin so that it comprises two ovals one inside the other.

Problem A2
Does the series $\sum_2^\infty 1/\ln n!$ converge? Does the series $1/3 + 1/(3\ 3^{1/2}) + 1/(3\ 3^{1/2}\ 3^{1/3}) + ... + 1/(3\ 3^{1/2}\ 3^{1/3}\ ...\ 3^{1/n}) + ...$ converge?

Solution
(1) $\ln n! < n \ln n$, so $\sum_2^\infty 1/\ln n! > \sum_2^\infty 1/(n \ln n)$, which diverges. For $\int 1/(x \ln x) = \ln \ln x$.

(2) We have $1 + 1/2 + 1/3 + ... + 1/n$ is approx $\ln n$, and $1/3^{\ln n}$ is approx $1/n^{\ln 3}$. But $\ln 3 > 1$, so we expect this to converge. In fact $1 + 1/2 + 1/3 + ... + 1/n > \int_1^{n+1} dx / x = \ln(n + 1) > \ln n$. So $1/3 + 1/(3\ 3^{1/2}) + 1/(3\ 3^{1/2}\ 3^{1/3}) + ... + 1/(3\ 3^{1/2}\ 3^{1/3}\ ...\ 3^{1/n}) + ... < 1/3^{\ln 1} + 1/3^{\ln 2} + 1/3^{\ln 3} + ... = 1/1 + 1/2^k + 1/3^k + ...$, where k = ln 3, which converges.

Problem A3
The sequence a_n is defined by $a_0 = \alpha$, $a_1 = \beta$, $a_{n+1} = a_n + (a_{n-1} - a_n)/(2n)$. Find $\lim a_n$.

Solution
$a_{n+1} - a_n = -1/2n\ (a_n - a_{n-1})$, hence $a_{n+1} - a_n = (-1)^n/(2\cdot4\cdot6\ ...\ .2n)\ (a_1 - a_0)$. Then $a_n = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + ... + (a_1 - a_0) + a_0$.
Hence the limit is $\beta + (\beta - \alpha)/2 - (\beta - \alpha)/2\cdot4 + (\beta - \alpha)/2\cdot4\cdot6 - ...$. But $1/2 - 1/2\cdot4 + 1/2\cdot4\cdot6 - ... = 1 - 1/\sqrt{e}$, so the limit is $\alpha + (\beta - \alpha)/\sqrt{e}$.

Problem A4
Do either (1) or (2)

(1) P is a prism with triangular base. A is a vertex. The total area of the three faces containing A is 3k. Show that if the volume of P is maximized, then each of the three faces has area k and the two lateral faces are perpendicular to each other.

(2) Let $f(x) = x + x^3/(1\cdot3) + x^5/(1\cdot3\cdot5) + x^7/(1\cdot3\cdot5\cdot7) + ...$, and $g(x) = 1 + x^2/2 + x^4/(2\cdot4) + x^6/(2\cdot4\cdot6) + ...$. Show that $\int_0^x \exp(-t^2/2)\ dt = f(x)/g(x)$.

Solution
(1) Let the sides of the base at A be a, b with angle between them θ. Let the height be h. Then the

volume is V = 1/2 abh sin θ and 3k = ah + bh + 1/2 ab sin θ. Now V^2 = (1/2 ab sin θ) (ah) (bh) 1/2 sin θ. By the arithmetic geometric mean theorem we have that (1/2 ab sin θ) (ah) (bh) ≤ k^3, with equality iff ah = bh = 1/2 ab sin θ. So $V^2 \le k^3/2$ sin θ. Hence $V \le \sqrt{(k^3/2)}$ with equality iff θ = π/2 and ah = bh = 1/2 ab.

(2) $\exp(-t^2/2) = 1 - t^2/2 + t^4/2\cdot4 - t^6 2\cdot4\cdot6 + \ldots$. We notice that this is similar to g(t). In fact g(t) = $\exp(t^2/2)$. So the required relation is $f(x) = \exp(x^2/2) \int_0^x \exp(-t^2/2)\, dt$. Differentiating gives f '(x) = x f(x) + 1. All that is just motivation.

We start from the series for f(x). By the ratio test it is absolutely convergent, so we may differentiate term by term to get $f'(x) = 1 + x^2/1 + x^4/1\cdot3 + x^6/1\cdot3\cdot5 + \ldots = x f(x) + 1$. Multiplying this by $\exp(-x^2/2)$ we get $\exp(-x^2/2) f'(x) - x \exp(-x^2/2) f(x) = \exp(-x^2/2)$. Integrating and using f(0) = 0, we get $\exp(-x^2/2) f(x) = \int_0^x \exp(-t^2/2)\, dt$ or $f(x)/g(x) = \int_0^x \exp(-t^2/2)\, dt$.

Problem A5
Let N be the set of natural numbers {1, 2, 3, ... }. Let Z be the integers. Define d **:** N → Z by d(1) = 0, d(p) = 1 for p prime, and d(mn) = m d(n) + n d(m) for any integers m, n. Determine d(n) in terms of the prime factors of n. Find all n such that d(n) = n. Define $d_1(m) = d(m)$ and $d_{n+1}(m) = d(d_n(m))$. Find $\lim_{n\to\infty} d_n(63)$.

Solution
$d(p^a) = a\, p^{a-1}$ by a trivial induction on a. Hence for $n = p^a q^b \ldots$ we have d(n) = n (a/p + b/q + ...) by a trivial induction on the number of primes.

Hence d(n) = n iff a/p + b/q + ... = 1. Multiplying through by all the prime denominators gives integral terms. All but the first are clearly divisible by p, so the first must be also. Hence a/p = 1 and b/q etc are zero. In other words, $n = p^p$ for some prime p.

We find $d_1(63) = 51$, $d_2(63) = 20$, $d_3(63) = 24$, $d_4(63) = 44$, $d_5(63) = 48$. Now suppose n is divisible by 16. Then $n = 2^k p^a q^b \ldots$, where p, q, ... are all odd and k >= 4. Hence d(n) = n(4/2 + a/p + b/q + ...). Now all of 2n, na/p, nb/q, ... are integral and multiples of 16. So d(n) is at least twice n and a multiple of 16. So if we have $d_k(m)$ divisible by 16, then $d_{h+k}(m) \ge 2^h$ which diverges. Hence $d_k(63)$ tends to infinity.

Problem A6
Let $f(x) = \sum_0^\infty a_n x^n$ and suppose that each a_n = 0 or 1. *Do either (1) or (2):*

(1) Show that if f(1/2) is rational, then f(x) has the form p(x)/q(x) for some integer polynomials p(x) and q(x).

(2) Show that if f(1/2) is not rational, then f(x) does not have the form p(x)/q(x) for any integer polynomials p(x) and q(x).

Solution
(1) f(1/2) is the binary expansion of some real in the closed interval [0, 1]. So if it is rational, then the binary expansion is periodic after some point. In other words, there are positive integers N and k such that $a_n = a_{n+k}$ for n > N. Hence $f(x) = a_0 + a_1 x + \ldots + a_N x^N + (a_{N+1} x^{N+1} + \ldots + a_{N+k-1} x^{N+k-1})(1 + x^k + x^{2k} + \ldots) = a_0 + a_1 x + \ldots + a_N x^N + (a_{N+1} x^{N+1} + \ldots + a_{N+k-1} x^{N+k-1})/(1 - x^k)$. Hence we can take $q(x) = 1 - x^k$ and $p(x) = (a_0 + a_1 x + \ldots + a_N x^N)(1 - x^k) + (a_{N+1} x^{N+1} + \ldots + a_{N+k-1} x^{N+k-1})$.
(2) is trivial. If f(x) = p(x)/q(x), then f(1/2) = p(1/2)/q(1/2) which is rational. The only slight complication is if q(1/2) = 0. But we can assume that p(x)/q(x) is in lowest terms, so if q(1/2) = 0,

then p(x) is non-zero, but then p(x) = f(x) q(x). We know that f(1/2) ≤ 1/2 + 1/4 + 1/8 + ... = 1, so f(1/2) q(1/2) = 0. Hence p(1/2) = 0. Contradiction.

Problem B1
Given n, not necessarily distinct, points P_1, P_2, ... , P_n on a line. Find the point P on the line to minimize $\sum |PP_i|$.

Solution
Assume that the points are in the order given. For two points A and B, |PA| + |PB| = AB for P on the segment AB and |PA| + |PB| > AB for P outside it. Thus we minimise $|PP_1| + |PP_n|$ by placing P between P_1 and P_n. Similarly, we minimise $|PP_2| + |PP_{n-1}|$ by placing P between P_2 and P_{n-1}. And so on. But note that if we minimise $|PP_2| + |PP_{n-1}|$ then we also minimise $|PP_1| + |PP_n|$. Thus for n odd we must take P to be the central point, for n even we can take P to be any point on the segment between the two central points.

Problem B2
An ellipse with semi-axes a and b has perimeter length p(a, b). For b/a near 1, is π(a + b) or $2\pi\sqrt{(ab)}$ the better approximation to p(a, b)?

Solution
Answer: π(a + b).
Put $b^2 = a^2(1 - \varepsilon)$. The perimeter is $4\int_0^{\pi/2} (a^2\sin^2 t + b^2\cos^2 t)^{1/2}\, dt = 4a\int (1 - \varepsilon\cos^2 t)^{1/2}\, dt = 4a\int (1 - 1/2\, \varepsilon\cos^2 t - 1/8\, \varepsilon^2\cos^4 t + O(\varepsilon^4)\, dt = 2\pi a (1 - \varepsilon/4 - 3\varepsilon^2/64 + O(\varepsilon^4))$.

$b = a(1 - \varepsilon)^{1/2} = a(1 - \varepsilon/2 - \varepsilon^2/8 + O(\varepsilon^3))$. Hence $\pi (a + b) = 2\pi a (1 - \varepsilon/4 - 4\varepsilon^2/64 + O(\varepsilon^3))$. Whilst $2\pi\sqrt{(ab)} = 2\pi a (1 - \varepsilon/2 - \varepsilon^2/8 + O(\varepsilon^3))^{1/2} = 2\pi a (1 - \varepsilon/4 - 6\varepsilon^2/64 + O(\varepsilon^3))$.

Hence the error in π (a + b) is $\pi a \varepsilon^2/32 + O(\varepsilon^3)$, and the error in $2\pi\sqrt{(ab)}$ is $3\pi a \varepsilon^2/32 + O(\varepsilon^3)$, which is roughly three times as large for small ε.

Problem B3
Leap years have 366 days; other years have 365 days. Year n > 0 is a leap year iff (1) 4 divides n, but 100 does not divide n, or (2) 400 divides n. n is chosen at random from the natural numbers. Show that the probability that December 25 in year n is a Wednesday is not 1/7.

Solution
365 = 52.7 + 1, so in non-leap years Christmas day is a day later in the week than the previous year. In leap years it is two days later. Now consider a period of 400 years. There are 100 multiples of 4, 3 or which are multiples of 100 but not 400, so there are 97 leap years. Thus after 400 years the day has advanced $303\cdot 1 + 97\cdot 2 = 497$ days or exactly 41 weeks. In other words after 400 years Christmas is on the same day of the week. This cycle then repeats indefinitely. But 400 is not a multiple of 7, so each day cannot occur with exactly probability 1/7.

If we look in more detail at the 400 years we find that Su, Tu, Fr each occur 58 times, We and Th each occur 57 times, and Mo and Sa each occur 56 times (compared with the expected 57 1/7).

Problem B4
A long, light cylinder has elliptical cross-section with semi-axes a > b. It lies on the ground with its main axis horizontal and the major axes horizontal. A thin heavy wire of the same length as the cylinder is attached to the line along the top of the cylinder. [We could take the cylinder to be the surface $|x| \le L$, $y^2/a^2 + z^2/b^2 = 1$. Contact with the ground is along $|x| \le L$, y = 0, z = -b. The wire is along $|x| \le L$, y = 0, z = b.] For what values of b/a is the cylinder in stable equilibrium?

Solution
Take the ellipse to be $x^2/a^2 + y^2b^2 = 1$. We need the normal to the ellipse at a point near (0, -b) to meet the y-axis above (0, b) (because then the heavy wire will be the correct side of the vertical through the point of contact to right the cylinder).

The gradient at (a cos t, b sin t) is -b/a cot t, so the normal has slope a/b tan t, so its equation is (y - b sin t) = a/b tan t (x - a cos t). This meets the y-axis at y = b sin t - a^2/b sin t. For points near (0, -b), sin t is -(1 - ε) so we need a^2/b - b > b or a > b√2.

Problem B5
Do either (1) or (2):
(1) Show that if $\sum(a_n + 2a_{n+1})$ converges, then so does $\sum a_n$.
(2) Let S be the surface $2xy = z^2$. The surface S and the variable plane P enclose a cone with volume $\pi a^3/3$, where a is a positive real constant. Find the equation of the envelope of P. What is the envelope in the case of a general cone?

Solution
(1) Suppose $a_n + 2a_{n+1}$ converges to 3k. We show that $a_n$converges to k.
Given any ε > 0, take N so that $a_n + 2a_{n+1}$ is within ε of 3k for all n ≥ N. Take a positive integer M such that a_N is within $(2^M + 1)\varepsilon$ of k.

Then a_{N+1} is within $((2^M + 1)\varepsilon + \varepsilon)/2 = (2^{M-1} + 1)\varepsilon$ of (3k - k)/2 = k. By a trivial induction a^{N+M} is within 2ε of k. Then a^{N+M+1} is within (2ε + ε)/2, and hence within 2ε, of k. So by a trivial induction, a_n is within 2ε of k for all n > N + M.

(2) Answer: the 2-sheet hyperboloid $u^2 = v^2 + z^2 + a^2$.
The nature of the surface S becomes much clearer if we change coordinates to u, v, z with z unchanged and the x, y axes rotated through π/4 to u = (x + y)/√2, v = (-x + y)/√2 (or inversely, x = (u - v)/√2, y = (u + v)/√2). Now S becomes $u^2 - v^2 = z^2$ or $v^2 + z^2 = u^2$, which is evidently a right circular cone with vertex the origin and axis the u-axis.

Evidently the plane u = a forms a cone volume 1/3 πa^3 with the surface (the base is a circle radius a and the height is a). If the plane moves round to be almost tangent to the sides of the cone it should still cut off a cone with the same volume, so we might suspect that the envelope is a two sheet hyperboloid asymptotic to the cone and passing through the points u = ±a, v = z = 0. This has equation $u^2 = v^2 + z^2 + a^2$. This is a surface of revolution formed by rotating the hyberbola $u^2 = v^2 + a^2$ about the u-axis, so it is sufficient to look at tangent planes that are parallel to (ie do not intersect) the z-axis. We need to show that these cut off cones volume 1/3 πa^3.

The tangent to the hyperbola at u = a cosh t, v = a sinh t is (v - a sinh t) = coth t (u - a cosh t) or v sinh t - u cosh t + a = 0 (*). It cuts v = u at u = v = a/(cosh t - sinh t) and it cuts v = -u at u = -v = a/(cosh t + sinh t). So the corresponding plane cuts the hyperboloid in an ellipse with major axis length 2A, where 2A is the distance between these two points of intersection, which is 2a cosh 2t. At the centre of the ellipse (which we find as midway between the two points of intersection) we have u = a cosh t, v = a sinh t, so the extremities of minor axis have $z^2 = u^2 - v^2 = a^2$. Thus the minor axis has length 2a. Hence the area of the ellipse is $\pi a^2\sqrt{(2 \cosh 2t)}$. The distance of the origin from the plane is its distance from the line (*) which is a/√(2 cosh 2t). Hence the volume of the cone is 1/3 area of base x height = 1/3 πa^3, as we had guessed.
Finally, if we have a general cone (which does not have circular cross-section or right-angle), we can transform it into a right circular cone by an affine transformation. Such transformations preserve the ratios of distances and hence the ratios of volumes. They also preserve tangency, so the envelope in this more general case will still be a hyperboloid asymptotic to the cone.

Problem B6

(1) The convex polygon C' lies inside the polygon C. Is it true that the perimeter of C' is no longer than the perimeter of C?

(2) C is the convex polygon with shortest perimeter enclosing the polygon C'. Is it true that the perimeter of C is no longer than the perimeter of C' ?

(3) The closed convex surface S' lies inside the closed surface S. Is it true that area S' $\leq$ area S?

(4) S is the smallest convex surface containing the closed surface S'. Is it true that area S $\leq$ area S'?

Solution

(1) True. On each side of C' take a semi-infinite strip perpendicular to the side and extending outwards from C'. Since C' is convex, these strips are all disjoint. But the intersection of C with each strip is at least as long as the corresponding side of C', hence the total length of C that intersects the strips is at least as long as the perimeter of C'.

(2) True. Take C as the convex hull of C'. If is sufficient to prove that the perimeter of C is no longer than the perimeter of C'. Evidently the vertices of C form a subset of the vertices of C'. If C' includes a vertex P which is not a vertex of C, then removing it does not increase the perimeter of C' (by the triangle inequality) and leaves C unaffected. So we can assume that C and C' have the same vertices. If C' just has the vertices of C permuted cyclically (or in reverse order), then its perimeter is the same. If not, then there are vertices A, B which are adjacent in C' but not in C. We show that in this case we can permute the vertices of C' so as to shorten the perimeter. We can repeat this process, shortening the perimeter each time. But there are only finitely many permutations so we must eventually reach a perimeter with the same length as C.

Suppose C' has vertices in the order $X_1X_2 \ldots X_n$. wlog we may assume that X_1 and X_2 are not adjacent in C, so we can find X_i on one side of X_1X_2 and X_j on the other. Then one of the pairs X_iX_{i+1}, $X_{i+1}X_{i+2}$, ... , $X_{j-1}X_j$ must straddle X_1X_2. Suppose it is X_kX_{k+1}. Consider the quadrilateral $X_1X_kX_2X_{k+1}$. Suppose its diagonals X_1X_2 and X_kX_{k+1} meet at O. Then $X_1X_2 + X_kX_{k+1} = X_1O + OX_2 + X_kO + OX_{k+1} = (X_1O + OX_k) + (X_2O + OX_{k+1}) > X_1X_k + X_2X_{k+1}$. So if we take C'' to have vertices in the order $X_1X_kX_{k-1} \ldots X_3X_2X_{k+1}X_{k+2} \ldots X_n$, then C'' has shorter perimeter.

(3) True. The same argument as (1) works. Take semi-infinite columns on each face of the inner polyhedron.

(4) False. Take the 4 vertices of a regular tetrahedron and its centre. We can find a surface S' with arbitrarily small area which contains these 5 points, but the convex hull S is the tetrahedron.

11th Putnam 1951

Problem A1
A is a skew-symmetric real 4 x 4 matrix. Show that det A ≥ 0.

Solution
Let A =

```
 0  a  b  c
-a  0  d  e
-b -d  0  f
-c -e -f  0
```

Then det A = $a^2f^2 + 2acdf - 2abef + b^2e^2 - 2bcde + c^2d^2$.
At this point it is helpful to notice that a only appears with f, b with e, and c with d. So putting X = af, Y = cd, Z = be, we have that det A = $X^2 + Y^2 + Z^2 + 2XY - 2XZ - 2YZ$. This easily factorizes as $(X + Y - Z)^2$.

Problem A2
k is a positive real and $P_1, P_2, \ldots, P_n$ are points in the plane. What is the locus of P such that $\sum PP_i^2 = k$? State in geometric terms the conditions on k for such points P to exist.

Solution
Let P_i have coordinates (a_i, b_i). Take the origin at the centroid of the points, so that $\sum a_i = \sum b_i = 0$. Then if P has coordinates (x, y), we have $\sum PP_i^2 = \sum((x - a_i)^2 + (y - b_i)^2) = n (x^2 + y^2) + \sum(a_i^2 + b_i^2)$. Let the origin be O, so that $\sum(a_i^2 + b_i^2) = \sum OP_i^2$. Then if $\sum OP_i^2 > k$, the locus is empty. Otherwise it is the circle with centre at the centroid of the n points and radius $\sqrt{((k - \sum OP_i^2)/n)}$.

Problem A3
Find $\sum_0^\infty (-1)^n/(3n + 1)$.

Solution
We have $\ln(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$ (1). Put $\omega = e^{2\pi i/3}$, so that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 1$. Then $\omega^2 \ln(1 + \omega x) = x - \omega x^2/2 + \omega^2 x^3/3 - x^4/4 + \ldots$ (2) and $\omega \ln(1 + \omega^2 x) = x - \omega^2 x^2/2 + \omega x^3/3 - x^4/4 + \ldots$ (3).

Adding (1), (2), (3) we get $3(x - x^4/4 + x^7/7 - \ldots) = \ln(1 + x) + \omega^2 \ln(1 + \omega x) + \omega \ln(1 + \omega^2 x)$. Hence the required series sums to $1/3 (\ln 2 + \omega^2 \ln(1 + \omega) + \omega \ln(1 + \omega^2))$.

If $k = \ln(1 + \omega)$, then $e^k = 1 + \omega = - \omega^2 = e^{i\pi/3}$, so $k = i\pi/3$. Hence $\omega^2 \ln(1 + \omega) = -(1 + i\sqrt{3})/2\ i\pi/3 = \pi/2\sqrt{3} - i\pi/6$. Similarly, $\omega \ln(1 + \omega^2) = -i\pi/3\ (-1/2 + i\sqrt{3}/2) = \pi/2\sqrt{3} + i\pi/6$. Hence the series sums to $1/3 \ln 2 + \pi/3\sqrt{3}$.

Problem A4
Sketch the curve $y^4 - x^4 - 96y^2 + 100x^2 = 0$.

Solution
The graph is in three parts. It is symmetrical on reflection in the x-axis and on reflection in the y-axis. The central part is a figure of 8 with crossing point at the origin and touching the lines $y/x = \pm\sqrt{(100/96)}$. The top and bottom of the figure of 8 are at $(0, \pm\sqrt{96})$. The right hand part is a sort of dimpled hyperbola with asymptotes $y/x = \pm 1$. It comes in along each asymptote and touches the line

$x = 8$ at $y = \pm\sqrt{48}$, then it dimples back away from the origin to cut the x-axis at $x = 10$. The left hand part of the curve is just the reflection of this in the y-axis.

Problem A5
Show that a line in the plane with rational slope contains either no lattice points or an infinite number. Show that given any line L of rational slope we can find $\delta > 0$, such that no lattice point is a distance k from L where $0 < k < \delta$.

Solution
The first part is obvious. If the line passes through (m, n) and has slope a/b (where m, n, a, b are all integers), then it also passes through (m + kb, n + ka) for any integer a. Although not required, we note that there are lines with rational slope passing through no lattice points, for example, $y = 1/2$ and $y = x + 1/2$.

Suppose L is $bx - ay + c = 0$. The distance of L from (x, y) is $|bx - ay + c|/\sqrt{(a^2 + b^2)}$. If (x, y) is a lattice point then $bx - ay$ is an integer. Hence $|bx - ay + c|$ is either zero or at least 1 if c is an integer, or at least the distance of c from the nearest integer if c is not an integer. So we can find $k > 0$, so that $|bx - ay + c| > k$ for all lattice points not on L. Hence the distance $|bx - ay + c|//\sqrt{(a^2 + b^2)}$ is at least $k/\sqrt{(a^2 + b^2)}$ for all lattice points not on L. So we can take δ to be $k/\sqrt{(a^2 + b^2)}$.

Problem A6
Let C be a parabola. Take points P, Q on C such that (1) PQ is perpendicular to the tangent at P, (2) the area enclosed by the parabola and PQ is as small as possible. What is the position of the chord PQ?

Solution
We may take the parabola as $y = x^2$ and P as (a, a^2). The tangent at P has slope 2a, so the normal has slope $-1/2a$ and equation $(y - a^2) + 1/2a\,(x - a) = 0$. This meets the parabola at $x^2 + x/2a - a^2 - 1/2 = 0$, so the other root is $x = -1/2a - a$ and Q is $(-1/2a - a, (a + 1/2a)^2)$. The area under the curve between P and Q is $1/3\,(a^3 + (a + 1/2a)^3\,)$. The area under the chord PQ is $(2a + 1/2a)\,(a^2 + (a + 1/2a)^2)/2 = 2a^3 + 3a/2 + 1/2a + 1/16a^3$. Hence the area enclosed is $4/3\,a^3 + a + 1/4a + 1/48a^3$. We wish to minimise this. The gradient is $4a^2 + 1 - 1/4a^2 - 1/16a^4 = 1/16a^4\,(4a^2 + 1)^2(2a - 1)(2a + 1)$. Hence there is a minimum at $a = 1/2$ (and another at $a = -1/2$). In the minimum position P is (1/2, 1/4) or (-1/2, 1/4), so the minimum positions are the intersection of the parabola with the perpendicular to the axis through the focus (0, 1/4).

Problem A7
Show that if $\sum a_n$ converges, then so does $\sum a_n/n$.

Solution
Put $s_n = a_1 + a_2 + \ldots + a_n$. Then $s_n - s_{n-1} = a_n$, and so $a_1/1 + a_2/2 + a_3/3 + \ldots + a_n/n = s_1(1/1 - 1/2) + s_2(1/2 - 1/3) + \ldots + s_n(1/n - 1/n+1) + s_n/n+1$. Now $s_1(1/1 - 1/2) + s_2(1/2 - 1/3) + \ldots + s_n(1/n - 1/n+1) + \ldots$ is absolutely convergent, because the sequence $|s_n|$ is bounded, say by B. Hence $|s_1(1/1 - 1/2)| + |s_2(1/2 - 1/3)| + \ldots + |s_n(1/n - 1/n+1)| <= B(\,(1/1 - 1/2) + (1/2 - 1/3) + \ldots + (1/n - 1/n+1)\,) = B(1 - 1/n+1) < B$. Obviously $s_n/n+1$ tends to zero, so $\sum a_n/n$ is absolutely convergent and hence convergent.

Problem B1
R is the reals. f, g **:** $R^2 \rightarrow R$ have continuous partial derivatives of all orders. What conditions must they satisfy for the differential equation $f(x, y)\,dx + g(x, y)\,dy = 0$ to have an integrating factor $h(xy)$?

Solution
If h(xy) is an integrating factor, then f(x, y) h(xy) dx + g(x, y) h(xy) dy is an exact form. In other words, $\partial/\partial y$ (f(x, y) h(xy)) = $\partial/\partial x$ (g(x, y) h(xy). So h $\partial f/\partial y$ + x f h' = h $\partial g/\partial x$ + y g h'. Hence h'/h = ($\partial f/\partial y$ - $\partial g/\partial x$) / (yg - xf).

The lhs is a function of xy, so the rhs must be also. Equally if the rhs is a function of xy, then we can immediately integrate to get h(xy). Thus the required necessary and sufficient condition for the existence of an integrating factor is that ($\partial f/\partial y$ - $\partial g/\partial x$) / (yg - xf) should be a function of xy.

Problem B2
R is the reals. Find an example of functions f, g **:** R → R, which are differentiable, not identically zero, and satisfy (f/g)' = f '/g' .

Solution
Suppose we take f = gh. Then we require h' = h + g/g' h'. If we take g/g' = 2, then h' = -h. So this suggests $f(x) = e^{-x/2}$, $g(x) = e^{x/2}$. Checking, we see that that works.

Problem B3
Show that ln(1 + 1/x) > 1/(1 + x) for x > 0.

Solution
We have the well-known $e^y > 1 + y$ for all y not equal to 1. [Prove, for example, by differentiating.] So ln(1 + y) < y. Put y = -1/(1 + x), so 1 + y = x/(1 + x). Hence 1/(1 + x) < - ln(x/(1+x)) = ln(1 + 1/x).

Problem B4
Can we find four distinct concentric circles all touching an ellipse?

Solution
Answer: yes, we can find four such circles for any ellipse except the circle.

We show first that we can find four such circles for any non-circular ellipse. Let the ellipse have centre O, major axis AB (length 2a) and minor axis CD (length 2b). Take a point P with OP < (a - b)/2. Then using the triangle inequality we have PA > OA - OP = (a + b)/2, PB > OB - OP = (a + b)/2. PC < OC + OP = (a + b)/2, PD < OD + OP = (a + b)/2. Put k = (a + b)/2. Then, if Q is a point on the ellipse, (PQ - k) is positive at A and B and negative at C and D. Hence there must be at least four points Q on the ellipse at which it is stationary and hence at which PQ is normal to the ellipse. There is a point A' near A which is a maximum and hence has PA' > k. Similarly, there is a point B' near B with PB' > k. There is a point C' near C with PC' < k and a point D' near D with PD' < k. Let us assume that P is in the quadrant AOC and not on either axis. The reflection D" of D' in the line AB lies at or near C' and hence has PD" >= PC' (which is known to be a minimum). Put P lies closer to D" than to D'. Hence PC' < PD'. A similar argument shows that PB' > PA'. So we have PB' > PA' > k > PD' > PC'. Thus the circles centre P through A', B', C' and D' are all distinct and all touch the ellipse.

Note that we cannot find 4 circles in the case when the ellipse is circular. For given a point P and a circle E centre O, a circle centre P can only touch E on the line OP, but that implies there can only be two such circles.

Problem B5
T is a torus, center O. The plane P contains O and touches T. Prove that P ∪ T is two circles.

Solution
We may take the torus to be $(\sqrt{(x^2 + y^2)} - a)^2 + z^2 = b^2$, where a > b. Viewed in the x-z plane the torus appears as two circles each radius b and centred at x = ±a, z = 0. The plane P appears as the line through the origin tangent to both circles and hence having equation c z = b x, where $c = \sqrt{(a^2 - b^2)}$. P also contains the y-axis which is perpendicular to the line z = k x. Hence the equation of the plane P is z = k x.

It is far from obvious that P meets T in two circles. If we draw the plane P we see that T contains the four points on the y-axis, y = ±a ± b, and also the points a distance c either side of the origin. The obvious way to put these onto two circles is to take circles radius c, centred at (x, y, z) = (0, ±b, 0). These can be written as (x, y, z) = (c cos t, ±b + a sin t, b cos t). Hence $\sqrt{(x^2 + y^2)} = a \pm b \sin t$. So $(\sqrt{(x^2 + y^2)} - a)^2 = b^2\sin^2 t$ and $(\sqrt{(x^2 + y^2)} - a)^2 + z^2 = b^2\sin^2 t + b^2\cos^2 t = b^2$. Thus these circles do indeed lie on the torus.

We may write the equation of T as $2a \sqrt{(x^2 + y^2)} = x^2 + y^2 + z^2 + c^2$. Hence $4a^2(x^2 + y^2) = (x^2 + y^2 + z^2 + c^2)^2$. So $(x^2 + y^2 + z^2 - c^2)^2 = 4a^2x^2 + 4a^2y^2 - 4c^2(x^2 + y^2 + z^2) = 4b^2x^2 + 4b^2y^2 - 4c^2z^2$. Hence $(x^2 + y^2 + z^2 - c^2)^2 - (2by)^2 = 4b^2x^2 - 4b^2c^2$. So $(x^2 + (y + b)^2 + z^2 - a^2)(x^2 + (y - b)^2 + z^2 - a^2) = 4(bx - cz)(bx + cz)$.

So if a point lies on the plane P, which has equation bx - cz = 0 and on the torus T, then it also lies on one of the two spheres $(x^2 + (y + b)^2 + z^2 - a^2) = 0$ and $(x^2 + (y - b)^2 + z^2 - a^2) = 0$. But a plane always intersects a sphere in a circle (or not at all) so the intersection of P and T is at most two circles. We have already established that it contains two circles, so it cannot contain any other points.

Problem B6
The real polynomial $p(x) \equiv x^3 + ax^2 + bx + c$ has three real roots $\alpha < \beta < \gamma$. Show that $\sqrt{(a^2 - 3b)} < (\gamma - \alpha) \leq 2 \sqrt{(a^2/3 - b)}$.

Solution
$a^2 - 3b = (\alpha+\beta+\gamma)^2 - (\alpha\beta + \beta\gamma + \gamma\alpha) = (\gamma - \alpha)^2 - (\gamma - \beta)(\beta - \alpha) < (\gamma - \alpha)^2$. Hence $\sqrt{(a^2 - 3b)} < (\gamma - \alpha)$.

Also $4(a^2 - 3b) - 3 (\gamma - \alpha)^2 = (\gamma - \alpha)^2 - 4(\gamma - \beta)(\beta - \alpha) = ((\gamma - \beta) + (\beta - \alpha))^2 - 4(\gamma - \beta)(\beta - \alpha) = ((\gamma - \beta) - (\beta - \alpha))^2 \geq 0$. Hence $(\gamma - \alpha) \leq 2 \sqrt{(a^2/3 - b)}$.

Problem B7
In 4-space let S be the 3-sphere radius r: $w^2 + x^2 + y^2 + z^2 = r^2$. What is the 3-dimensional volume of S? What is the 4-dimensional volume of its interior?

Solution
Answer: $2\pi^2 r^3$, $1/2 \pi^2 r^4$.

A hyperplane perpendicular to the x-axis will cut S in a sphere. If the hyperplane is at a distance k from the origin, then the radius of the sphere will be $\sqrt{(r^2 - k^2)}$. So the 4-dimensional volume $= \int_{-r}^{r} 4\pi/3 (r^2 - x^2)^{3/2} dx = 4\pi r^4/3 \int_{-1}^{1} (1 - t^2)^{3/2} dt$.

Let $K = (1 - t^2)^{3/2} dt$. We find that $A = t(1 - t^2)^{3/2}$ differentiates to $K - 3t^2(1 - t^2)^{1/2}$. $B = t(1 - t^2)^{1/2}$ differentiates to $(1 - t^2)^{1/2} - t^2(1 - t^2)^{-1/2}$. $C = \sin^{-1}t$ differentiates to $(1 - t^2)^{-1/2}$. Hence B + C differentiates to $2(1 - t^2)^{1/2}$, so 2A + 3(B + C) differentiates to 8K. So $\int_{-1}^{1} (1 - t^2)^{3/2} dt = (\text{terms involving powers of } (1 - t^2)^{1/2} + 3/8 \sin^{-1}t)|_{-1}^{1} = 3 \pi /8$. Hence the 4-dimensional volume is $\pi^2 r^4/2$.

The volume of the 3-dimensional surface is the derivative wrt r of the 4-dimensional volume (because the 4-dimensional volume for r + δr is the 4-dimensional volume for r + the 3-dimensional volume of the bounding sphere times δr). Hence the 3-dimensional volume is $2 \pi^2 r^3$.

12th Putnam 1952

Problem A1
p(x) is a polynomial with integral coefficients. The leading coefficient, the constant term, and p(1) are all odd. Show that p(x) has no rational roots.

Solution
Suppose there is a rational root p/q. Let $p(x) = \sum_0^n a_r x^r$. Then $\sum a_r p^r q^{n-r} = 0$. So p divides a_0 and hence is odd, and q divides a_n and must also be odd. But now $\sum a_r p^r q^{n-r} = a_r = 1 \pmod 2$. Contradiction.

Problem A2
Show that the solutions of the differential equation $(9 - x^2) (y')^2 = 9 - y^2$ are conics touching the sides of a square.

Solution
We have $dy/dx = \pm \sqrt{(1 - (y/3)^2)} / \sqrt{(1 - (x/3)^2}$. Integrating, $\sin^{-1}y/3 = \pm \sin^{-1}x/3 + A$, where A is a constant. Hence $y/3 = \sin A \cos(\pm\sin^{-1}x/3) \pm x/3 \cos A = \sin A \sqrt{(1 - (x/3)^2)} \pm x/3 \cos A$. So $y = k (9 - x^2) \pm h x$, where $h^2 + k^2 = 1$. So $y^2 \pm 2hxy + x^2 = 9(1 - h^2)$. This is evidently a family of conics. Note that h is not restricted to being less than 1. By taking a suitable complex constant A we can get any value for h. But since negative values are possible, there is no benefit in retaining the ± sign. So we can write the family of conics simply as $y^2 + 2hxy + x^2= 9(1 - h^2)$.

We might guess that the square is $x = \pm3$, $y = \pm3$. If we substitute $x = 3$ in the equation for the conic, we get $y^2 + 6hy + 9 = 9 - 9 h^2$, or $(y - 3h)^2 = 0$. This has a repeated root $y = 3h$, which shows that the line $x = 3$ touches the conic. Similarly for the other four sides of the square.

Problem A3
Let the roots of the cubic $p(x) \equiv x^3 + ax^2 + bx + c$ be α, β, γ. Find all (a, b, c) so that $p(\alpha^2) = p(\beta^2) = p(\gamma^2) = 0$.

Solution
Answer: (0, 0, 0) roots 0, 0, 0
(-1, 0, 0) roots 0, 0, 1
(-2, 1, 0) roots 0, 1, 1
(0, -1, 0) roots 0, 1, -1
(1, 1, 0) roots 0, ω, ω^2, where $\omega = e^{2\pi i/3}$
(-3, 3, -1) roots 1, 1, 1
(-1, -1, 1) roots 1, 1, -1
(1, -1, -1) roots 1, -1, -1
(0, 0, -1), roots 1, ω, ω^2
(i, 1, -i), roots 1, -1, -i
(-i, 1, i), roots 1, -1, i
(1-ω, 1-ω, -ω, roots ω, ω, ω^2
(1-ω^2, 1-ω^2, -ω^2), roots ω, ω^2, ω^2
(1+ω, 1+ω, ω), roots ω, -ω, ω^2
(1+ω^2, 1+ω^2, ω^2), roots ω, ω^2, -ω^2
((1-i$\sqrt{7}$)/2, (-1-i$\sqrt{7}$)/2, -1), roots k, k^2, k^4, where $k = e^{2i\pi/7}$
((1+i$\sqrt{7}$)/2, (-1+i$\sqrt{7}$)/2, -1), roots k^3, k^5, k^6.

$p(x) = 0$ has only three roots, so the squares must all be found in $\{ \alpha, \beta, \gamma \}$. Let α be the root with the largest modulus, then $|\alpha|^2 > |\alpha|$ unless $|\alpha| \le 1$. Let γ be the root with the smallest non-zero

modulus, then $|\gamma|^2 < |\beta|$ unless $|\gamma| \geq 1$. Hence all non-zero roots have modulus 1.
All roots zero is possible and gives (a, b, c) = (0, 0, 0).

Suppose $\beta = \gamma = 0$ and α is non-zero. Then $\alpha^2 = \alpha$, so $\alpha = 1$. This gives (a, b, c) = (-1, 0, 0).

Suppose $\gamma = 0$, and α, b are non-zero. If $^{\alpha 2} = \alpha$, then $\alpha = 1$. Then $\beta^2 = 1$ or β, so $\beta = 1$ or -1. This gives (a, b, c) = (-2, 1, 0) or (0, -1, 0). If α^2 is not α (and β^2 not β), then we must have $\alpha^4 = \alpha$, so α and β are ω and ω^2, where $\omega = e^{2i\pi/3}$. This gives (a, b, c) = (1, 1, 1).

The remaining possibility is that all roots are non-zero (and have modulus 1). There might be 3, 2, 1 or 0 roots equal to 1. The first case gives (-3, 3, -1). In the second case the third root must satisfy $\gamma^2 = 1$ or γ and hence must be -1, giving (a, b, c) = (-1, -1, 1).

We consider the third case (just one root equal to 1). If both the roots not equal to 1 have square 1, then they must both be -1 and (a, b, c) = (1, -1, -1). If one has square 1 and one not. Then they must be -1 and $\pm i$. This gives (a, b, c) = (i, 1, i) or (-i, 1, -i). If neither have square 1, then we must have $\beta^2 = \gamma$, $\gamma^2 = \beta$, hence $\beta^3 = 1$ and β and γ are ω and ω^2.

The final case is that no roots are 1. If we have $\alpha^2 = \beta$ and $\beta^2 = \alpha$, then α and β are ω and ω^2 and $\gamma^2 = \omega$ or ω^2, so $\gamma = \pm\omega$ or $\pm\omega^2$. So the roots are ω, ω, ω^2 or ω, ω^2, ω^2 or ω, $-\omega$, ω^2 or ω, ω^2, $-\omega^2$. This gives (a, b, c) = $(1-\omega, 1-\omega, \omega)$ or $(1-\omega^2, 1-\omega^2, \omega^2)$ or $(1+\omega, 1+\omega, \omega)$ or $(1+\omega^2, 1+\omega^2, \omega^2)$.

The final possibility is $\alpha^2 = \beta$, $\beta^2 = \gamma$, $\gamma^2 = \alpha$. Hence $\alpha^7 = 1$. If we put $k = e^{i2\pi/7}$, then the roots could be k, k^2, k^4 or k^6, k^5, k^3. This corresponds to (a, b, c) = $((1-i\sqrt{7})/2, (-1-i\sqrt{7})/2, -1)$, $((1+i\sqrt{7})/2, (-1+i\sqrt{7})/2, -1)$.

Problem A4
A map represents the polar cap from latitudes -45^o to 90^o. The pole (latitude 90^o) is at the center of the map and lines of latitude on the globe are represented as concentric circles with radii proportional to (90^o - latitude). How are east-west distances exaggerated compared to north-south distances on the map at a latitude of -30^o?

Solution
Answer: too high by a factor $4\pi/\sqrt{27} = 2.42$.

Let the globe have radius R. Then the distance from the pole to the circle of latitude at -30 is $2\pi R/3$. The circumference of the circle is $2\pi R\ (\sqrt{3})/2$. The ratio of circumference to distance is $(\sqrt{27})/2$. On the map the corresponding ratio is 2π. Thus the map overstates by a factor $4\pi/\sqrt{27}$.

Problem A5
a_i are reals $\neq 1$. Let $b_n = 1 - a_n$. Show that $a_1 + a_2b_1 + a_3b_1b_2 + a_4b_1b_2b_3 + ... + a_nb_1b_2 ... b_{n-1} = 1 - b_1b_2 ... b_n$.

Solution
Induction on n. Obvious for n = 1. Suppose true for n. Then $a_1 + a_2b_1 + a_3b_1b_2 + a_4b_1b_2b_3 + ... + a_{n+1}b_1b_2 ... b_n = 1 - b_1b_2 ... b_n + a_{n+1}b_1b_2 ... b_n = 1 - b_1b_2 ... b_n(1 - a_{n+1}) = 1 - b_1 ... b_{n+1}$, which is the result for n+1.

Problem A6
Prove that there are only finitely many cuboidal blocks with integer sides a x b x c, such that if the block is painted on the outside and then cut into unit cubes, exactly half the cubes have no face painted.

Solution
It is sufficient to show that abc = 2(a - 2)(b -2)(c - 2) has only finitely many solutions in integers $c \le b \le a$. If $c \le 4$, then $c \ge 2(c - 2)$, but $ab > (a - 2)(b - 2)$, so there are no solutions. If $c \ge 10$, then $c/(c - 2) \le 10/8$, so $D \le 125/64 < 2$, where for convenience we have written abc/((a - 2)(b - 2)(c - 2)) as D, so there are no solutions. Hence c = 5, 6, 7, 8, or 9.

If $c = 5$ and $b \ge 24$, then $D \le 5/3\ (24/22)^2 < 2$, so there are no solutions. Now a/(a - 2) is strictly monotonic decreasing, so there is at most one solution for given b, c. Hence there are only finitely many solutions for c = 5.

Similarly, for c = 6, we find that are only solutions for $b < 16$; for c = 7, we find $b < 14$; for c = 8, $b < 12$ and for c = 9, $b < 12$. Hence there are only finitely many solutions in each case.

Problem A7
Let O be the center of a circle C and P_0 a point on the circle. Take points P_n on the circle such that angle $P_nOP_{n-1} = +1$ for all integers n. Given that π is irrational, show that given any two distinct points P, Q on C, the (shorter) arc PQ contains a point P_n.

Solution
All the points must be distinct, for if we had $P_n = P_m$ for $n < m$, then the integer m - n would be a multiple of 2π and hence π would be rational.

Choose N so that the arc length $PQ < 2\pi R/N$, where R is the radius of the circle. Divide the circle into N equal arcs length $2\pi R/N$. Then at least two of the points $P_1, P_2, \ldots, P_{N+1}$ must lie in the same arc. Suppose the points are P_n and P_{n+m} (with m, n positive), so that P_nP_{n+m} is an arc of length less than $2\pi R/N$. Hence the arcs $P_{n+m}P_{n+2m}$, $P_{n+2m}P_{n+3m}$ ... have the same length. So for some k, P_{n+km} will lie in the arc PQ.

Problem B1
ABC is a triangle with, as usual, AB = c, CA = b. Find necessary and sufficient conditions for $b^2c^2/(2bc \cos A) = b^2 + c^2 - 2bc \cos A$.

Solution
Answer: BC = AB or CA.

We have that $a^2 = b^2 + c^2 - 2bc \cos A$. So the condition implies that $(b^2 + c^2 - a^2)/(2bc) = \cos A = bc/(2a^2)$. Hence, $b^2c^2 = a^2(b^2 + c^2 - a^2)$, so $(a^2 - b^2)(a^2 - c^2) = 0$. Hence a = b or c.

Conversely, if a = b, then the altitude from C meets AB at its midpoint and so $\cos A = \cos B = c/(2a) = bc/(2a^2)$. Hence $a^2 = bc/(2 \cos A)$, so $(b^2 + c^2 - 2bc \cos A) = b^2c^2/(2bc \cos A)$.

Problem B2
Find the surface comprising the curves which satisfy dx/(yz) = dy/(zx) = dz/(xy) and which meet the circle $x = 0, y^2 + z^2 = 1$.

Solution
We have x dx = y dy = z dz. Integrating, $y^2 = x^2 + h$, $z^2 = x^2 + k$ (*). We are told that some point of the circle $x = 0, y^2 + z^2 = 1$ belongs to the curve, so h and k must be non-negative with sum 1. Hence the curve (*) lies in the surface $2x^2 + 1 = y^2 + z^2$, which is a one-sheet hyperboloid with the x-axis as an axis of symmetry. However, not all points of this surface lie on a curve (*). Clearly we require $|y| >= |x|$ and $|z| \ge |x|$. Equally, it is clear that this is a sufficient condition for we can then find h, k to satisfy (*).

Problem B3
Let A(x) = be the matrix

$$\begin{matrix} 0 & a-x & b-x \\ -a-x & 0 & c-x \\ -b-x & -c-x & 0 \end{matrix}$$

For which (a, b, c) does det A(x) = 0 have a repeated root in x?

Solution
Multiplying out, we find that det A(x) = $-2x^3 + 2(bc - ac + ab)x$. This has roots 0 and $\pm\sqrt{(ab + bc - ac)}$. This has a repeated root iff ac = ab + bc.

Problem B4
The solid S consists of a circular cylinder radius r, height h, with a hemispherical cap at one end. S is placed with the center of the cap on the table and the axis of the cylinder vertical. For some k, equilibrium is stable if r/h > k, unstable if r/h < k and neutral if r/h = k. Find k and show that if r/h = k, then the body is in equilibrium if any point of the cap is in contact with the table.

Solution
Let O be the centre of the base of the hemispherical cap. As the solid rolls, O is always a height r above the table. Consider a point P on the axis a distance d from O and inside the cap. If the solid is rolled to a position where the axis is at an angle θ to the vertical, then P ends up a vertical distance d cos θ below O, so its height has increased. Similarly, a point on the axis a distance d above O would end up a vertical distance d cos θ above O, so its height would decrease. So equilibrium will be stable, neutral or unstable according as the centre of mass is below, at or above O.

We start by finding the position of the centre of mass of the cap. Take the x-axis along the axis with the origin at O. Assume the density is ρ. The moment about an axis perpendicular to the x-axis is $\int_0^r \pi(r^2 - x^2)\rho\, x\, dx = \pi \rho r^4/4$. The centre of mass of the cylindrical portion of the solid is obviously a distance h/2 from O, so we require $(\pi \rho r^2 h)\, h/2 = \pi \rho r^4/4$ or r/h = √2 get get the centre of mass at O. Thus k = √2.

If the centre of mass is at O, then the centre of mass is always at the same height above the table, so the solid is in equilibrium however far it is tilted from the vertical.

Problem B5
The sequence a_n is monotonic and $\sum a_n$ converges. Show that $\sum n(a_n - a_{n+1})$ converges.

Solution
The sum of the first n terms is $(a_1 - a_2) + 2(a_2 - a_3) + ... + n(a_n - a_{n+1}) = a_1 + a_2 + a_3 + ... + a_n - n\, a_{n+1}$. We are given that $\sum a_n$ converges, so it is sufficient to show that the sequence $n\, a_{n+1}$ converges to zero.

But since $\sum a_n$ converges, $|a_{n+1} + a_{n+2} + ... + a_{2n}|$ is arbitrarily small for n sufficiently large. Since a_n is monotonic, this implies that $n\, a_{n+1}$ is arbitrarily small for n sufficiently large.

Problem B6
A, B, C are points of a fixed ellipse E. Show that the area of ABC is a maximum iff the centroid of ABC is at the center of E.

Solution
Take the ellipse as $x^2/a^2 + y^2b^2 = 1$. Take the three points as A (a cos u, b sin u), B (a cos v, b sin v), C (a cos w, b sin w). If the area is a maximum then the tangent at A must be parallel to BC (otherwise we could keep B and C fixed and move A to increase the altitude and hence the area). So we have - b/a cot u = (b sin v - b sin w)/(a cos v - a cos w) and hence cos(u - v) = cos(u - w). Similarly, cos(v - u) = cos(v - w).

Hence 2u = v + w (mod 2π). Similarly 2v = u + w (mod 2π), so 3u = 3v (mod 2π). Hence u, v, w are (in some order and for some k) k, k + 2π/3, k - 2π/3. Hence the centroid is at x = a cos k + a cos(k+2π/3) + a cos(k-2π/3) = 0, y = b sin k + b sin(k+2π/3) + b sin(k-2π/3) = 0.

Conversely, if the centroid of A, B, C is at the centre, then cos u + cos v + cos w = 0, sin u + sin v + sin w = 0. Hence cos(u - v) = cos u (- cos u - cos w) + sin u (- sin u - sin w) = -1 - cos(u - w). Hence cos(u - v) + cos(u - w) = -1. Similarly, cos(v - u) + cos(v - w) = -1, hence cos(u - w) = cos(v - w). So cos w (cos u - cos v) = sin w (sin v - sin u), so -b/a cot w = (b sin v - b sin u)/(a cos v - a cos u), and hence the tangent at C is parallel to AB. Simiarly, the tangent at A is parallel to BC and the tangent at B is parallel to AC. Hence the area is a maximum.

Problem B7
Let R be the reals. Define a_n by $a_1 = \alpha \in R$, $a_{n+1} = \cos a_n$. Show that a_n converges to a limit independent of α.

Solution
$|\cos x| \le 1$, so $-1 \le a_2 \le 1$. Hence $0 < a_3 < 1$ and so $0 < a_n < 1$ for $n \ge 3$.

cos x is strictly monotonic decreasing over the range 0 to 1, so if $a_n < a_{n+1}$ then $a_{n+2} < a_{n+1}$. Now the gradient of cos x is greater than -1 throughout the interval (0, 1), so if $a_n < a_{n+1}$ then $\cos a_n - \cos a_{n+1} < a_{n+1} - a_n$. Hence $a_n < a_{n+2} < a_{n+1}$. Similarly, if $a_n > a_{n+1}$, then $a_n > a_{n+2} > a_{n+1}$. Thus either a_{2n} is an increasing sequence bounded above and a_{2n+1} is a decreasing sequence bounded below or vice versa. Hence both a_{2n} and a_{2n+1} converge.

By the mean value theorem $|a_{n+1} - a_{n+2}| = |\cos a_n - \cos a_{n+1}| = \sin k\, |a_n - a_{n+1}|$ for some k in (0, 1). But $\sin k < \sin 1 < 0.9$. So a_{2n} and a_{2n+1} must converge to the same limit. Suppose this limit is h. Then h = cos h. But cos x is strictly decreasing in (0, 1) and x is strictly increasing, so they have a single point of intersection in (0, 1). Thus h must be the unique point in (0, 1) at which h = cos h.

13th Putnam 1953

Problem A1
Show that $(2/3)\ n^{3/2} < \sum_1^n \sqrt{r} < (2/3)\ n^{3/2} + (1/2)\ \sqrt{n}$.

Solution
The gradient of $\sqrt{x}$ is falling from x = 0 to x = n, so $2/3\ n^{3/2} = \int_0^n \sqrt{x}\ dx < \sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n}$. That gives the first inequality.

To get the second, we note that the chords joining $(r, \sqrt{r})$ to $(r+1, \sqrt{(r+1)})$ all lie under the curve, so if we subtract the area of the little triangles from $\sum_1^n \sqrt{r}$ then we get something less than the integral. The triangles have area $1/2\ \sqrt{1} + 1/2\ (\sqrt{2} - \sqrt{1}) + 1/2\ (\sqrt{3} - \sqrt{2}) + ... + 1/2\ (\sqrt{n} - \sqrt{(n-1)}) = 1/2\ \sqrt{n}$. That gives the second inequality.

Problem A2
The complete graph with 6 points and 15 edges has each edge colored red or blue. Show that we can find 3 points such that the 3 edges joining them are the same color.

Solution
Take any point A. It has 5 edges, so at least 3 of them must be the same color. wlog it is red. So we have B, C, D with AB, AC, AD all red. Now if any of the three edges BC, CD, DB is red, then that gives us a red triangle. But if they are all blue, then BCD is a blue triangle.

Problem A3
a, b, c are real, and the sum of any two is greater than the third. Show that $2(a + b + c)(a^2 + b^2 + c^2)/3 > a^3 + b^3 + c^3 + abc$.

Solution
wlog we may take $a \geq b \geq c$. c (and hence also a and b) must be positive. For if $c \leq 0$, then $a \geq b + c$. Contradiction.

Multiplying out the required relation and cancelling, we must prove that $2(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) > a^3 + b^3 + c^3 + 3abc$. But we have $a^2(b + c) > a^3$, $b^2(c + a) > b^3$, $c^2(a + b) > c^3$. So it is sufficient to prove that $a^2b + a^2c + b^2a + b^2c + c^2a + c^2b > 3abc$. But $a^2b \geq abc$ and $ab^2 \geq abc$. Also $b^2c + bc^2 = bc(b + c) > abc$. Similarly, $ac(a + c) > abc$. So we have in fact proved the slightly stronger result with abc replaced by 4abc/3.

Problem A4
Using $\sin x = 2 \sin x/2 \cos x/2$ or otherwise, find $\int_0^{\pi/2} \ln \sin x\ dx$.

Solution
Let $I = \int_0^{\pi/2} \ln \sin x\ dx$. Making the suggested substitution, we get $I = \pi/2 \ln 2 + \int_0^{\pi/2} \ln \sin x/2 \cos x/2\ dx$. Putting $y = x/2$ we get $\int_0^{\pi/2} \ln \sin x/2 \cos x/2\ dx = 2 \int_0^{\pi/4} \ln \sin y + \ln \cos y\ dy$. But $\cos y = \sin(\pi/2 - y)$, so $\int_0^{\pi/4} \ln \cos y\ dy = \int_{\pi/4}^{\pi/2} \ln \sin y\ dy$. Hence $I = \pi/2 \ln 2 + 2I$, so $I = - \pi/2 \ln 2$.

Problem A5
S is a parabola with focus F and axis L. Three distinct normals to S pass through P. Show that the sum of the angles which these make with L less the angle which PF makes with L is a multiple of π.

Solution
We start by finding a formula for tan(A+B+C+D). Applying $\tan(A+B) = (\tan A + \tan B)/(1 - \tan A$

tan B) twice and writing a = tan A, b = tan B, c = tan C, d = tand D, we get tan(A+B+C+D) = (∑a - ∑abc)/(1 - ∑ab + abcd).

Take the parabola as $y = x^2$. The gradient at (k, k^2) is 2k, so the normal has slope -1/2k. The normal has equation $2k^3 + (1 - 2y)k - x = 0$. For given x, y this is a cubic in k, so has 1 or real 3 roots. Their slopes satisfy the cubic $4x\ h^3 + 2(1 - 2y)\ h^2 + 1 = 0$ (*).

The focus is at (0, 1/4), so the line joining (x, y) has slope (y - 1/4)/x and negative is (1/4 - y)/x. The quartic with this root and the 3 roots of (*) is $(h + (y - 1/4)/x\ )(4x\ h^3 + 2(1 - 2y)\ h^2 + 1)$ (**). So the angles the normals make with L less the angle which PF makes with L is a multiple of π iff (∑a = ∑abc), where a, b, c, d are the slopes of the four lines and hence iff the coefficients of h^3 and h in (**) are the same. But the coefficient of h^3 is 2(1 - 2y) + 4x (y - 1/4)/x = 1 and the coefficient of h is 1.

Problem A6
Show that √7, √(7 - √7), √(7 - √(7 + √7)), √(7 - √(7 + √(7 - √7))), ... converges and find its limit.

Solution
Answer: 2.
Let x_n be the nth number in the sequence. We have $x_{n+2} = √(7 - √(7 + x_n))$. Hence if $x_n < 2$, then $x_{n+2} > 2$, and if $x_n > 2$, then $x_{n+2} < 2$. So if the sequence converges, its limit must be 2.

If $x_n = 2 + ε$, with $0 < ε < 1$, then $9 + ε < 9 + ε + ε^2/36 = (3 + ε/6)^2$, so $x_{n+2} > √(7 - (3 + ε/6)) = √(4 - ε/6)$. But certainly $ε/6 < ε/3 - ε^2/144$, so $√(4 - ε/6) > 2 - ε/12$. Thus x_{n+2} differs from 2 by less than ε/12. Similarly, if $x_n = 2 - ε$, with $0 < ε < 1$, then $9 - ε > 9 - ε/2 + ε^2/144 = (3 - ε/12)^2$, so $x_{n+2} < √(4 + ε/12) < 2 + ε/48$.

So we have established that if $|x_n - 2| < 1$, then $|x_{n+2} - 2| < 1/12\ |x_n - 2|$. But we certainly have $|x_1 - 2| < 1$, and $|x_2 - 2| < 1$, so x_n converges to 2.

Problem A7
$p(x) ≡ x^3 + ax^2 + bx + c$ has three positive real roots. Find a necessary and sufficient condition on a, b, c for the roots to be cos A, cos B, cos C for some triangle ABC.

Solution
If $A + B + C = 180^o$, then $\cos A = -\cos(B + C) = \sin B \sin C - \cos B \cos C$. Squaring, we get $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$. So a necessary condition is $a^2 - 2b - 2c = 1$.

Conversely, suppose that this condition holds. Then if the roots are p, q, r, we have $p^2 + q^2 + r^2 + 2pqr = 1$ (*). We are given that the roots are all positive, so $2pqr > 0$, hence $p^2 < 1$ and so $p < 1$. Similarly for q and r. So we can find angles A, B, C greater than 0 and less than 90^o such that p = cos A, q = cos B, r = cos C. Now we can rewrite (*) as $(1 - \cos^2 B)(1 - \cos^2 C) = \cos^2 A + 2 \cos A \cos B \cos C + \cos^2 B \cos^2 C = (\cos A + \cos B \cos C)^2$. But $(1 - \cos^2 B) = \sin^2 B$, $(1 - \cos C) = \sin^2 C$, so we have $\sin B \sin C = ±(\cos A + \cos B \cos C)$. But we know that A, B, C are between 0 and 90^o, so cos A, cos B, cos C, sin B, sin C are all positive. Hence we must use the + sign and we have $\cos A = \sin B \sin C - \cos B \cos C$, so $A + B + C = 180^o$. Hence the condition is also sufficient.

Problem B1
Does $\sum_1^\infty 1/n^{1 + 1/n}$ converge?

Solution
Answer: no.
$x < e^x$ for all $x \geq 0$, so $x^{1/x} < e < 3$. Hence $1/n^{1 + 1/n} > 1/3n$. But $\sum 1/n$ diverges.

Problem B2
p(x) is a real polynomial of degree n such that p(m) is integral for all integers m. Show that if k is a coefficient of p(x), then n! k is an integer.

Solution
Note that n! is best possible, because 1/n! x(x + 1) ... (x + n - 1) is always integral for integral x (it is the binomial coefficient (x+n-1)Cn).

We need a standard result from the calculus of differences. Let $\Delta f(x) = f(x + 1) - f(x)$. Then $p(x) = p(0) + \Delta_1 x + 1/2! \Delta_2 x(x - 1) + 1/3! \Delta_3 x(x - 1)(x - 2) + ... + 1/n! \Delta_n x(x - 1) ... (x - n + 1)$ (*), where $\Delta_m = \Delta^m p(0)$ (thus $\Delta_1 = p(1) - p(0)$, $\Delta_2 = p(2) - 2p(1) + p(0)$ etc).

Assume this is true. Then if p(x) is integral for all integral x, all the Δ_m must be integral. So the result above gives n! p(x) = an integral combination of integral polynomials. Hence all the coefficients of n! p(x) are integral.

To prove the result, notice first that if $f(x) = x(x - 1)(x - 2) ... (x - m + 1)$, then $\Delta^r f(0) = m!$ for r = m and 0 otherwise. So if we call the rhs of (*) q(x) , then $\Delta^m q(0)$ is a sum of terms which are all zero except for $\Delta^m(1/m! \Delta^m x(x - 1) ... (x - m + 1))(0) = \Delta_m$. Hence $\Delta^m p(0) = \Delta^m q(0)$ for m = 1, 2, ... , n. Also p(0) = q(0), so by a simple induction p(m) = q(m) for m = 0, 1, 2, ... , n. But q(x) is a polynomial of degree at most n. If polyomials of degree at most n agree at n+1 points, then they must be identical. Hence p(x) = q(x).

Problem B3
k is real. Solve the differential equations $y' = z(y + z)^k$, $z' = y(y + z)^k$ subject to y(0) = 1, z(0) = 0.

Solution
Adding, $(y + z)' = (y + z)^{k+1}$. Integrating $(y + z)^k = 1/(1 - kx)$ (1).

Multiplying opposite sides of the two given equations together, we get $yy'(y + z)^k = zz' (y + z)^k$. Hence yy' = zz'. Integrating $y^2 - z^2 = 1$. Hence $(y + z)^k(y - z)^k = 1$, so $(y - z)^k = 1 - kx$ (2).

For k non-zero, we can immediately solve (1) and (2) to get $y = 1/2 ((1 - kx)^{1/k} + 1/(1 - kx)^{1/k})$, $z = 1/2 ((1 - kx)^{1/k} - 1/(1 - kx)^{1/k})$.

For k = 0, the original equations simplify to y' = z, z'= y. So y'' = y. So y = A cosh x + B sinh x, z = A cosh x + B sinh x. Applying the initial conditions, y = cosh x, z = sinh x.

Problem B4
R is the reals. S is a surface in R^3 containing the point (1, 1, 1) such that the tangent plane at any point $P \in S$ cuts the axes at three points whose orthocenter is P. Find the equation of S.

Solution
Consider a plane cutting the axes at **a** = (a, 0, 0), **b** = (0, b, 0), **c** = (0, 0, c). If the orthocentre is at **p** = (x, y, z), then we have (**p** - **a**).(**b** - **c**) = (**p** - **b**).(**a** - **c**) = 0. But **a**.**b** = **b**.**c** = **c**.**a** = 0, so we have **p**.(**b** - **c**) = **p**.(**a** - **c**) = 0. In other words the line from the origin (0, 0, 0) to (x, y, z) is normal to the plane. So the surface satisfies the condition that all its normals pass through the origin and it passes through (1, 1, 1). This implies that it is the sphere $x^2 + y^2 + z^2 = 3$.

Note, however, that for points with one coordinate zero, the tangent plane will meet one axis at infinity, so we should arguable exclude all such points. That divides the sphere into 8 disconnected pieces. The piece containing (1, 1, 1) is that in the positive octant ($x > 0$, $y > 0$, $z > 0$).

Problem B5
The coefficients of the complex polynomial $z^4 + az^3 + bz^2 + cz + d$ satisfy $a^2d = c^2 \neq 0$. Show that the ratio of two of the roots equals the ratio of the other two.

Solution
We start with a straight slog. Let the roots be p, q, r, s. We have $a^2d - c^2 = (p + q + r + s)^2pqrs - (pqr + pqs + prs + qrs)^2$. The terms like $2p^2q^2rs$ all cancel, leaving $p^3qrs + pq^3rs + pqr^3s + pqrs^3 - p^2q^2r^2 - p^2q^2s^2 - p^2r^2s^2 - q^2r^2s^2$.

The trick now is to factorise this. We might suspect that pq - rs is a factor. But in that case pr - qs and ps - qr would presumably also be factors. (pq - rs)(pr - qs)(ps - qr) has degree 6, as required. It also has the correct number of terms (8). So we try multiplying it out and find that it is the same.

So (pq - rs)(pr - qs)(ps - qr) = 0. But that means that at least one factor must be zero, which gives the result. Note that the only reason for giving us that a^2d is non-zero, is because that implies that none of the roots are zero (their product d is non-zero) and so having got pq = rs, we can divide to get p/r = s/q.

Problem B6
A and B are equidistant from O. Given k > OA, find the point P in the plane OAB such that OP = k and PA + PB is a minimum.

Solution
Let C be the circle centre O radius k. Take A' on the ray OA such that $OA \cdot OA' = k^2$ and B' on the ray OB such that $OB \cdot OB' = k^2$. If A'B' intersects the circle C, then the points of intersection give the positions where PA + PB is a minimum. If not then the nearest point of C to A'B' (which is on the perpendicular bisector of AB) gives the minimum.

For given P on C, the triangles OAP and OPA' are similar, so PA = (AO/PO) PA'. Similarly, PB = (BO/PO) PB', so PA + PB = (AO/k) (PA' + PB'), so minimising PA + PB is equivalent to minimising PA' + PB'. If A'B' intersects C, then clearly the points of intersection minimise. If not, let Q be the point on C closest to A'B'. Let L be the tangent to the circle at that point (so that L is parallel to A'B'). Then QA + QB < RA + RB for other points R on L. Given another point P on C, take the perpendicular to L through P. If it intersects L at R, then PA + PB > RA + RB > QA + QB.

Problem B7
Show that we can express any irrational number $\alpha \in (0, 1)$ uniquely in the form $\sum_1^\infty (-1)^{n+1} 1/(a_1a_2 \ldots a_n)$, where a_i is a strictly monotonic increasing sequence of positive integers. Find a_1, a_2, a_3 for $\alpha = 1/\sqrt{2}$.

Solution
Answer: $a_1 = 1$, $a_2 = 3$, $a_3 = 8$.

Let s_n be the sum of the first n terms. The terms alternate in sign and decrease in absolute value, so the odd terms of the sequence s_n decrease and the even terms increase. Every odd term exceeds every even term, so the odds and the evens must each converge. But $s_n - s_{n+1} < 1/2^n$ which tends to zero, so they tend to a common limit.

Choose a_n as follows. Take a_1 to be the smallest integer whose inverse exceeds α. Having chosen a_{2n-1}, take a_{2n} to be the largest integer such that $s_{2n} < \alpha$. Having chosen a_{2n}, take a_{2n+1} to be the largest integer such that $s_{2n+1} > \alpha$.

We have to show that these choices are always possible, or, in other words, that they yield a strictly increasing sequence a_n. This depends on the relation $1/k - 1/k(k+1) = 1/(k+1)$ (*). For suppose we have just chosen a_{2n-1}. Then we know that $s_{2n-1} > \alpha$, but that if increased a_{2n-1} by 1, then s_{2n-1} would be $< \alpha$. Hence, using (*), taking $a_{2n} = a_{2n-1} + 1$ certainly gives $s_{2n} < \alpha$. On the other hand, if we take a_{2n} to be sufficiently large, then s_{2n} will be close to s_{2n-1} and hence exceed α (note that α is irrational so it cannot equal any s_m). So the choice of a_{2n} will exceed a_{2n-1}. A similar argument shows that a_{2n+1} exceeds a_{2n}.

So we have established that we can find a sequence a_n such that all the odd partial sums s_n exceed α and all the even partial sums are less than α. But we have also established that s_n tends to a limit, so that limit must be α. That establishes existence.

Suppose there is another expansion so that $\alpha = 1/a_1 - 1/a_1a_2 + ... = 1/b_1 - 1/b_1b_2 + ...$. As above, we have that $1/(a_1 + 1) < \alpha < 1/a_1$ and also $1/(b_1 + 1) < \alpha < 1/b_1$. But since a_1 and b_1 are both integers that implies that $a_1 = b_1$. Suppose now that we have established that $a_i = b_i$ for $i \leq n$. Then we have that $\beta = (-1)^n a_1 ... a_n(\alpha - 1/a_1 + 1/a_1a_2 - ... + (-1)^n/a_1...a_n) = 1/a_{n+1} - 1/a_{n+1}a_{n+2} + ...$. But we also have $\beta = 1/b_{n+1} - 1/b_{n+1}b_{n+2} + ...$. We now argue as before that β lies between $1/(a_{n+1} + 1)$ and $1/a_{n+1}$ and also between $1/(b_{n+1} + 1)$ and $1/b_{n+1}$. Hence $a_{n+1} = b_{n+1}$. That establishes uniqueness.

Finally, consider $\alpha = 1/\sqrt{2}$. We have $1/2 < 1/\sqrt{2} < 1$, so $a_1 = 1$. We must pick a_2 as the largest integer so that $1 - 1/a_2 < 1/\sqrt{2}$, or $a_2 < 2 + \sqrt{2} = 3.4$. So $a_2 = 3$. We must pick a_3 as the largest integer so that $1 - 1/3 + 1/3a_3 > 1/\sqrt{2}$ or $2 + 1/x > 3/\sqrt{2}$ or $x < 3\sqrt{2} + 4 = 8.2$. So $a_3 = 8$.

14th Putnam 1954

Problem A1
Let N be the set {1, 2, ... , n}, where n is an odd integer. Let f **:** N x N → N satisfy: (1) f(r, s) = f(s, r) for all r, s; (2) {f(r, s) **:** s ∈ N} = N for each r. Show that {f(r, r) **:** r ∈ N} = N.

Solution
We have a square array of numbers a_{ij} = f(i, j). Each member of N occurs just once in each row (by (2)), so it occurs n times in all. But the array is symmetric (by (1)), so each member occurs an even number of times off the main diagonal. Hence it must occur an odd number of times, and hence at least once, on the main diagonal. But the main diagonal only has n entries, so if each of n numbers occurs at least once, then each must occur exactly once.

Problem A2
Given any five points in the interior of a square side 1, show that two of the points are a distance apart less than k = 1/√2. Is this result true for a smaller k?

Solution
Let the square be ABCD and O its midpoint. Let the midpoints of AB, BC, CD, DA be P, Q, R, S respectively. Then ABCD is the union of the 4 smaller squares: APOS, BQOP, CROQ, DSOR, each with diameter k. At least one of the smaller squares must contain two points. So the two points are a distance apart ≤ k. However, they cannot be a distance k apart, because the only pairs of points in a square realizing the diameter distance are opposite corners and they are not in the interior of ABCD. Hence the two points are a distance < k.
Take one point at O and the others on the main diagonals a distance ε from the corners. Then shortest distance between two points is k - ε which can be made arbitarily close to k.

Problem A3
Let S be the set of all curves satisfying y' + a(x) y = b(x), where a(x) and b(x) are never zero. Show that if C ∈ S, then the tangent at the point x = k on C passes through a point P_k which is independent of C.

Solution
Let C_h be the curve with y(k) = h. Then the gradient of C_h at (k, h) is b(k) - a(k) h, so the tangent at (k, h) is (y - h) = (b(k) - a(k) h)(x - k). This may be written as (a(k) x - a(k) k - 1) h = (b(k) (x - k) - y). Evidently, this always passes through the point (k + 1/a(k), b(k)/a(k)) whatever the value of h.

Problem A4
A uniform rod length 2a is suspended in midair with one end resting against a smooth vertical wall at X and the other end attached by a string length 2b to a point on the wall above X. For what angles between the rod and the string is equilibrium possible?

Solution
Answer: always 0; additionally, if b > a > b/2, $\cos^{-1}((b^2 + 2a^2)/(3ab))$.

Let the other end of the rod be Y. Let the string be ZY (with Z on the wall) and M its midpoint. The line along which the rod's weight acts passes through the midpoint of the rod and hence also through M. So a necessary condition for equilibrium is that the normal force at X also passes through M, in other words that MX is normal to the wall.
By the cosine rule: $MX^2 = b^2 + 4a^2 - 4ab \cos\theta$ (where θ = angle XYZ). Similarly, $XZ^2 = 4b^2 + 4a^2 - 8ab \cos\theta$. But $MX^2 + XZ^2 = b^2$, so solving, $\cos\theta = (b^2 + 2a^2)/(3ab)$ (*).
Hence this is also a sufficient condition for equilibrium, because the geometry is now fixed. So we

can solve for the tension in terms of the weight (resolving vertically) and then for the normal force (resolving horizontally). Hence the three forces have vector sum zero and the rod is in equilibrium. Returning to (*), we know that $0 < a < b$. Put $x = a/b$, then $\cos\theta = (1 + 2x^2)/(3x)$. But it is easy to see that for $(1 + 2x^2)/(3x) < 1$, we require $x > 1/2$.

Problem A5
R is the reals. f **:** $(0, 1) \to R$ satisfies $\lim_{x\to 0} f(x) = 0$, and $f(x) - f(x/2) = o(x)$ as $x \to 0$. Show that $f(x) = o(x)$ as $x \to 0$.

Solution
We wish to show that given $\delta > 0$, $|f(x)| < x\,\delta$ for sufficiently small x. Certainly we can find $\varepsilon > 0$, such that for $x < \varepsilon$, $|f(x/2^r) - f(x/2^{r+1})| < x/2^{r+2}\,\delta$. Summing, $|f(x)| < x\,\delta\,(1/4 + 1/8 + \ldots) + |f(x/2^n)| = x\,\delta/2 + |f(x/2^n)|$. But since $f(x/2^n)$ tends to zero, for any given x we can take n sufficiently large that $|f(x/2^n)| < x\,\delta/2$ and hence $|f(x)| < x\,\delta$ as required.

Problem A6
The real sequence a_n satisfies $a_n = \sum_{n+1}^{\infty} a_k^2$. Show $\sum a_n$ does not converge unless all a_n are zero.

Solution
Clearly $a_n \geq 0$. If any $a_n = 0$, then all subsequent a_i must be zero, and, by a trivial induction, all previous a_i. So assume no $a_n = 0$.
Notice that $a_{n+1} = a_n^2 + a_n$. But we have just shown that $a_n^2 > 0$, so $a_{n+1} > a_n$.
If the sum converges, then we can take n sufficiently large that $a_{n+1} + a_{n+2} + a_{n+3} + \ldots < 1$. Then $a_n = a_{n+1}^2 + a_{n+2}^2 + a_{n+3}^2 + \ldots < a_n(a_{n+1} + a_{n+2} + a_{n+3} + \ldots) < a_n$. Contradiction. So the sum does not converge.

Problem A7
Prove that the equation $m^2 + 3mn - 2n^2 = 122$ has no integral solutions.

Solution
If m, n is a solution, then $4m^2 + 12mn - 8n^2 = 488$, so $(2m + 3n)^2 - 17n^2 = 488$, so $(2m + 3n)^2 = 12 \pmod{17}$. But 12 is not a quadratic residue of 17 [check: $0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2 = 0, 1, 4, 9, 16, 8, 2, 15, 13 \pmod{17}$].

Problem B1
Show that for any positive integer r, we can find integers m, n such that $m^2 - n^2 = r^3$.

Solution
We notice that $m^2 - n^2 = (m + n)(m - n)$. This suggests taking $m + n = r^2$, $m - n = r$. This works: $m = r(r + 1)/2$, $n = r(r - 1)/2$.

Problem B2
Let $a_n = \sum_1^n (-1)^{i+1}/i$. Assume that $\lim_{n\to\infty} a_n = k$. Rearrange the terms by taking two positive terms, then one negative term, then another two positive terms, then another negative term and so on. Let b_n be the sum of the first n terms of the rearranged series. Assume that $\lim_{n\to\infty} b_n = h$. Show that $b_{3n} = a_{4n} + a_{2n}/2$, and hence that $h \neq k$.

Solution
If we simply write out $a_{4n} = (1 - 1/2 + 1/3 - 1/4 + \ldots - 1/4n)$ and $a_{2n}/2 = (1/2 - 1/4 + 1/6 - 1/8 + \ldots - 1/4n)$ and add term by term we find that $a_{4n} + a_{2n}/2 = b_{3n}$.
Taking the limit, we have immediately that $h = 3k/2$. But clearly $k > 0$, since if we group the terms in pairs, each pair is positive. Hence $h \neq k$.

Problem B3
Let S be a finite collection of closed intervals on the real line such that any two have a point in common. Prove that the intersection of all the intervals is non-empty.

Solution
Let [A, B] be the interval with the largest left-hand endpoint, and let [a, b] be the interval with the smallest right-hand endpoint. Then since [A, B] and [a, b] overlap, we must have $A \le b$, so [A, b] is non-empty.

Now given any interval [x, y] in S, we have $x \le A$ and $y \ge b$, so $[A, b] \subseteq [x, y]$.

Problem B4
Let F be a point, and L and D lines, in the plane. Show how to construct the point of intersection (if any) between L and the parabola with focus F and directrix D.

Solution
If L and D are parallel and a distance d apart, then take the circle center F radius d. If it meets L, then the point(s) of intersection are the required points. If it does not, then there are none.
So assume L and D meet at O. If F lies on D, then the (degenerate) parabola is the line through F perpendicular to D. If it meets L, then the point of intersection is the required point. So assume F does not lie on D.

If L and D are perpendicular, then there is just one point. Take any point P (except O) on L and let the circle center P radius PO cut the line OF again at Q. Let the line through F parallel to PQ cut L at G. Then G is the required point (because an expansion center O takes the circle center P radius PO into a circle center G radius GO = GF).

So assume L and D are neither perpendicular nor parallel and meet at O. Take D' through O making the same angle with L as D. D and D' divide the plane into 4 sectors. If F lies in one of the two sectors containing L, then the required points exist. There are two required points unless F lies on D', when there is just one. [Any circle with center on L, touching D, must lie entirely within these two sectors, and hence F must also if the points exist.]

Take any point P on L (except O) and draw a circle center P touching D. Let the circle meet the line OF at Q. Let the line through F parallel to PQ cut L at G. Then G is the required point. If F does not lie on D', then there will be two possibilities for Q and hence two possibilities for G.

Problem B5
Let R be the reals. Let $f : (-1, 1) \to R$ be a function with a derivative at 0. Let a_n be a sequence in (-1, 0) tending to zero and b_n a sequence in (0, 1) tending to zero. Show that $\lim_{n\to\infty} (f(b_n) - f(a_n))/(b_n - a_n) = f'(0)$.

Solution
From the definition of derivative, we have that $(f(b_n) - f(0))/b_n \to f'(0)$, and $(f(0) - f(a_n))/-a_n \to f'(0)$. So $f(b_n) - f(0)$ lies within $\varepsilon/2$ of $b_n f'(0)$ for sufficiently large n, and $f(0) - f(a_n)$ lies within $\varepsilon/2$ of $-a_n f'(0)$ for sufficiently large n. Hence, adding, $f(b_n) - f(a_n)$ lies within ε of $(b_n - a_n)f'(0)$ for sufficiently large n, which is what we need.

Problem B6
If x is a positive rational, show that we can find distinct positive integers $a_1, a_2, \ldots, a_n$ such that $x = \sum 1/a_i$.

Solution
Rather surprisingly, the simplest possible algorithm - the greedy algorithm - works: take a_n to be the smallest integer not already chosen so that the sum does not exceed x. This terminates after a finite number of steps.

Note that the numbers involved can be quite large even in simple cases. For example: 4 = 1/1 + 1/2 + 1/3 + ... + 1/30 + 1/200 + 1/77706 + 1/16532869712 + 1/3230579689970657935732 + 1/368029065225163751156397359905205029546527 00

The key idea is that the numerator of the difference gets smaller each time and hence the process must terminate.

First, we have to get the difference under 1. So take m such that $1 + 1/2 + ... + 1/m \leq x < 1 + 1/2 + ... + 1/(m+1)$. This is possible since the harmonic series diverges. If we have equality, then we are home. If not then the difference $x - (1 + 1/2 + ... + 1/m) < 1/(m+1)$.

Let the difference $x - (1/a_1 + 1/a_2 + ... + 1/a_n)$ be $d_n = r_n/s_n$ (where r_n and s_n are integers).
So we may assume that we have picked $a_1, ... , a_n$ and that $d_n < 1/(a_n + 1)$. We now take a_{n+1} so that $1/a_{n+1} \leq d_n < 1/(a_{n+1} - 1)$. Now $d_{n+1} = d_n - 1/a_{n+1} = (r_n a_{n+1} - s_n)/(s_n a_{n+1})$. This expression may not be in lowest terms, but it is certainly *sufficient* to show that $(r_n a_{n+1} - s_n) < r_n$ or $d_n a_{n+1} - 1 < d_n$. But that is true since $d_n(a_{n+1} - 1) < 1$. Finally, we notice that $d_{n+1} < 1/(a_{n+1} + 1)$ is equivalent to $d_n - 1/a_{n+1} < 1/(a_{n+1} + 1)$, which is true since $1/(a_{n+1} - 1) - 1/a_{n+1} < 1/(a_{n+1} + 1)$, since $a_{n+1} > 1$.

Problem B7
Let α be a positive real. Let $a_n = \Sigma_1^n (\alpha/n + i/n)^n$. Show that $\lim a_n \in (e^\alpha, e^{\alpha+1})$.

Solution
The largest term is $(1 + \alpha/n)^n$ which tends to e^α. The next largest is $(1 + (\alpha - 1)/n)^n$ which tends to $e^{\alpha - 1}$ and so on. All terms are positive, so the limit is at least $e^\alpha + e^{\alpha - 1} > e^\alpha$. Also the limit is at most $e^\alpha(1 + 1/e + 1/e^2 + ...) < e^\alpha(1 + 1/2 + 1/4 + ...) = 2e^\alpha < e^{\alpha + 1}$.

15th Putnam 1955

Problem A1
Prove that if a, b, c are integers and $a\sqrt{2} + b\sqrt{3} + c = 0$, then $a = b = c = 0$.

Solution
$\sqrt{3}$ is irrational. For if not, let $\sqrt{3} = r/s$ with r and s having no common factor. But now $3s^2 = r^2$, so 3 divides r^2 and hence r. Hence 3 divides s^2 and hence s. So r and s have a common factor 3. Contradiction. Similarly $\sqrt{2}$ is irrational.

If $a\sqrt{2} + b\sqrt{3} + c = 0$, then $2a^2 - 3b^2 - c^2 = 2bc\sqrt{3}$. But $\sqrt{3}$ is irrational, so b or c = 0. If b = 0, then $a\sqrt{2} + c = 0$. But $\sqrt{2}$ is irrational, so a = c = 0 also.

Hence c = 0. So $a\sqrt{2} + b\sqrt{3} = 0$. We may take a and b to be relatively prime (divide out any common factor). Squaring: $2a^2 = 3b^2$.

Problem A2
O is the center of a regular n-gon $P_1P_2 \ldots P_n$ and X is a point outside the n-gon on the line OP_1. Show that $XP_1 \, XP_2 \ldots XP_n + OP_1^n = OX^n$.

Solution
Evidently it is sufficient to prove the result for the case $OP_1 = 1$. So represent P_k by the complex number ω^k, where $\omega = e^{i\,2\pi/n}$ an nth root of unity. Represent X by the real number r. Then we have to show that $|r - 1||r - \omega| \ldots |r - \omega^{n-1}| = r^n - 1$, but that follows immediately because $r^n - 1 = (r - 1)(r - \omega) \ldots (r - \omega^{n-1})$.

Problem A3
a_n is a sequence of monotonically decreasing positive terms such that $\sum a_n$ converges. S is the set of all $\sum b_n$, where b_n is a subsequence of a_n. Show that S is an interval iff $a_{n-1} \le \sum_n^\infty a_i$ for all n.

Solution
The condition is certainly necessary. For suppose $a_{n-1} = k_2 > k_3 = \sum_n^\infty a_i$. Let $k_1 = \sum_1^{n-2} a_i$. Then k_1 and $k_1 + k_2$ belong to S, but no number in the non-empty interval $(k_1 + k_3, k_1 + k_2)$ belongs to S, so S cannot be an interval.

Now assume that the condition holds. Let $k = \sum a_n$. We show that S = [0, k]. We get the endpoints by taking the subsequence to be the empty set or the whole sequence. So take $h \in (0, k)$. Define b_n to be the earliest member of the sequence not so far chosen such that $\sum_1^n b_i \le h$. This is clearly possible since $a_m \to 0$. If at any point we get equality we are home.

So assume that the resulting $\{b_n\}$ is infinite. Clearly $\sum b_n \le h$. If $\{b_n\}$ is missing infinitely many members of $\{a_n\}$, then given any $\varepsilon > 0$, we can find $a_m < \varepsilon$ missing from the subsequence. But that means that for some n, we rejected a_m because $b_1 + b_2 + \ldots + b_n + a_m > h$. So $\sum b_i \ge b_1 + b_2 + \ldots + b_n > h - \varepsilon$. Hence $\sum b_i = h$.

It remains to consider the case where only finitely many members are missing from the subsequence. Let a_m be the largest such. Then for some n we have that $b_1 + b_2 + \ldots + b_n + a_m > h$. We also have that $b_1 + b_2 + \ldots + b_n + \sum_{m+1}^\infty a_i \le h$. But $a_m \le \sum_{m+1}^\infty a_i$. So we have a contradiction and this case cannot occur.

Problem A4
n vertices are taken on a circle and all possible chords are drawn. No three chords are concurrent (except at a vertex). How many points of intersection are there (excluding vertices)?

Solution
Answer: nC4.

Every 4 vertices correspond to a unique point of intersection (the intersection of the two diagonals defined by the 4 points).

[Actually, I got this the hard way. Take one of the vertices. Consider the diagonals from it. The diagonal to the next vertex but one has one vertex on one side and (n - 3) on the other. So it has 1(n - 3) points of intersection on it. The next diagonal has 2(n - 4) and so on up to (n - 3)1. We repeat for each vertex. That gives a total of n[1(n - 3) + 2(n - 4) + ... + (n - 3)1] points of intersection. But we count each diagonal twice and then each point of intersection twice. So the number of points of intersection is $n/4 \sum_1^{n-3} r(n - 2 - r) = n/4 [(n - 2) 1/2 (n - 3)(n - 2) - 1/6 (n - 3)(n - 2)(2n - 5)] = 1/24 n(n - 2)(n - 3) [3(n - 2) - (2n - 5)] = nC4$.
Of course, at this point, one wonders why the answer has such a nice form!]

Problem A5
Given a parabola, construct the focus (with ruler and compass).

Solution
You need to know two facts:
(1) Rays from infinity come to a focus at the focus. In other words, if a line L parallel to the axis of the parabola meets it at X and the focus is F, then L and XF are equally inclined to the normal at X;
(2) The line joining the midpoints of two parallel chords is parallel to the axis.

The construction is then fairly obvious. Take two parallel chords. Join their midpoints and extend to meet the parabola at X. Take a line M through X parallel to the chords. Then M is a tangent. The line perpendicular to M through X is the normal. Hence find the line XF (although not yet F). Repeat for another pair of parallel chords. The two lines interesect at F.

Problem A6
For what positive integers n does the polynomial $p(x) \equiv x^n + (2 + x)^n + (2 - x)^n$ have a rational root.

Solution
Answer: n = 1.

There are no roots at all if n is even. If n = 1, then -4 is the root. So suppose n is odd and at least 3. We can use a similar argument to that showing that $\sqrt{2}$ is irrational. Suppose x = r/s with r and s having no common factor is a root. Then $r^n + (2s + r)^n + (2s - r)^n = 0$. The last two terms expand and add to give terms with even coefficients, so r must be even. So set r = 2t and we have $t^n + (s + t)^n + (s - t)^n = 0$. The same argument shows that t must be even. But now we have $t^n + 2(s^n + nC2\ s^{n-2}\ t^2 + ...) = 0$. Since $n \geq 3$, s must be even also. Contradiction. So there are no rational roots for n odd ≥ 3.

Problem A7
k is a real constant. y satisfies $y'' = (x^3 + kx)\ y$ with initial conditions y = 1, y' = 0 at x = 0. Show that the solutions of y = 0 are bounded above but not below.

Solution
For some x_0, $x^3 + kx > 0$ for all $x > x_0$. [For example, if $k \geq 0$, then we may take $x_0 = 0$. If $k < 0$,

then we may take $x_0 = -k$.] Now suppose that $a < b$ are two consecutive zeros greater than x_0. If $y(x) > 0$ on (a, b), then y'' must be negative for at least part of the interval (a, b), but that is impossible, since $(x^3 + kx)\, y > 0$ for the whole interval. Similarly, if $y(x) < 0$ on (a, b), then y'' must be positive for at least part of the interval (a, b), but that is impossible, since $(x^3 + kx)\, y < 0$ for the whole interval. So there is at most one root greater than x_0 and that provides a bound. [For completeness, we note that we require y not to be identically zero on a non-zero interval, but that is not possible because $y(0)$ is non-zero.]

Suppose the second part is false. Then we can find x_2 such that for any $x < x_2$, $y(x)$ is non-zero. There are two cases. Suppose first that $y(x) > 0$ for all $x < x_2$. We can take $x_3 < x_2$ such that $x^3 + kx < -2$ for all $x < x_3$. Hence, in particular, $y''(x) < 0$ for $x < x_3$. Take arbitary $a < b < x_3$, then as usual we can find $\xi \in (a, b)$ such that $y(a) = y(b) - (b - a)\, y'(b) + 1/2\, (b - a)^2\, y''(\xi) < y(b) - (b - a)\, y'(b)$. If $y'(b) > 0$, then for $(b - a)$ sufficiently large $y(a) < 0$ (contradiction). So $y'(b) \geq 0$. So $y(a) \leq y(b) + 1/2\, (b - a)^2\, y''(\xi)$. But now we use the stronger assumption that $\xi^3 + k\xi < -2$ and hence $y(a) < y(b)(1 - (b - a)^2)$, which becomes negative for sufficiently large $(b - a)$. Contradiction.

The remaining case is $y(x) < 0$ and $y''(x) > -2\, y(x)$ for all $x < x_3$. In particular, $y''(x) > 0$ for $x < x_3$. So arguing as before, $y'(b) \leq 0$. As before $y(a) \geq -y(b)\, (\, (b - a)^2 - 1)$, which becomes positive for sufficiently large $(b - a)$. Contradiction.

Problem B1
The lines L and M are horizontal and intersect at O. A sphere rolls along supported by L and M. What is the locus of its center?

Solution
Let the angle between the two lines be 2θ. Evidently the center rolls above the angle bisector. Let its projection onto the angle bisector have length x and let it be a height y above the angle bisector. Suppose the sphere has radius r. Then $(x \sin \theta)^2 + y^2 = r^2$. So the locus is an ellipse with semi-axes r and $r/\sin \theta$.

If we assume the rolling is under gravity, then the sphere falls off when it reaches $x = r/\sin \theta$, so the locus is just the upper half of the ellipse. If we assume the sphere is somehow kept in contact with the lines, then it can roll back underneath and so the locus is the whole ellipse. That is the ambiguity.

The trap is that it can also roll the other way, over the other angle bisector, giving another ellipse (or upper half) with semi-axes r and $r/\cos \theta$.

Problem B2
Let R be the reals. $f : R \to R$ is twice differentiable, f'' is continuous and $f(0) = 0$. Define $g : R \to R$ by $g(x) = f(x)/x$ for $x \neq 0$, $g(0) = f'(0)$. Show that g is differentiable and that g' is continuous.

Solution
The only issue is $x = 0$, because elsewhere we have simply $g'(x) = f'(x)/x - f(x)/x^2$.

Given x, we can find ξ such that $f(x) = f(0) + x\, f'(0) + 1/2\, x^2\, f''(\xi) = x\, f'(0) + 1/2\, x^2\, f''(\xi)$. Hence $\lim_{x \to 0} (g(x) - f'(0)\,)/x = \lim_{x \to 0} 1/2\, f''(\xi) = 1/2\, f''(0)$. So $g'(0)$ exists.

Now $\lim_{x \to 0} g'(x) = \lim_{x \to 0} (x\, f'(x) - f(x))/x^2$. But $f'(x) = f'(0) + x\, f''(\zeta)$, and $f(x) = x\, f'(0) + 1/2\, x^2\, f''(\xi)$ with $\xi, \zeta \in (0, x)$. So $\lim_{x \to 0} g'(x) = \lim_{x \to 0} f''(\zeta) - 1/2\, f''(\xi) = 1/2\, f''(0) = g'(0)$. So g' is continuous at 0.

Problem B3
Let S be a spherical cap with distance taken along great circles. Show that we cannot find a distance preserving map from S to the plane.

Solution
Let O be the center of the cap and C its perimeter. Then C is a circle center O. Its image must also be a circle with the same radius R, since the distance between each point of C and O is preserved. The circumference of the circle is also preserved. But the circumference is not equal to $2\pi R$.

Problem B4
Can we find n such that $\mu(n) = \mu(n + 1) = ... = \mu(n + 1000000) = 0$? [The Möbius function $\mu(r) = 0$ iff r has a square factor > 1.]

Solution
Answer: yes.

Let p_n be the nth prime. We show how to find a_n such that p_1^2 divides a_n, p_2^2 divides $a_n + 1$, p_3^2 divides $a_n + 2$, ... , p_n^2 divides $a_n + n - 1$.

Evidently we can take $a_1 = 4$. We now use induction. $a_n + k\, p_1^2 ... p_n^2$ has the same property as a_n. So we need to pick k such that $a_n + k\, p_1^2 ... p_n^2 = 0 \pmod{p_{n+1}^2}$. But that is always possible (by Euclid's algorithm, for example) since p_{n+1}^2 is relatively prime to $p_1^2 ... p_n^2$.

Problem B5
n is a positive integer. An infinite sequence of 0s and 1s is such that it only contains n different blocks of n consecutive terms. Show that it is eventually periodic.

Solution
We use induction on n. It is obvious for n = 1. Suppose it is true for n. Let S be a sequence with just n + 1 different (n+1)-blocks. Consider the n-blocks formed by taking the first n elements of each (n+1)-block. These evidently form all the possible n-blocks in S. So if there are at most n of them, then we are home by induction.

But if there are n + 1 of them (all distinct), then the first n elements of each (n+1)-block determine the last element. So an n-block determines the entire sequence from that point on (because having fixed the (n + 1)th element, we move the n-block along one place and hence fix the (n + 2)nd element and so on). But there are only finitely many possible n-blocks, so there are two the same (in S). That gives a period, since the elements r places after the start of each block must be the same.

Note that the result is the best possible in the sense that we can construct a non-periodic sequence with just n + 1 different n-blocks. Take, for example, the n-blocks to be those with no or just one 1. Then if we take 1 followed by n + 1 zeros, followed by a 1, followed by n + 2 zeros, followed by a 1, followed by n + 3 zeros and so on, we have a non-periodic sequence involving only those n + 1 n-blocks.

Problem B6
Let N be the set of positive integers and R^+ the positive reals. $f : N \to R^+$ satisfies $f(n) \to 0$ as $n \to \infty$. Show that there are only finitely many solutions to $f(a) + f(b) + f(c) = 1$.

Solution
We show first that there are only finitely many solutions to $f(a) + f(b) = k$ (> 0). Take a to be < b. Since $f(n) \to 0$ we can find N such that $f(n) < k/2$ for $n > N$. So a must be $\le N$. Now for each such

a_0, we can find M such that $f(n) < k - f(a_0)$ for $n \geq M$, so there are only finitely many b such that $f(a_0) + f(b) = k$. Hence there are only finitely many solutions to $f(a) + f(b) = k$.

Now consider $f(a) + f(b) + f(c) = 1$. We can find N such that $f(n) < 1/3$ for $n > N$. So at least one of a, b, c must be $\leq N$. But for each such a we then have only finitely many solutions to $f(b) + f(c) = 1 - f(a)$. Hence there are only finitely many solutions in total.

Problem B7
A three-dimensional solid acted on by four constant forces is in equilibrium. No two lines of force are in the same plane. Show that the four lines of force are rulings on a hyperboloid.

Solution
Let the lines be L_1, L_2, L_3, L_4. If the line L intersects each of L_1, L_2, L_3, then the moment of the first three forces about L is zero. Hence the moment of the fourth force about L is also zero and so L must also intersect L_4. This is sufficient to prove that L_1, L_2, L_3, L_4 lie in a ruled quadric.

In fact the union of all the lines which intersect L_1, L_2, L_3 is a ruled quadric. Let P be a point on L_4. For a point X and a line L not containing it we take XL to be the unique plane containing X and L. If L and L' are skew lines and X does not lie on either, then $XL \cap XL'$ is the unique line through X which meets L and L'. So $PL_1 \cap PL_2$ is the unique line through P which meets L_1 and L_2. Suppose it meets L_1 at Q. Now $QL_2 \cap QL_3$ meets L_1, L_2 and L_3 so it must also meet L_4. But there is a unique line through Q meeting L_2 and L_4, so $QL_2 \cap QL_3 = PL_1 \cap PL_2$. Hence P lies on the line $QL_2 \cap QL_3$ which is a line of the quadric. P was arbitrary, so we have shown that L_4 lies in the quadric.

16th Putnam 1956

Problem A1
$\alpha \neq 1$ is a positive real. Find $\lim_{x\to\infty} ((\alpha^x - 1)/(\alpha x - x))^{1/x}$.

Solution
$\lim_{x\to\infty} (1/x)^{1/x} = \lim_{y\to 0} y^y = 1$. If $\alpha > 1$, then $\lim 1/(\alpha - 1)^{1/x} = 1$, and $\lim (\alpha^x - 1)^{1/x} = \lim (\alpha^x)^{1/x} = \alpha$, so the whole expression tends to α. If $\alpha < 1$, then $(\alpha^x - 1)/(\alpha - 1)$ tends to $1/(1 - \alpha)$, so the whole expression tends to 1.

Problem A2
Given any positive integer n, show that we can find a positive integer m such that mn uses all ten digits when written in the usual base 10.

Solution
Let n have d digits. Take N = 12345678900 ... 0, ending in d+1 digits. One of N+1, N+2, ... , N+n must be a multiple of n and its first 10 digits are all different.

Problem A3
Find the trajectory of a particle which moves from rest in a vertical plane under (constant) gravity and a force kv perpendicular to its velocity v.

Solution
Take the x-axis horizontal and the y-axis vertically downwards. Then the equations of motion are: $y'' = g - kx'$, $x'' = ky'$. If $k = 0$, then we have the usual constant acceleration downwards ($x = 0$, $y = 1/2\ gt^2$). So suppose $k \neq 0$.

By inspection, $x = A (\cos kt - 1) + B (\sin kt - kt)$ (we have chosen the terms - A and -kBt to give $x(0) = x'(0) = 0$). Hence (integrating $x'' = ky'$) we have $y = - A \sin kt + B(\cos kt - 1)$. But $y'(0) = 0$, so $A = 0$. Finally, to satisfy $y'' = g - kx'$, we $B = -g/k^2$. So $x = gt/k - g/k^2 \sin kt$, $y = g/k^2 (1 - \cos kt)$. This is the equation of a cycloid. Note that the particle is never able to fall more than a finite distance, it keeps being returned to its starting level.

Problem A4
Let p(x) be a real polynomial of degree n with leading coefficient 1 and all roots real. Let R be the reals and f **:** [a, b] → R be an n times differentiable function with at least n + 1 distinct zeros. Show that p(D) f(x) has at least one zero on [a, b], where D denotes d/dx.

Solution
The basic tool is Rolle's theorem which tells us that there is a zero of f '(x) between any two zeros of f(x). But we would like a zero of (D - α)f between any two zeros of f. For that, notice that the zeros of f(x) are also zeros of $e^{-\alpha x}$ f(x). So if f(x) has n + 1 zeros, then the derivative of $e^{-\alpha x}$ f(x) has at least n zeros. But the dervative of $e^{-\alpha x}$ f(x) is $e^{-\alpha x}$ times (D - α) f(x). $e^{-\alpha x}$ is never zero, so (D - α) f(x) has at least n zeros.

That (more than) proves the result for p(x) of degree 1. We now use induction on n, the degree of p(x). Suppose it is true for n. Let p(x) be a real polynomial with real roots and degree n + 1. Then $p(x) = q(x) (x - \alpha)$ for some α and q(x) of degree n. Let f be an n+1 times differentiable real function on [a, b] with at least n + 2 distinct zeros. Then we proved above that (D - α) f(x) has at least n + 1 distinct zeros. It is also at least n times differentiable. So, by the inductive hypothesis, q(D) (D - α) f(x) has at least one zero, which proves the result is true for n + 1.

Problem A5
Show that there are just (n-k+1)Ck subsets of {1, 2, ... , n} with k elements and not containing both i and i+1 for any i.

Solution
There is a bijection between subsets of {1, 2, ... , n} with k elements not containing both i and i+1 for any i and subsets of {1, 2, ... , n - k + 1} with k elements. Namely, associate $r_1 < r_2 < ... < r_k$ and $r_1, r_2 - 1, r_3 - 2, ... , r_k - k + 1$. But there are obviously just (n-k+1)Ck such subsets of {1, 2, ... , n - k + 1}.

Problem A6
Let R be the reals. Find f **:** $R \to R$ which preserves all rational distances but not all distances. Show that if f **:** $R^2 \to R^2$ preserves all rational distances then it preserves all distances.

Solution
Let f(x) = x for x rational, x + 1 for x irrational. If x - y is rational, then either both x and y are rational in which case f(x) - f(y) = x - y, or neither are in which case f(x) - f(y) = (x + 1) - (y + 1) = x - y. So f preserves rational distances. But f does not preserve all distances. For example $f(\sqrt{2}) - f(0) \neq \sqrt{2} - 0$.

Given points A and B in the plane a distance k apart, and any $\varepsilon > 0$, we can find a point C such that AC and BC are rational and $|AC - k| < \varepsilon/2$, $|BC| < \varepsilon/2$. Let A' = f(A), B' = f(B), C' = f(C). Then |A'C'| = |AC|, |B'C'| = |BC|, and hence |A'B'| lies within ε of k. This is true for all $\varepsilon > 0$, so |A'B'| = k and f preserves all distances.

Problem A7
Show that for any given positive integer n, the number of odd nCm with $0 \leq m \leq n$ is a power of 2.

Solution
(2n)C(2m+1) = 2n/(2m+1) (2n-1)C2m, which is even. There are n even factors in the numerator of 2nC2m and n even factors in its denominator. If we cancel 2 from each of these even factors, then we get nCm. So 2nC2m has the same parity as nCm. Hence there are the same number of odd binomial coefficients for 2n as for n.

(2n+1)C2m =2nC(2m-1) + 2nC2m, so (2n+1)C2m has the same parity as 2nC2m. Similarly, (2n+1)C(2m+1) = 2nC2m + 2nC(2m+1), so (2n+1)C(2m+1) has the same parity as 2nC2m. Hence there are twice as many odd binomial coefficients for 2n+1 as for 2n.
The required result now follows by a trivial induction.

Problem B1
The differential equation a(x, y) dx + b(x, y) dy = 0 is homogeneous and exact (meaning that a(x, y) and b(x, y) are homogeneous polynomials of the same degree and that $\partial a/\partial y = \partial b/\partial x$). Show that the solution y = y(x) satisfies x a(x, y) + y b(x, y) = c, for some constant c.

Solution
d(x a + y b) = (a dx + b dy) + x($\partial a/\partial x$ dx + $\partial a/\partial y$ dy) + y($\partial b/\partial x$ dx + $\partial b/\partial y$ dy).

Using exactness, this becomes: (a dx + b dy) + x($\partial a/\partial x$ dx + $\partial b/\partial x$ dy) + y($\partial a/\partial y$ dx + $\partial b/\partial y$ dy). Homogeneity implies that for some integer k we have x $\partial a/\partial x$ + y $\partial a/\partial y$ = k a, and x $\partial b/\partial x$ + y $\partial b/\partial y$ = k b. So we get finally d(x a + y b) = (a dx + b dy) + k (a dx + b dy). So if y is a solution, then d(x a + y b) = 0 and hence x a + y b = c for some constant c.

Problem B2
Let P be the set of all subsets of the plane. f **:** P → P satisfies f(X ∪ Y) ⊇ f(f(X)) ∪ f(Y) ∪ Y for all X, Y ∈ P (*). Show that (1) f(X) ⊇ X, (2) f(f(X)) = f(X) , (3) if X ⊇ Y, then f(X) ⊇ f(Y), for all X, Y ∈ P. Show conversely that if f **:** P → P satisfies (1), (2), (3), then f satisfies (*).

Solution
Taking Y = X in (*) gives immediately that f(X) = f(X ∪ Y) ⊇ f(f(X)) ∪ f(X) ∪ X ⊇ X, which proves (1).
It also shows that f(X) ⊇ f(f(X)). Now put Y = f(X) in (*). Then X ∪ Y = f(X) (by (1)), so (*) gives f(f(X)) ⊇ Y = f(X). So we have proved (2).

Finally, if X ⊇ Y, then X ∪ Y = X, so f(X) = f(X ∪ Y) ⊇ f(Y), which proves (3).

Now assume (1), (2) and (3) and take any X, Y. Then X ∪ Y ⊇ Y, so f(X ∪ Y) ⊇ f(Y) (using (3)). But f(Y) ⊇ Y (using (1)), so also f(X ∪ Y) ⊇ Y. Similarly, f(X ∪ Y) ⊇ f(X). But f(X) = f(f(X)) (using (2)), so f(X ∪ Y) ⊇ f(f(X)), which establishes (*).

Problem B3
ABCD is an arbitrary tetrahedron. The inscribed sphere touches ABC at S, ABD at R, ACD at Q and BCD at P. Show that the four sets of angles {ASB, BSC, CSA}, {ARB, BRD, DRA}, {AQC, CQD, DQA}, {BPC, CPD, DPB} are the same.

Solution
The key is to notice that the two angles in the sets subtended by the same side of the tetrahedron are the same. For example ∠ASC = ∠AQC. Let O be the centre of the sphere and r its radius. Then $AQ^2 + r^2 = AO^2 = AS^2 + r^2$, so AQ = AS. Similarly, CQ = CS. So the triangles AQC, ASC are similar.

Now the sum of the angles in each set is the same. But ∠ASB in the first set equals ∠ARB in the second set, so ∠BSC + ∠CSA = ∠BRD + ∠DRA. Similarly, ∠CQD in the third set equals ∠CPD in the fourth, so ∠AQC + ∠DQA = ∠BPC + ∠DPB. Adding these two equations and using ∠BSC = ∠BPC, ∠CSA = ∠AQC, ∠BRD = ∠DPB, ∠DRA = ∠DQA gives ∠CSA = ∠BRD. In other words, the angles in the set subscribed by opposite sides of the tetrahedron are also the same.
That gives us all we need: ∠ASC = ∠BRD = ∠AQC = ∠BPD; ∠ASB = ∠ARB = ∠CQD = ∠CPD; ∠BSC = ∠ARD = ∠AQD = ∠BPC.

Problem B4
Show that for any triangle ABC, we have sin A cos C + A cos B > 0.

Solution
The result is obviously true unless B or C is obtuse. If C is obtuse, then |cos C| < cos B (because |cos C| = cos(A + B) < cos B). But sin A < A, so the result follows. So assume that B is obtuse.

We have A acute and hence A < tan A. So A cos A < sin A. But C is acute, so cos C is positive and A cos A cos C < sin A cos C. Now A sin A sin C is positive, so A cos(A + C) = A cos A cos C - A sin A sin C < A cos A cos C < sin A cos C. But cos(A + C) = -cos B, so - A cos B < sin A cos C.

Problem B5
Show that a graph with 2n points and $n^2 + 1$ edges necessarily contains a 3-cycle, but that we can find a graph with 2n points and n^2 edges without a 3-cycle.

Solution
Induction. For n = 2, the result is obviously true, because there is only one graph with 4 points and 5 edges and it certainly contains a triangle. Suppose the result is true for n. Consider a graph G with 2n + 2 points and $n^2 + 2n + 2$ edges. Take any two points P and Q joined by an edge. We now consider two cases. If there are less than 2n + 1 other edges to P and Q, then if we remove P and Q, we get a graph with 2n points and at least $n^2 + 1$ edges, which must contain a triangle, so G does also. If there are at least 2n + 1 other edges then at least one point must be joined to both P and Q, but that gives a triangle.

Let G and H be disjoint sets each with n points. Join every point of G to every point of H by an edge. Then the resulting graph has n^2 edges, but it does not contain any triangles, because two points of any triangle must belong to G or to H and in either case there is no edge connecting them.

Problem B6
The sequence a_n is defined by $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. Show that any pair of values in the sequence are relatively prime and that $\sum 1/a_n = 1$.

Solution
We show by induction on k that $a_{n+k} = 1 \pmod{a_n}$. Obviously true for k = 1. Suppose it is true for k. Then for some m, $a_{n+k} = m\, a_n + 1$. Hence $a_{n+k+1} = a_{n+k}(m\, a_n + 1 - 1) + 1 = a_{n+k} m\, a_n + 1 = 1 \pmod{a_n}$. So the result is true for all k. Hence any pair of distinct a_n are relatively prime.

We show by induction that $\sum_1^n 1/a_r = 1 - 1/(a_{n+1} - 1)$. For n = 2, this reduces to $1/2 = 1 - 1/(3 - 1)$, which is true. Suppose it is true for n. Then $\sum_1^{n+1} 1/a_r = 1 - k$, where $k = 1/(a_{n+1} - 1) - 1/a_{n+1} = 1/(a_{n+1}^2 - a_{n+1}) = 1/(a_{n+2} - 1)$, so it is true for n + 1. But $a_n \to \infty$, so $\sum 1/a_n = 1$.

Problem B7
p(z) and q(z) are complex polynomials with the same set of roots (but possibly different multiplicities). p(z) + 1 and q(z) + 1 also have the same set of roots. Show that $p(z) \equiv q(z)$.

Solution
This is much easier than it looks. We just have to consider p - q and p' - q'. Suppose that p has A roots, of which B are distinct, and that p(z) + 1 has A roots, of which C are distinct. Without loss of generality we may assume the degree of q is at most A, so that p - q has at most A roots. Clearly the roots of p(z) and p(z) + 1 do not overlap. The B distinct roots of p(z) and the C distinct roots of p(z) + 1 must all be roots of p - q. So $B + C \le A$. On the other hand, p' has at least A - B roots in common with p and at least A - C in common with p(z) + 1, so it has at least 2A - (B + C) in total. But its degree is A - 1, so $(B + C) \ge A + 1$. Contradiction.

17th Putnam 1957

Problem A1
A surface S in 3-space is such that every normal intersects a fixed line L. Show that we can find a surface of revolution containing S.

Solution
Unfortunately, this is false.

It is not hard to paste together pieces of different surfaces of revolution. A simple example is as follows. Take a cylinder terminated at one end by a circle C. For part of the circumference of C continue the cylinder. Leave a gap either side and then on the rest join a portion of a cone (so that the surface is angled outwards away from the central axis).
It is possible, but hard, certainly too hard for a qu. 1, to prove that S is locally a surface of revolution. For details see Gleason et al.

Problem A2
k is a real number greater than 1. A uniform wire consists of the curve $y = e^x$ between x = 0 and x = k, and the horizontal line $y = e^k$ between x = k - 1 and x = k. The wire is suspended from $(k - 1, e^k)$ and a horizontal force applied at the other end, (0, 1) to keep it in equilibrium. Show that the force is directed towards increasing x.

Solution
If d is the density per unit length, then the total mass of wire is $d(1 + \int_0^k \sqrt{(1 + e^{2x})}\, dx)$. The moment about the y-axis is $d(k - 1/2) + d \int_0^k x \sqrt{(1 + e^{2x})}\, dx$. We require that the x-coordinate of the centre of mass exceeds k - 1. In other words, $k - 1/2 + \int_0^k x \sqrt{(1 + e^{2x})}\, dx > (k - 1) (1 + \int_0^k \sqrt{(1 + e^{2x})}\, dx)$.

Set $f(k) = \int_0^k x \sqrt{(1 + e^{2x})}\, dx - (k - 1) \int_0^k \sqrt{(1 + e^{2x})}\, dx$. Then we require f(k) > -1/2 for all k > 1. Clearly f(0) = 0. We try differentiating, and get $f'(k) = \sqrt{(1 + e^{2k})} - \int_0^k \sqrt{(1 + e^{2x})} dx$.

One possibility is to evaluate the integral explicitly (substitute $y = e^{2x}$, then $z^2 = 1 + y$, we end up with a messy but doable integral). But it is maybe easier to approximate. We have $1 + e^{2x} < 1 + e^{2x} + e^{-2x}/4 = (e^x + e^{-x}/2)^2$. So $\int_0^k \sqrt{(1 + e^{2x})} dx < \int_0^k (e^x + e^{-x}/2)\, dx = (e^x - e^{-x}/2) |_0^k = e^k - e^{-k}/2 - 1/2 < e^k < \sqrt{(1 + e^{2k})}$. So f '(k) > 0. Hence f(k) > f(0) > 0 > -1/2.

Problem A3
A and B are real numbers such that $\cos A \neq \cos B$. Show that for any integer n > 1, $|\cos nA \cos B - \cos A \cos nB| < (n^2 - 1) |\cos A - \cos B|$.

Solution
It is not clear how to use induction, expanding $\cos (n+1)A = \cos nA \cos A - \sin nA \sin A$ gives a profusion of sines and it is not clear how to get rid of them. So some other approach is needed.

An ingenious manipulation puts the expression entirely in terms of sines. Put x = (A + B)/2, y = (A - B)/2. Then A = x + y, B = x - y. Now $2 \cos nA \cos B = \cos(nA + B) + \cos(nA - B)$. Similarly, $2 \cos A \cos nB = \cos(nB + A) + \cos(nB - A)$. Switch to x and y and then go back to products of cosines. We have $\cos(nA + B) - \cos(nB + A) = \cos((n+1)x + (n-1)y) - \cos((n+1)x - (n-1)y) = -2 \sin(n+1)x \sin(n-1)y$. Similarly, $\cos(nA - B) - \cos(nB - A) = - 2 \sin(n-1)x \sin(n+1)y$. Hence $|\cos nA \cos B - \cos A \cos nB| = |\sin(n+1)x \sin(n-1)y + \sin(n-1)x \sin(n+1)y|$. Obviously $|\cos A - \cos B| = |2 \sin x \sin y|$, so we have to prove that $|\sin(n+1)x \sin(n-1)y + \sin(n-1)x \sin(n+1)y| < 2(n^2 - 1) |\sin x \sin y|$.

If would evidently be sufficient to show that |sin mx| < m |sin x|. Unfortunately, that is not true. We have equality if sin x = 0 or if m = 1. However, we can easily prove by induction that we have strict inequality in all other cases.

For m = 2, we have sin 2x = 2 sin x cos x, which establishes the result since |cos x| < 1 for sin x non-zero. Suppose it is true for m. Then sin(m+1)x = sin mx cos x + sin x cos mx, so |sin(m+1)x| <= |sin mx| |cos x| + |sin x| |cos mx| <= |sin mx| + |sin x| < (m+1) |sin x|, so it is true for m+1. Now if sin x = 0, then A + B is a multiple of 2π, so cos A = cos B. Similarly if sin y = 0, then A - B is a multiple of 2π, so cos A = cos B. But we are told that cos A $\neq$ cos B, so sin x and sin y are both non-zero. Also we are given that n > 1, so we have strict inequality on |sin(n+1)x| > (n+1) |sin x| and |sin(n+1)y| > (n+1) |sin y|, whilst |sin(n-1)x| $\geq$ (n-1) |sin x| and |sin(n-1)y| $\geq$ (n-1) |sin y|. Thus we get the required strict inequality.

Problem A4
p(z) is a polynomial of degree n with complex coefficients. Its roots (in the complex plane) can be covered by a disk radius r. Show that for any complex k, the roots of n p(z) - k p'(z) can be covered by a disk radius r + |k|.

Solution
Let the roots of p(z) be $a_1, a_2, \ldots, a_n$. Suppose they all lie in the disk centre c, radius r. Then $|c - a_n| \leq r$. Suppose that |c - w| > r + |k|. We show that w is not a root of n p(z) - k p'(z). We have $|w - a_i| \geq |w - c| - |c - a_i| > r + |k| - r = |k|$. Now $p'(z)/p(z) = \sum 1/(z - a_i)$ (note that this is still true if we have repeated roots), so $|p'(w)/p(w)| < n/|k|$ and hence $|k\, p'(w)/p(w)| < n$. So $|n - k\, p'(w)/p(w)| > 0$. But |p(w)| > 0 (since w lies outside the disk containing all the roots of p(z)), so $|n\, p(w) - k\, p'(w)| = |p(w)|\, |n - k\, p'(w)/p(w)| > 0$.

Problem A5
Let S be a set of n points in the plane such that the greatest distance between two points of S is 1. Show that at most n pairs of points of S are a distance 1 apart.

Solution
Induction on n. Obviously true for $n \leq 3$. Suppose it is true for n. Take n+1 points. If no point is a distance 1 from more than 2 points, then we are done. So assume that A, B and C are all a distance 1 from P. wlog the largest of the three angles APB, APC, BPC is APB. It must be at most 60^o, since $AB \leq 1$. So the ray PC lies between the rays PA and PB.

Now suppose there is another point D (apart from P) such that CD = 1. Then CD must intersect PA, because otherwise one of CP, CA, DP and DA would exceed 1. Similarly, it must intersect PB. But that is impossible. So there is no such point D. Hence if we remove C we lose only one realisation of the distance 1. But the remaining n points have at most n realisations, so the result is established.

Problem A6
Define a_n by $a_1 = \ln \alpha$, $a_2 = \ln(\alpha - a_1)$, $a_{n+1} = a_n + \ln(\alpha - a_n)$. Show that $\lim_{n\to\infty} a_n = \alpha - 1$.

Solution
Unfortunately, this does not quite work as stated. If α is too small then $a_2 > \alpha$ and hence a_3 is undefined. The limit is just under 0.3442. So assume that α is sufficiently large that $a_2 < \alpha$.

For all non-zero x we have $e^x > 1 + x$. Hence for all positive x not equal to 1 we have ln x < x - 1.

Suppose that $a_2 = \alpha - 1$. Then $a_n = \alpha - 1$ for all n > 2, which gives the result. So assume a_2 is not equal to $\alpha - 1$. Then for all $n \geq 2$ we have $\ln(\alpha - a_n) < \alpha - a_n - 1$, so $a_{n+1} = a_n + \ln(\alpha - a_n) < \alpha - 1$.

Also for n >= 3, we have that $\alpha - a_n > 1$, so $\ln(\alpha - a_n) > 0$, so $a_{n+1} > a_n$. Thus for $n \geq 3$, a_n is a monotonic increasing sequence bounded above by $\alpha - 1$. So it must converge.

But $a_{n+1} - a_n = \ln(\alpha - a_n)$, so $\ln(\alpha - a_n)$ tends to 0 and hence a_n tends to $\alpha - 1$.

Problem A7
Show that we can find a set of disjoint circles such that given any rational point on the x-axis, there is a circle touching the x-axis at that point. Show that we cannot find such a set for the irrational points.

Solution
Two disjoint circles touching the x-axis at A and B and each with radius r cannot have AB < 2r. Now suppose a circle radius $R \geq r$ touches at A and a disjoint circle radius r touches at B. The circle radius r touching at A does not extend outside the circle radius R, so the circle at B must also be disjoint from it. Hence we still have $AB \geq 2r$. Hence there can only be countably many disjoint circles radius r or more touching the x-axis. But a countable set of countable sets is still countable, so there can only be countably many disjoint circles touching the x-axis (any such circle has radius > 1/n for some n). There are uncountably many irrational points, so we cannot have disjoint circles touching at all the irrational points.

For the rational points take the circle touching at m/n (in lowest terms) to have radius $1/(3n^2)$. Now suppose the circles at m/n and a/b have centres P and Q. We have $PQ^2 = (m/n - a/b)^2 + (1/(3n^2) - 1/(3b^2))^2$. If they intersect, then $PQ \leq (1/(3n^2) + 1/(3b^2))$ and hence $(m/n - a/b)^2 \leq (1/(3n^2) + 1/(3b^2))^2 - (1/(3n^2) - 1/(3b^2))^2 = 4/9\ 1/(n^2b^2)$, so $(mb - an)^2 < 4/9$. But (mb - an) must be integral, so mb - an = 0 and hence m/n = a/b. So distinct circles do not intersect.

Problem B1
Let A be the 100 x 100 matrix with $a_{mn} = mn$. Show that the absolute value of each of the 100! products in the expansion of det A is congruent to 1 mod 101.

Solution
Each product is 100! 100! . But 101 is prime, so the numbers 1, 2, ... , 100 can be divided into pairs with the product of each pair being 1 mod 101.

Problem B2
The sequence a_n is defined by its initial value a_1, and $a_{n+1} = a_n(2 - k\ a_n)$. For what real a_1 does the sequence converge to 1/k?

Solution
Answer: For a_1 strictly between 0 and 2/k.

Suppose a_n has the opposite sign to k. Then $2 - k\ a_n$ is positive, and so a_{n+1} also has the opposite sign to k, so we cannot get convergence. Similarly, if $a_n = 0$, then $a_{n+1} = 0$ and we cannot get convergence. So it is a necessary condition that a_1 should be non-zero and have the same sign as k.

If k is positive and $a_1 > 2/k$, then a_2 is negative and so the sequence does not converge to 1/k (as above). Similarly, if k is negative and $a_1 < 2/k$, then a_2 is positive and so we do not have convergence. Thus it is a necessary condition for a_1 to lie strictly between 0 and 2/k.

Suppose $a_n = 1/k + h$. Then $a_{n+1} = (1/k + h)(2 - (1 + kh)) = 1/k - kh^2$ (*). This is all we need. But to spell it out, consider first k positive. If a_1 lies strictly between 0 and 1/k, then h lies strictly between -1/k and 1/k. Now (*) shows that for $n \geq 2$, $a_n \leq 1/k$ and is monotonic increasing. Hence it tends to a

positive limit. If this limit is L, then referring to the original equation, L = L(2 - kL), so L = 1/k. Similarly for k negative.

Problem B3
R^+ is the positive reals, f **:** [0, 1] → R^+ is monotonic decreasing. Show that $\int_0^1 f(x)\,dx \int_0^1 x\,f(x)^2\,dx \le \int_0^1 x\,f(x)\,dx \int_0^1 f(x)^2\,dx$.

Solution
$(f(x) - f(y))(y - x) \ge 0$. Also $f(x), f(y) \ge 0$, so $\int\int f(x)\,f(y)\,(f(x) - f(y))(y - x)\,dx\,dy \ge 0$. But $f(x)\,f(y)\,(f(x) - f(y))(y - x) = (f(x)^2\,y\,f(y) - f(x)\,y\,f(y)^2) + (f(y)^2\,x\,f(x) - f(y)\,x\,f(x)^2)$. So if we take the same limits of integration, 0 and 1, for both integrals, then we have $\int\int f(x)\,f(y)\,(f(x) - f(y))(y - x)\,dx\,dy = 2\int f(x)^2\,y\,f(y) - f(x)\,y\,f(y)^2\,dx\,dy = 2\int f(x)^2\,dx \int y\,f(y)\,dy - 2\int f(x)\,dx \int y\,f(y)^2\,dy$.

Problem B4
Show that the number of ways of representing n as an ordered sum of 1s and 2s equals the number of ways of representing n + 2 as an ordered sum of integers > 1. For example: 4 = 1 + 1 + 1 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 (5 ways) and 6 = 4 + 2 = 2 + 4 = 3 + 3 = 2 + 2 + 2 (5 ways).

Solution
We can establish a bijection as follows. Given a representation of n as an ordered sum of 1s and 2s, proceed as follows. Add a final 2 at the end. Now group together all summands up to and including the first 2, then all following summands up to and including the next 2, and so on. Add the members in each group. This gives us an ordered set of integers each at least 2 and summing to n+2.

Conversely, given a representation of n+2, write a summand m as m-2 1s followed by a 2. This gives an ordered sum of 1s and 2s. Finally remove the final 2. This gives us a representation of n. It is clear that these two operations are inverse and hence each is a bijection.

Problem B5
Let S be a set and P the set of all subsets of S. f **:** P → P is such that if $X \subseteq Y$, then $f(X) \subseteq f(Y)$. Show that for some K, f(K) = K.

Solution
Let K be the union of all subsets A of S such that $A \subseteq f(A)$. We show that $K \subseteq f(K)$. Take any x in K. Then x belongs to some A which satisfies $A \subseteq f(A)$. But $A \subseteq K$, so $f(A) \subseteq f(K)$. So x belongs to $A \subseteq f(A) \subseteq f(K)$, so x belongs to f(K).
Since $K \subseteq f(K)$, we have $f(K) \subseteq f(f(K))$. Hence by definition $f(K) \subseteq K$.

Problem B6
y is the solution of the differential equation $(x^2 + 9)\,y'' + (x^2 + 4)y = 0$, y(0) = 0, y'(0) = 1. Show that y(x) = 0 for some $x \in [\sqrt{(63/53)}\,\pi, 3\pi/2]$.

Solution
We compare the equation given with (1) z'' = -4/9 z, z(0) = 0, z'(0) = 1, and (2) w'' = -53/63 w, w(0) = 0, w'(0) = 1.

For all x > 0 we have $(x^2 + 4)/(x^2 + 9) > 4/9$, so $y''z - yz'' = (4/9 - (x^2 + 4)/(x^2 + 9))\,yz$. Integrating gives $y'z - yz'|_0^{3\pi/2} = \int (4/9 - (x^2 + 4)/(x^2 + 9))\,yz\,dx$ (*). Now y(0) = z(0) = 0, z = sin 2x/3, so $z(3\pi/2) = 0$, $z'(3\pi/2) = -2/3$, hence $y'z - yz'|_0^{3\pi/2} = 2/3\,y(3\pi/2)$. Now if y has no zeros on the open interval $(0, 3\pi/2)$, then $y(3\pi/2) \ge 0$ and the integrand $(4/9 - (x^2 + 4)/(x^2 + 9))\,yz$ is negative for the

entire interval. Hence lhs (*) ≥ 0 and rhs (*) < 0. Contradiction. So y has at least on zero on (0, $3\pi/2$).

Turning to (2), we note that $\pi^2 < 10$, so $(3\pi/2)^2 < 22.5$. Hence on the interval $[0, 3\pi/2]$ we have $(x^2 + 4)/(x^2 + 9) < 26.5/31.5 = 53/63$. Now $w = \sin(\sqrt{(53/63)}\, x)$, so $w(x) > 0$ on the interval $(0, k)$, where $k = \sqrt{(63/53)}\, \pi$.

Now suppose that y has a root in $(0, k]$. Let h be the smallest such root, so that y(x) is positive on (0, h) and $y(h) = 0$, $y'(h) \leq 0$. Integrate $w''y - w\, y'' = ((x^2 + 4)/(x^2 + 9) - 53/63)$. The lhs is $w'y - w\, y' |_0^h = w(h)\, y'(h) \geq 0$. The rhs is $\int_0^h ((x^2 + 4)/(x^2 + 9) - 53/63)\, y\, w\, dx$. But the integrand is negative on (0, h), so the rhs < 0. Contradiction. Hence y has no roots on $(0, k]$. So it must have a root on $(k, 3\pi/2)$.

Problem B7
Let P be a regular polygon and its interior. Show that for any $n > 1$, we can find a subset S_n of the plane such that we cannot translate and rotate P to cover S_n but we can translate and rotate P to cover any n points of S_n.

Solution
Let the radius of the inscribed circle in P be 1. Let P have m sides. We take S_n to be a circle of radius slightly greater than 1. Clearly it cannot fit inside P. Place it so that its centre is at the centre of P. Suppose it cuts each side a distance $\tan^{-1}\theta$ either side of the midpoint, where θ is small. Then the arcs lying outside P subtend a total angle $2m\theta$ at the centre. Now rotate the circle about its centre. Let φ measure the angle of rotation from the starting point. In order to keep a given point of the circle inside P, φ must avoid intervals of total length $2m\theta$. So to keep any given n points of the circle inside P, φ must avoid intervals of total length $2mn\theta$. The worst case is that these intervals are all disjoint, but provided $\theta < \pi/mn$ they cannot exhaust the available 2π, so we can find angles φ for which all n points are inside P.

18th Putnam 1958

Problem A1
Show that the real polynomial $\sum_0^n a_i x^i$ has at least one real root if $\sum a_i/(i + 1) = 0$.

Solution
Integrate from 0 to 1. We get zero. Hence the polynomial has at least one zero between 0 and 1.

Problem A2
A rough sphere radius R rests on top of a fixed rough sphere radius R. It is displaced slightly and starts to roll off. At what point does it lose contact?

Solution
Let the top of the fixed sphere be T and its centre O. Let X be the point of contact. Take angle XOT = θ. Let the mass be M and the normal force between the two spheres N. Resolving radially, $MR(d<\theta/dt)^2 = Mg \cos\theta - N$ (1). Taking moments about an axis through X, $MgR \sin\theta = I\, d^2\theta/dt^2$ (2), where I is the moment of inertia of the sphere about X = the moment about a diameter + $MR^2 = 7/5\ MR^2$.

Integrating equation (2) gives $g(1 - \cos\theta) = 7/10\ R\ (d\theta/dt)^2$. Substituting in (1) gives $N = g \cos\theta - 10/7\ (1 - \cos\theta)$. Contact is lost when N becomes 0. In other words at the angle $\theta = \cos^{-1} 10/17$.

Problem A3
A sequence of numbers $\alpha_i \in [0, 1]$ is chosen at random. Show that the expected value of n, where $\sum_1^n \alpha_i > 1$, $\sum_1^{n-1} \alpha_i \le 1$ is e.

Solution
We can consider the possible values of $\alpha_1, \alpha_2, \ldots, \alpha_n$ as points of the n-cube. The points corresponding to sum at most 1 are those in the corner at the origin, bounded by the hyperplane through (1, 0, ... , 0), (0, 1, 0, ... , 0) , ... , (0, ... , 0, 1). By an easy induction this has volume 1/(n-1)! . Let p_n = the prob that the sum of the first n numbers is at most 1. We have just shown that $p_n = 1/(n-1)!$.

Now the required expected value is $(1/1! - 1/2!)\ 2 + (1/2! - 1/3!)\ 3 + (1/3! - 1/4!)\ 4 + \ldots = 2 + (3 - 2)/2! + (4 - 3)/3! + (5 - 4)/4! + \ldots = e$.

Problem A4
$z_1, z_2, \ldots, z_n$ are complex numbers with modulus a > 0. Let f(n, m) denote the sum of all products of m of the numbers. For example, $f(3, 2) = z_1z_2 + z_2z_3 + z_3z_1$. Show that $|f(n, m)|\ /a^m = |f(n, n-m)|\ /a^{n-m}$.

Solution
It is obviously sufficient to take a = 1. Then if $z = e^{i\theta}$ we have $1/z = e^{-i\theta}$ = the complex conjugate of z. Now $f(n, m)/(z_1z_2 \ldots z_n)$ = the sum of the inverses of the terms in f(n, n-m) = the sum of the complex conjugates of the terms in f(n, n-m) = complex conjugate of f(n, n-m). But a number and its complex conjugate have the same modulus.

Problem A5
Let R be the reals. Show that there is at most one continuous function $f : [0, 1]^2 \to R$ satisfying $f(x, y) = 1 + \int_0^x \int_0^y f(s, t)\, dt\, ds$.

Solution
If there are two, then their difference d(x, y) satisfies $d(x, y) = \int_0^x \int_0^y d(s, t)\, dt\, ds$ (*).

The domain is compact and d is continuous, so |d| has some upper bound k. Now applying (*) gives $|d(x, y)| \leq k\, xy$. Applying (*) again gives $|d(x, y)| \leq k\, x^2/2\, y^2/2$. So by a simple induction $|d(x, y)| \leq k\, x^n/n!\, y^n/n!$. Hence $|d(x, y)| \leq k/(n!\, n!)$ for all n, and so d(x, y) = 0.

Problem A6
Assume that the interest rate is r, so that capital of k becomes $k(1 + r)^n$ after n years. How much do we need to invest to be able to withdraw 1 at the end of year 1, 4 at the end of year 2, 9 at the end of year 3, 16 at the end of year 4 and so on (in perpetuity)?

Solution
To meet the withdrawal at the end of year n we need to invest $n^2/(1+r)^n$. Thus the total investment required is $1/(1+r) + 4/(1+r)^2 + 9/(1+r)^3 + 16/(1+r)^4 + \ldots$. Now we know that $1 + x + x^2 + \ldots = 1/(1 - x)$. Differentiating, we get: $x/(1 - x)^2 = x + 2x^2 + 3x^3 + \ldots$. Differentiating again: $x(1 + x)/(1 - x)^3 = 1^2x + 2^2x^2 + 3^2x^3 + \ldots$. Substituting x = 1/(1+r) gives $(1 + r)(2 + r)/r^3$ as the required investment.

Problem A7
Show that we cannot place 10 unit squares in the plane so that no two have an interior point in common and one has a point in common with each of the others.

Solution
Note that we can place 9 squares - arrange them in a regular 3 x 3 array. Then the centre square touches 4 at the corners and the other 4 along the sides.

Consider first placing squares to touch a line. If neither square has a side in contact with the line, then it is easy to see that the two points of contact are more than 1 apart. If square U has a side in contact and square V does not, then together they occupy more than a length 1 of the line (from the far side of U to the point of contact of V).

So the only way to get three squares in contact with AB, a side of a fixed square S = ABCD, is to place one square so that its side is AB. One can then place a second square with its corner touching A and a third with its corner touching B.

So to get more than 8 = 4 x 2 squares touching S, we must get three squares to touch one side as above. Call them T2 in contact with AB, T1 touching at A and T3 touching at B. Now consider the side BC. T3 has a side BD. The most favourable case is that angle CBD = 90^o - T3 may be placed so that angle CBD is less than 90^o. But certainly any square touching BC (apart from T3) cannot extend beyond the ray BD which is perpendicular to BC. But it is easy to see that this means we can get only one additional square to touch BC unless we place T4 with so that its side is BC and T5 touching C at a corner, which gives 2.

In particular, if 3 squares touch AB, then at most 2 (additional) squares touch each of BC and DA.

Hence at most 2 sides, which must be opposite, can have three squares touching. But in that case it is easy to see that the other two sides each have at most 1 additional square touching, so we get only 8 in total. Hence the only way to get 9 is to have 3 squares touching AB and 2 (additional) squares touching each of the other three sides. But those touching BC and DA must then be placed as indicated above and it is easy to see that we then get at most 1 (additional) square touching CD.

Problem B1
Do both (1) and (2):

(1) Given real numbers a, b, c, d with a > b, c, d, show how to construct a quadrilateral with sides a, b, c, d and the side length a parallel to that length b. What conditions must a, b, c, d satisfy?

(2) H is the foot of the altitude from A in the acute-angled triangle ABC. D is any point on the segment AH. BD meets AC at E, and CD meets AB at F. Show that $\angle AHE = \angle AHF$.

Solution
(1) A necessary and sufficient condition is that we should be able to form a triangle with sides c, d, (a - b). Hence we require $b + c + d \geq a$, $a + d \geq b + c$, and $a + c \geq b + d$. Construct two such triangles, one at each end of the segment a, then join the two tops to get the required quadrilateral.

(2) Take a line through A parallel to BC and project CF, HF, HE, BE to meet it in W, X, Y, Z. Then using similar triangles, we deduce that $AY/CH = AZ/BC = BH \cdot AW/CH\ (1/BC) = BH/CH\ AW/BC = BH/CH\ AX/BH = AX/CH$, so AX = AY. But AH is perpendicular to XY, so the result follows.

Problem B2
Let n be a positive integer. Prove that n(n + 1)(n + 2)(n + 3) cannot be a square or a cube.

Solution
$(n+1)(n+2) = n^2 + 3n + 2$ and $n(n+3) = n^2 + 3n$. So their product is $(n^2 + 3n + 1)^2 - 1$. Hence n(n + 1)(n + 2)(n + 3) is 1 less than a square, so it cannot be a square.

One of n+1, n+2 must be odd. Suppose it is n+1. Then n+1 has no factor in common with n(n + 2)(n + 3), so $n(n + 2)(n + 3) = n^3 + 5n^2 + 6n$ must be a cube. But $(n + 1)^3 = n^3 + 3n^2 + 3n + 1 < n^3 + 5n^2 + 6n < n^3 + 6n^2 + 12n + 8 = (n + 2)^3$, so $n^3 + 5n^2 + 6n$ cannot be a cube.

Similarly, suppose n + 2 is odd. Then it has no factor in common with $n(n + 1)(n + 3) = n^2 + 4n^2 + 3n$, so $n^2 + 4n^2 + 3n$ must be a cube. But for $n \geq 2$, $(n + 1)^3 < n^2 + 4n^2 + 3n < (n + 2)^3$, so $n^2 + 4n^2 + 3n$ cannot be a cube for $n \geq 2$. The case n = 1 is checked by inspection: 24 is not a cube.

Problem B3
In a tournament of n players, every pair of players plays once. There are no draws. Player i wins w_i games. Prove that we can find three players i, j, k such that i beats j, j beats k and k beats i iff $\sum w_i^2 < (n - 1)n(2n - 1)/6$.

Solution
Suppose there are no such three players. Let A be the (or one of the) top-scoring players. We claim that A has n-1 wins. Suppose not, then he is beaten by some B. Now if every player beaten by A is also beaten by B, then B would have a higher score than A, so we must be able to find C who is beaten by A, but who beats B (who beats A). But we assumed there were no such triads. So A has n-1 wins. But we can now consider the remaining players. The same argument shows that the top scoring player amongst them beat n-2 of them. He did not beat A (who beat everyone), so his score is n-2. The argument repeats to show that the scores are n-1, n-2, ... , 1, 0. The sum of the squares is thus (n-1)n(2n-1)/6.

Conversely, it is clear that if the scores are n-1, n-2, ... , 1, 0 then there can be no triad. For the player with n-1 beats everyone, so he cannot be part of a triad. But now the player with n-2 beats everyone else, so he cannot either and so on.

Thus we have shown that there is a triad iff the scores are not n-1, n-2, ... , 1, 0. It remains to show that if the scores are not n-1, n-2, ... , 1, 0, then sum of the squares is less than (n-1)n(2n-1)/6. But if the scores are not n-1, n-2, ... , 1, 0, then two players must have the same score m. Changing the

scores to m-1, m+1 (by changing the result of the match between the two players) increases the sum of the squares by 2. After a finite number of repetitions we must arrive at n-1, n-2, ... , 1, 0, so the starting sum must be strictly less.

Problem B4
Let S be a spherical shell radius 1. Find the average straight line distance between two points of S. [In other words S is the set of points (x, y, z) with $x^2 + y^2 + z^2 = 1$).

Solution
Answer: 4/3.

It is sufficient to fix one point and to find the average distance of the other points from it. Take the point as P and the centre as O. Now consider a general point Q on the surface. Let angle POQ = θ. The distance PQ is 2 sin θ/2 and this is the same for all points in a band angular width dθ at the angle θ. The band has radius sin θ. Hence the average distance is $1/4\pi \int_0^\pi (2\pi \sin \theta) (2 \sin \theta/2) \, d\theta = \int 4 \sin^2\theta/2 \, d(\sin \theta/2) = 4/3$.

Problem B5
S is an infinite set of points in the plane. The distance between any two points of S is integral. Prove that S is a subset of a straight line.

Solution
Suppose not. Take three non-collinear points A, B, C. Suppose AB = n. Any point P has PA + AB ≥ PB, so PB - PA ≤ n. Similarly PB - PA ≥ -n. But PB - PA is integral, so it must take one of the values -n, -(n-1), ... , 0, 1, ... , n. So P must lie on one of a finite number of hyperbolae |PB - PA| = k (regarding the allowed line pair as a degenerate hyperbola). Similarly it must lie on one of a finite number of hyperbolae |PA - PC| = h. But each pair of hyperbolae intersect in at most 4 points, so the number of points in addition to A, B, C is finite. Contradiction.

Problem B6
A particle of unit mass moves in a vertical plane under the influence of constant gravitational force g and a resistive force which is in the opposite direction to its velocity and with magnitude a function of its speed. The particle starts at time t = 0 and has coordinates (x, y) at time t. Given that x = x(t) and is not constant, show that $y(t) = - g x(t) \int_0^t ds/x'(s) + g \int_0^t x(s) / x'(s) \, ds + a x(t) + b$, where a and b are constants.

Solution
We can write the equations of motion as x'' = - f(x',y') x', y'' = - f(x', y') y' - g. If x'(t) = 0 at some t, then a possible solution to the equations would be x = constant (and y found by integrating y'' = - f(0, y') y' - g). But solutions are unique, so this would be *the* solution. But we are told that x is not constant. So x'(t) is never zero.

We do not know anything about f, so we resolve perpendicular to it to get x'' sin θ - y'' cos θ = g cos θ, or x'' tan θ - y'' = g. But tan θ = dy/dx = y'/x', so x'' y'/x' - y'' = g (*).

Dividing by x' gives $x'' y'/(x')^2 - y''/x' = g/x'$. Integrating, $y'/x' = A - g \int dt/x'$. So $y' = Ax' - g x' \int dt/x'$. Integrating again, $y(T) = A x + B - g \int_0^T x' \int_0^t ds/x' \, dt$. Now we can integrate the last integral by parts: $\int_0^T x' \int_0^t ds/x' \, dt = \int_{t=0}^T \int_0^t ds/x' \, dx = x(T) \int_0^T ds/x' - \int_0^T x/x' \, dt$. So we get finally the expression in the question.

Problem B7
R is the reals. f **:** [a, b] → R is continuous and $\int_a^b x^n f(x)\,dx = 0$ for all non-negative integers n. Show that f(x) = 0 for all x.

Solution
We deduce immediately that $\int_a^b x^n f(x)\,p(x)\,dx = 0$ for any polynomial p(x). We now need the Weierstrass approximation theorem: we can uniformly approximate any continuous function on a compact set by polynomials. In other words, we can find a polynomial p(x) such that $|f(x) - p(x)| < \varepsilon$ for all x in [a, b]. Also, since [a, b] is compact, f must be bounded, say by k, so $|f(x)| < k$ for all x in [a, b]. Now we have that $|\int_a^b f(x)^2\,dx| = |\int f(x)\,(f(x) - p(x))\,dx| \leq \int |f(x)|\,|f(x) - p(x)|\,dx \leq (b - a)\,k\,\varepsilon$, which can be made arbitrarily small by choosing ε sufficiently small. Hence $\int_a^b f(x)^2\,dx = 0$, so f(x) = 0 throughout the interval.

19th Putnam 1958

A1. Define f(i, j) as follows: f(i, 1) = f(1, j) = 1, f(i+1, j+1) = f(i, j+1) + f(i+1, j) + f(i, j). Let d(n) = f(1, n-1) + f(2, n-2) + ... + f(n-1, 1). Show that d(n + 2) = d(n) + 2 d(n + 1).

A2. Define $a_1 = 1$, $a_{n+1} = 1 + n/a_n$. Show that $\sqrt{n} \le a_n < 1 + \sqrt{n}$.

A3. Assuming that there is a unique function f(x) satisfying f(0) = 1, $f'(x) = f(x) + \int_0^1 f(t)\,dt$, find it.

A4. Find the general solution in real numbers a, b, c, d to the inequalities 2a > a + b > a + c > 2b > b + c > a + d > 2c > b + d > c + d > d + d. Find the smallest solution in positive integers.

A5. Let A = (a_{ij}) be the n x n matrix with $a_{ij} = 1$ if $i \ne j$, and $a_{ii} = 0$. Show that the number of non-zero terms in the expansion of det A is $n! \sum_0^n (-1)^i/i!$.

A6. R is the reals. $\alpha \in [0, 1)$. f **:** [0, 1] → [0, α] and g **:** [0, 1] → R are continuous. β satisfies β = max (g(x) + f(x) β). When do we have β = max g(x)/(1 - f(x))?

A7. m, n are relatively prime positive integers with n even. Given any positive integer r, define f(r) to be the integer which minimizes |f(r)/r - m/n|. Show that $\lim_{k\to\infty} \sum_1^k |f(r)/r - m/n|\, r/k = 1/4$.

B1. Let $a_n = \sum_0^n 1/nCr$. Show that $a_n = 1 + a_{n-1}\,(n + 1)/(2n)$. Deduce that lim $a_n = 2$.

B2. Let X be the set {1, 2, 3, ... , 2n}, take Y ⊆ X with |Y| = n + 1. Show that we can find a, b ∈ Y with a dividing b.

B3. Show that if X is a square side 1 and X = A ∪ B, then A or B has diameter at least √5 /2. Show that we can find A and B both having diameter ≤ √5 /2.

B4. Let R be the reals. f **:** R → R is three times differentiable. As x → ∞, f(x) tends to a finite limit, and f '''(x) tends to zero. Show that f '(x) and f ''(x) also tend to zero.

B5. A sequence of points P_n in the plane is defined by: P_0 is at the origin; P_1 is at (1, 0); and $P_{n-1}P_n$ is length 1/n and at an angle θ to the previous segment. Find the coordinates of lim P_n.

B6. A graph has n vertices {1, 2, ... , n} and a complete set of edges. Each edge is oriented, as either i → j or j → i. Show that we can find a permutation of the vertices a_i so that $a_1 \to a_2 \to a_3 \to \ldots \to a_n$.

B7. Let N = {1, 2, ... , n}. Given a permutation f **:** N → N, define d(f) = no. of i such that f(i) > f(j) for all j > i. Find the mean of d(f) over all permutations f on N.

20th Putnam 1959

Problem A1
Prove that we can find a real polynomial p(y) such that $p(x - 1/x) = x^n - 1/x^n$ (where n is a positive integer) iff n is odd.

Solution
Take n odd. For n = 1 the result is obvious. Now suppose the result is true for odd $n \le 2m - 1$. Expand $(x - 1/x)^{2m+1}$ by the binomial theorem. Since (2m+1)C0 = (2m+1)C(2m+1), (2m+1)C1 = (2m+1)C2m, (2m+1)C2 = (2m+1)C(2m-1), ... , (2m+1)Cm = (2m+1)C(m+1), we may group these pairs of terms together to get: $(x - 1/x)^{2m+1} = (x^{2m+1} - x^{-(2m+1)}) - (2m+1)C1\ (x^{2m-1} - x^{-(2m-1)}) + ... + (-1)^m$ (2m+1)Cm (x - 1/x). This gives the polynomial for n = 2m+1 in terms of the lower order polynomials.

Note that x - 1/x has the same value 3/2 for x = -1/2 and x = 2. But $x^{2m} - 1/x^{2m}$ is negative for x = -1/2 and positive for x = 2, so we cannot express $x^{2m} - 1/x^{2m}$ as a function of (x - 1/x), polynomial or otherwise.

Problem A2
Let $\omega^3 = 1$, $\omega \ne 1$. Show that z_1, z_2, $-\omega z_1 - \omega^2 z_2$ are the vertices of an equilateral triangle.

Solution
Let A be the point z_1, B the point z_2. Then $z_2 - z_1$ represents the vector from A to B. Now $-\omega^2$ has unit length and makes an angle $\pm\pi/3$ with the positive real axis, so multiplying $z_2 - z_1$ by it rotates AB through an angle $\pi/3$ (clockwise or counterclockwise). Adding the result to $z_2 - z_1$ gives a point C such that AC is at an angle $\pi/3$ to AB. In other words ABC is equilateral. It is easily checked that C is then $z_1 - \omega^2(z_2 - z_1) = -\omega z_1 - \omega^2 z_2$ (since $1 + \omega + \omega^2 = 0$).

Problem A3
Let C be the complex numbers. f **:** C → C satisfies f(z) + z f(1 - z) = 1 + z for all z. Find f.

Solution
Putting z = 1 - w we have f(1 - w) + (1 - w) f(w)= 2 - w. Hence w f(1 - w) + (w - w^2) f(w) = 2w - w^2. Hence (1 + w - f(w)) + (w - w^2) f(w) = 2w - w^2, giving (w^2 - w + 1) f(w) = (w^2 - w + 1). So provided w is not $(1 \pm i\sqrt{3})/2$, we have f(w) = 1.

But at these two values f can be different. We can take one of them to be arbitrary. For example, take $f((1 + i\sqrt{3})/2) = k$, any complex number. Then $f((1 - i\sqrt{3})/2) = 1 + (1 - k)(1 - i\sqrt{3})/2$.

Problem A4
R is the reals. f, g **:** [0, 1] → R are arbitary functions. Show that we can find x, y such that $|xy - f(x) - g(y)| \ge 1/4$.

Solution
It is enough to consider the values at 0 and 1. If the pairs (0, 0), (0, 1), (1, 0) do not work for (x, y), then we have $|f(0) + g(0)| < 1/4$, $|f(0) + g(1)| < 1/4$, and $|f(1) + g(0)| < 1/4$. Hence $f(1) + g(1) \le f(1) + g(0) + f(0) + g(1) - (f(0) + g(0)) < 1/4 + 1/4 + 1/4$. So $1 - f(1) - g(1) > 1 - 3/4 = 1/4$. Hence the pair (1, 1) does work.

Problem A5
At a particular moment, A, T and B are in a vertical line, with A 50 feet above T, and T 100 feet

above B. T flies in a horizontal line at a fixed speed. A flies at a fixed speed directly towards B, B flies at twice T's speed, also directly towards T. A and B reach T simultaneously. Find the distance traveled by each of A, B and T, and A's speed.

Solution
Answer: A travels $25(3 + \sqrt{73})/3 = 96.2$ ft, B travels $400/3$ ft = 133.3 ft, T $200/3$ ft = 66.7 ft, A's speed is $(3 + \sqrt{73})/8 = 1.443$ times T's speed.

Take the x-axis vertical and the y-axis horizontal, so that at t = 0, the target is at (0, 0) and the pursuer is at (a, 0). Assume the target (T) has speed v and the pursuer (A or B) has speed kv with k > 1. At time t, the target is at (0, vt). The equations of motion are: $(dx/dt)^2 + (dy/dt)^2 = kv$ and $y' = (y - vt)/x$. The first equation gives $kv/(dx/dt) = -\sqrt{(1 + y'^2)}$ (negative because x decreases with time).

Differentiating the second equation gives $xy'' = - v/(dx/dt)$. So $kxy'' = \sqrt{(1 + y'^2)}$. Integrating, $\ln(y' + \sqrt{(1 + y'^2)}) = \ln x + \text{const}$. At x = a, y = 0, so $(x/a)^{1/k} = y' + \sqrt{(1 + y'^2)}$. Hence $2y' = (x/a)^{1/k} - (x/a)^{-1/k}$. Integrating again, $2y = a/(1+1/k)\ (x/a)^{1+1/k} - a/(1-1/k)\ (x/a)^{1-1/k} - a/(1+1/k) + a/(1-1/k)$, since y = 0 at x = a. When pursuit ends, x = 0 and hence $y = ak/(k^2 - 1)$.

For B, k = 2, so $y = 2a/3 = 200/3$ ft. That is the distance travelled by T. B is travelling at twice the speed, so B travels a distance 400/3 ft. For A reaches T at the same y, hence $200/3 = 50\ k/(k^2 - 1)$, so $4k^2 - 3k - 4 = 0$, so $k = (3 + \sqrt{73})/8$.

Problem A6
Given any real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta$, show that for m, n > 1 we can find m real n x n matrices $A_1, \ldots, A_m$ such that $\det A_i = \alpha_i$, and $\det(\sum A_i) = \beta$.

Solution
Start by setting A_i to be the matrix with 1, 1, ..., 1, α_i down the main diagonal and zeros elsewhere. Modify A_1 by changing the n, n-1 element to 1. Modify A_m by changing the n-1, n element to $m(\alpha_1 + \ldots + \alpha_m) - \beta/m^{n-2}$. It is clear that this gives $\det A_i = \alpha_i$.

$A_1 + \ldots + A_m$ has m, m, ... , m, $(\alpha_1 + \ldots + \alpha_m)$ down the main diagonal. The only other non-zero elements are n, n-1, which is 1 and n-1, n, which is $m(\alpha_1 + \ldots + \alpha_m) - \beta/m^{n-2}$. Hence its determinant evaluates to β.

Problem A7
Let R be the reals. Let f **:** [a, b] → R have a continuous derivative, and suppose that if f(x) = 0, then $f'(x) \neq 0$. Show that we can find g **:** [a, b] → R with a continuous derivative, such that $f(x)g'(x) > f'(x)g(x)$ for all $x \in [a, b]$.

Solution
We note that the derivative of f/g is $-(fg' - f'g)/g^2$ so if we take g so that f/g is decreasing, then we are almost home. Not quite, because of the difficulty that f may be zero. For example, if we take g(x) = x f(x), then fg' - f 'g is zero whenever f(x) = 0.

f has only finitely many zeros, for otherwise the zeros would have a limit point c and then by continuity we would have f(c) = f '(c) = 0. So we may take a polynomial p(x) such that p(x) f '(x) = -1 at each zero of f. Now consider g(x) = x f(x) + k p(x). We have $f(x)\ g'(x) - f'(x)\ g(x) = f(x)^2 + k(f(x)\ p'(x) - f'(x)\ p(x))$. Now f(x) p'(x) - f '(x) p(x) = 1 at each zero of f, so we can find an open set K containing all these zeros such that f(x) p'(x) - f '(x) p(x) > 1/2 on K. Now [a, b] - K is compact and contains no zeros of $f(x)^2$, which is continuous, so we can find $\varepsilon > 0$, so that $f(x)^2 > \varepsilon$ on [a, b] - K. But [a, b] is compact, so $|f(x)\ p'(x) - f'(x)\ p(x)| <$ some M on [a, b]. Take $k < \varepsilon/M$. Then on [a, b]

- K, $f(x)^2 > \varepsilon$ and $k|f(x)\,p'(x) - f'(x)\,p(x)| < \varepsilon$, so $f(x)\,g'(x) - f'(x)\,g(x) > 0$. On K, $f(x)^2$ is non-negative and $k(f(x)\,p'(x) - f'(x)\,p(x))$ is positive, so $f(x)\,g'(x) - f'(x)\,g(x) > 0$.

Problem B1
Join each of m points on the positive x-axis to each of n points on the positive y-axis. Assume that no three of the resulting segments are concurrent (except at an endpoint). How many points of intersection are there (excluding endpoints)?

Solution
Answer: mn(m-1)(n-1)/4.

Let us call the points on the y-axis $y_1 < y_2 < ... < y_n$ and the points on the x-axis $x_1 < x_2 < ... < x_m$. Let us join first y_1 to each x_i, then y_2 and so on. Joining y_1 to each of the x_i creates no intersections. Joining y_2 to x_1 creates m-1 intersections. Joining it to x_2 creates m-2 intersections and so on. So, in all, joining y_2 gives $1 + 2 + ... + m-1 = m(m-1)/2$ intersections.

Similarly joining y_3 to x_1 gives 2(m-1) intersections. Joining it to x_2 gives 2(m-2) and so on. So, in all, y_3 gives 2 x m(m-1)/2. Similarly, y_{r+1} gives r x m(m-1)/2. So in total we get $m(m-1)/2\,(1 + 2 + ... + n-1) = mn(m-1)(n-1)/4$.

Problem B2
Show that any positive real can be expressed in infinitely many ways as a sum $\sum 1/(10\,a_n)$, where $a_1 < a_2 < a_3 < ...$ are positive integers.

Solution
$\sum 1/n$ diverges, so $\sum 1/(10n)$ diverges. Let k be any positive real. Take N such that $1/(10N) < k$. Let M be any integer > N. We show how to find an expression $k = \sum 1/(10a_n)$ with $a_1 = M$. Take enough terms $a_2 = 1/(10(M+1))$, $a_3 = 1/(10(M+2))$, ... so that $\sum 1/(10a_n) < k$, but adding another term would give a sum $\geq k$. Let the difference $k - \sum 1/(10a_n)$ be $k_1 > 0$. Now take M_1 so that $1/(10\,M_1) < k_1$ and repeat. [In other words, take as many terms $1/(10\,M_1) + 1/(10(M_1+1)) + ...$ as we can whilst keeping the sum $< k_1$.] Let k_2 be the new difference, and so on. This process gives a sum which converges to k. Each such sum is different because it has a different starting term.

Problem B3
Find a continuous function f **:** $[0, 1] \to [0, 1]$ such that given any $\beta \in [0, 1]$, we can find infinitely many α such that $f(\alpha) = \beta$.

Solution
We use an adapted space-filling curve. Let g(x) be a piecewise linear function with period 2: g(x) = 0 on [0, 1/3], 1 on [2/3, 4/3] and 0 on [5/3, 2] with linear connecting pieces. For t any real in the range $0 \leq t \leq 1$ let $f(t) = g(t)/2 + g(9t)/4 + g(81t)/8 + g(729t)/16 + ...$. The series is obviously uniformly convergent, so f is continuous.

Now suppose that in base 3 we have $t = 0.c_1c_2c_3 ...$ with all $c_n = 0$ or 2. Then $9^n t$ = even integer + $0.c_{2n+1}c_{2n+2}...$. Now $0.c_{2n+1}c_{2n+2}...$ lies in [0, 1/3] if $c_{2n+1} = 0$ and in [2/3, 1] if $c_{2n+1} = 2$, so $g(9^n t) = 0$ if $c_{2n+1} = 0$, 1 if $c_{2n+1} = 2$. Thus f(t) has the binary expansion $0.a_1a_2a_3 ...$, where $a_n = c_{2n-1}/2$. So given any s in [0, 1] we can find infinitely many t with f(t) = s, for odd n pick c_n to give the correct binary expansion and for even n pick c_n arbitrarily.

Problem B4
A is the 5 x 5 array:

```
11 17 25 19 16
24 10 13 15  3
12  5 14  2 18
23  4  1  8 22
 6 20  7 21  9
```

Pick 5 elements, one from each row and column, whose minimum is as large as possible (and prove it so).

Solution
Answer: 25, 23, 20, 18, 15.

We can only pick one of 16, 17, 19, 25; only one of 23, 24; only one of 18, 22; and only one of 20, 21. That only gives 4 elements, so the minimum cannot be larger than 15. The answer shows that 15 can be achieved.

Problem B5
L_1 is the line { (t + 1, 2t - 4, -3t + 5) **:** t real } and L_2 is the line { (4t - 12, -t + 8, t + 17) **:** t real }. Find the smallest sphere touching L_1 and L_2.

Solution
The sphere has diameter PQ, where P(s + 1, 2s - 4, -3s + 5) is on L_1, and Q(4t - 12, -t + 8, t + 17) is on L_2. PQ is as short as possible. It can also be characterised as perpendicular to L_1 and L_2.

$PQ^2/2 = 7s^2 + st + 9t^2 + 25s - 52t + 457/2$. At the minimum the two partial derivatives must be zero, so 14s + t + 25 = 0, s + 18t - 52 = 0. Solving s = -2, t = 3. This gives P (-3, -12, 11), Q(0, 5, 20), the radius (= PQ/2) as 1/2 √251 and the centre as (-3/2, -7/2, 31/2).

Problem B6
α and β are positive irrational numbers satisfying 1/α + 1/β = 1. Let $a_n = [n\ \alpha]$ and $b_n = [n\ \beta]$, for n = 1, 2, 3, ... Show that the sequences a_n and b_n are disjoint and that every positive integer belongs to one or the other.

Solution
β positive implies α > 1, so [α (n+1)] > [α n] and the sequence does not contain any integers twice. Similarly for [βn].

If k appears in both sequences, then for some m, n we have: k < n α < k + 1, k < m β < k + 1. Hence k/α < n < (k+1)/α, k/β < m < (k+1)/β. Adding gives k < m+n < k + 1. Contradiction. So an integer appears in at most one sequence.

Suppose k does not appear in the sequence [n α]. Then for some n we have n α < k, and k + 1 < (n + 1) α. The first inequality implies that n < k/α = k - k/β. Hence (k - n) β > k. The second inequality implies that n + 1 > (k + 1)/α = k + 1 - (k + 1)/β. Hence (k - n) β < k + 1. So [(k - n) β] = k.

Problem B7
Given any finite ordered tuple of real numbers X, define a real number [X], so that for all x_i, α:
(1) [X] is unchanged if we permute the order of the numbers in the tuple X;

(2) $[(x_1 + \alpha, x_2 + \alpha, \ldots, x_n + \alpha)] = [(x_1, x_2, \ldots, x_n)] + \alpha$;

(3) $[(-x_1, -x_2, \ldots, -x_n)] = -[(x_1, x_2, \ldots, x_n)]$;

(4) for $y_1 = y_2 = \ldots = y_n = [(x_1, x_2, \ldots, x_n)]$, we have $[(y_1, y_2, \ldots, y_n, x_{n+1})] = [(x_1, x_2, \ldots, x_{n+1})]$.
Show that $[(x_1, x_2, \ldots, x_n)] = (x_1 + x_2 + \ldots + x_n)/n$.

Solution
It is convenient to write $[(x_1, \ldots, x_n)]$ simply as $[x_1, \ldots, x_n]$. We use induction on n. For n = 1, we have [0] = - [0], so [0] = 0. Then (3) gives that [x] = x, which establishes the result for n = 1.

[x, 0] = - [-x, 0] = - [0, -x] = - [x, 0] + x, so [x, 0] = x/2. Hence [x, y] = y + [x - y, 0] = y + (x - y)/2 = (x + y)/2, which establishes the result for n = 2. Now suppose it is true for n.

We have that [x, -x, 0, 0, ... ,0] = - [-x, x, 0, 0, ... , 0] by (3), = - [x, -x, 0, 0, ... ,0] by (1), so [x, -x, 0, 0, ... ,0] = 0. Now suppose $x_1 + x_2 + \ldots + x_{n+1} = 0$, then the n numbers $x_1, x_2, \ldots, x_n$ and the n numbers $0, 0, \ldots, 0, -x_{n+1}$ have the same sum, so by (4) we have for any z that $[x_1, \ldots, x_n, z] = [0, 0, \ldots, 0, -x_{n+1}, z]$. Take $z = x_{n+1}$, then $[0, \ldots 0, -x_{n+1}, x_{n+1}] = 0$, so $[x_1, \ldots, x_n, x_{n+1}] = 0$.

Finally, given arbitrary $x_1, \ldots, x_{n+1}$, let k be their mean, then $x_1 - k, x_2 - k, \ldots, x_{n+1} - k$ have zero sum, so $[x_1 - k, x_2 - k, \ldots, x_{n+1} - k] = 0$. Hence $[x_1, \ldots, x_{n+1}] = k$, as required.

21st Putnam 1960

Problem A1
For n a positive integer find f(n), the number of pairs of positive integers (a, b) such that ab/(a + b) = n.

Solution
ab = n(a + b) implies a(b - n) = nb > 0, so a and b must both exceed n. Let b = n + d, then a = n(n+d)/d = n + n^2/d. This is a solution iff d divides n^2. So f(n) = the number of divisors of n^2.

Problem A2
Let S be the set consisting of a square with side 1 and its interior. Show that given any three points of S, we can find two whose distance apart is at most √6 - √2.

Solution
Suppose we can find three points of S so that their three distances exceed k = √6 - √2. If one of the points is not on the top side of S and another on the bottom side, we can expand S by a combination of translation and expansion in the direction perpendicular to those sides to get three points with larger minimum distance. Similarly, if one of the points is not on each of the other two sides of S, we can expand S by a combination of expansion and translation to get three points with larger minimum distance. So we end up with a point on each side. But there are only three points, so one point must be at a vertex of the square. The other points on the two sides meeting at the vertex are all a distance <= 1 < k from the vertex, so the other two points must be on the other two sides. Let the vertices be A, B, C, D (in order). Assume the vertex point is at A. Let X be the point on BC a distance 2 - √3 from B. Then AX = k, so the point on BC must lie on the segment XC. Similarly, let Y be the point on DC a distance 2 - √3 from D. Then AY = k, so the point on DC must lie on the segment YC. But XY = k, so the two points will be closer than k unless one is at X and the other at Y but in that case the distances AX, AY do not exceed k. So it is not possible to arrange for the three points to have all distances > k.

Problem A3
Let α, β, γ, δ, ε be arbitary reals. Show that (1 - α) e^{α} + (1 - β) $e^{\alpha+\beta}$ + (1 - γ) $e^{\alpha+\beta+\gamma}$ + (1 - δ) $e^{\alpha+\beta+\gamma+\delta}$ + (1 - ε) $e^{\alpha+\beta+\gamma+\delta+\varepsilon} \le k_4$, where $k_1 = e$, $k_2 = k_1^e$, $k_3 = k_2^e$, $k_4 = k_3^e$ (so k_4 is 10^k with k approx 1.66 million).

Solution
Differentiating shows that the maximum of (1 - x)e^x is 1 and occurs at x = 0. Now consider (1 - α) e^{α} + (1 - β) $e^{\alpha+\beta} = e^{\alpha}$ (1 - α + (1 - β) e^{β}). For any given α, the maximum over β is 1, so we have to maximise e^{α}(2 - α). Differentiating shows that this has maximum $k_1 = e$ at α = 1.

Now consider (1 - α) e^{α} + (1 - β) $e^{\alpha+\beta}$ + (1 - γ) $e^{\alpha+\beta+\gamma} = e^{\alpha}$ (1 - α + (1 - β) e^{β} + (1 - γ) $e^{\beta+\gamma}$). We have just shown that the maximum of (1 - β) e^{β} + (1 - γ) $e^{\beta+\gamma}$ is k_1 at β = 1, γ = 0. So we have to maximise (1 - α + k_1) e^{α}. Differentiating shows that this is k_2 at α = k_1.

Similarly, the maximum of (1 - α) e^{α} + (1 - β) $e^{\alpha+\beta}$ + (1 - γ) $e^{\alpha+\beta+\gamma}$ + (1 - δ) $e^{\alpha+\beta+\gamma+\delta}$ is k_3 at α = k_2, β = k_1, γ = 1, δ = 0. Finally, the maximum of the desired expression is k_4 at α = k_3, β = k_2, γ = k_1, δ = 1, ε = 0.

Problem A4
Given two points P, Q on the same side of a line *l*, find the point X which minimises the sum of the distances from X to P, Q and *l*.

Solution
wlog take P to the right of Q and Q to be closer to the line than P. Take A on the line so that PA makes a 30^o angle with the line and A is to the left of P. Similarly, take B on the line so that QB makes a 30^o angle with the line and B is to the right of Q. There are three cases to consider.
(1) If the two segments PA and QB intersect, then their point of intersection X is the required point.
(2) If A is to the right of B, then join P to Q' the reflection of Q in the line. X is the point at which PQ' meets the line.
(3) If PA and QB do not meet, but A is to the left of B, then take X = Q.

We must now prove these statements. It is convenient to take the line *l* to be the x-axis. Take Q to be (a, b) and P to be (c, d). We assume that d > b > 0. Notice first that the sum of the three distances from the point X with coordinates (x, y) , which we will write as f(x, y), is a continuous and differentiable function of x and y. Also it tends to infinity as x or y tends to infinity. Hence the minimum must occur either (A) at a point where grad f = 0, or (B) on the boundary of the allowed region, in other words on y = 0, or (C) at a point where grad f does not exist.

We have $f(x, y) = y + \sqrt{((x - a)^2 + (y - b)^2)} + \sqrt{((x - c)^2 + (y - d)^2)}$. Hence grad f = ((x - a)/QX + (x - c)/PX, 1 + (y - b)/QX + (y - d)/PX) = **r** + **q** + **p**, where **r** is the unit vector along the perpendicular line from the line *l* to X, **q** is the unit vector from Q to X, and **p** is the unit vector from P to X. By resolving along the line perpendicular to **r** we find that the angles between **q** and **r** and between **p** and **r** must be equal. Similarly, the other pairs of angles. Hence the angle between each pair of **p**, **q**, **r** is 120^o. But it is clear that the only possible such point is the intersection H of the segments PA and QB (if it exists). H is thus the only candidate of type (A).

It is obvious that for points X on the line *l*, f(x, y) is minimised at K, the intersection of the line with PQ' (joining P to the reflection of Q), so that is the only candidate of type (B). Finally, the only points were the existence of grad f is doubtful are P and Q and indeed it is easy to see that grad f is not well defined there (for example, the limits as we approach parallel to the two axes are different). But clearly the distance sum is lower at Q than at P, so Q is the only candidate of type (C).

We now look separately at cases (1), (2) and (3). In case (1) all three candidates H, K and Q are available. We show first that H is better than Q. Let the perpendicular from Q to *l* meet *l* at C and the perpendicular from H to *l* meet *l* at D. Then angle HQC is 60^o, so QC = HD + QH/2. By the cosine rule we have $PQ^2 = PH^2 + QH^2 + PH \cdot QH > (PH + QH/2)^2$. Hence PQ + QH/2 > PH + QH and so PQ + QC = PQ + QH/2 + HD > PH + QH + HD. Next we show that H is better than K. As before let HD be the perpendicular to *l*. It is sufficient to show that PH + QH + HD < PD + QD, since we know that PD + QD <= PK + QK. Now $PD^2 = PH^2 + HD^2 + PH \cdot HD > (PH + HD/2)^2$, so PD > PH + HD/2. Similarly, QD > QH + HD/2. Adding gives the required result. That shows that H minimises the distance sum for case (1).

In case (2) only candidates K and Q are available. Take QC and PE as the perpendiculars to *l*. Then $PQ^2 = CE^2 + (PE - QC)^2$, $PQ'^2 = CE^2 + (PE + QC)^2$. We wish to show that PQ' < PQ + QC, or $CE^2 + (PE + QC)^2 < CE^2 + (PE - QC)^2 + QC^2 + 2\ QC\ \sqrt{(CE^2 + (PE - QC)^2)}$, or $(4\ PE - QC)^2 < 4(CE^2 + PE^2 - 2\ PE\ QC + QC^2)$, or $4\ CE^2 > 12\ PE^2 - 3\ QC^2$. That is certainly true since we have CE > (PE + QC) √3.

Finally, in case (3) the only candidates are K and Q. So as in the previous paragraph we wish to show that $4\ CE^2 < 12\ PE^2 - 3\ QC^2$. Now we have CE < √3 (PE - QC) from which the inequality follows.

Problem A5
The real polynomial p(x) is such that for any real polynomial q(x), we have p(q(x)) = q(p(x)). Find p(x).

Solution
Take q(x) = x + k and set x = 0. Then we have p(k) = p(0) + k. This is true for all k, so p(x) must be x + c for some c. Now take $q(x) = x^2$. We get $x^2 + c = (x + c)^2 = x^2 + 2cx + c^2$. Hence c = 0 and p(x) = x.

Problem A6
A player throws a fair die (prob 1/6 for each of 1, 2, 3, 4, 5, 6 and each throw independent) repeatedly until his total score ≥ n. Let p(n) be the probability that his final score is n. Find lim p(n).

Solution
Answer: 2/7.

For i = 1, 2, 3, 4, 5, let $p_i(n)$ = prob that the final score is n+i if the player stops when his total score is at least n. We note that p(n) is also the probability that the player's total equals n on some throw if he throws repeatedly. Now we can see that $p_5(n) = 1/6\ p(n-1)$, because the only way to achieve a final score of n+5 without passing through n, n+1, n+2, n+3, n+4 is to reach n-1 and then throw a 6. Similarly, $p_4(n) = 1/6\ p(n-1) + 1/6\ p(n-2)$, because to reach n+4 without passing through n, n+1, n+2, n+3 you must either to through n-1, which requires reaching n-1 and then throwing a 5, or not, in which case you must reach n-2 and then throw a 6. Similarly, $p_3(n) = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3)$, $p_2 = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3) + 1/6\ p(n-4)$, and $p_1(n) = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3) + 1/6\ p(n-4) + 1/6\ p(n-5)$. Adding, we get: $1 = p(n) + p_1(n) + p_2(n) + p_3(n) + p_4(n) + p_5(n) = p(n) + 5/6\ p(n-1) + 4/6\ p(n-2) + 3/6\ p(n-3) + 2/6\ p(n-4) + 1/6\ p(n-5)$.

In the limit, p(n-5) = p(n-4) = ... = p(n). Hence we get: $p_1(n) = 5/6\ p(n)$, $p_2(n) = 4/6\ p(n)$, $p_3(n) = 3/6\ p(n)$, $p_4(n) = 2/6\ p(n)$, $p_5(n) = 1/6\ p(n)$. But they sum to 1, so p(n) = 2/7.

We have (as above) $p(n) = 1/6\ p(n-1) + 1/6\ p(n-2) + 1/6\ p(n-3) + 1/6\ p(n-4) + 1/6\ p(n-5) + 1/6\ p(n-6)$ (*). Let m(n) = min{ p(n-1), p(n-2), p(n-3), p(n-4), p(n-5), p(n-6) }. Then (*) establishes that p(n) ≥ m(n), and so m(n+1) ≥ m(n). Similarly, let M(n) = max{ p(n-1), p(n-2), p(n-3), p(n-4), p(n-5), p(n-6) }. Then (*) shows that p(n) ≤ M(n), so M(n+1) ≤ M(n). Thus m(n) is a monotonic increasing sequence and M(n) is a monotonic decreasing sequence. But m(n) is obviously bounded above by any M(m), and M(n) is bounded below by any m(m). So both sequences converge. Suppose they converged to different limits. So m(n) converges to m and M(n) converges to M with M - m > 36k > 0. Take n sufficiently large that m(n) > m - 6k. At least one of the terms on the rhs of (*) must equal M(n) and the others are at least m(n), so p(n) ≥ 5/6 m(n) + 1/6 M(n) > 5/6 (m - 6k) + 1/6 M > 5/6 m - 5k + 1/6 (m + 36k) = m+ k. But that means that m(n) > m+k for all sufficiently large n. Contradiction. Hence M and m are the same and p(n) must have the same limit.

Problem A7
Let f(n) be the smallest integer such that any permutation on n elements, repeated f(n) times, gives the identity. Show that f(n) = p f(n - 1) if n is a power of p, and f(n) = f(n - 1) if n is not a prime power.

Solution
A permutation on n elements can be written as a product of disjoint cycles of length <= n. So f(n) is the smallest number divisible by 2, 3, ... , n.

If n is not a power of a prime, then we can write n = rs, with r and s relatively prime and each greater than 1. Then r, s < n - 1, so r and s divide f(n-1) and hence n divides f(n-1). So f(n) = f(n-1). If $n = p^{m+1}$, then p^m divides f(n-1), but not p^{m+1}, so f(n) = p f(n).

Problem B1
Find all pairs of unequal integers m, n such that $m^n = n^m$.

Solution
Answer: (2, 4), (4, 2), (-2, -4), (-4, -2).

Suppose first that m and n are both positive. Assume m > n. Then we can put m = n + k with k > 0. Hence $(1 + k/n)^n = n^k$. But for x > 1 we have $1 + x < e^x$ (the derivative of $f(x) = e^x - x - 1$ is positive and f(0) = 0) and hence $(1 + k/n)^n < e^k$. So there are no solutions for n > 2. If n = 1, then $n^m = 1$ and hence m = 1, contradicting the fact that m and n are unequal. If n = 2, then m must be a power of 2. Suppose $m = 2^h$. Then we find h = 1 or 2. h = 1 is invalid (because m and n are unequal), so m = 4. There is also the corresponding solution with m < n.

If n < 0 and m > 0, then $n^m = 1/m^{-n}$. So m divides 1 and hence m = 1. But m must be even for to make n^m positive. Contradiction. So there are no solutions of this type. If m and n are both negative, then -m, -n is a solution, so the only possibilities are (-2, -4) and (-4, -2) and it is readily checked that these are indeed solutions.

Problem B2
Let $f(m, n) = 3m+n+(m+n)^2$. Find $\sum_0^\infty \sum_0^\infty 2^{-f(m, n)}$.

Solution
Answer: 4/3.

The obvious approach is to sum over n and then over m (or vice versa). But it is not obvious how to sum x^N where N runs through the squares. So we wonder just what values f(m, n) can take. A little experimentation suggests that it takes each even value just once.

In fact, it is clear that r(r+1) is always even and that the difference between r(r+1) and the next number in the sequence, (r+1)(r+2) is 2r+2, so any positive even number can be uniquely expressed as r(r+1) + 2s with $0 \le s \le r$. But taking m = s, n = r-s, this is equivalent to the statement that any positive even number can be uniquely expressed as $3m+n+(m+n)^2$.
Hence the sum is just $1 + 1/4 + 1/4^2 + ... = 4/3$.

Problem B3
Fluid flowing in the plane has the velocity $(y + 2x - 2x^3 - 2xy^2, - x)$ at (x, y). Sketch the flow lines near the origin. What happens to an individual particle as $t \to \infty$?

Solution
The key insight is that the unit circle centre the origin is one possible trajectory. [The velocity is $(y + 2x - 2xr^2, - x) = (y, -x)$ on r = 1.] Hence fluid starting inside this circle remains inside it and fluid starting outside it remains outside. Differentiating $r^2 = x^2 + y^2$ gives $r\, dr/dt = 2x^2(1 - r^2)$. Hence if r < 1, then dr/dt is positive, so points inside the unit circle move outwards towards it. Similarly, if r > 1, then dr/dt is negative, so points outside the unit circle move inwards towards it.

To find the behaviour close to the origin, we drop the cubic terms. The equations are then linear and can be solved to get $x = (at + b)e^t$, $y = (-at + a - b)e^t$. This gives S shaped curves centred on the origin at a 45^o angle touching the line y = -x.

Problem B4
Show that if an (infinite) arithmetic progression of positive integers contains an nth power, then it contains infinitely many nth powers.

Solution
Let the difference between adjacent terms of the progression be d. Suppose that N^n is an nth power in the progression. Then $(N+d)^n$ is a larger nth power in the progression (it is obvious from the binomial expansion that it has the form N + kd).

Problem B5
Define a_n by $a_0 = 0$, $a_{n+1} = 1 + \sin(a_n - 1)$. Find $\lim (\sum_0^n a_i)/n$.

Solution
Answer: 1.

Note that $1 - a_{n+1} = \sin (1 - a_n)$. Put $c_n = 1 - a_n$. Note that c_0 belongs to the interval (0, 1]. Now for x in (0, 1] we have that $\sin x < x$ and $\sin x$ is also in (0 , 1]. So it follows (by a trivial induction) that $1 = c_0 > c_1 > c_2 > ... > c_n > 0$. So c_n is a monotonically decreasing sequence bounded below by 0. Hence it must tend to a limit $c \geq 0$. But c must satisfy $c = \sin c$. Hence $c = 0$. Hence a_n converges to 1. Hence $(\sum_0^n a_i)/n$ converges to 1 also.

Problem B6
Let $2^{f(n)}$ be the highest power of 2 dividing n. Let $g(n) = f(1) + f(2) + ... + f(n)$. Prove that $\sum \exp(- g(n))$ converges.

Solution
If n is odd, then f(n) = 0. If n is even, then $f(n) \geq 1$. Hence $g(n+2) \geq g(n) + 1$. So the series is dominated by $1 + (1/e + 1/e) + (1/e^2 + 1/e^2) + ... < 2/(1 - 1/e)$. So the sums are bounded above. All terms are positive, so it must converge.

Problem B7
Let R' be the non-negative reals. Let f, g **:** R' → R be continuous. Let a **:** R' → R be the solution of the differential equation: a' + f a = g, a(0) = c. Show that if b **:** R' → R satisfies b' + f b ≥ g for all x and b(0) = c, then $b(x) \geq a(x)$ for all x. Show that for sufficiently small x the solution of $y' + f y = y^2$, y(0) = d, is $y(x) = \max (d\, e^{-h(x)} - \int_0^x e^{-(h(x)-h(t))} u(t)^2 dt)$, where the maximum is taken over all continuous u(t), and $h(t) = \int_0^t (f(s) - 2u(s)) ds$.

Solution
Put $F(x) = \int_0^x f(t) dt$. Put $A(x) = a(x) e^{F(x)}$, $B(x) = b(x) e^{F(x)}$. Then $A' = (a' + f a) e^F = g e^F$, and $B' = (b' + f b) e^F \geq g e^F$, since $e^F > 0$. Hence $B' \geq A'$ for all x. But B(0) = A(0), so $B(x) \geq A(x)$ for all x and hence $b(x) \geq a(x)$ for all x.
Let u be any continuous function on R'. Put $U(x) = \int_0^x u(t) dt$. Put $h(x) = F(x) - 2\, U(x) = \int_0^x f(t) - 2u(t) dt$. Then h(0) = 0. We have $(y - u)^2 \geq 0$, so $y^2 - 2uy \geq -u^2$. Now put $z = y e^h$, so that $z(0) = y(0) e^{h(0)} = d$.

$z' = (y' + h'y) e^h = (y' + (f - 2u)y) e^h = (y^2 - 2uy) e^h \geq -u^2 e^h$. Hence $z(x) = z(0) + \int_0^x z'(t) dt \geq d - \int_0^x u(t)^2 e^{h(t)} dt$. Hence $y(x) = z(x) e^{-h(x)} \geq d e^{-h(x)} - \int_0^x e^{-(h(x) - h(t))} u(t)^2 dt$. So $z(x) \geq$ the maximum over all u.
If we put u = y (the solution), then we get equality.

22nd Putnam 1961

Problem A1
The set of pairs of positive reals (x, y) such that $x^y = y^x$ form the straight line y = x and a curve. Find the point at which the curve cuts the line.

Solution
Put y = kx. Then we get $x = k^{1/(k-1)}$, $y = k^{k/(k-1)}$. This gives a curve which cuts y = x at $x = \lim k^{1/(k-1)}$, where the limit is taken as k tends to 1. Put k = 1 + 1/n, then the limit is $(1 + 1/n)^n$, which is e. So the point of intersection is (e, e).

Problem A2
For which real numbers α, β can we find a constant k such that $x^\alpha y^\beta < k(x + y)$ for all positive x, y?

Solution
Answer: α, β ≥ 0 and α + b = 1.

If α < 0, then taking x sufficiently small and y = 1 gives $x^\alpha y^\beta > k(x + y)$ for any k. Similarly for β < 0. If α + b > 1, then taking x = y sufficiently large gives $x^\alpha y^\beta > k(x + y)$ for any k. On the other hand, if α + β < 1, then taking x = y sufficiently small gives $x^\alpha y^\beta > k(x + y)$ for any k. Suppose, on the other hand that α, b ≥ 0 and α + β = 1. Take x ≥ y. Then $x^\alpha y^\beta \le x^{\alpha+\beta} = x$, so the result holds with k = 1.

Problem A3
Find $\lim_{n\to\infty} \sum_1^N n/(N + i^2)$, where $N = n^2$.

Solution
As usual, we try integration. We can write the sum as $1/n \sum_1^N 1/(1 + (i/n)^2)$. This is a Riemann sum for the integral $\int_0^n 1/(1 + x^2)\, dx$ and hence tends to $\int_0^\infty 1/(1 + x^2)\, dx = \tan^{-1}x \,|_0^\infty = \pi/2$.

Problem A4
If $n = \prod p^r$ be the prime factorization of n, let $f(n) = (-1)^{\sum r}$ and let $F(n) = \sum_{d|n} f(d)$. Show that F(n) = 0 or 1. For which n is F(n) = 1?

Solution
Answer: F(n) = 1 for n square.

If some r is odd, then the factors of n can be grouped into pairs $p^a m$, $p^{r-a} m$ with $f(p^a m)$, $f(p^{r-a} m)$ having opposite signs. So in this case F(n) = 0.

If all r are even, so that n is a square, then we may use induction on the number N of prime factors. For N = 1, we have $n = p^{2m}$, so there are m+1 factors p^{2r} with f = 1 and m factors p(2r+1) with f = -1. So the result is true for N = 1. Suppose it is true for N. Take $n = p^{2m}P$, where P has m prime factors. Each divisor of n has the form $p^{2r}s$ or $p^{2r+1}s$, where s is a divisor of P. For the first type $f(p^{2r}s) = f(s)$, so summing over s gives 1. Then summing over r gives m+1. For the second type $f(p^{2r+1}s) = -f(s)$, so summing over s gives -1. Then summing over r gives -m. Hence F(n) = 1 and the result is true for N+1.

Problem A5
Let X be a set of n points. Let P be a set of subsets of X, such that if A, B ∈ P, then X - A, A ∪ B, A ∩ B ∈ P. What are the possible values for the number of elements of P?

Solution
Answer: 2, 4, 8, ... , 2^n.

Take $0 \le r < n$. Let $A_1 = \{1\}$, $A_2 = \{2\}$, ... , $A_r = \{r\}$, $A_{r+1} = \{r+1, r+2, ... , n\}$. Consider the collection of all unions of sets A_i. This collection has 2^{r+1} elements and satisfies the conditions. So this shows that we can achieve the values 2, 4, 8, ... , 2^n.

To prove the converse we use induction on n. It is obviously true for n = 1. Suppose it is true for all $n < N$. Let X have N points and P be a set of subsets of X satisfying the condition. If $|P| = 2$, then there is nothing to prove. Otherwise we can find a member Y of P other than the empty set and X which has no non-trivial subsets in P. Now consider the collection Q of subsets of X - Y which are in P. It is easy to check that (1) P is just the collection of all A and $A \cap Y$, where A is in Q, so that $|P| = 2|Q|$, and (2) Q satisfies the conditions, so |Q| is a power of 2 by induction.

Problem A6
Consider polynomials in one variable over the finite field F_2 with 2 elements. Show that if n + 1 is not prime, then $1 + x + x^2 + ... + x^n$ is reducible. Can it be reducible if n + 1 is prime?

Solution
Let n+1 = ab, then $1 + x + x^2 + ... + x^n = (1 + x + x^2 + ... + x^{a-1})(1 + x^a + x^{2a} + ... + x^{ab-a})$. Note that this does not depend upon the field having two elements.

Yes. For example, $(1 + x + x^3)(1 + x^2 + x^3) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$.

Problem A7
S is a non-empty closed subset of the plane. The disk (a circle and its interior) $D \supseteq S$ and if any disk $D' \supseteq S$, then $D' \supseteq D$. Show that if P belongs to the interior of D, then we can find two distinct points $Q, R \in S$ such that P is the midpoint of QR.

Solution
The existence of D imposes strong restrictions on S. In fact, we will show that S must contain the entire perimeter C of D. For suppose it does not contain a point X on C. Then since S is closed, it does not contain a neighbourhood of X. So we can find a small circle C' centre X so that the disk D' enclosed by C' lies entirely outside S. Let C' meet C at A and B. Take a circle C'' through A and B with centre on the same side of AB as the centre of C, but further away (so that it has a larger radius than C). Then C'' contains all points of D - D' and hence all points of S. But it does not contain all points of D. Contradiction.

Finally, note that any point in the interior of D lies on the midpoint of some chord of C.

Problem B1
a_n is a sequence of positive reals. $h = \lim (a_1 + a_2 + ... + a_n)/n$ and $k = \lim (1/a_1 + 1/a_2 + ... + 1/a_n)/n$ exist. Show that $h k \ge 1$.

Solution
Apply the arithmetic-geometric mean theorem to each sum. We get that $(a_1 + a_2 + ... + a_n)/n \ge (a_1a_2 ... a_n)^{1/n}$, $(1/a_1 + 1/a_2 + ... + 1/a_n)/n = 1/(a_1a_2 ... a_n)^{1/n}$. Hence their product is at least 1. Hence the product of the limits also.

Problem B2
Two points are selected independently and at random from a segment length β. What is the probability that they are at least a distance α (< β) apart?

Solution
Consider which points (x, y) of the square x = 0 to β, y = 0 to β have $|x - y| \geq \alpha$. Evidently the acceptable points lie in the two right-angled triangles: (0, α), (0, β), (β-α, β) and (α, 0), (β, 0), (β, β-α). These fit together to give a square side β-α, so the area of the acceptable points is $(\beta-\alpha)^2$ out of a total area of β^2. Thus the probability is $(\beta-\alpha)^2/(\beta)^2$.

Problem B3
A, B, C, D lie in a plane. No three are collinear and the four points do not lie on a circle. Show that one point lies inside the circle through the other three.

Solution
If the convex hull is a triangle, then the fourth point lies inside its circumcircle. So suppose ABCD is convex. One pair of opposite angles must have sum greater than 180^o (otherwise the points would lie on a circle). Suppose they are A and C. Then we claim that C lies inside the circle through A, B, D. Take a point C' on the ray DC on the far side of C from D such that $\angle CBC' = \angle A + \angle C - 180^o$. Then $\angle C' = 180^o - \angle A$. So C' lies on the circle. Hence C which lies on the segment C'D lies inside the circle.

Problem B4
Given $x_1, x_2, \ldots, x_n \in [0, 1]$, let $s = \Sigma_{1\leq i<j\leq n} |x_i - x_j|$. Find f(n), the maximum value of s over all possible $\{x_i\}$.

Solution
Answer: $f(2m) = m^2$ (half the points 0, half 1); $f(2m+1) = m^2 + m$ (m+1 points 0, m points 1).

It remains to prove that we cannot do better than this. We prove that this is best possible by induction. It is obvious for n = 2, 3. Suppose it is true for n. Take n+2 points. Let A be the leftmost point and B the rightmost point. Then the new terms in the sum the n+2 points are AB and AC, CB for each existing C. But these 2n + 1 terms sum to at most n + 1 (AC + CB = AB ≤ 1). So $f(n+2) \leq f(n) + n + 1$. For n = 2m this gives $f(2m+2) \leq m^2 + 2m + 1 = (m + 1)^2$. For n = 2m + 1, it gives $f(2m+3) \leq m^2 + m + 2m + 1 + 1 = (m+1)^2 + (m+1)$.

Problem B5
Let n be an integer greater than 2. Define the sequence a_m by $a_1 = n$, a_{m+1} = n to the power of a_m. *Either* show that $a_m <$ n!! ... ! (where the factorial is taken m times), *or* show that $a_m >$ n!! ... ! (where the factorial is taken m-1 times).

Solution
If it was true that $n^k < k!$. then the upper limit on a_m would be almost trivial. Unfortunately, $n^k > k!$ for small k, which results in a few complications. Let n_m = n! ... ! (m times). We will prove that $a_m < n_m$ by induction on m. For m = 1, the result just states that n < n!, which is obviously true. Suppose now that n ≥ 5. Then $n! \geq n(n-1)3 > 2n^2$. So, a fortiori, $n_m > 2n^2$ for m ≥ 1. It is easy to check that for n = 3, $a_2 = 27 < n_2 = 3!! = 720$, so the result is true for m = 2. Also $n_2 > 2n^2 = 18$. Similarly, for n = 4, $a_2 = 256 < n_2 = 4!! = 24!$ which establishes the result for m = 2. Also $n_2 > 2n^2 = 32$. So for all n we have a starting case m for which also $n_m > 2n^2$.

If $k \geq 2n^2$, then half the terms in k! are at least n^2, so their product is at least $n^{2\,k/2} = n^k$. In other words $k! > n^k$. So suppose that $a_m < n_m$, (*). Then since $n_m > 2n^2$, we may take (*) to the power of n to get $a_{m+1} = n^{lhs} < n^{rhs} < n_{m+1}$. Hence the result is true for all m.

The lower limit is harder because it is not true that $n^k > k!$, so the result for m+1 does not follow in an immediate way from that for m.

Let $f_0(n) = n$, $f_1(n) = f_0(n)\, n!$, $f_2(n) = f_1(n)\, n!!$, $f_3(n) = f_2(n)\, n!!!$, ... We show that $f_{m+1}(n) < n$ to the power of $f_m(n)$. We use induction on m. For m = 0, this reads $n\, n! < n^n$, which is obviously true. Now assume it is true for m ***and all n***. Then we have $f_{m+1}(n) = n\, f_m(n!) < n$ (n! to the power of $f_{m-1}(n!)$) < (n n!) to the power of $f_{m-1}(n!) < n^n$ to the power of $f_{m-1}(n!) = n$ to the power of $f_m(n)$.

It is now an easy induction that $f_m(n) < a_{m+1}$. For m = 1, this just states that $n\, n! < n^n$. Suppose it is true for m, then $f_{m+1}(n) < n$ to the power of $f_m(n) < n$ to the power of $a_{m+1} = a_{m+2}$. But obviously n! ... ! (! taken m times) < $f_m(n)$ and hence < a_{m+1}.

Problem B6
Let y be the solution of the differential equation $y'' = -(1 + \sqrt{x})\, y$ such that $y(0) = 1$, $y'(0) = 0$. Show that y has exactly one zero for $x \in (0, \pi/2)$ and find a positive lower bound for it.

Solution
Let z be the solution of $z'' = -z$ with $z(0) = 1$, $z'(0) = 0$. Suppose $y > 0$ for $0 <= x < \pi/2$. Then since $y''z - yz'' = -(1 + \sqrt{x})\, yz + yz = -\sqrt{x}\, yz$, we have $\int_0 \pi/2 - \sqrt{x}\, yz\, dx = \int_0^{\pi/2} (y''z - yz'')\, dx = (y'z - yz') \big|_0^{\pi/2} = y(\pi/2) \geq 0$. But $-\sqrt{x}\, yz < 0$ throughout the range, so $\int_0^{\pi/2} - \sqrt{x}\, yz\, dx < 0$. Contradiction. Hence y has at least one zero in $(0, \pi/2)$.

Let w be the solution of $w'' = -3w$ with $w(0) = 1$, $w'(0) = 0$. Then $w = \cos(\sqrt{3})x$. Suppose y had a zero in $(0, \pi/(2\sqrt{3}))$. Then the same argument would show that w had a root in $(0, \pi/(2\sqrt{3})$ which is false. So $\pi/(2\sqrt{3})$is a positive lower bound for any zero of y.

Finally, suppose that y has more than one zero in $(0, \pi/2)$. Let the first be at x = h and the second at x = k. Take v(x) to be $A \cos((\sqrt{3})\, x + B)$. We select A and B so that $v(h) = 0$ and $v'(h) = y'(h)$. Now v cannot have another zero in (h, k), so we can establish a contradiction by a similar argument to the above. We have $v'' = -3v$. Hence $y''v - yv'' = (-(1 + \sqrt{x}) + 3)\, yv$ which is strictly positive on (h, k). Hence $\int_h^k (y''v - yv'')\, dx > 0$. But $\int_h^k (y''v - yv'')\, dx = (y'v - yv') \big|_h^k = y'(k)v(k) < 0$. [y' must be positive since y is zero and y(x) < 0 for x just less than k, whilst we know that v(k) is negative). Contradiction. Hence y has exactly one zero.

Problem B7
The sequence of non-negative reals satisfies $a_{n+m} \leq a_n a_m$ for all m, n. Show that $\lim a_n^{1/n}$ exists.

Solution
$a_n \leq a_1 a_{n-1} \leq a_1^2 a_{n-2} \leq ... \leq a_1^n$. So $a_n^{1/n} \leq a_1$. All a_n are non-negative, so $a_n^{1/n} \geq 0$. Thus we have established that $\{a_n^{1/n}\}$ is bounded.

Fix n. Take any N > n. Then we may write N = nq + r, with $0 \leq r < n$. We have $a_N \leq a_n^q\, a_r$, so $a_N^{1/N} \leq a_n^s\, a_r^{1/N}$, where s = q/N = (1 - r/N)/n. But (1 - r/N) tends to 1 as N tends to infinity, as does $a_r^{1/N}$. Hence lim sup $a_N^{1/N} \leq a_n^{1/n}$. This is true for all n, so the sequence cannot have more than one limit point and hence converges.

23rd Putnam 1962

Problem A1
5 points lie in a plane, no 3 collinear. Show that 4 of the points form a convex quadrilateral.

Solution
If the convex hull has 4 or 5 vertices, then we are done. If not, then two of the points, say D and E, must lie inside the triangle ABC formed by the other three. DE must intersect just two of AB, BC and CA. Suppose that it does not intersect AB. Then the four points A, B, D, E form a convex quadrilateral.

Problem A2
Let R be the reals. Find all f **:** K → R, where K is [0, ∞) or a finite interval [0, a), such that $(1/k \int_0^k f(x)\,dx)^2 = f(0)\,f(k)$ for all k in K.

Solution
Answer: for [0, ∞) $f(x) = b/(cx + 1)^2$ with c non-negative. For [0, a), $f(x) = b/(cx + 1)^2$ with c >= -1/a.

Put $y = \int_0^x f(t)\,dt$. Then the given equation becomes $y^2/x^2 = by'$, where b = f(0). Integrating, b/y = c + 1/x, or y = bx/(cx + 1). Differentiating gives $f(x) = y' = b/(cx + 1)^2$.

f(-1/c) is undefined, so if K is the half-line, then c cannot be negative. If K is the finite interval [0, a), then we require $c \geq -1/a$. Note that we can have equality because k must be strictly less than a. On the other hand, b can have any value.

Problem A3
ABC is a triangle and k > 0. Take A' on BC, B' on CA, C' on AB so that AB' = k B'C, CA' = k A'B, BC' = k C'A. Let the three points of intersection of AA', BB', CC' be P, Q, R. Show that the area PQR $(k^2 + k + 1)$ = area ABC $(k - 1)^2$.

Solution
There does not seem to be a geometric solution, so one is faced with a messy algebraic calculation. There are various ways of doing it. The worst is probably to use ordinary Cartesian coordinates.

Let us use points in the Argand diagram. Take A to be the origin and C to be the real point 1+k. Take B to be z(1+k). Then B' is k and A' is zk+1 (check: ((zk+1) - (1+k))/((z(1+k) - (zk+1)) = (zk-k)/(z-1) = k). We want to find R, the intersection of AA' and BB'. It lies on AA', so it is r(zk+1) for some 0 < r < 1. But it lies on BB', so it is k + s(zk+z-k) for some 0 < s < 1. After some manipulation, we get $r = (k^2+k)/(k^2+k+1)$.

Now take k < 1. Then area ARB = (AR/AA') area AA'B = r area AA'B (this is where we need k < 1, because if k > 1, then the geometry changes). But area AA'B = (A'B/CB) area ABC, so area ARB = r/(1+k) area ABC = $k/(k^2+k+1)$ area ABC. Similarly, taking P as the intersection of BB' and CC' and Q as the intersection of AA' and CC', we have that area AQC = $k/(k^2+k+1)$ area ABC, and area CPB = $k/(k^2+k+1)$ area ABC. Adding we get (area ABC - area PQR), so area PQR/area ABC = $1 - 3k/(k^2+k+1) = (1 - k)^2/(k^2+k+1)$.

If k > 1, then we get the result with 1/k replacing k, but multiplying through by k^2 gives the same formula.

Problem A4
R is the reals. [a, b] is an interval with $b \geq a + 2$. f **:** [a, b] $\rightarrow$ R is twice differentiable and $|f(x)| \leq 1$ and $|f''(x)| \leq 1$. Show that $|f'(x)| \leq 2$.

Solution
Take the interval to be [-1, 1]. Taylor's formula gives $f(1) = f(x) + (1 - x) f'(x) + (1 - x)^2 f''(h)/2$, $f(-1) = f(x) + (-1 - x) f'(x) + (-1 - x)^2 f''(k)/2$ for some h, k in the interval (note that we are expanding about x).

Subtracting, $f(1) - f(-1) = 2 f'(x) + (1 - x)^2 f''(h)/2 - (1 + x)^2 f''(k)/2$. Hence $2|f'(x)| \leq |f(1)| + |f(-1)| + (1 - x)^2|f''(h)|/2 + (1 + x)^2|f''(k)|/2 \leq 2 + (1 - x)^2/2 + (1 + x)^2/2 = 3 + x^2 \leq 4$. So $|f'(x)| \leq 2$.

Problem A5
Find $nC1\ 1^2 + nC2\ 2^2 + nC3\ 3^2 + ... + nCn\ n^2$ (where nCr is the binomial coefficient).

Solution
Answer: $n(n+1)2^{n-2}$.
Differentiate $(1 + x)^n$, multiply by x and differentiate again. Set x = 1.

Problem A6
X is a subset of the rationals which is closed under addition and multiplication. $0 \notin X$. For any rational $x \neq 0$, just one of x, $-x \in X$. Show that X is the set of all positive rationals.

Solution
Either x or -x belongs to X. X is closed under multiplication, so the square $x^2 = (-x)^2$ belongs to X. In particular, 1 belongs to X. Hence by repeated addition all positive integers must belong to X. Suppose positive rational m/n does not belong to X. Then -m/n does, and hence by repeated addition -m. So m does not belong to X. Contradiction. So X contains all positive rationals. But if x is in X, -x is not, so X does not contain any negative rationals and hence X is just the set of all positive rationals.

Problem B1
Define $x^{(n)} = x(x - 1)(x - 2) ... (x - n + 1)$ and $x^{(0)} = 1$. Show that $(x + y)^{(n)} = nC0\ x^{(0)}y^{(n)} + nC1\ x^{(1)}y^{(n-1)} + nC2\ x^{(2)}y^{(n-2)} + ... + nCn\ x^{(n)}y^{(0)}$.

Solution
Induction on n. It is obviously true for n = 1. Suppose it is true for n. We now multiply the rhs by (x + y - n). We now write:

$nC0\ x^{(0)}y^{(n)}(x + y - n) = (nC0\ x^{(0)}y^{(n)}\ (y-n)\ ) + (nC0\ x^{(0)}y^{(n)}\ x\ );$
$nC1\ x^{(1)}y^{(n-1)}\ (x + y - n) = (nC1\ x^{(1)}y^{(n-1)}\ (y-n+1)\ ) + (nC1\ x^{(1)}y^{(n-1)}\ (x-1)\ );$
$nC2\ x^{(2)}y^{(n-2)}\ (x + y - n) = (nC2\ x^{(2)}y^{(n-2)}\ (y-n+2)\ ) + (nC2\ x^{(2)}y^{(n-2)}\ (x-2)\ );$
...
$nCn\ x^{(n)}y^{(0)}\ (x + y - n) = (nCn\ x^{(n)}y^{(0)}\ y\ ) + (nCn\ x^{(n)}y^{(0)}\ (x-n)\ ).$

Now the first term in the 1st equation gives $(n+1)C0\ x^{(0)}y^{(n+1)}$. The second term in the 1st equation and the first term in the 2nd equation give $(n+1)C1\ x^{(1)}y^{(n)}$. Then the second term in the 2nd equation and the first in the 3rd equation gives $(n+1)C2\ x^{(2)}y^{(n-1)}$ and so on. Finally the second term in the last equation gives $(n+1)C(n+1)\ x^{(n+1)}y^{(0)}$. So we have the result for n+1.

Problem B2
Let R be the reals, let N be the set of positive integers, and let P = {X **:** X $\subseteq$ N}. Find f **:** R $\rightarrow$ P such that f(a) $\subset$ f(b) (and f(a) $\neq$ f(b)) if a < b.

Solution
Let Q be the rationals. Define g: Q $\rightarrow$ N as follows. For m/n where m and n are positive integers without any common factor, let g(m/n) = 2^m3^n and let g(-m/n) = 2^m5^n. Now define f: R $\rightarrow$ P by f(x) = { g(r) : r <= x is rational }. We claim that f has the required property.

Clearly g is injective. Now suppose x < y are reals. We can find a rational r such that x < r < y. So g(r) belongs to f(y) but not to f(x). But S belongs to f(x), then S = g(q) for some q in Q with q < x. So q < y and hence S belongs to f(y) also. So f(x) is a subset of f(y).

Problem B3
Show that a convex open set in the plane containing the point P, but not containing any ray from P, must be bounded. Is this true for any convex set in the plane?

Solution
Answer: no.
Let S be the convex set. Take any ray R from P. We can find X on R not in S. Since S is open we can take a small segment AB perpendicular to R through P with AP = PB which is entirely contained in S. Take A' on the line AX, the opposite side to A, and B' on BX the opposite side to B, with A'B' perpendicular to R. Then no point C' on the segment A'B' can be in S. For if it was, then we could project C'X to meet the segment AB in C and X would lie between two points of S and hence would have to be in S. But the angle A'PB' is greater than zero. So we have found an open interval about the direction PX (measured as a polar angle at P) such that S extends at most a distance AA' for directions within that interval.

The interval [0, 2π] is compact and is covered by these open intervals, so a finite number of them cover it. Hence there is a finite distance d such that we can find a point not in S within a distance d of P in *any* direction. That means that no point Q a distance greater than d from P can be in S (otherwise all points on the segment QP would be in S and there would be no point not in S within a distance d in that direction). So S lies inside the disk radius d on P and hence is bounded.

Take the slab 0 < y < 1 together with the point (0, 0). It is convex and unbounded but does not contain any ray through the origin.

Problem B4
A finite set of circles divides the plane into regions. Show that we can color the plane with two colors so that no two adjacent regions (with a common arc of non-zero length forming part of each region's boundary) have the same color.

Solution
Color points inside an odd number of circles blue and points inside an even number of circles red. Then we change color whenever we cross a circle.

Problem B5
Show that for n > 1, (3n + 1)/(2n + 2) < $\sum_1^n r^n/n^n$ < 2.

Solution
Suppose m <= n, then expanding by the binomial theorem, m^n = ((m-1) + 1)^n = $(m-1)^n$ + n $(m-1)^{n-1}$ + further positive terms > $(m-1)^n$ + n $(m-1)^{n-1}$ > 2 $(m-1)^n$. Hence n^n > $(n-1)^n$ + $(n-1)^n$ > $(n-1)^n$ + (n-

$2)^n + (n-2)^n > ... > (n-1)^n + (n-2)^n + ... + 1^n$. So $2\,n^n > \sum_1^n r^n$, which is one of the two required inequalities.

The other is slightly harder. We approximate the integral of $f(x) = x^n$ between 0 and 1. We have to do slightly better than the usual Riemann sums. We also take into account the small triangles. The curve has increasing gradient, so the area under the curve between (r-1)/n and r/n is less than the area under the chord joining ((r-1)/n, f((r-1)/n)) and (r/n, f(r/n)). Summing those areas gives: $(1/n) \sum_1^n (r/n)^n - (1/2n) \sum_1^n ((r/n)^n - ((r-1)/n)^n = (1/n)\,K - (1/2n)\,(n/n)^n$, where $K = \sum_1^n (r/n)^n$. The integral is just 1/(n+1), so we have: $1/(n+1) < K/n - 1/(2n)$, or $K > n/(n+1) + 1/2 = (3n+1)/(2n+2)$.

Problem B6
$f : [0, 2\pi) \to [-1, 1]$ satisfies $f(x) = \sum_0^n (a_j \sin jx + b_j \cos jx)$ for some real constants a_j, b_j. Also $|f(x)| = 1$ for just 2n distinct values in the interval. Show that $f(x) = \cos(nx + k)$ for some k.

Solution
We call f(x) a trigonometric sum of degree n. Let $x_1, x_2, ... , x_{2n}$ be the points at which f(x) = 1 or -1. Two insights are needed. The first is that $f'(x_i) = 0$. For if the value was non-zero, then f(x) would lie outside [-1, 1] for points sufficiently close (and on the correct side). [Note that there is a minor complication at 0. If f(0) = 1 and $f'(0) < 0$, then the aberrant points would be just less than 2π.]

The second is that if a trigonometric sum of degree n has 2n roots (counting multiplicities), then it is determined up to a multiplicative constant (and if it has more than 2n roots then it is identically zero). We will establish that later. But $1 - f(x)^2$ has the 2n roots x_i. Also, they all have multiplicity at least 2 since $f'(x_i) = 0$. So we have at least 4n roots (counting multiplicities). Using the familiar formulae cos(A + B) etc we can write $1 - f(x)^2$ in the same form as f(x) but with the sum from 0 to 2n.

Similarly, f '(x) is a trigonometric sum of degree n with the 2n roots x_i. If we square it, each root is doubled, so $f'(x)^2$ is a trigonometric sum of degree 2n with the 4n roots x_i (each counted twice). Hence we must have $f'(x)^2$ = multiple of $(1 - f(x)^2)$. Both are non-negative, so it must be a positive multiple. Hence we can take the square root to get $f'(x) = N \sqrt{(1 - f(x)^2)}$ for some fixed N. Solving, we get f(x) = cos(Nx + k). But for this to have just 2n values ±1 we must have N = n.

To prove the result about trigonometric polynomials put $z = e^{ix}$, then $f(x) = z^{-n} p(z)$, where p is a polynomial in z of degree 2n. Hence p has at most 2n roots and is determined by those roots up to a multiplicative constant. Note that if $f'(x) = 0$ and f(x) = 0, then p(z) = 0 and since $0 = f'(x)$ = combination of p(z) and p'(z), then p'(z) = 0 also, so double roots of f have corresponding roots of multiplicity at least 2 in p.

24th Putnam 1963

Problem A1
Dissect a regular 12-gon into a regular hexagon, 6 squares and 6 equilateral triangles. Let the regular 12-gon have vertices P_1, P_2, ... , P_{12} (in that order). Show that the diagonals P_1P_9, $P_{12}P_4$ and P_2P_{11} are concurrent.

Solution
Take squares on the inside of alternate sides of the 12-gon. Each angle of the 12-gon is 150^o, so the remaining angles after placing the squares are all 60^o, so we have equilateral triangles on the inside of the remaining 6 sides of the 12-gon. Now the opposite sides of the squares form a hexagon. (As a check we may notice that if A, B, C, D are adjacent vertices of the 12-gon and we place squares ABPQ on AB and CDRS on CD, then S and P coincide and the equilateral triangle ABP and the two squares take $60^o + 90^o + 90^o$ out of the 360^o angle at P leaving 120^o, which is the angle between sides of a regular hexagon.)

Let O be the centre of the 12-gon. P_1P_9 and $P_{12}P_4$ have the same length and one is obtained from the other by rotation about O through 90^o. Thus they intersect on the perpendicular bisector of $P_{12}P_1$ a distance $P_{12}P_1/2$ from $P_{12}P_1$. $P_{11}P_2$ is parallel to $P_{12}P_1$ and its midpoint lies on the perpendicular bisector of $P_{12}P_1$. The angle $P_{11}P_2P_1$ is 180^o - angle $P_{12}P_1P_2 = 30^o$, so the distance between of the midpoint along the bisector is $P_{12}P_1 \sin 30^o = P_{12}P_1/2$. Hence the three lines intersect at this point.

Problem A2
The sequence a_1, a_2, a_3, ... of positive integers is strictly monotonic increasing, $a_2 = 2$ and $a_{mn} = a_m a_n$ for m, n relatively prime. Show that $a_n = n$.

Solution
The key observation is that if $a_N = N$ for N odd, then $a_{2N} = 2N$, and hence $a_{N+1} = N+1$, $a_{N+2} = N+2$, ... , $a_{2N-1} = 2N-1$ (because a_n is strictly increasing, so there are only N-1 holes for N-1 pegs. But we can now repeat using 2N-1 (provided 2N-1 > N, or N > 1) and the result is established for all n.

So we just need to get started with a_3. Suppose $a_3 = m$. Then $a_6 = 2m$. So $m+2 \le a_5 \le 2m-1$. Hence $a_{10} \le 4m-2$ and $a_9 \le 4m-3$. So $a_{18} \le 8m-6$ and hence $a_{15} \le 8m-9$. But $a_{15} = a_3a_5 \ge m^2+2m$, so $m^2+2m \le 8m-9$ or $(m-3)^2 \le 0$. Hence $m = 3$.

Problem A3
Let D be the differential operator d/dx and E the differential operator $xD(xD - 1)(xD - 2) ... (xD - n)$. Find an expression of the form $y = \int_1^x g(t)\,dt$ for the solution of the differential equation $Ey = f(x)$, with initial conditions $y(1) = y'(1) = ... = y^{(n)}(1) = 0$, where f(x)is a continuous real-valued function on the reals.

Solution
The first step is to try to simplify Ey. Experimenting with small n, we soon suspect that $Ey = x^{n+1}y^{(n+1)}$ and that is easily proved by induction.

So we seek a solution to $x^{n+1}y^{(n+1)} = f(x)$, subject to $y(1) = y'(1) = ... = y^{(n)}(1) = 0$. The trick is to use the Taylor expansion: $y(x) = y(1) + (x-1)y'(1) + (x-1)^2/2!\, y''(1) + ... + (x-1)^n y^{(n)}(1) + \int_1^x g(t)\,dt$, where $g(t) = (x - t)^n y^{n+1}(t)/n! = (x - t)^n f(t)/(n!\, t^{n+1})$.

Problem A4
Show that for any sequence of positive reals, a_n, we have $\limsup_{n\to\infty} n((a_{n+1} + 1)/a_n - 1) \ge 1$. Show that we can find a sequence where equality holds.

Solution
Suppose lim sup < 1. Then we cannot find an infinite subsequence with $n((a_{n+1} + 1)/a_n - 1) \geq 1$, so we must have $n((a_{n+1} + 1)/a_n - 1) < 1$ for all sufficiently large n. Suppose it is true for all $n \geq N$. Then $(a_{N+1} + 1)/a_N < (N + 1)/N$ and hence $a_N/N > (a_{N+1} + 1)/(N + 1) = a_{N+1}/(N+1) + 1/(N+1)$. But similarly, $a_{N+1}/(N+1) > a_{N+2}/(N+2) + 1/(N+2)$ and so on. Hence $a_N \geq 1/(N+1) + 1/(N+2) + \ldots$, which is impossible since the rhs diverges. So we have established that lim sup ≥ 1.

We can experiment with various series. $a_n = n$ gives 2. $a_n = n^2$ also gives 2. Higher powers give higher values. So we try looking at exponents between 1 and 2. $a_n = n^{1+k}$ gives 1+k. So by taking k arbitrarily small we can get close to 1, but we cannot reach it. It is often helpful to think of log n as n^k with k infinitesimally small. So we try $a_n = n \log n$. That gives $n((a_{n+1} + 1)/a_n - 1) = (1 + n \log (1 + 1/n) + \log(n + 1))/\log n = (1 + n \log (1 + 1/n) + (\log(1 + 1/n))/\log n + 1$. But $1 + n \log(1 + 1/n) + \log (1 + 1/n)$ is bounded and so $n((a_{n+1} + 1)/a_n - 1)$ has limit 1 (and hence also lim sup = 1).

Problem A5
R is the reals. f **:** $[0, \pi] \rightarrow R$ is continuous and $\int_0^\pi f(x) \sin x \, dx = \int_0^\pi f(x) \cos x \, dx = 0$. Show that f is zero for at least two points in $(0, \pi)$. Hence or otherwise, show that the centroid of any bounded convex open region of the plane is the midpoint of at least three distinct chords of its boundary.

Solution
$\sin x > 0$ for all x in $(0, \pi)$, so if f did not change sign, then we could not have $\int_0^\pi f(x) \sin x \, dx = 0$. If there was only one sign change, at k say, then $\sin(x - k)$ would also have only one sign change and $f(x) \sin(x - k)$ would not change sign in the interval, so $\int_0^\pi f(x) \sin(x - k) \, dx = \cos k \int_0^\pi f(x) \sin x \, dx - \sin k \int_0^\pi f(x) \cos x \, dx$ would be non-zero. Contradiction. So f must have at least two sign changes and hence at least two zeros in the interval.

Take polar coordinates with the centroid as origin. Let the boundary be $f(\theta)$ for $0 \leq \theta \leq \pi$ and $-g(\theta)$ for $0 > -\theta > -\pi$.

Taking moments about the x-axis gives $\int_0^\pi f(\theta)\sin \theta \, d\theta = \int_0^\pi g(\theta) \sin \theta \, d\theta = \int_0^\pi h(\theta) \sin \theta \, d\theta$, where $h(\theta) = g(\pi - \theta)$. So $\int (f(\theta) - h(\theta)) \sin \theta \, d\theta = 0$.

Similarly, taking moments about the y-axis gives $\int f(\theta) \cos \theta \, d\theta + \int g(\theta) \cos \theta \, d\theta = 0$ or $\int (f(\theta) - h(\theta)) \, d\theta = 0$. So the result proved in the first part gives that $f(\theta) = g(\pi-\theta)$ for at least two values in $(0, \pi)$. In other words, the centroid is the midpoint of at least two chords. But we could take the x-axis to be one of these chords and then repeat the result to get two more, giving three in total.

Problem A6
M is the midpoint of a chord PQ of an ellipse. A, B, C, D are four points on the ellipse such that AC and BD intersect at M. The lines AB and PQ meet at R, and the lines CD and PQ meet at S. Show that M is also the midpoint of RS.

Solution
The key is to generalise and think of the ellipse and the line pair as conics. Take M as the origin and PQ as the x-axis. The ellipse has equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. If P is (k, 0), then Q must be (-k, 0), so $ak^2 + dk + f = ak^2 - dk + f = 0$. Hence d = 0. The line AC must have equation $y = gx$ for some g and the line BD must have equation $y = hx$ for some h. So the line pair AC, BD is the conic with equation $(y - gx)(y - hx) = 0$. The ellipse and the line pair are two distinct conics through A, B, C, D, so any conic through those four points has equation $r(ax^2 + bxy + cy^2 + ey + f) + s(y - gx)(y - hx)$. This conic meets PQ (which is y = 0) at the points given by $r(ax^2 + f) + sgh x^2$. This has no x term, so the two points of intersection are equally spaced about M. In particular, this is true for the line pair AB, CD.

Problem B1
Find all integers n for which $x^2 - x + n$ divides $x^{13} + x + 90$.

Solution
Answer: n = 2.

If n is negative or zero, then the quadratic has two real roots. But we can easily check that the other polynomial has derivative everywhere positive and hence only one real root. So n must be positive.

If $x^2 - x + n$ divides $x^{13} + x + 90$, then $x^{13} + x + 90 = p(x) (x^2 - x + n)$, where p(x) is a polynomial with integer coefficients. Putting x = 0, we see that n must divide 90. Putting x = 1, we see that it must divide 92. Hence it must divide (92 - 90) - 2. So the only possibilities are 1 and 2. Suppose n = 1. Then putting x = 2, we have that 3 divides $2^{13} + 92$. But 2^{odd} is congruent to 2 mod 2, so $2^{13} + 92$ is congruent to 1 mod 3. So n cannot be 1.

To see that n = 2 is possible, we write explicitly: $(x^2 - x + 2) (x^{11} + x^{10} - x^9 - 3 x^8 - x^7 + 5 x^6 + 7 x^5 - 3 x^4 - 17 x^3 - 11 x^2 + 23 x + 45) = x^{13} + x + 90$.

Problem B2
Is the set { $2^m 3^n$**:** m, n are integers } dense in the positive reals?

Solution
Answer: yes.

It is easier to consider the set X = { m log 2 + n log 3: m, n integers }. We note first that there is no pair of integers m, n (except (0, 0)) such that m log 2 + n log 3 = 0. For we would then have $2^m 3^n = 1$. That is clearly impossible if m and n have the same sign. If m and n have opposite signs, then it would imply that $2^{|m|} = 3^{|n|}$, but a power of 2 is not divisible by 3.

If X has a least positive member k, then log 2 must be an integral multiple of k (for we can write log 2 = q k + r for some integer q and 0 <= r < k, but then r must be zero). Similarly log 3. So if log 2 = nk and log 3 = mk, then m log 2 - n log 3 = 0, contradicting what we just proved.

X contains the positive member log 2. But log 2 cannot be the least member, so it contains another member in (0, log 2). That cannot be the least member, so we can find another smaller positive member, and so on. So we get a countable infinity of members in (0, log 2). Hence for any ε > 0, there are infinitly many members in a subinterval of (0, log 2) of length ε. The difference between any two of these must also lie in X and will lie in the interval (0, ε). Now the multiples of this number all lie in X and form a grid of mesh less than ε, so at least one of them is within ε of any given real.

That establishes that X is dense in the reals. But the map e^x maps the reals to the positive reals and is continuous, so the set { $2^m 3^n$**:** m, n are integers } is dense in the positive reals.

Problem B3
R is the reals. Find all f **:** R → R which are twice differentiable and satisfy: $f(x)^2 - f(y)^2 = f(x + y) f(x - y)$.

Solution
Differentiate wrt x: $2 f(x) f'(x) = f'(x + y) f(x - y) + f(x + y) f'(x - y)$. Differentiate the result wrt y: $0 = f''(x + y) f(x - y) - f'(x + y) f'(x - y) + f'(x + y) f'(x - y) - f(x + y) f''(x - y)$. Put X = x + y, Y = x - y. Then we have $f''(X) f(Y) = f(X) f''(Y)$.

If $f(X) = 0$ for all X, then we have a solution. So suppose for some X_0, $f(X_0)$ is non-zero and put $k = f''(X_0)/f(X_0)$, then $f''(Y) = k f(Y)$. Now we consider separately $k = 0$, $k \leq 0$ and $k > 0$. If $k = 0$, then integrating gives $f(Y) = AY + B$. But putting $y = 0$ in the original relation gives immediately that $f(y) = 0$, so $B = 0$ and we have the solution $f(Y) = AY$. This includes the solution $f(Y) = 0$ noticed earlier.

If $k < 0$, put $k = - a_2$. Then $f(Y) = A \sin aY + B \cos aY$. But $f(0) = 0$, so $B = 0$ and we have the solution $f(Y) = A \sin aY$. If $k > 0$, put $k = a_2$. Then $f(Y) = A \sinh aY + B \cosh aY$. Again $B = 0$ and we have the solution $f(Y) = A \sinh aY$.

It remains to check that these are solutions. It is obvious that $f(Y) = AY$ is a solution. If $f(Y) = A \sin aY$, then $f(x + y)f(x - y) = A^2(\sin ax \cos ay + \cos ax \sin ay)(\sin ax \cos ay - \cos ax \sin ay) = A^2(\sin^2 ax \cos^2 ay - \cos^2 ax \sin^2 ay) = A^2(\sin^2 ax - \sin^2 ay) = f(x)^2 - f(y)^2$, as required. Similarly for the sinh expression.

Problem B4
Γ is a closed plane curve enclosing a convex region and having a continuously turning tangent. A, B, C are points of Γ such that ABC has the maximum possible perimeter p. Show that the normals to Γ at A, B, C are the angle bisectors of ABC. If A, B, C have this property, does ABC necessarily have perimeter p? What happens if Γ is a circle?

Solution
Let L be the external bisector of angle C (so that it is perpendicular to the internal bisector of angle ACB). Let B' be the reflection of B in the line L. If X is any point on the other side of L from ABC, then $BX > B'X$, so $AX + BX > AX + B'X$. But $AX + B'X \geq AB'$ (with equality iff X lies on AB'), and $AB' = AC + CB$. So $AX + BX > AC + CB$. Thus the perimeter of AXB > p. Hence X cannot belong to the region. But if L does not coincide with the tangent at C then it will intersect the region in further points and so the region will contain points on both sides of L. So L must be the tangent.

No, this is not a sufficient condition. For example, take the curve Γ to be an equilateral triangle with rounded corners. Then take A, B, C to be the midpoints of the sides. In this case the perimeter is certainly not maximal.

If Γ is a circle, then the achieve the maximum by taking ABC to be equilateral.

Problem B5
The series $\sum a_n$ of non-negative terms converges and $a_i \leq 100a_n$ for $i = n, n + 1, n + 2, \ldots, 2n$. Show that $\lim_{n\to\infty} na_n = 0$.

Solution
We need to invert the inequality given. We are given a collection of a_i which are less than a fixed a_n. We want to fix a_n and find a collection of a_j such that $a_n \leq 100\, a_j$.

Evidently, $a_{2n} \leq 100\, a_{2n-1}$, $a_{2n} \leq 100\, a_{2n-2}$, $a_{2n} \leq 100\, a_{2n-3}$, ... , $a_{2n} \leq 100\, a_n$. Adding and multiplying by two, $2n\, a_{2n} \leq 200\, (a_n + a_{n+1} + \ldots + a_{2n-1})$. But $\sum a_n$ converges, so $(a_n + a_{n+1} + \ldots + a_{2n-1}) < \varepsilon/200$ for all sufficiently large n, and hence $2n\, a_{2n} < \varepsilon$ for sufficiently large n. Similarly for $(2n+1)\, a^{2n+1}$. So $n\, a_n$ tends to zero.

Problem B6
Let $S = S_0$ be a set of points in space. Let $S_n = \{ P : P$ belongs to the closed segment AB, for some $A, B \in S_{n-1}\}$. Show that $S_2 = S_3$.

Solution
We have to show that S_2 is already the convex hull of S. We can define the convex hull of points $\mathbf{x_i}$ i = 1, 2, ... , n as the collection of all points $\sum \lambda_i \mathbf{x_i}$ with all λ_i non-negative and $\sum \lambda_i = 1$. Given two such points $P = \sum \lambda_i \mathbf{x_i}$ and $Q = \sum \mu_i \mathbf{x_i}$, any point on the closed segment PQ can be written as $\lambda P + \mu Q$ with λ, μ non-negative and $\lambda + \mu = 1$, but $\lambda P + \mu Q = \sum (\lambda \lambda_i + \mu \mu_i) \mathbf{x_i}$ and $\sum (\lambda \lambda_i + \mu \mu_i) = 1$. So it follows immediately that all S_n are subsets of the convex hull.

Now we claim that any point in the convex hull can be written as a sum $\sum \lambda_i \mathbf{x_i}$ with at most 4 non-zero terms. Suppose not. Then some point P requires at least $m \geq 5$ terms. wlog we may take these to be the first m terms, so that $P = \sum_1^m \lambda_i \mathbf{x_i}$. But now we can find μ_i not all zero so that $\sum_1^m \mu_i \mathbf{x_i} = 0$ (3 equations, one for each coordinate) and $\sum \mu_i = 0$ (1 equation). But now $P = \sum_1^m (\lambda_i + k \mu_i) \mathbf{x_i}$ for all k and we still have $\sum (\lambda_i + k \mu_i) = 1$. Take the smallest λ_i/μ_i and set k as its negative. Then $\lambda_i + k \mu_i = 0$ but the other terms $\lambda_j + k \mu_j$ remain non-negative, so we have expressed P as a sum of less than m terms. Contradiction.

So given any P in the convex hull we may write (wlog) $P = \lambda_1 \mathbf{x_1} + ... + \lambda_4 \mathbf{x_4}$ for some non-negative λ_i with sum 1. If two or three of the λ_i are zero, then this shows that P is in S_1. So suppose λ_1, λ_2, λ_3 are all positive. Then $P = (\lambda_1 + \lambda_2) (\lambda_1 \mathbf{x_1}/(\lambda_1 + \lambda_2) + \lambda_2 \mathbf{x_2}/(\lambda_1 + \lambda_2)) + (\lambda_3 + \lambda_4) (\lambda_3 \mathbf{x_3}/(\lambda_3 + \lambda_4) + \lambda_4 \mathbf{x_4}/(\lambda_3 + \lambda_4))$, which shows that P is in S_2.

25th Putnam 1964

Problem A1
Let $A_1, A_2, A_3, A_4, A_5, A_6$ be distinct points in the plane. Let D be the longest distance between any pair, and d the shortest distance. Show that $D/d \geq \sqrt{3}$.

Solution
Let ABC be a triangle with sides $a = BC \geq b = CA \geq c = AB$. Suppose that angle $A \geq 2\pi/3$. It follows immediately from the cosine formula we have $a^2 = b^2 + c^2 - 2bc \cos A \geq b^2 + c^2 + bc \geq 3c^2$, and hence that $a/c \geq \sqrt{3}$. So it is sufficient to find a triangle with an angle of at least $2\pi/3$.

If the 6 points form a convex hexagon, then the six angles of the hexagon sum to 4π so at least one is at least $2\pi/3$. If not then one point is inside the convex hull of the others. Draw diagonals to triangulate the hull. Then the inside point must lie inside (or on) one of the triangles. But if P lies inside (or on) the triangle ABC, then at least one of the angles APB, BPC, CPA is at least $2\pi/3$.

Problem A2
α is a real number. Find all continuous real-valued functions $f : [0, 1] \to (0, \infty)$ such that $\int_0^1 f(x)\,dx = 1$, $\int_0^1 x f(x)\,dx = \alpha$, $\int_0^1 x^2 f(x)\,dx = \alpha^2$.

Solution
We have $\int_0^1 (\alpha - x)^2 f(x)\,dx = \alpha^2 - 2\alpha^2 + \alpha^2 = 0$. But the integrand is positive, except possibly for one point of the range, so the integral must also be positive. Contradiction. So there are no functions with this property.

Problem A3
The distinct points x_n are dense in the interval (0, 1). $x_1, x_2, \ldots, x_{n-1}$ divide (0, 1) into n sub-intervals, one of which must contain x_n. This part is divided by x_n into two sub-intervals, lengths a_n and b_n. Prove that $\sum a_n b_n (a_n + b_n) = 1/3$.

Solution
The trick is to notice that $3a_n b_n (a_n + b_n) = (a_n + b_n)^3 - a_n^3 - b_n^3$. The first n-1 points $x_1, \ldots, x_{n-1}$ divide the interval into n sub-intervals. Let c_n be the sum of the cubes of these sub-intervals. Then $3 \sum_1^{n-1} a_i b_i (a_i + b_i) = 1 - c_n$. So it is sufficient to prove that c_n tends to zero.

Take $\varepsilon > 0$ and < 1. Since the points x_n are dense, we can take N so that for $n > N$ all the subintervals are smaller than ε. Then $c_n < \varepsilon^2$ times the sum of the sub-interval lengths $= \varepsilon^2 < \varepsilon$.

Problem A4
The sequence of integers u_n is bounded and satisfies $u_n = (u_{n-1} + u_{n-2} + u_{n-3}u_{n-4})/(u_{n-1}u_{n-2} + u_{n-3} + u_{n-4})$. Show that it is periodic for sufficiently large n.

Solution
This is almost trivial. The u_n are bounded and integral, so there are only *finitely* many possible values. Hence there are only finitely many possible values for the 4-tuples $(u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4})$. So we must eventually get a repeat. Suppose the t-tuples are the same for n = N and n = N+M. Then the recurrence relation implies that they must also be the same for n = N+1 and n = N+M+1, and for n = N+2 and N+M+2 and so on. In other words the sequence is periodic (with period M) from this point onwards.

Problem A5
Find a constant k such that for any positive a_i, $\sum_1^\infty n/(a_1 + a_2 + \ldots + a_n) \leq k \sum_1^\infty 1/a_n$.

Solution
Let us write $s_n = a_1 + a_2 + ... + a_n$ and $b_n = n/s_n$. If $\sum 1/a_n$ diverges, then there is nothing to prove. So assume it converges. Hence a_n must diverge, so there are only finitely many values less than any given K, so we can arrange the terms in increasing order $c_1 \le c_2 \le c_3$ ... . Since all the terms are positive $\sum 1/a_n$ is absolutely convergent and equal to $\sum 1/c_n$. Also we have $s_n \ge c_1 + c_2 + ... + c_n$. The trick is to notice that half the c_i are at least $c_{n/2}$. It is convenient to consider separately n = 2m and n = 2m+1. So $s_{2m} \ge m\ c_m$. Hence $b_{2m} < 2/c_m$. Similarly, $s_{2m-1} > m\ c_m$, so $b_{2m-1} < 2/c_m$. Hence $\sum_1^{2m} b_i \le 4 \sum_1^m 1/c_i$, which gives the required result with k = 4.

Problem A6
S is a finite set of collinear points. Let k be the maximum distance between any two points of S. Given a pair of points of S a distance d < k apart, we can find another pair of points of S also a distance d apart. Prove that if two pairs of points of S are distances a and b apart, then a/b is rational.

Solution
Since we are only interested in ratios, we may take k = 1 and the points to have coordinates $x_1 = 0$, x_2 ... , x_{n-1}, $x_n = 1$. Let these points generate a vector space V of dimension m over Q.

If m = 1, then every x_i is a rational multiple of x_n and hence rational, so the ratio of any two distances is rational.

Suppose that m > 1. Take a basis $b_1, b_2, ... , b_m$ for V as follows. Take $b_1 = 1$. Let $b_2 = x_t$, for some irrational x_t, then extend $\{b_1, b_2\}$ to a basis. Then each x_i is a unique rational linear combination of the b_j. In particular, $x_1 = 0$, $x_t = b_2$ and $x_n = b_1$. Now define a linear map f from V to Q as follows. Let $f(b_2) = r\ b_2$, where r is rational and sufficiently large that $f(b_2) > 1$, and $f(b_i) = b_i/2$ for all other i.

Now suppose we take two distinct points in S, we can write them as $\sum r_i b_i$ and $\sum s_i b_i$, where all r_i and s_i are rational. Hence their images under f are $r\ r_2 b_2 + \sum r_i b_i/2$ and $r\ s_2 b_2 + \sum s_i b_i/2$, where the summations exclude i = 2. These must be unequal, otherwise we would have a linear combination of b_i with rational coefficients which was zero.

So there is a unique point x in S with f(x) maximum and a unique point y in S with f(y) minimum. Then f(x - y) is greater than f(d) where d is any other difference $x_i - x_j$. But every difference except $x_n - x_1$ and $x_1 - x_n$ occurs at least twice, so x and y must be the endpoints. Now $f(x_1) = 0$ and $f(x_n) = 1/2$, so f(x - y) = 1/2. But $f(x_t - x_1) = f(x_t) = f(b_2) > 1$. Contradiction.

Problem B1
a_n are positive integers such that $\sum 1/a_n$ converges. b_n is the number of a_n which are <= n. Prove lim $b_n/n = 0$.

Solution
If not, then for some k > 0 we can find arbitrarily large N such that $b_N > kN$. Then at least kN of the integers $a_1, ... , a_N$ do not exceed N. Let S be the set of indices i from 1, 2, ... N with $a_i \le N$. Then since S has at least kN members, the sum over S of $1/a_i$ is at least k.
The idea is to take N' > N with $b_{N'} > kN'$ and to ensure that half the indices i with $a_i \le N'$ were not included in S. Take S' to be these new indices. Then since S' does not overlap S, the sum over S and S' of $1/a_i$ is at least k + k/2. We then repeat always choosing the new set of indices not to overlap any of the previous sets. So at each stage we will add k/2 to the sum of $1/a_i$, which must therefore diverge, establishing a contradiction.

We can do this by taking the successor N' to N sufficiently big. In fact, it is sufficient to take $N'/2 > N/k$ because we know that all of the previous index sets are contained in $\{1, 2, \ldots, N\}$, so to avoid them we do not have to drop more than $N < kN'/2$ indices, leaving at least $kN'/2$ new indices.

Problem B2
S is a finite set. A set P of subsets of S has the property that any two members of P have at least one element in common and that P cannot be extended (whilst keeping this property). Prove that P contains just half of the subsets of S.

Solution
For any subset X of S, P can contain at most one of X and S-X, so P cannot contain more than half the subsets.

Suppose P contains less than half the subsets. Then for some X, neither X nor S-X are in P. Hence there must be Y in P such that X and Y have no elements in common (otherwise we could extend P to include X). So Y is a subset of S-X. Similarly, P must contain a subset Z of X (otherwise we could extend P to include S-X). But that means Y and Z cannot overlap, contradicting the fact that as members of P they must have an element in common.

Problem B3
R is the reals. $f : R \to R$ is continuous and for any $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. Prove $\lim_{x\to\infty} f(x) = 0$.

Solution
Take $\varepsilon > 0$. Let $A_n = \{x : |f(nx)| \le \varepsilon\}$. Let B_n be the intersection of all A_m with $m \ge n$. The function f is continuous, so each A_n is closed. Hence also each B_n. Now suppose we can show that an open interval (a, b) is contained in some B_N. That means that for any x in (a, b) and any $n > N$ we have $|f(nx)| < \varepsilon$. Hence $|f(x)| < \varepsilon$ for any x in (na, nb). But the intervals (na, nb) expand in size as they move to the right, so for n sufficiently large they overlap and we have that their union includes all sufficiently large x. In other words, for any sufficiently large x we have $|f(x)| < \varepsilon$. But ε was arbitrary, so we have established that $\lim f(x) = 0$ as x tends to infinity.

It remains to show that we can find such an open interval (a, b). Now we are given that for any given x, $\lim f(nx) = 0$ (as n tends to infinity), so x belongs to some B_n for n sufficiently large. In other words, the union of the B_n is the entire line.

We now need the Baire category theorem which states that if a union of closed sets covers the line, then one of the sets contains an open interval. This is bookwork. [But straightforward. Assume none of the B_n contain an open interval. Take a point not in B_1. Since B_1 is closed we may take a closed interval C_1about the point, not meeting B_1. Having chosen C_n, there must be a point in it not in B_{n+1} (or B_{n+1} would contain an open interval - the interior of C_n). Hence we may take C_{n+1}, a closed subinterval of C_n which does not meet B_{n+1}. Now C_n is a nested sequence of closed intervals, so there must be a point X in all the C_n But B_n cover the line, so X must be in some B_n. Contradiction.].

Problem B4
n great circles on the sphere are in general position (in other words at most two circles pass through any two points on the sphere). How many regions do they divide the sphere into?

Solution
Answer: $n^2 - n + 2$.

We use the well-known formula E + 2 = V + F, where E is the number of edges, V the number of vertices and F the number of faces. It is true for a sphere provided V > 1, so certainly for n > 1. Each circle intersects every other circle in 2 vertices, so V = n(n - 1). Each vertex has degree 4, so E = 2V. Hence F = V + 2 = n^2 - n + 2. It is easy to check that the formula also holds for n = 1.

Problem B5
Let a_n be a strictly monotonic increasing sequence of positive integers. Let b_n be the least common multiple of $a_1, a_2, \ldots, a_n$. Prove that $\sum 1/b_n$ converges.

Solution
We need a crude upper bound on the number of divisors for N. N is too crude. But we can do better by noticing that N has at most $\sqrt{N}$ divisors $\leq \sqrt{N}$ and hence at most $2\sqrt{N}$ divisors in total (every divisor $d > \sqrt{N}$ has a matching divisor $N/d < \sqrt{N}$). Now we know that b_n has at least n distinct divisors (namely $a_1, \ldots, a_n$). Hence $b_n \geq n^2/4$. But we know that $\sum 1/n^2$ converges.

Problem B6
D is a disk. Show that we cannot find congruent sets A, B with $A \cap B = \varnothing$, $A \cup B = D$.

Solution
It is easy to find two congruent sets which overlap only at the centre O of the disk, so this suggests that the centre is the key.

Wlog we may assume O is in A and that the radius is 1. Let O' be the corresponding point in B. Let PQ be the diameter perpendicular to OO'. Then PO' and QO' are both greater than 1. But $OX \leq 1$ for any point X in A, so $O'Y \leq 1$ for any point Y in B. So P and Q must both be in A. Let P' and Q' be the corresponding points in B. Then P'Q' = PQ = 2, so P'Q' must be a diameter. But O'P' = OP = 1 and O'Q' = OQ = 1, so O' must be the midpoint of P'Q' and hence the centre of the disk. So O' = O. Contradiction.

26th Putnam 1965

Problem A1
The triangle ABC has an obtuse angle at B, and angle A is less than angle C. The external angle bisector at A meets the line BC at D, and the external angle bisector at B meets the line AC at E. Also, BA = AD = BE. Find angle A.

Solution
Answer: 12 degrees.

Let angle BAC = k. Then since BA = BE, angle BEA = k. Take B' on BA the opposite side of B to A. Then angle B'BE = 2k. Angle B'BC is bisected by BE, so angle CBE = 2k. Hence angle ACB = 3k. So angle DBA = 4k. But AD = BA, so angle BDA = 4k. But AD is the exterior bisector, so angle BAD = 90 - k/2. The angles in BAD must sum to 180 deg, so k = 12 deg.

Problem A2
Let k = [(n - 1)/2]. Prove that $\sum_0^k ((n - 2r)/n)^2 (nCr)^2 = 1/n (2n-2)C(n-1)$ (where nCr is the binomial coefficient).

Problem A3
$\{a_r\}$ is an infinite sequence of real numbers. Let $b_n = 1/n \sum_1^n \exp(i a_r)$. Prove that $b_1, b_2, b_3, b_4, ...$ converges to k iff $b_1, b_4, b_9, b_{16}, ...$ converges to k.

Solution
If a sequence converges to k, then any subsequence also converges to k.

So suppose that the square terms converge to k. Let $c_r = \exp(i a_r)$. Take two consecutive squares $N = n^2$ and $N' = n^2 + 2n + 1$ and m between them. Then $|b_N - b_m| \le |b_N - 1/N \sum c_r| + |(1/N - 1/m) \sum c_r|$, where the sums are over m terms. But $|b_N - 1/N \sum c_r| \le 1/N \sum_{N+1}^m |c_r| = (m - N)/N \le 2n/n^2 = 2/n$. Also $|(1/N - 1/m) \sum c_r| \le (1/N - 1/m) \sum |c_r| < (1/n^2 - 1/(n+1)^2) (n+1)^2 = (2n+1)/n^2 < 3/n$. So the difference $|b_m - b_N|$ is arbitrarily small for n sufficiently large. Thus if we take m sufficiently large, then $|b_m - b_N| < \varepsilon$ and $|b_N - k| < \varepsilon$. So $|b_m - k| < 2\varepsilon$ and the sequence converges.

Problem A4
S and T and finite sets. U is a collection of ordered pairs (s, t) with $s \in S$ and $t \in T$. There is no element $s \in S$ such that all possible pairs $(s, t) \in U$. Every element $t \in T$ appears in at least one element of U. Prove that we can find distinct $s_1, s_2 \in S$ and distinct $t_1, t_2 \in T$ such that $(s_1, t_1), (s_2, t_2) \in U$, but $(s_1, t_2), (s_2, t_1) \notin U$.

Solution
Suppose that we cannot find such s_i, t_i. We will establish a contradiction.

Take t in T. Suppose that there are n distinct s in S such that (s, t) is in U. Suppose n > 0. Then take a specific s' such that (s', t) is in U. There must be some t' such that (s', t') is not in U. Now consider whether we have (s, t') in U. If (s, t) is not in U, then (s, t') cannot be in U (or we would have found s_i, t_i). But there are at most n-1 distinct s such that (s, t') is in U (the only candidates are the cases for which (s, t) is in U, and one of those, namely s', does not work).

Iterating, we must eventually get some x in T for which there is no s in S with (s, x) in U. Contradiction.

Problem A5
How many possible bijections f on {1, 2, ... , n} are there such that for each i = 2, 3, ... , n we can find j < n with f(i) - f(j) = ± 1?

Solution
Answer: 2^{n-1}.

Consider the last element f(n). Suppose it is m, not 1 or n. Then the earlier elements fall into two non-empty sets A = {1, 2, ... , m-1} and B = {m+1, m+2, ... , n}. But the difference between an element of A and an element of B is at least 2. So if f(1) is in A, then the first time we get an element of B it has only elements of A preceding it. Contradiction. Similarly, if f(1) is in B.

So we conclude that the last element is always 1 or n. We can now prove the result by induction. Clearly given an arrangement for n we can derive one for n+1 by adding n+1 at the end. We can also derive one for n+1 by increasing each element by 1 and adding 1 at the end. Equally it is clear that all these are distinct and that there are no other arrangements for n+1 that end in 1 or n+1. So thee are twice as many arrangements for n+1 as for n.

Problem A6
α and β are positive real numbers such that $1/\alpha + 1/\beta = 1$. Prove that the line mx + ny = 1 with m, n positive reals is tangent to the curve $x^\alpha + y^\alpha = 1$ in the first quadrant ($x, y \geq 0$) iff $m^\beta + n^\beta = 1$.

Solution
Suppose mx + ny = 1 is tangent to the curve. Suppose it touches at (a, b). Differentiating, we see that the tangent at (a, b) is $a^{\alpha-1}x + b^{\beta-1} = 1$, so $m = a^{\alpha-1}$, $n = b^{\beta-1}$. Hence, using $\alpha\beta - \beta = \alpha$, we have that $m^\beta + n^\beta = a^\alpha + b^\alpha = 1$.

Conversely, suppose that $m^\beta + n^\beta = 1$. Take $a = m^{\beta/\alpha}$, $b = n^{\beta/\alpha}$. Then $a^\alpha + b^\alpha = 1$, so (a, b) lies on the curve in the first quadrant. Its tangent is Mx + Ny = 1, where $M = a^{\alpha-1}$, $N = b^{\beta-1}$. But $a = m^{\beta/\alpha}$ and $\beta/\alpha\ (\alpha - 1) = 1$, so M = m. Similarly, N = n. Thus we have established that mx + ny = 1 is tangent to the curve in the first quadrant as required.

Problem B1
X is the unit n-cube, $[0, 1]^n$. Let $k_n = \int_X \cos^2(\pi(x_1 + x_2 + ... + x_n)/(2n))\, dx_1 ... dx_n$. What is $\lim_{n\to\infty} k_n$?

Solution
Let $y_i = 1 - x_i$ and change variables from x_i to y_i. The sum $(x_1 + x_2 + ... + x_n)/(2n)$ becomes $1/2 - (y_1 + y_2 + ... + y_n)/(2n)$, so the integrand becomes $\sin^2(\pi(y_1 + y_2 + ... + y_n)/(2n))$. So the integral is identical to the original except that cos has been changed to sin. Thus we can add it to the original to get $2k_n = \int_X dx_1\, dx_2 ... dx_n = 1$. So $k_n = 1/2$ for all n.

Problem B2
Every two players play each other once. The outcome of each game is a win for one of the players. Player n wins a_n games and loses b_n games. Prove that $\sum a_n^2 = \sum b_n^2$.

Solution
Suppose there are N players in total. Each player plays N-1 games, so $b_n = N - 1 - a_n$. Hence $\sum b_n^2 = \sum (N - 1)^2 - 2(N - 1) \sum a_n + \sum a_n^2 = N(N - 1)^2 - 2(N - 1) \sum a_n + \sum a_n^2$.

Each game is won by just one player, so $\sum a_n$ = no. of games = N(N - 1)/2. Hence $\sum b_n^2 = N(N - 1)^2 - 2(N - 1)N(N - 1)/2 + \sum a_n^2 = \sum a_n^2$.

Problem B3
Show that there are just three right angled triangles with integral side lengths $a < b < c$ such that $ab = 4(a + b + c)$.

Solution
Answer: 12, 16, 20; 10, 24, 26; 9, 40, 41.

We need the result that for some integral d, m, n we have $c = d(m^2 + n^2)$ and $b = 2dmn$, $a = d(m^2 - n^2)$ or $b = d(m^2 - n^2)$, $a = 2dmn$ (*).

It follows that $4(a + b + c) = 4d(2m^2 + 2mn)$, $ab = 2d^2mn(m^2 - n^2)$. Hence, $4 = dn(m - n)$. So we must have n = 1, 2 or 4 and (d, m, n) = (1, 5, 1), (2, 3, 1), (4, 2, 1), (1, 4, 2), (2, 3, 2) or (1, 5, 4), giving the three answers above.

It remains to prove (*). Let d be the gcd of a and b. It follows that d also divides c. Put $a = dA$, $b = dB$, $c = dC$, so that A, B, C are relatively prime in pairs. C cannot be even, for then A and B would both be odd and hence $A^2 + B^2$ would be congruent to 2 mod 4, which is impossible, since C is a square. So C must be odd and just one of A, B must be even. Assume A is odd. Then $A^2 = (C - B)(C + B)$. If an odd prime p divides A, then it must divide C - B or C + B. It cannot divide both, for then it would also divide B and C. So C - B and C + B must both be odd squares. Say $C + B = h^2$, $C - B = k^2$. Then $A = hk$, $B = (h^2 - k^2)/2$, $C = (h^2 + k^2)/2$, with h and k odd. Put $m = (h + k)/2$, $n = (h - k)/2$, then $h = m + n$, $k = m - n$ and $A = m^2 - n^2$, $B = 2mn$, $C = m^2 + n^2$, so we have put a, b, c, in the form (*). On the other hand, it is obvious that if a, b, c have this form then they satisfy $a^2 + b^2 = c^2$.

Problem B4
Define $f_n : R \to R$ by $f_n(x) = (nC0 + nC2\ x + nC4\ x^2 + \ldots) / (nC1 + nC3\ x + nC5\ x^2 + \ldots)$, where nCm is the binomial coefficient. Find a formula for $f_{n+1}(x)$ in terms of $f_n(x)$ and x, and determine $\lim_{n\to\infty} f_n(x)$ for all real x.

Solution
It is almost obvious that $f_{n+1} = (f_n + x)/(f_n + 1)$ (*). So if $f_n(x)$ tends to a limit k(x), then $k(x) = (k(x) + x)/(k(x) + 1)$, and hence $k(x) = \sqrt{x}$.

Obviously, $f_n(0) = 1/n$, so $k(0) = 0$. We notice also that $f_n(x) = (\sqrt{x})\ N/D$, where $N = (1 + \sqrt{x})^n + (1 - \sqrt{x})^n$ and $D = (1 + \sqrt{x})^n - (1 - \sqrt{x})^n$.

Suppose $0 < x \le 1$. Then put $y = (1 - \sqrt{x})/(1 + \sqrt{x})$. We have $0 \le y < 1$ and $N/D = (1 + y^n)/(1 - y^n)$ which tends to 1, so in this case also $k(x) = \sqrt{x}$.

Similarly if $x > 1$, put $y = (\sqrt{x} - 1)/(1 + \sqrt{x})$ and we again get $k(x) = \sqrt{x}$.
It is clear from (*), that for $x < 0$, $f_n(x)$ does not tend to a limit.

Problem B5
Let S be a set with $n > 3$ elements. Prove that we can find a collection of $[n^2/4]$ 2-subsets of S such that for any three distinct elements A, B, C of the collection, $A \cup B \cup C$ has at least 4 elements.

Solution
This is just another form of finding a graph without triangles. If $n = 2m$, then take S to be the disjoint union of U and V, where U and V each have m elements. Now take K to be the collection all sets {u, v} with u in U and v in V. Evidently K has $m^2 = [n^2/4]$ members. Suppose we have three distinct elements A, B, C of K. $A \cup B \cup C$ cannot have less than three elements, for then two of

A, B, C would be identical. So if it has less than 4 elements, then it has just three elements. But that means that for some a, b, c, we have A = {b, c}, B = {a, c}, C = {a, b}. Suppose a is in U. Then, considering B and C, c and b must be in V, but then A has two elements in V. Contradiction. Similarly if a is in V. So K has the required property.

Similarly, if n = 2m+1, take S to be the disjoint union of U and V, where U has m+1 elements and V has m elements. Then K, the set of all {u, v} with u in U and v in V has $m(m+1) = [n^2/4]$ elements and has the required property by the same argument.

Problem B6
Four distinct points A_1, A_2, B_1, B_2 have the property that any circle through A_1 and A_2 has at least one point in common with any circle through B_1 and B_2. Show that the four points are collinear or lie on a circle.

Solution
The trick is to take *concentric* circles. If A_1A_2 is not parallel to B_1B_2 then their perpendicular bisectors must intersect at some point O. Take circles centre O through A_1, A_2 and through B_1, B_2. These must either coincide, in which case the 4 points lie on a circle, or have no points in common.

In the case where A_1A_2 is parallel to B_1B_2. Assume they do not coincide (otherwise the 4 points would be collinear). Then take a point O on the perpendicular bisector of A_1A_2 on the opposite side of A_1A_2 to B_1B_2 and sufficiently distant from A_1A_2 that the circle through A_1, A_2 centre O only extends less than halfway towards the line B_1B_2. Similarly, take a circle through B_1, B_2 which extends less than halfway towards A_1A_2. Then these circles will not meet.

27th Putnam 1966

Problem A1
Let $f(n) = \sum_1^n [r/2]$. Show that $f(m + n) - f(m - n) = mn$ for $m > n > 0$.

Solution
It is a trivial induction to show that $f(2k) = k^2$, $f(2k+1) = k^2 + k$. So if m and n have the same parity, then $f(m+n) + f(m-n) = (m+n)^2/4 + (m-n)^2/4 = mn$. If m and n have opposite parity, then $f(m+n) + f(m-n) = (m+n-1)(m+n+1)/4 + (m-n-1)(m-n+1)/4 = mn$.

Problem A2
A triangle has sides a, b, c. The radius of the inscribed circle is r and $s = (a + b + c)/2$. Show that $1/(s - a)^2 + 1/(s - b)^2 + 1/(s - c)^2 \geq 1/r^2$.

Solution
Let $A = s-a$, $B = s-b$, $C = s-c$. Then $(A-B)^2 \geq 0$ with equality iff $a = b$. Hence $2/(AB) \leq 1/A^2 + 1/B^2$. Similarly for $2/(BC)$ and $2/(CA)$. Hence $1/(BC) + 1/(CA) + 1/(AB) \leq 1/A^2 + 1/B^2 + 1/C^2$ with equality iff the triangle is equilateral.

Now $1/(BC) + 1/(CA) + 1/(AB) = (A + B + C)/(ABC)$. But $A + B + C = s$. By Heron's theorem, $sABC = k^2$, where k is the area of the triangle. Also (dividing the triangle into three by connecting the incentre to each vertex, and considering the area of each part) $k = rs$. Hence $s/(ABC) = 1/r^2$, giving the required result.

Problem A3
Define the sequence $\{a_n\}$ by $a_1 \in (0, 1)$, and $a_{n+1} = a_n(1 - a_n)$. Show that $\lim_{n\to\infty} n\, a_n = 1$.

Solution
We show first that $a_n < 1/(n+1)$ for all $n > 1$. We have $a_1(1 - a_1) = 1/4 - (a_1 - 1/2)^2 < 1/3$, so it is true for $n = 2$. Also, the quadratic expression shows that $a_n(1 - a_n)$ is an increasing function of a_n for $a_n < 1/2$. Hence if $a_n < 1/(n+1)$, then $a_{n+1} < (1 - 1/(n+1))/(n+1) = n/(n_2 + 2n + 1) < n/(n_2 + 2n) = 1/(n + 2)$. Hence, by induction it is true for all n.

Suppose that $1/2 > a_n > 1/(k + \sqrt{k})$. Then $a_{n+1} > (1/(k + \sqrt{k}))(1 - 1/(k + \sqrt{k}))$. We show that this is greater than $1/(k+1 + \sqrt{(k+1)})$ for k sufficiently large. We require $k^2 - 1 + (k + 1)\sqrt{k} + (k - 1)\sqrt{(k + 1)} + \sqrt{(k^2 + k)} > k^2 + k\sqrt{k} + k$. Obviously $\sqrt{(k^2 + k)} > k$, so it is sufficient to show that $(k - 1)\sqrt{(k + 1)} > (k - 1)\sqrt{k} + 1$. But this is almost obvious since $\sqrt{(k + 1)} > \sqrt{k} + 1/(3\sqrt{k})$, and $(k - 1) > 3\sqrt{k}$ for $k > 11$.

Now $a_1 > 0$, so $a_1 > 1/(k + \sqrt{k})$ for some k. Increasing k if necessary, we can take $k > 11$. It then follows by induction that $a_n > 1(n + k + \sqrt{(n + k)})$ for all n, where k is fixed. Hence $n\, a_n$ lies between $n/(n + k + \sqrt{(n + k)})$ and $n/(n + 1)$. But both $n/(n + k + \sqrt{(n + k)})$ and $n/(n + 1)$ converge to 1, so $n\, a_n$ does also.

Problem A4
Delete all the squares from the sequence 1, 2, 3, ... Show that the nth number remaining is n + m, where m is the nearest integer to $\sqrt{n}$.

Solution
Any integer N in the sequence lies between k^2 and $(k + 1)^2$ for some k, and hence $N = k^2 + h$ for some k and some h such that $1 \leq h \leq 2k$.
Since there are just k squares smaller than N, N is the $(N - k)$th number in the sequence. But $(k -$

$1/2)^2 < k^2 - k + 1 \le N - k \le k^2 + k < (k + 1/2)^2$, so the nearest integer to the square root of (N - k) is k and N = n + m, where n = N - k and m is the nearest integer to the square root of n.

Problem A5
Let S be the set of continuous real-valued functions on the reals. φ **:**S → S is a linear map such that if f, g □ S and f(x) = g(x) on an open interval (a, b), then φf = φg on (a, b). Prove that for some h □ S, (φf)(x) = h(x)f(x) for all f and x.

Solution
Obviously if f = 0 on an open interval (a, b) then φf = 0 on (a, b). But we need the stronger result that if $f(x_0) = 0$, then $\varphi f(x_0) = 0$.

So take any f and any x_0 such that $f(x_0) = 0$. Let L(x) = f(x) for $x < x_0$ and 0 for $x \ge x_0$, and le R(x) = 0 for $x \le x_0$ and f(x) for $x > x_0$. Then f = L + R. Also φL(x) = 0 for any x > 0. But φL is continuous, so $\varphi L(x_0) = 0$. Similarly, $\varphi R(x_0) = 0$. Hence $\varphi f(x_0) = 0$.

Let u be the function which has value 1 for all x. Let h = φu. Then h is continuous. Also for any f and any x_0 the function $f - f(x_0)u$ is zero at x_0 and hence $\varphi f(x_0) = h(x_0)f(x_0)$.

Problem A6
Let $a_n = \sqrt{(1 + 2 \sqrt{(1 + 3 \sqrt{(1 + 4 \sqrt{(1 + 5 \sqrt{(... + (n - 1) \sqrt{(1 + n)} ...)))))}$. Prove $\lim a_n = 3$.

Solution
Clearly $a_n < a_{n+1}$. Also, if we replace the final (1 + n) in a_n by $(1 + n)^2$, then a simple induction shows that the resulting expression simplifies to 3. Hence $a_n < 3$. An increasing sequence which is bounded above must converge. So a_n tends to a limit which is at most 3.

However, it is harder to show that the limit is 3. We need a new idea. Put $f(x) = \lim \sqrt{(1 + x \sqrt{(1 + (x+1) \sqrt{(1 + (x+2) \sqrt{(1 + (x+3) \sqrt{(... + (x+n-1) \sqrt{(1 + x+n)} ...)))))}$. Then we may guess that f(x) = x + 1. The same idea as before shows that f(x) exists and is at most x + 1. Also, we have that $f(x)^2 = x\, f(x+1) + 1$ (*).

The trick is that a crude lower limit works, because we can use (*) to refine it repeatedly. Removing all the 1s and replacing (x+1), (x+2) ... by x gives that $f(x) \ge \lim \sqrt{(x \sqrt{(x \sqrt{(x \sqrt{(x ...))))} = x > 1/2 (x + 1)$. Now (*) gives $f(x)^2 \ge 1/2 (x^2 + 2x) + 1 > 1/2 (x + 1)^2$, so $f(x) > (x + 1)\sqrt{(1/2)}$. Using (*) again gives $f(x) > (x + 1) (1/2)^{1/4}$ and so on. Hence $f(x) \ge x + 1$ as required.

Problem B1
A convex polygon does not extend outside a square side 1. Prove that the sum of the squares of its sides is at most 4.

Solution
Form a right-angled triangle on each side of the polygon (and outside it), by taking the other two sides parallel to the sides of the square. The sum of the squares of the polygon's sides equals the sum of the squares of the non-hypoteneuse sides of the triangles. Because the polygon is convex, these triangle sides form 4 sets, one for each side of the square, and each set having lengths totalling less than 1 (the side of the square). So the sum of the squares in each set is less than 1 ($\sum x^2 < (\sum x)^2 = 1$).

Problem B2
Prove that at least one integer in any set of ten consecutive integers is relatively prime to the others in the set.

Solution
There are 5 odd numbers in the set. At most 2 are multiples of 3, at most 1 is a multiple of 5 and at most 1 is a multiple of 7. So there is at least one odd number, k, that is not divisible by 3, 5 or 7. Now if k has a common factor with another member in the set, then that factor must divide their difference, which is at most 9. But the common factor cannot be divisible by 2, 3, 5 or 7, so it must be 1.

Problem B3
a_n is a sequence of positive reals such that $\sum 1/a_n$ converges. Let $s_n = \sum_1^n a_i$. Prove that $\sum n^2 a_n/s_n^2$ converges.

Solution
Let $A = (\sum 1/a_n)^{1/2}$ and $B_N = \sum_1^N n^2 a_n/s_n^2$.
Since all a_n are positive, $s_{n-1} < s_n$ and hence $B_N < \sum_1^N n^2 a_n/(s_n s_{n-1})$. But $a_n = s_n - s_{n-1}$, so for $n > 1$ we may write the summand as $n^2(1/s_{n-1} - 1/s_n)$. Hence $B_N < 1/a_1 + (4/s_1 - 4/s_2) + (9/s_2 - 9/s_3) + ... + (N^2/s_{N-1} - N^2/s_N) < 5/a_1 + 5/s_2 + 7/s_3 + 9/s_4 + ... + (2N-1)/s_{N-1} - N^2/s_N < 2/a_1 + 3(1/s_1 + 2/s_2 + 3/s_3 + ... + N/s_N)$.

Now $\sum_1^N n/s_n = \sum (1/\sqrt{a_n})\,(n(\sqrt{a_n})/s_n) <= (\sum 1/a_n)^{1/2} (\sum n^2 a_n/s_n^2)^{1/2} < A\, B_N^{1/2}$. So we have $B_N < 2/a_1 + 3A\, B_N^{1/2}$. That implies that B_N is bounded above. For example, we certainly have $B_N < (1 + 3/a_1 + 3A)^2$. But any increasing sequence which is bounded above must converge.

Problem B4
Given a set of (mn + 1) unequal positive integers, prove that we can either (1) find m + 1 integers $b_i$in the set such that b_i does not divide b_j for any unequal i, j, or (2) find n+1 integers a_i in the set such that a_i divides a_{i+1} for i = 1, 2, ... , n.

Solution
Given any a_1 in the set, let $f(a_1)$ be the length of the longest sequence $a_1, a_2, ... , a_k$ taken from the set such that $a_1|a_2, a_2|a_3, ... , a_{k-1}|a_k$. If $f(a_1) > n$, then we are done, so assume that for all a in the set, $f(a) <= n$. Thus f can only have n possible values (1, 2, ... , n). There are mn+1 members in the set, so there must be at least m+1 members $b_1, b_2, ... , b_{m+1}$ with the same value of f, say h. Now we cannot have any one of these members divide another, for if $b_i|b_j$, then we could extend the sequence length h for b_j to give a sequence length h+1 for b_i (in which each element divides the next).

Problem B5
Given n points in the plane, no three collinear, prove that we can label them P_i so that $P_1P_2P_3 ... P_n$ is a simple closed polygon (with no edge intersecting any other edge except at its endpoints).

Solution
Take arbitrary x, y axes. Take P to be the point with the smallest x-coordinate, or if there are two such, the one with the smaller y-coordinate. Take Q to be the point with the largest x-coordinate, or if there are two such the one with the smaller y-coordinate. Join P to Q by a path p along the lower part of the convex hull of the points (so that all other points are above the path).

Now order the remaining points not in the path according to the size of their x-coordinate (with the largest first). Continue the path from Q back to P by taking the remaining points in this order. We

can never intersect the existing path p without going outside the convex hull and we cannot intersect the later part of the path because it has smaller x-coordinate.

Problem B6
$y = f(x)$ is a solution of $y'' + e^x y = 0$. Prove that $f(x)$ is bounded.

Solution
We have $2 e^{-x} y' y'' + 2 y y' = 0$. Integrating from 0 to k gives $y(k)^2 = y(0)^2 - 2 \int_0^k e^{-x} y' y'' dx$.
Integrating by parts gives $2 \int_0^k e^{-x} y' y'' dx = e^{-x} (y')^2|_0^k + \int_0^k (y')^2 e^{-x} dx = e^{-k} y'(k)^2 - y'(0)^2 + \int_0^k (y')^2 e^{-x} dx = A - y'(0)^2$, where $A > 0$.

Hence $y(k)^2 = y(0)^2 + y'(0)^2 - A < y(0)^2 + y'(0)^2$, which establishes that y is bounded.

28th Putnam 1967

Problem A1
We are given a positive integer n and real numbers a_i such that $|\sum_1^n a_k \sin kx| \le |\sin x|$ for all real x. Prove $|\sum_1^n k\, a_k| \le 1$.

Solution Put $f(x) = \sum_1^n a_k \sin kx$. We note that $\sum_1^n k\, a_k = f'(0)$.

We also have $f'(0) = \lim (f(x) - f(0))/x = \lim f(x)/x = \lim f(x)/\sin x \lim (\sin x)/x = \lim f(x)/\sin x$. But $|f(x)| \le |\sin x|$, so $|f(x)/\sin x| \le 1$ and hence $|f'(0)| = |\lim f(x)/\sin x| \le 1$.

Problem A2
Let u_n be the number of symmetric n x n matrices whose elements are all 0 or 1, with exactly one 1 in each row. Take $u_0 = 1$. Prove $u_{n+1} = u_n + n\, u_{n-1}$ and $\sum_0^\infty u_n x^n/n! = e^{f(x)}$, where $f(x) = x + (1/2)\, x^2$.

Solution
There is an obvious bijection between (1) n x n matrices satisfying the conditions and (2) (n+1) x (n+1) matrices satisfying the conditions which have 1 at the top left. [Just delete the first row and column to get from (2) to (1)).

Similarly for any i = 2, 3, ... or n+1, there is an obvious bijection between (1) (n-1) x (n-1) matrices satisfying the conditions and (2) (n+1) x (n+1) matrices satisfying the conditions which have a 1 in row 1, col i (and hence also in row i, col 1). Just delete rows 1 and i and cols 1 and i to get from (2) to (1).
That establishes that $u_{n+1} = u_n + n\, u_{n-1}$. Also, we are given that $u_0 = 1$ and it is clear that $u_1 = 1$.

We have that $e^{f(x)} = 1 + (x + 1/2\, x^2) + (x + 1/2\, x^2)^2/2! + \ldots$ which is clearly $1 + v_1 x/1! + v_2 x^2/2! + \ldots$ for some v_n. Differentiating, we get that $(1 + x)(1 + v_1 x/1! + v_2 x^2/2! + \ldots) = v_1 + v_2 x/1! + v_3 x^2/2! + \ldots$. Hence $v_1 = 1$, $v_2 = v_1 + 1$, $v_{n+1} = v_n + n\, v_{n-1}$. A trivial induction now shows that $v_n = u_n$.

Problem A3
Find the smallest positive integer n such that we can find a polynomial $nx^2 + ax + b$ with integer coefficients and two distinct roots in the interval (0, 1).

Solution
Answer: n = 5 with equation $5x^2 - 5x + 1$ with roots $(1 \pm \sqrt{(1/5)})/2$.
The product of the roots lies in the interval (0, 1), so b must be 1, 2, 3, ... or n-1 (1). The larger root is $(-a + \sqrt{(a^2 - 4bn)})/(2n)$. This must be less than 1, so $-a < b + n$ (2). The roots are real and distinct, so $a^2 > 4bn$ (3).

Putting (2) and (3) together we get: $(n + b - 1)^2 \ge a^2 \ge 4bn + 1$. So if b = 1, then $n \ge 5$. If b = 2, then $n \ge 6$. If b = 3, then $n \ge 8$. If b = 4, then $n \ge 10$. Thus there are no solutions for $n < 5$ (which requires $b < 4$ by (1)). If n = 5, then the only possible solution is $5x^2 - 5x + 1$, which is easily verified to be a solution.

Problem A4
Let $1/2 < \alpha \in R$, the reals. Show that there is no function $f : [0, 1] \to R$ such that $f(x) = 1 + \alpha \int_x^1 f(t)\, f(t - x)\, dt$ for all $x \in [0, 1]$.

Solution
Suppose there is such a function. Let $K = \int_0^1 f(x)\, dx$. Then $K = 1 + \alpha \int_0^1 \int_x^1 f(t)\, f(t - x)\, dt\, dx$.
Interchanging the order of integration gives $\int_0^1 \int_x^1 f(t)\, f(t - x)\, dt\, dx = \int_0^1 \int_0^t f(t)\, f(t - x)\, dx\, dt = \int_0^1 f(t)$

$\int_0^t f(x)\,dx\,dt$ (*).
Put $g(x) = \int_0^x f(t)\,dt$. Then $g'(x) = f(x)$, so (*) gives $\int_0^1 g'(t)\,g(t)\,dt = 1/2\,g(1)^2 - 1/2\,g(0)^2$. But $g(1) = K$ and $g(0) = 0$. Thus we have $K = 1 + \alpha/2\,K^2$, or rearranging $(K - 1/\alpha)^2 = -2/\alpha^2\,(\alpha - 1/2)$. But that is impossible for $\alpha > 1/2$.

Problem A5
K is a convex, finite or infinite, region of the plane, whose boundary is a union of a finite number of straight line segments. Its area is at least $\pi/4$. Show that we can find points P, Q in K such that PQ = 1.

Solution
Suppose the result is false, so that the maximum distance is $2d < 1$. Take a diameter of K (in other words two points A, B for which AB = 2d). Let the x-axis lie along this diameter with the origin at its midpoint. Take the y-axis perpendicular to the x-axis as usual. Then K lies entirely between x = -d and x = +d. Let t(x) be the top boundary of K (above the x-axis) and -b(x) be the bottom boundary of K (below the x-axis).

The area A of K is the area under the curve t(x) plus the area between b(x) and the x-axis. In other words $A = \int_{-d}^{d} t(x)\,dx + \int_{-d}^{d} b(x)\,dx = \int_{-d}^{d} t(x)\,dx + \int_{-d}^{d} b(-x)\,dx = \int_{-d}^{d}(\,t(x) + b(-x)\,)\,dx$. Now $(t(x) + b(-x))^2 = D^2 - (2x)^2$, where D is the distance between the two points (x, t(x)) and (-x, b(-x)).
Certainly, $D < 1$, so $\int_{-d}^{d}(\,t(x) + b(-x)\,)\,dx < \int_{-d}^{d} \sqrt{(\,1 - 4x^2\,)}\,dx$.
The indefinite integral is $1/2\,\sqrt{(\,1 - 4x^2)} + (1/4)\,\sin^{-1}(2x)$, so integrated between -1/2 and 1/2 it gives $\pi/4$. The integrand is positive, so between -d and d it gives a smaller result. In other words we have established that if $2d < 1$, then $A < \pi/4$. Contradiction.

Problem A6
a_i and b_i are reals such that $a_1b_2 \neq a_2b_1$. What is the maximum number of possible 4-tuples (sign x_1, sign x_2, sign x_3, sign x_4) for which all x_i are non-zero and x_i is a simultaneous solution of $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ and $b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0$. Find necessary and sufficient conditions on a_i and b_i for this maximum to be achieved.

Solution
Solving in terms of x_3, x_4 gives $x_1 = s_{23}/s_{12}\,x_3 + s_{24}/s_{12}\,x_4$, $x_2 = s_{31}/s_{12}\,x_3 + s_{41}/s_{12}\,x_4$, $x_3 = x_3$, $x_4 = x_4$, where $s_{ij} = (a_ib_j - a_jb_i)$. Plot the 4 lines $s_{23}/s_{12}\,x_3 + s_{24}/s_{12}\,x_4 = 0$, $x_2 = s_{31}/s_{12}\,x_3 + s_{41}/s_{12}\,x_4 = 0$, $x_3 = 0$, $x_4 = 0$ in the x_3, x_4 plane. We get 4 lines through the origin. Evidently x_1 changes sign if we cross the first, x_2 changes sign if we cross the second, x_3 changes sign if we cross the third and x_4 changes sign if we cross the fourth. So we have a different combination of signs in each sector, but the same combination throughout any given sector.

Thus the maximum is achieved when the four lines are distinct, giving 8 sectors (and hence 8 combinations). This requires that s_{23} and s_{24} are non-zero (otherwise the first line coincides with one of the last two) and that s_{31} and s_{41} are non-zero (otherwise the second line coincides with one of the last two). Finally, the first two lines must not coincide with each other. That requires that s_{31}/s_{41} is not equal to s_{23}/s_{24}. After some slightly tiresome algebra that reduces to s_{34} non-zero. So a necessary and sufficient condition to achieve 8 is that all s_{ij} are non-zero.

Problem B1
A hexagon is inscribed in a circle radius 1. Alternate sides have length 1. Show that the midpoints of the other three sides form an equilateral triangle.

Solution
Each side length 1 forms an equilateral triangle with the centre. Let the other three sides subtend

angles 2A, 2B, 2C at the centre. The midpoints of the sides subtending 2A and 2B are distances cos A and cos B from the centre and the line joining them subtends an angle A+B+60 at the centre. So, using the cosine formula, the square of the distance between them is $\cos^2 A + \cos^2 B - 2 \cos A \cos B \cos(A+B+60)$ (*). But A + B + C = 90, so $\cos(A+B+60) = \cos(150-C) = -\sqrt{3}/2 \cos C + 1/2 \sin C$. Hence we may write (*) as $(\cos^2 A + \cos^2 B + \cos^2 C) + \sqrt{3} \cos A \cos B \cos C - \cos A \cos B \sin C - \cos^2 C$. But we can write cos C as sin(A + B) = sin A cos B + cos A sin B and hence $\cos^2 C$ as sin A cos B cos C + cos A sin B cos C. Thus (*) has the symmetrical form $(\cos^2 A + \cos^2 B + \cos^2 C) + \sqrt{3}$ cos A cos B cos C - (cos A cos B sin C + cos A sin B cos C + sin A cos B cos C), which establishes that the triangle is equilateral.

Problem B2

A, B $\in$ [0, 1] and we have $ax^2 + bxy + cy^2 \equiv (A x + (1 - A)y)^2$, $(A x + (1 - A)y)(B x + (1 - B)y) \equiv dx^2 + exy + fy^2$. Show that at least one of a, b, c $\geq$ 4/9 and at least one of d, e, f $\geq$ 4/9.

Solution

For the first part, $a = A^2$, $b = 2 A (1 - A)$, $c = (1 - A)^2$. If a < 4/9, then A < 2/3. If c < 4/9 then A > 1/3. But b > 4/9 for 1/3 < A < 2/3.

For the second part, d = A B, f = (1 - A)(1 - B), e = A + B - 2 AB.

Measure A along the x-axis and B along the y-axis. Consider the regions of the square for which each of d, e, f are $\geq$ 4/9. d $\geq$ 4/9 for points above the hyperbola y = 4/(9x) which passes through the points (4/9, 1), (2/3, 2/3), (1, 4/9). Similarly f $\geq$ 4/9 for points below the hyperbola y = 1 - 4/(9 - 9x), which passes through the points (0, 5/9), (1/3, 1/3), (5/9, 0). Finally, e > 4/9 for points lying between the two branches of the hyperbola y = (4 - x)/(9 - 18x). The bottom branch passes through (0, 4/9), (1/3, 1/3), (4/9, 0) and the top branch passes through (5/9, 1), (2/3, 2/3), (1, 5/9). Thus the bottom branch is entirely below the d = 4/9 hyperbola, except at the point of intersection (1/3, 1/3), and the top branch is entirely above the f = 4/9 hyperbola, except at the point of intersection (2/3, 2/3). This is easily checked by solving d = e = 4/9 and d = f = 4/9. Thus the three areas d $\geq$ 4/9, e $\geq$ 4/9, f $\geq$ 4/9 cover the unit square.

Problem B3

R is the reals. f, g are continuous functions R $\rightarrow$ R with period 1. Show that $\lim_{n\rightarrow\infty} \int_0^1 f(x)\, g(nx)\, dx = (\int_0^1 f(x)\, dx)\, (\int_0^1 g(x)\, dx)$.

Solution

The idea is to split the integration range into n equal parts. Thus we get $\int_0^1 f(x)\, g(nx)\, dx = \sum \int_{r/n}^{r/n+1/n} f(x)\, g(nx)\, dx$. For large n, f is roughly constant over the range, so we get $\cdot \quad f(r/n) \int_{r/n}^{r/n+1/n} g(nx)\, dx$. Changing the integration variable to t = nx, gives $\sum f(r/n)\, 1/n \int_0^1 g(t)\, dt$ since g is periodic. But lim $\sum f(r/n)\, 1/n$ is just $\int_0^1 f(x)\, dx$, so we get the required $(\int_0^1 f(x)\, dx)\, (\int_0^1 g(x)\, dx)$.

It remains to look at the error involved in approximating f. The function f is continuous and [0, 1] is compact, so it must be uniformly continuous on [0, 1]. Thus we given any $\varepsilon > 0$, we can find N such that for n > N, we have $|f(x) - f(r/n)| < \varepsilon$ on [r/n, r/n + 1/n] for each of r = 0, 1, 2, ... , n-1. So the error is at most $\sum \int_{r/n}^{r/n+1/n} |f(x) - f(r/n)|\, |g(nx)|\, dx < \sum \varepsilon\, 1/n \int_0^1 |g(t)|\, dt = \varepsilon \int_0^1 |g(t)|\, dt$, which can be made arbitrarily small.

Problem B4

We are given a sequence $a_1, a_2, \ldots, a_n$. Each a_i can take the values 0 or 1. Initially, all $a_i = 0$. We now successively carry out steps 1, 2, ... , n. At step m we change the value of a_i for those i which are a multiple of m. Show that after step n, $a_i = 1$ iff i is a square. Devise a similar scheme give $a_i =$ 1 iff i is twice a square.

Solution
a_i is changed once for each divisor of i. If i is non-square then it has an even number of divisors (they come in pairs d, i/d), so a_i ends as 0. If i is square it has an odd number of divisors and so a_i ends as 1.
We change those that a multiple of 2, then those that are a multiple of 4, then those that are a multiple of 6 and so on. This only affects the even numbers and indeed a_{2i} is changed once for each divisor of i, so the same argument as before shows that a_{2i} ends as 1 iff i is a square. Note that a_{2i+1} is never changed, so it remains 0.

Problem B5
The first n terms of the exansion of $(2 - 1)^{-n}$ are 2^{-n} (1 + n/1! (1/2) + n(n + 1)/2! $(1/2)^2$ + ... + n(n + 1) ... (2n -2)/(n - 1)! $(1/2)^{n-1}$). Show that they sum to 1/2.

Solution
Consider a random walk on a two dimensional lattice. The particle starts at (0, 0). At each step it moves up 1 step with probability 1/2 and right one step with probability 1/2. Suppose that we stop when the particle reaches x = n or y = n.

Suppose the particle first reaches x = n at y = m ($0 \le m < n$). Then its final move must be from (n-1, m) to (n, m). In order to reach (n-1, m) it must make n+m-1 moves, m of which must be upwards. So the probability is (n+m-1Cm)$(1/2)^{n+m}$. Thus the probability that it reaches x = n is (n-1)C0 $(1/2)^n$ + nC1 $(1/2)^{n+1}$ + ... + (2n-2)C(n-1) $(1/2)^{2n-1}$. The probability that it reaches y = n is the same. Note that it cannot reach the point (n, n) (because it reaches (n, n-1) or (n-1, n) first and stops). So each series must total 1/2.

Problem B6
R is the reals. D is the closed unit disk $x^2 + y^2 = 1$ in R^2. The function f **:** D → R has partial derivatives $f_1(x, y)$ and $f_2(x, y)$ and all $f(x, y) \in [-1, 1]$. Show that there is a point (a, b) in the interior of D such that $f_1(a, b)^2 + f_2(a, b)^2 \le 16$.

Solution
Consider $f(x, y) + 2x^2 + 2y^2$. It is at least 1 on the entire boundary of D and at most 1 at the centre. So it is either constant at an interior point of D or has a minimum at an interior point. In either case, there is an interior point (a, b) at which its two partial derivatives are zero. So $f_1(a, b) = -4a$, $f_2(a, b) = -4b$ and $f_1^2 + f_2^2 = 16(a^2 + b^2) < 16$.

29th Putnam 1968

Problem A1
Prove that $\int_0^1 x^4(1 - x)^4/(1 + x^2)\, dx = 22/7 - \pi$.

Solution
Divide the denominator by the numerator to get: $(x^8 - 4x^7 + 6x^6 - 4x^5 + x^4) = (1 + x^2)(x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - 4$.
Now we can integrate to get $(x^7/7 - 2x^6/3 + x^5 - 4x^3/3 + 4x)|_0^1 - \int_0^1 4dx/(1 + x^2) = 22/7 - \pi$.

Problem A2
Given integers a, b, c, d such that $ad - bc \neq 0$, integers m, n and a real $\varepsilon > 0$, show that we can find rationals x, y, such that $0 < |ax + by - m| < \varepsilon$ and $0 < |cx + dy - n| < \varepsilon$.

Solution
Take k rational such that $0 < k < \varepsilon$. Now solve the simultaneous equations: $ax + by = m + k$, $cx + dy = n + k$. We get $x = (dm - bn + dk - bk)/(ad - bc)$, $y = (an - cm + ak - ck)/(ad - bc)$. Clearly x and y are rational and satisfy the required inequalities.

Problem A3
S is a finite set. P is the set of all subsets of S. Show that we can label the elements of P as A_i, such that $A_1 = \varnothing$ and for each $n >= 1$, either $A_{n-1} \subset A_n$ and $|A_n - A_{n-1}| = 1$, or $A_{n-1} \supset A_n$ and $|A_{n-1} - A_n| = 1$.

Solution
Let $N = 2^n$ and let $A_1, \ldots, A_N$ be a solution for n with $A_N = \{n\}$. Now take $A_{N+m} = A_{2N+1-m}$ union $\{n+1\}$ for $m = N+1, \ldots, 2N$. This gives a solution for n+1.

Problem A4
Let S_2 be the 2-sphere $\{ (x, y, z) : x^2 + y^2 + z^2 = 1\}$. Show that for any n points on S_2, the sum of the squares of the $n(n - 1)/2$ distances between them (measured in space, not in S_2) is at most n^2.

Solution
Let the points be (x_i, y_i, z_i) for $i = 1, 2, \ldots n$. The square of the distance between the pair i, j is $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = (x_i^2 + y_i^2 + z_i^2) + (x_j^2 + y_j^2 + z_j^2) - 2(x_ix_j + y_iy_j + z_iz_j) = 2 - 2(x_ix_j + y_iy_j + z_iz_j)$. Hence the sum of the squares of the distances is $n(n-1) - 2 \sum (x_ix_j + y_iy_j + z_iz_j)$.

Now $(x_1 + x_2 + \ldots + x_n)^2 = \sum x_i^2 + 2 \sum x_ix_j$. So $2 \sum (x_ix_j + y_iy_j + z_iz_j) = X^2 + Y^2 + Z^2 - \sum (x_i^2 + y_i^2 + z_i^2) = X^2 + Y^2 + Z^2 - n$, where $X = \sum x_i$, $Y = \Sigma y_i$, $Z = \sum z_i$. Hence the sum of the squares is $n^2 - (X^2 + Y^2 + Z^2)$, which is at most n^2.

Problem A5
Find the smallest possible α such that if $p(x) \equiv ax^2 + bx + c$ satisfies $|p(x)| \leq 1$ on $[0, 1]$, then $|p'(0)| \leq \alpha$.

Solution
Answer: 8. Extreme case is $\pm(8x^2 - 8x + 1)$.
Note that $|p'(0)| = |b|$. So the question is how large we can make $|b|$ and still be able to find a and c such that p(x) lies between -1 and 1 on [0 ,1].

wlog a > 0. Write $p(x) = a(x + b/2a)^2 + c$. If $-b/2a \le 0$, then p(x) has its minimum on [0, 1] at 0 and its maximum at 1, so we require $p(1) - p(0) \le 2$ (we can then adjust c to get $|p(x)| \le 1$ on the entire interval). But $p(1) - p(0) = a + b$, so $b \le 2$ (and the extreme case is $p(x) = 2x - 1$).

If $-b/2a \ge 1$, then p(x) has its maximum at 0 and its minimum at 1 (on the interval). So we require $p(0) - p(1) < 2$ and hence $a + b \ge -2$. But $-b \ge 2a$, so $a \le 2$ and $b \ge -4$ (and the extreme case is $2x^2 - 4x + 1$).
If $0 < -b/2a \le 1/2$, then p(x) has its minimum at $-b/2a$ and its maximum at 1. So we require $a + b + b^2/4a \le 2$ or $(b + 2a)^2/4a \le 2$. But $0 < -b \le a$, so $(b + 2a)^2 \ge a^2$ and hence $a^2/4a \le 2$ or $a \le 8$ and $0 < -b \le 8$.
Finally, if $1/2 \le -b/2a < 1$, the p(x) has its minimum at $-b/2a$ and its maximum at 0. So we require $b^2/4a \le 2$. So again $a^2 \le b^2/4a \le 2$. Hence $a \le 8$. But $b^2 \le 8a$, so $-b < 8$. Hence $-8 \le b < 0$.

Problem A6
Find all finite polynomials whose coefficients are all ±1 and whose roots are all real.

Solution
Answer: $\pm(x + 1)$, $\pm(x - 1)$, $\pm(x^2 + x - 1)$, $\pm(x^2 - x - 1)$, $\pm(x^3 + x^2 - x - 1)$, $\pm(x^3 - x^2 - x + 1)$.

The linear and quadratic polynomials are easy to find (and it is easy to show that they are the only ones).
Suppose the polynomial has degree $n \ge 3$. Let the roots be k_i. Then $\sum k_i^2 = (\sum k_i)^2 - 2 \sum k_i k_j = a^2 \pm 2b$, where a is the coefficient of x^{n-1} and b is the coefficient of x^{n-2}. Hence the arithmetic mean of the squares of the roots is $(a^2 \pm 2b)/n$. But it is at least as big as the geometric mean which is 1 (because it is an even power of c, the constant term). So we cannot have $n > 3$. For $n = 3$, we must have b the opposite sign to the coefficient of x^n and it is then easy to check that the only possibilities are those given above.

Problem B1
The random variables X, Y can each take a finite number of integer values. They are not necessarily independent. Express prob(min(X, Y) = k) in terms of p_1 = prob(X = k), p_2 = prob(Y = k) and p_3 = prob(max(X, Y) = k).

Solution
Let q_1 = prob(X = k, Y = k), q_2 = prob(X = k, Y > k), q_3 = prob(X > k, Y = k), q_4 = prob(X = k, Y < k), q_5 = prob(X < k, Y = k). These are all probabilities for disjoint events, so we can add them freely to get: prob(min(X, Y) = k) = $q_1 + q_2 + q_3$, $p_1 = q_1 + q_2 + q_4$, $p_2 = q_1 + q_3 + q_5$, $p_3 = q_1 + q_4 + q_5$.
Hence prob(min(X, Y) = k) = $p_1 + p_2 - p_3$.

Problem B2
(G, *) is a finite group with n elements. K is a subset of G with more than n/2 elements. Prove that for every $g \in G$, we can find $h, k \in K$ such that $g = h * k$.

Solution
Take any g in G. Let H be the set of elements of the form $h^{-1}*g$ where h is in K. Since G is a group, H has the same number of elements as K. Hence it must overlap K (since between them they have more than n elements). So for some h, k in K we have $k = h^{-1}*g$, or $g = h*k$.

Problem B3
Given that a 60^o angle cannot be trisected with ruler and compass, prove that a $120^o/n$ angle cannot be trisected with ruler and compass for n = 1, 2, 3, ...

Solution
This is a trap.

The obvious answer is that if we could trisect an angle $120^o/n$, that would give us an angle $40^o/n$ and hence $60^o/3$. But we are told we cannot trisect 60^o.
This does not work for a slightly subtle reason. We are given the angle $120^o/n$. That ***in itself*** may help us to trisect 60^o. For example, it certainly allows us to trisect 60^o if n = 6.

So we have to use the standard Galois theory approach. It is not hard to show that a straight-edge allows us to get rational distances and compasses allow us to get any finite number of square roots. So we can construct distances which are in an extension K of the rationals of degree 2^n for some n. Constructing an angle is equivalent to constructing its cosine.

So if we are given an angle A, then we can construct distances which are in any extension K of degree 2^nover Q(cos A). Now if K is an extension of L of degree r and L is an extension of F of degree s, then K is an extension of degree rs of F. So suppose Q(cos A) has degree u and Q(cos(A/3) has degree v. If we can construct A/3 given A, then cos(A/3) must lie in an extension K of Q(cos A) of degree 2^n. But K is also an extension of Q(cos(A/3)) so v/u must be a power of 2.

Finally, it is well-known that Q(cos(360/m)) has degree φ(m) over Q, where φ is Euler's function (so that φ(m) is the number of 1, 2, ... m-1 relatively prime to m). So we have to show that φ(9n)/φ(3n) is not a power of 2. But $\varphi(m) = m \prod(1 - 1/p)$, where the product is taken over all primes p dividing m. But 3n and 9n have the same prime divisors, so φ(9n) = 3 φ(3n), and 3 is not a power of 2.

Problem B4
R is the reals. f **:** R → R is continuous and $L = \int_{-\infty}^{\infty} f(x)\,dx$ exists. Show that $\int_{-\infty}^{\infty} f(x - 1/x)\,dx = L$.

Solution
Substitute x = y - 1/y. As y increases from -∞ to 0, x increases (monotonically) from -∞ to +∞. Also $dx = (1 + 1/y^2)\,dy$, so we have $L = \int_{-\infty}^{0} f(y - 1/y)(1 + 1/y^2)\,dy = \int_{-\infty}^{0} f(y - 1/y)\,dy + \int_{-\infty}^{0} f(y - 1/y)\,1/y^2\,dy$.

In the last integral we may substitute z = -1/y to get $\int_0^{\infty} f(z - 1/z)\,dz$ or, using y instead of z, $\int_0^{\infty} f(y - 1/y)\,dy$.

Problem B5
Let F be the field with p elements. Let S be the set of 2 x 2 matrices over F with trace 1 and determinant 0. Find |S|.

Solution
Answer: $p^2 + p$. Let the matrix be

a b
c d

If we take any a not 0 or 1, then d is fixed as 1 - a and is also not 0. Hence bc is fixed as a(1 - a) and is non-zero. So we can take any b not 0 and d is then fixed. Altogether that gives us (p - 2)(p - 1) possibilities.

If we take a as 0 or 1, the d is fixed as 1 - a and bc must be 0. That gives 2p-1 possibilities for bc, or 4p-2 possibilities in all.
So in total we have $(p - 2)(p - 1) + 4p - 2 = p^2 + p$.

Problem B6
A *compact* set of real numbers is closed and bounded. Show that we cannot find compact sets A_1, A_2, A_3, ... such that (1) all elements of A_n are rational and (2) given any compact set K whose members are all rationals, $K \subseteq$ some A_n.

Solution
Use a diagonalisation argument.

Suppose we can find such A_n. Let S_n be the interval $[1/2^{2n}, 1/2^{2n-1}]$, and let T_n be the set of rational points in S_n. Then the closure of T_n is S_n, which contains irrational points, so there must be points of T_n which are not in A_n. Let x_n be one such. Now consider $\{x_n\}$. Its only limit point is 0, so K, the union of $\{x_n\}$ and $\{0\}$, is a compact set of rationals. But K is not contained in any of the A_n, because it has a member x_n not in A_n for each n. Contradiction.

30th Putnam 1969

Problem A1
R^2 represents the usual plane (x, y) with $-\infty < x, y < \infty$. p: $R^2 \to R$ is a polynomial with real coefficients. What are the possibilities for the image $p(R^2)$?

Solution
Answer: [k, k], [k, inf), (-inf, k], (-inf, inf), (k, inf), (-inf, k) for all real k.

The first four possibilities given in the answer are easily realised: p(x, y) = k gives [k, k]; p(x, y) = x^2 + k gives [k, inf); p(x, y) = k - x^2 gives (-inf, k]; p(x, y) = x gives (-inf, inf).
If p is not constant, then wlog there is a positive power of x. Let n be the highest such power. Fix y so that the p becomes a polynomial in x with a non-zero term in x^n (this must be possible since only finitely many values of y can give a zero term). As x tends to inf, this term will dominate and tend to +inf or -inf. The domain R^2 is connected and p is continuous, so the image must be connected also. So certainly there are no other possibilities apart from those given in the Answer.

It is tempting to think that we cannot get (k, inf), but attempts to prove it fail. The trick is to use two square terms which cannot be zero simultaneously. For example, $(xy - 1)^2 + x^2 + k$. We can make x arbitrarily small and choose y to make the first term zero, but if we make x zero, then the first term is 1.

Problem A2
A is an n x n matrix with elements $a_{ij} = |i - j|$. Show that the determinant $|A| = (-1)^{n-1} (n - 1) 2^{n-2}$.

Solution
For i = 1, 2, 3, ... , n-2 subtract twice row i+1 from row i and add row i+2 to row i. For i < n-1, row i becomes all 0s except for a 2 in column i+1.
Now expand successively by the first, second, third ... rows to get $(-2)^{n-2}$ times the 2 x 2 determinant with first row n-2, 1 and second row n-1, 0. This 2 x 2 determinant has value -(n-1), so the $|A| = (-1)^{n-1} (n-1) 2^{n-2}$.

Problem A3
An n-gon (which is not self-intersecting) is triangulated using m interior vertices. In other words, there is a set of N triangles such that: (1) their union is the original n-gon; (2) the union of their vertices is the set consisting of the n vertices of the n-gon and the m interior vertices; (3) the intersection of any two distinct triangles in the set is either empty, a vertex of both triangles, or a side of both triangles. What is N?

Solution
Answer: N = n + 2m - 2.

We use the well-known relation F = E + 2 - V (*), where F is the number of faces, E the number of edges and V the number of vertices. In this case, F = N + 1, because there are N triangles and one n-gon (the exterior polygon). So 3N + 1.n gives each edge twice, in other words 2E = 3N + n. Clearly V = m + n. So (*) gives: N + 1 =3N/2 + n/2 + 2 - m - n. Hence N = n + 2m - 2.

Problem A4
Prove that $\int_0^1 x^x \, dx = 1 - 1/2^2 + 1/3^3 - 1/4^4 + \ldots$.

Solution
The rhs is a series, so this suggests that we should expand the integrand as a series and integrate

term by term.
We have that $x^x = e^{x \ln x}$, so the obvious approach is to expand this as a series: $1 + (x \ln x) + (x \ln x)^2/2! + \ldots$. For this to work we need that $\int_0^1 (x \ln x)^n \, dx = n! \, (-1)^n/(n+1)^{n+1}$ (*).
The integral is easy to evaluate by parts. Each step reduces the exponent on the log term without affecting the exponent on the x term and the non-integral term always vanishes at both endpoints. In other words, we have $\int_0^1 x^n \ln^m x \, dx = -m/(n+1) \int_0^1 x^n \ln^{m-1} x \, dx$, which gives (*).

Problem A5
u(t) is a continuous function. x(t), y(t) is the solution of x' = -2y + u(t), y' = -2x + u(t) satisfying the initial condition $x(0) = x_0$, $y(0) = y_0$. Show that if $x_0 \neq y_0$, then we do not have x(t) = y(t) = 0 for any t, but that given any $x_0 = y_0$ and any T > 0, we can always find some u(t) such that x(T) = y(T) = 0.

Solution
Subtracting, we have z' = 2z, where z = x - y. So $z(t) = Ae^{2t}$. If $x_0 \neq y_0$, then $A \neq 0$ and so $z(t) \neq 0$ for any t. Hence, in particular, we do not have x(t) = y(t) = 0 for any t.

If $x_0 = y_0 = k$, then z(0) = 0, so A = 0, so x(t) = y(t) for all t. Take (for example) $u(t) = -(k/T) e^{-2t}$. Then we may solve x' + 2x = u with x(0) = k to get $x(t) = k(1 - t/T) e^{-2t}$. This has x(T) = 0 as required.

Problem A6
The sequence $a_1 + 2a_2$, $a_2 + 2a_3$, $a_3 + 2a_4$, ... converges. Prove that the sequence a_1, a_2, a_3, ... also converges.

Solution
Note that the result is not true for $a_i + a_2$, $a_2 + a_3$, ... or for $2a_1 + a_2$, $2a_2 + a_3$, ... In the first case, we could have 1, -1, 1, -1, 1, ... In the second case, we could have 1, -2, 4, -8, 16, -32, ...

Suppose $a_n + 2a_{n+1}$ converges to 3k. We show that $a_n$converges to k.
Given any ε > 0, take N so that $a_n + 2a_{n+1}$ is within ε of 3k for all n >= N. Take a positive integer M such that a_N is within $(2^M + 1)\varepsilon$ of k.
Then a_{N+1} is within $((2^M + 1)\varepsilon + \varepsilon)/2 = (2^{M-1} + 1)\varepsilon$ of (3k - k)/2 = k. By a trivial induction a^{N+M} is within 2ε of k. Then a^{N+M+1} is within (2ε + ε)/2, and hence within 2ε, of k. So by a trivial induction, a_n is within 2ε of k for all n > N + M.

Problem B1
The positive integer n is divisible by 24. Show that the sum of all the positive divisors of n - 1 (including 1 and n - 1) is also divisible by 24.

Solution
Let n = 24m. We show first that if d is a divisor of n - 1 = 24m - 1, then $d^2 - 1$ is divisible by 24. Clearly d is not a multiple of 3 (because n is), so 3 must divide $d^2 - 1$. Also d must be odd (like n - 1), so d - 1 and d + 1 are consecutive even numbers, so one must be a multiple of 4, and there product $d^2 - 1$ must be a multiple of 8.

Now 24m - 1 cannot be a square (because squares are congruent to 0 or 1 mod 4), so its divisors come in pairs d, (24m - 1)/d. But d + (24m - 1)/d is divisible by 24 (because $d^2 - 1$ and 24m are and no factor of 24 can divide d). Hence the sum of all the divisors of n - 1 is divisible by 24.

Problem B2
G is a finite group with identity 1. Show that we cannot find two proper subgroups A and B (≠ {1} or G) such that $A \cup B = G$. Can we find three proper subgroups A, B, C such that $A \cup B \cup C = G$?

Solution
We have |A| divides |G| and hence $|A| \le |G|/2$. Similarly, $|B| \le |G|/2$. But 1 belongs to both A and B, so |A union B| < |G|.

For three subgroups, sometimes we can and sometimes we cannot. For example, if G is the group of order 4 defined by ba = ab, $a^2 = b^2 = 1$, then we can: A = {1, a}, B = {1, b}, C = {1, ab}. On the other hand, if G is a cyclic group of prime order, then it has no proper subgroups. [But these are far from the only exceptions. For example take G to be cyclic of order 15. Then any proper subgroup must have order 3 or 5. All subgroups contain 1, so any three subgroups contain at most 13 elements between them.]

Problem B3
The sequence $a_1, a_2, a_3, \ldots$ satisfies $a_1a_2 = 1$, $a_2a_3 = 2$, $a_3a_4 = 3$, $a_4a_5 = 4$, ... Also, $\lim_{n\to\infty} a_n/a_{n+1} = 1$. Prove that $a_1 = \sqrt{(2/\pi)}$.

Solution
Let $a_1 = 1/k$. Then we deduce successively that $a_2 = k$, $a_3 = 2/k$, $a_4 = (3/2)\,k$. By a trivial induction, $a_{2n} = (3.5.7 \ldots 2n\text{-}1)/(2.4.6 \ldots 2n\text{-}2)\,k$ and $a_{2n+1} = (2.4.6 \ldots 2n)/(3.5.7 \ldots 2n\text{-}1)\,1/k$. Hence $a_{2n+1}/a_{2n+2} = (2/1)\,(2/3)\,(4/3)\,(4/5)\,(6/5) \ldots (2n/2n\text{-}1)\,1/k^2$. We are given that this has limit 1, so (2/1) (2/3) (4/3) (4/5) (6/5) ... (2n/2n-1) has limit k^2.
So we need to establish that (2/1) (2/3) (4/3) (4/5) (6/5) (6/7) ... (2n/2n-1) (2n)/(2n+1) ... = $\pi/2$.
This is the well-known Wallis product. It is usually established by proving the product representation for sin z, but that requires relatively advanced complex analysis, which is outside the Putnam syllabus, so we need a simpler approach. The following is rather unmotivated unless you have seen it before.

Let $I_n = \int_0 \pi/2\, \sin^n x\, dx$. Integrating by parts, we have that $I_n = -\int \sin^{n-1}x\, d(\cos x) = (n\text{-}1)\, I_{n-2} + (n\text{-}1)\, I_n$. Hence $I_n = (n\text{-}1)/n\, I_{n-2}$. But $I_0 = \pi/2$, $I_1 = 1$. So we find that $I_{2n} = (1/2)(3/4)(5/6) \ldots (2n\text{-}1)/2n\, \pi/2$, $I_{2n+1} = (2/3)(4/5)(6/7) \ldots (2n/2n+1)$.
Now $0 < \sin x < 1$ on $(0, \pi/2)$, so $I_{2n-1} < I_{2n} < I_{2n+1}$. Dividing by I_{2n+1}, we get $(2n+1)/2n > I_{2n}/I_{2n+1} > 1$. So I_{2n}/I_{2n+1} tends to 1 as n tends to infinity, which estabishes the Wallis product.

Problem B4
Γ is a plane curve of length 1. Show that we can find a closed rectangle area 1/4 which covers Γ.

Solution
Let the endpoints of the curve be A and B. Take lines p, q parallel to AB and on opposite sides of it so that the band between them just covers the curve. Similarly take lines r, s perpendicular to AB so that the band between them just covers the curve (and so that r is nearer to A than to B).

Reflect A in q to get A_1. Reflect A_1 in r to get A_2. Similarly, reflect B in p to get B_1, then B_1 in s to get B_2. Consider the line segment A_2B_2. By reflection we can derive from it a curve from A to B consisting of five straight line segments which intersects p, q, r and s. Moreover it is evidently the shortest curve with this property.

Let X be the distance between r and s, Y the distance between p and q, and Z the distance between A and B. We have $(X - 2Y)^2 + (X - Z)^2 + 2X(X - Z) \ge 0$ (*). Hence $(2X - Z)^2 + (2Y)^2 \ge 4XY$, which tells us that the square of the length A_2B_2 is greater than or equal to four times the area of the rectangle formed by the intersection of the two bands. Thus if the curve Γ has length 1, then A_2B_2 must have length at least 1 and hence we have found a rectangle with area at most 1/4 which covers Γ. Note that (*) shows the area will be less than 1/4 unless Z = X (so that r goes through A and s through B) and X = 2Y.

Problem B5
The sequence a_i, i = 1, 2, 3, ... is strictly monotonic increasing and the sum of its inverses converges. Let f(x) = the largest i such that $a_i < x$. Prove that f(x)/x tends to 0 as x tends to infinity.

Solution
Suppose not. Then for some fixed k > 0, we can find arbitrarily large x such that f(x) > kx. So take a sequence $x_1 < x_2 < x_3 < ...$ such that (1) $f(x_i) > kx_i$, (2) $kx_i/2$ gt; $f(x_{i-1})$. Then by (1) at least kx_i members of the sequence a_n are less than x_i. By (2) at most $kx_i/2$ are less than x_{i-1}. So at least $kx_i/2$ must lie between x_{i-1} and x_i.

So $\sum 1/a_n$ has at least $kx_i/2$ terms between $1/x_i$ and $1/x_{i-1}$. These terms sum to at least k/2. All terms are positive, so the series diverges. Contradiction.

Problem B6
M is a 3 x 2 matrix, N is a 2 x 3 matrix. MN = $\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}$ Show that NM = $\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$

Solution
The key observation is that $(MN)^2 = 9$ MN. [Of course, we expect this to be true since NM = 9 I, and it is easy to verify.]

It is also easy to check that MN has rank 2. The rank of NM must be at least as big as M(NM)N = 9 MN, so NM is non-singular. Now $(NM)^3 = N(MN)^2M = N(9\ MN)M = 9\ (NM)^2$. Multiplying by the inverse of NM twice gives that NM = 9 I.

31st Putnam 1970

Problem A1
$e^{bx} \cos cx$ is expanded in a Taylor series $\sum a_n x^n$. b and c are positive reals. Show that either all a_n are non-zero, or infinitely many a_n are zero.

Solution
$e^{bx} \cos cx = \text{Re}\, e^{(b+ic)x}$, so $n!\, a_n = \text{Re}((b + ic)^n)$. Let $b + ic = k e^{i\theta}$, then $n!\, a_n/k^n = \cos n\theta$. So $a_n = 0$ iff $n\theta = (2m+1)\pi/2$ for some integer m. So if there are any zeros then there are infinitely many.

Problem A2
$p(x, y) = a x^2 + b x y + c y^2$ is a homogeneous real polynomial of degree 2 such that $b^2 < 4ac$, and q(x, y) is a homogeneous real polynomial of degree 3. Show that we can find $k > 0$ such that $p(x, y) = q(x, y)$ has no roots in the disk $x^2 + y^2 < k$ except (0, 0).

Solution
The disk is a strong hint that one should use polar coordinates, so put $x = r \cos\theta$, $y = r \sin\theta$. Then we get $r = p(\cos\theta, \sin\theta)/q(\cos\theta, \sin\theta)$. Now $|\cos\theta|$ and $|\sin\theta| \le 1$, so $|q(\cos\theta, \sin\theta)| \le h$, where h is the sum of the absolute values of the coefficients of q. So we are home if we can establish some inequality $|p(\cos\theta, \sin\theta)| \ge h'$, because we can then take $k = h'/h$.

Without loss of generality $a > 0$ and hence also $c > 0$. Now $p(\cos\theta, \sin\theta) = 1/2\,(a + c)(\cos^2\theta + \sin^2\theta) + 1/2\,(a - c)(\cos^2\theta - \sin^2\theta) + 1/2\, b \sin 2\theta = 1/2\,(a + c) + 1/2\,(a - c)\cos 2\theta + 1/2\, b \sin 2\theta$. Put $d = \sqrt{(b^2 + (a - c)^2)}$ and take φ so that $\cos\varphi = (a - c)/d$, $\sin\varphi = b/d$. Then $2p(\cos\theta, \sin\theta) = (a + c) + d(\cos\varphi \cos 2\theta + \sin\varphi \sin 2\theta) = (a + c) + d\cos(\varphi - 2\theta) >= (a + c) - d$.

But $4ac > b^2$, so $(a + c)^2 > (a - c)^2 + b^2$, so $(a + c) > d$, and we are done.

Problem A3
A perfect square has *length* n if its last n digits (in base 10) are the same and non-zero. What is the longest possible length? What is the smallest square achieving this length?

Solution
Answer: 3, $38^2 = 1444$.

All squares end in the digit 0, 1, 4, 9, 6, or 5. A square ending in 11, 99, 66, 55 would be congruent to 3, 3, 2, 3 mod 4, but squares are congruent to 0 or 1 mod 4. So for length greater than 1 the square must end in 4. For example, $12^2 = 144$.
A square ending in 4444 would be congruent to 12 mod 16, but squares are congruent to 0, 1, 4 or 9 mod 16. So the maximum length is 2 or 3.

If n^2 ends in 4, then n must end in 2 or 8. $(100a + 10b + 2)^2 = 10000a^2 + 1000\,(2ab) + 100\,(4a + b^2) + 10\,(4b) + 4$. So if this ends in 44, then b = 1 or 6. If b = 1, then $4a + b^2$ is odd, so the square cannot end in 444. If b = 6, then the square is $1000k + (4a + 38)\,100 + 44$. This will end in 444 if we take a = 4 or 9. Thus the smallest numbers ending in 2 whose square ends in 444 are 462 (square 213444) and 962 (square 925444).

$(100a + 10b + 8)^2 = 1000k + 100\,(16a^2 + b^2) + 10\,(16b + 6) + 4$. So if this ends in 44, then b = 3 or 8. $(100a + 38)^2 = 1000k + 100\,(76a) + 1444$, so this ends in 444 if a = 0. This must be the smallest solution.

Problem A4

The real sequence $a_1, a_2, a_3, ...$ has the property that $\lim_{n\to\infty} (a_{n+2} - a_n) = 0$. Prove that $\lim_{n\to\infty} (a_{n+1} - a_n)/n = 0$.

Solution
Suppose we have any series a_n satisfying the condition. Take a new series b_n defined by $b_{2n+1} = 0$ and $b_{2n} = a_{2n}$. Then b_n also satisfies the condition, but $(b_{2n+1} - b_{2n})/n = - a_{2n}/n$, so we must have the apparently stronger result that $a_{2n}/(2n)$ tends to zero (and similarly that $a_{2n+1}/(2n+1)$ tends to zero).

Thus we need to prove that if c_n is any sequence such that $c_{n+1} - c_n$ tends to zero, then c_n/n tends to zero.

Given any positive k, we have $|c_{n+1} - c_n| < k$ for all n > some N. Now for any M > N, we have $c_M = c_N + (c_{N+1} - c_N) + ... + (c_M - c_{M-1})$. Hence $|c_M| \le |c_N| + (M-N)k$. If we take M sufficiently large that $|b_N|/M < k$, then $|c_M|/M < 2k$, which establishes that c_n/n tends to zero.

Problem A5
Find the radius of the largest circle on an ellipsoid with semi-axes a > b > c.

Solution
Answer: b.

A circle lies in a plane, so we consider planes cutting the ellipsoid. The intersection of any plane P with the ellipsoid is an ellipse (assuming it cuts the ellipsoid in more than one point). That is fairly obvious. Substitute the linear equation for the plane into the quadratic equation for the ellipsoid and we get a quadratic equation for the projection of the intersection onto one of the coordinate planes. This must be a conic and, since bounded, an ellipse. Projecting back onto P shows that the intersection is an ellipse.

It is less obvious, but true, that parallel planes give similar ellipses. So to maximise the size we take the plane P through the centre of the ellipsoid. Suppose its intersection with the ellipsoid is a circle K. P cannot be normal to one of the semi-axes, because then its intersection is certainly an ellipse with unequal semi-axes. So P must meet the bc plane in a line. This line is a diameter of K. But it is also a diameter of an ellipse with semi-major axes b and c, so it has length at most 2b, so K has diameter is at most 2b. Similarly P meets the ab plane in a line, which has length at least 2b, so the diameter of K is at least 2b. So if K has radius b.

It remains to show that some planes do intersect the ellipsoid in a circle. Consider a plane through the semi-major axis b of the ellipsoid. The intersection K clearly has one diameter 2b. Moreover, the ellipse K is symmetrical about this diameter, so it must be one of its two semi-major axes. If we start with the plane also containing the semi-major axis c of the ellipsoid, then K has another diameter perpendicular to the first and length 2c. This must be the other semi-major axis of K. As we rotate the plane through a right-angle, the length of this diameter increases continuously to 2a. So at some angle it must be 2b. But an ellipse with equal semi-major axes is a circle.

Problem A6
x is chosen at random from the interval [0, a] (with the uniform distribution). y is chosen similarly from [0, b], and z from [0, c]. The three numbers are chosen independently, and $a \ge b \ge c$. Find the expected value of min(x, y, z).

Solution
Answer: $c/2 - c^2(1/(6a) + 1/(6b)) + c^3/(12ab)$.

Let m(k) = prob(x ≥ k and y ≥ k and z ≥ k). We have that prob(x ≥ k) = 1 for k < 0, 1 - k/a for 0 ≤ k ≤ a, 0 for k > a. Similarly for y and z. So m(k) is 1 for k < 0, and (1 - k/a)(1 - k/b)(1 - k/c) for 0 ≤ k ≤ c, and 0 for k ≥ a.

Now the probability that the minimum lies between k and k + δk is m(k) - m(k + δk), so the probability density for min(x, y, z) to equal k is -m'(k) which is: (1/a + 1/b + 1/c) -2k(1/(ab) + 1/(bc) + 1/(ca)) + $3k^2$/(abc) for 0 ≤ k ≤ c. Thus the expected value is the integral from 0 to c of k(1/a + 1/b + 1/c) -$2k^2$(1/(ab) + 1/(bc) + 1/(ca)) + $3k^3$/(abc) which is c^2(1/a + 1/b + 1/c)/2 -$2c^3$(1/(ab) + 1/(bc) + 1/(ca))/3 + $3c^4$/(4abc).

Problem B1
Let f(n) = $(n^2 + 1)(n^2 + 4)(n^2 + 9) ... (n^2 + (2n)^2)$. Find $\lim_{n\to\infty} f(n)^{1/n}/n^4$.

Solution
Rearranging slightly, we may take g(n) = $(1 + (1/n)^2)(1 + (2/n)^2)(1 + (3/n)^2) ... (1 + (2n/n)^2)$. We have to find lim $g(n)^{1/n}$.

It is a mistake to approach this algebraically. It is not hard to show that the product of the pair of terms r and 2n+1-r is at least 4. There are n such pairs, so certainly $g(n)^{1/n} > 4$, but it is hard to get any further.

The key is to take logs. We then see immediately that $g(n)^{1/n}$ becomes a standard Riemann sum for $\int_0^2 \log(1 + x^2)\, dx$. So the limit is simply the integral.

x log(1 + x^2) differentiates to log(1 + x^2) + $2x^2$/(1 + x^2). A little reflection then suggests using 2 $\tan^{-1}$x, which differentiates to 2/(1 + x^2). So the complete integral is x log(1 + x^2) + 2 $\tan^{-1}$x - 2x. Evaluating between 0 and 2 gives 2 log 5 + 2 $\tan^{-1}$2 - 4 = k, say. Then the original limit in the question is e^k. That is approximately 4.192.

Problem B2
A weather station measures the temperature T continuously. It is found that on any given day T = p(t), where p is a polynomial of degree ≤ 3, and t is the time. Show that we can find times $t_1 < t_2$, which are independent of p, such that the average temperature over the period 9am to 3pm is ($p(t_1)$ + $p(t_2)$ / 2. Show that t_1 = 10:16am, t_2 = 1:44pm.

Solution
Let t be time after 9am and T be 6 hrs after 9am, so that t = 0 represents 9am and t = T represents 3pm. We may write the polynomial p(t) as $at^3 + bt^2 + ct + d$ for some a, b, c, d. The average temperature is $1/T \int_0^T (at^3 + bt^2 + ct + d)\, dt = T^3$ a/4 + T^2 b/3 + T c/2 + d. We wish to find t_1, t_2 so that this equals ($t_1^3 + t_2^3$) a/2 + ($t_1^2 + t_2^2$) b/2 + ($t_1 + t_2$) c/2 + d for all a, b, c, d.

The terms in d match. The term in c matches provided we take $t_1 + t_2 = T$. The term in b matches provided we take $t_1^2 + t_2^2 = 2T^2/3$. These two equations already determine t_1 and t_2. In fact, solving the quadratic, we get $t_1 = T(1 - 1/\sqrt{3})/2$ and $t_2 = T(1 + 1/\sqrt{3})/2$.

For the terms in a to match we need $t_1^3 + t_2^3 = T_3/2$. But we can easily check that the values above satisfy this relation also.

Finally note t_1 = 6(1/2 - (√3)/6) hrs = 3 - √3 = 3 - 1.732 = 1.268 hrs = 1 hr 16 min (approx), so t_1 represents about 10:16am and t_2 about 1 hr 16 min before 3pm or 1:44pm.

Problem B3
S is a closed subset of the real plane. Its projection onto the x-axis is bounded. Show that its projection onto the y-axis is closed.

Solution
Let Y be the projection onto the y-axis and X the projection on the x-axis. Let $\{ y_n \}$ be any Cauchy sequence in Y. Then $\{ y_n \}$ must converge to some real y. We have to show that y is in Y. For each n take any x_n such that (x_n, y_n) is in S. Then $\{ x_n \}$ lies in X, which is bounded, so $\{ x_n \}$ is bounded. But any bounded sequence has a convergent subsequence. So we can take a subsequence $\{ u_n \}$ of $\{ x_n \}$ which is convergent and therefore Cauchy. Let $\{ v_n \}$ be the corresponding subsequence of $\{ y_n \}$. Then $\{ v_n \}$ is also Cauchy and hence the sequence of points $P_n = (u_n, v_n)$ in S is Cauchy. But S is closed so P converges to some (u, v) in S. Hence $\{ v_n \}$ converges to v, which is in Y (it is the projection of (u, v)). But since $\{ y_n \}$ converges to y, its subsequence $\{ v_n \}$ must also converge to y. Hence y = v and y is in Y.

Problem B4
A vehicle covers a mile (= 5280 ft) in less than a minute, starting and ending at rest and never exceeding 90 miles/hour. Show that its acceleration or deceleration exceeded 6.6 ft/sec^2.

Solution
Plot velocity (v in ft/sec) against time (t in sec) .

The graph never gets above the line v = 132 (ft/sec =90 mph).

If the vehicle's acceleration never exceeds 6.6 ft/sec^2, then in particular, its acceleration in the first 20 seconds does not exceed 6.6 ft/sec and hence the velocity curve never gets above the line v = 6.6 t, which cuts the line v = 132 at t = 20.

Similarly, if the vehicle's deceleration never exceeds 6.6 ft/sec^2, then its deceleration in the last 20 seconds never exceeds 6.6 ft/sec, so the velocity curve it never gets above the line v = - 6.6 (t - 60), which represents constant deceleration at the maximum 6.6 ft/sec^2 from t = 40 until the finish at t = 60 when it is stationary. In fact the curve must lie strictly under this line, since the vehicle finishes in less than a minute.

Hence the area under the curve, which represents distance travelled, is less than the area of the quadrilateral bounded by t = 0, v = 6.6t, v = 132 and v = -6.6(t - 60), which is 1/2 20. 132 + 20.132 + 1/2 20.132 = 40.132 = 5280 = 1 mile. But we know that the distance travelled was 1 mile. Hence either it accelerated at more than 6.6 ft/sec^2 at some point in the first 20 seconds, or it decelerated at more than 6.6 ft/sec^2 at some point in the last 20 seconds.

Problem B5
$k_n(x) = -n$ on $(-\infty, -n]$, x on $[-n, n]$, and n on $[n, \infty)$. Prove that the (real valued) function f(x) is continuous iff all $k_n(f(x))$ are continuous.

Solution
This is apparently almost trivial.
The k_n are continuous, so if f is continuous, then so are all $k_n f$.

If f is not continuous, then there must be a point x_0 and an infinite sequence of points x_m in the interval $(x_0 - 1, x_0 + 1)$ but not equal to x_0, such that x_m tend to x_0, but $f(x_m)$ do not tend to $f(x_0)$. Take n sufficiently large that $(x_0 - 1, x_0 + 1)$ lies inside (-n, n). Then $k_n f(x_i) = f(x_i)$, so $k_n f$ is not continuous at x_0.

Problem B6
The quadrilateral Q contains a circle which touches each side. It has side lengths a, b, c, d and area $\sqrt{(abcd)}$. Prove it is cyclic.

Solution
It helps a lot to know the following result about the area of quadrilaterals: the area of a quadrilateral side a, b, c, d is maximised by making it cyclic, in which case its area is $\sqrt{((s - a)(s - b)(s - c)(s - d))}$ (*), where $2s = a + b + c + d$.

If we assume this result, then the problem is fairly easy. Assume that the sides are in the order a, b, c, d, so that the side length a is opposite that length c. If Q contains a circle C which touches each side, then $a + c = b + d$. [The distances from a vertex or the quadrilateral to the two points of contact of its two sides are equal. So for some w, x, y, z we have $a = w + x$, $b = x + y$, $c = y + z$, $d = z + w$ and hence $a + c = b + d$.]

But this means that $s - a = (- a + c + b + d)/2 = (-a + c + a + c)/2 = c$. Similarly, $s - b = d$, $s - c = a$ and $s - d = b$, so the maximum possible area of the quadrilateral is $\sqrt{(abcd)}$, with equality iff it is cyclic. But we are given that the area is $\sqrt{(abcd)}$, so it must be cyclic.

To establish the result about quadrilaterals, drop the assumption that $a + c = b + d$, and let the angle between a and b be θ and the opposite angle be φ. Let the area be A. Then $2A = ab \sin \theta + cd \sin \varphi$, so $16A^2 = 4(a^2b^2 \sin^2\theta + c^2d^2 \sin^2\varphi) + 8abcd \sin \theta \sin \varphi$ (*).

By the cosine rule we have $a^2 + b^2 - 2ab \cos \theta = c^2 + d^2 - 2cd \cos \varphi$, so $(a^2 + b^2 - c^2 - d^2)^2 = 4(a^2b^2 \cos^2\theta + c^2d^2 \cos^2\varphi) - 8abcd \cos \theta \cos \varphi$. Adding to (*) gives $16A^2 = 4(a^2b^2 + c^2d^2) - (a^2 + b^2 - c^2 - d^2)^2 - 8abcd \cos(\theta + \varphi) \leq 4(a^2b^2 + c^2d^2) - (a^2 + b^2 - c^2 - d^2)^2 + 8abcd$ (**), with equality iff $\cos(\theta + \varphi) = -1$, in other words, iff the quadrilateral is cyclic.

But we can easily check that $(-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d) = 4(a^2b^2 + c^2d^2) - (a^2 + b^2 - c^2 - d^2)^2 + 8abcd$.

32nd Putnam 1971

Problem A1
Given any 9 lattice points in space, show that we can find two which have a lattice point on the interior of the segment joining them.

Solution
We can divide the points into 8 categories according to the parity of each coordinate. There must be at least 2 points in the same category. The midpoint of the line joining them is then also a lattice point.

Problem A2
Find all possible polynomials f(x) such that f(0) = 0 and $f(x^2 + 1) = f(x)^2 + 1$.

Solution
Answer: f(x) = x.
f(0) = 0. $f(1) = f(0)^2 + 1 = 1$. $f(1^2 + 1) = f(1)^2 + 1 = 2$. Similarly, by an easy induction we can get an arbitrarily large number of integers n for which f(n) = n. So if f has degree m, we can find at least m+1 integers on which it agrees with the polynomial p(x) = x. Hence it is identically equal to p.

Problem A3
The vertices of a triangle are lattice points in the plane. Show that the diameter of its circumcircle does not exceed the product of its side lengths.

Solution
Let the side lengths be a, b, c and the circumradius R. Let θ be the angle opposite side a. Then the area of the triangle A = 1/2 bc sin θ. The side a subtends an angle 2θ at the centre of the circumcircle, so a = 2R sin θ. Hence 2A = abc/(2R). So we have to show that A ≥ 1/2.

This follows at once from the well-known Pick's theorem: the area of any (non-self-intersecting) polygon whose vertices are lattice points is v/2 + i - 1, where v is the number of lattice points on the perimeter and i is the number of lattice points inside the polygon (so since for a triangle v ≥ 3, and i ≥ 0, we have area at least 3/2 - 1 = 1/2).

Problem A4
k lies in the open interval (1, 2). Show that the polynomial formed by expanding $(x + y)^n(x^2 - k\,xy + y^2)$ has positive coefficients for sufficiently large n. Find the smallest such n for k = 1.998.

Solution
Answer: 1999.

The coefficient of $x^{n-r}y^{r+2}$ is (nCr - k nCr+1 + nCr+2). We show that the worst case is r near n/2 and that even this is positive for sufficiently large n.
(nCr - k nCr+1 + nCr+2) = n!/(r+2! n-r!) ((r+1)(r+2) - k(r+2)(n-r) + (n-r-1)(n-r)). Let f(r) = ((r+1)(r+2) - k(r+2)(n-r) + (n-r-1)(n-r)). Then f '(r) = 2r+3 - kn +2kr + 2k - 2n + 2r + 1. So f '(r) = 0 iff (2k+4)r = (k+2)n - (2k+4) or r = n/2 - 1. Also it is clear that f '(r) is negative for smaller values and positive for larger values, so this represents a minimum. So if n = 2m the minimum occurs at r = m - 1 and is 2mCm-1 - k 2mCm + 2mCm+1. Now 2mCm-1 = 2mCm+1 = m/(m+1) 2mCm, so the minimum value is positive iff 2m/(m+1) > k which is certainly true for all sufficiently large m since k < 2.

Similarly if n = 2m+1, then the minimum occurs at r = m-1 or m. In either case, the minimum value is 2m+1Cm-1 - k 2m+1Cm + 2m+1Cm+1. But 2m+1Cm = 2m+1Cm+1 and 2m+1Cm-1 = m/(m+2) 2m+1Cm, so the minimum value is (m/(m+2) - (k-1)) 2m+1Cm, which is certainly positive for all sufficiently large m since k-1 < 1.

We have 2m/(m+1) > 1.998 for m > 999, so the smallest even n with all coefficients positive is 2000. On the other hand, m/(m+2) > 0.998 for m > 998, so the smallest odd n with all coefficients positive is 1999. Thus the smallest n is 1999.

Problem A5
A player scores either A or B at each turn, where A and B are unequal positive integers. He notices that his cumulative score can take any positive integer value except for those in a finite set S, where |S| =35, and 58 ∈ S. Find A and B.

Solution
Answer: A, B = 8, 11.

Let us call a number *green* if it can be expressed as a non-negative multiple of A plus a non-negative multiple of B. Assume A > B. A and B must be coprime. Otherwise there would be infinitely many non-green numbers. The set of integers 0, A, 2A, ... , (B-1)A are all incongruent mod B (since A and B are coprime), so they form a complete set of residues mod B. Hence any integer ≥ (B-1)A can be expressed as the sum of a multiple of B and one of the members of the set and is thus green. In fact, the none of the numbers (B-1)A - B + 1, (B-1)A - B + 2, ... , (B-A)A - 1is a multiple of A and they are are incongruent to (B-1)A mod B, so they must equal one of the numbers 0, A, 2A, ... , (B-2)A plus a multiple of B. Thus any integer greater than (B-1)A - B is green. On the other hand, (B-1)A - B itself cannot because it is (B-1)A mod B, so it is incongruent to kA mod B for k < (B-1) (and it cannot be (B-1)A plus a multiple of B because it is too small).

So we need to look at the numbers ≤ AB - A - B. Exactly [A/B] of them are congruent to A mod B, but are not green. For if A = [A/B] B + r (with 0 < r < B), then they are r, r + B, r + 2B, ... , r + ([A/B] - 1)B. Similarly, [2A/B] are congruent to 2A mod B, but not green. So the total number of non-green numbers is: [A/B] + [2A/B] + ... + [(B-1)A/B] (*). But kA/B cannot be integral for k = 0, 1, ... , (B-1) since A and B are coprime, so [A/B] + [(B-1)A/B] = A-1, [2A/B] + [(B-2)A/B] = A-1, ... If B - 1 is even, then this establishes that (*) is 1/2 (A-1)(B-1). If B - 1 is odd, then the central term is [A/2]. B is even, so A must be odd, so [A/2] = 1/2 (A - 1). So in this case also (*) is 1/2 (A-1)(B-1).

So we have (A-1)(B-1) = 2·35 = 70 = 2·5·7. So A = 11, B = 8, or A = 15, B = 6, or A = 36, B = 3. But the last two cases are ruled out because they do not have A, B coprime. So we have A = 11, B = 8. We can easily check that 58 is not green in this case.

Problem A6
α is a real number such that 1^{α}, 2^{α}, 3^{α}, ... are all integers. Show that α ≥ 0 and that α is an integer.

Solution
Note first that α is obviously not negative, because then n^{α} would be between -1 and 0 for sufficiently large n.

It is also fairly obvious that α cannot be rational (and non-integral). For if $2^{m/n} = A$, then $2^m = A^n$, so A must be a power of 2, but then m is a multiple of n. However, that line of approach gets us nowhere with irrational values.
The hint is to use some sort of mean value theorem. The idea is that this gives us another expression

for the difference between the value of f(x) at two points. We might then hope to show that the expression was non-integral, but the difference integral.

In principle, we have two choices for f(x): x^{α} or n^{x}. The latter does not seem promising because the derivative is $n^{x} \log n$. If x is allowed to range over some interval (as in the MVT) then we cannot be sure whether it is integral or non-integral. It is certainly not small.

On the other hand, the former is not immediately promising either. The derivative is $\alpha x^{\alpha-1}$ (*). But suppose $\alpha < 1$. Then $\alpha - 1 < 0$, so for sufficiently large x the derivative will be less than 1. It will still be positive, so it will be between 0 and 1 and hence non-integral. The MVT tells us that $f(n+1) - f(n) = f'(\xi)$ for some ξ between n and n+1, so if n is suffiently large f(n+1) - f(n) is an integer, but $f'(\xi)$ is not. Contradiction. So we can rule out $0 < \alpha < 1$.

In principle, we could get a contradiction for larger α if we could use a larger derivative. We have $f^{(n)}(x) = \alpha(\alpha-1) \ldots (\alpha-k+1) x^{\alpha-k}$, which will be between 0 and 1 provided that we choose k so that $k-1 < \alpha < k$ and take x sufficiently large.

So we need a MVT which involves $f^{(n)}(\xi)$ plus terms which are all integral. The usual generalisation is the Taylor series with remainder. That does not help, because the other terms are mainly non-integral - they involve the lower derivatives which all have a factor α.

However, there is another generalisation. Define $\Delta f(x) = f(x+1) - f(x)$. We can iterate D, so that $D^2 f(x) = \Delta f(x+1) - \Delta f(x) = f(x+2) - 2f(x+1) - f(x)$. We claim that $\Delta^2 f(x) = f^{(2)}(\xi)$ for some ξ in [x, x+2]. The proof is almost immediate by applying the MVT to $\Delta f(x)$. [Let $g(x) = \Delta f(x)$. Then by the MVT $\Delta g(x) = g'(\zeta)$ for some ζ in [x, x+1]. But $g'(\zeta) = f'(\zeta+1) - f'(\zeta)$. By the MVT this is $f''(\xi)$ for some ξ in $[\zeta, \zeta+1]$ and hence in [x, x+2].] By a simple induction, we can also show that $\Delta^n f(x) = f^{(n)}(\xi)$ for some ξ in [x, x+n].

This is the generalisation we need because $\Delta^k f(n)$ is an integral combination of values of f at integer points and hence integral. Whereas for suitable n and k we showed above that $f^{(k)}(n)$ is non-integral.

Problem B1
S is a set with a binary operation * such that (1) $a * a = a$ for all $a \in S$, and (2) $(a * b) * c = (b * c) * a$ for all $a, b, c \in S$. Show that * is associative and commutative.

Solution
b*a = (b*a)(b*a) using (1)
= ((b*a)*b)*a using (2) twice
But (b*a)*b = (b*b)*a using (2) twice
= b*a using (1)
So b*a = (b*a)*a
= (a*a)*b using (2) twice
= a*b
That shows that * is commutative. (2) now gives immediately that * is associative, because (a*b)*c = (b*c)*a = a*(b*c).

Problem B2
Let X be the set of all reals except 0 and 1. Find all real valued functions f(x) on X which satisfy $f(x) + f(1 - 1/x) = 1 + x$ for all x in X.

Solution
The trick is that $x \to 1 - 1/x \to 1/(1-x) \to x$. Thus we have:

$f(x) + f(1 - 1/x) = 1 + x$ (1);
$f(1 - 1/x) + f(1/(1-x)) = 2 - 1/x$ (2);
$f(1/(1-x)) + f(x) = 1 + 1/(1-x)$ (3).
Now (1) - (2) + (3) gives $2 f(x) = x + 1/x + 1/(1-x)$ or $f(x) = (x^3 - x^2 - 1) / (2x^2 - 2x)$. It is easily checked that this does indeed satisfy the relation in the question.

Problem B3
Car A starts at time $t = 0$ and, traveling at a constant speed, completes 1 lap every hour. Car B starts at time $t = \alpha > 0$ and also completes 1 lap every hour, traveling at a constant speed. Let $a(t)$ be the number of laps completed by A at time t, so that $a(t) = [t]$. Similarly, let $b(t)$ be the number of laps completed by B at time t. Let $S = \{t \geq \alpha : a(t) = 2\, b(t)\}$. Show that S is made up of intervals of total length 1.

Solution
Answer: $S = [\, [\alpha]+\alpha, 2[\alpha]+1 \,) \cup [\, 2[\alpha]+2, [\alpha]+\alpha+2 \,)$.

At all times in the interval $[\, [\alpha]+\alpha, 2[\alpha]+1 \,)$ car A has completed $2[\alpha]$ laps and car B has completed $[\alpha]$ laps. At all times in the interval $[\, 2[\alpha]+2, [\alpha]+\alpha+2 \,)$ car A has completed $2[\alpha]+2$ laps and car B has completed $[\alpha]+1$ laps. The first interval has length $1-\{\alpha\}$ and the second interval has length $\{\alpha\}$, where $\{\alpha\}$ denotes the fractional part of α.

During the interval $[\, 2[\alpha]-1, 2[\alpha] \,)$ car A has completed $2[\alpha]-1$ laps and car B either $[\alpha]-2$ or $[\alpha]-1$ laps, so car A has completed more than twice as many laps as car B. Similarly, during the interval $[\, 2[\alpha]-n, 2[\alpha]-n+1)$ car A has completed $2[\alpha]-n$ laps and car B $[\alpha]-n-1$ or $[\alpha]-n$ laps, so at all times before $2[\alpha]$ car A has completed more than twice as many laps as car B and these times do not form part of S.

During $[\, 2\alpha]+1, [\alpha]+\alpha+1 \,)$ car A has completed $2[\alpha]+1$ laps and car B $[\alpha]$ laps, which is less than half the number. During $[\, [\alpha]+\alpha+1, 2[\alpha]+2 \,)$ car A has completed $2[\alpha]+1$ laps and car B $[\alpha]+1$ laps, which is more than half the number. So no points in the gap between the two intervals of S belong in S.

Finally, during $[\, [\alpha]+\alpha+2, 2[\alpha]+3 \,)$ car A has completed $2[\alpha]+2$ laps and car B $[\alpha]+2$ laps, whilst during $[\, 2[\alpha]+n, 2[\alpha]+n+1 \,)$ car A has completed $2[\alpha]+n$ laps and car B either $[\alpha]+n-1$ or $[\alpha]+n$ laps. So at all times after $[\alpha]+\alpha+2$ car A has completed less than twice the number of laps completed by car B and none of these times belong in S.

Problem B4
A and B are two points on a sphere. S(A, B, k) is defined to be the set $\{P : AP + BP = k\}$, where XY denotes the great-circle distance between points X and Y on the sphere. Determine all sets S(A, B, k) which are circles.

Solution
Answer: S(A, B, k) is a circle iff $k = \pi$ and A and B are not antipodal, in which case S(A, B, k) is the great circle perpendicular to the great circle through A and B and so that A and B are on the same side of and equidistant from the plane containing it.

Wlog we may take the sphere to have radius 1. Let the great-circle distance between A and B be d. So $0 \leq d \leq \pi$. If $d = 0$, then clearly any k (in the range $0 < k < \pi$) gives a circle. If $d = \pi$, then any point P on the sphere has $PA + PB = \pi$, so S is either empty or the whole sphere. So let us assume that $0 < d < \pi$.

Let C be the great circle through A and B. Let O be the centre. Let QR be the diameter with angle AOQ = angle BOR. Let B' be the reflection of B in QR. Then AB' is a diameter. Let C' be the great circle through Q and R perpendicular to C. For any point P on the sphere we have AP + PB' = π. If P lies on C', then by symmetry BP = B'P, so AP + PB = π. If P lies in the open hemisphere containing B, then BP < B'P, so AP + BP < π. Similarly, if P lies in the open hemisphere containing B', then BP > B'P, so AP + BP > π. Thus S(A, B, π) is the circle C'.

Let us now assume 0 < k < π. If k < d, then S(A, B, k) is empty, since AP + BP cannot be less than AB. If k = d, then S(A, B, k) is the arc of C between A and B. So suppose d < k < π. Then there are two points on C' in S(A, B, k): Q' on the arc AQ and R' on the arc BR, with AQ' = BR'. Now S(A, B, k) must be symmetrical about the plane containing C', so if it is a circle C'', then it must be the circle diameter Q'R' and perpendicular to C'. It suffices to find a single point X on this circle with AX + BX > k. Let Y be the midpoint of the arc AB, and take X so that the great circle through X and Y is perpendicular to C'. Since we are assuming C'' is a circle we have YQ' = YX = YR'. Since the great circle through X and Y is perpendicular to C'', we must have AX > YX and BX > YX, so AX + BX > 2 YX = 2 YQ' = AQ' + BQ' = k, so Y is not in S(A, B, k) after all.

Finally consider k > π. Let A' be the antipodal point to B. Now for any point P on the sphere, AP = π - B'P and BP = π - A'P, so S(A, B, k) = S(A', B', 2π-k).

Problem B5
A *hypocycloid* is the path traced out by a point on the circumference of a circle rolling around the inside circumference of a larger fixed circle. Show that the plots in the (x, y) plane of the solutions (x(t), y(t)) of the differential equations x'' + y' + 6x = 0, y'' - x' + 6y = 0 with initial conditions x'(0) = y'(0) = 0 are hypocycloids. Find the possible radii of the circles.

Solution
Answer: if we take the radius of the rolling circle to be R, then the radius of the fixed circle is 5/2 R or 5/3 R.

Take x, y coordinates and let the fixed circle have centre at the origin O and radius cR (with c > 1). Take the rolling circle to have centre C, radius R and to be initially touching the fixed circle at x = cR, y = 0. Take this point to be the point P on the rolling circle whose motion we track. After time t, let OC make an angle θ with the x-axis. Then the rolling circle will have rolled through an angle cθ. We assume that the motion is uniform so that θ = bt.

At time t, the coordinates of C will be x = aR cos bt, y = aR sin bt, where a = c - 1. CP makes an angle cθ - θ = aθ with the x-axis (in the opposite sense to θ), so P has coordinates: x = aR cos bt + R cos abt, y = aR sin bt - R sin abt (*).
However, we could also take the y-axis in the opposite direction, in which case the equations would become: x = aR cos bt + R cos abt, y = -aR sin bt + R sin abt (**).

Turning to the differential equations given, take the first + i times the second and put z = x + i y. Then we get: z'' - i z + 6z = 0. This has general solution $z = A e^{3it} + B e^{-2it}$, or x = A cos 3t + B cos 2t, y = A sin 3t - B sin 2t. But we are told that x'(0) = y'(0) = 0, so 3A = 2B. Hence x = 2/3 B cos 3t + B cos(2/3 3t), y = 2/3 B sin 3t - B sin(2/3 3t). Comparing with (*) we see that it represents a hypocycloid with c = 5/3.

Alternatively, we can cast the solution into the form (**) by writing it as: x = 3/2 A cos 2t + A cos(3/2 2t), y = -3/2 A sin 2t + A sin(3/2 2t). Comparing with (**) we see that this represents a hypocycloid with c = 3/2 + 1 = 5/2.

Problem B6
$|f(1)/1 + f(2)/2 + ... + f(n)/n - 2n/3| < 1$, where f(n) is the largest odd divisor of n.

Solution
If n is odd, then f(n)/n = 1. If n is even, then f(n) = f(n/2). If n = 2m, then f(1)/1 + f(3)/3 + ... + f(2m-1)/(2m-1) = m, and f(2)/2 + f(4)/4 + ... + f(2m)/(2m) = 1/2 (f(1)/1 + f(2)/2 + ... + f(m)/m). So if we put g(n) = f(1)/1 + f(2)/2 + ... + f(n)/n, then we have the relations: g(2n+1) = g(2n) + 1 (*), and g(2n) = n + g(n)/2 (**). That allows us to establish by induction that $2n/3 < g(n) < 2n/3 + 5/6$ for n odd and $2n/3 < g(n) < 2n/3 + 1/2$ for n even.

For g(1) = 1 = 2/3 + 1/3, g(2) = 3/2 = 2/3 x 2 + 1/6, so the result is true for n = 1, 2. Suppose it is true for n < 2m. Then g(2m) = m + g(m)/2 > m + m/3 = 2/3 (2m), and g(2m) < m + m/3 + 1/2 (5/6) < 2/3 (2m) + 1/2. Also g(2m+1) = g(2m) + 1 > 2/3 (2m) + 2/3 = 2/3 (2m+1), and g(2m+1) < 2/3 (2m) + 1/2 + 1 = 2/3 (2m+1) + 5/6.

33rd Putnam 1972

Problem A1
Show that we cannot have 4 binomial coefficients nCm, nC(m+1), nC(m+2), nC(m+3) with n, m > 0 (and m + 3 ≤ n) in arithmetic progression.

Solution
Note that we can have three, for example: 7C1 = 7, 7C2 = 21, 7C3 = 35.
a, b, c are in arithmetic progression iff 2b = a + c. We have nCm = (m+1)/(n - m) nC(m+1), so 2 nCm+1 = nCm + nC(m+2) iff 2 = (n - m - 1)/(m + 2) + (m + 1)/(n - m). Simplifying, this becomes: m^2 - (n - 2)m + (n^2 - 5n + 2)/4 = 0. For any given n, this is a quadratic in m, so it has at most two solutions. If we have 4 consecutive coefficients in arithmetic progression, then the two solutions must differ by 1. If they are m_1 and m_2, then (m_1 - m_2) = 1 implies (m_1 - m_2)^2 = 1 and hence (m_1 + m_2)^2 - 4m_1m_2 = 1. So (n - 2)^2 - (n^2 - 5n + 2) = 1, and hence n = -1. Hence for n > 0 we cannot have four coefficients in arithmetic progression.

Alternatively, because nCm = nC(n-m), m and m+1 must be centrally placed (in other words n = 2m+1). Otherwise we would have four distinct solutions to the quadratic, which is impossible. But (2m+1)C(m-1), (2m+1)Cm, (2m+1)C(m+1), (2m+1)C(m+2) cannot be a solution because both (2m+1)C(m-1) and (2m+1)C(m+2) are less than (2m+1)Cm.

Problem A2
Let S be a set with a binary operation * such that (1) a * (a * b) = b for all a, b ∈ S, (2) (a * b) * b = a for all a, b ∈ S. Show that * is commutative. Give an example for which S is not associative.

Solution
Consider (a*b)*((a*b)*b). One can view it as c * (c * b), so that it is b by (1). Or one can consider first (a*b)*b, which is a by (2), so that the expression is (a*b)*a. Hence b = (a*b)*a. Multiplying on the right by a, we get b*a = ((a*b)*a)*a, which is (a*b) by (2). That proves * is commutative.

Take S to have three elements a, b, c. Let a*a = a, b*b = b, c*c = c, a*b = c, b*c = a, c*a = b and assume * is commutative. Then we easily check that the required conditions are met. But c = a*b = (a*a)*b, whereas a*(a*b) = a*c = b. So * is not always associative.

Problem A3
A sequence { x_i } is said to have a Cesaro limit iff $\lim_{n\to\infty}(x_1 + x_2 + ... + x_n)/n$ exists. Find all (real-valued) functions f on the closed interval [0, 1] such that { $f(x_i)$ } has a Cesaro limit iff { x_i } has a Cesaro limit.

Solution
Answer: the linear functions f(x) = Ax + B where A is non-zero. [We need A non-zero, because otherwise *all* { $f(x_i)$ } would have a Cesaro limit.]

It is straightforward to show that these functions satisfy the condition.
Suppose {x_n} has a Cesaro limit. Then we can find a limit k so that for any ε > 0 we have $|(x_1 + ... + x_n)/n - k| < \varepsilon/A$ for all sufficiently large n. Let h = f(k), we wish to show that $|(f(x_1) + ... + f(x_n))/n - h| < \varepsilon$ for all sufficiently large n. But $|(f(x_1) + ... + f(x_n))/n - h| = |A(x_1 + ... + x_n)/n - Ak| < \varepsilon$.

Similarly, suppose { $f(x_n)$ } has a Cesaro limit. Then we can find a limit h so that for any ε > 0 we have $|(f(x_1) + ... + f(x_n))/n - h| < A\varepsilon$ for all sufficiently large n. Take k so that f(k) = h. We wish to show that $|(x_1 + ... + x_n)/n - k| < \varepsilon$ for all sufficiently large n. But again this is obvious since we have $|(f(x_1) + ... + f(x_n))/n - h| = |A(x_1 + ... + x_n)/n - Ak|$.

It is harder to show the converse - that any function satisfying the condition must be linear. Note first that it is not true for ordinary limits. If $\{ x_n \} \rightarrow k$, then $f(x_n) \rightarrow f(k)$ for any continuous f (and it is not hard to show the converse). So the functions satisfying the corresponding condition for ordinary limits are just the homeomorphisms. The ordinary limit condition is stronger than Cesaro-summability - it is easy to show that if $x_n \rightarrow k$, then $(x_1 + x_2 + ... + x_n)/n \rightarrow k$ also. But it is easy to find sequences which do not tend to a limit but do have a Cesaro limit. For example, $x_n = (-1)^n$. So we need to use members of this wider class of sequences in order to prove the result.

Given any k in the open interval (0, 1), take a sequence x_n of 0s and 1s whose average $s_n = (x_1 + ... + x_n)/n$ tends to k. Then the average $F_n = (f(x_1) + ... + f(x_n))/n$ tends to $(1 - k) f(0) + k f(1) = h$, since a fraction (1 - k) of the n terms $f(x_i)$ are f(0) and a fraction k are f(1).

Now define a new sequence y_n as follows. Take enough terms from the first sequence to bring $G_n = (g(y_1) + ... + g(y_n))/n$ within 1/2 of h. Then take enough terms k to bring G_n within 1/4 of f(k). Then (starting where we left off before) take enough terms from the first sequence to bring G_n within 1/8 of h, then enough ks to bring it within 1/16 of f(k) and so on. Thus we have a subsequence of the G_ns which tends to h and another which tends to f(k). But the average $(y_1 + ... + y_n)/n$ tends to k, so G_n must tend to a single limit. Hence f(k) = h, or f(k) = a + bk, where a = f(0), b = f(1) - f(0).

Problem A4
Show that a circle inscribed in a square has a larger perimeter than any other ellipse inscribed in the square.

Solution
The first step is to show that if an ellipse touches all four sides of a square then its axes must lie along the diagonals of the square (in other words, it is symmetrically placed).

Let the ellipse have centre O, semi-major axis a and semi-minor axis b. Let the line A through O contain the ellipse's major axis and the line B through O contain its minor axis. Let S be the square in which the ellipse is inscribed. Let k be the similarity operation which leaves A invariant and expands by a factor a/b perpendicular to A. (So given a general point P, with X be the foot of the perpendicular from P to A, P goes to the point P' on the ray XP such that XP'/XP = a/b.)

k takes the ellipse into the circle centre O, radius a. It also takes parallel lines to parallel lines and touching curves to touching curves, so it takes the square S to a parallelogram whose sides all touch the circle. But that means the parallelogram must be a rhombus (with equal sides). But a line segment making an angle θ to the line A is taken by k to a line segment whose length is longer by a factor $\sqrt{(1 + ((a/b)^2 - 1) \sin^2\theta)}$, which is a strictly monotonically increasing function of θ. Since the equal sides of the square are taken to the equal sides of the rhombus, they must make equal angles to the line A. Hence they must be at 45^o to it and hence the line A must lie along a diagonal of the square.

The second step is to show that the ellipse with the largest perimeter is the circle. Recall that the perimeter of an ellipse is not an elementary function (it is a complete elliptical integral of the second kind). So it is not clear how to attack the problem. An analytic approach could evidently get somewhat messy.
But to pursue that a little, the tangent at the point (X, Y) of the ellipse has equation $xX/a^2 + yY/b^2 = 1$. If this is at 45^o (with X and Y both positive) then we have $Xb^2 = Ya^2$. The corresponding tangent at (X, -Y) is will evidently meet it on the x-axis at the point $(\sqrt{(a^2 + b^2)}, 0)$, so the side of the square has length $\sqrt{(2a^2 + 2b^2)}$.

The perimeter is $4a \int_0^{\pi/2} \sqrt{(1 - c \cos^2\theta)}\, d\theta$, where $c = (1 - (b/a)^2)$ (*). The easiest way to see this is to use the parametric equation of the ellipse $x = a \cos \theta$, $y = b \sin \theta$. Then if s represents arc length and ' differentiation wrt θ, we have $s' = \sqrt{((x')^2 + (y')^2)}$ and (*) follows almost immediately.

Now $a/\sqrt{(a^2 + b^2)} = 1/\sqrt{(2 - c)}$, so we have to maximise $f(c) = 1/\sqrt{(2 - c)} \int_0^{\pi/2} \sqrt{(1 - c \cos^2\theta)}\, d\theta$ over the range [0, 1], with c = 0 corresponding to b = a or the circle.
Putting the factor $1/\sqrt{(2 - c)}$ inside the integral, the integrand becomes $I(c, \theta) = \sqrt{((1 - c \cos^2\theta) / (2 - c))}$. We note that at $\theta = \pi/4$ this is always $\sqrt{(1/2)}$ irrespective of c. But for $\theta < \pi/4$, the integrand is a decreasing function of c and for $\theta > \pi/4$ it is an increasing function of c. So we cannot simply argue that the integrand is maximised by taking c = 0 and hence the integral also.

However, a slight elaboration of that argument does work. We can split the integral at $\pi/4$. Then for the range $\pi/4$ to $\pi/2$ we can make the substituion $\varphi = \pi/2 - \theta$ to get back to an integral over 0 to $\pi/4$ with integrand $J(c, \varphi) = \sqrt{((1 - c \sin^2\varphi) / (2 - c))}$. So now we have to maximise the integral over 0 to $\pi/4$ of $I(c, \theta) + J(c, \theta)$. But now it is true that c = 0 maximises the integrand at every point of the range.

We just have to square twice. So $I(c, \theta) + J(c, \theta) \le \sqrt{2}$ is equivalent to $(1 - c \cos^2\theta) + (1 - c \sin^2\theta) + 2 \sqrt{((1 - c \cos^2\theta)(1 - c \sin^2\theta))} <= 2(2 - c)$, or $\sqrt{((1 - c \cos^2\theta)(1 - c \sin^2\theta))} <= 1 - c/2$. Squaring again, that is equivalent to $1 - c + c \sin^2\theta \cos^2\theta \le 1 - c + c^2/4$ or $c^2 \sin 2\theta \le c^2$. But the last relation is certainly true with equality iff c = 0 or $\theta = \pi/4$. Hence for the integral we have equality iff c = 0.

Problem A5
Show that n does not divide $2^n - 1$ for n > 1.

Solution
Suppose that n does divide $2^n - 1$. Then n must be odd. Let p be the smallest prime dividing n. Then $2^{p-1} = 1 \pmod p$. Let m be the smallest divisor of p - 1 such that $2^m = 1 \pmod p$. Since m is smaller than p it must be coprime to n, so $n = qm + r$ with $0 < r < m$. Hence $2^r = 1 \pmod p$. Contradiction.

Problem A6
f is an integrable real-valued function on the closed interval [0, 1] such that $\int_0^1 x^m f(x)\, dx = 0$ for m = 0, 1, 2, ... , n - 1, and 1 for m = n. Show that $|f(x)| \ge 2^n(n + 1)$ on a set of positive measure.

Solution
The trick is to look at $(x - 1/2)^n f(x)$. Using the relations given in the question, we have immediately that $\int_0^1 (x - 1/2)^n f(x)\, dx = 1$. But we can also show that it must be small unless f(x) is large. For if $|f(x)| < 2^n(n + 1)$ except possibly on a set of zero measure, then $| \int | < 2^n(n + 1) \int_0^1 |x - 1/2|^n\, dx = 2^{n+1}(n + 1) \int_0^{1/2} x^n\, dx = 2^{n+1} x^{n+1} |_0^{1/2} = 1$. Contradiction.

Problem B1
Let $\sum_0^\infty x^n(x - 1)^{2n} / n! = \sum_0^\infty a_n x^n$. Show that no three consecutive a_n are zero.

Solution
Let $p(x) = x(x - 1)^2$, and $f(x) = e^{p(x)}$. Then f(x) has the expansion given. Differentiating $f ' = e^{p(x)} p'(x) = f p'$ (*). Now since p is a polynomial of degree 3, we have $p^{(n)} = 0$ for n > = 4. So differentiating (*) n times we get: $f^{(n+1)} = nC0\, f^{(n)} p' + nC1\, f^{(n-1)} p'' + nC2\, f^{(n-2)} p'''$. So if $f^{(n)}(0) = f^{(n-1)}(0) = f^{(n-2)}(0)$, then $f^{(m)}(0) = 0$ for all $m \ge n - 2$. In other words if three consecutive a_n are zero then all subsequent a_n are zero, and hence f(x) is a finite polynomial, which is impossible.

Problem B2
A particle moves in a straight line with monotonically decreasing acceleration. It starts from rest

and has velocity v a distance d from the start. What is the maximum time it could have taken to travel the distance d?

Solution
Answer: 2d/v.
Plot velocity u(t) against time. We have u(T) = v. The area under the curve between t = 0 and t = T is the distance d. But since the acceleration is monotonically decreasing, the curve is concave and hence the area under it is at least the area of the triangle formed by joining the origin to the point t = T, u = v (other vertices t = 0, u = 0 and t = T, u = 0). Hence $d \geq 1/2\ vT$, so $T \leq 2d/v$. This is achieved by a particle moving with constant acceleration.

Problem B3
A group has elements g, h satisfying: $ghg = hg^2h$, $g^3 = 1$, $h^n = 1$ for some odd n. Prove h = 1.

Solution
It is hard to get started on this. Or rather, it took me a long time to find the right way to get started. With hindsight, the correct approach is probably to work systematically through a set of increasingly complex expressions, trying to simplify them more than one way in order to get a new relation. It would not take long to reach hg^2hg^2h.
$gh^2 = (ghg)g^2h = (hg^2h)g^2h = (hg^2)(hg^2h) = hg^2(ghg) = h^2g$. From this point it is easy.

$gh^2g^2 = h^2$. Hence $gh^{2n}g^2 = h^{2n}$. Choose n so that $h^{2n} = h$. Then $ghg^2 = h$, or gh = hg. So the given relation implies $h = h^2$. Hence h = 1.

Problem B4
Show that for n > 1 we can find a polynomial p(a, b, c) with integer coefficients such that $p(x^n, x^{n+1}, x + x^{n+2}) \equiv x$.

Solution
By playing with small n, we soon find a general pattern:
$x = (x + x^{n+2})(1 - x^{n+1} + x^{2(n+1)} - ... + (-1)^{n-2}x^{(n-2)(n+1)}) + (-1)^{n+1}(x^n)^n$.
The proof is immediate.

Problem B5
A, B, C and D are non-coplanar points. $\angle ABC = \angle ADC$ and $\angle BAD = \angle BCD$. Show that AB = CD and BC = AD.

Solution
It is worth looking first at the coplanar case. If ABCD is convex, then opposite angles are equal, so it is a parallelogram and opposite sides are equal. In this case the result is true. But if ABDC is convex, then $\angle ABC = \angle ADC$ implies that A, B, D, C lie on a circle and hence that $\angle BAD = \angle BCD$. In this case it is certainly not true that AB = CD or that BC = AD.

Applying the cosine formula to angles ABC and ADC gives: $(AB^2 + BC^2 - AC^2)/(2\ AB\ BC) = (AD^2 + CD^2 - AC^2)/(2\ AD\ CD)$. Hence $(AB\cdot CD - AD\cdot BC)(AB\cdot AD - BC\cdot CD) + AC^2(AB.BC - AD.CD) = 0$ (*). Similarly, applying it to the other two angles gives: $(BC^2 + CD^2 - BD^2)/(2\ BC\ CD) = (AB^2 + AD^2 - BD^2)/(2\ AB\cdot AD)$ and hence $(AD\cdot BC - AB\cdot CD)(AB\cdot BC - CD\cdot AD) + BD^2(BC\cdot CD - AB\cdot AD) = 0$ (**).

Put $x = (AB\cdot CD - AD\cdot BC)$, $y = (AB\cdot AD - BC\cdot CD)$, $z = (AB\cdot BC - AD\cdot CD)$. Then (*) and (**) become $xy + AC^2z = 0$, $xz + BD^2y = 0$. So if either of y or z is 0, then the other is also. But if y and z are 0, then AB/CD = BC/AD = AD/BC, so BC = AD and hence also AB = CD. On the other hand,

if neither y nor z is 0, then we can deduce that $AC^2BD^2 = x^2$, and hence $AC \cdot BD = \pm x = \pm(AB \cdot CD - AD \cdot BC)$. Thus either $AC \cdot BD + AD \cdot BC = AB \cdot CD$ or $AC \cdot BD + AB \cdot CD = AD \cdot BC$.

In either case we can use Ptolemy's theorem which tells us that A, B, C, D must be (1) coplanar, and (2) concyclic (or collinear). But we are told that they are not coplanar. Hence BC = AD and AB = CD as required.

Problem B6
The polynomial p(x) has all coefficients 0 or 1, and p(0) = 1. Show that if the complex number z is a root, then $|z| \geq (\sqrt{5} - 1)/2$.

Solution
Let $h = (\sqrt{5} - 1)/2$. h is approx 0. 618, so certainly if $|z| \geq 1$, then $|z| > h$. So it is sufficient to show that if z is a root with $|z| < 1$, then $|z| \geq h$.

For such values of z, $1 + z + z^2 + ...$ converges, so we have $2 + z + z^2 + z^3 + ... - 2\ p(z) = \pm z \pm z^2 \pm z^3 \pm ...$. Now $|rhs| \leq |z| + |z^2| + ... = |z|/(1 - |z|)$. If z is a root, then p(z) = 0, so $lhs = 2 + z + z^2 + ... = 2 + z/(1 - z)$. If we could show that $|lhs| \geq (2 + |z|)/(1 + |z|)$, then we would have $|z|/(1 - |z|) \geq (2 + |z|)/(1 + |z|)$ and hence $|z|^2 + |z| - 1 \geq 0$. But $|z| \geq 0$, so $|z| \geq h$.

We require $|2 - z|\ (1 + |z|) \geq |1 - z|\ (2 + |z|)$. Squaring and using the polar form $z = r\ e^{i\theta}$, this is equivalent to $(4 - 4r \cos \theta + r^2)(1 + r)^2 \geq (1 - 2r \cos \theta + r^2)\ (2 + r)^2$, or after some simplification, $2r\ (1 - r^2)(1 + \cos \theta) \geq 0$, which is true.

34th Putnam 1973

Problem A1
ABC is a triangle. P, Q, R are points on the sides BC, CA, AB. Show that one of the triangles AQR, BRP, CPQ has area no greater than PQR. If BP ≤ PC, CQ ≤ QA, AR ≤ RB, show that the area of PQR is at least 1/4 of the area of ABC.

Solution
It is convenient to do the second part first. Let L be the midpoint of BC, M of AC and N of AB. Then LN is parallel to AC, so the ray PR meets the ray CA. Hence M is at least as close to PR as Q. So area PQR ≥ area PMR. Similarly, the ray MP meets the ray AB, so N is at least as close to PM as R. Hence area PMR ≥ area PMN. Finally MN is parallel to BC, so area PMN = area LMN. But area LMN = 1/4 area ABC, so area PQR ≥ 1/4 area ABC.

For the first part, there are two cases. Either each of AMN, BLN, CLM includes just one of P, Q, R, or one includes two of them. In the first case, we have the situation just considered (possibly with some relabeling of P, Q, R) , so area PQR ≥ 1/4 area ABC. If all three of AQR, BRP, CPQ had areas > area PQR, then in total their area plus that of PQR would exceed the area of ABC. Contradiction. Hence one of AQR, BRP, CPQ has area not greater than PQR.

In the second case, suppose AR <= AN and AQ <= AM. Let AP cut RM at X. Then AX ≤ AP/2, since RM cuts AP closer to A than MN, which bisects it (or possibly at the same point if R=N, Q=M). Hence AX ≤ PX. But area AQR/ area PQR = AX/PX, so area AQR ≤ area PQR.

Problem A2
$a_n = \pm 1/n$ and $a_{n+8} > 0$ iff $a_n > 0$. Show that if four of $a_1, a_2, \ldots, a_8$ are positive, then $\sum a_n$ converges. Is the converse true?

Solution
Answer: yes.

Consider $a_n + a_{n+1} + \ldots + a_{n+7}$. It is $(\pm(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7) \pm n(n+2)(n+3)(n+4)(n+5)(n+6)(n+7) \pm \ldots \pm n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)) / (n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7))$. If there are an equal number of plus and minus signs, then there is no n^7 term in the numerator, so the expression is $O(1/n^2)$. On the other hand if there are an unequal number of plus and minus signs, then there is an n^7 term in the numerator and the expression is $O(1/n)$. But $\sum 1/n$ diverges, and $\sum 1/n^2$ converges.

Let s_n be the sum of the first n terms. We have shown that with an unequal number of positive and negative signs s_{8n} diverges and hence s_n does not converge. With an equal number of signs, s_{8n} converges. But each term tends to zero, so $|s_{8n+i} - s_{8n}|$ tends to zero for i = 1, ... 7. Hence s_n converges.

Problem A3
n is a positive integer. Prove that $[\sqrt{(4n + 1)}] = [\min (k + n/k)]$, where the minimum is taken over all positive integers k.

Solution
We evaluate each side. It is probably easiest to tackle $[\sqrt{(4n + 1)}]$ first. Let $f(n) = [\sqrt{(4n + 1)}]$. If $n \le m^2 - 1$, then $4n + 1 \le 4m^2 - 3 < (2m)^2$, so $f(n) < 2m$. If $n \ge m^2$, then $4n + 1 > 4m^2$, so $f(n) \ge 2m$. If $n < m(m+1)$, then $4n + 1 < 4m^2 + 4m + 1 = (2m+1)^2$, so $f(n) < 2m+1$. If $n \ge m(m+1)$, then $4n+1 \ge$

$(2m+1)^2$, so $f(n) \geq 2m+1$. Hence for $n = m^2, m^2 + 1, m^2 + 2, \ldots, m^2 + m - 1$, we have $f(n) = 2m$. For $n = m^2 + m, m^2 + m + 1, m^2 + m + 2, \ldots, m^2 + 2m$ we have $f(n) = 2m+1$.

Let $g(n)$ be the minimum value of $[k + n/k]$, where k is a positive integer. Consider $h(x) = x + n/x$, with x real and positive. $h'(x) <0, =0, >0$ according as $x <\sqrt{n}, =\sqrt{n}, >\sqrt{n}$. So for positve real x, the smallest value of $x + n/x$ occurs at $x = \sqrt{n}$ and the smallest value with x restricted to be an integer must occur at $[\sqrt{n}]$ or $[\sqrt{n}] + 1$.

For $n = m^2, m^2 + 1, m^2 + 2, \ldots, m^2 + m - 1$, we have $[\sqrt{n}] = m$. Also $0 < (m-1)/m < 1$, so $[m + n/m] = 2m$. Now $n > m^2 - 1$, so $n/(m+1) > m - 1$ and hence $[(m+1) + n/(m+1)] \geq [m+1 + m-1] = 2m$. So for these values of n we have $g(n) = 2m = f(n)$.

Now consider $n = m^2 + m, m^2 + m + 1, m^2 + m + 2, \ldots, m^2 + 2m$. For these values of n also we have $[\sqrt{n}] = m$. For all except the last value of n, we have $[m + n/m] = 2m+1$. For the last we have $[m + n/m] = 2m+2$. But for all of them we have $[(m+1) + n/(m+1)] = 2m+1$. Hence for these values of n also $g(n) = 2m+1 = f(n)$.

Problem A4
How many real roots does $2^x = 1 + x^2$ have?

Solution
Answer: 3. $x = 0$, $x = 1$ and a value just over 4.

Clearly there are no roots for negative x, since for such x, $2^x < 1$, whereas $1 + x^2 > 1$. There are certainly roots at $x = 0$ and 1. Also $2^4 < 4^2 + 1$, whereas $2^5 > 5^2 + 1$, so there is a root between 4 and 5. We have to show that there are no other roots. Put $f(x) = 2^x - x^2 - 1$. Then $f''(x) = (\ln 2)^2 2^x - 2$. This is strictly increasing with a single zero. $f'(0) > 0$, so $f'(x)$ starts positive, decreases through zero to a minimum, then increases through zero. So it has just two zeros. Hence $f(x)$ has at most three zeros, which we have already found.

Problem A5
An object's equations of motion are: $x' = yz$, $y' = zx$, $z' = xy$. Its coordinates at time $t = 0$ are (x_0, y_0, z_0). If two of these coordinates are zero, show that the object is stationary for all t. If $(x_0, y_0, z_0) = (1, 1, 0)$, show that at time t, $(x, y, z) = (\sec t, \sec t, \tan t)$. If $(x_0, y_0, z_0) = (1, 1, -1)$, show that at time t, $(x, y, z) = (1/(1 + t), 1/(1 + t), -1/(1 + t))$. If two of the coordinates x_0, y_0, z_0 are non-zero, show that the object's distance from the origin $d \to \infty$ at some finite time in the past or future.

Solution
You are meant to assume the usual uniqueness theorems. So the only non-trivial part of the question is the last. For the first three parts, one just has to verify that the three solutions given satisfy the equations.

We have $xx' = yy' = zz' = xyz$. So integrating $y^2 = x^2 + A$, $z^2 = x^2 + B$. We may assume that y_0 and z_0 are non-zero and that $|x_0| \leq |y_0|$ and $|z_0|$. So A and B are positive and $x' = ((x^2 + A) (x^2 + B))^{1/2}$. Suppose we take the positive square root so that x increases with time. Then $t = \int ((x^2 + A) (x^2 + B))^{1/2} dx$. But the integral converges as x tends to infinity, so x reaches infinity at a finite (future) time. If we had taken the negative root, then we conclude that x reaches infinity at a finite (past) time.

Problem A6
Show that there are no seven lines in the plane such that there are at least six points which lie on the intersection of just three lines and at least four points which lie on the intersection of just two lines.

Solution
From 7 lines we can choose just 21 pairs. An intersection of 3 lines accounts for 3 distinct pairs of lines, an intersection of 2 lines for 1 pair. Hence the configuration given would have at least 6 x 3 + 4 x 2 = 22 distinct pairs of lines.

Problem B1
S is a finite collection of integers, not necessarily distinct. If any element of S is removed, then the remaining integers can be divided into two collections with the same size and the same sum. Show that all elements of S are equal.

Solution
Note that we need the collections to be both the same size and the same sum. Otherwise we could take S = {1, 1, 1, 1, 3}. If we remove a 1 we have: 3 = 1 + 1 + 1, and if we remove the 3 we have: 1 + 1 = 1 + 1.

The key is to notice that if we transform each element x of S to ax + b, then the new elements still satisfy the property given. In particular, by taking b sufficiently large we may make each element of S positive. Let f(S) denote the sum of the elements of S. Let us say that a finite set has the property X if (1) all its elements are positive and integral, (2) it has the property given in the question, and (3) not all its elements are equal. So we have just established that if there is a set with the property given and not all elements equal, then there is a set with property X. We show that given such a set S we can find another such set T with $f(T) < f(S)$.

For each element x of S, f(S) - x is even, so all elements of S have the same parity. If they are all even, we may take T to be {x/2 for x in S}. If they are all odd, we may take T to be { (x + 1)/2 for x in S}. Either way, the transformed elements are all integral and positive and have the property given. Also unequal elements transform into unequal elements. So T has the property X. We always have $x/2 < x$, and we have $(x+1)/2 < x$ unless x = 1, in which case (x+1)/2 = x. The elements cannot all be 1 (because we are assuming they are not all equal). So at least one must be reduced by the transformation, and hence the $f(T) < f(S)$.

But this process could now be repeated indefinitely. That gives a contradiction, since positive integers cannot be indefinitely reduced and remain positive. Hence our assumption that all members were unequal must be wrong.

Problem B2
The real and imaginary parts of z are rational, and z has unit modulus. Show that $|z^{2n} - 1|$ is rational for any integer n.

Solution
We may put $z = \cos\theta + i \sin\theta$. Then $z^{2n} - 1 = (\cos 2n\theta - 1) + i \sin 2n\theta$, so $|z^{2n} - 1|^2 = 2 - 2 \cos 2n\theta = 4 \sin^2 n\theta$. Hence $|z^{2n} - 1| = 2 |\sin n\theta|$. But $\sin n\theta$ is the imaginary part of $(\cos\theta + i \sin\theta)^n$. If we expand by the binomial theorem we get a series of terms $\cos^r\theta \sin^s\theta$ with integer coefficients. Each term is rational since $\cos\theta$ and $\sin\theta$ are rational. Hence $\sin n\theta$ is rational.

Problem B3
The prime p has the property that $n^2 - n + p$ is prime for all positive integers less than p. Show that there is exactly one integer triple (a, b, c) such that $b^2 - 4ac = 1 - 4p$, $0 < a \le c$, $-a \le b < a$.

Solution
Answer: (1, -1, p) is the only solution.

$b^2 = 4ac + 1 - 4p$, so b^2 is odd and hence b is odd (but not necessarily positive). Put $b = 2n - 1$. Then $4n^2 - 4n + 1 = 4ac + 1 - 4p$, so $n^2 - n + p = ac$. That is promising, because if n was in the required range we could conclude that a or c = 1 and, since $0 < a \le c$, that a = 1.
We are given that $n^2 - n + p$ is prime for n = 1, 2, ... , p-1. But it is obviously also prime for n = 0 and for n = -1, -2, ... , -(p-2) since $(-m)^2 - (-m) = m^2 + m = (m+1)^2 - (m+1)$ (so truth for m+1 implies truth for -m). So the required range is $-(p-2) \le n \le (p-1)$ or $(-2p+3) \le b \le (2p - 3)$.

$ac \ge a^2 \ge b^2$, so $b^2 - 4ac \le -3b^2$. If $|b| \ge p$, then $-3b^2 \le -3p^2 < 1 - 4p$ (since $(3p-1)(p-1) > 0$). But we know that $b^2 - 4ac = 1 - 4p$, so certainly $|b| < p$. But we know b is odd, so $|b| <= p - 2$. That is sufficient to guarantee that b is in the required range since $(p - 2) < (2p - 3)$.
So we can conclude that a = 1. But now b is odd and satisfies $-1 \le b < 1$, so b must be -1. Finally $b^2 - 4ac = 1 - 4p$ gives c = p.

Problem B4
f is defined on the closed interval [0, 1], f(0) = 0, and f has a continuous derivative with values in (0, 1]. By considering the inverse f^{-1} or otherwise, show that $(\int_0^1 f(x)\,dx)^2 \ge \int_0^1 f(x)^3\,dx$. Give an example where we have equality.

Solution
We might suspect that there is nothing special about the upper limit 1. So put $g(y) = (\int_0^y f(x)\,dx)^2 - \int_0^y f(x)^3\,dx$. Then certainly g(0) = 0. So the obvious approach is to show that $g'(y) \ge 0$.
Differentiating, we get $2 f(y) \int_0^y f(x)\,dx - f(y)^3$. We know that $f(y) \ge 0$ (because it is 0 at y = 0 and has non-negative derivative). So we need to show that $2 \int_0^y f(x)\,dx - f(y)^2 \ge 0$. Certainly it is 0 at y = 0, so we try differentiating again, hoping that the derivative will be non-negative. Differentiating gives $2 f(y) - 2 f(y) f'(y)$. But that is non-negative, because $f(y) \ge 0$ and $f'(y) \le 1$.

It is easy to see that f(x) = x gives equality. [Careful, f(x) = 0 is not an acceptable solution because the derivative is not strictly positive.]

[The suggestion about f^{-1} was probably intended as follows. Let if y = f(x), let x = g(y). Then $\int f(x)\,dx = \int y\,dx/dy\,dy = \int y\,g'(y)\,dy$. Now think of the square as a double integral and use $g'(y) \ge 1$. But the straightforward solution above seems easier.]

Problem B5
If x is a solution of the quadratic $ax^2 + bx + c = 0$, show that, for any n, we can find polynomials p and q with rational coefficients such that $x = p(x^n, a, b, c) / q(x^n, a, b, c)$. Hence or otherwise find polynomials r, s with rational coefficients so that $x = r(x^3, x + 1/x) / s(x^3, x + 1/x)$.

Solution
The first part of this is somewhat confusing. Clearly we can find particular values of a, b, c for which we do not have $x = p(x^n, a, b, c) / q(x^n, a, b, c)$. For example, take a = 1, b = 0, c = -2. Then x is irrational, whereas any p and q would have rational values.

One can carry out a careful analysis of what values of a, b, c are allowed, but that does not seem to be what the question is after. It seems to want something much simpler.

An easy induction shows that $x^n = (x\,p_1(a, b, c) + p_2(a, b, c))/ p_3(a, b, c)$, where p_1, p_2 and p_3 are polynomials with integer coefficients. Just start with $x^2 = (- bx - c)/a$ and at each stage multiply through by x and substitute. Now p_1 cannot be identically zero because then both solutions to the original quadratic would have the same |x|, whereas it is easy to see that this is not true for most values of a, b, c. So we may divide by p_1 to get the required relation with $p(x^n, a, b, c) = x^n\,p_3(a, b, c) - p_2(a, b, c)$ and $q(x^n, a, ,b\,c) = p_1(a, b, c)$.

For the second part, it is simpler to start from scratch. $(x + 1/x)^2 = x^2 + 1/x^2 + 2$. So $x (x + 1/x)^2 = x^3 + 1/x + 2x$. Hence $x (x + 1/x)^2 - x = x^3 + (x + 1/x)$. That is essentially the relation we want: $x = (x^3 + y) / (y^2 - 1)$, where $y = x + 1/x$.

Problem B6
Show that $\sin^2 x \sin 2x$ has two maxima in the interval $[0, 2\pi]$, at $\pi/3$ and $4\pi/3$. Let $f(x)$ = the absolute value of $\sin^2 x \sin^3 4x \sin^3 8x \ldots \sin^3 2^{n-1}x \sin 2^n x$. Show that $f(\pi/3) >= f(x)$. Let $g(x) = \sin^2 x \sin^2 4x \sin^2 8x \ldots \sin^2 2^n x$. Show that $g(x) \leq 3^n/4^n$.

Solution
Let $h(x) = \sin^2 x \sin 2x$. Then $h(x) = 2 \sin^3 x \cos x$, and $h'(x) = 2 \sin^2 x (3 - 4 \sin^2 x)$. So $h'(x) = 0$ at $x = 0, \pi/3, 2\pi/3, 4\pi/3, 5\pi/3, 2\pi$ in the interval $[0, 2\pi]$. We easily see that $h(x)$ has maxima at $\pi/3$ and $4\pi/3$, minima at $2\pi/3$ and $5\pi/3$ and inflexions at $0, 2\pi$. At the inflexions $h(x) = 0$, at the other stationary values, $|h(x)| = (3\sqrt{3})/8$.

Notice that, for $n > 1$, $2^n\pi/3 = 2\pi/3$ or $4\pi/3 \pmod{2\pi}$. So at $x = 2^n\pi/3$, $|h(x)|$ attains its maximum value.

Let $f_n(x) = \sin^2 x \sin^3 4x \sin^3 8x \ldots \sin^3 2^{n-1}x \sin 2^n x$. Then $f_2(x) = h(x) h(2x)$ and in general $f_n(x) = f_{n-1}(x) h(2^{n-1}x)$, so that $f_n(x) = h(x) h(2x) h(4x) \ldots h(2^{n-1}x)$. At $x = \pi/3$ each of $|h(2^i x)|$ is maximised separately and hence $|f_n(x)|$ is maximised.

We have in fact that $|f_n(x)| \leq ((\sqrt{3})/2)^{3n}$. Hence also $|\sin x| |f_n(x)| |\sin^2 2^n x| \leq ((\sqrt{3})/2)^{3n}$. But $|\sin x| |f_n(x)| |\sin^2 2^n x| = g(x)^{3/2}$, so $|g(x)| \leq ((\sqrt{3})/2)^{2n} = (3/4)^n$, and hence $g(x) \leq (3/4)^n$ (g is certainly non-negative).

35th Putnam 1974

Problem A1
S is a subset of {1, 2, 3, ... , 16} which does not contain three integers which are relatively prime in pairs. How many elements can S have?

Solution
Answer: 11

{2, 4, 6, 8, 10, 12, 14, 16, 3, 9, 15} has 11 elements. Any three integers from it include two which are multiples of 2 or two which are multiples of 3, so it does not contain three integers which are relatively prime in pairs.
On the other hand, any subset of 12 elements must include at least 3 members of {1, 2, 3, 5, 7, 11, 13} and those 3 will be relatively prime in pairs.

Problem A2
C is a vertical circle fixed to a horizontal line. P is a fixed point outside the circle and above the horizontal line. For a point Q on the circle, $f(Q) \in (0, \infty]$ is the time taken for a particle to slide down the straight line from P to Q (under the influence of gravity). What point Q minimizes f(Q)?

Solution
Answer: Let X be the lowest point of C. Q is the (other) point at which PX intersects C.

Let k be the angle which PQ makes to the horizontal. Then the acceleration of the particle along PQ is g sin k. Hence the time squared is proportional to PQ/sin k. So we minimise the time by minimising PQ/sin k. Let PQ meet the circle again at R and let PT be tangent to the circle. Then $PQ \cdot PR = PT^2$, a constant. So we minimise the time by maximising PR sin k. But PR sin k is just the vertical distance of R below P. That is clearly maximised by taking R = X.

Problem A3
Which odd primes p can be written in the form $m^2 + 16n^2$? In the form $4m^2 + 4mn + 5n^2$,where m and n may be negative? [You may assume that p can be written in the form $m^2 + n^2$ iff p = 1 (mod 4).]

Solution
Answer: p = 1 (mod 8); p = 5 (mod 8).

(A) $m^2 + 16n^2$.
We show first that $m^2 + 16n^2 = 1 \pmod 8$. Clearly m is odd, otherwise $m^2 + 16n^2$ would be even. But odd squares are = 1 (mod 8) (because $(2N+1)^2 = 4N(N+1) + 1$ and N(N+1) is even). Hence $m^2 + 16n^2 = 1 \pmod 8$.

Conversely, suppose p = 1 (mod 8). Take, $p = M^2 + N^2$. Without loss of generality, M is odd and N even. Hence $N^2 = 0 \pmod 8$, so N is a multiple of 4.

(B) $4m^2 + 4mn + 5n^2$.
n must be odd (otherwise p would be even). But $4m^2 + 4mn + 5n^2 = (2m + n)^2 + (2n)^2$. The first term is congruent to 1 and the second to 4 mod 8. Hence p = 5 (mod 8).

Conversely, if $p = M^2 + N^2 = 5 \pmod 8$, then wlog M is odd and N even. Take N = 2n. Since $M^2 = 1 \pmod 8$, $N^2 = 4 \pmod 8$ and hence n is odd. So M + n is even and we may set m = (M - n)/2. Now M = 2m + n, so $M^2 = 4m^2 + 4mn + n^2$ and $p = 4m^2 + 4mn + 5n^2$, as required.

Problem A4
Find $1/2^{n-1} \sum_1^{[n/2]}$ (n - 2i) nCi, where nCi is the binomial coefficient.

Solution
Answer: n((n-1)C[n/2] - 1)/2^{n-1}

It suffices to show that $\sum_0^{[n/2]}$ (n - 2i) nCi = n ((n-1)C[n/2]).

We need two obvious facts: i nCi = n (n-1)C(i-1) (use factorials) and $\sum_0^{[n/2]}$ nCi = 2^{n-1} if n is odd, or 2^{n-1} + 1/2 nC[n/2] if n is even (use nCr = nC(n-r)).

So consider first n odd. $\sum_0^{[n/2]}$ (n - 2i) nCi = n 2^{n-1} - 2 $\sum_0^{[n/2]}$ i nCi = n 2^{n-1} - 2n $\sum_1^{[n/2]}$ (n-1)C(i-1). Taking out the factor n we have 2^{n-1} - 2 $\sum_0^{[n/2]-1}$ (n-1)Ci = 2^{n-1} - (2^{n-1} - (n-1)C[n/2]) = (n-1)C[n/2], as required.

Similarly for n even, we have $\sum_0^{[n/2]}$ (n - 2i) nCi = n (2^{n-1} + 1/2 nC[n/2]) - 2n $\sum_0^{[n/2]-1}$ (n-1)Ci = n/2 (nC[n/2]) = n! / ([n/2}! ([n/2]-1)!) = n (n-1)C[n/2], as required.

Problem A5
The parabola $y = x^2$ rolls around the fixed parabola $y = -x^2$. Find the locus of its focus (initially at x = 0, y = 1/4).

Solution
Answer: horizontal line y = 1/4.

The tangent to the fixed parabola at (t, $-t^2$) has gradient -2t, so its equation is $(y + t^2) = -2t(x - t)$. We wish to find the reflection of the focus of the fixed parabola in this line, because that is the locus of the rolling parabola when they touch at (t, $-t^2$). The line through the focus of the fixed parabola (0, -1/4) and its mirror image is perpendicular to the tangent and so has gradient 1/(2t). Hence its equation is (y + 1/4) = x/(2t). Hence it intersects the tangent at the point with $y + t^2 = -4t^2(y + 1/4) + 2t^2$ or $y(1 + 4t^2) = 0$, hence at the point (t/2, 0). So the mirror image is the point (t, 1/4). t can vary from -∞ to +∞, so the locus is the horizontal line through (0, 1/4).

Problem A6
Let f(n) be the degree of the lowest order polynomial p(x) with integer coefficients and leading coefficient 1, such that n divides p(m) for all integral m. Describe f(n). Evaluate f(1000000).

Solution
Answer: f(n) is smallest integer N such that n divides N! and f(1000000) = 25.

Put $p_N(x) = x(x + 1)(x + 2) ... (x + N - 1)$. Note first that $p_N(m) / N! = (m + N - 1)! / ((m - 1)! N!)$ which is a binomial coefficient and hence integral, so $p_N(m)$ is certainly divisible by N! for all positive integers m. If m = 0, -1, ... , -(N - 1), then $p_N(m)$ is zero, which can also be regarded as a multiple of N!. If m < -(N - 1), then we may put m = -m' - (N - 1) where m' is positive. Then $p_N(m) = (-1)^N p_N(m')$ which is divisible by N!. So certainly if we take N to be the smallest integer such that n divides N!, then we can find a polynomial with leading coefficient 1 (namely $p_N(x)$) such that n divides p(m) for all integral m.

Now suppose p(x) is any polynomial with this property. We may write $p(x) = p_M(x) + a_1 p_{M-1}(x) + ... + a_{M-1} p_1(x) + a_M$, where M is the degree of M and a_i are integers (just choose successively, a_1 to match the coefficient of x^{M-1}, then a_2 to match the coefficient of x^{M-2} and so on).

Taking x = n, we conclude that n must divide a_M, so there is another polynomial of the same degree, namely p(x) - a_M with the same property. In other words, we may take $a_M = 0$. Now take x = n - 1. Then $p_2(x)$, $p_3(x)$, ... are all divisible by n. So $a_{M-1}p_1(n - 1)$ must also be divisible by n. But n and n - 1 are relatively prime, so a_{M-1} must be divisible by n. Hence the polynomial $p_M(x) + a_1p_{M-1}(x) + ... + a_{M-2}p_2(x)$ has the same property.

This argument continues on until we conclude that $p_M(x)$ has the same property, but a little care is needed, because in general n and $p_i(n - i)$ are not relatively prime. However, their greatest common divisor is also the gcd of n and i! (just multiply out $p_i(n - i) = (n - 1)(n - 2)...(n - i)$ - all terms except the i! term have a factor n). But i! divides $p_i(m)$ for all m (proved above), so the fact than n divides $a_{M-i}p_i(n - i)$ allows us to conclude that n divides $a_{M-i}p_i(m)$ for all m and hence that we can drop the term.

So we have finally that $p_M(x)$ also has the property. In particular, n must divide $p_M(1) = M!$ so M must be at least as big as N, the smallest number with this property. That establishes that f(n) is just the smallest N such that n divides N!.

Finally we want $f(1000000) = f(2^6 5^6)$. Evidently 25 is the smallest N such that 5^6 divides N! (25! includes 5 x 10 x 15 x 20 x 25 which is just the 6 powers of 5 needed). Certainly at least 2^{12} divides 25! (which includes 12 even numbers in the product: 2, 4, 6, ... , 24), so f(1000000) = 25.

Problem B1
P, Q, R, S, T are points on a circle radius 1. How should they be placed to maximise the sum of the perimeter and the five diagonals?

Solution
Answer: equally spaced (at the vertices of a regular pentagon).

We consider separately the questions of maximising the perimeter and the sum of the diagonals. We show first that if X lies on the arc AB, then we maximise AX + BX by taking X as the midpoint of the arc.

Let O be the centre of the circle and let the angle subtended at O by the arc by 2k. If the radius is R and angle AOX is x, then $AX + BX = 2R \sin(x/2) + 2R \sin(k - x/2) = 2R ((1 - \cos k) \sin(x/2) + \sin k \cos(x/2)) = 2R(2 \sin^2(k/2) \sin(x/2) + 2 \sin(k/2) \cos(k/2) \cos(x/2)) = 4R \sin(k/2) \cos(k/2 - x/2)$, which is uniquely maximised by taking x = k.

It follows immediately that equal spacing maximises the perimeter, for if the points are not equally spaced, then some side (say QR) is not equal to its counter-clockwise neighbouring side (say PQ). But then by moving Q to the midpoint of the arc PR we increase the perimeter, so the arrangement was not maximal.

But a similar (albeit slightly more complicated) argument shows that equal spacing also maximises the sum of the diagonals. Notice first that the relation $AX + BX = 4R \sin(k/2) \cos(k/2 - x/2)$ shows that the sum AX + BX is strictly decreasing as X moves away from the midpoint until it hits one of the points A, B. So if X is not at the midpoint then *any non-zero* move towards the midpoint increases the sum AX + BX.

Label the points so that the diagonals are PQ, QR, RS, ST, TP (and the order of the points around the circle is P, R, T, Q, S, moving counter-clockwise, say). Then if PQ = QR and QR = RS and RS = ST and ST = TP and TP = PQ, it follows that all the diagonals are equal and the points equally spaced. So if the points are not equally spaced then, without loss of generality, PQ does not equal

QR. Now if no two points are coincident, it follows immediately that the arrangement is not maximal, because we can move Q towards the midpoint (of the arc PR on which it lies) and increase the sum.

However, if we allow coincident points, this argument fails, because if Q coincides with T (say) then it may be blocked by T from moving closer to the midpoint - if we move it beyond T, then QR ceases to be a diagonal and becomes a side.

Let us examine this case more closely. Since Q is blocked by T, we have that arc RT > arc PQ. Let QQ' be a diameter. Then P cannot coincide with Q' (since arc RT would then not exceed arc PQ). So if we move P closer to Q' we will increase QP + PT (= 2QP). If we move counter-clockwise around the circle from T we must reach P after Q' (otherwise arc PQ > arc Q'Q > arc RT, contradiction). So we can move P closer to Q' (and hence establish that the configuration is not maximal) unless it is blocked by R.

Now consider RS + ST. S is certainly not blocked from moving closer to the midpoint, because the two possible blocking points P and Q coincide with R and T. So if the configuration is to be maximal, then S must already be at the midpoint. Finally, consider QR + RS. The arc RT is greater than a semicircle so the midpoint must lie on it. P does not block because it lies on the other side. T does not block because it coincides with Q. So by moving R we can increase the sum, contradicting maximality.
Thus equal spacing maximises the perimeter and diagonal sum separately. So, a fortiori, it maximises their sum.

Problem B2
f(x) is a real valued function on the reals, and has a continuous derivative. $f'(x)^2 + f(x)^3 \to 0$ as $x \to \infty$. Show that f(x) and $f'(x) \to 0$ as $x \to \infty$.

Solution
The key to getting started is to notice that if f ' = 0 for arbitrarily large values of x then the result is certainly true. Suppose $f'(x_n) = 0$ and $x_n \to \infty$. Then since $f'^2 + f^3 \to 0$, we have $f(x_n) \to 0$. But f is monotonic on the interval $[x_n, x_{n+1}\}$ since its derivative does not change sign, hence $f \to 0$. Hence also $f' \to 0$. So we may assume that for sufficiently large x, f ' does not change sign.

Now suppose f tends to a limit as $x \to \infty$. Then f ' must also tend to a limit. If that limit is non-zero, then f increases or decreases faster than some non-constant linear function for sufficiently large x and so cannot tend to a limit. Hence f ' must tend to zero. Hence f also.

So we may assume that either (1) for sufficiently large x, f is strictly monotonic increasing and tends to infinity, or (2) for sufficiently large x, f is strictly monotonic decreasing and tends to minus infinity.

The first case is impossible, because then f^3 and hence also $f^3 + f'^2$ would tend to infinity.

Showing that the second case is impossible needs a little more work. Suppose that for $x \geq X$, we have $f(x) < -1$ and $1/4 > f(x)^3 + f'(x)^2 > -1/4$. Then $1/2\ f(x)^3 < -1/2$, so $-1/2\ f(x)^3 > 1/2$. Hence $f'(x)^2 + 1/2\ f(x)^3 > -1/4 - 1/2\ f(x)^3 > 1/4 > 0$. So $f'(x)^2 > -1/2\ f(x)^3$. f '(x) is negative, so $f'(x) < -1/2\ |f(x)|^{3/2}$ (*).

Now define g(x) to satisfy g(X) = -1, $g' = -1/2\ |g|^{3/2}$. Solving, we get $g(x) = (1 - (x - X)/4)^{-2}$ for x >= X. (*) shows that we must have $f(x) < g(x)$ for $x \geq X$. But $g(x) \to -\infty$ as $x \to 5X$, so f must be discontinuous on the interval (X, 6X). Contradiction.

Problem B3
Prove that $(\cos^{-1}(1/3))/\pi$ is irrational.

Solution
Let $x = \cos^{-1}(1/3)$. If $x = m/n\ \pi$ for some integers m, n, then $\cos nx = \cos m\pi = \pm 1$. But we show that $\cos nx$ cannot be ± 1. It follows that x/π must be irrational as required.

As usual, we have $\cos nx = nC0\ c^n - nC2\ c^{n-2}s^2 + nC4\ c^{n-4}s^4 - \ldots$, where $c = \cos x$, $s = \sin x$. We may put $s^2 = 1 - c^2$ to get $\cos nx$ = a polynomial of degree n in c with integer coefficients. The coefficient of $c^n = nC0 + nC2 + nC4 + \ldots = 2^{n-1}$. But $c = 1/3$, so $\cos nx = 2^{n-1}/3^n + k/3^{n-1} = (2^{n-1} + 3k)/3^n$ for some integer k. This must be in its lowest terms since 2^{n-1} is not divisible by 3. In particular, it cannot be ± 1.

[A variant on this is to consider $\cos(2^n x)$. By a simple induction using $\cos 2y = 2\cos^2 y - 1$, we show that $\cos(2^n x) = a_n/b_n$, where a_n is not a multiple of 3 and $b_n$is 3 to the power of 2^n. It follows that as n runs through the natural numbers, all the values $\cos(2^n x)$ are distinct. But if x/π was rational, there would only be finitely many distinct values.]

Problem B4
R is the reals. $f : R^2 \to R$ is such that $f_{x0} : R \to R$ defined by $f_{x0}(x) = f(x_0, x)$ is continuous for every x_0 and $g_{y0} : R \to R$ defined by $g_{y0}(x) = f(x, y_0)$ is continuous for every y_0. Show that there is a sequence of continuous functions $h_n : R^2 \to R$ which tend to f pointwise.

Solution
Let M_n be the set of vertical lines at $1/2_n$ spacing, including the y-axis, ie all lines $y = m/2^n$ for integral m. Define h_n to agree with f on M_n and to be horizontally interpolated elsewhere, so if $x = m/2^n + x'$, where $0 \le x' < 1/2^n$, then $h_n(x, y) = h_n(x, y) + (x'/2^n)(h_n(x + 1/2^n, y) - h_n(x, y))$.

We show first that h_n tends to f pointwise. Take any point P in R^2. f is continuous along the horizontal line L through P, so given any $\varepsilon > 0$, we may take $\delta > 0$ so that points on L within δ of P are taken by f to points within ε of f(P). Take n sufficiently large so that the there are adjacent lines of M_n flanking P and closer than δ to it. Let these lines cut L at Q and R. Then $h_n(P)$ lies between the values of f(Q) and f(R) and hence within ε of f(P).

It remains to show that h_n is continuous. This is slightly tricky to nail down rigorously.Take any point $P = (x_0, y_0)$ and any $\varepsilon > 0$. We need to show that h_n takes poins close to P to values close to $h_n(P)$. Let L be the horizontal line through P. Assume that P does not lie on a line of M_n, so take H to be the first line to the left of P, and K the first to the right of P. Let H meet L at Q, and K meet L at R. Note that the separation of Q and R is non-zero and fixed, so the values f(Q) and f(R) may differ by a large amount. But we may take $\delta > 0$ so that points on H within δ of Q are taken by f to values within $\varepsilon/2$ of f(Q) and points on K within δ of R are taken by f to values within $\varepsilon/2$ of f(R). We want to show that for sufficiently small $\delta' > 0$, h_n takes points inside the rectangle $|x - x_0| < \delta'$, $|y - y_0| < \delta$ to values within ε of $h_n(P)$.

It is now easiest to think geometrically. Let us retain the x-axis to represent values of x, but now use the y-axis to represent possible values of h_n. So at Q we have a vertical bar height ε centred on f(Q) and at R we have another vertical bar height ε centred on f(R). Joining the tops of these bars and joining the bottoms gives a parallelogram. A vertical line through this parallelogram at x gives the possible values of $h_n(x, y)$, where $|y - y_0| < \delta$ (because to get that value we must interpolate linearly between some point on the left-hand edge of the parallelogram and some point on the right-hand edge). Now suppose the top and bottom edges of the parallelogram make an angle θ with the y-axis. Then a thin vertical slice of the parallelogram centred on y_0 width δ' will project onto a segment

length $\varepsilon + \delta'\cot\theta$. By taking δ' sufficiently small we can make this less than 2ε. In other words the spread of values of $h_n(x, y)$ for $|x - x_0| < \delta'$ and $|y - y_0| < \delta$ is less than 2ε, which is the statement that h_n is continuous.

Finally, note that if P lies on a line of M_n, then we can deal separately with points to the left and right of P, using the same argument.

Problem B5
Let $f_n(x) = \sum_0^n x^i/i!$. Show that $f_n(n) > e^n/2$. [Assume $e^x - f_n(x) = 1/n! \int_0^x (x - t)^n e^t\, dt$, and $\int_0^\infty t^n e^{-t}\, dt = n!$]

Solution
Using the two relations in the square brackets, we find almost immediately that it suffices to show that $\int_0^n t^n e^{-t}\, dt < \int_n^\infty t^n e^{-t}\, dt$.

We notice that the integrand (which we will call g(t)) is positive for all real t and has a maximum at t = n (differentiate). It would certainly be sufficient to show that $g(n + x) > g(n - x)$ for x in the interval (0, n). This turns out to be true. Taking logs, we need to show that $n \ln(n + x) - x > n \ln(n - x) + x$. Putting $h(x) = n \ln(n + x) - n \ln(n - x) - 2x$, we need to show that $h(x) > 0$. Differentiating, $h'(x) = 2x^2/(n^2 - x^2) > 0$ for x in (0, n). But $h(0) = 0$, which gives the result.

Problem B6
Let S be a set with 1000 elements. Find a, b, c, the number of subsets R of S such that $|R| = 0, 1, 2 \pmod 3$ respectively. Find a, b, c if $|S| = 1001$.

Solution
Answer: $[2^{1000}/3]$, $[2^{1000}/3]$, $[2^{1000}/3] + 1$; $[2^{1001}/3] + 1$, $[2^{1001}/3]$, $[2^{1001}/3] + 1$.

Let $f_0(n)$, $f_1(n)$, $f_2(n)$ be the number of subsets of a set with n elements whose size is $= 0, 1, 2 \pmod 3$ respectively.

Let S be a set with n elements and X another element not in S, so that $S' = S \cup \{X\}$ has n+1 elements. Then the subsets of S' are just the subsets of S and the subsets of S with X adjoined. So $f_0(n+1) = f_0(n) + f_2(n)$. Similarly, $f_1(n+1) = f_1(n) + f_0(n)$ and $f_2(n+1) = f_2(n) + f_1(n)$.

These recurrence relations and the obvious initial values $f_0(1) = f_1(1) = 1$, $f_2(1) = 0$ are sufficient to determine f_0, f_1 and f_2. By looking at low values of n we quickly conjecture that $f_i(n) = [2^n/3]$ for $n = 2i+2, 2i+3, 2i+4 \pmod 6$ and $[2^n/3] + 1$ for $n = 2i-1, 2i, 2i+1 \pmod 6$. Proving that is an easy (but slightly monotonous) induction. The required answer then follows at once.

36th Putnam 1975

Problem A1
A triangular number is a positive integer of the form n(n + 1)/2. Show that m is a sum of two triangular numbers iff 4m + 1 is a sum of two squares.

Solution
$4 (a(a+1)/2 + b(b+1)/2) + 1 = 2a^2 + 2a + 2b^2 + 2b + 1 = (a - b)^2 + (a + b + 1)^2$.

If $A^2 + B^2 = 1 \pmod 4$, then one of A, B must be odd and the other even. Hence (A + B - 1) and (A - B - 1) are both even. Put C = (A + B - 1)/2, D = (A - B - 1)/2. Then $1/2\ C(C + 1) + 1/2\ D(D + 1) = 1/8 ((A + B - 1)(A + B + 1) + (A - B - 1)(A - B + 1)) = 1/8 ((A + B)^2 - 1 + (A - B)^2 - 1) = 1/4 (A^2 + B^2 - 1)$. So if $A^2 + B_2 = 4m + 1$, then $m = 1/2\ C(C + 1) + 1/2\ D(D + 1)$.

Problem A2
For what region of the real (a, b) plane, do both (possibly complex) roots of the polynomial $z^2 + az + b = 0$ satisfy $|z| < 1$?

Solution
Answer: the interior of the triangle with vertices (±2, 1), (0, -1).

Clearly we require $|a| \le 2$, $|b| \le 1$. But that is not sufficient. For example, if b = 0, then we require $|a| \le 1$.
We consider first the region where $a \ge 0$. If $b > a^2/4$, then the roots are $z = a/2 \pm i \sqrt{(b - a^2/4)}$, so these satisfy $|z| < 1$ iff $(a/2)^2 + (b - a^2/4) < 1$ or $b < 1$. This gives the quasi-triangular region bounded by the two lines a = 0 and b = 1 on two sides and the curve $4b = a^2$ on the third side.

If $b < a^2/4$, then the roots are $z = a/2 \pm \sqrt{(a^2/4 - b)}$. These satisfy $|z| < 1$ iff $a/2 + \sqrt{(a^2/4 - b)} < 1$ which gives $a - b < 1$. This gives the quasi-triangular region bounded by the lines a = 0 and a - b = 1 on two sides and the curve $4b = a^2$ on the third. But since the line a - b = 1 touches the curve $b = a^2/4$ at a = 2, the two regions fit together to give the interior of triangle with vertices (a, b) = (0, 1), (0, -1) and (2, 1).

The region for $a \le 0$ is evidently the mirror image in the b-axis, so we get in total the (interior of the) triangular region with vertices (2, 1), (-2, 1) and (0, -1).

Problem A3
Let $0 < \alpha < \beta < \gamma \in R$, the reals. Let $K = \{ (x, y, z) : x^\beta + y^\beta + z^\beta = 1$, and $x, y, z \ge 0\} \in R^3$. Define $f : K \to R$ by $f(x, y, z) = x^\alpha + y^\beta + z^\gamma$. At what points of K does f assume its maximum and minimum values?

Solution
Clearly for any (x, y, z) in K we have $x, y, z \le 1$. Hence $x^\alpha \ge x^\beta$. In fact $x^\alpha - x^\beta$ is zero at x = 0, increases to a maximum and then reduces to zero again at x = 1. Differentiating, we find the maximum is at $x = (\alpha/\beta)^{1/(\beta-\alpha)}$. Similarly $z^\gamma - z^\beta$ is non-negative, with a minimum at $z = (\beta/\gamma)^{1/(\gamma-\beta)}$.

To achieve a maximum for f(x, y, z), any non-zero value of z is unhelpful, whilst y is neutral, so the maximum is achieved at $x = (\alpha/\beta)^{1/(\beta-\alpha)}$, $y = (1 - (\alpha/\beta)^{\beta/(\beta-\alpha)})^{1\beta}$, z = 0. [The value of y is chosen so that $x^\beta + y^\beta = 1$.] Similarly, to achieve a minimum, we must take x = 0, $y = (1 - (\beta/g)^{\beta/(g-\beta)})^{1/\beta}$, $z = (\beta/\gamma)^{1/(\gamma-\beta)}$.

Problem A4
m > 1 is odd. Let n = 2m and $\theta = e^{2\pi i/n}$. Find a finite set of integers $\{a_i\}$ such that $\sum a_i \theta^i = 1/(1 - \theta)$.

Solution
We have $\theta^m = -1$. Since m is odd we have $0 = (\theta^m + 1) = (\theta + 1)(\theta^{m-1} - \theta^{m-2} + ... - \theta + 1)$. θ is not - 1, so $\theta^{m-1} - \theta^{m-2} + ... - \theta + 1 = 0$ (*).

Since m is odd, we may write (*) as: $1 - \theta(1 - \theta)(1 + \theta^2 + \theta^4 + ... + \theta^{m-3} = 0$, or $1/(1 - \theta) = \theta + \theta^3 + \theta^5 + ... + \theta^{m-2}$.

Problem A5
Let I be an interval and f(x) a continuous real-valued function on I. Let y_1 and y_2 be linearly independent solutions of y" = f(x) y, which take positive values on I. Show that from some positive constant k, $k \sqrt{(y_1 y_2)}$ is a solution of $y'' + 1/y^3 = f(x) y$.

Solution
Answer: $k = \sqrt{(2/d)}$ where d is the Wronksian.

The key is to show that $y_1y_2' - y_1'y_2$ is a constant. This expression is known as the Wronskian and it is well-known that it is non-zero iff y_1 and y_2 are linearly independent.
The derivative is $y_1y_2'' - y_1y_2'' = f\ y_1y_2 - f\ y_1y_2 = 0$, which shows that the expression is constant. Let its value be d.

Let us write $z = k \sqrt{(y_1y_2)}$, where k is a constant to be determined later. It is also convenient to put $c = k^2/2$, so that $z^2 = 2c\ y_1y_2$. Differentiating we get: $z\ z' = c(y_1y_2' + y_1'y_2)$ (*). Differentiating again: $z\ z'' + (z')^2 = c\ y_1y_2'' + c\ y_1''y_2 + 2c\ y_1'y_2' = 2c\ f\ y_1y_2 + 2c\ y_1'y_2' = f\ z^2 + 2c\ y_1'y_2'$.

Multiplying by z^2: $z^3z'' + (z\ z')^2 = f\ z^4 + 2z^2\ c\ y_1'y_2' = f\ z^4 + 4\ c^2\ y_1y_2y_1'y_2'$. Using (*), $z^3\ z'' + c^2(y_1y_2' + y_1'y_2)^2 = f\ z^4 + 4c^2\ y_1y_2y_1'y_2'$. Hence $z^3z'' + c^2(y_1y_2' - y_1'y_2)^2 = f\ z^4$, or $z^3\ z'' + c^2d^2 = f\ z^4$. So if we set c = 1/d, then $z'' + 1/z^3 = f\ z$.

Problem A6
Given three points in space forming an acute-angled triangle, show that we can find two further points such that no three of the five points are collinear and the line through any two is normal to the plane through the other three.

Solution
Let the points be A, B, C. Let the two additional points be D, E. There are three cases to consider:

(1) the plane ABC and the line DE; (2) the plane ADE and the line BC, and the two similar configurations; (3) the plane ABD and the line CE, and the five similar configurations.
Clearly (1) works provided the line DE is normal to ABC.

Now consider (2). Let the line DE intersect ABC at X. Let AX meet BC at Y. If BC is normal to ADE, then it is perpendicular to any line in ADE, so in particular it must be perpendicular to AY. In other words, AY must be an altitude of the triangle. But that is also sufficient. For we know that BC is perpendicular to DE (and if it is perpendicular to two non-parallel lines of the plane, then it must be normal to the plane). Similarly, X must lie on the other two altitudes. So this case works provided X is the orthocenter of ABC.

Take X as the origin. Let the vectors XA, XB, XC, XD, XE be **a**, **b**, **c**, **d**, k **d** respectively. A necessary and sufficient condition for AE to be normal to the plane BCD is that AE be

perpendicular to BC and BD. We already know that it is perpendicular to BC (we have just shown that BC is normal to the plane ADE which contains it). So a necessary and sufficient condition is that AE be perpendicular to BD, or in vector language, (k **d** - **a**).(**b** - **d**) = 0, or k $\mathbf{d}^2$ = - **a**.**b**. This fixes k. Note that k is positive (so that D and E are on the same side of ABD, since **a**.**b** is negative). Now since X is the orthocenter of ABC, we have **a**.**b** = **b**.**c** = **c**.**a**. So with this choice of k we also have k $\mathbf{d}^2$ = - **b**.**c** = - **c**.**a**. It is easily checked that this is sufficient for the other five configurations also.

Finally, note that since ABC is acute, X lies strictly inside the triangle and hence **a**.**b** is non-zero, so k is non-zero and hence E is not collinear with any two of A, B, C. The only remaining point to check on non-collinearity is that D and E do not coincide. But then we would have AD perpendicular to BD, which is clearly false since AX is perpendicular to BX and D lies above X.

Problem B1
Let G be the group { (m, n) **:** m, n are integers } with the operation (a, b) + (c, d) = (a + c, b + d). Let H be the smallest subgroup containing (3, 8), (4, -1) and (5, 4). Let H_{ab} be the smallest subgroup containing (0, a) and (1, b). Find a > 0 such that H_{ab} = H.

Solution
Answer: 7.

(3,8) = 3 (1,5) - (0,7), (4,-1) = 4 (1,5) - 3 (0,7), (5,4) = 5 (1,5) - 3 (0,7).
(1,5) = (5,4) - (4,-1), (0,7) = - 4 (3,8) - 7 (4,-1) + 8 (5,4).
This shows that { (3,8), (4,-1), (5,4) } and { (1,5), (0,7) } generate the same subgroups.

Problem B2
A *slab* is the set of points strictly between two parallel planes. Prove that a countable sequence of slabs, the sum of whose thicknesses converges, cannot fill space.

Solution
A slab thickness d intersects a sphere radius R in a volume less than $\pi R^2 d$. So the entire set of slabs fill less than $\pi R^2 D$ of the sphere, where D is the sum of their thicknesses. If we take R > D this is less than the volume of the sphere, so the slabs cannot even fill the sphere.

Problem B3
Let n be a fixed positive integer. Let S be any finite collection of at least n positive reals (not necessarily all distinct). Let $f(S) = (\sum_{a \in S} a)^n$, and let g(S) = the sum of all n-fold products of the elements of S (in other words, the nth symmetric function). Find $\sup_S g(S)/f(S)$.

Solution
Answer: 1/n!

For any n elements a, b, ... , w of S, the coefficient of $a^1 b^1 ... w^1$ in the multinomial expansion of f(s) is just n!/(1! 1! ... 1!) = n!. In other words, f(S) = n! g(S) + other terms. But the other terms are all positive, so f(S) > n! g(S). Hence g(S)/f(S) < 1/n! . This establishes that 1/n! is an upper bound.

Take S to be m elements all 1. Then $f(S) = m^n$ and g(S) = mCn. Hence g(S)/f(S) = (m/m) ((m-1)/m) ((m-2)/m) ... ((m-n)/m) 1/n! . As m tends to infinity, each of the n terms (m-r)/m tends to 1 and hence g(S)/f(S) tends to 1/n! . Hence 1/n! is the least upper bound.

Problem B4
Does a circle have a subset which is topologically closed and which contains just one of each pair of diametrically opposite points?

Solution
Answer: no.

The map taking each point to the diametrically opposite point is a homeomorphism. [It is obviously (1,1) and its own inverse. So it is sufficient to prove it continuous. But that is almost obvious using an ε δ argument.] Homeomorphisms take closed sets to closed sets. So if the subset was closed, then its complement would also be closed. So the circle would be the disjoint union of two closed sets and hence not connected. But it is connected.

Problem B5
Define $f_0(x) = e^x$, $f_{n+1}(x) = x\, f_n'(x)$. Show that $\sum_0^\infty f_n(1)/n! = e^e$.

Solution
The trick is to use the power series for e^x. Then we have immediately that $f_n(x) = \sum r^n x^r/r!$ Hence, $\sum f_n(x)/n! = 1 + x/1! (1 + 1/1! + 1^2/2! + \dots) + x^2/2! (1 + 2/1! + 2^2/2! + \dots) + \dots = 1 + xe/1! + (xe)^2/2! + \dots = e^{ex}$. [We assume the usual theorems about rearranging terms in absolutely convergent series.]

Problem B6
Let $h_n = \sum_1^n 1/r$. Show that $n - (n - 1)\, n^{-1/(n-1)} > h_n > n(n + 1)^{1/n} - n$ for $n > 2$.

Solution
Consider the numbers: $1 + 1/1, 1 + 1/2, 1 + 1/3, \dots , 1 + 1/n$. Their arithmetic mean is $(n + h_n)/n$. The geometric mean is $(\prod (1 + 1/r))^{1/n}$. But $\prod (1 + 1/r) = 2/1\ 3/2\ 4/3 \dots (n+1)/n = n + 1$. So the geometric mean is $(n + 1)^{1/n}$. The numbers are not all equal, so the arithmetic mean is strictly greater than the geometric mean. That gives the right inequality.

Similarly, consider the numbers: $1 - 1/2, 1 - 1/3, \dots , 1 - 1/n$. [One must be careful not to include $1 - 1/1$, because that would make the geometric mean 0.] The arithmetic mean is $1 - (h_n - 1)/(n-1)$. The geometric mean is $n^{-1/(n-1)}$ (the terms telescope is a similar way). Rearranging gives the left inequality.

37th Putnam 1976

Problem A1
Given two rays OA and OB and a point P between them. Which point X on the ray OA has the property that if XP is extended to meet the ray OB at Y, then XP·PY has the smallest possible value.

Solution
Answer: take OX = OY.

Let M be the foot of the perpendicular from P to OA, and N be the foot of the perpendicular from P to OB. Obviously X must lie further from O than M - otherwise moving it closer to M would reduce both PX and PY. Similarly Y must lie further from O than N. Let φ be the angle MPX and let θ be the angle MON. Then the angle NPY is $\theta-\varphi$. Hence $PX\cdot PY = PM\cdot PN/(\cos\varphi \cos(\theta-\varphi))$. So we minimise PX.PY by maximising $(\cos\varphi \cos(\theta-\varphi))$. But $(\cos\varphi \cos(\theta-\varphi)) = 1/2\,(\cos(\varphi+\theta-\varphi) + \cos(2\varphi-\theta))$ $= 1/2 \cos\theta + 1/2 \cos(2\varphi-\theta)$. This is obviously maximised by taking $\varphi = \theta-\varphi$. But that corresponds to angle PXM = angle PYN and hence OX = OY.

Alternatively, PX·PY may remind us of the elementary result that if XX'YY' is cyclic with diagonals XY, X'Y' meeting at P, then PX·PY = PX'·PY'. The problem is to find a suitable circle. With hindsight, it is fairly obvious. Take the circle which touches OA at X and OB at Y. Then any other line through P intersects the circle at X' and Y' before it intersects OA and OB (at X" and Y" say). Hence PX" > PX', PY" > PY' and PX'·PY' = PX·PY. However, I failed to find the right circle until after I had solved the problem trigonometrically.

Problem A2
Let a(x, y) be the polynomial $x^2y + xy^2$, and b(x, y) the polynomial $x^2 + xy + y^2$. Prove that we can find a polynomial $p_n(a, b)$ which is identically equal to $(x + y)^n + (-1)^n (x^n + y^n)$. For example, $p_4(a, b) = 2b^2$.

Solution
Let us write $E(n) = (x + y)^n + (-1)^n (x^n + y^n)$. We use induction on n. We have E(1) = 0, E(2) = 2b, E(3) = 3a.
We find that $(x + y)^{n+3} = b\,(x + y)^{n+1} + a\,(x + y)^n$ and $x^{n+3} + y^{n+3} = b\,(x^{n+1} + y^{n+1}) - a\,(x^n + y^n)$. So it follows that E(n+3) = b E(n+1) + a E(n). That completes the induction.

Problem A3
Find all solutions to $p^n = q^m \pm 1$, where p and q are primes and $m, n \geq 2$.

Solution
Answer: $2^3 = 3^2 - 1$.

p^n and q^m have opposite parity, so one must be even. So without loss of generality we may take p = 2. Now $q^m \pm 1$ factors as $(q \pm 1)$ and a sum of m powers of q, each of which is odd. If m is odd, then the sum is odd, but that is impossible, since 2^n has no odd factors. So m is even. Take it to be 2M. Consider first the case $2^n = q^{2M} - 1 = (q^M + 1)(q^M - 1)$. But $q^M + 1$ and $q^M - 1$ are successive even numbers, so one of them has an odd factor (which is impossible) unless then are 4 and 2. Thus the only solution of this type is $2^3 = 3^2 - 1$.
Finally consider $2^n = q^{2M} + 1$. q^M is odd, so put $q^M = 2N + 1$, then $q^{2M} + 1 = 4N^2 + 4N + 2$, which is not divisible by 4. So n = 1 and q = 1, which is not a solution, since 1 is not prime.

Problem A4
Let $p(x) \equiv x^3 + ax^2 + bx - 1$, and $q(x) \equiv x^3 + cx^2 + dx + 1$ be polynomials with integer coefficients. Let α be a root of $p(x) = 0$. $p(x)$ is irreducible over the rationals. $\alpha + 1$ is a root of $q(x) = 0$. Find an expression for another root of $p(x) = 0$ in terms of α, but not involving a, b, c, or d.

Solution
Answer: the other roots are $-1/(\alpha+1)$ and $-(\alpha+1)/\alpha$.

Let the roots of $p(x)$ be α, β, γ. The polynomial $q(x+1)$ has α as one of its roots. Suppose it is different from $p(x)$. Then by subtracting we get either a quadratic or a linear equation which also has α as a root. It cannot be a linear equation, because then α would be rational and hence $p(x)$ would not be irreducible over the rationals. If it was quadratic, then by multiplying it by a suitable rational factor $rx + s$ and subtracting from $p(x)$ we would get a linear factor $ux + v$ which also had α as a root. If this factor is zero, then we have reduced $p(x)$ over the rationals (as $(rx + s)$ times the quadratic). If not, then α is a root of $ux + v$ and hence rational, so that $p(x)$ is still reducible. So we must have that $p(x)$ and $q(x+1)$ are identical. So the three roots of $q(x)$ must be $\alpha+1, \beta+1, \gamma+1$. Hence we have the relations:

$\alpha\beta\gamma = 1$, $(\alpha+1)(\beta+1)(\gamma+1) = -1$ (derived from the constant terms of $p(x)$ and $q(x)$ respectively).

Hence $\alpha(\beta+\gamma)+(\beta+\gamma)+\alpha+1+\beta\gamma+\alpha\beta\gamma = -1$, so $(\beta+\gamma)(\alpha+1) = -(3+\alpha+1/\alpha)$, or $\beta+\gamma = -(3\alpha+\alpha^2+1) / (\alpha(\alpha+1))$. Hence $(\beta-\gamma)^2 = (\beta+\gamma)^2 - 4\beta\gamma = (\alpha^4+2\alpha^3+3\alpha^2+2\alpha+1)/(\alpha^2(\alpha+1)^2)$. So $\beta-\gamma = \pm(\alpha^2+\alpha+1)/(\alpha(\alpha+1))$ and hence $\beta = -1/(\alpha+1)$ or $-(\alpha+1)/\alpha$.

Problem A5
Let P be a convex polygon. Let Q be the interior of P and $S = P \cup Q$. Let p be the perimeter of P and A its area. Given any point (x, y) let d(x, y) be the distance from (x, y) to the nearest point of S. Find constants α, β, γ such that $\int_U e^{-d(x,y)}\, dx\, dy = \alpha + \beta p + \gamma A$, where U is the whole plane.

Solution
Answer: $A + p + 2\pi$.

For any point in S we have $d(x, y) = 0$. Hence the integral over S is just A. The locus of points a distance z from S is a set of segments parallel to the sides of P and displaced a distance z outwards, together with a set of arcs joining them. Each arc is centered on a vertex of P and has radius z. Together the arcs can be translated to form a complete circle radius z. Thus the set of points a distance z to $z + \delta z$ from S is a strip of area $p\,\delta z + 2\pi z\,\delta z$. Thus the integral outside S is just $\int_0^\infty \exp(-z)(p + 2\pi z)\, dz$. This evaluates easily to $p + 2\pi$.

Problem A6
Let R be the real line. $f : R \to [-1, 1]$ is twice differentiable and $f(0)^2 + f'(0)^2 = 4$. Show that $f(x_0) + f''(x_0) = 0$ for some x_0.

Solution
Let $k(x) = f(x)^2 + f'(x)^2$. By the mean value theorem for some a in the interval (0, 2) we have $f'(a) = 1/2\,(f(2) - f(0))$. But $|f(x)| \le 1$ for all x, so $|f'(a)| \le 1$. Hence $k(a) \le 1 + 1 = 2$. Similarly, we can find b in the interval (-2, 0) with $k(b) \le 2$. We are given that $k(0) = 4$. Hence $k(x)$ has a maximum at some interior point of (-2, 2). Let this point be c. Then certainly $k(c) \ge k(0) = 4$, $f(c)^2 \le 1$, so $|f'(c)| > 0$. We have $k'(c) = 0$. But $k'(c) = 2 f'(c)(f(c) + f''(c))$. We have just shown that $f'(c)$ is non-zero, so $f(c) + f''(c) = 0$.

Problem B1
Show that $\lim_{n\to\infty} 1/n \sum_1^n ([2n/i] - 2[n/i]) = \ln a - b$ for some positive integers a and b.

Solution
Answer: ln 4 - 1.

The expression inside the limit is a partial sum for $\int_0^1 [2/x] - [1/x]\, dx$ (taking the points x = 1/n, 2/n, ... , 1). So the limit is $\int_0^1 [2/x] - [1/x]\, dx$.
Evidently the integrand is non-zero on the intervals (1/(n+1), 1/(n + 1/2)], for if x belongs to such an interval then n + 1/2 <= x < n + 1, so [1/x] = n, whilst 2n + 1 ≤ x ≤ 2n + 2, so [2/x] = 2n+1. A similar argument shows it is zero on the complementary intervals (1/(n + 1/2), 1/n]. Thus the integral evaluates to (1/(1 + 1/2) - 1/2) + (1/(2 + 1/2) - 1/2 + ... = 2(1/3 - 1/4 + 1/5 - ...) = 2(1 - 1/2 + 1/3 - 1/4 + ...) - 1 = 2 ln 2 - 1 = ln 4 - 1.

Problem B2
G is a group generated by the two elements g, h, which satisfy $g^4 = 1$, $g^2 \neq 1$, $h^7 = 1$, $h \neq 1$, $ghg^{-1}h = 1$. The only subgroup containing g and h is G itself. Write down all elements of G which are squares.

Solution
Answer: 1, g^2, h, h^2, h^3, h^4, h^5, h^6.

Obviously $g^{-1} = g^3$, $h^{-1} = h^6$. So we have $gh = h^6g$ (*). This allows us to write any element generated by g and h in the form h^ng^m (with n = 0, 1, 2, ... , 6; m = 0, 1, 2, 3). Applying (*) we find that $(h^ng)^2 = g^2$, $(h^ng^2)^2 = h^{2n}$, $(h^ng^3)^2 = g^2$. Thus the only possible squares are 1, g^2, h^1 [for example it is the square of h^4], h^2, ... , h^6. Moreover, these are all distinct. For if two powers of h were equal then we could deduce h = 1 (a contradiction). We are given that g^2 is not 1. If g^2 equalled a power of h, then the square of that power would be 1 and hence h would be 1 (a contradiction).

Problem B3
Let $0 < \alpha < 1/4$. Define the sequence p_n by $p_0 = 1$, $p_1 = 1 - \alpha$, $p_{n+1} = p_n - \alpha p_{n-1}$. Show that if each of the events $A_1, A_2, \ldots, A_n$ has probability at least 1 - α, and A_i and A_j are independent for $| i - j | > 1$, then the probability of all A_i occurring is at least p_n. You may assume all p_n are positive.

Solution
This was a rare Putnam disaster - the question is wrong. Let q_n be the probability that all of $A_1, \ldots, A_n$ occur. The idea was that one can more or less write down that $q_{n+1} \geq q_n - \alpha q_{n-1}$ (*). The problem is then to show that it follows that $q_n \geq p_n$ (despite the awkward minus sign in (*)). Unfortunately, (*) requires that A_{n+1} is independent of the event (all of $A_1, \ldots, A_{n-1}$ occur) and that does not follow from the fact that A_{n+1} is independent of each of $A_1, \ldots, A_{n-1}$.

Let us assume first that the question was correctly worded. Let E_n be the event that all of $A_1, A_2, \ldots, A_n$ occur. We write p(E) as the probability that event E occurs. We write ~E as the event (not E). We assume that E_{n-1} and A_{n+1} are independent. We may partition the event E_n into the disjoint events E_{n+1} and $(E_n \,\&\, \sim A_{n+1})$. So $q_n = q_{n+1} + p(E_n \,\&\, \sim A_{n+1})$. But E_{n-1} implies E_n, so $p(E_{n-1} \,\&\, \sim A_{n+1}) \geq p(E_n \,\&\, \sim A_{n+1})$. But E_{n-1} and A_{n+1} are assumed to be independent, so $p(E_{n-1} \,\&\, \sim A_{n+1}) = q_{n-1}\, p(\sim A_{n+1}) \leq \alpha q_{n-1}$. Hence $p(E_n \,\&\, \sim A_{n+1}) \leq \alpha q_{n-1}$. Hence $q_{n+1} \geq q_n - \alpha q_{n-1}$. I have spelt that out at somewhat tedious length, but one should in fact be able to write the conclusion straight down.

Now let us assume that $q_{n+1} = q_n - \alpha q_{n-1} + \beta_{n+1}$, where $\beta_{n+1} \geq 0$. A simple induction establishes that $q_n = p_n + p_{n-2}\beta_1 + p_{n-3}\beta_2 + \ldots + p_0\beta_{n-1} + \beta_n$, so $q_n \geq p_n$ as required.

Finally, we need to show that the conclusion can be false if we assume no more than stated in the question. Take: $p(A_1 \& A_2 \& A_3 \& A_4 \& A_5) = p(\sim A_1 \& A_2 \& A_3 \& A_4 \& A_5) = p(A_1 \& \sim A_2 \& A_3 \& A_4 \& A_5) = p(A_1 \& A_2 \& \sim A_3 \& A_4 \& A_5) = p(A_1 \& A_2 \& A_3 \& \sim A_4 \& A_5) = p(A_1 \& A_2 \& A_3 \& A_4 \& \sim A_5) = 4/25$, $p(\sim A_1 \& \sim A_2 \& \sim A_3 \& \sim A_4 \& \sim A_5) = 1/25$.

We can easily check that $p(A_1) = p(A_2) = p(A_3) = p(A_4) = 20/25$, so we may take $\alpha = 1/5$. We can also check that the prob of any pair, such as $p(A_1 \& A_2) = 16/25$, so an independence assumption stronger than that in the question is satisfied. But $p(A_1 \& A_2 \& A_3 \& A_4) = 4/25 \leq p_5$ (= 6/25 - 1/125).

Problem B4
Let an ellipse have center O and foci A and B. For a point P on the ellipse let d be the distance from O to the tangent at P. Show that $PA \cdot PB \cdot d^2$ is independent of the position of P.

Solution
Answer: a^2b^2.

Let the ellipse be $x^2/a^2 + y^2b^2 = 1$. Then the foci are at $(\pm ae, 0)$ where the eccentricity e is given by $b^2 = a^2(1 - e^2)$. It is also a well-known property that PA + PB = 2a. Thus we may express the product $2\ PA \cdot PB$ as $(PA + PB)^2 - PA^2 - PB^2 = 4a^2 - (x + ae)^2 - y^2 - (x - ae)^2 - y^2 = 4a^2 - 2x^2 - 2y^2 - 2a^2e^2 = 2a^2 + 2b^2 - 2x^2 - 2y^2$.

The tangent at (x, y) is $x/b^2\ Y + x/a^2\ X = 1$, so it meets the axes at $(a^2/x, 0)$, $(0, b^2/y)$. These two points and the origin form a right-angled triangle with the origin a height d from the hypoteneuse. So we may calculate its area as $1/2\ a^2b^2/(xy)$ or as 1/2 d times hypoteneuse. Hence $d^2 = a^4b^4/(xy)^2\ (a^4x^2 + b^4/y^2) = a^4b^4/(a^4y^2 + b^4x^2)$. Using the fact that (x, y) lies on the ellipse, we have $a^4y^2 = a^4b^2 - a^2b^2x^2$, and $b^4x^2 = a^2b^4 - a^2b^2y^2$, so $a^4y^2 + b^4x^2 = a^2b^2\ (a^2 + b^2 - x^2 - y^2) = a^2b^2PA \cdot PB$. Hence $d^2 = a^2b^2/(\ PA \cdot PB\ )$.

Problem B5
Find $\sum_0^n (-1)^i\ nCi\ (\ x - i\ )^n$, where nCi is the binomial coefficient.

Solution
Answer: n!

Given a polynomial p(x), define $\Delta p(x) = p(x) - p(x-1)$. If p(x) is of order n with leading coefficient a x^n, then $\Delta p(x)$ is of order n-1 with leading coefficient a n x^{n-1}. [Proof: let $p(x) = a\ x^n + b\ x^{n-1}$ + terms in x^{n-2} and below. Then $\Delta p(x) = a(x^n - (x-1)^n) + b(x^{n-1} - (x-1)^{n-1})$ + terms in x^{n-2} and below = $a(x^n - x^n + n\ x^{n-1}) + b(x^{n-1} - x^{n-1})$ + terms in x^{n-2} and below = a n x^{n-1} + lower terms.]
Hence $\Delta^n\ p(x) = a\ n!$ But the expression given is simply $\Delta^n p(x)$ with $p(x) = x^n$, so it evaluates to n!

That is the slick solution. The more plodding solution is to observe that the coefficient of x^{n-m} is $(-1)^m\ nCm\ (\ -\ nC1\ 1^m + nC2\ 2^m - nC3\ 3^m + ...\ )$. But the expression in parentheses is just the value at x = 1 of $(x\ d/dx)^m\ (1 - x)^n$ (*). For m < n, all terms in (*) will have a non-zero power of (1 - x) and hence will evaluate to zero. For m = n, the only term not evaluating to zero will be $x^n\ n!\ (-1)^n$ which gives $n!\ (-1)^n$. Hence the expression in the question has the coefficient of x^r zero, except for the constant term, which is $(-1)^n\ nCn\ n!\ (-1)^n = n!$.

Problem B6
Let σ(n) be the sum of all positive divisors of n, including 1 and n. Show that if σ(n) = 2n + 1, then n is the square of an odd integer.

Solution
If n is odd and non-square, then the divisors can be arranged in pairs d, n/d. The components of each pair are odd, so their sum is even and hence $\sigma(n)$ is even and cannot equal 2n+1. It is more difficult to show that n cannot be even.

We have $\sigma(n) = \prod(1 + p + p^2 + ... + p^k)$, where the product is taken over all primes p dividing n and k is the highest power of p dividing n. If p and k are odd, then the factor $(1 + p + ... + p^k)$ is even, and hence $\sigma(n)$ is even. So if $\sigma(n) = 2n+1$, all k corresponding to odd primes must be even. In other words, we have $n = 2^aN^2$, for some odd integer N. Hence $\sigma(n) = (2^{a+1}-1)\sigma(N^2) = 2^{a+1}N^2 + 1$. So $N^2 = (2^{a+1}-1)(\sigma(N^2) - N^2) - 1$. Hence if q is any odd prime dividing $2^{a+1}-1$, then $N^2 = -1 \pmod q$. But -1 is a quadratic non-residue of primes of the form 4r+3. So every prime factor of $2^{a+1} - 1$ must be of the form 4r+1. But that is not possible if $a \geq 1$ (a product of numbers congruent to 1 mod 4 is congruent to 1, not -1). Hence a = 0 and n is an odd square.

38th Putnam 1977

Problem A1
Show that if four distinct points of the curve $y = 2x^4 + 7x^3 + 3x - 5$ are collinear, then their average x-coordinate is some constant k. Find k.

Solution
Answer: - 7/8

Suppose the common line is $y = ax + b$, the the x-coordinates satisfy $2x^4 + 7x^3 + (3 - a)x - (5 + b) = 0$. This has at most 4 distinct roots. The arithmetic mean of the roots is 1/4 (-7/2) = -7/8.
The only other possibility is that the line is x = a, but that only meets the curve in one point.

Problem A2
Find all real solutions (a, b, c, d) to the equations a + b + c = d, 1/a + 1/b + 1/c = 1/d.

Solution
Answer: a, b arbitary; c = - a, d = b.

Take a, b arbitary. We then have: -c + d = a + b; cd = -ab. So -c and d are the roots of the quadratic $x^2 - (a + b)x + ab = 0$. Solving, the roots are a, b. So either c = -a, d = b, or c = -b, d = a.

Problem A3
R is the reals. f, g, h are functions R → R. f(x) = (h(x + 1) + h(x - 1))/2, g(x) = (h(x + 4) + h(x - 4))/2. Express h(x) in terms of f and g.

Solution
h(x) = g(x) - f(x - 3) + f(x - 1) + f(x + 1) - f(x + 3).

Problem A4
Find polynomials p(x) and q(x) with integer coefficients such that $p(x)/q(x) = \sum_0^\infty x^{2n}/(1 - x^{2n+1})$ for $x \in (0, 1)$.

Solution
It is an easy induction that the sum of the first n terms is:
$(x + x^2 + ... + x^{2n-1})/(1 - x^{2n})$.

But that may be written as $[(1 - x^{2n})/(1 - x) - 1] /(1 - x^{2n}) = 1/(1 - x) - 1/(1 - x^{2n})$. Since $x \in (0, 1)$, the second term tends to 1 as $n \to \infty$. So the result is 1/(1 - x) - 1 = x/(1 - x).

Problem A5
p is a prime and $m \geq n$ are non-negative integers. Show that (pm)C(pn) = mCn (mod p), where mCn is the binomial coefficient.

Solution
Let f(n) be the highest power of p dividing n. The multiples of p in (mp)! are mp, (m-1)p, ... , 2p, p. Hence $f((mp)!) = p^m f(m!)$. Similarly, $f((np)!) = p^n f(n!)$ and $f(((m-n)p)!) = p^{m-n} f((m-n)!)$. Hence f(mCn) = f((mp)C(np)).

Problem A6
R is the reals. X is the square [0, 1] x [0, 1]. f **:** X → R is continuous. If $\int_Y f(x, y)\, dx\, dy = 0$ for all

squares Y such that (1) $Y \subseteq X$, (2) Y has sides parallel to those of X, (3) at least one of Y's sides is contained in the boundary of X, is it true that $f(x, y) = 0$ for all x, y?

Solution
Answer: yes.

Given any square Z inside X with sides parallel to X, we can find squares Y_1, Y_2 satisfying the conditions in the question such that $Z = Y_1 \cap Y_2$. Hence $\int_Z f(x, y)\, dx\, dy = 0$ for all Z.

If $f(x, y) > 0$ for any interior point (x, y) of X, then by continuity we can find a square Z such that $f(x, y) > 0$ on Z. Contradiction. Similarly for $f(x, y) < 0$ for an interior point of X. Hence $f(x, y) = 0$ on the interior of X and hence (by continuity) on the whole of X.

Problem B1
Find $\prod_2^\infty (n^3 - 1)/(n^3 + 1)$.

Solution
Answer: 2/3.

If we factor: $n^3 - 1 = (n - 1)(n^2 + n + 1)$, $n^3 + 1 = (n + 1)(n^2 - n + 1)$, then most of the terms cancel. Take the product up to n = N. Then the numerator is $1 \cdot 2 \cdot 3 \ldots (N - 1) \cdot 7 \cdot 13 \cdot 21 \cdot 31 \ldots (N^2 + N + 1)$ and the denominator is $3 \cdot 4 \cdot 5 \ldots (N + 1) \cdot 3 \cdot 7 \cdot 13 \cdot 21 \cdot 31 \ldots (N^2 - N + 1)$. Hence the product up to n = N is $2/(N(N + 1))\ (N^2 + N + 1)/3 = 2/3\ (N^2 + N + 1)/(N(N + 1))$, which tends to 2/3 as N tends to infinity.

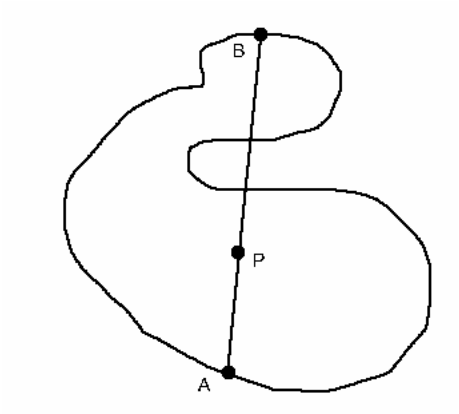
Problem B2
P is a plane containing a convex quadrilateral ABCD. X is a point not in P. Find points A', B', C', D' on the lines XA, XB, XC, XD respectively so that A'B'C'D' is a parallelogram.

Solution
Let X' be the intersection of AC and BD. Take A', C' so that XA'X'C' is a parallelogram. Similarly take B', D' so that XB'X'D' is a parallelogram. Then both A'C' and B'D' have their midpoint at the midpoint of XX'. Hence A'B'C'D' is a parallelogram.

Problem B3
Let S be the set of all collections of 3 (not necessarily distinct) positive irrational numbers with sum 1. If $A = \{x, y, z\} \in S$ and $x > 1/2$, define $A' = \{2x - 1, 2y, 2z\}$. Does repeated application of this operation necessarily give a collection with all elements < 1/2?

Solution
Answer: no.

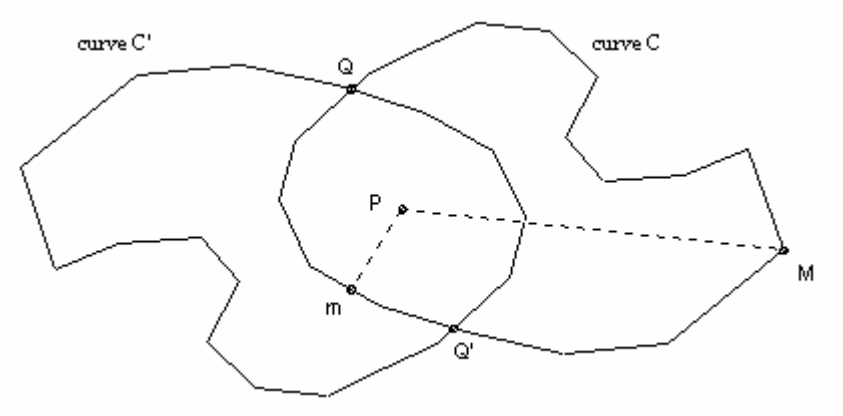
Write the three numbers in binary: $x = 0.x_1x_2x_3 \ldots$, $y = 0.y_1y_2y_3 \ldots$, $z = 0.z_1z_2z_3 \ldots$, where every x_i, y_i, z_i is 0 or 1. Then after n operations (assuming a number > 1/2 at all stages) the three numbers are simply $x = 0.x_{n+1}x_{n+2}x_{n+3} \ldots$, $y = 0.y_{n+1}y_{n+2}y_{n+3} \ldots$, $z = 0.z_{n+1}z_{n+2}z_{n+3} \ldots$. So we have to choose the x_i, y_i, z_i so that (1) for each i, exactly one of x_i, y_i, z_i is 1, (2) x, y, z are irrational. To achieve (2) we just have to ensure that there is no periodicity. So, for example, we could take: $x_i = 1$ for i = 1, 4, 9, 16, ... ; $y_i = 1$ for i = 2, 5, 10, 17, ...; $z_i = 1$ if i is not a square or a square plus 1.
[If the triples are not required to be irrational, we have the even simpler solution: 1/7, 2/7, 4/7.]



Problem B4
Let P be a point inside a continuous closed curve in the plane which does not intersect itself. Show that we can find two points on the curve whose midpoint is P.

Solution
Take an arbitrary chord AB through P. If P is the midpoint then we are done. So assume it is not. Let A' complete a circuit of the curve starting at A and returning to it. Let the chord through A' and P be A'B'. Let f(A') = A'P/B'P. Then f is a continuous function and $f(A_{start}) = 1/f(A_{finish})$. So for some point C on the curve f must assume the intermediate value 1, which means that P is the midpoint of this chord.

Let C be the curve and C' the curve obtained by rotating C through 180^o about P. Let m be a point on C closest to P, and M a point on C furthest from P. Then m must lie inside or on C', and M must lie outside or on C'. Hence C and C' must intersect. Take Q to be any common point. Then the point Q' obtained by rotating Q through 180^o must also lie on C and C'. Now P is the midpoint of QQ'.



Problem B5
$a_1, a_2, \ldots, a_n$ are real and $b < (\sum a_i)^2/(n - 1) - \sum a_i^2$. Show that $b < 2a_ia_j$ for all distinct i, j.

Solution
It is sufficient to show that $(\sum a_i)^2/(n - 1) - \sum a_i^2 \leq 2a_1a_2$. But this follows immediately from the Cauchy inequality for the two n-1 tuples:
$a_1 + a_2, a_3, a_4, \ldots, a_n$; and $1, 1, \ldots, 1$:

$(\sum a_i)^2 <= (n - 1)((a_1 + a_2)^2 + a_3^2 + \ldots + a_n^2) = (n - 1) \sum a_i^2 + (n - 1) 2a_1a_2$.

Problem B6
G is a group. H is a finite subgroup with n elements. For some element $g \in G$, $(gh)^3 = 1$ for all elements $h \in H$. Show that there are at most $3n^2$ distinct elements which can be written as a product of a finite number of elements of the coset Hg.

Solution
It is essential to notice first that $g^3 = 1$ (since $1 \in H$).

A little messing around should now convince us that we can simplify finite products of elements of the form hg. In fact, we show that they can always be written as (1) h_1gh_2, (2) $h_1g^2h_2$, or (3) $h_1g^2h_2g$.

Clearly hg is of the form (1), so it is sufficient to show that given an element k of form (1), (2) or (3), then k (hg) is also of one of these forms.

It is convenient to note that: $ghg = h^{-1}g^2h^{-1}$ (*) (post-multiply ghghgh = 1 successively by h^{-1}, g^2, h^{-1}); and $g^2hg^2 = h^{-1}gh^{-1}$ (**) (pre-multiply $gh^{-1}gh^{-1}gh^{-1} = 1$ successively by g^2, h, g^2).

So dealing with the three cases in turn: $(h_1gh_2)\ h_3g = h_1h_3^{-1}h_2^{-1}g^2h_3^{-1}h_2^{-1}g$, which is of form (2).

$(h_1g^2h_2)\ h_3g$ is obviously of form (3).

$(h_1g^2h_2g)\ h_3g = h_1g^2h_2h_3^{-1}g^2h_3^{-1} = h_1h_3h_2^{-1}gh_3h_2^{-1}h_3^{-1}$, which is of form (1).

39th Putnam 1978

Problem A1
Let S = {1, 4, 7, 10, 13, 16, ... , 100}. Let T be a subset of 20 elements of S. Show that we can find two distinct elements of T with sum 104.

Solution
Not best possible: we can make do with 19 elements. Note that the numbers have the form 3n + 1 for n = 0, 1, ... , 33. We seek 3n + 1, 3m + 1 so that n + m = 34. Evidently n = 0 and n = 17 do not help. The other 32 numbers form 16 pairs with the required sum. So if we take 19 numbers then we are sure to get two from the same pair.

Problem A2
Let A be the real n x n matrix (a_{ij}) where $a_{ij} = a$ for $i < j$, b ($\neq a$) for $i > j$, and c_i for $i = j$. Show that $\det A = (b\,p(a) - a\,p(b))/(b - a)$, where $p(x) = \prod (c_i - x)$.

Solution

$$\begin{vmatrix} c_1 & a & a & \dots & a \\ b & c_2 & a & \dots & a \\ b & b & c_3 & \dots & a \\ \dots & & & & \\ b & b & b & \dots & c_n \end{vmatrix} = \begin{vmatrix} a & a & a & \dots & a \\ b & c_2 & a & \dots & a \\ b & b & c_3 & \dots & a \\ \dots & & & & \\ b & b & b & \dots & c_n \end{vmatrix} + \begin{vmatrix} c_1-a & 0 & 0 & \dots & 0 \\ b & c_2 & a & \dots & a \\ b & b & c_3 & \dots & a \\ \dots & & & & \\ b & b & b & \dots & c_n \end{vmatrix}$$

To evaluate the first determinant on the right, we subtract the first column from each of the others. Then expanding by the top row we get a D, where D has zeros below the main diagonal and hence is just the product of the elements on its diagonal. In other words, the first determinant is just a $\prod_2^n (c_i - b)$.

If we expand the second determinant by the top row we get $(c_1 - a) \det A'$, where A' is the (n-1) x (n-1) matrix formed by deleting the first row and column of A. So we can use induction. The result is trivial for n = 1. So assume it is true for n - 1. Then for n we have a $\prod_2^n (c_i - b) + b/(b - a) \prod_1^n (c_i - a) - a/(b - a)\,(c_1 - a) \prod_2^n (c_i - b)$. Adding the first and third terms we get: $a\,(1 - (c_1 - a)/(b - a)) \prod_2^n (c_i - b) = - a/(b - a) \prod_1^n (c_i - b)$. So the result is true for n.

Problem A3
Let $p(x) = 2(x^6 + 1) + 4(x^5 + x) + 3(x^4 + x^2) + 5x^3$. Let $a = \int_0^\infty x/p(x)\,dx$, $b = \int_0^\infty x^2/p(x)\,dx$, $c = \int_0^\infty x^3/p(x)\,dx$, $d = \int_0^\infty x^4/p(x)\,dx$. Which of a, b, c, d is the smallest?

Solution
Answer: b.

Let $a_1 = \int_0^1 x/p(x)\,dx$, $a_2 = \int_1^\infty x/p(x)\,dx$. Similarly, b_1, b_2, c_1, c_2, d_1, d_2. Also define $e_1 = \int_0^1 1/p(x)\,dx$. Using the subtitution $y = 1/x$ we find that $a_1 = c_2$, $a_2 = c_1$, $b_1 = b_2$, and $e_1 = d_2$. Hence, in particular, $a = a_1 + a_2 = c_2 + c_1 = c$.

But $x - 2x^2 + x^3 = x(x - 1)^2 > 0$ on (0, 1), so $a_1 - 2b_1 + c_1 > 0$. Hence $a = a_1 + a_2 > b_1 + b_2 = b$. Hence also $c > b$.

Similarly, $1 - 2x^2 + x^4 = (x^2 - 1)^2 > 0$ on (0, 1). So $e_1 - 2b_1 + d_1 > 0$. Hence $d = d_1 + d_2 > b_1 + b_2 = b$. So b is the smallest.

Problem A4
A binary operation (represented by multiplication) on S has the property that (ab)(cd) = ad for all a, b, c, d. Show that: (1) if ab = c, then cc = c; (2) if ab = c, then ad = cd for all d. Find a set S, and such a binary operation, which also satisfies: (A) a a = a for all a; (B) ab = a ≠ b for some a, b; (C) ab ≠ a for some a, b.

Solution
(1): (ab)(ab) = ab, so cc = c. (2): (ab)d = ((ab)(ab)) (dd) = (ab)(dd) = ad. Note that an exactly similar argument gives a(bc) = ac. So the operation is in fact associative and $a_1a_2... a_n = a_1a_n$.

In passing we note various possible special cases: (A) ab = k for all a, b (where k is fixed); (B) ab = a for all a, b; (C) ab = b for all a, b. The extra conditions are presumably designed to rule out these special cases.

So we need some operation which preserves something of the first element and something of the second. The easiest is to take S to consist of pairs (r, s) and to define the operation as: (r, s) (u, v) = (r, v). The simplest such example is S = { (0, 0), (0, 1), (1, 0), (1, 1) }. This obviously satisfies all the required conditions. Writing a = (0, 0), b = (0, 1), c = (1, 0), d = (1, 1), we have:

	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Problem A5
Let $a_1, a_2, ... , a_n$ be reals in the interval (0, π) with arithmetic mean μ. Show that $\prod (\sin a_i)/a_i \le ((\sin \mu)/\mu)^n$.

Solution
Products are intractable, whereas sums are easy, so we take logs. Let f(x) = ln ((sin x)/x) = ln sin x - ln x. The required relation is now that $1/n \sum f(a_i) \le f(\mu)$. This is true if the curve is concave, in other words if f "(x) ≤ 0.

Differentiating, we have: f '(x) = cot x - 1/x, f "(x) = - $\text{cosec}^2 x + 1/x^2$. But sin x ≤ x on [0, π], so f "(x) ≤ 0 on [0, π].

Problem A6
Given n points in the plane, prove that less than $2n^{3/2}$ pairs of points are a distance 1 apart.

Solution
Label the points 1, 2, ... , n. Let n_i be the number of points a distance 1 from point i. We wish to show that $\sum n_i < 4 n^{3/2}$. We can assume that all $n_i \ge 2$. [Keep removing points until those that are left have $n_i \ge 2$. If we exhaust the points before this happens then the number of pairs ≤ n < $2n^{3/2}$. Otherwise, suppose we remove k points. Then the result follows from the result for n - k points since $k + 2(n - k)^{3/2} < 2n^{3/2}$.

The points a distance 1 from point i must all lie on C_i the circle radius 1, center point i. Each pair of circles meets in 0, 1 or 2 points. So the total number of points of intersection of two circles is at most n(n - 1) (where a point of intersection arises in more than one way we count one for each way it arises). Some of these points of intersection will be members of the original set of n points. Point i has n_i circles through it, so it arises as a point of intersection in $1/2\ n_i(n_i - 1)$ ways. Thus all the

points together give rise to $\sum 1/2\ n_i(n_i - 1)$ points of intersection. Hence $\sum n_i(n_i - 1) \le 2n(n - 1)$. Since $n_i \ge 2$, $n_i \le 2(n_i - 1)$, so we have that $\sum n_i^2 \le 2 \sum n_i(n_i - 1) \le 4n(n - 1) < 4n^2$. Now Cauchy's inequality gives that $n \sum n_i^2 \ge (\sum n_i)^2$. So $\sum n_i < 2\ n^{3/2}$.

Problem B1
A convex octagon inscribed in a circle has 4 consecutive sides length 3 and the remaining sides length 2. Find its area.

Solution
Let the radius be R. The area is made up of four triangles sides 3, R, R and four triangles sides 2, R, R. The area is unchanged if we rearrange these triangles to form an octagon with sides alternately 2 and 3. But the new octagon is a square side $3 + 2\sqrt{2}$ with four corners lopped off, each sides 2, $\sqrt{2}$, $\sqrt{2}$. Hence its area is $(17 + 12\sqrt{2}) - 4 = 13 + 12\sqrt{2}$.

Problem B2
Find $\sum_1^\infty \sum_1^\infty 1/(i^2j + 2ij + ij^2)$.

Solution
Answer: 7/4.

Let us fix i and sum over j. The term is $1/(i(i + 2))\ (1/j - 1/(j + i + 2))$. So, summing over j, all the terms cancel except for the first i + 2, giving: $1/(i(i + 2))\ (1/1 + 1/2 + ... + 1/(i + 2))$. Also $1/(i(i + 2)) = 1/2\ (1/i - 1/(i + 2))$.

So summing over i (and multiplying by 2) we get:
(1/1 - 1/3) (1 + 1/2 + 1/3) +
(1/2 - 1/4) (1 + 1/2 + 1/3 + 1/4) +
(1/3 - 1/5) (1 + 1/2 + 1/3 + 1/4 + 1/5) +
(1/4 - 1/6) (1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6) + ...

Each of the minus terms partially cancels with the corresponding plus term two lines lower, so we get:
1/1(1 + 1/2 + 1/3) + 1/2(1 + 1/2 + 1/3 + 1/4) + 1/3(1/4 + 1/5) + 1/4(1/5 + 1/6) + ...
= 1 + 1/4 + 1/2 + (1/1 1/2 + 1/2 1/3 + 1/3 1/4 + ...) + (1/1 1/3 + 1/2 1/4 + 1/3 1/5 + ...)
= 7/4 + ((1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + ...) + 1/2 ((1 - 1/3) + (1/2 - 1/4) + (1/3 - 1/5) + ...)
= 7/4 + 1 + 1/2 (1 + 1/2) = 7/2.
Hence the answer is 7/4.

Problem B3
The polynomials $p_n(x)$ are defined by $p_1(x) = 1 + x$, $p_2(x) = 1 + 2x$, $p_{2n+1}(x) = p_{2n}(x) + (n + 1)\ x\ p_{2n-1}(x)$, $p_{2n+2}(x) = p_{2n+1}(x) + (n + 1)\ x\ p_{2n}(x)$. Let a_n be the largest real root of $p_n(x)$. Prove that a_n is monotonic increasing and tends to zero.

Solution
It is fairly obvious that $a_n < 0$ and that the sequence is strictly monotonic increasing. The problem is proving it tends to zero.

Trivial inductions show that $p_n(0) = 1$ and $p_n(x) > 0$ for $x > 0$, so $a_n < 0$. Also we note that $a_1 = -1$, $a_2 = -1/2$. If $x > a_n$, then $p_n(x) > 0$ (otherwise there would be a root greater than a_n).

$p_{2n+1}(a_{2n}) = (n + 1)\ a_{2n}\ p_{2n-1}(a_{2n})$. By induction $p_{2n-1}(a_{2n}) > 0$, so $p_{2n+1}(a_{2n}) < 0$ and hence $a_{2n+1} > a_{2n}$. Similarly $a_{2n+2} > a_{2n+1}$. So, as claimed, a_n is strictly monotonic increasing.

To show it tends to zero, it suffices to prove that: $a_{2n-1} > -1/(n-1)$ and $a_{2n} > -1/n$. This is true for n = 1. Suppose it is true for n. We have already established monotonicity, so $a_{2n+1} > -1/n$. Also $p_{2n-1}(-1/(n+1)) > 0$. But $p_{2n+2}(-1/(n+1)) = p_{2n+1}(-1/(n+1)) - p_{2n}(-1/(n+1)) = - p_{2n-1}(-1/(n+1)) < 0$, so $a_{2n+2} > -1/(n+1)$ and we have established the inductive hypothesis for n+1.

Problem B4
Show that we can find integers a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = abc + abd + acd + bcd$, and the smallest of a, b, c, d is arbitarily large.

Solution
If b, c, d are fixed, then the relation gives a quadratic in a whose two roots have sum bc + bd + cd. So if we have a solution a, b, c, d, then we can derive another solution a', b, c, d with a' = (bc + bc + cd) - a (*). If we take a < b < c < d, and all of a, b, c, d positive, then clearly a' > d.

So starting with any positive solution we may derive solutions with successively larger smallest members. In fact, starting with (1, 1, 1, 1), we get successively (1, 1, 1, 2), (1, 1, 2, 4), (1, 2, 4, 13), (2, 4, 13, 85). From this point on, the smallest member is increased each time.

Problem B5
Find the real polynomial p(x) of degree 4 with largest possible coefficient of x^4 such that p([-1, 1]) $\in$ [0, 1].

Solution
Let $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. Then $a_4x^4 - a_3x^3 + a_2x^2 - a_1x + a_0$ also satisfies the conditions and hence also $a_4x^4 + a_2x^2 + a_0$. So we may take $a_3 = a_1 = 0$.

Now p(0), p(1/2), p(1) belong to [0, 1], so $0 \le a_0 \le 1$, $0 \le a_4/4 + a_2/2 + a_0 \le 1$, $0 \le a_4 + a_2 + a_0 \le 1$. Hence: $-2 \le -2a_0 \le 0$, $0 \le a_4 + 2a_2 + 4a_0 \le 4$, $-2 \le -2a_4 - 2a_2 - 2a_0 \le 0$. Adding: $-4 \le -a_4 < 4$. So the maximum value of a_4 is at most 4.

This can be achieved by $4(x^2 - 1/2)^2 = 4x^4 - 4x^2 + 1$.

Problem B6
a_{ij} are reals in [0, 1]. Show that $(\sum_{i=1}^{n} \sum_{j=1}^{mi} a_{ij}/i)^2 \le 2m \sum_{i=1}^{n} \sum_{j=1}^{mi} a_{ij}$.

Solution
The question looks incredibly complicated, but is actually easy. We just use induction and the fact that each $a_{ij} \le 1$.

Put $b_i = \sum_{1}^{mi} a_{ij}$. Notice that since each $a_{ij} \le 1$, we have $b_i \le mi$. We use induction on n. For n = 1, the required result is: $b_1^2 \le 2mb_1$. But $b_1 \le m$ and $b_1 \ge 0$, so this is certainly true.

Now assume the result is true for n.
For n + 1, the lhs is $(b_1/1 + b_2/2 + ... + b_n/n)^2 + 2(b_1/1 + ... + b_n/n)\, b_{n+1}/(n+1) + b_{n+1}^2/(n+1)^2$, and the rhs is $2m(b_1 + ... + b_n) + 2mb_{n+1}$. Given the result for n, it is sufficient to show that: $2(b_1/1 + ... + b_n/n)\, b_{n+1}/(n+1) + b_{n+1}^2/(n+1)^2 \le 2mb_{n+1}$. Divding by b_{n+1} and using b_i/i le; m, we need: $2nm/(n+1) + m/(n+1) \le 2m$, which is clearly true.

40th Putnam 1979

Problem A1
Find the set of positive integers with sum 1979 and maximum possible product.

Solution
For $n > 4$, $2(n - 2) > n$, so the maximum product cannot include any integer greater than 4. Also, $2^3 < 3^2$, so it cannot include more than two 2s. Since $4 = 2 \cdot 2$, it cannot include both a 2 and a 4. It obviously does not include a 1, since $n + 1 > n \times 1$. So the maximum product must be made up mainly of 3s, with either no, one or two 2s (or equivalently one 4). Hence for $1979 = 2 + 659 \cdot 3$, the maximum product is $3^{659}2$.

Problem A2
R is the reals. For what real k can we find a continuous function $f : R \to R$ such that $f(f(x)) = k\ x^9$ for all x.

Solution
Evidently $k^{1/4}\ x^3$ works for any $k \geq 0$.

For $k \neq 0$, f must be (1, 1) [otherwise we would get $x^9 = y^9$ for $x \neq y$, contradiction.] But f is continuous, so it is strictly monotonic. If it is strictly monotonic increasing, then so is f(f(x)). But if it is strictly monotonic decreasing, then f(f(x)) is strictly monotonic *increasing*. So, either way, f(f(x)) is strictly monotonic increasing. If $k < 0$, then $k\ x^9$ is strictly monotonic decreasing, so we cannot have $k < 0$.

Problem A3
a_n are defined by $a_1 = \alpha$, $a_2 = \beta$, $a_{n+2} = a_n a_{n+1}/(2a_n - a_{n+1})$. α, β are chosen so that $a_{n+1} \neq 2a_n$. For what α, β are infinitely many a_n integral?

Solution
A trivial induction shows that $a_{n+2} = \alpha b/((n + 1)\alpha - n\beta) = \alpha\beta/(n(\alpha - \beta) + \alpha)$.

So we must have $\alpha = \beta$, otherwise the denominator grows without limit. But if $\alpha = \beta$, then all $a_n = \alpha$. So infinitely many a_n are integral iff $\alpha = \beta =$ integer.

Problem A4
A and B are disjoint sets of n points in the plane. No three points of $A \cup B$ are collinear. Can we always label the points of A as $A_1, A_2, \ldots, A_n$, and the points of B as $B_1, B_2, \ldots, B_n$ so that no two of the n segments A_iB_i intersect?

Solution
Answer: yes.

It is easy to waste a lot of time failing to find inductive arguments (can we find a line with the same number of A-points and B-points on one side of it etc).

The trick is to find another property which cannot be satisfied if two segments intersect. Take the labeling which minimises the total length of the segments (obviously exists, since there are only finitely many labelings). If A_iB_i intersects A_jB_j at X. Then $A_iB_j < A_iX + XB_j$, $A_jB_i < A_jX + XB_i$. So $A_iB_j + A_jB_i < A_iB_i + A_jB_j$. So the labeling was not minimal. Contradiction. Hence no pair of segments can intersect.

Problem A5
Show that we can find two distinct real roots α, b of $x^3 - 10x^2 + 29x - 25$ such that we can find infinitely many positive integers n which can be written as $n = [r\alpha] = [s\beta]$ for some integers r, s.

Solution
The polynomial has value -25, -5, 1, -1, -5, -5, 5 at x = 0, 1, 2, 3, 4, 5, 6. So one root lies between 1 and 2, another between 2 and 3, and another between 5 and 6. In particular, all the roots are greater than 1, so if α is a root then all the values [α], [2α], [3α], [4α] ... are different. Hence just [n/α] of them lie in the range 1 to n. Similarly, if the other roots are β, γ, then [n/β] values [mβ] lie in the range 1 to n, and [n/γ] values [mγ]. So at least [n/α] + [n/β] + [n/γ] - n integers in the range 1 to n must have at least two of the representations [mα], [m'β], [m''γ]. But 1/α + 1/β + 1/γ > 1/2 + 1/3 + 1/6 = 1. So as n tends to infinity, the number of integers <= n with at least two representations tends to infinity. In other words, infinitely many positive integers have at least two representations. But there are only finitely many possibilities, so we must be able to find two roots α, β such that infinitely many integers have the representations [mα] and [m'β].
Interestingly, this argument does not tell us which two roots!

Problem A6
Given n reals $\alpha_i \in [0, 1]$ show that we can find $\beta \in [0, 1]$ such that $\sum 1/|\beta - \alpha_i| \le 8n \sum_1^n 1/(2i - 1)$.

Solution
The trick is to pick several candidates β_j and then to take $\sum_j 1/|\beta_j - \alpha_i|$.

Divide [0, 1] into 2n equal intervals, each width 1/2n. Then at least n of them do not contain any α_i. Take the β_j to be the midpoints of n empty subintervals. Then certainly $|\beta_j - \alpha_i| \ge 1/4n$. If we fix j, then we cannot say more than that, so we get $\sum_i 1/|\beta_j - \alpha_i| \le n\ 4n$, which is not good enough.

But if we fix i, then we can say more: at most two β_j can be at the minimum distance 1/4n, at most two more at 3/4n, at most two more at 5/4n and so on. We now sum over j. We do not know whether there are one or two β_j at each stage (because if α_i is off-center, then fairly soon we run into the endpoint going one way, and so start getting just one β_j). But it is certainly conservative to assume that we have two at each stage and go on for n pairs. So summing over j gives: $\sum_j 1/|\beta_j - \alpha_i| < 8n \sum 1/(2i - 1)$. Now summing over all a_i multiplies the result by n. At least one b_j must give a result that is not above average, so we can find β with the sum at most $8n \sum 1/(2i - 1)$.

Problem B1
Can we find a line normal to the curves y = cosh x and y = sinh x?

Solution
Answer: no.

The gradient of y = cosh x at x = a is sinh a, so the equation of the normal is sinh a (y - cosh a) + (x - a) = 0. Similarly, the normal to y = sinh x at x = b is cosh b (y - sinh b) + (x - b) = 0. For these two equations to be the same (so that the normals coincide) we require: sinh a = cosh b and a + sinh a cosh a = b + sinh b cosh b, or b - a = sinh a cosh a - sinh b cosh b = cosh a cosh b - sinh a sinh b = cosh (b - a). But that is impossible since cosh x > x for all x.

Problem B2
Given $0 < \alpha < \beta$, find $\lim_{\lambda \to 0} \left(\int_0^1 (\beta x + \alpha(1 - x))^\lambda \, dx \right)^{1/\lambda}$.

Solution
Answer: $\beta^{\beta/(\beta - \alpha)}/(e\ \alpha^{\alpha/(\beta - \alpha)})$

Let $t = \beta x + \alpha(1 - x)$. Then the integral becomes $\int_\alpha^\beta t^\lambda \, dt/(\beta - \alpha) = 1/(1 + \lambda) \, (\beta^{\lambda+1} - a^{\lambda+1})/(\beta - \alpha)$. We evaluate the limits of $1/(1 + \lambda)^\lambda$ and $(\beta^{\lambda+1} - a^{\lambda+1})^\lambda/(\beta - \alpha)^\lambda$ separately. Put $k = 1/\lambda$. The first limit is $\lim 1/(1 + 1/k)^k = 1/e$. To evaluate the second, note that $\beta^x = e^{x \ln \beta} = (1 + x \ln \beta + O(x^2))$, so the expression is $(1 + 1/k \, (\beta \ln \beta - \alpha \ln \alpha)/(\beta - \alpha) + O(1/k^2))^k$, which tends to the limit $\exp((\beta \ln \beta - \alpha \ln \alpha)/(\beta - \alpha))$ as k tends to infinity, in other words, $\beta^{\beta/(\beta - \alpha)}/ \alpha^{\alpha/(\beta - \alpha)}$.

Problem B3
F is a finite field with n elements. n is odd. $x^2 + bx + c$ is an irreducible polynomial over F. For how many elements $d \in F$ is $x^2 + bx + c + d$ irreducible?

Solution
Since n is odd, $h \neq -h$ for any $h \in F$. So there are exactly $(n + 1)/2$ quadratic residues. Since n is odd, we may write b as 2k, and so we are interested in d for which $(x + k)^2 = k^2 - c - d$ is irreducible. In other words, d for which $k^2 - c - d$ is a quadratic non-residue. But $k^2 - c - d$ runs through all the elements of F, so there are just $(n - 1)/2$ values of d which give a quadratic non-residue.

Problem B4
Find a non-trivial solution of the differential equation $F(y) \equiv (3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 0$.
$y = f(x)$ is the solution of $F(y) = 6(6x + 1)$ such that $f(0) = 1$, and $(f(-1) - 2)(f(1) - 6) = 1$. Find a relation of the form $(f(-2) - a)(f(2) - b) = c$.

Solution
Answer: $a = 6$, $b = 14$, $c = 1$.

The differential equation looks fairly horrible; there is no obvious systematic way of solving it. So we try various types of solutions. Trying simple polynomials leads to $x^2 + x$. Trying exponentials leads to e^{3x}. Obviously, a particular solution to $F(y) = 36x + 6$ is $y = 2$, so the general solution is $y = 2 + Ae^{3x} + Bx + Bx^2$. $f(0) = 1$ implies $A = -1$. $(f(-1) - 2)(f(1) - 6) = (- e^{-3})(-4 - e^3 + 2B) = 1$, so $B = 2$.

$f(-2) = 2 - e^{-6} + 4$, $f(2) = 2 - e^6 + 12$, so we need $a = 6$, $b = 14$, $c = 1$ (and then the relation reduces to $(- e^{-6})(- e^{6}) = 1$, as required).

Problem B5
A convex set S in the plane contains (0, 0) but no other lattice points. The intersections of S with each of the four quadrants have the same area. Show that the area of S is at most 4.

Solution
Let A, B, C, D be the points (1, 0), (0, 1), (-1, 0), (0, -1). These all lie outside S, so we may take support lines S_A, S_B, S_C, S_D through each of them. If S_A meets the x-axis in the interval [-1, 1], then we are home, because the part of S in the first quadrant lies inside a triangle area < 1. So assume each support line is either parallel to the other axis or meets it a distance more than 1 from the origin. Hence S_A and S_B meet at W in the first quadrant, S_B and S_C meet at X in the second quadrant, S_C and S_D meet at Y in the third quadrant, and S_D and S_A meet at Z in the fourth quadrant. S lies entirely inside WXYZ. At least one of the angles of this quadrilateral must be $\leq 90^o$. Suppose it is W.

Now we show that the area of the quadrilateral OAWB (where O is the origin) is at most 1. The required result then follows immediately. Area OAWB = area OAB + area WAB, and area OAB = 1/2. So we need to show that area WAB <= 1/2. But that is almost immediate. The locus of W given angle W is an arc through A, W, B. We maximise the area of the triangle by taking W the midpoint

of the arc. But this triangle lies entirely inside the triangle AW'B with AW' = W'B and angle AW'B = 90^o, which has area 1/2.

Problem B6
z_i are complex numbers. Show that $|\text{Re}[(z_1^2 + z_2^2 + ... + z_n^2)^{1/2}]| \le |\text{Re } z_1| + |\text{Re } z_2| + ... + |\text{Re } z_n|$.

Solution
Let $z_k = x_k + i\, y_k$, and let $(z_1^2 + z_2^2 + ... + z_n^2)^{1/2} = a + ib$.

Then, squaring, $a^2 - b^2 = \sum x_k^2 - \sum y_k^2$ (1), $ab = \sum x_k y_k$ (2). The Cauchy inequality gives that $|\sum x_k y_k| \le (\sum x_k^2)^{1/2} (\sum y_k^2)^{1/2}$, so from (1) we have $|ab| \le (\sum x_k^2)^{1/2} (\sum y_k^2)^{1/2}$ (3). If $|a| > (\sum x_k^2)^{1/2}$, then from (3) $|b| < (\sum y_k^2)^{1/2}$. But then $a^2 - b^2 > \sum x_k^2 - \sum y_k^2$, contradicting (1). So we must have that $|a| \le (\sum x_k^2)^{1/2}$. The required result follows immediately since obviously $(\sum |x_k|)^2 \ge \sum x_k^2$.

41st Putnam 1980

Problem A1
Let $f(x) = x^2 + bx + c$. Let C be the curve $y = f(x)$ and let P_i be the point $(i, f(i))$ on C. Let A_i be the point of intersection of the tangents at P_i and P_{i+1}. Find the polynomial of smallest degree passing through $A_1, A_2, \ldots, A_9$.

Solution
$y = f(x) - 1/4$.

The answer suggests there ought to be a one-line solution, but I cannot see it.
The equation of the tangent at $x = i$ is $y - (i^2 + ib + c) = (2i + b)(x - i)$, or $y - (2i + b)x = c - i^2$.
Solving, we find that A_i is $(i + 1/2, i^2 + i + bi + b/2 + c)$. Clearly these do not lie on a straight line, so the degree is at least 2. But we see at once that *any* A_i lies on the second degree $y = x^2 + bx + c - 1/4$.

Problem A2
Find f(m, n), the number of 4-tuples (a, b, c, d) of positive integers such that the lowest common multiple of any three integers in the 4-tuple is $3^m 7^n$.

Solution
Answer: $(6m^2 + 3m + 1)(6n^2 + 3n + 1)$.

Each of a, b, c, d must be of the form $3^h 7^k$ with $h \le m$, $k \le n$. We can consider h and k separately. At least two of a, b, c, d must have $h = m$. So there are three cases: (1) all of a, b, c, d have $h = m$ (1 possibility); (2) three of a, b, c, d have $h = m$ and the other has $0 \le h < m$ ($3m$ possibilities); (3) two of a, b, c, d have $h = m$ and the other two have $0 \le h < m$ ($6m^2$ possibilities). So, in all, there are $6m^2 + 3m + 1$ possibilities for h. Similarly, there are $6n^2 + 3n + 1$ possibilities for k. Each of the possibilities for h can be combined with each of the possibilities for k, so in all there are $(6m^2 + 3m + 1)(6n^2 + 3n + 1)$ possible 4-tuples.

Problem A3
Find $\int_0^{\pi/2} f(x)\,dx$, where $f(x) = 1/(1 + \tan^{\sqrt{2}}x)$.

Solution
Answer: $\pi/4$.

This involves a trick. $f(\pi/2 - x) = 1/(1 + \cot^{\sqrt{2}}x) = \tan^{\sqrt{2}}x/(1 + \tan^{\sqrt{2}}x) = 1 - f(x)$. Hence $\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_{\pi/4}^{\pi/2} f(x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_0^{\pi/4} f(\pi/2 - x)\,dx = \int_0^{\pi/4} f(x)\,dx + \int_0^{\pi/4} (1 - f(x))\,dx = \int_0^{\pi/4} dx = \pi/4$.
Note that the $\sqrt{2}$ is irrelevant - the argument works for any exponent.

Problem A4
Show that for any integers a, b, c, not all zero, and such that $|a|, |b|, |c| < 10^6$, we have $|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}$. But show that we can find such a, b, c with $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

Solution
This needs a trick: $(a + b\sqrt{2} + c\sqrt{3})(a + b\sqrt{2} - c\sqrt{3})(a - b\sqrt{2} + c\sqrt{3})(a - b\sqrt{2} - c\sqrt{3}) = a^4 + 4b^4 + 9c^4 - 4a^2b^2 - 6c^2a^2 - 12b^2c^2$ which is an integer. Moreover, it is non-zero by a slight extension of the argument used to prove the irrationality of $\sqrt{2}$.

For suppose $a + b \sqrt{2} = c \sqrt{3}$ (*). Squaring: $2ab \sqrt{2} = 3c^2 - a^2 - 2b^2$. But $\sqrt{2}$ is irrational, so a or b is zero. But b cannot be zero since $\sqrt{3}$ is irrational, so a must be zero. Take b and c relatively prime (by dividing out any common factor if necessary). We have $2b^2 = 3c^2$. By the usual argument 2 divides both b and c. Contradiction. Hence (*) is impossible.

So the identity $(a + b \sqrt{2} + c \sqrt{3})(a + b \sqrt{2} - c \sqrt{3})(a - b \sqrt{2} + c \sqrt{3})(a - b \sqrt{2} - c \sqrt{3}) = a^4 + 4b^4 + 9c^4 - 4a^2b^2 - 6c^2a^2 - 12b^2c^2$ gives that $|a + b \sqrt{2} + c \sqrt{3}| \geq 1/|(a + b \sqrt{2} - c \sqrt{3})(a - b \sqrt{2} + c \sqrt{3})(a - b \sqrt{2} - c \sqrt{3})|$. But each of the three numbers in this product is at most 10^7, so $|a + b \sqrt{2} + c \sqrt{3}| \geq 10^{-21}$.

For the other inequality we use a pigeon-hole argument. Take the 10^{18} numbers $a + b \sqrt{2} + c \sqrt{3}$ with $0 \leq a, b, c < 10^6$. They all lie in the interval $[0, 5 \times 10^6]$, so if we divide this interval into 6×10^{17} equal parts, then at least two numbers must lie in the same part. Subtracting them gives a number with absolute value at most $5/6\ 10^{-11}$.

Problem A5
Let p(x) be a polynomial with real coefficients of degree 1 or more. Show that there are only finitely many values α such that $\int_0^\alpha p(x) \sin x\, dx = 0$ and $\int_0^\alpha p(x) \cos x\, dx = 0$.

Solution
Note that either equation alone has infinitely many zeros.

Let $d(x) = p(x) - p''(x) + p^{(4)}(x) - \ldots$. Then we can easily check that $\int p(x) \sin x\, dx = -d(x) \cos x + d'(x) \sin x$, so the definite integral is $-d(\alpha) \cos \alpha + d'(\alpha) + d(0)$. Similarly, the second definite integral is $d(\alpha) \sin \alpha + d'(\alpha) \cos \alpha - d'(0)$. So any roots α of both equations also satisfy $d(\alpha) = d(0) \cos \alpha + d'(0) \sin \alpha$ (*). Let $k = |d(0)| + |d'(0)|$, then $|\text{rhs (*)}| \leq k$. But d(x) has the same degree as p(x), so $|d(x)| > k$ for all $x >$ some k'. In other words all the roots of (*) must satisfy $|\alpha| \leq k'$. By Rolle's theorem, if f(x) has only finitely many zeros, then so does $\int_0^x f(t)\, dt$. Both p(x) and sin x have only finitely many zeros in any finite range $|x| \leq k'$, so $\int_0^\alpha p(x) \sin x\, dx$ has only finitely many zeros in the range $|\alpha| \leq k'$. Hence result.

Problem A6
Let R be the reals and C the set of all functions $f : [0, 1] \to R$ with a continuous derivative and satisfying $f(0) = 0$, $f(1) = 1$. Find $\inf_C \int_0^1 | f'(x) - f(x) |\, dx$.

Solution
If it were not for the condition $f(0) = 0$, we could arrange for the infimum to be zero (which is obviously the lowest possible value), by taking $f(x) = e^{x-1}$. It is tempting to think that we can adjust this solution (and keep zero inf) by taking $f(x) = e^{x-1}$ for $[\varepsilon, 1]$ and kx for $[0, \varepsilon]$. However, this does not work because $\int_0^\varepsilon f'(x)\, dx = f(\varepsilon) = 1/e$.

We need the following (familiar) trick: $(f(x) e^{-x})' = (f'(x) - f(x)) e^{-x}$. So $\int_0^1 | f'(x) - f(x) |\, dx = \int_0^1 | e^x (f(x) e^{-x})' |\, dx \geq \int_0^1 | (f(x) e^{-x})' |\, dx \geq \int_0^1 (f(x) e^{-x})'\, dx = f(1) e^{-1} - f(0) e^0 = 1/e$.

Finally, we just need to improve slightly our opening example. The integral of f(x) over $[0, \varepsilon]$ will be arbitrarily small for small ε. But derivative is discontinuous at ε. However, we can fix that by taking f(x) to be an arbitrary monotonic increasing function with the required values at 0 and ε and with the required derivative at ε.

Problem B1
For which real k do we have $\cosh x \leq \exp(k x^2)$ for all real x?

Solution
Answer: $k \geq 1/2$.

$\cosh x = 1 + x^2/2! + x^4/4! + ... + x^{2n}/(2n)! + ...$ and $\exp(x^2/2) = 1 + x^2/2 + x^4/8 + ... + x^{2n}/(2^n n!)$. But $2^n n! = 2n.(2n - 2).(2n - 4) ... 2 < (2n)!$ for $n > 1$, so $\cosh x < e^{x^2/2}$ for all x. Hence the inequality holds for $k \geq 1/2$.

$\cosh x = 1 + x^2/2 + o(x)$, $\exp(k x^2) = 1 + kx^2 + o(x)$. So if $k < 1/2$, then $\cosh x > \exp(k x^2)$ for sufficiently small x. Thus the inequality does not hold for $k < 1/2$.

Problem B2
S is the region of space defined by $x, y, z \geq 0$, $x + y + z \leq 11$, $2x + 4y + 3z \leq 36$, $2x + 3z \leq 24$. Find the number of vertices and edges of S. For which a, b is $ax + by + z \leq 2a + 5b + 4$ for all points of S?

Solution
Answer: 7 vertices, 11 edges. Condn on a, b is $2/3 \leq a \leq 1$ and $b = 2 - a$.

The face of S in the xy plane has vertices (0,0,0), (11,0,0), (4,7,0) and (0,9,0). The face of S in the yz plane has vertices (0,0,0), (0,9,0), (0,3,8) and (0,0,8). The face of S in the zx plane has vertices (0,0,0), (11,0,0), (0,0,8) and (9,0,2). There are no other vertices. So 7 vertices in total. Note that (11,0,0), (4,7,0), (0,3,8) and (9,0,2) lie on $x + y + z = 11$; (0,3,8), (0,9,0) and (4,7,0) lie on $2x + 4y + 3z = 36$; and (0,0,8), (0,3,8) and (9,0,2) lie on $2x + 3z = 24$.

There are 6 faces (the six planes x=0, y=0, z=0, $x + y + z = 11$, $2x + 4y + 3z = 36$ and $2x + 3z = 24$). Hence the number of edges is $6 + 7 - 2 = 11$. [Explicitly: (0,0,0) to (11,0,0), (0,9,0) and (0,0,8); (0,3,8) to (0,0,8), (0,9,0), (4,7,0) and (9,0,2); (0,9,0) to (4,7,0) to (11,0,0) to (9,0,2) to (0,0,8).]

The point (2,5,4) is the midpoint of the edge (0,3,8) to (4,7,0). If we fix a, b then $2a + 4b + 5$ has some value k. The condition $ax + by + z \leq k$ is then the condition that (x,y,z) lies on or to one side of the plane $ax + by + z = k$. But the point (2, 5, 4) lies on the plane, so we require that the plane is a support plane of S. Since (2, 5, 4) lies on an edge of S, that edge must lie in the plane. The two extreme positions are evidently $x + y + z = 11$ and $2x + 4y + 3z = 36$, or equivalently $2/3\ x + 4/3\ y + z = 12$, each of which contain a face of S including the edge.

So the acceptable a, b are $2/3 \leq a < 1$, and $b = 2 - a$. [Check: this gives the condition $ax + (2 - a)y + z \leq (14 - 3a)$. For the vertex (0,9,0) we have $18 - 9a < 14 - 3a$, which is equivalent to $a \geq 2/3$. For the vertex (11,0,0) we have $11a < 14 - 3a$, which is equivalent to $a \leq 1$.]

Problem B3
Define a_n by $a_0 = \alpha$, $a_{n+1} = 2a_n - n^2$. For which α are all a_n positive?

Solution
Answer: $\alpha \geq 3$.

The trick is that we can solve the recurrence relation. A particular solution is obviously a polynomial with leading term n^2, so we soon find $a_n = n^2 + 2n + 3$. The general solution is then $a_n = n^2 + 2n + 3 + k2^n$. The initial condition $a_0 = \alpha$ gives that $k = \alpha - 3$. A necessary and sufficient condition for all a_n to be positive is evidently $k \geq 0$, or $\alpha \geq 3$.

Problem B4
S is a finite set with subsets $S_1, S_2, \ldots, S_{1066}$ each containing more than half the elements of S. Show that we can find $T \subseteq S$ with $|T| \le 10$, such that all $T \cap S_i$ are non-empty.

Solution
Let $|S| = n$. Consider the number of pairs (x, X) with $x \in S$ and X one of the 1066 subsets. There are $> n/2$ pairs (x, X_0) for a given X_0, and hence more than 533n in total. Hence there is an x_1 which is in at least 534 subsets. That leaves ≤ 532 subsets. Each of these contains $> n/2$ elements of $S - \{x_1\}$, so by the same argument we can find x_2 so that the number of subsets not containing either x_1 or x_2 is at most 265. Similarly, we can find x_3, so that at most 132 subsets do not intersect $\{x_1, x_2, x_3\}$. Continuing, the sequence goes 65, 32, 15, 7, 3, 1, 0 for ten elements.

Problem B5
R^{0+} is the non-negative reals. For $\alpha \ge 0$, C_α is the set of continuous functions $f : [0, 1] \to R^{0+}$ such that: (1) f is convex [$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y)$ for $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$]; (2) f is increasing; (3) $f(1) - 2 f(2/3) + f(1/3) \ge \alpha (f(2/3) - 2 f(1/3) + f(0))$. For which α is C_α is closed under pointwise multiplication?

Solution
Answer: $\alpha \le 1$.

The function $f(x) = x$ always belongs to C_α because it has $f(1) - 2f(2/3) + f(1/3) = f(2/3) - 2f(1/3) + f(0) = 0$. So if C_α is closed under pointwise multiplication, then $f(x) = x^2$ is also a member. But this requires $1 - 8/9 + 1/9 \ge \alpha (4/9 - 2/9 + 0)$ and hence $\alpha \le 1$.

Now if f and g are convex and increasing, then the pointwise product f*g is also convex. For if $x \le y$, we have $(f(y) - f(x))(g(y) - g(x)) \ge 0$ and hence $\lambda\mu(f(x)g(y) + f(y)g(x)) \le \lambda\mu(f(x)g(x) + f(y)g(y))$ (*).

But $f*g(\lambda x + \mu y) = f(\lambda x + \mu y)g(\lambda x + \mu y) \le (\lambda f(x) + \mu f(y))(\lambda g(x) + \mu g(y)) = \lambda^2 f(x)g(x) + \lambda\mu(f(x)g(y) + f(y)g(x)) + \mu^2 f(y)g(y)$. So, using (*), we get $f*g(\lambda x + \mu y) \le (\lambda^2 + \lambda\mu)f(x)g(x) + (\lambda\mu + \mu^2)f(y)g(y) = \lambda\, f*g(x) + \mu\, f*g(y)$.

So it remains to show that if $\alpha \le 1$ and f, g are in C_α, then f*g satisfies condition (3). This is not easy - one has to find the right collection of inequalities using the fact that f, g are increasing, convex and satisfy (3).

Since $f(2/3) > f(1/3)$ and $g(1) - 2g(2/3) + g(1/3) \ge \alpha (g(2/3) - 2g(1/3) + g(0))$, we have:

$f(2/3) (g(1) - 2g(2/3) + g(1/3)) \ge \alpha f(1/3) (g(2/3) - 2g(1/3) + g(0))$ **(1)**

Similarly, we have:

$(f(1) - 2f(2/3) + f(1/3)) g(2/3) \ge \alpha (f(2/3) - 2f(1/3) + f(0)) g(1/3)$ **(2)**

Since f and g are convex, we have $(f(1) - f(2/3)) \ge (f(2/3) - f(1/3))$ and $(g(1) - g(2/3)) \ge (g(2/3) - g(1/3))$. So $(f(1) - f(2/3)) (g(1) - g(2/3) \ge (f(2/3) - f(1/3)) (g(2/3) - g(1/3))$. The rhs is non-negative and $\alpha \le 1$, so $(f(2/3) - f(1/3)) (g(2/3) - g(1/3)) \ge \alpha (f(2/3) - f(1/3)) (g(2/3) - g(1/3))$. Hence:

$(f(1) - f(2/3)) (g(1) - g(2/3) \ge \alpha (f(2/3) - f(1/3)) (g(2/3) - g(1/3))$ **(3)**

Similarly:

$(f(2/3) - f(1/3)) (g(2/3) - g(1/3)) \geq \alpha (f(1/3) - f(0)) (g(1/3) - g(0))$ **(4)**

Adding **(1)**, **(2)**, **(3)** and **(4)** gives the required result:

$f(1)g(1) - 2f(2/3)g(2/3) + f(0)g(0) \geq \alpha (f(2/3)g(2/3) - 2f(1/3)g(1/3) + f(0)g(0))$.

Problem B6
An array of rationals f(n, i) where n and i are positive integers with i > n is defined by f(1, i) = 1/i, f(n+1, i) = (n+1)/i (f(n, n) + f(n, n+1) + ... + f(n, i - 1)). If p is prime, show that f(n, p) has denominator (when in lowest terms) not a multiple of p (for n > 1).

Solution
Generating functions are always a technique to bear in mind for sequences. They are strongly suggested here by the form of the recurrence relation. So let us put $f_n(x) = f(n, n) x^n + f(n, n+1) x^{n+1} + f(n, n+2) x^{n+2} + ...$. We can get the sum f(n, n) + f(n, n+1) + ... + f(n, i - 1) by multiplying by $(1 + x + x^2 + x^3 + ...)$. For then the coefficient of x^{i-1} is f(n, n) + f(n, n+1) + ... + f(n, i-1). We can easily get the factor (n+1) by multiplying by it. This gives us a series where the coefficient of x^{i-1} is i f(n+1, i). In other words, it gives us $f'_{n+1}(x)$. So $f'_{n+1}(x) = (n+1) f_n(x) (1 + x + x^2 + ...)$.

Let us examine the special case n = 1: $f'_2(x) = 2 (x + x^2/2 + x^3/3 + ...)(1 + x + x^2 + ...)$. At this point we need to spot that $(1 + x + x^2 + ...)$ is in fact the *derivative* of $(x + x^2/2 + x^3/3 + ...)$. So integration gives simply $(x + x^2/2 + x^3/3 + ...)^2$. In fact it is now obvious that we can carry out a simple induction to get $f_n(x) = (x + x^2/2 + x^3/3 + ...)^n$.

This is all that we need. For the coefficient of x^p is a sum of terms a/b, where a is an integer and b is a product of n integers taken from {1, 2, ..., p-n+1}. Hence we may add these terms to get a fraction with denominator $((p-n+1)!)^n$. This has no factor p. A fortiori, when reduced to lowest terms.

42nd Putnam 1981

Problem A1
Let the largest power of 5 dividing $1^1 2^2 3^3 \ldots n^n$ be $5^{f(n)}$. What is $\lim_{n\to\infty} f(n)/n^2$?

Solution
Answer: 1/8.

Let 5^k be the largest power of 5 not exceeding n.
We proceed by looking first at the contribution of the single power of 5 in all multiples of 5, then at the additional contribution from the second power of 5 in all multiples of 25 and so on. A single power of 5 in m appears m times in m^m, so the first power of 5 in 5, 10, 15, 20, ... contributes to f(n): $5 + 10 + 20 + \ldots + 5[n/5] = 5(1 + 2 + \ldots + [n/5]) = 1/2\ 5\ ([n/5]^2 + [n/5])$. Similarly, the second power of 5 in 25, 50, 75, ... contributes an additional $1/2\ 5^2\ ([n/5^2]^2 + [n/5^2])$. And so on. Hence
$2 f(n) = 5 ([n/5]^2 + [n/5]) + 5^2 ([n/5^2]^2 + [n/5^2]) + \ldots + 5^k([n/5^k]^2 + [n/5^k])$.

Let $[n/5^i] = n/5^i + \delta_i$. Then $5^i ([n/5^i]^2 + [n/5^i]) = 5^i ((n/5^i - \delta_i)^2 + (n/5^i - \delta_i) = n^2/5^i - 2n\ \delta_i + 5^i\delta_i^2 + n - 5^i\delta_i$.
Hence $2 f(n) = n^2\sum 1/5^i - 2n \sum \delta_i + \sum 5^i(\delta_i^2 - \delta_i) + nk$.

Now $k = [\log_5 n]$, so $k/n \to 0$ as $n \to \infty$. Also $|\delta_i|$ and $|\delta_i^2 - \delta_i| < 1$. Hence $\lim 1/n^2 |- 2n \sum \delta_i + \sum 5^i(\delta_i^2 - \delta_i) + nk| <= \lim 2k/n + \lim k/n + \lim 1/n^2 \sum 5^i = 0 + 0 + \lim 1/n^2\ (1/4\ (5^{k+1} + 1) = 0$ (since the last term is less than 5/4n).
Hence $\lim f(n)/n^2 = 1/2\ (1/5 + 1/5^2 + \ldots) = 1/2\ 1/5\ 1/(1 - 1/5) = 1/8$.

Problem A2
We can label the squares of an 8 x 8 chess board from from 1 to 64 in 64! different ways. For each way we find D, the largest difference between the labels of two squares which are adjacent (orthogonally or diagonally). What is the smallest possible D?

Solution
Answer: 9.

Consider the straightforward ordering 1, 2, 3, 4, 5, 6, 7, 8 for the first row, 9, 10, ... , 16 for the second row, ... , 57, 58, ... , 64 for the last row. Adjacent squares in the same row have difference 1, adjacent squares in different rows have difference 7, 8 or 9. So for this ordering D = 9.
If we take any two squares on a chessboard there is a path from one to the other of length at most 7, where each step of the path is to an adjacent square [if the squares are at opposite corners of an m x n rectangle with m >= n, then take m - 1 steps, n - 1 of them diagonal and the rest along the longest side.] So there is a path of at most 7 steps from 1 to 64. At least one step of that path must have a difference of at least 9 (since 7 x 9 = 64 - 1). So we always have $D \geq 9$. Hence the minimal D is 9.

Problem A3
Evaluate: $\lim_{k\to\infty} e^{-k} \int_R (e^x - e^y) / (x - y)\ dx\ dy$, where R is the rectangle $0 \leq x, y, \leq k$.

Solution
Answer: It diverges to plus infinity.

We use L'Hôpital's rule. Let $f(k) = \int_R (e^x - e^y) / (x - y)\ dx\ dy$, then the limit is the same as the limit of $f'(k)/e^k$. To work out f'(k), note that $f(k + \delta k) - f(k) = \int_{\delta R} (e^x - e^y) / (x - y)\ dx\ dy$, where δR is the strip from x = 0 to x = k at y = k, width δk and the strip from y = 0 to k at x = k width δk. The integral

over the two strips is obviously the same, so we get $f'(k) = 2 \int_0^k (e^k - e^x) / (k - x)\, dx$. Hence $f'(k)/e^k = 2 \int_0^k (1 - e^{-(k-x)}) / (k - x)\, dx$. Changing the integration variable to $z = k - x$, we get $2 \int_0^k (1 - e^{-z}) / z\, dz$. The integrand is always positive and for $z >= 1$ is at least $1 - 1/e > 1/2$. So the integral is at least $\int_1^k 1/z\, dz = \ln k$. This tends to infinity with k.

Problem A4
A particle moves in a straight line inside a square side 1. It is reflected from the sides, but absorbed by the four corners. It starts from an arbitrary point P inside the square. Let c(k) be the number of possible starting directions from which it reaches a corner after traveling a distance k or less. Find the smallest constant a_2, such that from some constants a_1 and a_0, $c(k) \le a_2k^2 + a_1k + a_0$ for all P and all k.

Solution
Answer: π.

By a series of reflections we can transform the path into a straight line. The particle reaches a corner iff the transformed line passes through a lattice point. So the number of directions which lead to absorption in a distance k or less is the number of lattice points in a circle radius k. At least that would be true except for the shadowing problem - the starting point may lie on a line of lattice points in which case it could never reach the ones further out. Shadowing reduces the number of lattice points in the circle that we should count, so the number of lattice points is still an upper bound.
Evidently what we need are some estimates for the number of lattice points in the circle. They do not need to be good estimates - we only need the highest power to be correct. We can regard each lattice point as the centre of a unit square. The resulting unit squares tile the plane. The circle radius k has area πk^2, so it contains around πk^2 of the unit squares and hence around πk^2 lattice points. However, the squares around the edge may be incomplete and the incomplete squares may or may not include their centres. The maximum distance between two points of a unit square is $\sqrt{2}$, so the circle radius $k + \sqrt{2}$ contains the entirety of any square which overlaps the circle radius k. Hence the number of lattice points in the circle radius k is at most $\pi(k + \sqrt{2})^2$. Thus if we take $a_2 = \pi$, $a_1 = 2\pi\sqrt{2}$, $a_0 = 2\pi$, then $c(k) \le a_2k^2 + a_1k + a_0$ for all P and k.
Similarly, any unit square overlapping the circle radius $k - \sqrt{2}$ is entirely contained in the circle radius k. There are at least $\pi(k - \sqrt{2})^2$ such squares, so the circle radius k contains at least $\pi(k - \sqrt{2})^2$ lattice points. Take a coordinate system with axes parallel to the sides of the squares. Then all lines joining lattice points have rational slope, so if we take the point P to have one coordinate rational and the other irrational, it cannot lie on any such lines. Thus for these points there is no shadowing problem and the number of directions is at least $\pi(k - \sqrt{2})^2$. This shows that π is the minimal value of a_2 (because for any smaller value and any a_1, a_0 we would have $a_2k^2 + a_1k + a_0 < \pi(k - \sqrt{2})^2$ for sufficiently large k).

Problem A5
p(x) is a real polynomial with at least n distinct real roots greater than 1. [To be precise we can find at least n distinct values $a_i > 1$ such that $p(a_i) = 0$. It is possible that one or more of the a_i is a multiple root, and it is possible that there are other roots.] Put $q(x) = (x^2 + 1)\, p(x)\, p'(x) + x\, p(x)^2 + x\, p'(x)^2$. Must q(x) have at least 2n - 1 distinct real roots?

Solution
Answer: yes.

Notice that $q(x) = (x\, p(x) + p'(x))(p(x) + x\, p'(x))$. The second factor is $(x\, p(x))'$. Now $x\, p(x)$ has at least n + 1 distinct roots (those of p(x) plus the root x = 0), so its derivative has at least n distinct zeros. To be precise assume that p(x) has exactly $m \ge n$ values $a_i > 1$ at which it is zero: $1 < a_1 < a_2 <$

... < a_m. Then (x p(x))' has at least m zeros, one in the interval (0, a_1) and one in each of the intervals (a_i, a_{i+1}).
We would like to make a similar argument for x p(x) + p'(x). This needs a trick. p(x) exp(x^2/2) also has zeros at a_i. Hence its derivative, exp(x_2/2) (x p(x) + p'(x)) has at least m-1 roots, one in each interval (a_i, a_{i+1}). But exp(x^2/2) is positive for all real x, so (x p(x) + p'(x)) has a root in each interval (a_i, a_{i+1}).

So we are done unless a root k of (x p(x))' is also a root of (x p(x) + p'(x)). But in this case k belongs to some open interval (a_i, a_{i+1}) and we have k p(k) + p'(k) = k p'(k) + p(k). Hence (k - 1)(p(k) - p'(k)) = 0. But k > 1, so p'(k) = p(k). Hence kp(k) + p(k) = 0, so p(k) = 0. But this contradicts the assumption that the a_i are the only roots greater than 1. Hence there are at least 2m - 1 (which is ≥ 2n - 1) roots.

Problem A6
A, B, C are lattice points in the plane. The triangle ABC contains exactly one lattice point, X, in its interior. The line AX meets BC at E. What is the largest possible value of AX/XE?

Solution
Answer: 5.

Take A = (2, 4), B = (0, 0), C = (2, 1). This contains a single lattice point X (1, 1). B is a distance 2 from the line AC, and AC = 3, so ABC has area 3. B is a distance 1 from the line CX, and CX = 1, so XBC has area 1/2. So A is 6x the distance of X from BC. Hence AX = 5 XE. So the value 5 can certainly be achieved.

Let L, M, N be the midpoints of BC, CA, AB. Let B have the coordinates (b_1, b_2) and similarly for the other points. Take B' to be such that X is the midpoint of BB'. Then it is also a lattice point, since its coordinates are ($2x_1 - b_1$, $2x_2 - b_2$). So it cannot lie in the interior of the triangle ABC (since X is the only such lattice point). So X cannot lie in the interior of the triangle BNL (which has dimensions half that of BAC and is similar to it). Similarly, X cannot lie in the interior of CLM.
Now consider the point L' on the ray LX such that LL' = 3LX. It is also a lattice point, since its coordinates are ($3x_1 - (b_1 + c_1)$, $3x_2 - (b_2 + c_2)$). So it cannot lie in the interior of ABC. A fortiori, it cannot lie in the interior of LMN. Take M' on LM such that LM' = 1/3 LM, and N' on LN such that LN' = 1/3 LN. Then LM'N' is similar to LMN and 1/3 the dimensions. So X cannot lie inside LM'N' (otherwise L' would lie within LMN).
It follows that X must lie above the extended line M'N' (which is parallel to BC). But the distance of this line above BC is 1/6 the distance of A above BC. So if the line AX cuts it at F (and BC at E), then AF/FE = 5. Hence AX/XE ≤ 5.

Problem B1
Evaluate $\lim_{n\to\infty} 1/n^5 \sum (5 r^4 - 18 r^2 s^2 + 5 s^4)$, where the sum is over all r, s satisfying 0 < r, s ≤ n.

Solution
Answer: - 1.

This seems curiously difficult for a qu 1, which makes me think I am missing something!
We need: $\sum_1^n r^2 = 1/3\ n^3 + 1/2\ n^2 + O(n)$, and $\sum_1^n r^4 = 1/5\ n^5 + 1/2\ n^4 + O(n^3)$.
Applying these we get that the sum given is $5\ (1/5\ n^5 + 1/2\ n^4 + O(n^3)\ )\ n - 18\ (1/3\ n^3 + 1/2\ n^2 + O(n)\ )^2 + 5\ (1/5\ n^5 + 1/2\ n^4 + O(n^3)\ )\ n = (n^6 + 5/2\ n^5 + O(n^4)\ ) - (2n^6 + 6n^5 + O(n^4)\ ) + (n^6 + 5/2\ n^5 + O(n^4)\ ) = - n^5 + O(n^4)$. So the required limit is - 1.
The tricky part is proving the relation for $\sum r^4$. A fairly trivial induction gives that $\sum r(r+1)(r+2)...(r+s) = 1/(s+2)\ r(r+1)...(r+s+1)$. Hence $\sum r^k = 1/(k+1)\ n^{k+1} + O(n^k)$. Also $\sum(r^4 + 6r^3 +$

$O(r^2)$) = 1/5 (n^5 + 10 n^4 +$O(n^3)$). Hence $\sum r^4$ = 1/5 n^5 + $2n^4$ + $O(n^3)$ - 6(1/4 n^4 + $O(n^3)$) - $\sum O(n^2)$ = 1/5 n^5 + 1/2 n^4 + $O(n^3)$. The relation for $\sum r^2$ can be obtained similarly, although most people probably remember the exact formula 1/6 n(n+1)(2n+1), which is easy to prove by induction.

Problem B2
What is the minimum value of $(a - 1)^2 + (b/a - 1)^2 + (c/b - 1)^2 + (4/c - 1)^2$, over all real numbers a, b, c satisfying $1 \le a \le b \le c \le 4$.

Solution
Answer: $12 - 8\sqrt{2}$.

Let A = a, B = b/a, C = c/b, D = 4/c. Then we require the minimum of $(A-1)^2 + (B-1)^2 + (C-1)^2 + (D-1)^2$ subject to ABCD = 4 and A, B, C, D >= 1.
We consider first the simpler case of minimising $(A-1)^2 + (B-1)^2$ subject to $AB = k^2$ and A, B > 1. We show that it is achieved by taking A = B. This is routine. Set $f(x) = (x - 1)^2 + (k^2/x - 1)^2$ and set f'(x) = 0, leading to a quartic $(x - k)(x + k)(x^2 - x + k^2) = 0$ with real roots k, -k. Note that f'(x) < 0 for x just less than k, and > 0 for x just greater than k, so x = k is a minimum.
Now it follows that all four of A, B, C, D must be equal. For if any two were unequal, we could keep the others fixed and reduce the sum of squares by equalising the unequal pair whilst keeping their product fixed. Thus the minimum value is achieved with $A = B = C = D = \sqrt{2}$, which gives value $12 - 8\sqrt{2}$.

Problem B3
Prove that infinitely many positive integers n have the property that for any prime p dividing $n^2 + 3$, we can find an integer m such that (1) p divides $m^2 + 3$, and (2) $m^2 < n$.

Solution
I started by finding some small n with the required property. This led to: (1) $5^2 + 3 = 2^2 7$, with $7 \mid 2^2 + 3$, $2 \mid 1^2 + 3$; (2) $12^2 + 1 = 7^2 3$, with $3 \mid 3^2 + 3$; (3) $23^2 + 1 = 2^2 7\ 19$ with $19^2 \mid 4^2 + 3$.
That suggested as the first line of attack looking at special n such as $2^a 7^b$. That led nowhere.

My second line of attack was to look for relations of the type $(a^2 + 3)(b^2 + 3) = f(a, b)^2 + 3$. After some playing around, I obtained $(m^2 + 3)((m+1)^2 + 3) = M^2 + 3$, where $M = (m^2 + m + 3)$. We are almost there. Any prime factor of $M^2 + 3$ which divides $m^2 + 3$ has the required property since $m^2 < M$. But that is not true for prime factors of $(m+1)^2 + 3$. However, all we have to do is to iterate. We need to split the larger factor:
$(m^2 + 3)((m+1)^2 + 3)((m^2+m+2)^2 + 3) = ((m^2+m+2)^2 + (m^2+m+2) + 3)^2 + 3$.

This shows that any n of the form $((m^2+m+2)^2 + (m^2+m+2) + 3)$ has the required property.

Problem B4
A is a set of 5 x 7 real matrices closed under scalar multiplication and addition. It contains matrices of ranks 0, 1, 2, 4 and 5. Does it necessarily contain a matrix of rank 3?

Solution
Answer: no.

The 5 x 7 is something of a red herring. Note that we would expect the answer to be no, because addition and scalar multiplication do not impose any mixing on the matrix elements.
Consider the 5 x 5 matrix with a in the first 4 positions on the diagonal, c is the last position, b at positions (4,5) and (5,4) and zeros elsewhere. Taking (a, b, c) = (1, 0, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0) gives matrices of rank 5, 4, 2, 1, 0 respectively. If a = 0, then all the entries in rows 1, 2 and

3 are zero, so the rank is at most 2. If a is non-zero, then there is certainly a 4 x 4 unit submatrix, so the rank is at least 4. Thus no member of the set has rank 3.
If we add two columns of zeros to every member of the set, then we get a counter-example for the 5 x 7 case.

Problem B5
f(n) is the number of 1s in the base 2 representation of n. Let $k = \sum f(n) / (n + n^2)$, where the sum is taken over all positive integers. Is e^k rational?

Solution
Answer: yes.

Obviously f(2n+1) = f(2n) + 1 (because the base 2 representation for 2n+1 is the same as that for 2n with an additional 1 at the end). Note also that f(2n) = f(n).
Let us look at the adjacent terms f(2n) /(2n(2n + 1)), and f(2n + 1) / ((2n+1)(2n+2)). They sum to 1/ ((2n+1)(2n+2)) + 2 f(2n) / (2n(2n+2)) = 1/ ((2n+1)(2n+2)) + 1/2 f(n) / (n(n+1)). Thus we get $k = f(1)/2 + \sum_1^\infty$ (1/ ((2n+1)(2n+2)) + 1/2 f(n) / (n(n+1)) = (1 - 1/2 + 1/3 - 1/4 + ...) + 1/2 k [since 1/((2n+1)(2n+2)) = 1/(2n+1) - 1/(2n+2)]. Thus k = 2 ln 2 = ln 4. So $e^k = 4$.

Problem B6
Let P be a convex polygon each of whose sides touches a circle C of radius 1. Let A be the set of points which are a distance 1 or less from P. If (x, y) is a point of A, let f(x, y) be the number of points in which a unit circle center (x, y) intersects P (so certainly $f(x, y) \geq 1$). What is sup $1/|A| \int_A f(x, y)\, dx\, dy$, where the sup is taken over all possible polygons P?

Solution
Answer: 8/3.

Construct a rectangle on each side of P, with the other dimension of each rectangle 1. Between each adjacent pair of rectangles construct a sector radius 1, center the common vertex. Then A comprises the interior of P, the rectangles and the sectors. By joining each vertex of P to the center of C we divide the interior of P into triangles, each with area 1/2 x length of side. So the area of P = k/2, where k is the perimeter of P. Obviously, the area of the rectangles is k and since the sectors form a circle their area is π. So $|A| = \pi + 3k/2$.The center of a unit circle which cuts a segment length δx must lie in one of two crescents centered on δx, which have total area 4δx. We may now integrate around P to get the area 4k. In this integration points are counted multiple times if the unit circles centered on them intersect P multiple times. In fact, the integral is exactly $\int_A f(x, y)\, dx\, dy$. Note that most unit circles meeting P will meet it twice, but those with centres near a vertex may meet it 4 times.

Thus $1/|A| \int_A f(x, y)\, dx\, dy = 4k/(\pi + 3k/2) = 8/(3 + 2\pi/k)$. So clearly 8/3 is an upper bound. But we can make k arbitrarily large (by taking one vertex of P off to infinity), so 8/3 is also the least upper bound.

43rd Putnam 1982

Problem A1
Let S be the set of points (x, y) in the plane such that $|x| \le y \le |x| + 3$, and $y \le 4$. Find the position of the centroid of S.

Solution
Answer: (0, 13/5).

Let T be the right-angled triangle with hypoteneuse (-4, 4), (4, 4) and third vertex the origin. Let T ' be the small right-angled triangle (0, 3), (1, 4), (-1, 4). Then S is T with T ' removed. The centroid of T is (0, 8/3) and the centroid of T ' is (0, 11/3). The area of T is 16 times the area of T ', so the centroid of S is (0, y) with 15y + 11/3 = 16 8/3. Hence result.

Problem A2
Let $B_n(x) = 1^x + 2^x + ... + n^x$ and let $f(n) = B_n(\log_n 2) / (n \log_2 n)^2$. Does f(2) + f(3) + f(4) + ... converge?

Solution
Answer: yes.

This looks much worse than it is. A crude estimate for $B_n(\log_n 2)$ is sufficient. We know that $\sum 1/n$ only just fails to converge. So it is likely that $\sum 1/n\ 1/\log^2 n$ will converge. In that case we need only prove that $B_n(\log_n 2) \le O(n)$. But it has n terms, the biggest of which is 2. So certainly $B_n(\log_n 2) < 2n$.
We can easily check that $\sum 1/n\ 1/\log^2 n$ converges by the integral test (the indefinite integral is -1/log n).

Problem A3
Evaluate $\int_0^\infty (\tan^{-1}(\pi x) - \tan^{-1} x) / x\ dx$.

Solution
Answer: $\pi/2 \ln \pi$.

$\int_0^\infty (\tan^{-1}(\pi x) - \tan^{-1} x) / x\ dx = \ln x\ (\tan^{-1}\pi x - \tan^{-1} x) \big|_0^\infty - \int_0^\infty (\pi(1 + \pi^2 x^2) - 1/(1 + x^2)) \ln x\ dx$.
Putting $y = \pi x$, $\int_0^\infty (\pi(1 + \pi^2 x^2)\ dx = \int_0^\infty (\ln y - \ln \pi)/(1 + y^2)\ dy$. So $- \int_0^\infty (\pi(1 + \pi^2 x^2) - 1/(1 + x^2)) \ln x\ dx = \ln \pi \int_0^\infty dy/(1 + y^2) = \ln \pi \tan^{-1} y \big|_0^\infty = \pi/2 \ln \pi$.

It remains to show that the first term is zero. The Taylor series for $\tan^{-1} x = x + O(x^2)$, so the value at the lower limit is the limit as $x \to 0$ of $((\pi - 1) x + O(x^2)) \ln x$. But $x \ln x \to 0$ as $x \to 0$, so the lower limit gives zero.

We also have the expansion $\tan^{-1} x = \pi/2 - 1/x + O(1/x^2)$. So the value at the upper limit is the limit of $(1/x\ (1 - 1/\pi) + O(1/x^2)) \ln x$ as $x \to \infty$, which is zero since $(\ln x)/x \to 0$.

Problem A4
Given that the equations $y' = -z^3$, $z' = y^3$ with initial conditions y(0) = 1, z(0) = 0 have the unique solution y = f(x), z = g(x) for all real x, prove f(x) and g(x) are both periodic with the same period.

Solution
We have $y^3\ y' + z^3\ z' = 0$. Integrating, $y^4 + z^4 =$ constant. But y(0) = 1, z(0) = 0, so the constant is 1 and $y^4 + z^4 = 1$. So as x varies, (y, z) moves along the oval. Let s be the length of the curve from the

point (y, z) = (1, 0). Then s' = $\sqrt{((y')^2 + (z')^2)} = \sqrt{(y^6 + z^6)}$. If |y| and |z| are both <= 0.8, then $|y^4 + z^4|$ < 0.9. Contradiction. So at least one of |y|, |z| > 0.8. Hence $|y^6 + z^6| > 0.8^6 > 1/4$. So s' > 1/4. Let L be the perimeter of the oval. Then L is finite (for example, L is convex and lies inside the square side 2, so L < 8). Hence the point (y, z) completes a circuit of the oval for some value k < 4L of x. But s' depends solely on y and z. So (y(x+k), z(x+k)) = (y(x), z(x)).

Problem A5
a, b, c, d are positive integers satisfying a + c ≤ 1982 and a/b + c/d < 1. Prove that 1 - a/b - c/d > $1/1983^3$.

Solution
Let us consider the general case with 1982 replaced by N. We have a < N, so 1 - a/b > 1/N. So there is no benefit in making d too large. Certainly, for example, we must have d < 2Nc < $2N^2$. Similarly for b. So there are only finitely many candidates for a, b, c, d. Hence there is a set of values which minimises 1 - a/b - c/d. Let us adopt these values.

Evidently 1 - a/b - c/d = k/bd for some positive integer k. We may assume that b ≥ d. The fact that a, b, c, d is an optimal set means that k cannot be too large. Multipying across, k = bd - ad - bc. So if k > d, we could increase a by 1. That would reduce 1 - a/b - c/d to (k-d)/bd, contradicting minimality. So k ≤ d.
Now bd = k + ad + bc <= k + ab + bc = k + b(a + c) <= k + bN. Hence d <= N + k/b <= N + 1.
But (k/bd + a/b) = 1 - c/d ≥ 1/d ≥ 1/(N+1). We have (k/bd + a/b) = 1/b (a + k/d) ≤ 1/b (a + 1) <= 1/b N. So 1/b ≥ 1/(N(N+1)) > $1/(N+1)^2$. Hence 1 - a/b - c/d = k/bd > $k/(N+1)^3 ≥ 1/(N+1)^3$.

Problem A6
a_i are real numbers and $\sum_1^\infty a_i = 1$. Also $|a_1| > |a_2| > |a_3| > \ldots$. f(i) is a bijection of the positive integers onto itself, and $|f(i) - i|\ |a_i| \to 0$ as $i \to \infty$. Prove or disprove that $\sum_1^\infty a_{f(i)} = 1$.

Solution
Answer: it is false.

Take a_i to be a sequence of terms with alternating signs, so that the sum of the odd terms diverges and the sum of the even terms diverges. To be specific, let us take the odd terms positive and the even terms negative. Now take the sequence f(1), f(2), f(3), ... to have runs of odd numbers followed by the corresponding even numbers. So it might start with 1, 3, 5, 7. The corresponding even numbers would then follow: 1, 3, 5, 7, 2, 4, 6, 8. At this point f(1), f(2), ... , f(8) is just a rearrangement of 1, 2, ... , 8. We then have another run of odd numbers, maybe 9, 11, 13, 15, 17, 19. Then the corresponding evens, 10, 12, ... , 20, so that f(1), f(2), ... , f(20) is just a rearrangement of 1, 2, ... , 20. No matter how long the runs, we will always have |f(n) - n| <= n for all n. So we can satisfy $|f(n) - n|\ |a_n| \to 0$, provided that $n\ a_n \to 0$.
Now since the sum of the odd terms diverges, we can take each run of odds long enough that its sum is at least 2. The sum up to the start of the run is just $\sum_1^N a_i$ for some large N and so is close to 1. In particular, for the mth run of odds, the sum up to the start of the run will certainly be at least 1/2 for all sufficiently large m. Hence the sum up to the end of the run is more than 2. So we get arbitrarily large partial sums which differ from $\sum a_i$ by at least 1. So we cannot have $\sum a_{f(i)} = \sum a_i$. In fact $\sum a_{f(i)}$ does not even converge, since we also have arbitrarily large partial sums (up to the start of each run of odds) which converge to 1.
It remains to exhibit a_n which satisfy $|a_n| \to 0$. A suitable example is $a_{n-1} = (-1)^n\ 1/(n \ln n)$.

Problem B1
ABC is an arbitary triangle, and M is the midpoint of BC. How many pieces are needed to dissect AMB into triangles which can be reassembled to give AMC?

Solution
Answer: 2.

Let N be the midpoint of AB and L the midpoint of AC. Cut AMB along MN into AMN and BMN. Then AMN is congruent to MAL and BNM is congruent ot MLC, so the two pieces can be reassembled to AMC. Clearly one piece is not sufficient unless AB = AC.

Problem B2
Let a(r) be the number of lattice points inside the circle center the origin, radius r. Let $k = 1 + e^{-1} + e^{-4} + ... + \exp(-n^2) + ...$. Express $\int_U a(\sqrt{(x^2 + y^2)}) \exp(-(x^2+y^2)) dx\, dy$ as a polynomial in k, where U represents the entire plane.

Solution
Answer: $\pi(4k^2 - 4k + 1)$.

a(r) is constant except at certain values $\sqrt{n}$ where it increases. Let a_n be the increase at $\sqrt{n}$, and $a_0 = 1$, so that $a(r) = a_0 + a_1 + ... + a_n$ on the interval $(\sqrt{n}, \sqrt{(n+1)})$.

In the integral given, change variables to r, θ. The Jacobian is r, so we get $2\pi \int_0^\infty r\, a(r) \exp(- r^2) dr = 2\pi \sum_0^\infty \int_{\sqrt{n}}^{\sqrt{(n+1)}} r\, a(r) \exp(- r^2) dr = 2\pi \sum_0^\infty \int_{\sqrt{n}}^{\sqrt{(n+1)}} r\, (a_0 + a_1 + ... + a_n) \exp(- r^2) dr = \pi (a_0(1 - e^{-1}) + (a_0 + a_1)(e^{-1} - e^{-2}) + (a_0 + a_1 + a_2)(e^{-2} - e^{-3}) + ...) = \pi(a_0 + a_1e^{-1} + a_2e^{-2} + ...)$.

Let $p(x) = 1 + x + x^4 + x^9 + x^{16} + ...$. Then the coefficient of x^n in $p(x)^2$ is the number of lattice points in the first quadrant on the circle radius $\sqrt{n}$ centre the origin. The total number of lattice points on the circle is not $4p(x)^2$ because that double-counts the points on the axes, it is $4p(x)^2 - 4p(x) + 1$ (the final 1 is to put back the point at the origin). Hence $a_0 + a_1e^{-1} + a_2e^{-2} + ... = 4p(1/e)^2 - 4p(1/e) + 1 = 4k^2 - 4k + 1$ and the integral is $\pi(4k^2 - 4k + 1)$.

Problem B3
Let p_n be the probability that two numbers selected independently and randomly from {1, 2, 3, ... , n} have a sum which is a square. Find $\lim_{n\to\infty} p_n \sqrt{n}$.

Solution
Answer: $4(\sqrt{2} - 1)/3$.

There are $m^2 - 1$ ordered pairs with sum m^2 for $m^2 <= n$, and $2n - m^2 + 1$ for $m^2 > n$ (and $\le 2n$).
Hence the total number of ordered pairs taken from {1, 2, ... , n} which sum to a square is
$\sum_1^{[\sqrt{n}]} (m^2 - 1) + \sum_{[\sqrt{n}+1]}^{[\sqrt{(2n)}]} (2n - m^2 + 1)$.
Now $\sum_1^N m^2 = 1/3\, N^3 + O(N^2)$. Also removing the [ ] from the limits gives an error of O(n). So we get:

$1/3\, n^{3/2} + 2(\sqrt{2} - 1)n^{3/2} - 1/3 (\sqrt{(2n)^3} - \sqrt{n^3}) + O(n)$.

$p_n = 1/n^2$ x no. of pairs which sum to a square. So $p_n \sqrt{n} = (1/3 + 2(\sqrt{2} - 1) - 1/3 (2\sqrt{2} - 1)) + O(n)/n^{3/2}$ which tends to the limit $(1/3 + 2(\sqrt{2} - 1) - 1/3 (2\sqrt{2} - 1)) = 4(\sqrt{2} - 1)/3$.

Problem B4
A set S of k distinct integers n_i is such that $\prod n_i$ divides $\prod (n_i + m)$ for all integers m. Must 1 or -1 belong to S? If all members of S are positive, is S necessarily just {1, 2, ... , k}?

Solution
Answer: yes, yes.

m is allowed to be *negative*, so we can use the weaker condition that $\prod n_i \geq \prod (n_i + m)$. Suppose neither 1 nor -1 is in S. Then taking m = 1 and -1 gives non-zero products. Moreover the product of their absolute values is $\prod |n_i + 1|\,|n_i - 1| = \prod |n_i^2 - 1| < (\prod n_i)^2$. So at least one of the absolute values is less than $|\prod n_i|$ (and hence cannot be divisible by it). Contradiction.
Now consider the case where all members of S are positive. Let m be the smallest positive integer not in S. Then $\prod n_i = |\prod (n_i - m)|$, where we take the product over the members of S less than m. For any member x of S greater than m, we obviously have $0 < x - m < m$. So if there are any such members, then $\prod n_i > |\prod (n_i - m)|$ (taking the products over all members of S). Contradiction. Hence S must be just 1, 2, ... , m - 1.

Of course, the question becomes more interesting if we restrict m to be positive.

Problem B5
Given $x > e^e$, define the sequence f(n) as follows: f(0) = e, f(n+1) = (ln x)/(ln f(n)). Prove that the sequence converges. Let the limit be g(x). Prove that g is continuous.

Solution
We establish that f(2n) is an increasing sequence, and f(2n-1) a decreasing sequence. Specifically we show that $e < f(2n) < f(2n+2) < f(2n+1) < f(2n-1)$ (*). We need the preliminary result that if $e < x < y$, then $x^y > y^x$ (**). We also use repeatedly the obvious result that if $1 < x < y$ and $x^a = y^b$, then $a > b$ (***).
To prove (**), let k(x) = (ln x)/x, then $k'(x) = (1 - \ln x)/x^2 < 0$ for x > e. Hence for $e \leq x < y$, (ln x)/x > (ln y)/y, and hence y ln x > x ln y, and hence $x^y > y^x$.
We establish (*) by induction. By definition, we have $x = f(n)^{f(n+1)}$. In particular, $x = e^{f(1)}$. We are given that $x > e^e$, so certainly f(1) > e. Applying (**), $e^{f(1)} > f(1)^e$. But $x = e^{f(1)} = f(1)^{f(2)}$, so f(2) > e. Also since e < f(1) and $e^{f(1)} = f(1)^{f(2)}$ we have f(1) > f(2). So we have established that e < f(2) < f(1).

Now assume that e < f(2n) < f(2n-1). By definition, $x = f(2n-1)^{f(2n)} = f(2n)^{f(2n+1)}$. But f(2n-1) > f(2n), so by (***) f(2n) < f(2n+1). Also, using (**), f(2n) < f(2n-1) implies $x = f(2n-1)^{f(2n)} < f(2n)^{f(2n-1)}$. Hence f(2n+1) < f(2n-1). So we have established that f(2n) < f(2n+1) < f(2n-1).

We now use $f(2n)^{f(2n+1)} = f(2n+1)^{f(2n+2)}$. Since f(2n) < f(2n+1), (***) gives f(2n+1) > f(2n+2). Also, using (**), $f(2n)^{f(2n+1)} > f(2n+1)^{f(2n)}$, so f(2n+2) > f(2n). So we have established that f(2n) < f(2n+2) < f(2n+1) < f(2n-1). That completes the induction.
Hence f(2n) is an increasing sequence which is bounded above. So it must converge to some limit h. Similarly, f(2n+1) is a decreasing sequence which is bounded below. So it must converge to some limit k. Also we have h, k > e. But it follows that $\lim f(2n)^{f(2n+1)} = h^k$ and $\lim f(2n+1)^{f(2n+2)} = k^h$. But both these sequences have the constant value x. Hence $h^k = k^h$. If h and k were unequal, then (**) would imply that h^k and k^h were unequal. Hence h = k.
So we have established that the sequence f(n) converges to some limit g(x). We have also shown that $g(x)^{g(x)} = x$. But y^y is continuous and strictly increasing, so its inverse g(x) is continuous.

Problem B6
Let A(a, b, c) be the area of a triangle with sides a, b, c. Let $f(a, b, c) = \sqrt{A(a, b, c)}$. Prove that for any two triangles with sides a, b, c and a', b', c' we have $f(a, b, c) + f(a',b',c') \leq f(a + a', b + b', c + c')$. When do we have equality?

Solution
Answer: equality iff the triangles are similar (a/a' = b/b' = c/c').

We prove first that for any positive reals A, B, A', B' we have $\sqrt{(AB)} + \sqrt{(A'B')} \leq \sqrt{((A + A')(B + B'))}$ with equality iff A/A' = B/B' . We have $(\sqrt{(AB')} - \sqrt{(A'B)})^2 \geq 0$, with equality iff A/A' = B/B' .

Hence $2\sqrt{(ABA'B')} \le AB' + A'B$. Hence $(AB + A'B') + 2\sqrt{(ABA'B')} \le (AB + A'B') + AB' + A'B$. Now the lhs is $(\sqrt{(AB)} + \sqrt{(A'B')})^2$. The rhs is $(A + A')(B + B')$. So taking the square root, we have $\sqrt{(AB)} + \sqrt{(A'B')} \le \sqrt{((A + A')(B + B'))}$ with equality iff $A/A' = B/B'$.

Now set $A = \sqrt{(ab)}$, $B = \sqrt{(cd)}$, $A' = \sqrt{(a'b')}$, $B' = \sqrt{(c'd')}$. Then the relation becomes $(abcd)^{1/4} + (a'b'c'd')^{1/4} \le \sqrt{((\sqrt{(ab)} + \sqrt{(a'b')})(\sqrt{(cd)} + \sqrt{(c'd')})}$. Applying the relation again to each of the inner parentheses on the rhs, we get $(\sqrt{(ab)} + \sqrt{(a'b')})(\sqrt{(cd)} + \sqrt{(c'd')} \le \sqrt{((a + a')(b + b')(c + c')(d + d'))}$.

So we arrive finally at $(abcd)^{1/4} + (a'b'c'd')^{1/4} \le ((a + a')(b + b')(c + c')(d + d'))^{1/4}$. We have equality iff $A/A' = B/B'$, $a/a' = b/b'$ and $c/c' = d/d'$ and hence iff $a/a' = b/b' = c/c' = d/d'$.

Applying this result to Heron's formula gives the required result immediately, with equality iff $a/a' = b/b' = c/c'$.

44th Putnam 1983

Problem A1
How many positive integers divide at least one of 10^{40} and 20^{30}?

Solution
Answer: 2301.

Factors of 10^{40} have the form 2^m5^n with 0 <= m, n <= 40. So there are $41^2 = 1681$ such factors. Factors of $20^{30} = 2^{60}5^{30}$ not dividing 10^{40} have the form 2^m5^n with $41 \le m \le 60$ and $0 \le n \le 30$, so there are $20.31 = 620$ such factors.

Problem A2
A clock's minute hand has length 4 and its hour hand length 3. What is the distance between the tips at the moment when it is increasing most rapidly?

Solution
Answer: $\sqrt{7}$.

Let the angle between the hands be θ. Then the distance between the tips is $\sqrt{(25 - 24 \cos \theta)}$. Differentiating, the rate of increase is $12\, d\theta/dt \sin \theta / \sqrt{(25 - 24 \cos \theta)}$. Differentiating again, this is a maximum when $\cos \theta (25 - 24 \cos \theta) = 12 \sin^2\theta$, and hence when $12 \cos^2\theta - 25 \cos \theta + 12 = 0$, or $(3 \cos \theta - 4)(4 \cos \theta - 3) = 0$. We cannot have $\cos \theta > 1$, so the maximum is when $\cos \theta = 3/4$ and the distance is $\sqrt{(25 - 24.3/4)} = \sqrt{7}$.

Problem A3
Let $f(n) = 1 + 2n + 3n^2 + ... + (p - 1)n^{p-2}$, where p is an odd prime. Prove that if $f(m) = f(n) \pmod p$, then $m = n \pmod p$.

Solution
Clearly if $n = 0 \pmod p$, then $f(n) = 1 \pmod p$. Similarly, if $n = 1 \pmod p$, then $f(n) = 1 + 2 + ... + (p - 1) = p(p - 1)/2 = 0 \pmod p$.
Otherwise we have $(1 - n) f(n) = 1 + n + n^2 + ... + n^{p-2} - (p-1)n^{p-1}$. Hence $(1 - n)^2 f(n) = 1 - pn^{p-1} + (p - 1)n^p = 1 + (p-1)n^p = 1 + (p-1)n = 1 - n \pmod p$. Since we are assuming $n \neq 1$, $1 - n$ is invertible and we have $f(n) = (1 - n)^{-1}$. Note that this cannot have the value 1 (because $(1 - n)^{-1} = 1$ implies $1.(1 - n) = 1$ implies $n = 0$) or 0 (because we cannot have $0.(1 - n) = 1$).

So suppose $f(n) = f(m) \pmod p$. If the common value is 0 or 1, then n and m are both $1 \pmod p$ or both $0 \pmod p$ and hence equal. If the common value is not 0 or 1, then $(1 - n)^{-1} = (1 - m)^{-1} \pmod p$, hence $(1 - n) = (1 - m) \pmod p$ and $n = m \pmod p$.

Problem A4
Prove that for $m = 5 \pmod 6$, $mC2 - mC5 + mC8 - mC11 + ... - mC(m-6) + mC(m-3) \neq 0$.

Solution
Answer: $(-3)^{(m-1)/2} + 1$.

Let $m = 6n+5$. We have $x(1 - x)^m = mC0\, x - mC1\, x^2 + mC3\, x^3 - ... - mCm\, x^{m+1}$ (*). Let ω be a complex cube root of 1. Substitute $x = 1$, ω and ω^2 in (*) and add. The terms in x^{3r+1} and x^{3r+2} give zero since $1 + \omega + \omega^2 = 0$. So we get $mC2 - mC5 + ... - mCm = (\omega(1 - \omega)^m + \omega^2(1 - \omega^2)^m)/3$.
Now $(1 - \omega^2)^m = (1 - \omega)^m(1 + \omega)^m = (1 - \omega)^m(-\omega^2)^m = - \omega(1 - \omega)^m$ (since $m = 5 \bmod 6$). So $mC2 -$

$mC5 + ... - mCm = (\omega(1 - \omega)^m - (1 - \omega)^m)/3 = - 1/3 (1 - \omega)^{m+1}$. But $(1 - \omega)^2 = 1 - 2\omega + \omega^2 = - 3\omega$. So $- 1/3 (1 - \omega)^{m+1} = (-3)^{(m-1)/2}$.

The series in the question is missing the final term - mCm, so it sums to $1 + (-3)^{(m-1)/2}$, which is clearly never zero for $m \geq 5$.

Problem A5
Does there exist a positive real number α such that $[\alpha^n] - n$ is even for all integers $n > 0$?

Solution
Answer: yes.

The idea is that an interval length 1 is acceptable for α^n. This corresponds to an interval length α for α^{n+1}. So if α is sufficiently large (in fact > 3), this will always contain an acceptable subinterval of length 1 for α^{n+1}. Translated back to intervals for α itself this gives to a sequence of nested intervals, which must have non-zero intersection.

So, being slightly more precise, define $I_1 = [5, 5.9]$. Then any point of I_1 has integral part 5, which is odd. This corresponds to the interval $I_1' = [25, 34.8]$ for α^2. This contains the interval [26, 26.9] every point of which has integral part 26, even. It corresponds to $I_2 = [26^{1/2}, 26.9^{1/2}]$ which is a subinterval of I_1 (it is roughly [5.10, 5.19]. It corresponds to $I_2' = [26^{3/2}, 26.9^{3/2}]$, or roughly [132.6, 139.5] for α^3. This has the subinterval [133, 133.9] every point of which has integral part 133, odd. It corresponds to $I_3 = [133^{1/3}, 133.9^{1/3}]$ (roughly [5.10, 5.12]), which is a subinterval of I_2. This process can be continued indefinitely, because I_n' always has length $0.9 \alpha > 4$ and hence contains at least 3 consecutive integers. We pick one of the two smaller, so that it is even or odd as required. This gives us an acceptable subinterval $[m, m+0.9]$ of I_n'.

Note that we could have started with [3, 4) and used half-open intervals of length 1, but then the argument is slightly more complicated because an intersection of non-closed intervals is not guaranteed to be non-empty.

Problem A6
Let T be the triangle with vertices (0, 0), (a, 0), and (0, a). Find $\lim_{a\to\infty} a^4 \exp(-a^3) \int_T \exp(x^3+y^3)\, dx\, dy$.

Solution
Answer: 2/9.

The corresponding indefinite integral is obviously intractable and the definite integral clearly diverges. But $\exp(a^3)/a^4$ also diverges. So we have the ratio of two divergent quantities. This calls for l'Hôpital's rule: to evaluate lim f(a)/g(a) we take lim f'(a)/g'(a). The first step is to rearrange the integral so that a only occurs as an integration limit for one of the variables (thus making it easier to differentiate with respect to it).

After a little experimentation we take $s = x + y$, $t = x - y$. The Jacobian is 1/2 and so we get $1/2 \int_{s=0}^{a} \int_{-s}^{s} \exp(s^3/4 + 3s\, t^2/4)\, dt\, ds$. Differentiating wrt a, we get $1/2 \int_{-a}^{a} \exp(a^3/4 + 3a/4\, t^2)\, dt = 1/2 \exp(a^3/4) \int_{-a}^{a} \exp(3a/4\, t^2)\, dt$. Similarly, differentiating the denominator gives $(3/a^2 - 4/a^5) \exp(a^3)$. We can cancel out the $\exp(a^3/4)$ to get $1/2 \int_{-a}^{a} \exp(3a/4\, t^2)\, dt$ and $(3/a^2 - 4/a^5) \exp(3a^3/4)$, but both these still diverge. Accordingly, we must apply the rule again.

We would like to eliminate the a in the integrand to make differentiation simpler. This can be achieved by setting $s = a^{1/2} t$. Notice that the integrand has the same value for t and -t (or s and -s) so

we can further simplify by taking the integration from 0 to a and doubling. Thus we get for the numerator $a^{-1/2} \int_0^{a\sqrt{a}} \exp(3s^2/4)\, ds$. We can cancel the $a^{-1/2}$. Now differentiating the numerator gives $\exp(3a^3 4)\, 3a^{1/2}/2$. Differentiating the denominator gives $(27a^{1/2}/4$ + lower powers of a) $\exp(3a^3/4)$. This evaluates to (3/2)/(27/4) = 2/9 as a tends to infinity.

Problem B1
Let C be a cube side 4, center O. Let S be the sphere center O radius 2. Let A be one of the vertices of the cube. Let R be the set of points in C but not S, which are closer to A than to any other vertex of C. Find the volume of R.

Solution
Answer: $8 - 4\pi/3$.

The set of points of C closer to A than to any other vertex is the cube side 2 with A and O as opposite vertices. Evidently the intersection of this cube with S has 1/8 the volume of S. So R has volume $8 - 1/8\ 4/3\ \pi\ 2^3 = 8 - 4\pi/3$.

Problem B2
Let f(n) be the number of ways of representing n as a sum of powers of 2 with no power being used more than 3 times. For example, f(7) = 4 (the representations are 4 + 2 + 1, 4 + 1 + 1 + 1, 2 + 2 + 2 + 1, 2 + 2 + 1 + 1 + 1). Can we find a real polynomial p(x) such that f(n) = [p(n)] ?

Solution
Answer: f(n) = [(n + 2)/2].

We prove this relation by induction. It is obviously true for n = 1. Any representation of 2m can include at most two 1s (because a representation with three 1s is necessarily odd). So from any representation of 2m we can derive a representation of 2m+1 by adding a 1. Similarly, any representation of 2m+1 must include at least one 1 (since it is odd), so we can derive a representation of 2m by removing a 1. Hence f(2m) = f(2m+1). So if the relation is true for 2m then it is true for 2m+1.

Similarly, the representations of 2m with no 1s correspond to the representations of m (just double every component) and the representations of 2m with two 1s correspond to the representations of m - 1 (double every component and add two 1s). So if the relation is true for $n < 2m$, then f(2m) = [(m+2)/2] + [(m+1)/2]. Just one of m+2 and m+1 is odd, so this is (m+2)/2 + (m+1)/2 - 1/2 = m+1 = [(n+2)/2], and the relation holds for n.

Problem B3
y_1, y_2, y_3 are solutions of $y''' + a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ such that $y_1^2 + y_2^2 + y_3^2 = 1$ for all x. Find constants α, β such that $y_1'(x)^2 + y_2'(x)^2 + y_3'(x)^2$ is a solution of $y' + \alpha\, a(x)\, y + \beta c(x) = 0$.

Solution
Answer: α = 2/3, β = -2/3.

Let $z = y_1'(x)^2 + y_2'(x)^2 + y_3'(x)^2$. Multiplying the differential equation by y, we see that y_1 satisfies $y_1 y_1''' + a\, y_1 y_1'' + b\, y_1 y_1' + c\, y_1^2 = 0$. Similarly for y_2 and y_3. Adding, we get $(y_1 y_1''' + y_2 y_2''' + y_3 y_3''') + a\,(y_1 y_1'' + y_2 y_2'' + y_3 y_3'') + b(y_1 y_1' + y_2 y_2' + y_3 y_3') + c\,(y_1^2 + y_2^2 + y_2^2) = 0$ (*). We now use the relation $y_1^2 + y_2^2 + y_3^2 = 1$ (**) to obtain alternative expressions for each of the terms in parentheses.
We already have the term following c. Differentiating (**) we get $y_1 y_1' + y_2 y_2' + y_3 y_3' = 0$, so the term following b vanishes. Differentiating again gives $(y_1 y_1'' + y_2 y_2'' + y_3 y_3'') = -z$, which gives the

term following a. Differentiating again, we get $(y_1'y_1'' + y_2'y_2'' + y_3'y_3'') + (y_1y_1''' + y_2y_2''' + y_3y_3''') = - z'$. The first parenthesis here is just z'/2, so we get $(y_1y_1''' + y_2y_2''' + y_3y_3''') = -3/2\ z'$, which deals with the first term in (*). Putting all this together we have $-3/2\ z' - az + c = 0$, or $z' + 2/3\ az - 2/3\ c = 0$.

Problem B4
Let $f(n) = n + [\sqrt{n}]$. Define the sequence a_i by $a_0 = m$, $a_{n+1} = f(a_n)$. Prove that it contains at least one square.

Solution
Suppose $m = n^2 + r$, with $1 \le r \le n$. Then $[\sqrt{m}] = n$, so $a_1 = n^2 + n + r$. We still have $[\sqrt{(n^2 + n + r)}] = n$ (because the next square up is $n^2 + 2n + 1$), so $a_2 = n^2 + 2n + r = (n + 1)^2 + (r - 1)$. Thus if $r = 1$, then a_2 is a square, and by a simple induction in the general case a_{2r} is a square.
Now if $n+1 \le r < 2n+1$, then we still have $a_1 = n^2 + n + r = (n + 1)^2 + (r - n - 1)$ and $0 \le (r - n - 1) < n$. If $r = 0$, then we have a square. If not, then a further $2(r - n - 1)$ steps give a square.

Problem B5
Define $\|x\|$ as the distance from x to the nearest integer. Find $\lim_{n\to\infty} 1/n \int_1^n \|n/x\|\ dx$. You may assume that $\prod_1^\infty 2n/(2n-1)\ 2n/(2n+1) = \pi/2$.

Solution
Answer: $\ln(4/\pi)$.

Note that on the interval $(2n/2n, 2n/(2n-1))$ the integrand is $n - n/x$; on $(2n/(2n-1), 2n/(2n-2))$ it is $n/x - (n-1)$; on $(2n/(2n-2), 2n/(2n-3))$ it is $(n-1) - n/x$; ... ; on $(2n/5, 2n/4)$ it is $n/x - 2$; on $(2n/4, 2n/3)$ it is $2 - n/x$; and on $(2n/3, 2n/2)$ it is $n/x - 1$.
Integrating the constant term over the interval $(2n/(2r+1), 2n/2r)$ gives $- r\ (2n/2r - 2n/(2r+1)) = - n/(2r + 1)$. Integrating it over the adjacent interval $(2n/(2r+2), 2n/(2r+1))$ gives $+ (r + 1)\ (2n/(2r+1) - 2n/(2r+2)) = n/(2r + 1)$. So over the entire interval (1, n) the constant terms integrate to zero.

Integrating the 1/x term over $(2n/(2r+1), 2n/2r)$ gives $n \ln((2r+1)/2r)$, whilst over $(2n/2r, 2n/(2r-1))$ gives $- n \ln(2r/(2r-1))$. So over the combined interval $(2n/(2r+1), 2n/(2r-1))$ we get $- n \ln (2r/(2r-1)\ 2r/(2r+1))$. Summing we get that integral over $(2n/(2n-1), 2n/3)$ is $- n \ln P_n$, where $P_n = \prod_2^{n-1} 2r/(2r-1)\ 2r/(2r+1)$. That leaves the intervals at each end which integrate to $n \ln 3/2$ and $- n \ln(2n/(2n-1))$.

So multiplying by 1/n and taking the limit we get $- \ln(3/4\ \pi/2) + \ln 3/2 = \ln(4/\pi)$.

Problem B6
Let α be a complex $(2^n + 1)$th root of unity. Prove that there always exist polynomials p(x), q(x) with integer coefficients, such that $p(\alpha)^2 + q(\alpha)^2 = -1$.

Solution
Note that we have $\alpha + \alpha^2 + \alpha^3 + ... + \alpha^N = -1$ where we use $N = 2^n$ (because of the shortcomings of HTML which does not allow iterated exponents). We give an inductive procedure for constructing $p_n(x)$ and $q_n(x)$ so that $p_n(x)^2 + q_n(x)^2 = x^2 + x^4 + x^6 + ... + x^{2N}$. For $x = \alpha$ this becomes $\alpha^2 + \alpha^4 + ... + \alpha^N + \alpha + \alpha^3 + ... + \alpha^{N-1} = -1$.

Evidently for n = 1, we may take $p(x) = \alpha$ and $q(x) = \alpha^2$. Given $p_n(x)$ and $q_n(x)$, take $p_{n+1}(x) = p_n(x) + x^N q_n(x)$, $q_{n+1}(x) = q_n(x) - x^N p_n(x)$. Then $p_{n+1}(x)^2 + q_{n+1}(x)^2 = (p_n(x)^2 + q_n(x)^2)(1 + x^{2N}) = x^2 + x^4 + ... + x^{4N}$ as required.

45th Putnam 1984

Problem A1
S is an a x b x c brick. T is the set of points a distance 1 or less from S. Find the volume of T.

Solution
Answer: abc + 2(ab + bc + ca) + π(a + b + c + 4/3).

T comprises: S, volume abc; a slab on each face, two volume ab, two volume bc, two volume ca; a quarter cylinder on each edge, four with length a, total volume πa, four length b, total volume πb, and four length c, total volume πc; and an eighth sphere radius 1 at each vertex, total volume 4/3 π.

Problem A2
Evaluate $6/((9 - 4)(3 - 2)) + 36/((27 - 8)(9 - 4)) + ... + 6^n/((3^{n+1} - 2^{n+1})(3^n - 2^n)) + ...$.

Solution
Answer: 2.

An easy induction on n shows that the sum of the first n terms is $6(3^n - 2^n)/(3^{n+1} - 2^{n+1})$. We may write this as $2 (1 - (2/3)^n)/(1 - (2/3)^{n+1})$, which shows that it tends to 2 as n tends to infinity.

Problem A3
Let A be the 2n x 2n matrix whose diagonal elements are all x and whose off-diagonal elements $a_{ij} = a$ for i + j even, and b for i + j odd. Find $\lim_{x \to a} \det A/(x - a)^{2n-2}$.

Solution
Answer: $n^2(a^2 - b^2)$

Subract the first row from the third, fifth and other odd numbered rows. Subtract the second row from the fourth, sixth and other even numbered rows. This gives all rows from 3 onwards a factor (x - a). Having extracted these factors, we may set x = a, to get $\lim_{x \to a} \det A/(x - a)^{2n-2}$ =

```
 a   b   a   b  ...  a   b
 b   a   b   a  ...  b   a
-1   0   1   0  ...  0   0
 0  -1   0   1  ...  0   0
...
 0  -1   0   0  ...  0   1
```

Now add the first row to the second row. This gives a factor (a + b). Remove it and add cols 3, 5 , ... to col 1 and subtract cols 2, 4, 6, ... from col 1. This gives n(a - b) in position 1, 1 and zeros elsewhere in col 1, so we may take out a factor n(a - b). Now add cols 4, 6, ... to col 2. This gives nb in position 1, 2 and n in position 2, 2 with zeros elsewhere in col 2. So we may take out a factor n, leaving

```
 1   b   a   b  ...  a   b
 0   1   1   1  ...  1   1
 0   0   1   0  ...  0   0
 0   0   0   1  ...  0   0
...
 0   0   0   0  ...  0   1
```

Expanding by the first col gives

```
 1    1    1   ...   1    1
 0    1    0   ...   0    0
 0    0    1   ...   0    0
...
 0    0    0   ...   0    1
```

Expanding again by the first col gives 1. So $\lim_{x \to a} \det A/(x - a)^{2n-2} = n^2(a^2 - b^2)$.

Problem A4
A convex pentagon inscribed in a circle radius 1 has two perpendicular diagonals which intersect inside the pentagon. What is the maximum area the pentagon can have?

Solution
Answer: $1 + (3\sqrt{3})/4$.

Let the pentagon be ABCDE with AC and BD the perpendicular diagonals. It is clear that E should be taken as the midpoint AD (since this maximises the area of ADE without affecting the area of ABCD).
There are evidently two independent variables, so in principle all we have to do is to choose the variables, express the area as a function of the two variables and set its partial derivatives to zero. The difficult part is choosing the variables. If, for example, we take the two lengths AC, BD, then we get a simple expression for the area of ABCD, but a messy expression for the area of ADE.

The best choice seems to be the angles subtended by the sides at O, the center of the circle. Note that if AB subtends α and CD subtends β, then angle ACB = $\alpha/2$ and angle CBD = $\beta/2$. But if AC and BD are perpendicular, then these two angles must sum to $\pi/2$. So α and β sum to π. Thus we may take as our two independent variables α and θ, the angle AOE. Then 2 x area ABCDE = 2 area AOB + 2 area BOC + 2 area COD + 2 area DOE + 2 area EOA = $\sin \alpha + \sin(2\theta) + \sin \alpha + \sin \theta \cos \theta$.

We maximise $\sin \alpha$ at 1 by taking $\alpha = \pi/2$, and we maximise $\sin \theta + \sin \theta \cos \theta$ at $(3\sqrt{3})/4$ by taking $\theta = \pi/3$ (for example, write $k = \sin \theta$, so $\sqrt{(1 - k^2)} = \cos \theta$ and differentiate).

Problem A5
V is the pyramidal region x, y, z >= 0, x + y + z <= 1. Evaluate $\int_V x\, y^9\, z^8\, (1 - x - y - z)^4\, dx\, dy\, dz$.

Solution
Answer: 9! 8! 4! / 25! .

The most straightforward approach is simply to carry out the integrations. This gets slightly messy, gives no insight into what is going on, and carries the risk of error. On the other hand, it is not *that* bad and it gets the answer reliably and quickly.
Integrate first over x, from 0 to 1 - y - z. The indefinite integral has two terms - $x(1 - x - y - z)^5 y^9 z^8/5 - (1 - x - y - z)^6 y^9 z^8/30$. The first gives zero and the second gives $y^9 z^8 (1 - y - z)^6/30$.

Now integrate over z, from 0 to 1 - y. Ignoring the constant factor $y^9/30$, we get 7 terms $z^9(1 - y - z)^6/9 + ... + z^{15}$ (6! 8!)/15! . The first 6 integrate to zero. The last gives that the required integral I = 6! 8! /(15! 30) $\int_0^1 y^9 (1 - y)^{15}\, dy$. The integral from 0 to 1 is 9! 15! / 25! Hence the result.

Problem A6
Let f(n) be the last non-zero digit in the decimal representation of n! . Show that for distinct integers

$a_i \geq 0$, $f(5^{a}_1 + 5^{a}_2 + ... + 5^{a}_r)$ depends only on $a_1 + ... + a_r = a$. Write the value as g(a). Find the smallest period for g, or show that it is not periodic.

Solution
Answer: period 4.

Let ν(n) denote the last non-zero digit of n. Evidently ν(10n) = ν(n) and if mn is not a multiple of 10, then ν(mn) = ν(ν(m)ν(n)) . We have ν((5n+1)(5n+2)) = ν((5n+3)(5n+4)) = 2. Hence f(5n) = $\nu(4^n 5^n n!) = \nu(2^n n!) = \nu(2^n f(n))$ (*).

We now show by induction that $f(5^a + 5^b + ...) = \nu(2^{a+b+...})$ for $a > b > ... \geq 0$. It is clearly true for $5^a + 5^b + ... = 1$ or 5. So suppose it is true for all values < $N = 5^a + 5^b + ...$ There are two cases to consider. If the smallest power of 5 in N is 5^0, then N - 1 is a multiple of 5, so N ends in a 6 or a 1. (N - 1)! is even, so f(N-1) = 2, 4, 6 or 8. In any case its last digit is unchanged on multiplying by 1 or 6, so f(N) = f(N-1). The result is true for N - 1 and the a + b + ... is not changed by adding 0, so it is also true for N.

So it remains to consider the case where N is divisible by 5. In other words N = 5M, where $M = 5^{a-1} + 5^{b-1} + ...$. By (*), we have that $f(N) = \nu(2^M f(M))$. Now $\nu(2^5) = 2$, so by a simple induction $\nu(2^k) = 2$ if k is any power of 5. Hence $\nu(2^M) = \nu(2^n)$, where n is the number of terms in $M = 2^{a-1} + 2^{b-1} + ...$. By induction $f(M) = \nu(2^{a-1+b-1+...})$, hence $f(N) = \nu(2^{a+b+...})$ as required.
Finally, we find immediately that $\nu(2^1) = 2$, $\nu(2^2) = 4$, $\nu(2^3) = 8$, $\nu(2^4) = 6$, $\nu(2^5) = 2$. Hence the period is 4.

Problem B1
Define $f(n) = 1! + 2! + ... + n!$. Find a recurrence relation f(n + 2) = a(n) f(n + 1) + b(n) f(n), where a(x) and b(x) are polynomials.

Solution
f(n+2) = (n+3) f(n+1) - (n+2) f(n).

Problem B2
Find the minimum of $f(x, y) = (x - y)^2 + (\sqrt{(2 - x^2)} - 9/y)^2$ in the half-infinite strip $0 < x < \sqrt{2}$, $y > 0$.

Solution
Answer: 8.

Setting the partial derivatives to zero gives: $y^2 = 9x/\sqrt{(2 - x^2)}$ and $-x + y + 9/y^2 \sqrt{(2 - x^2)} - 81/y^3$. Using the first equation to substitute $9x/y^2$ for $\sqrt{(2 - x^2)}$, the second equation becomes $(x - y)(81/y^4 - 1)$, so x = y or y = 3. Substituting back in the first equation, we find that x = y does not give a stationary point, whereas y = 3 gives x = 1, y = 3. This gives the value 8. Finally, we check that we cannot get a minimum on the edges y = 0, $x = \sqrt{2}$, and that the value is a minimum not a maximum or saddle point by comparing the value to two nearby values.

Problem B3
Let S_n be a set with n elements. Can we find a binary operation * on S which satisfies (1) right cancellation: a*c = b*c implies a = b (for all a, b, c), and (2) non-associativity: $a * (b * c) \neq (a * b) * c$ for all a, b, c? Note that we are not just requiring that * is not associative, but that it is *never* associative.

Solution
Answer: yes.

Let the elements be a_1, ... , a_n. Define $a_i*b = a_{i+1}$ for all b (taking a_{n+1} to be a_1). Then we certainly have right cancellation, because if $a_i*c = a_j*c$, then $a_{i+1} = a_{j+1}$, so i = j. We also have non-associativity, because $a_i*(b*c) = a_{i+1}$, but $(a_i*b)*c = a_{i+1}*c = a_{i+2}$.

Problem B4
Find all real valued functions f(x) defined on [0, ∞), such that (1) f is continuous on [0, ∞), (2) f(x) > 0 for x > 0, (3) for all $x_0 > 0$, the centroid of the region under the curve y = f(x) between 0 and x_0 has y-coordinate equal to the average value of f(x) on [0, x_0].

Solution
Answer: $f(x) = A\, x^{1+\sqrt{2}}$.

The condition is $\int_0^x 1/2\, f^2(t)\, dt = 1/x\, (\int_0^x f(t)\, dt)^2$. Put $y = \int_0^x f(t)\, dt$ and differentiate to get $1/2\, (y')^2 = 2yy'/x - y^2/x^2$, or $x^2(y')^2 - 4xyy' + 2y^2 = 0$.
At this point it is easy to get stuck. The key is to factorize the equation: $(x\, y' - (2 + \sqrt{2})\, y\,)\, (x\, y' - (2 - \sqrt{2})\, y\,) = 0$. Hence $y = Ax^{2+\sqrt{2}}$ or $Ax^{2-\sqrt{2}}$. Hence $f(x) = Bx^{1+\sqrt{2}}$ or $Bx^{1-\sqrt{2}}$. But f(0) is defined, so $f(x) = Bx^{1+\sqrt{2}}$ at least for small x. There remains the possibility that we might switch to $Cx^{1-\sqrt{2}}$ at some x_0. But it is easy to see that if we fix C so that B is continuous, then for $x > x_0$, $\int_0^x f(t)\, dt$ will have a non-zero constant term as well as an $x^{2-\sqrt{2}}$ term and hence we will not satisfy the required equation. So the only possible solutions are those given.

Problem B5
Let f(n) be the number of 1s in the binary expression for n. Let $g(m) = \pm 0^m \pm 1^m \pm 2^m \ldots \pm (2^m - 1)^m$, where we take the + sign for k^m iff f(k) is even. Show that g(m) can be written in the form $(-1)^m\, a^{p(m)}\, (q(m))!$ where a is an integer and p(x) and q(x) are polynomials.

Solution
Answer: a = 2, p(m) = m(m-1)/2, q(m) = m.

Put $h(n) = (-1)^{f(n)}$. We show that $h(0)\, 0^m + h(1)\, 1^m + h(2)\, 2^m + \ldots + h(2^n - 1)\, (2^n - 1)^m = 0$ for n > m and $(-1)^m\, m!\, 2^{m(m-1)/2}$ for n = m. We use induction on m. This answers the question since the case n = m is just g(m).

For m=0 in the case n = m the sum is just h(0) = 1 which is $(-1)^0\, 0!\, 2^0$, as required. For the case n > m the sum is just $h(0) + h(1) + \ldots + h(2^n - 1)$. But we always have f(2r+1) = f(2r) + 1 and hence h(2r) + h(2r+1) = 0, so the sum is zero as required.

Now assume the result is true for values ≤ m. Put $N(r) = 2^r - 1$. Since $h(2^m + k) = - h(k)$ for $k < 2^m$, we have that $\sum_0^{N(m+1)} h(k)\, k^{m+1} = - \sum_0^{N(m)} h(k)\, (\, (k + 2^m)^{m+1} - k^{m+1}) = - (m+1)\, 2^m \sum_0^{N(m)} h(k)\, k^m +$ other terms which are zero by the induction hypothesis $= -(m+1)\, 2^m\, (-1)^m\, m!\, 2^{m(m-1)/2}$ by induction $= (-1)^{m+1}\, (m+1)!\, 2^{m(m+1)/2}$ as required.

Similarly, for the sum to N(n), where n > m+1, the binomial expansion of the summand $(\, (k + 2^m)^{m+1} - k^{m+1})$ gives a series of terms all of which sum to zero. So the case m+1 is established.

Problem B6
Define a sequence of convex polygons P_n as follows. P_0 is an equilateral triangle side 1. P_{n+1} is obtained from P_n by cutting off the corners one-third of the way along each side (for example P_1 is a regular hexagon side 1/3). Find $\lim_{n\to\infty} \text{area}(P_n)$.

Solution
Answer: (√3)/7.

n = 0 and n = 1 are untypical cases. Later polygons have unequal sides and sides that do not touch the circle inscribed in P_0. The key is to look at a slightly more general case. Suppose that just before the nth series of cuts we have two adjacent sides, represented by vectors **a**, **b**. Then the new side formed when we cut off the corner is (**a**+**b**)/3, so the three new sides are **a**/3, (**a**+**b**)/3, **b**/3, and the area of corner removed is (**a x b**)/2. At step n+1 we remove two corners, with areas (**a**/3 **x** (**a**+**b**)/3 + (**a**+**b**)/3 **x b**/3)/2 = (**a x b**)/9 = 2/9 x area removed at step n.

The area removed in going from the triangle to the hexagon is 1/3 area triangle. So the total area removed is 1/3 $(1 + 2/9 + (2/9)^2 + ...)$ = 3/7 area triangle. Hence the area remaining is 4/7 $(\sqrt{3})/4$ = $(\sqrt{3})/7$.

46th Putnam 1985

Problem A1
How many triples (A, B, C) are there of sets with union $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A \cap B \cap C = \varnothing$?

Solution
Answer: 6^{10}.

Careful. The answer is not 3^{10} (this trap is the only reason I do not rate the problem trivial!). There are six possibilities for each element: just A, just B, just C, not A (meaning B and C, but not A), not B, or not C.

Problem A2
ABC is an acute-angled triangle with area 1. A rectangle R has its vertices R_i on the sides of the triangle, R_1 and R_2 on BC, R_3 on AC and R_4 on AB. Another rectangle S has it vertices on the sides of the triangle AR_3R_4, two on R_3R_4 and one one each of the other two sides. What is the maximum total area of R and S over all possible choices of triangle and rectangles?

Solution
Answer: 2/3.

Let $k = AR_3/AC$. The triangle AR_4R_3 is similar to ABC with all dimensions a factor k smaller. So its area is k^2. Similarly, the two triangles BR_4R_1 and CR_3R_2 together have area $(1-k)^2$. So the area of the bottom rectangle is $1 - k^2 - (1-k)^2 = 2k(1-k)$. Defining h in a similar way, the area of the top rectangle is $2h(1-h)$ times the area of triangle $AR_4R_3 = 2h(1-h)k^2$.
For any given choice of k, we maximise the area of the top rectangle by taking h = 1/2. The area of the two rectangles together is then $2k(1-k) + k^2/2 = 2/3 - 3/2(k - 2/3)^2$. This evidently has maximum value 2/3, obtained by taking k = 2/3. Note that the maximum is independent of the choice of triangle.

Problem A3
x is a real. Define $a_{i\,0} = x/2^i$, $a_{i\,j+1} = a_{i\,j}{}^2 + 2\,a_{i\,j}$. What is $\lim_{n\to\infty} a_{n\,n}$?

Solution
Answer: $e^x - 1$.

$a_{n+1\,n}$ for x is the same as $a_{n\,n}$ for x/2. So if we write $a_{n\,n}$ as $p_n(x)$ then evidently p_n is a polynomial and we have the relation $p_{n+1}(x) = 2p_n(x/2) + p_n(x/2)^2$ (*) and also $p_0(x) = x$.

If we add 1 to both sides of (*), then it becomes $1 + p_{n+1}(x) = (1 + p_n(x/2))^2$. Iterating: $1 + p_n(x) = (1 + x/N)^N$, where $N = 2^n$. But $\lim (1 + x/n)^n = e^x$ as $n \to \infty$, so $p_n(x) \to e^x - 1$.

Problem A4
Let a_n be the sequence defined by $a_1 = 3$, $a_{n+1} = 3^k$, where $k = a_n$. Let b_n be the remainder when a_n is divided by 100. Which values b_n occur for infinitely many n?

Solution
Answer: 87.

We find fairly quickly that $3^{20} = 1 \bmod 100$. So $a_3 = 3^{27} = 3^7 = 87 \bmod 100$. Hence $a_4 = 3^{87} = 3^7 = 87 \bmod 100$. So $b_n = 87$ for $n > 2$.

Problem A5
Let $f_n(x) = \cos x \cos 2x \ldots \cos nx$. For which n in the range 1, 2, ... , 10 is the integral from 0 to 2π of $f_n(x)$ non-zero?

Solution
Answer: 3, 4, 7, 8.

Note first that since $\cos x = \cos(2\pi - x)$, the integral is twice the integral over 0 to π, so it is sufficient to consider the smaller range.
$\cos(n\pi/2 - x) = (-1)^n \cos(n\pi/2 + x)$. So $\cos(n\pi/2 - x)) = \cos(n\pi/2 + x)$ for n even, and $-\cos(n\pi/2 + x)$ for n odd. Hence for n = 1, 2, 5, 6, 9, 10, $f_n(\pi/2 - x) = -f_n(\pi/2 + x)$ and so the integral over 0 to π is zero. It remains to consider the cases n = 3, 4, 7, 8.

I cannot see a neat way of proving them non-zero. What follows is an inelegant proof using $\cos a \cos b = (\cos(a+b) + \cos(a-b))/2$ repeatedly. So $\cos x \cos 3x = 1/2 \cos 4x + 1/2 \cos 2x$. So $f_3(x) = 1/2 \cos^2 2x + 1/2 \cos 2x \cos 4x = 1/2 \cos^2 2x + 1/4 \cos 2x + 1/4 \cos 6x$. The integral of the first term is positive and the integrals of the second and third terms are zero. So f_3 has non-zero integral.

Similarly, $f_4(x) = (\cos x \cos 2x)(\cos 3x \cos 4x) = 1/4 (\cos x + \cos 3x)(\cos x + \cos 7x) = 1/4 \cos^2 x + 1/8 (\cos 2x + \cos 4x + \cos 6x + \cos 8x + \cos 4x + \cos 10x)$. All terms except the first integrate to zero and the first has a non-zero integral. Hence f_4 has non-zero integral.

Similarly, $f_7(x) = \cos x (\cos 2x \cos 3x)(\cos 4x \cos 5x)(\cos 6x \cos 7x) = 1/8 \cos x (\cos x + \cos 5x)(\cos x + \cos 9x)(\cos x + \cos 13x) = 1/8 \cos x(\cos^2 x + \cos x \cos 5x + \cos x \cos 9x + \cos 5x \cos 9x)(\cos x + \cos 13x) = 1/16 \cos x (1 + \cos 2x + \cos 4x + \cos 6x + \cos 8x + \cos 10x + \cos 4x + \cos 14x)(\cos x + \cos 13x) = 1/32 \cos x (\cos x + \cos x + \cos 3x + 2\cos 3x + 2\cos 5x + \cos 5x + \cos 7x + \cos 7x + \cos 9x + \cos 9x + \cos 11x + \cos 13x + \cos 15x + \cos 11x + \cos 15x + 2 \cos 9x + 2 \cos 17x + \cos 7x + \cos 19x + \cos 5x + \cos 21x + \cos 3x + \cos 23x + \cos x + \cos 27x) = 1/32 (3\cos^2 x +$ other terms), where the other terms all have the form cos ax cox bx with a and b unequal and hence integrate to zero. So f_7 has non-zero integral.

Finally, $f_8(x) = 1/16 (\cos x + \cos 3x)(\cos x + \cos 7x)(\cos x + \cos 11x)(\cos x + \cos 15x) = 1/32 (1 + 2 \cos 2x + 2\cos 4x + \cos 6x + \cos 8x + \cos 10x)(\cos x + \cos 11x)(\cos x + \cos 15x) = (4 \cos x + 5 \cos 3x + 4 \cos 5x + 4 \cos 7x + 4 \cos 9x + 2 \cos 11x + 2 \cos 13x + 2 \cos 15x + \cos 17x + \cos 19x + \cos 21x)(\cos x + \cos 15x) = 1/64 (6 +$ other terms). So f_8 has non-zero integral.

Problem A6
Find a polynomial f(x) with real coefficients and f(0) = 1, such that the sums of the squares of the coefficients of $f(x)^n$ and $(3x^2 + 7x + 2)^n$ are the same for all positive integers n.

Solution
Answer: $6x^2 + 5x + 1$.

The sums of the squares of the coefficients of $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ equals the coefficient of x^n in $(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0)(a_0 x^n + a_1 x^{n-1} + \ldots + a_n)$. But $(3x^2 + 7x + 2)(2x^2 + 7x + 3) = (6x^2 + 5x + 1)(x^2 + 5x + 6) = 6x^4 + 35x^3 + 62x^2 + 25x + 6$. So $6x^2 + 5x + 1$ has the required property.
How did we find it? Well, Note that $3x^2 + 7x + 2 = (3x + 1)(x + 2)$, so we pick $(3x + 1)(2x + 1)$.

Problem B1
p(x) is a polynomial of degree 5 with 5 distinct integral roots. What is the smallest number of non-zero coefficients it can have? Give a possible set of roots for a polynomial achieving this minimum.

Solution
Answer: 3. For example: 0, ±1, ±2.

The example given has three non-zero coefficients: $x(x^2 - 1)(x^2 - 4) = x^5 - 5x^3 + 4x$. So we have to show that two coefficients are not possible.
The polynomial cannot be $x^5 + ax^n$ with $n > 1$, because it would then have 0 as a repeated root and hence less than 5 distinct roots. $x^5 + ax$ has at most 3 real roots, and $x^5 + a$ has only one real root.

Problem B2
The polynomials $p_n(x)$ are defined as follows: $p_0(x) = 1$; $p_{n+1}'(x) = (n + 1)\, p_n(x + 1)$, $p_{n+1}(0) = 0$ for $n \geq 0$. Factorize $p_{100}(1)$ into distinct primes.

Solution
Answer: $(101)^{99}$.

We show that $p_n(x) = x\,(x + n)^{n-1}$ for $n > 0$. In fact, this is an easy induction. We have $p_{n+1}'(x) = (n + 1)\,(x + n + 1)^n - n(n + 1)(x + n + 1)^{n-1}$. Integrating: $p_{n+1}(x) = (x + n + 1)^{n+1} - (n + 1)(x + n + 1)^n = x(x + n + 1)^n$.

Problem B3
$a_{i\,j}$ is a positive integer for i, j = 1, 2, 3, ... and for each positive integer we can find exactly eight $a_{i\,j}$ equal to it. Prove that $a_{i\,j} > ij$ for some i, j.

Solution
The idea is to show that for some N there are more than 8N pairs (i, j) with $ij \leq N$. For then if we always had $a_{i\,j} \leq ij$ we would have $a_{i\,j} \leq N$ for more than 8N pairs (i, j) and hence some value in {1, 2, ... , N} would be shared by more than 8 $a_{i\,j}$, contradicting the premise.

The number of pairs (i, j) with $ij \leq N$ is $[N/1] + [N/2] + ... + [N/N] = N(1 + 1/2 + 1/3 + ... + 1/N) - N$. But $\sum 1/n$ diverges, so it is certainly greater than 9 for N sufficiently large.

Problem B4
Let C be the circle radius 1, center the origin. A point P is chosen at random on the circumference of C, and another point Q is chosen at random in the interior of C. What is the probability that the rectangle with diagonal PQ, and sides parallel to the x-axis and y-axis, lies entirely inside (or on) C?

Solution
Answer: $4/\pi^2$.

Given P, let R be the rectangle inscribed in C with sides parallel to the axes. Then the rectangle with diagonal PQ lies entirely inside C iff Q lies inside R.
Let O be the center of C. Let the diameter of C parallel to the y-axis make an angle θ with OP. Then the area of R is $4 \sin\theta \cos\theta$. So the required probability is $2/\pi \int_0^{\pi/2} (4 \sin\theta \cos\theta)/\pi \, d\theta = 4/\pi^2 \int_0^{\pi/2} \sin 2\theta \, d\theta = 4/\pi^2$.

Problem B5
Assuming that $\int_{-\infty}^{\infty} e^{sq(x)}\, dx = \sqrt{\pi}$ (where $sq(x) = x^2$), find $\int_0^{\infty} x^{-1/2}\, e^{-1985(x + 1/x)}\, dx$.

Solution
A little experimentation suggests that we are not going to solve the problem by substitutions like $t = \sqrt{x}$. That substitution gives $I = 2 \int_0^\infty \exp(-k(x^2 + 1/x^2))\, dx$ (*).
Another standard approach is to introduce an additional variable y into the integrand and then to differentiate. The hope is that we can solve the resulting differential equation. We still need an initial condition, so we must choose the variable so that the integration simplifies for y = 0 (or some other suitable value).

Starting from (*), we put $f(y) = \int_0^\infty \exp(-kx^2 - y/x^2)\, dx$. Hence $f'(y) = - \int_0^\infty 1/x^2 \exp(-kx^2 - y/x^2)\, dx$. If we substitute $s = 1/x$, then we get $f'(y) = - \int_0^\infty \exp(-k/s^2 - ys^2)\, ds$. Now putting $s = t \sqrt{(k/y)}$ gives $f'(y) = - \sqrt{(k/y)} \int_0^\infty \exp(-kt^2 - y/t^2)\, dt = - \sqrt{(k/y)}\, f(y)$. Integrating, we get $f(y) = f(0) \exp(- 2 \sqrt{(ky)})$. But $f(0) = 1/2 \sqrt{(\pi/k)}$, so $I = 2f(k) = \sqrt{(\sqrt{/k)}}\, e^{-2k}$.

Problem B6
G is a finite group consisting of real n x n matrices with the operation of matrix multiplication. The sum of the traces of the elements of G is zero. Prove that the sum of the elements of G is the zero matrix.

Solution
Let the elements of G be $A_1, A_2, \ldots, A_m$. Let $A = \sum A_i$. Then $A_iA = A$ (since G is a group the multiplication just rearranges the order of the terms in the sum). Hence $A^2 = mA$ (*).
Now if we allow complex values, A has m eigenvalues (some possibly repeated). Similarly, if we allow complex vectors, then each eigenvalue has an eigenvector. But if $A\mathbf{v} = \lambda\mathbf{v}$, then (*) implies that $\lambda^2 = m\lambda$, so $\lambda = 0$ or m. But the trace of A is the sum of its eigenvalues, so m cannot be an eigenvalue. In other words, det(A - mI) is non-zero and A - mI is invertible. Suppose its inverse is B. We may write (*) as A(A - mI) = 0. Hence 0 = A(A - mI)B = A.

47th Putnam 1986

Problem A1
S is the set {x real st $x^4 - 13x^2 + 36 \le 0$}. Find the maximum value of $f(x) = x^3 - 3x$ on S.

Solution
Answer: 18.

$x^4 - 13x^2 + 36 = (x^2 - 4)(x^2 - 9)$, so S is the union of the intervals [-3, -2] and [2, 3]. But f(x) is negative on the first interval and positive on the second, so the maximum value occurs in the second interval. But f(x) is strictly increasing on the second interval (it is the product of x and $x^2 - 3$, both strictly increasing and positive), so the maximum value occurs at the endpoint x = 3 and is 18.

Problem A2
What is the remainder when the integral part of $10^{20000}/(10^{100} + 3)$ is divided by 10?

Solution
Answer: 3.

We start with the identity $1 = (1 - (3/10^{100}) + (3/10^{100})^2 - (3/10^{100})^3 + ...) (1 + 3/10^{100})$. Multiplying through by 10^{20000} and dividing by $(10^{100} + 3)$ we get: $10^{20000}/(10^{100} + 3) = \sum \{10^{19900} (-3/10^{100})^n\}$ (*), and the early terms in this sum are all integral. In fact they are integral for $n < 200$ and these terms sum to $10^{19900}(1 - (3/10^{100})^{200}) /(1 + 3/10^{100}) = (10^{20000} - 3^{200})/(10^{100} + 3)$. Multiplying by $(10^{100} + 3)$ and subracting from 10^{20000} gives $3^{200} = 9^{100}$ which is less than $10^{100} + 3$. So the integral part of $10^{20000}/(10^{100} + 3)$ is indeed given by the first 200 terms, $0 \le n < 200$, of (*). The first 199 terms are all divisible by 10. The last term is $(-3)^{199}$. But $3^2 = -1 \pmod{10}$, so $(-3)^{198} = 3^{198} = (-1)^{99} = -1 \pmod{10}$, and hence $(-3)^{199} = 3 \pmod{10}$.

Problem A3
Find $\cot^{-1}(1) + \cot^{-1}(3) + ... + \cot^{-1}(n^2+n+1) + ...$, where the $\cot^{-1}(m)$ is taken to be the value in the range $(0, \pi/2]$.

Solution
Answer: $\pi/2$.

The problem is trivial once you see that $\cot^{-1}(n^2 + n + 1) = \cot^{-1}n - \cot^{-1}(n+1)$. [Because then the sum telescopes to $\cot^{-1}0 = \pi/2$.]
But how do you get that insight? We know that Putnam problems are broadly in order of increasing difficulty and that the 3rd problem, whilst not trivial, is usually fairly easy, so we expect this the solution to be fairly short and straightforward. The simplest thing is to try to evaluate $\cot^{-1}m$ explicitly. But whilst $\cot^{-1}1 = \pi/4$, but there do not look to be any simple expressions for $m > 1$.

The next simplest thing is likely to be something involving some $\cot^{-1}a \pm \cot^{-1}b$ formula. We probably do not know any such formulae, but we do know $\cot(A \pm B)$, or can easily work it out from the formulae for $\sin(A + B)$ and $\cos(A + B)$: $\cot(A + B) = (\cot A \cot B - 1)/(\cot A + \cot B)$ and $\cot(A - B) = (\cot A \cot B + 1)/(\cot B - \cot A)$. To turn that into something involving $\cot^{-1}$, we need to put $A = \cot^{-1}a$, $B = \cot^{-1}b$, giving $\cot^{-1}a + \cot^{-1}b = \cot^{-1}((ab - 1)/(a + b))$, and $\cot^{-1}a - \cot^{-1}b = \cot^{-1}((ab + 1)/(b - a))$ (*).

That may not be enough of a clue. Putting $a = n^2 + n + 1$, $b = (n+1)^2 + (n+1) + 1$, for example, just gives a mess. But we might think of telescoping, or just of using the formula the other way around

to get $\cot^{-1}(n^2 + n + 1)$ as a sum or difference of two $\cot^{-1}$. With a little playing around that should lead to using the difference with a = n, b = n+1 in (*).

An alternative approach is to notice that for large m, $\cot^{-1}m$ is about 1/m. But $1/(n^2 + n + 1)$ is about $1/(n^2 + n) = 1/n - 1/(n+1)$ and hence about $\cot^{-1}n - \cot^{-1}(n+1)$. That might trigger the insight. Alternatively, we might speculate about why $n^2 + n + 1$ had been chosen. If we noticed that $n^2 + n + 1 = n(n + 1) + 1$, we might think about the relation to the cot(A + B) formulae (remembering perhaps the cot A cot B + 1). That could give us the solution.

Problem A4
Let r(n) be the number of n x n matrices $A = (a_{ij})$ such that: (1) each $a_{ij} = -1$, 0, or 1; and (2) if we take any n elements a_{ij}, no two in the same row or column, then their sum is the same. Find rational numbers a, b, c, d, u, v, w such that $r(n) = a\,u^n + b\,v^n + c\,w^n + d$.

Solution
Answer: $r(n) = 2.3^n + 4^n - 4.2^n + 1$.

The key insight is that the elements in any two rows have a fixed difference between corresponding elements. In other words, if the first row is $a_1, a_2, \ldots, a_n$, then the ith row is $a_1 + b, a_2 + b, \ldots, a_n + b$ for some b (*). Indeed, a little thought shows that this is a necessary and sufficient condition for (2) to hold. Because if we have $r \neq s$ and $c \neq d$, then given a sum involving a_{rc} and a_{sd}, we have another sum which is identical except that these two elements are replaced by a_{rd} and a_{sc}. Hence $a_{rc} - a_{sc} = a_{rd} - a_{sd}$. So as we vary d the difference $a_{rd} - a_{sd}$ remains fixed.

We now examine the following disjoint cases: (A) all elements of the first row are the same; (B) both -1 and 1 appear somewhere in the first row; (C) 1 and 0, but not -1, appear in the first row; (D) -1 and 0, but not 1, appear in the first row.
In case (A), (*) implies that the same must be true for the other rows, and then (1) is automatically satisfied. We have 3 choices for each row, so 3^n possibilities in all.

In case (B), the other rows must all be identical to the first, because if the difference b in (*) is non-zero then we get an element that does not satisfy (1). In other words each column has all elements the same. So we have to find the number of possibilities for the first row. The total number of possibilities is 3^n. There are 2^n possibilities where 1 is not present, 2^n where -1 is not present, and 1 where neither 1 nor -1 is present. Hence the required number is $3^n - 2.2^n + 1$.

In case (C), b in (*) must be 0 or -1. There are $2^n - 2$ ways of choosing the first row so that it includes 1 or 0, but not -1. We then have 2 choices for the difference b for each of the other n-1 rows, or 2_{n-1} difference choices in all, giving a total of $2^{2n-1} - 2^n$ possibilities in total. Similarly, for case (D). So in (C) and (D) together there are $4^n - 2.2^n$ possibilities.

Adding up, we get $4^n + 2.3^n - 4.2^n + 1$. As a check, this is correct for n = 1.
In a sense, everything after the first paragraph above is fairly routine (once you grasp that the small number of available values for the elements places severe restrictions on b). But *care* is needed to avoid double counting.

Notice also the typical Putnam misdirection of *rational* values.
Finally, a formula of this type should instantly suggest recurrence relations. But I did not find that a productive approach. Does anyone else have a solution of that type?

Problem A5
$\mathbf{f}:R^n \to R^n$ is defined by $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and the n functions $f_i:R^n \to R$ have continuous 2nd order partial derivatives and satisfy $\partial f_i/\partial x_j - \partial f_j/\partial x_i = c_{ij}$ (for all $1 \le i, j \le n$) for some constants c_{ij}. Prove that there is a function $g:R^n \to R$ such that $f_i + \partial g/\partial x_i$ is linear (for all $1 \le i \le n$).

Solution
Well, it *is* easy, provided one can assume the standard result of the vector calculus that if $\partial f_i/\partial x_j = \partial f_j/\partial x_i$ for all i, j, then there is a function $g:R^n \to R$ such that $\mathbf{f} = \nabla g$.

But can one assume this result? Well, it is pure bookwork, and relatively time-consuming to prove, so presumably yes. Of course, it is a *bad* question, because it is unclear that you can assume it. The question does not *tell* you that you can assume it, because that would make it trivial - the only difficulty of the question is that it is slightly inverted, apparently asking about some linear expression rather than about g.

Let $h_i(\mathbf{x}) = (c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{in}x_n)/2 - \mathbf{f}(\mathbf{x})$. Then $\partial h_i/\partial x_j - \partial h_j/\partial x_i = c_{ij}/2 - c_{ji}/2 - c_{ij} = 0$, since it follows directly from the definition that $c_{ij} = -c_{ji}$.

Problem A6
Let p(x) be the real polynomial $1 + \alpha_1 x^{m}_1 + \alpha_2 x^{m}_2 + \ldots + \alpha_n x^{m}_n$, where $0 < m_1 < m_2 < \ldots < m_n$. For some real polynomial q(x) we have $p(x) = (1 - x)^n q(x)$. Find q(1) in terms of $m_1, \ldots, m_n$.

Solution
Answer: $m_1 m_2 \ldots m_n/n!$

Notice that n occurs in two places: p(x) has n terms apart from 1, and it also has an n-fold root x = 1. The first key (but elementary) point is the fact that p(x) has a factor $(1 - x)^n$ iff $p(1) = p'(1) = \ldots = p^{(n-1)}(1) = 0$. This gives us n linear equations for the n variables α_i, so the α_i are completely determined in terms of the m_i. Moreover, differentiating p(x) one more time (n in all) gives $p^{(n)}(1) = (-1)^n n!\, q(1)$. So having solved for the α_i we just have to substitute them into $p^{(n)}(1)$ to get the answer. That is all straightforward, the only difficulty of the question is actually carrying out that programme. If you are not familiar with determinants, it can seem sufficiently intimidating to make you give up and try another approach.

It is convenient to have some notation for $m_1(m_1 - 1) \ldots (m_1 - r + 1)$. The traditional notation is Pochhammer's symbol $(m_1)_r$, but Graham et al are pushing $m_1^{\underline{r}}$ (read as m_1 to the r falling) [see Ronald Graham, Donald Knuth, Oren Patashnik, Concrete Mathematics, 2nd Ed 1994, Addison-Wesley, ISBN 0201558025, especially section 2.6 Finite Calculus], which I also prefer. So, we have:

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = -1;$$
$$m_1^{\underline{1}}\alpha_1 + m_2^{\underline{1}}\alpha_2 + \ldots + m_n^{\underline{1}}\alpha_n = 0;$$
$$m_1^{\underline{2}}\alpha_1 + m_2^{\underline{2}}\alpha_2 + \ldots + m_n^{\underline{2}}\alpha_n = 0;$$
$$\cdots$$
$$m_1^{\underline{n-1}}\alpha_1 + m_2^{\underline{n-1}}\alpha_2 + \ldots + m_n^{\underline{n-1}}\alpha_n = 0;$$

The above n equations determine the α_i, and we then substitute them into:

$$m_1^{\underline{n}}\alpha_1 + m_2^{\underline{n}}\alpha_2 + \ldots + m_n^{\underline{n}}\alpha_n = (-1)^n n!\, q(1); \quad (*)$$

Let D be the n x n determinant with first row 1, 1, ... ,1; and ith row $m_1^{\underline{i-1}}$, $m_2^{\underline{i-1}}$, ... , $m_n^{\underline{i-1}}$; etc) for i > 1. Let D_i be the determinant derived from D by replacing the ith column by the values on the other side of the n simultaneous equations (-1 in the top row and 0s below). [Sorry, that is hard to typeset in the type of HTML that browsers currently accept]. Then we know that $\alpha_i = D_i/D$. (**)

It is also convenient to define E as the determinant derived from D by replacing the first row by $m_1^{\underline{n}}$, $m_2^{\underline{n}}$, ... , $m_n^{\underline{n}}$ and F as the determinant derived from D by deleting its first row and adding $m_1^{\underline{n}}$, $m_2^{\underline{n}}$, ... , $m_n^{\underline{n}}$ as a new bottom row. Note that $E = (-1)^{n-1}F$ since it is derived from it by n-1 row transpositions.

Having got the solution (**) we just substitute the values into (*). But notice that because the ith column of D_i is all zeros except for -1 in the first row, its value is just $-D_{1i}$, where D_{1i} is the cofactor of the (1, i) element of D [D_{1i} is $(-1)^{1+i}$ times the value of the (n-1) x (n-1) determinant obtained from D by deleting the 1st row and ith column]. So the left hand side of (*) is just: $-(m_1^{\underline{n}} D_{11} + m_2^{\underline{n}} D_{12} + ... + m_n^{\underline{n}} D_{1n}) / D$. But the expression in parentheses is just the expansion of the determinant E, so (remembering that $E = (-1)^{n-1} F$) we get that $q(1)\, n! = F/D$.

$m^{\underline{k}}$ is a polynomial in m with highest term m^k and no constant term m^0 (unless i > m, in which case it is zero). So by adding multiples of rows 2, 3, ... , (k - 1) to row k successively for k = 2, 3, ... , n we can convert D into the Vandermonde determinant V which has $m_i^{\underline{k}}$ replaced by m_i^k. Since such row operations do not change the value of a determinant, we have D = V. Similarly, F = W, the determinant derived from F by replacing $m_i^{\underline{k}}$ by m_i^k. Now observe that every element in column i of W has a factor m_i. We can take out these factors to get that $W = m_1 m_2 ... m_n V$. Hence result.

Problem B1
ABCD is a rectangle. AEB is isosceles with E on the opposite side of AB to C and D and lies on the circle through A, B, C, D. This circle has radius 1. For what values of |AD| do the rectangle and triangle have the same area?

Solution
Answer: 2/5.

The height of E above AB is 1 - AD/2. Hence (1 - AD/2) = 2 AD.

Problem B2
x, y, z are complex numbers satisfying x(x - 1) + 2yz = y(y - 1) + 2zx = z(z - 1) + 2xy. Prove that there are only finitely many possible values for the triple (x - y, y - z, z - x) and enumerate them.

Solution
Answer: (0, 0, 0), (1, 0, -1), (-1, 1, 0), (0, -1, 1).

Rearranging x(x - 1) + 2yz = y(y - 1) + 2zx we get $(x^2 - y^2) = (x - y) + 2z(x - y)$, so either x = y or (y - z) = 1 + (z - x) (1). Similarly, either y = z or (z - x) = 1 + (x - y) (2).
So we must look at 4 possibilities. Obviously if x = y and y = z, then the triple is (0, 0, 0). If x = y and (z - x) = 1 + (x - y), then (x - y) = 0, (z - x) = 1, (y - z) = -1, so the triple is (0, -1, 1). If (y - z) = 1 + (z - x) and y = z, then (y - z) = 0, (z - x) = -1, (x - y) = 1, so the triple is (1, 0, -1). The final case is (y - z) = 1 + (z - x) and (z - x) = 1 + (x - y), or rearranging: 1 + 2z = x + y, 1 + 2x = y + z.
Subtracting: 3(z - x) = 0, so the triple is (-1, 1, 0).

Problem B3
We use the congruence notation for polynomials (in one variable) with integer coefficients to mean that the corresponding coefficients are congruent. Thus if $f(x) = a_k x^k + ... + a_0$, and $g(x) = b_k x^k + ...$

+ b_0, then f = g (mod m) means that $a_i = b_i$ (mod m) for all i. Let p be a prime and f(x), g(x), h(x), r(x), s(x) be polynomials with integer coefficients such that r(x)f(x) + s(x)g(x) = 1 (mod p) and f(x)g(x) = h(x) (mod p). Prove that for any positive integer n we can find F(x) and G(x) with integer coefficients such that F(x) = f(x) (mod p), G(x) = g(x) (mod p) and F(x)G(x) = h(x) (mod p^n).

Solution
Suppose that r(x)f(x) + s(x)g(x) = 1 + p u(x) and f(x)g(x) = h(x) + p v(x). The obvious thing to try is F(x) = f(x) - p v(x) s(x), G(x) = g(x) - p v(x) r(x). That gives: F(x)G(x) = f(x)g(x) - p v(x) (f(x)r(x) + g(x)s(x)) + p^2 $v(x)^2$r(x)s(x) = h(x) + terms divisible by p^2. That does not quite work. Upping the power of p does not help, because we need p^1 to get h(x). However, it does move us from (mod p) to (mod p^2), which suggests that induction could complete the proof.

So suppose r(x)f(x) + s(x)g(x) = 1 + p^m u(x) and f(x)g(x) = h(x) + p^m v(x). Then put F(x) = f(x) - p^m v(x) s(x), G(x) = g(x) - p^m v(x) r(x). That gives: F(x)G(x) = h(x) (mod p^{2m}), which is sufficient to give the induction.

Problem B4
For real r > 0, define m(r) =min{ $|r - \sqrt{(m^2+2n^2)}|$ for m, n integers}. Prove or disprove: (1) $\lim_{r\to\infty}$ m(r) exists; and (2) is zero.

Solution
Answer: true, true.

Two heuristic arguments. First, if we think of lattice points at (m, n/√2), it seems likely that large circles will pass increasingly close to lattice points. So the results look true.

Second, let $x_{m,n} = \sqrt{(m^2+2n^2)}$, and let us confine our attention to the values $x_{m,n}$ lying between m and m + 1. There are about √m of them in a distance 1 and roughly equally spaced, so any real number between m and m+1 should lie within about 1/√m of one of them.

In making this rigorous, the important thing is to keep it simple. We do not need a good approximation, just an adequate one. Observe first that √(1 + x) < 1 + x/2, so √(a + x) - √a < x/(2 √a). So given any r > 0, take r ∈ (m, m+1], and the largest n st $m^2 + 2n^2 < r^2$, so that r ∈ ($\sqrt{(m^2 + 2n^2)}$, $\sqrt{(m^2 + 2(n+1)^2)}$]. The distance of r from the nearest lattice point is certainly less than the width of this interval, d. Also $2n^2 < 2m + 1 < 8m$, so n < 2√m. So, using our earlier observation, d < $(2n + 1)/\sqrt{(m^2 + 2n^2)} < (2n + 1)/m < (4\sqrt{m} + 1)/m$ which tends to zero as $r \to \infty$.

Problem B5
Let $f(x, y, z) = x^2 + y^2 + z^2 + xyz$. a(x, y, z), b(x, y, z), b(x, y, z) are polynomials with real coefficients such that f(a(x, y, z), b(x, y, z), c(x, y, z)) = f(x, y, z). Prove or disprove that a(x, y, z), b(x, y, z), c(x, y, z) must be ±x, ±y, ±z in some order (with an even number of minus signs).

Solution
False. Another solution is x, -y, xy + z.

It is not immediately obvious whether this is true or false. The obvious line of attack is to try solutions which are almost x, y, z, but that cuts down the complexity. Specifically, we might try a(x, y, z) = x, b(x, y, z) = y. That requires $c^2 + xy\,c - z^2 - xyz = 0$, which is a quadratic in c. We know one solution is z, so the other is - xy - z (sum of roots is -xy, for example). It looks slightly more elegant if we switch two of the signs.

Of course, if this had not worked (the only solution turned out to be c(x, y, z) = z), then we would have achieved nothing, but we were lucky.

Problem B6
A, B, C, D are n x n matrices with entries in some field F. The transpose of a matrix X is denoted as X' (defined as $X'_{ij} = X_{ji}$). Given that A B' and C D' are symmetric, and A D' - B C' = 1, prove that A'D - C'B = 1.

Solution
Notice first three obvious facts: X is symmetric iff X = X'; X'' = X, (XY)' = Y'X'. So we are given that AB' = BA', CD' = DC', and AD' - BC' = 1. A little playing around shows that the result does not follow trivially. So we need some additional piece.
The commonest non-trivial fact about matrices is that one-sided inverses are two-sided inverses. In other words, if XY = 1, then YX = 1. But it is not immediately obvious how to use it.Playing around may lead us to write AB' = BA' as AB' - BA' = 0. Further playing around might lead us to take transposes of AD' - BC' = 1 to get DA' - CB' = 1. Some inspiration might lead us to notice that we can write the relations as:

AD' - BC' = 1
AB' - BA' = 0
CD' - DC' = 0
-CB' + DA' = 1

If we are lucky these formulae may remind us of 2 x 2 matrix multiplication (the only obstacle to instant recognition is the minus signs). It is a well-known trick that one can group matrix elements into blocks and iterate to get the matrix product (the proof is trivial). So we could summarise the four equations as: (A, B; C, D) (D', -B'; -C', A') = 1, where (A, B; C, D) is the 4 x 4 matrix with the upper left 2 x 2 block as A, the upper right 2 x 2 block as B, the lower left 2 x 2 block as C, and the lower right 2 x 2 block as D.

Now we can use the fact noted at the start to get (D', -B'; -C', A') (A, B; C, D) and hence four more relations, including, for the bottom right blocks, the required -C'B + A'D = 1.

48th Putnam 1987

Problem A1
Four planar curves are defined as follows: $C_1 = \{(x, y): x^2 - y^2 = x/(x^2 + y^2)\}$, $C_2 = \{(x, y): 2xy + y/(x^2 + y^2) = 3\}$, $C_3 = \{(x, y): x^3 - 3xy^2 + 3y = 1\}$, $C_4 = \{(x, y): 3yx^2 - 3x - y^3 = 0\}$. Prove that $C_1 \cap C_2 = C_3 \cap C_4$.

Solution
The wrong way to go about this is to attempt to plot the curves or identify the intersections. The curves are obviously quite complicated, and we know that the first Putnam question is always easy!

So there must be some way of deriving one pair of equations from the other. A little playing soon shows that this is true. Write the equations as:
(1): $x^2 - y^2 = x/(x^2 + y^2)$
(2): $2xy - 3 = - y/(x^2 + y^2)$
(3): $x^3 - 3xy^2 + 3y = 1$
(4): $3yx^2 - 3x - y^3 = 0$

Then x (1) - y (2) gives (3), and y (1) + x (2) gives (4). So any point on the first two is certainly on the second two. Similarly, $(x (3) + y (4))/(x^2 + y^2)$ gives (1), and $(-y (3) + x (4))/(x^2 + y^2)$ gives (2), so any point on the second two is on the first two, except possibly the point (0, 0). But it is easy to check that that is not on (3) (and hence not in the intersection of the second two).

Problem A2
An infinite sequence of decimal digits is obtained by writing the positive integers in order: 1234567891011121314151617181920 21 ... . Define f(n) = m if the 10^n th digit forms part of an m-digit number. For example, f(1) = 2, because the 10th digit is part of 10, and f(2) = 2, because the 100th digit is part of 55. Find f(1987).

Solution
Answer: 1984.

The first 9 numbers have 1 digit, the next 90 2 digits, the next 900 3 digits and so on. So the m-digit numbers and below constitute the first $9(1 + 2.10 + ... + m.10^{m-1}) = 9(1 - (m+1) 10^m + m 10^{m+1})/(1 - 10)^2 = 1/9 (1 - (m+1) 10^m + m 10^{m+1})$ digits. For m = 1983 this is about $0.2 \cdot 10^{1987}$ and for m = 1984 about $2 \cdot 10^{1987}$. So the 10^{1987}th digit is in a 1984-digit number.

Problem A3
y = f(x) is a real-valued solution (for all real x) of the differential equation $y'' - 2y' + y = 2e^x$ which is positive for all x. Is f '(x) necessarily positive for all x? y = g(x) is another real valued solution, which satisfies g'(x) > 0 for all real x. Is g(x) necessarily positive for all x?

Solution
(a) no, (b) yes.

We know that the general solution of $y'' - 2y' + y = 0$ is $A e^x + B x e^x$. So the particular solution to $y'' - 2y' + y = 2e^x$ must be of the form $C x^2 e^x$, and, indeed, differentiating, we find C = 1. So the general solution is $f(x) = x^2 e^x + A e^x + B x e^x$. That is all pure book work.

It is now convenient to rewrite this in a slightly different form: $f(x) = ((x + a)^2 + b) e^x$, where a and b are still arbitary constants. Then $f'(x) = ((x + a + 1)^2 + b - 1) e^x$. The answers are now obvious.

f(x) is always positive iff b > 0, and f '(x) is always positive iff b > 1. So the answer to (a) is No, and the answer to (b) is Yes.

Problem A4
p(x, y, z) is a polynomial with real coefficients such that: (1) $p(tx, ty, tz) = t^2 f(y - x, z - x)$ for all real x, y, z, t (and some function f); (2) p(1, 0, 0) = 4, p(0 ,1, 0) = 5, and p(0, 0, 1) = 6; and (3) $p(\alpha, \beta, \gamma) = 0$ for some complex numbers α, β, γ such that $|\beta - \alpha| = 10$. Find $|\gamma - \alpha|$.

Solution
$5\sqrt{30}/3$.

The key observation is that f must be a polynomial with each term of degree 2. This is well-known and could be quoted. It is also almost obvious. For put u = x, v = y - x, w = z - x. Then x = u, y = u + v, z = u + w, so we can write p(x, y, z) = q(u, v, w), where q is a polynomial. But we are given that f(v, w) = q(u, v, w) for all u, v, w, so u cannot appear in q (otherwise we could vary it to get a contradiction). Hence f is actually a polynomial. But now the relation with t implies that each term has degree 2.

So $p(x, y, z) = a(y - x)^2 + b(y - x)(z - x) + c(z - x)^2$ for some a, b, c. But we are given three values for p which allows us to solve for a, b, c: a = 5, b = -7, c = 6. We have a quadratic in (y - x)/(z - x) which has roots $(7 \pm \sqrt{71}\, i)/10$, so any roots α, β, γ of p must satisfy $(\beta - \alpha) / (\gamma - \alpha) = (7 \pm \sqrt{71}\, i)/10$, and hence $|(\beta - \alpha)| / |(\gamma - \alpha)| = \sqrt{120}/10$. Hence if a root satisfies $|\beta - \alpha| = 10$, then it also satisfies $|\gamma - \alpha| = 100/\sqrt{120} = 5\sqrt{30}/3$.

Problem A5
f: $R^2 \to R^3$ (where R is the real line) is defined by $\mathbf{f}(x, y) = (-y/(x^2 + 4y^2), x/(x^2 + 4y^2), 0)$. Can we find **F**: $R^3 \to R^3$, such that:
(1) if $\mathbf{F} = (F_1, F_2, F_3)$, then F_i all have continuous partial derivatives for all $(x, y, z) \neq (0, 0, 0)$;
(2) $\nabla \times \mathbf{F} = \mathbf{0}$ for all $(x, y, z) \neq 0$;
(3) $\mathbf{F}(x, y, 0) = \mathbf{f}(x, y)$?

Solution
Answer: no.

We look first at (2). We know that $\nabla \times \mathbf{F} = \mathbf{0}$ iff $\mathbf{F} = \nabla g$ for some g: $R^3 \to R$. Remembering that $\tan^{-1} k$ has derivative $1/(1 + k^2)$ and playing around a little, we soon hit on $g(x, y, z) = 1/2 \tan^{-1}(2y/x) + h(z)$. That looks promising. In fact, it *almost* works. We might, for example take $h(z) = 1/2\, z^2$, so that $F(x, y, z) = (-y/(x^2 + 4y^2), x/(x^2 + 4y^2), z)$. That satisfies (2) and (3), and almost satisfies (1). The difficulty is that it, and its partial derivatives wrt x and y, are discontinuous (and arguably undefined) on the entire line x = 0, y = 0 (not just at the origin).

So we suspect that we *cannot* find F to satisfy all the conditions. The obvious approach is to assume we can and to seek a contradiction using Stokes' theorem ($\int_S \nabla \times \mathbf{F} \cdot d\mathbf{a} = \int_\Gamma \mathbf{F}.d\mathbf{s}$, where S is a surface bounded by a closed curve Γ). We want S to cut the z axis at $z \neq 0$, because we believe the discontinuities are on the z-axis and so that will ensure that the surface integral is non-zero, whereas (2) would imply that it was zero. We want a curve Γ on which the line integral is easy to calculate. It clearly must lie entirely within the x, y plane (so that we can use (3) to give us F - otherwise we do not know what F is). It also looks sensible to take an ellipse $x^2 + 4y^2 = k$, because then f is drastically simplified. The line integral is then $\int_\Gamma \mathbf{F}.d\mathbf{s} = \int_\Gamma \mathbf{f}.d\mathbf{s} = \int_\Gamma (-y\, dx + x\, dy)/k = 2/k$ area ellipse $\neq 0$. If (2) was satisfied, then the surface integral would be zero, since the surface avoids the origin. Contradiction.

Problem A6
Define f(n) as the number of zeros in the base 3 representation of the positive integer n. For which positive real x does $F(x) = x^{f(1)}/1^3 + x^{f(2)}/2^3 + ... + x^{f(n)}/n^3 + ...$ converge?

Solution
Answer: $x < 25$.

To get a handle on the problem, we collect a few terms and find that $F(x) = (1/1^3 + 1/2^3 + 1/4^3 + 1/5^3 + 1/7^3 + 1/8^3 + ...) + x(1/3^3 + 1/6^3 + 1/10^3 + 1/11^3 + 1/12^3 + 1/15^3 + 1/19^3 + 1/20^3 + 1/21^3 + 1/24^3 + ...) + x^2(1/9^3 + 1/18^3 + 1/28^3 + 1/29^3 + 1/30^3 + 1/33^3 + ...) + x^3(1/27^3 + 1/54^3 + ...) + ...$. This does not obviously help.

Let us focus on the m-digit numbers (in base 3). There are obviously 2 possibilities for m-1 zeros (namely 10 ... 0 and 20 ... 0). For r zeros, we can place the zeros in m-1Cr ways (the leading digit must be non-zero) and then each of the remaining m-r digits can be 1 or 2, a total of 2^{m-r} m-1Cr possibilities [aCb represents the binomial coefficient a! / (b! a-b!)]. The m-digit numbers have a limited range of values, so that suggests bounds for F(x).

The smallest m-digit number is 3^{m-1} and the largest is $3^m - 1$. So we get a lower bound by replacing $1/n^3$ by $1/3^{3m} = 1/27^m$. In fact, the lower bound for the sum over the m-digit numbers is $1/27^m (x^0 2^{m-0}$ m-1C0 + $x^1 2^{m-1}$ m-1C1 + ... + $x^{m-1} 2^{m-(m-1)}$ m-1Cm-1) = $2^m/27^m (1 + x/2)^{k-1} = 2/27 ((x + 2)/27)^{k-1}$. Hence the lower bound for the complete sum is $2/27 (1 + y + y^2 + y^3 + ...)$, where $y = (x + 2)/27$, which clearly diverges for $y \geq 1$ or $x \geq 25$. The upper bound is 27 times larger, since we replace $1/n^3$ by $1/3^{3m-3}$ and it converges for $y < 1$ or $x < 25$.

Problem B1
Evaluate $\int_2^4 \ln^{1/2}(9 - x)/(\ln^{1/2}(9 - x) + \ln^{1/2}(x + 3)) dx$.

Solution
Answer: 1.

The integrand looks horrible. A little playing around, differentiating things like $\ln^{1/2}(9 - x)$, soon confirms this impression. But we know the first question in each Putnam session is always easy, so there must be some trick!
We might notice that as x goes from 2 to 4, 9 - x goes from 7 to 5, and x + 3 goes from 5 to 7. That might suggest substituting y such that 9 - y = x + 3 (and hence also y + 3 = 9 - x). dy = - dx, the limits interchange and we get $\int_2^4 \ln^{1/2}(y + 3)/(\ln^{1/2}(9 - y) + \ln^{1/2}(y + 3)) dy$. We are home, because we can add the two integrals to get $\int_2^4 1 dx = 2$.

Problem B2
Let n, r, s be non-negative integers with $n \geq r + s$, prove that $\sum_{i=0}^{s}$ sCi / nC(r+i) = (n+1)/((n+1-s) (n-s)Cr), where mCn denotes the binomial coefficient.

No solution

Problem B3
F is a field in which $1 + 1 \neq 0$. Define $P_\alpha = ((\alpha^2 - 1)/(\alpha^2 + 1), 2\alpha/(\alpha^2 + 1))$. Let $A = \{ (\beta, \gamma) : \beta, \gamma \in F$, and $\beta^2 + \gamma^2 = 1\}$, and let $B = \{ (1, 0) \} \cup \{ P_\alpha : \alpha \in F$, and $\alpha^2 \neq -1\}$. Prove that A = B.

Solution
Obviously $B \subseteq A$. The problem is to find some expression for α in terms of β and γ.
We start from $\beta = (\alpha^2 - 1)/(\alpha^2 + 1)$, and get $\alpha^2 = (1 + \beta) / (1 - \beta)$. (*)

We cannot take square roots. So it may not be obvious what to do next. But we do have one square root available, because we know that $\gamma^2 = 1 - \beta^2$. It is now obvious that we multiply top and bottom of (*) by $(1 + \beta)$. That gives $\alpha = (1 + \beta)/\gamma$.

So we can now start again to show that $A \subseteq B$. Given $(\beta, 0) \in A$. Then $\beta^2 - 1 = 0$, so $(\beta + 1)(\beta - 1) = 0$, so $\beta = \pm 1$. $(1, 0)$ is specifically included in B, and $(-1, 0) = P_0$. Now for $(\beta, \gamma) \in A$ with $\gamma \neq 0$, we also have $\beta \neq 1$. Let $\alpha = (1 + \beta)/\gamma$. Then $\alpha^2 = (1 + \beta)^2/\gamma^2 = (1 + \beta)^2/((1 + \beta)(1 - \beta)) = (1 + \beta)/(1 - \beta)$. Hence $\alpha^2 + 1 = 2/(1 - \beta)$, and $\alpha^2 - 1 = 2\beta/(1 - \beta)$, so $P_\alpha = (\beta, \gamma)$.

Problem B4
Define the sequences x_i and y_i as follows. Let $(x_1, y_1) = (0.8, 0.6)$ and let $(x_{n+1}, y_{n+1}) = (x_n \cos y_n - y_n \sin y_n, x_n \sin y_n + y_n \cos y_n)$ for $n \geq 1$. Find $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$.

Solution
Put $y_0 = \sin^{-1} 0.6$. Then the relations give immediately $x_{n+1} = \cos(y_0 + y_1 + ... + y_n)$, $y_{n+1} = \sin(y_0 + y_1 + ... + y_n)$.

Since $\sin x < x$ for $x > 0$, we have $\sin x < \pi - x$ for $0 < x < \pi$. Hence $y_0 + y_1 + ... + y_n < \pi$ implies $y_0 + y_1 + ... + y_{n+1} < \pi$. So all y_n are positive. Hence $\sum y_n$ is monotone increasing and bounded above. So it converges. Hence $y_n \to 0$. But that implies that $\sum y_n \to \pi$, and hence $x_n \to -1$.

Problem B5
A is a complex 2n x n matrix such that if **z** is a real 1 x 2n row vector then $\mathbf{z}\,\mathbf{A} \neq \mathbf{0}$ unless $\mathbf{z} = \mathbf{0}$. Prove that given any real 2n x 1 column vector **x** we can always find a complex n x 1 column vector **z** such that the real part of $\mathbf{A}\,\mathbf{z} = \mathbf{x}$.

Solution
Let $\mathbf{A} = \mathbf{B} + i\mathbf{C}$, where **B** and **C** are real matrices. Let **D** be the 2n x 2n real matrix (**B C**). Then since there is no non-zero real row vector **z** such that $\mathbf{zA} = \mathbf{0}$, it follows that there is no non-zero real row vector **z** such that $\mathbf{zD} = \mathbf{0}$. In other words **D** is invertible. So given any real 2n x 1 column vector **x** we can find a real 2n x 1 column vector **y** such that $\mathbf{Dy} = \mathbf{x}$. Now define a complex n x 1 column vector **z** by $z_1 = y_1 - iy_{n+1}$, $z_2 = y_2 - iy_{n+2}$ etc. Then the real part of **zA** is **x** as required.

Problem B6
F is a finite field with p^2 elements, where p is an odd prime. S is a set of $(p^2 - 1)/2$ distinct non-zero elements of F such that for each $a \in F$, just one of a and -a is in S. Prove that the number of elements in $S \cap \{2a: a \in S\}$ is even.

Solution
Put $q=(p^2-1)/2$, so S has q elements. Since p is odd, q is even. For each b there is just one c such that $b = 2c$. Just one of c, -c is in S. So for each b in S we can find a unique f(b) in S such that $b = 2f(b)k(b)$ where $k(b) = \pm 1$. We cannot have $f(a) = f(b)$ unless $a = b$, so as b runs through the elements of S, so does f(b). Hence $\prod a = 2^q k \prod a$, where the product is taken over all elements of S and k is the number of negative k(a). So $k = 2^q \pmod p$. k(a) is positive iff a belongs to the required intersection. The number of elements in S is even, so the number of positive k(a) is even iff the number of negative k(a) is even. Hence we wish to show that $k = 1$. But $q = (p-1) \times (p+1)/2$. We know that $2^{p-1} = 1 \pmod p$, so $2^q = 1 \pmod p$ as required.

Putnam 1988

Problem A1
Let S = { (x, y) : |x| - |y| ≤ 1 and |y| ≤ 1 }. Sketch S and find its area.

Solution
Answer: 6.

The region is the rectangle (-2, 1), (2, 1), (2, -1), (-2, -1) less the two triangles (2, 1), (1, 0), (2, -1) and (-2, 1), (-1, 0), (-2, -1). Hence the area is 8 - 2.1 = 6.

Problem A2
Find an open interval I and a non-zero function g(x) on I such that (fg)' = f 'g' on I, or prove that they do not exist.

Solution
Answer: $\sqrt{(2x - 1)}e^x$ on (1, 2) (or many other intervals).

Put $h(x) = e^{x^2}g(x)$. We require $g'(x)/g(x) = 2x/(2x-1)$. Integrating: $\ln g(x) = x + 1/2 \ln(2x - 1) +$ const, so $g(x) = A \sqrt{(2x-1)}\, e^x$.

Problem A3
For what real numbers α does $(1/ 1 \operatorname{cosec}(1) - 1)^\alpha + (1/2 \operatorname{cosec}(1/2) - 1)^\alpha + ... + (1/n \operatorname{cosec}(1/n) - 1)^\alpha + ...$ converge?

Solution
Answer: α > 1/2.

cosec 1/n is about n + 1/(6n). So the series is roughly $1/6 \sum 1/n^{2\alpha}$, which converges iff α > 1/2. We just have to tighten that up slightly.

$n \sin 1/n = 1 - 1/(6n^2) + 1/(120n^4) - ...$, so $1/n \operatorname{cosec}(1/n) = 1/(1 - 1/(6n^2) + 1/(120n^4) - ...) = 1 + 1/(6n^2)$ + terms in $1/n^4$ and higher. Hence $(1/n \operatorname{cosec}(1/n) - 1) = 1/n^2$ (1/6 + terms which tend to zero as $n \to \infty$). So for n sufficiently large $(1/n \operatorname{cosec}(1/n) - 1)$ lies between $1/(12n^2)$ and $1/(3n^2)$ and hence the nth term of the series lies between $k/n^{2\alpha}$ and $k'/n^{2\alpha}$. But $\sum 1/n^\beta$ converges for β > 1 and diverges for β ≤ 1. So the comparison test gives the result.

Problem A4
The plane is divided into 3 disjoint sets. Can we always find two points in the same set a distance 1 apart? What about 9 disjoint sets?

Solution
Answer: yes, no.

Suppose it is impossible. Take a triangle ABC with AB = AC = √3, BC = 1. A and B must be in the same set, because we can take two points X, Y such that AXY and BXY are equilateral triangles with side 1 (and then X and Y use up two sets, so A and B must both be the third set). Similarly A and C must be in the same set. But then B and C are in the same set and a distance 1 apart.

Tile the plane with squares side 0.7. Take all the points in a tile to be in the same set and label the tile 1 to 9 accordingly. Labels the tiles as follows:
. . .

```
...123123123...
...456456456...
...789789789...
...123123123...
...456456456...
...
```
Two points in the same tile are at most a distance $0.7\sqrt{2} < 1$ apart, and two points in different tiles with the same color are at least a distance $2 \times 0.7 = 1.4 > 1$ apart.

Problem A5
R^+ denotes the positive reals. Prove that there is a unique function $f : R^+ \to R^+$ satisfying $f(f(x)) = 6x - f(x)$ for all x.

Solution
Answer: unique solution is $f(x) = 2x$.

To get a handle on the problem, we try taking f(x) linear. Solving gives the solutions $f(x) = -3x$ or $f(x) = 2x$, but the first is ruled out because the values must be positive. So we have to prove that $f(x) = 2x$ is the only solution.
It is obviously the only polynomial solution, but we are not even given that f is continuous. In principle, it could be an entirely arbitary function. So we must somehow use the relation $f(f(x)) = 6x - f(x)$ directly to get a contradiction. The only obvious approach is to define a sequence $x_{n+1} = f(x_n)$ and hope that we eventually get x_n negative. A little experimentation suggests that that is true.

So suppose we have $f(x_0) = (2 - \lambda_0)x_0$ for some x_0 with $0 < \lambda_0 < 2$. Then $f((2 - \lambda_0)x_0) = (4 + \lambda_0)x_0$, and hence $f((4 + \lambda_0)x_0) = (8 - 7\lambda_0)x_0 = (2 - 9\lambda_0/(4 + \lambda_0)) (4 + \lambda_0)x_0$. In other words, we have found x_1 and $\lambda_1 = 9\lambda_0/(4 + \lambda_0)$ such that $f(x_1) = (2 - \lambda_1)x_1$. But $\lambda_1 > 3/2\ \lambda_0$. If $\lambda_1 > 2$, then we have a contradiction (because $f(x_1) < 0$). If not, then we can repeat to get an x_2 with $f(x_2) = (2 - \lambda_2)x_2$ and $\lambda_2 > 3/2\ \lambda_1 > (3/2)^2\ \lambda_0$. And so on. So if we can find x_0 with $f(x_0) < 2x_0$ then we have a contradiction.

If we can find x_0 with $f(x_0) > 2x_0$, then $f(x_0) = (2 + \lambda)x_0$ for some $\lambda > 0$. So $f((2 + \lambda)x_0) = (4 - \lambda)x_0 < 2\ (2 + \lambda)x_0$. In other words, we have found $x_1 = (2 + \lambda)x_0$ such that $f(x_1) < 2x_1$ and hence have a contradiction. So we must have $f(x) = 2x$ for all x.

Problem A6
V is an n-dimensional vector space. Can we find a linear map $A : V \to V$ with n+1 eigenvectors, any n of which are linearly independent, which is not a scalar multiple of the identity?

Solution
Answer: no.

Let the n+1 corresponding eigenvalues have sum k. Given any eigenvector with eigenvalue λ, the remaining n eigenvectors form a basis (since they are linearly independent) which diagonalises the matrix of the linear map. The resulting matrix has trace $k - \lambda$. But trace is independent of basis, so all the λ are the same and the diagonalised matrix is just a multiple λ of the identity. In other words, the linear map is just a scalar multiple of the identity.

Problem B1
If $n > 3$ is not prime, show that we can find positive integers a, b, c, such that $n = ab + bc + ca + 1$.

Solution
If n is not prime, then we have n = AB with A, B > 1. But AB = (A - 1)(B - 1) + (B - 1)1 + 1(A - 1) + 1.

Problem B2
α is a non-negative real. x is a real satisfying $(x + 1)^2 \geq \alpha(\alpha + 1)$. Is $x^2 \geq \alpha(\alpha - 1)$?

Solution
Yes.

Suppose $\alpha(\alpha+1) = (x+1)^2$ and x >= 0. Then since $(x + 1/2)(x + 3/2) = (x + 1)^2 - 1/4$, we have $\alpha > x + 1/2$. Hence $\alpha(\alpha - 1) = \alpha(\alpha + 1) - 2\alpha < (x + 1)^2 - 2(x + 1/2) = x^2$.

If $\alpha(\alpha+1) < (x+1)^2$ and $x \geq 0$, then take $\beta > \alpha$ with $\beta(\beta + 1) = (x+1)^2$. Clearly $\alpha - 1 < \beta - 1$ and since α is positive, $\alpha(\alpha - 1) < \alpha(\beta - 1) < \beta(\beta - 1)$ which is $< x^2$.
So it just remains to consider the case x < 0. But in this case $-|x| - 1 < x + 1 < |x| + 1$, so $(x + 1)^2 < (|x| + 1)^2$ and $x^2 = |x|^2$, and the result follows from the result for α, |x|.

Problem B3
α_n is the smallest element of the set $\{ |a - b\sqrt{3}| : a, b$ non-negative integers with sum $n\}$. Find sup α_n.

Solution
Answer: $(\sqrt{3} + 1)/2$.

Evidently the elements of $\{ |a - b\sqrt{3}| : a, b$ non-negative integers with sum $n\}$ are spaced a distance $(\sqrt{3} + 1)$ apart, so either smallest positive or the largest negative lies in the interval $(-(\sqrt{3} + 1)/2, (\sqrt{3} + 1)/2)$. Thus $(\sqrt{3} + 1)/2$ is certainly an upper bound.

Let a_n, b_n be the pair with $|a_n - b_n \sqrt{3}| = \alpha_n$. All the values $(a_n - b_n \sqrt{3})$ lie in the interval $(- (\sqrt{3} + 1)/2, (\sqrt{3} + 1)/2))$, so for any ε > 0, we can find two distinct values, corresponding to say m and n, which differ by less than ε. In other words $\alpha_{m-n} < \varepsilon$. But now consider the sequence of values $\alpha_{m-n}, \alpha_{2(m-n)}, \alpha_{3(m-n)}, \ldots$ For $k\alpha_{m-n} < (\sqrt{3} + 1)/2$ we have $\alpha_{k(m-n)} = k\,\alpha_{m-n}$. So we get a sequence of increasing values with spacing less than ε. Thus the sequence gives a value within ε of $(\sqrt{3} + 1)/2$, which is therefore the least upper bound.

Problem B4
α_n are positive reals, and $\beta_n = \alpha_n^{n/(n+1)}$. Show that if $\sum \alpha_n$ converges, then so does $\sum \beta_n$.

Solution
Note that $\beta_n > 2\alpha_n$ iff $(1/\alpha_n)^{1/(n+1)} > 2$ or $\alpha_n < 1/2^{n+1}$. So either $\beta_n < 2\alpha_n$ or $\beta_n < 1/2^n$. But both these series converge absolutely.

Problem B5
Find the rank of the 2n+1 x 2n+1 skew-symmetric matrix with entries given by $a_{ij} = 1$ for $(i - j) = -2n, -(2n-1), \ldots, -(n+1)$; -1 for $(i - j) = -n, -(n-1), \ldots, -1$; 1 for $(i - j) = 1, 2, \ldots, n$; -1 for $(i - j) = n+1, n+2, \ldots, 2n+1$. In other words, the main diagonal is 0s, the n diagonals immediately below the main diagonal are 1s, the n diagonals below that are -1s, the n diagonals immediately above the main diagonal are -1s, and the n diagonals above that are 1s.

Solution
Answer: 2n.

The determinant is zero, since each column sums to zero. Hence the rank is not 2n+1. However, we will show that the matrix D_{2n} formed by deleting the first row and the first column has determinant ±1, which is non-zero. Hence the rank is 2n.

We proceed by induction. The result is clearly true for n = 1, since $D_2 = 1$. Suppose it is true for n-1. We carry out row and column operations on D_{2n} to show that its determinant is $\pm D_{2n-2}$.

Adding the row n+2 to row 1 gives 0s except in for the -1 in col 1 and the 1 in col n+2. Adding or subracting the new row 1 from all others except row n+2 gives 0s in col 1 except for -1 in rows 1 and n+2. It also gives 0s in col n+2 except for 1 in row 1. Note that the submatrix formed by deleting rows 1 and n+2 and cols 1 and n+2 has not been changed by these operations and is D_{2n-2}.

We may now expand by col n+2. This gives a single term ± the determinant formed by deleting row 1 and col n+2. This has only one non-zero entry in col 1, that in the old row n+2. So expanding gives $\pm D_{2n-2}$ as required.

Problem B6
The triangular numbers are 1, 3, 6, 10, ... , n(n+1)/2, ... Prove that there are an infinite number of pairs a_i, b_i such that m is triangular iff $a_i m + b_i$ is triangular.

Solution
Let $T_n = n(n + 1)/2$. Let us try to express T_{an+b} in the form $AT_n + B$. Evidently this works if a = 2b + 1, with $A = (2b + 1)^2$ and B = b(b + 1)/2.

The only if part is not quite so obvious. Suppose that $T = (2b + 1)^2 t + b(b+1)/2$ is triangular. We must show that t is triangular. Put k = b(b+1). Then T = (4k+1)t + k/2. Since T is triangular, 8T+1 = 8(4k+1)t + (4k+1) is an odd square. But $4k+1 = (2b+1)^2$ is an odd square, so 8t+1 is an odd square and hence t is triangular also.

50th Putnam 1989

Problem A1
Which members of the sequence 101, 10101, 1010101, ... are prime?

Solution
Answer: 101.

Let k_n represent the member of the sequence with n 1s. It is obvious that 101 divides k_{2n}. So we need only consider k_{2n+1}.

But $k_{2n+1} = 1 + 10^2 + 10^4 + ... + 10^{4n} = (10^{4n+2} - 1)/99 = (10^{2n+1} + 1)/11\ (10^{2n+1} - 1)/9$. Each of these is integral: the first is $1 - 10 + 10^2 - ... + 10^{2n}$, the second is 11...1 (2n+1 1s).

Problem A2
Let R be the rectangle $0 \le x \le a$, $0 \le y, \le b$. Evaluate $\int_R e^{f(x, y)}\, dx\, dy$, where $f(x, y) = \max(b^2x^2, a^2y^2)$.

Solution
Divide the rectangle into two by the diagonal from (0, 0) to (a, b). The required value is evidently twice the integral over the lower half of $e^{g(x)}$, where $g(x) = b^2x^2$. But integrating over y just gives a factor bx/a, and the integration over x is then trivial to give $(e^k - 1)/(2ab)$, where $k = a^2b^2$. So the required value is twice this or $(e^k - 1)/(ab)$.

Problem A3
Prove that all the roots of $11z^{10} + 10i\, z^9 + 10i\, z - 11 = 0$ have unit modulus.

Solution
Note that $z^9 = (11 - 10iz)/(11z + 10i)$. But $|11 - 10iz|^2 = (121 + 100|z|^2) + 220$ Im z, and $|11z - 10iz|^2 = (100 + 121|z|^2) + 220$ Im z. In other words, if $|z^9| = N/D$, then $N - D = 21(1 - |z|^2)$. So if $|z| > 1$, then $N < D$ and hence $|z^9| < 1$. Similarly if $|z| < 1$, then $|z^9| > 1$. A contradiction in both cases. Hence $|z| = 1$.

Problem A4
A player plays the following game. At each turn a fair coin is tossed (probability 1/2 of heads, and all tosses independent), and, depending on the results of the tosses to date, (1) the game ends and the player wins, (2) the game ends and the player loses, or (3) the coin is tossed again. Given an irrational p in the interval (0, 1), can we find a rule such that (A) the player wins with probability p, and (B) the game ends after a finite number of tosses with probability 1?

Solution
Answer: yes.

Let the binary expansion of p be $0.p_1p_2p_3$ ... . The coin is tossed repeatedly until it comes up heads. If the first heads is on toss n then the player wins iff $p_n = 1$.

Note that the prob that the game ends after toss n is $1/2^n$. So the probability that the player wins is $\sum p_n/2^n = p$, as required.

Finally, the probability that the game requires more than n tosses is $1/2^n$, so the probability that only finitely many tosses are required is 1.

Problem A5
Show that we can find $\alpha > 0$ such that, given any point P inside a regular polygon with an odd number of sides which is inscribed in a circle radius 1, we can find two vertices of the polygon whose distance from P differs by less than $1/n - \alpha/n^3$, where the polygon has $2n + 1$ sides.

Solution
There are 2n+1 distances, all less than 2 (the diameter of the circle), so by the usual pigeon-hole argument there must be two distances differing by less than 2/2n. This is almost strong enough and suggests that we just need to refine this argument slightly.

The obvious refinement is that the diameter is slightly less than 2. In fact, it is a long diagonal of the polygon, length $2\cos(\pi/(4n+2))$. So we need to show that $\cos(\pi/(4n+2)) < 1 - \alpha/n^2$ for some positive α.

But it is easy to show that $\cos x < 1 - x^2/8$ for $0 < x < \pi/2$ and so the result follows.

Problem A6
Let $\alpha = 1 + a_1x + a_2x^2 + \ldots$ where $a_n = 1$ if every block of zeros in the binary expansion of n has even length, 0 otherwise. Prove that, if we calculate in the field of two elements, then $\alpha^3 + x\alpha + 1 = 0$. [For example, calculating in this field, $(1 + x)^2 = 1 + x + x + x^2 = 1 + x^2$.]

Solution
$\alpha^2 = 1 + \sum a_n^2x^{2n}$ because 2 = 0. Also $a_n^2 = a_n$, so $\alpha^2 = 1 + \sum a_nx^{2n}$. The expression for α^3 is not so convenient, but we notice that if we multiply the relation in the question by α then we get the powers 1, 2, 4 and clearly $\alpha^4 = 1 + \sum a_nx^{4n}$.

So we now consider $\alpha^4 + x\alpha^2 + \alpha$. It is convenient to consider separately the coefficients of x^{4n}, x^{4n+2} and x^{2n+1}.

Evidently the coefficient of x^{4n} is $a_n + a_{4n}$. But $a_{4n} = a_n$, since the binary expansion of 4n is just that for n with two zeros added at the end. Hence $a_n + a_{4n} = 0$. [This does not deal with the coefficient of x^0, but that is clearly 1 + 1 = 0 also.]
The coefficient of x^{4n+2} is just a_{4n+2}. But the binary expansion of 4n+2 ends ... 10, so $a_{4n+2} = 0$.

Finally, the coefficient of x^{2n+1} is $a_{2n+1} + a_n$. But the binary expansion of 2n+1 is the same as that for n with an extra 1 added at the end, so $a_{2n+1} = a_n$ and $a_{2n+1} + a_n = 0$ also.

We have established that $\alpha^4 + x\alpha^2 + \alpha = 0$, but $\alpha \neq 0$, so $\alpha^3 + x\alpha + 1 = 0$ as required.

Problem B1
S is a square side 1. R is the subset of points closer to its center than to any side. Find the area of R.

Solution
Answer: $(4\sqrt{2} - 5)/3$.

Take axes through the center of the square. A point is closer to the center than to the side y = 1/2 if it lies below the parabola $y = 1/4 - x^2$. A similar result applies in the other three directions. So the area is 4 times that of the region S bounded by the parabola and the lines y = x, y = - x. The line y = x cuts the parabola at $x = 1/\sqrt{2} - 1/2$ (= k for convenience). The area S is the area under the parabola between x = -k and k, less the area under the two diagonals. So the area S is $2\int_0^k (1/4 - x^2)\,dx - k^2 = k/2 - k^2 - 2k^3/3$. We have $k^2 = 1/4 - k$, so $k^3 = 5k/4 - 1/4$ and $k/2 - k^2 - 2k^3/3 = 2k/3 - 1/12 = (\sqrt{2})/3 - 5/12$. Thus the required area is $(4\sqrt{2} - 5)/3$.

Problem B2
Let S be a non-empty set with a binary operation (written like multiplication) such that: (1) it is associative; (2) ab = ac implies b = c; (3) ba = ca implies b = c; (4) for each element, the set of its powers is finite. Is S necessarily a group?

Solution
Answer: yes.

Let a be any element. We show that for some $n > 1$ we have $a^n = a$. The set of its powers is finite, so for some $r > s$ we have $a^r = a^s$. If $s = 1$, we are done. If not, put $b = a^{s-1}$, then $b\, a^{r-s+1} = b\, a$, so we may cancel to get $a^n = a$ with $n = r - s + 1 > 1$. Now put $e = a^{n-1}$. Then we have $ea = ae = a$.

Now take any b. We have $a(eb) = (ae)b = ab$, and cancelling gives $eb = b$. Similarly, $(be)a = b(ea) = ba$, so $be = b$. Hence e is an identity.
Also a has an inverse. If $n - 1 = 1$, then $a = e$, so a is its own inverse. If $n - 1 > 1$, then a^{n-2} is its inverse.

Now if b is any other element, we may use the same argument to find another identity f and an element c such that $cb = bc = f$. But we have $e = ef = f$, so the identity is unique and c is an inverse for b.

Problem B3
Let R be the reals and R* the non-negative reals. $f: R^* \to R$ satisfies the following conditions: (1) it is differentiable and $f'(x) = -3 f(x) + 6 f(2x)$ for $x > 0$; (2) $|f(x)| \le e^{-\sqrt{x}}$ for $x \ge 0$. Define $u_n = \int_0^\infty x^n f(x)\, dx$ for $n \ge 0$. Express u_n in terms of u_0, prove that the sequence $u_n 3^n/n!$ converges, and show that the limit is 0 iff $u_0 = 0$.

Solution
Integrating by parts and using the relation given, we get $u_n = 3/(n+1)\, u_{n+1} - 3/(n+1)\, (1/2)^{n+1}\, u_{n+1}$. So $u_n = u_0\, n!/3^n \prod_1^n 1/(1 - 1/2^r)$.

The product $\prod (1 - 1/2^r)$ converges to a positive value since $\sum 1/2^r$ converges. Hence $3^n/n!\, u_n$ converges to a positive multiple of u_0. Hence the limit is zero iff u_0 is zero.

Problem B4
Does there exist an uncountable set of subsets of the positive integers such that any two distinct subsets have finite intersection?

Solution
Answer: yes.

Consider first the set Q of rationals. For each positive real in the interval (0, 1), take a sequence of rationals converging to it. This gives an uncountable collection of subsets of Q. Clearly two collections have at most finitely many points in common. To get the corresponding subsets of the postive integers simply take a bijection between the positive integers and the rationals.

Problem B5
A quadrilateral is inscribed in a circle radius 1. Two opposite sides are parallel. The difference between their lengths is $d > 0$. The distance from the intersection of the diagonals to the center of the circle is h. Find sup d/h and describe the cases in which it is attained.

Solution
The tricky part is deciding what to use as variables. Take x, y coordinates centered on the center of the circle. Assume that the diagonals intersect at (0, h) and that one of the diagonals has gradient m, so that it has equation $y = mx + h$. Then it cuts the circle at x-coordinates x_1, x_2 which are the roots of the quadratic: $x^2 + (mx + h)^2 = 1$, or $(m^2 + 1)x^2 + 2mhx - (1 - h^2) = 0$. Since one root is positive and the other negative we have that d = 2 x sum of roots = $4mh/(m^2 + 1)$. So $d/h = 2.\ 2m/(m^2 + 1)$. But $(m - 1)^2 \geq 0$ implies $2m/(m^2 + 1) \geq 1$, with equality iff m = 1. Hence sup d/h = 2, with equality when the diagonals are perpendicular.

Problem B6
f is a continuous real-valued function on the closed interval [0, 1] such that f(1) = 0. A point (a_1, a_2, ... , a_n) is chosen at random from the n-dimensional region $0 < x_1 < x_2 < ... < x_n < 1$. Define $a_0 = 0$, $a_{n+1} = 1$. Show that the expected value of $\sum_0^n (a_{i+1} - a_i) f(a_{i+1})$ is $\int_0^1 f(x)\, p(x)\, dx$, where p(x) is a polynomial of degree n which maps the interval [0, 1] into itself (and is independent of f).

Solution
The expected value is simply $\int S\, dv / \int dv$, where the integral is taken over the allowed region and S is the sum given.

$\int dv = 1/n!$ [First integrate 1 from $x_1 = 0$ to x_2, giving $x_2/1!$. Then integrate that from $x_2 = 0$ to x_3, giving $x_3^2/2!$ and so on.]

We now have to attack the main integral. We can take the sum outside the integral, giving a sum of n integrals. Now there at most two variables in each integral. The trick is to use a different order of integration for each integral, so that one integrates last over the variables in the integrand. Starting at the x_1 end gives us a factor $x^i/i!$ and starting at the x_n end gives us a factor $(1 - x)^j/j!$. We are then left with the a single final integration from 0 to 1 of a term like $f(x)\, x^{n-i}(1 - x)^i/((n - i)!\, i!)$.

At this point we switch the order of summation and integration again to get:

$\int_0^1 x^n/n! + x^{n-1}(1 - x)/(n - 1!\, 1!) + x^{n-2}(1 - x)^2/(n - 2!\, 2!) + ... + x\,(1 - x)^{n-1}/(1!\, n - 1!)\, dx$.

After dividing by the other integral (equivalent to multiplying by n!) we get an integrand which is simply $(x + 1 - x)^n - (1 - x)^n = 1 - (1 - x)^n$. So we have the required result with $p(x) = 1 - (1 - x)^n$.

51st Putnam 1990

Problem A1
Prove that the sequence $a_0 = 2, 3, 6, 14, 40, 152, 784, ...$ with general term $a_n = (n+4)\, a_{n-1} - 4n\, a_{n-2} + (4n-8)\, a_{n-3}$ is the sum of two well-known sequences.

Solution
Answer: $n! + 2^n$.

This is not a nice problem. We know the answer is easy (because A1 is almost always easy), so we are looking for something simple. Just try subtracting off various simple series until you recognize the result. I was lucky: 152, 784 vaguely reminded me of 120, 720.

Problem A2
Can we find a subsequence of $\{ n^{1/3} - m^{1/3} : n, m = 0, 1, 2, ... \}$ which converges to $\sqrt{2}$?

Solution
Answer: yes.

There is obviously nothing special about $\sqrt{2}$. In fact a little thought suggests that the set $\{ n^{1/3} - m^{1/3} : n, m \text{ integers} \}$ is dense in the reals.

To prove this note that we can find arbitrarily small values because certainly for $n > 64$, we have $(n^{1/3} + 1/n)^3 < n + 3/4 + 3/1024 + 1/64^3 < n + 1$. Also if α is in the set, then so are all integral multiples of α, because $(n^3N)^{1/3} - (n^3M)^{1/3} = n(N^{1/3} - M^{1/3})$. So by taking a suitable multiple of a value less than ε we can get within ε of any desired value.

Problem A3
A convex pentagon has all its vertices lattice points in the plane (and no three collinear). Prove that its area is at least 5/2.

Solution
We need Pick's theorem: given a polygon (whose sides are not self-intersecting, but which is not necessarily convex) whose vertices are lattice points, its area is A + B/2 - 1, where A is the number of interior lattice points and B is the number of lattice points on the perimeter.

So we have to show that a *convex* pentagon either has an interior lattice point or has at least two lattice points on its sides (apart from its vertices).

We can divide the coordinates into four parity classes: (odd, odd), (odd, even), (even, odd) and (even, even). At least two of the vertices must belong to the same class. But then their midpoint must be a lattice point. If the vertices are not adjacent, then the midpoint is an interior lattice point and we are done. So suppose the two vertices are A and B, adjacent and with midpoint M, also a lattice point. Let the other vertices be C, D, and E. A, C, D and E must all have different parity classes, otherwise we get another lattice point and we are done. But now M must have the same parity class as one of A, C, D and E, which gives us another lattice point.
Note that the result is the best possible. For example, (0, 0), (1, 0), (2, 2), (1, 2), (0, 1) has area exactly 5/2.

Problem A4
Given a point P in the plane, let S_P be the set of points whose distance from P is irrational. What is the smallest number of such sets whose union is the entire plane?

Solution
We clearly need at least 3 sets, since given any two points P and Q we can find a point R whose distance from each of P and Q is rational (indeed integral).

However, 3 suffice. Take any two distinct points P and Q. There are at most two points with given rational distances x from P and y from Q, so there are only countably many points X outside S_P and S_Q. Each of the these points X rules out as the third point countably many points on the line PQ (namely those which are a rational distance from X). So only countably many points are ruled out in total from a line with uncountably many points. Take one of the remaining points.

Problem A5
M and N are n x n matrices such that $(MN)^2 = 0$. Must $(NM)^2 = 0$?

Solution
Answer: yes for n = 2; no for n > 2.

For $n \geq 3$, Take $M_{12} = M_{31} = 1$, other elements 0, and $N_{11} = N_{33} = 1$, other elements 0. Then $(MN)_{31} = 1$ and its other elements are 0. So $(MN)^2 = 0$. $(NM)_{31} = (NM)_{12} = 1$ and its other elements are 0. So $(NM)^2$ is non-zero (its 3,2 element is 1). So we have a counter-example for $n \geq 3$.

Note that if $(MN)^2 = 0$, then $(NM)^3 = N(MN)^2M = 0$. But we can prove that if K is a 2 x 2 matrix and $K^3 = 0$, then $K^2 = 0$.

Suppose $K_{11} = a$, $K_{12} = b$, $K_{21} = c$, $K_{22} = d$. Then since $K^3 = 0$, we have: $a^3 + 2abc + bcd = 0$, (1); $a^2b + abc + b^2c + bd^2 = 0$, (2); $a^2c + bc^2 + acd + cd^2 = 0$, (3); $abc + 2bcd + d^3 = 0$, (4). Taking (1) - a/b (2) gives: $a + d = 0$ or $bc = ad$. We show that in fact $a + d = 0$ *and* $bc = ad$. For suppose $a + d = 0$, then (3) gives $bc = ad$. Conversely, if $bc = ad$, then (1) implies $a(a + d)^2 = 0$, so $a = 0$ or $a + d = 0$. If $a = 0$, then (4) implies $d = 0$ and so $a + d = 0$.
But $a + d = 0$ and $bc = ad$, gives immediately that $K^2 = 0$.

Problem A6
How many ordered pairs (A, B) of subsets of {1, 2, ... , 10} can we find such that each element of A is larger than |B| and each element of B is larger than |A|.

Solution
Answer: 17711.

Evidently we are meant to solve the general case of subsets of {1, 2, ... , N}. But the choice of N = 10 is poor, because it is sufficiently small to be tractable and in an exam it is certainly safer to slog it out for the specific case, rather than to risk failing to find a more elegant solution in the limited time available. Thus if A has 0 elements, then B can be anything, so we have 1024 possibilities. If A has 1 element, then we have 10C1 9C0 + 9C1 9C1 + 8C1 9C2 + ... + 1C1 9C9 = 2816 possibilities. If A has 2 elements, then we have 10C2 8C0 + 9C2 8C1 + ... + 2C2 8C8 = 4096 possibilities and so on.

Let u_N be the solution for the general case. Let v_N be the solution with the restriction that B does not contain the element N. Let w_N be the solution with the restriction that B does contain the element N. We seek recurrence relations.

First observe that $w_N = u_{N-1}$ because we have the correspondence (A, B) ↔ (A', B - {N}), where A' is the set derived from A by reducing each element by 1. Obviously $u_N = v_N + w_N$, so $v_N = u_N - u_{N-1}$.

Now consider u_{N+1}. There are four cases: (1) neither A nor B contains N+1, (2) A contains N+1, but B does not, (3) B contains N+1, but A does not, (4) both A and B contain N+1. Clearly, there are u_N possibilities in case (1). In case (2) we have the correspondence (A, B) ↔ (A - {N+1}, B'), where B' is derived from B by reducing each element by 1. This shows that there are v_N possibilities for case (2). Similarly for case (3). For case (4) we use the correspondence (A, B) ↔ (A', B') to show that there are u_{N-1} possibilities. Thus we have finally: $u_{N+1} = u_N + 2v_N + u_{N-1} = 3u_N - u_{N-1}$.

Clearly, $u_0 = 1$, $u_1 = 3$, so we may now use the relation to calculate successively 8, 21, 55, 144, 377, 987, 2584, 6765, 17711 (= u_{10}).

We may recognize these as alternate Fibonacci numbers and indeed a little thought shows that if $a_{n+2} = a_{n+1} + a_n$ then $a_{n+2} = 2a_n + a_{n-1} = 2a_n + (a_n - a_{n-2}) = 3a_n - a_{n-2}$.

Alternatively, we may notice that the relations above may be rewritten as $u_N = v_N + u_{N-1}$ and $v_N = u_{N-1} + v_{N-1}$, which (together with the starting values), shows that $v_0, u_0, v_1, u_1, \ldots$ is Fibonacci.

Problem B1
R is the real line. Find all possible functions f: R → R with continuous derivative such that $f(\alpha)^2 = 1990 + \int_0^\alpha (f(x)^2 + f'(x)^2)\, dx$ for all α.

Solution
Putting y = f(x) and differentiating the relationship gives $2y\,y' = y^2 + y'^2$ or $(y - y')^2 = 0$. So y = y' or y = -y'. Integrating, $y = Ae^x$ or $y = -Ae^x$. But $y(0) = \pm\sqrt{1990}$, so $f(x) = \pm\sqrt{1990}\, e^x$. The function is continuous, so we cannot "mix" the two solutions: either $f(x) = \sqrt{1990}\, e^x$ for all x, or $f(x) = -\sqrt{1990}\, e^x$ for all x.

Problem B2
Let $P_n(x, z) = \prod_1^n (1 - z\, x^{i-1}) / (z - x^i)$. Prove that $1 + \sum_1^\infty (1 + x^n)\, P_n(x, z) = 0$ for $|z| > 1$ and $|x| < 1$.

Solution
If we calculate a few partial sums, we soon find a pattern emerging. Let $S_n = 1 + \sum_1^n (1 + x^i)\, P_i(x, z)$. Then we find $S_1 = (1 - zx)/(z - x)$, $S_2 = S_1\,(1 - zx^2)/(z - x^2)$, $S_3 = S_2\,(1 - zx^3)/(z - x^3)$, ... The general case follows by an easy induction.

But $|x| < 1$, so $|1 - zx^n| \to 1$, and $|z - x^n| \to |z|$. But $|z| > 1$, so we can find positive $k < 1$ such that for n sufficiently large $|(1 - zx^n)/(z - x^n)| < k$. Hence $S_n \to 0$ as n tends to infinity.

Problem B3
Let S be the set of 2 x 2 matrices each of whose elements is one of the 15 squares 0, 1, 4, ... , 196. Prove that if we select more than $15^4 - 15^2 - 15 + 2$ matrices from S, then two of those selected must commute.

Solution
There are 225 diagonal matrices in the set. Any two of those commute. There also 14 (non-diagonal) matrices with equal elements and 14 other matrices with zeros on the diagonal and the other two elements equal. Any two of those 28 commute. Thus unless we leave out at least 224 + 27 matrices we will get two that commute. In other words, we are sure to get two that commute if we pick at least $15^4 - 15^2 - 25$, which is slightly stronger than the required result.

Problem B4
A finite group with n elements is generated by g and h. Can we arrange two copies of the elements

of the group in a sequence (total length 2n) so that each element is g or h times the previous element and the first element is g or h times the last?

Solution
Regard the n elements as the points of a directed graph. We have a line from A to B if B = gA or B = hA. Then every point A of the graph has an equal number of incoming lines (just 2, from $g^{-1}A$ and $h^{-1}A$) and outgoing lines (again just 2, to gA and hA). Also, the graph is connected since g and h generate the group. Hence it has an Eulerian cycle, which solves the problem.

The existence of such a cycle is a standard graph theory result. Any graph (connected or not) in which every point has equal number of incoming and outgoing lines (and there is at least one line) has a cycle. Start with a point P_1 with at least one outgoing line. Take a line to P_2. If $P_2 = P_1$ then we are done. If not, then P_2 must have an outgoing line. And so on. Having got to P_m, if P_m belongs to $\{P_1, \ldots, P_{m-1}\}$ then we have a cycle (if $P_m = P_j$, then the cycle is $P_jP_{j+1} \ldots P_m$). If not, then it must have an outgoing line. But the process must terminate since the graph has only finitely many points.

Now remove the lines in the cycle. The remaining graph has an equal number of incoming and outgoing lines at every point. So we can find another cycle. Continuing, we partition the lines of the original graph into disjoint cycles. We now build up the Eulerian cycle. Start from one of the cycles. If that exhausts the lines we are done. If not, there must be another cycle with a point P in common with it (otherwise the graph would be disconnected). Form a new cycle by traversing these two cycles successively. If this exhausts the lines, we are done. If not, we may extend it as before. The process terminates, since the graph has only finitely many lines.

Problem B5
Can we find a sequence of reals $a_i \neq 0$ such that each polynomial $p_n(x) = a_0 + a_1x + \ldots + a_nx^n$ has all its roots real and distinct?

Solution
Answer: yes.

Take $a_0 = -1$, $a_1 = 1$. We take even coefficients negative and odd coefficients positive. Suppose we have found $a_0, a_1, \ldots, a_n$, so that the polynomial $p_n(x)$ has roots at $x_1 < x_2 < \ldots < x_n$. Take points $y_0 < x_1$, y_1 between x_1 and x_2, y_2 between x_2 and x_3, ... , y_{n-1} between x_{n-1} and x_n, and $y_n > x_n$. Then $p_n(y_i)$ is positive for i odd and negative for i even. If we now take $|a_{n+1}|$ sufficiently small, then $p_{n+1}(y_i)$ will also be positive for i odd and negative for i even. Moreover, if we take y_{n+1} to be sufficiently large, then $p_{n+1}(y_{n+1})$ will have the opposite sign to $p_{n+1}(y_n)$. So $p_{n+1}(y_0)$, $p_{n+1}(y_1)$, ... , $p_{n+1}(y_{n+1})$ alternate in sign. Hence $p_{n+1}(x)$ has at least n+1 distinct roots, namely one between y_i and y_{i+1} for each of i = 0, 1, 2, ... , n.

Problem B6
C is a non-empty, closed, bounded, convex subset of the plane. Given a support line L of C and a real number $0 \leq \alpha \leq 1$, let B_α be the band parallel to L, situated midway between L and the parallel support line on the other side of C, and of width α times the distance between the two support lines. What is the smallest α such that $\cap B_\alpha$ contains a point of C, where the intersection is taken over all possible directions for the support line L?

Solution
Answer: 1/3

The key is to consider the centroid of C. First, we need the result that the centroid cannot lie too far from the midline between two parallel support lines. To be more precise, it can lie at most 1/3 of the

way from the midline to the support lines. Let L and L' be parallel support lines, with P a point in the intersection of L and C. Take a triangle PQR with base QR on L' so that the area above PQ in C equals the area below PQ not in C and similarly for PR. Then the centroid of C is above or level with that of PQR and so cannot be below 1/3 of the way from the midline towards L'. Similarly, it cannot be above 1/3 of the way from the midline towards L.

For a triangle the intersection is empty if $\alpha < 1/3$. If $\alpha \geq 1/3$, then the intersection contains at least the centroid.

52nd Putnam 1991

Problem A1
The rectangle with vertices (0, 0), (0, 3), (2, 0) and (2, 3) is rotated clockwise through a right angle about the point (2, 0), then about (5, 0), then about (7, 0), and finally about (10, 0). The net effect is to translate it a distance 10 along the x-axis. The point initially at (1, 1) traces out a curve. Find the area under this curve (in other words, the area of the region bounded by the curve, the x-axis and the lines parallel to the y-axis through (1, 0) and (11, 0)).

Solution
Answer: $7\pi/2 + 6$.

Draw a diagram. The required area is comprised of two quarter-disks radius $\sqrt{2}$, two quarter disks radius $\sqrt{5}$, four right-angle triangles sides 1, 1, $\sqrt{2}$, and four right-angle triangles sides 1, 2, $\sqrt{5}$. Hence area = $\pi + 5/2\ \pi + 2 + 4$.

Problem A2
M and N are real unequal n x n matrices satisfying $M^3 = N^3$ and $M^2N = N^2M$. Can we choose M and N so that $M^2 + N^2$ is invertible?

Solution
Answer: No.

The only difficulty is that we are not told whether or not it is true. So one has to (1) try experimenting a little to find invertible M, N satisfying the conditions, and (2) try proving you cannot find such M, N. If you have bad luck you will try in that order!

$(M^2 + N^2)M = M^3 + N^2M = N^3 + M^2N = (M^2 + N^2)N$. But now if $M^2 + N^2$ was invertible, we could multiply by its inverse to get M = N, whereas we are told M and N are unequal.

Problem A3
P(x) of is a polynomial of degree $n \geq 2$ with real coefficients, such that (1) it has n unequal real roots, (2) for each pair of adjacent roots a, b the derivative P'(x) is zero halfway between the roots (at x = (a + b)/2). Find all possible P(x).

Solution
Answer: quadratics with two (distinct) real roots: $A(x - \alpha_1)(x - \alpha_2)$.

The question is whether each maximum (and minimum) can be placed halfway between its two flanking zeros. Experience suggests that this is not, in general, possible. Clearly, it is possible for certain maxima. For example if we take the polynomial's graph to have a line of symmetry then the maximum (or minimum) on the line of symmetry will be centrally placed. But we expect the extreme stationary values to be closer to the extremes.

Let the roots be $\alpha_1 < \alpha_2 < ... < \alpha_n$. We show that for n > 2 the stationary value between α_{n-1} and α_n is closer to α_n than α_{n-1}.

Let n > 2, $\beta = (\alpha_{n-1} + \alpha_n)/2$ and let $P(x) = A(x - \alpha_1) ... (x - \alpha_n)$. Then $P'(x) = A(x - \alpha_2)(x - \alpha_3) ... (x - \alpha_n) + A(x - \alpha_1)(x - \alpha_3) ... (x - \alpha_n) + ... + A(x - \alpha_1) ... (x - \alpha_{n-2})(x - \alpha_n) + A(x - \alpha_1) ... (x - \alpha_{n-1})$. For $x = \beta$, the last two terms have equal magnitude and opposite sign, so they sum to zero, whilst the remaining terms are all $-A ((\alpha_n - \alpha_{n-1})/2)^2$ times (n - 2) positive numbers. In other words, the

remaining terms are all non-zero and all have the opposite sign to A. Thus $p'(\beta) \neq 0$. [Indeed, since it has the opposite sign as A, it must be smaller than the stationary value, as claimed.]

Problem A4
Can we find an (infinite) sequence of disks in the Euclidean plane such that: (1) their centers have no (finite) limit point in the plane; (2) the total area of the disks is finite; and (3) every line in the plane intersects at least one of the disks?

Solution
Answer: yes.

The fact that the centers have no limit point is a red herring - after all putting them close together would hardly help to ensure that the disks blocked all lines. The easiest strategy is evidently to find a continuous line of overlapping disks. That evidently requires that if their radii are r_i, then $\sum r_i$ diverges, but the sum of their areas is finite, so $\sum r_i^2$ must converge. That is easy to achieve. For example, take $r_i = 1/i$.

That leaves a little tidying up. We need the line to be infinite in both directions, and we need another line of disks to catch lines parallel to the first line of disks. But multiplying by four leaves convergence and divergence unaffected.

So, define $C_n = \sum_1^n 1/i$, $R_n = 1/n$. Define disk D_{4n} to have center $(C_n, 0)$ and radius R_n. That gives us a line of overlapping disks of finite total area, covering the positive x-axis. Similarly, define D_{4n+1} to have center $(-C_n, 0)$ and radius R_n; they cover the negative x-axis. Define D_{4n+2} to have center $(0, C_n)$, radius R_n, and D_{4n+3} to have center $(0, -C_n)$, radius R_n; they cover the y-axis. The complete set $D_1, D_2, D_3, \ldots$ have the required properties.

Problem A5
Let $f(z) = \int_0^z \sqrt{(x^4 + (z - z^2)^2)}\, dx$. Find the maximum value of f(z) in the range $0 \leq z \leq 1$.

Solution
Answer: 1/3.

The integrand is strictly positive for $0 < x \leq 1$ and $0 \leq z \leq 1$, so $f(z) > 0$ for $0 < z \leq 1$. However, it is not at all clear how to get the integral in closed form except for z = 0 or 1. So we have a strong suspicion that the maximum must be the higher of the two values we can actually calculate: f(1) = 1/3! To prove this we need to show that $f'(z) > 0$ for $z > 0$.

$f'(z) = \sqrt{(z^4 + z^2(1 - z)^2)} + z(1 - z)(1 - 2z) \int_0^z (x^4 + z^2(1 - z)^2)^{-1/2}\, dx$. The integrand is positive, so clearly $f'(z) > 0$ for $0 < z \leq 1/2$. The integrand is less than $1/\sqrt{(z^2(1 - z)^2)} = 1/z\ 1/(1 - z)$, so the integral is less than $1/(1 - z)$, and for $z > 1/2$, $f'(z) > \sqrt{(z^4 + z^2(1 - z)^2)} - z(2z - 1)$. But, squaring, we see that $z(2z - 1) < \sqrt{(z^4 + z^2(1 - z)^2)}$ for $1/2 < z < 1$, which completes the proof.

Problem A6
An n-sum of *type 1* is a finite sequence of positive integers $a_1, a_2, \ldots, a_r$, such that: (1) $a_1 + a_2 + \ldots a_r = n$; and (2) $a_1 > a_2 + a_3$, $a_2 > a_3 + a_4, \ldots, a_{r-2} > a_{r-1} + a_r$, and $a_{r-1} > a_r$. For example, there are five 7-sums of type 1, namely: 7; 6, 1; 5, 2; 4, 3; 4, 2, 1.

An n-sum of *type 2* is a finite sequence of positive integers $b_1, b_2, \ldots, b_s$ such that: (1) $b_1 + b_2 + \ldots + b_s = n$; (2) $b_1 \geq b_2 \geq \ldots \geq b_s$; (3) each b_i is in the sequence $1, 2, 4, \ldots, g_j, \ldots$ defined by $g_1 = 1$, $g_2 = 2$, $g_j = g_{j-1} + g_{j-2} + 1$; and (4) if $b_1 = g_k$, then $1, 2, 4, \ldots, g_k$ is a subsequence. For example, there are five 7-sums of type 2, namely: 4, 2, 1; 2, 2, 2, 1; 2, 2, 1, 1, 1; 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1.

Prove that for $n \geq 1$ the number of type 1 and type 2 n-sums is the same.

Solution
Let f_i be the usual Fibonacci sequence defined by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$. A trivial induction gives that $g_n = \sum_1^n f_i$.

We show that there is a bijection between n-sums of type 2 with $b_1 = g_k$ and n sums of type 1 with k members.

Since a type 2 n-sum with $b_1 = g_k$ is made up entirely of elements of $\{g_1, g_2, \ldots, g_k\}$ and includes each element at least once, there is a bijection between such n-sums and sequences of non-negative numbers $c_1, c_2, \ldots, c_k$ satisfying $(c_1 + 1)g_1 + (c_2 + 1)g_2 + \ldots + (c_k + 1)g_k = n$ (*).

Suppose a_i is a type 1 n-sum with k terms. Then condition (2) implies that $a_i \geq a_{i+1} + a_{i+2} + 1$, for $i < k - 1$, and $a_k \geq 1$, $a_{k-1} \geq 2$. So $a_i \geq g_i$. If we increase a_i above this minimum value, then that has a knock-on effect for a_j with $j < i$. In fact, we can write:

$a_k = g_1 + c_k f_1$;
$a_{k-1} = g_2 + c_k f_2 + c_{k-1} f_1$;
$a_{k-2} = g_3 + c_k f_3 + c_{k-1} f_2 + c_{k-2} f_1$;
...
$a_1 = g_k + c_k f_k + c_{k-1} f_{k-1} + \ldots + c_1 f_1$.

Since the sum of the a_i is n, the c_i satisfy (*). Conversely, if c_i are any non-negative numbers satisfying (*), then the relations above give a type 1 n-sum a_i. So we have a bijection between type 1 n-sums with k members and non-negative c_i satisfying (*). Putting the two bijections together gives the claimed bijection between type 2 n-sums with largest member g_k and type 1 n-sums with k members.

It follows that there is a bijection between type 1 and type 2 n-sums.

Problem B1
For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$. For what b_0 do we have b_i constant for sufficiently large i?

Solution
Answer: the squares.

If b_k is a square, then $d(b_k) = 0$, so $b_{k+1} = b_k$ and the sequence is constant from that point on.

If b_k is not a square, then for some m it lies between m^2 and $(m + 1)^2$, so we can write it as $m^2 + r$, where $1 \leq r \leq 2m$. So $b_{k+1} = m^2 + 2r$. But $m^2 < b_{k+1} < (m + 2)^2$ and $b_{k+1} \neq (m + 1)^2$, so b_{k+1} is not a square (and is $> b_k$).

Problem B2
R is the real line. f, g: $R \to R$ are non-constant, differentiable functions satisfying: (1) $f(x + y) = f(x)f(y) - g(x)g(y)$ for all x, y; (2) $g(x + y) = f(x)g(y) + g(x)f(y)$ for all x, y; and (3) $f'(0) = 0$. Prove that $f(x)^2 + g(x)^2 = 1$ for all x.

Solution
Let $f(0) = A$, $g(0) = B$. Setting $x = y = 0$ in (1) and (2) gives: $A = A^2 - B^2$, $B = 2AB$. The latter implies that $B = 0$ or $A = 1/2$. But the former shows that $A \neq 1/2$ (since B^2 cannot be negative).

Hence B = 0, and hence A = 0 or 1.
Putting y = 0 in (2) gives g(x) = Ag(x). So $A \neq 0$ (since g is not constant). Hence A = 1.

We now use the differentiability of f and g. Let g'(0) = C. Then $f(\delta x) = 1 + O(\delta x^2)$, $g(\delta x) = C\delta x + O(\delta x^2)$. Hence, using (1), $f(x + \delta x) = f(x) f(x + \delta x) - g(x) g(\delta x) = f(x) (1 + O(\delta x^2)) - g(x) (C \delta x + O(\delta x^2))$. So f '(x) = - C g(x). Similarly, from (2), we get g'(x) = C f(x). So the derivative of $f(x)^2 + g(x)^2$ is $2f(x)f'(x) + 2g(x)g'(x) = -2Cf(x)g(x) + 2Cf(x)g(x) = 0$ for all x, and hence $f(x)^2 + g(x)^2 = f(0)^2 + g(0)^2 = 1$ for all x.

Problem B3
Can we find N such that all m x n rectangles with m, n > N can be tiled with 4 x 6 and 5 x 7 rectangles?

Solution
Answer: yes.

If a and b are relatively prime then we can express any integer N > ab as N = ra + sb with r and s positive integers. For a proof, take a > b, and observe that N, N - b, N - 2b, N - (a - 1)b are all positive and incongruent mod a, so they must constitute a complete set of residues mod a. In particular, we can find one which is congruent to zero mod a.

We can tile 35 x 5 and 35 x 7 using just 5 x 7 rectangles. But 5 and 7 are relatively prime, so we can use 35 x 5 and 35 x 7 rectangles to tile any 35 x n rectangle for n > 35. Similarly, we can tile 24 x 4 and 24 x 6 using just 4 x 6 rectangles. 4 and 6 have greatest common divisor 2, so we can use 24 x 4 and 24 x 6 rectangles to tile any 24 x 2n rectangle for n > 24. But 24 and 35 are relatively prime so for any 2n > 48, and m > 840 we can use 35 x 2n and 24 x 2n rectangles to tile an m x 2n rectangle. Finally, given any m x (2n+1) rectangle with 2n+1 > 83 and m > 840, we can divide it into an m x 35 rectangle and an m x 2n rectangle, each of which can be tiled. So any m x n rectangle can be tiled provided m, n > 840.

Problem B4
p is a prime > 2. Prove that $\sum_{0 \leq n \leq p} pCn\ (p+n)Cn = 2^p + 1 \pmod{p^2}$. [aCb is the binomial coefficient a!/(b! (a-b)!).]

Solution
Let $D_n = pCn\ (p+n)Cn$, so that the required sum is $\sum D_n$. We show that $D_0 = pC0 \pmod{p^2}$, $D_p = pCp + 1 \pmod{p^2}$, and for the other terms $D_n = pCn \pmod{p^2}$. The result then follows since $\sum pCn = 2^p$.

For n = 0, it is obvious that $D_0 = 1 = pC0$.

n = p is slightly trickier. Notice that we can write (p-1)! (2p-1)C(p-1) as (1 + p)(2 + p)(3 + p) ... (p-1 + p). Expanding, we get $(p-1)! + (1 + 2 + ... + p-1)\ p + O(p^2) = (p-1)! \pmod{p^2}$, since (1 + 2 + ... + p-1) = p (p-1)/2. Hence, $(2p-1)C(p-1) = 1 \pmod{p^2}$. Hence $2pCp = 2p/p\ (2p-1)C(p-1) = 2 = 1 + pCp \pmod{p^2}$.

For 0 < n < p, pCn = 0 (mod p) [because all the factors in n!(p - n)! are < p and hence relatively prime to p, so the factor p in the denominator p! remains after dividing by n!(p - n)!]. Also $p + i = i \neq 0 \pmod{p}$, so (p + n) ... (p + 2)(p + 1) = n! (mod p) and hence (p+n)Cn = 1 (mod p) for 0 < n < p. In other words, (p+n)Cn = kp + 1 for some integer k. But $kp\ pCn = 0 \pmod{p^2}$, since p divides pCn, so $D_n = (p+n)Cn\ pCn = pCn \pmod{p^2}$.

Problem B5
p a prime > 2. How many residues mod p are both squares and squares plus one? [For example, for p = 7, there are two such residues: $1 = 1^2 = 0^2 + 1$ and $2 = 3^2 = 1^2 + 1$.]

Solution
Answer: [p/4] + 1.

We require (x + y)(x - y) = 1 (mod p). But this is just the statement that (x - y) is the inverse of (x + y). Any non-zero residue has a unique inverse mod p. So we may take k to be any of the p - 1 values 1, 2, ... , p - 1 and then (x + y) = k, (x - y) = 1/k (mod p). Thus there are p - 1 solutions pairs (x, y): ((k + 1/k)/2, (k - 1/k)/2) k = 1, 2, ... , p - 1.

We are not quite there yet, because we require the number of distinct h such that $h = x^2 = y^2 + 1$ (mod p), and some of the solution pairs for (x, y) give the same value h. In fact, $z^2 = s$ (mod p) has either two or zero solutions unless s = 0, when it has just 1. Hence there are in general 4 solution pairs $(\pm x_0, \pm y_0)$ corresponding to a given value of h. But there are always just two, namely (±1, 0), corresponding to h = 1. If -1 is a quadratic residue of p, then there are two solutions $(0, \pm y_0)$ corresponding to h = 0, otherwise none. But we show below that -1 is a quadratic residue iff p = 1 (mod 4). Hence if p = 1 (mod 4), then there are (p - 1 - 4)/4 + 2 = (p + 3)/4 = [p/4] + 1 solutions. Similarly, if p = 3 (mod 4), then there are (p - 1 - 2)/4 + 1 = (p + 1)/4 = [p/4] + 1 solutions.

It remains to show when -1 is a quadratic residue, although this could probably be assumed in the exam. In any case, it is pure bookwork. If p = 3 (mod 4) and -1 is a quadratic residue, then we would have $t^2 = -1$ (mod p) and hence $t^{p-1} = (-1)^{(p-1)/2} = -1$ (mod p). But $t^{p-1} = 1$ (mod p) [Fermat's theorem, proof: both t, 2t, 3t, ... , (p-1)t and 1, 2, 3, ... , p - 1 are complete sets of non-zero residues, so (p-1)! t^{p-1} = (p-1)! (mod p) and hence $t^{p-1} = 1$ (mod p)], and p does not divide 2. Contradiction. Conversely, if p = 1 (mod 4), then we show that $(1 \cdot 2 \cdot 3. \ldots . (p-1)/2)^2 = -1$ (mod p). But the square is just (p - 1)! so we have to prove Wilson's theorem. s is its own inverse mod p iff s = 1 or p - 1. So the other p - 3 residues divide into pairs with product 1 and hence (p - 1)! = p - 1 = - 1 (mod p).

Problem B6
Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that for all x with $0 < |x| \le c$ and for all α with $0 < \alpha < 1$, we have:
$a^{\alpha} b^{1-\alpha} \le a \sinh \alpha x/\sinh x + b \sinh x(1 - \alpha)/\sinh x$.

Solution
Answer: c = |ln a/b|.

Put β = 1 - α, so that α, b are both positive numbers less than 1 with sum 1. Let $g_\alpha(x) = \sinh \alpha x/\sinh x$, and $f(x) = a\, g_\alpha(x) + b\, g_\beta(x)$. For large positive x, $g_\alpha(x)$ is approximately $e^{\alpha x}/e^x = e^{-\beta x}$ which is a decreasing function of x, so we suspect that $g_\alpha(x)$ has a maximum at x = 0. To check we look at small x, where we have $g_\alpha(x) = \alpha - \alpha(1 - \alpha^2)/6\, x^2$. So we conjecture that $g_\alpha(x)$ has a maximum at x = 0, and similarly, of course, $g_\beta(x)$. This would imply that f(x) also had a maximum at x = 0 and indeed was strictly monotonic decreasing for x > 0 (and strictly monotonic increasing for x < 0, since f(-x) = f(x)). If we can prove the conjecture, then we just have to find the value of x for which $f(x) = a^{\alpha} b^{\beta}$.

So, with that motivation, we try to prove that $g_\alpha(x)$ has a maximum at x = 0. The derivative of $g_\alpha(x)$ is $\alpha \cosh \alpha x/\sinh x - \cosh x \sinh \alpha x/\sinh^2 x$. Multiplying by $\sinh^2 x/(\cosh x \cosh \alpha x)$ gives $h(x) = \alpha \tanh x - \tanh \alpha x$ and does not change the sign. It is not immediately clear that we are getting anywhere. But we try differentiating again: $h'(x) = \alpha \operatorname{sech}^2 x - \alpha \operatorname{sech}^2 \alpha x$. We have apparently been lucky, since it is clear that: for x > 0, αx < x, so sech x < sech αx, and hence h'(x) < 0. But h'(-x) =

h'(x), so h'(x) < 0 for all $x \neq 0$. Hence h(x) is strictly monotonic decreasing, as required. Hence $g_\alpha(x)$, and hence f(x), does indeed have a (global) maximum at x = 0. [The difficulty about this procedure is that it is not at all clear in advance that we are going to get anywhere, the expressions seem to get *more* complicated as we differentiate. Also, it is much worse if you fail to simplify by taking out the always positive factor $\sinh^2 x/(\cosh x \cosh \alpha x)$ after the first differentiation.]

We now have to find the value of x_0 at which $f(x_0) = a^\alpha b^\beta$. It is not at all clear that this value will be independent of α, but the question suggests that it is. So the first step is to find x_0 for some special value of α. It is convenient at this point to assume that $a \geq b$. α = 0 or 1 are unhelpful because in that case $f(x) = a^\alpha b^\beta$ for all non-zero x. So we try the next simplest: α = 1/2. Using $\sinh x_0 = 2 \sinh x_0/2 \cosh x_0/2$, we get that $\cosh x_0/2 = (\sqrt{(a/b)} + \sqrt{(b/a)}) / 2$. But the rhs is $\cosh (1/2 \ln (a/b))$, so we find that $x_0 = \ln (a/b)$.

The second step is to see if this works for all α. Noting that a = b (a/b), we see that $f(\ln \sqrt{(a/b)}) = b/(a/b - b/a) \{ (a/b) ((a/b)^\alpha - (a/b)^{-\alpha}) + (a/b)^{1-\alpha} - (a/b)^{-1+\alpha} \} = b/(a/b - b/a) \{ (a/b)^{1+\alpha} - (a/b)^{-1+\alpha} \}$ (the two middle terms canceled) $= b (a/b)^\alpha = a^\alpha b^\beta$, as required. So it does work for all α, and we are done. Note that there was no real loss of generality in assuming that $a \geq b$. If b> a, just use the result with a and b interchanged. Hence the | | in the answer.

Putnam 1992

Problem A1
Let Z be the integers. Prove that if f **:** Z → Z satisifies f(f(n)) = f(f(n+2) + 2) = n for all n, and f(0) = 1, then f(n) = 1 - n.

Solution
We are given that f(0) = 1 - 0 = 1. Hence f(1) = f(f(0)) = 0. Suppose f(r) = 1 - r for all r ≤ n. Then f(-(n-2)) = f(f(n-1)) = n-1. Hence f(n+1) = f(n-1 + 2) = f(f(-(n-2)) + 2) = -(n-2) - 2 = 1 - (n+1). So the result is true for all positive n. Hence f(-n) = f(f(n+1)) = n+1 = 1 - (-n), so the result is true for negative n also.

Problem A2
Let the coefficient of x^{1992} in the power series $(1 + x)^{\alpha} = 1 + \alpha x + ...$ be $C(\alpha)$. Find $\int_0^1 C(-y-1) \sum_{k=1}^{1992} 1/(y+k)\, dy$.

Solution
The usual Taylor series gives C(-y-1) = (y+1)(y+2) ... (y+1992)/1992! . Hence the integrand is simply the derivative of (y+1)(y+2) ... (y+1992)/1992! , so the integral evaluates to 1993 - 1 = 1992.

Problem A3
Find all positive integers a, b, m, n with m relatively prime to n such that $(a^2 + b^2)^m = (ab)^n$.

Solution
Answer: a, b, m, n = 2^r, 2^r, 2r, 2r+1 for any positive integer r.

$a^2 + b^2 \geq 2ab > ab$, so m < n.

Let d be the greatest common divisor of a and b. Set a=dA, b = dB. Then $(A^2 + B^2)^m = (AB)^n d^{2(n-m)}$. Hence if p divides A, then it must also divide B. But A and B are coprime, so A = 1. Similarly, B = 1. Now 2(n-m) > 1, so d divides $(A^2 + B^2)^m = 2^m$. So d is a power of 2. Hence a = b = 2^r for some r. So (2r+1)m = 2r n. But 2r+1 and 2r are coprime, so m = M·2r, n = N·(2r+1). Hence M = N. But m and n are coprime, so M = N = 1.

Problem A4
Let R be the reals. Let f **:** R → R be an infinitely differentiable function such that $f(1/n) = n^2/(n^2+1)$ for n = 1, 2, 3, ... Find the value of the derivatives of f at zero: $f^{(k)}(0)$ for k = 1, 2, 3, ...

Solution
Answer: the odd derivatives are zero; $f^{(2n)}(0) = (-1)^n$.

Clearly one possibility is $f(x) = 1/(1 + x^2)$. This has the power series expansion $1 - x^2 + x^4 - ...$, so in this case $f^{(k)}(0) = 0$ for k odd and $(-1)^n$ for k = 2n.

Suppose g(x) is another such function. Define h by h(x) = f(x) - g(x). Then h is infinitely differentiable and zero at x = 1/1, 1/2, 1/3, ... We show that all derivatives of h are zero at x = 0. Suppose otherwise. Take $h^{(k)}(0)$ as the non-zero derivative of lowest order. Then for some ε > 0, $h^{(k)}(x)$ is non-zero for all $0 \leq x < \varepsilon$. Take n sufficiently large that 1/n < ε. By the (nth order) mean value theorem, $h(1/n) = h^{(k)}(\varepsilon')/k!$ for some $\varepsilon' < 1/n < \varepsilon$. But the lhs is zero and the rhs is non-zero. Contradiction. So all the derivatives of h are zero at x = 0, and hence the derivatives of g at x = 0 are the same as those of $1/(1+x^2)$.

Problem A5
Let N be the positive integers. Define $f : N \to \{0, 1\}$ by f(n) = 1 if the number of 1s in the binary representation of n is odd and 0 otherwise. Show that there do not exist positive integers k and m such that $f(k + j) = f(k + m + j) = f(k + 2m + j)$ for $0 \le j < m$.

Solution
Note that $f(2n) \ne f(2n+1)$. So f(n), f(n+1), f(n+2) cannot all be equal, which establishes the case m = 1.

Note that the condition $f(k + j) = f(k + m + j) = f(k + 2m + j)$ for $0 \le j < m$ can alternatively be written as $f(k + j) = f(k + j + m)$ for $0 \le j < 2m$.

f(4n+1) = f(4n+2), so we cannot have f(a) = f(b) and f(a+1) = f(b+1) if a is even and b = 1 (mod 4). This is sufficient to establish the case m odd and > 1. For suppose k is even. Then either k+m or k+m+2 = 1 (mod 4). So taking a = k and b = whichever of k+m and k+m+2 is 1 mod 4, we cannot have f(a) = f(b) and f(a+1) = f(b+1). Similarly, if k is odd, then either k or k+2 is 1 mod 4 and both k+m and k+m+2 are even.

It remains to consider m even. But given a case with 2m we can always find another case with m. Repeating we would eventually reach a case with m odd. So even cases cannot exist either.

Since f(2n) = f(n) the case 2k, 2m immediately gives the case k, m. Just select the even j: $f(2k + 2j) = f(2k + 2j + 2m)$ for $0 \le j < m$ and deduce $f(k + j) = f(k + j + m)$ for $0 \le j < m$, which is the case k, m. Similarly, the case 2k+1, 2m gives the case k+1, m.

Problem A6
Four points are chosen independently and at random on the surface of a sphere (using the uniform distribution). What is the probability that the center of the sphere lies inside the resulting tetrahedron?

Solution
Answer: 1/8.

Having placed 3 points ABC, the 4th D will enclose the center in the tetrahedron iff it lies in the spherical triangle A'B'C', where P' is directly opposite to P (so that the center lies on PP'). The probability of this is the area of ABC divided by the area of the sphere. So taking the area of the sphere as 1, we want to find the expected area of ABC. But the 8 triangles ABC, A'BC, AB'C, ABC', A'B'C, AB'C', A'BC', A'B'C' are all equally likely and between them partition the surface of the sphere. So the expected area of ABC, and hence the required probability, is just 1/8.

Problem B1
Let R be the reals. Let $S \subseteq R$ have $n \ge 2$ elements. Let $A_S = \{ x \in R : x = (s + t)/2$ for some $s, t \in S$ with $s \ne t\}$. What is the smallest possible $|A_S|$?

Solution
Answer: 2n - 3.

{1, 2, ... , n} gives 2n - 3 averages, namely 1 1/2, 2, 2 1/2, ... , n - 1/2, so it remains to show that we cannot do better.

Suppose $a_1 < a_2 < ... < a_{n+1}$. Let S be the set $\{a_1, a_2, ... , a_n\}$ and S' the set $S \cup \{a_{n+1}\}$. Then $x = (a_n + a_{n+1})/2 > a_n$, so $x \notin A_S$. Similarly, $y = (a_{n+1} + a_{n-1})/2 > (a_n + a_{n-1})/2$, which is the largest element of A_S, so $y \notin A_S$. Hence $|A_{S'}| \geq |A_S| + 2$.

Problem B2
Show that the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$ is $\sum_{j=0}^{k}$ nCj nC(k-2j).

Solution
$1 + x + x^2 + x^3 = (1 + x)(1 + x^2)$. So the expression is $(\sum$ nCi $x^i)$ $(\sum$ nCj $x^{2j})$. To get a term in x^k we must multiply a term x^{2j} by a term x^{n-2j} for some j. The result follows.

Problem B3
Let S be the set of points (x, y) in the plane such that the sequence a_n defined by $a_0 = x$, $a_{n+1} = (a_n^2 + y^2)/2$ converges. What is the area of S?

Solution
Answer: $4 + \pi$.

We claim that:
(1) the sequence diverges for $|y| > 1$;
(2) the sequence diverges for $|y| < 1$ and $|x| > 1 + \sqrt{(1-y^2)}$;
(3) the sequence converges for $|y| < 1$ and $|x| < 1 + \sqrt{(1-y^2)}$.

It follows that the area S is the square with sides $x = \pm 1$, $y = \pm 1$ with a hemispherical cap on each of the vertical sides $x = \pm 1$, and hence that the area is $4 + \pi$.

Notice that $a_{n+1} - a_n = ((a_n - 1)^2 - (1 - y^2))/2$ (*). So if $|y| > 1$, then $a_{n+1} - a_n > (y^2 - 1)/2 > 0$, which establishes (1).

In case (2), take $\varepsilon > 0$, such that $|x| > 1 + \sqrt{(1 - y^2)} + \varepsilon$. Then $(x - 1)^2 > 1 - y^2 + \varepsilon^2$, so $a_1 - x > \varepsilon^2/2$ (using (*)). Hence $a_1 > x > 1 + \sqrt{(1 - y^2)} + \varepsilon$. So by a simple induction $a_{n+1} - a_n > \varepsilon^2/2$ for all n, which establishes (2).

In case (3), assume first that x is positive. $x < 1 + \sqrt{(1-y^2)}$ implies that $(x - 1)^2 < 1 - y^2$, so by (*) $a_1 < x$. So by a simple induction $a_{n+1} < a_n$ for all x. But all a_n are positive, so convergence follows. In the case x negative the sequence is the same for all later terms as that started by -x, so we have convergence in this case also.

Problem B4
p(x) is a polynomial of degree < 1992 such that p(0), p(1), p(-1) are all non-zero. The 1992th derivative of $p(x)/(x^3 - x) = f(x)/g(x)$ for polynomials f(x) and g(x). Find the smallest possible degree of f(x).

Solution
Answer: 3984.

We may write $p(x)/(x^3 - x) = q(x) + a/x + b/(x-1) + c/(x+1)$, where q(x) has degree less than 1992, and (since 0, 1, and -1 are not roots of p(x)) a, b, c are all non-zero. When we differentiate 1992 times, q(x) disappears and we get a non-zero constant times $(a/x^{1993} + b/(x-1)^{1993} + c/(x+1)^{1993})$. Hence f(x) is a multiple of $a(x^2-1)^{1993} + bx^{1993}(x+1)^{1993} + cx^{1993}(x-1)^{1993}$. We can zero the highest power $2 \cdot 1993 = 3986$ by taking $a + b + c = 0$. We can zero the next highest power 3985 by taking b

= c. That also zeros all the other odd powers. Now the coefficient of x^{3984} is -1992a + 1993.1992 b = 1992(1993 + 2)b which is non-zero. So the smallest possible degree of f(x) is 3984.

Problem B5
Let A_n denote the n-1 x n-1 matrix (a_{ij}) with $a_{ij} = i + 2$ for i = j, and 1 otherwise. Is the sequence (det A_n)/n! bounded?

Solution
Answer: no.

Subtract col 1 from each of the others. This reduces each element along the diagonal after the first by 1 (so that the diagonal becomes 3, 3, 4, ... , n). The first row becomes -2 apart from the first element, the first column remains 1 apart from the first element and all other elements become 0. We now subtract 1/3 of col 2 from col 1, 1/4 of col 3 from col 1, 1/5 of col 4 from col 1 and so on. This zeros all elements of col 1 except the first which becomes 3 + 2(1/3 + 1/4 + ... + 1/n). We may now expand by the first column to get n!/2 times 3 + 2(1/3 + 1/4 + ... + 1/n). So (det A_n)/n! = (3 + 2(1/3 + 1/4 + ... + 1/n))/2. But this diverges as n tends to infinity.

Problem B6
Let M be a set of real n x n matrices such that: (1) $1 \in M$; (2) if $A, B \in M$, then just one of AB, -AB is in M; (3) if $A, B \in M$, then either AB = BA or AB = -BA; (4) if $1 \neq A \in M$, then there is at least one $B \in M$ such that AB = - BA. Prove that M contains at most n^2 matrices.

Solution
For example, we might take the set:

```
I = (1   0)     A = (0   1)     B = (1   0)     C = (0 −1)
    (0   1)         (1   0)         (0 −1)         (1   0)
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We find $A^2 = B^2 = -C^2 = I$, AB = -BA =C, AC = -CA = B, CB = -BC = A.

We show first that if A is in M, then $A^2 = \pm I$. Take any B. Then BA = kAB where $k^2 = 1$, so $BA^2 = k\,ABA = k^2\,A^2B = A^2B$. So A^2 commutes with every element of M. Obviously $-A^2$ does also. But by (2) one of A^2 and $-A^2$ must be in M and (4) then guarantees a non-commuting element unless $A^2 = \pm I$.

If there are more than n^2 elements, then we can find a linear combination of elements which is zero: $\sum \lambda_A A = 0$ (*). If we multiply on both sides by B we get $\sum \pm\lambda_A A = 0$. We could add this to (*) to get a linear combination with fewer terms provided BA = AB for at least one A in (*) and BA = -AB for at least one A in (*).

The only way to guarantee a + sign is if A = I. But we can do that by multiplying by A, where A appears in the combination. The A term then becomes an I term. Guaranteeing a - sign is then easy. There must be at least one other term (apart from I). Suppose it is A. Then choose B such that AB = - BA.

So if there are more than n^2 terms we can find a linear relation which we can shorten without limit. Contradiction. So there are at most n^2 terms.

Putnam 1993

Problem A1
Let O be the origin. y = c intersects the curve y = 2x - $3x^3$ at P and Q in the first quadrant and cuts the y-axis at R. Find c so that the region OPR bounded by the y-axis, the line y = c and the curve has the same area as the region between P and Q under the curve and above the line y = c.

Solution
Answer: 4/9.

Suppose the line cuts the curve at x = a and x = b. Then $\int_0^b (2x - 3x^3)\,dx = bc$ if c has the correct value (draw a picture). So $b^2 - 3/4\ b^4 = b\ (2b - 3b^3)$ or b = 2/3 and hence c = 4/9.

Problem A2
The sequence a_n of non-zero reals satisfies $a_n^2 - a_{n-1}a_{n+1} = 1$ for $n \geq 1$. Prove that there exists a real number α such that $a_{n+1} = \alpha\, a_n - a_{n-1}$ for $n \geq 1$.

Solution
For $n \geq 2$ define $b_n = (a_n + a_{n-2})/a_{n-1}$. Then the relation given shows that $b_{n+1} = b_n$.

Problem A3
Let P be the set of all subsets of {1, 2, ... , n}. Show that there are $1^n + 2^n + ... + m^n$ functions f **:** P → {1, 2, ... , m} such that f(A ∩ B) = min(f(A), f(B)) for all A, B.

Solution
Let X = {1, 2, ... , n}, $X_i = X \setminus \{i\}$. Then f is completely determined by its value on X and the X_i. Its value on these sets can be any member of {1, 2, ... , m} except that we must have $f(X_i) \leq f(X)$. Thus there are k^n possible functions with f(X) = k and hence $1^n + 2^n + ... + m^n$ possible functions in all.

Problem A4
Given a sequence of 19 positive (not necessarily distinct) integers not greater than 93, and a set of 93 positive (not necessarily distinct) integers not greater than 19. Show that we can find non-empty subsequences of the two sequences with equal sum.

Solution
We establish the general case for $0 < x_1 \leq x_2 \leq ... \leq x_m \leq n$, $0 < y_1 \leq y_2 \leq ... \leq y_n \leq m \leq n$.

Let $a_h = x_1 + x_2 + ... + x_h$, $b_k = y_1 + y_2 + ... + y_k$. Suppose that $a_m \leq b_n$ (if not, we can just reverse the roles of x, a and y, b in what follows). For each i we have $b_n \geq a_m \geq a_i$, so we may define f(i) as the smallest j such that $b_j \geq a_i$. Now consider the set $\{b_{f(1)} - a_1, b_{f(2)} - a_2, ... , b_{f(m)} - a_m\}$. If any element is zero, then we are done, so we may assume that all elements are ≥ 1. Each element must be less than m, because if $b_{f(i)} \geq a_i + m$, then $b_{f(i)-1} >= a_i$, contradicting the minimality of f(i). So we have a set of m elements drawn from {1, 2, ... , m-1}. Hence there must be two elements, $b_{f(r)} - a_r$ and $b_{f(s)} - a_s$, which are equal. Assume r < s, then $b_{f(s)-f(r)} = a_{s-r}$.

Problem A5
Let U be the set formed as the union of three open intervals, U = (-100, -10) ∪ (1/101, 1/11) ∪ (101/100, 11/10). Show that $\int_U (x^2 - x)^2/(x^3 - 3x + 1)^2\,dx$ is rational.

Solution
Answer: (not required) 11131110/107634259.

It is not obvious how to integrate the integrand in closed form. So If we sketch the integrand we find that it is positive and close to zero except for poles near -2, 1/2 and 1 1/2. So poles fall between the three ranges of integration. However, it still seems odd that we are given a sum of three ranges. So we look for a substitution that might bring the ranges into coincidence or something like it. A little tinkering suggests y = 1/(1-x). This takes (1/101, 1/11) to (101/100, 11/10), and (101/100, 11/10) to (-100, -10). We also find that it takes $(x^2 - x)/(x^3 - 3x + 1)$ to $(y^2 - y)/(y^3 - 3y + 1)$.

Let $I(a, b) = \int_a^b (x^2 - x)^2/(x^3 - 3x + 1)^2 \, dx$. Then we get $I(1/101, 1/11) = \int_{101/100}^{11/10} (y^2 - y)^2/(y^3 - 3y + 1)^2 \, dy/y^2$. Applying the transformation again gives that $I(-100, -10) + I(1/101, 1/11) + I(101/100, 11/10) = \int_{-100}^{-10} (z^2 - z)^2/(z^3 - 3z + 1)^2 \, (1 + (2z^2 - 2z + 1)/((z - 1)^2 z^2)) \, dz = \int_{-10}^{-100} (z^2 - z + 1)^2/(z^3 - 3z + 1)^2 \, dz$.

At this point we are in a difficulty. It is still not obvious how to evaluate the integral explicitly. But at the same time, there is no other obvious approach. If it does have a closed form, then $(az^2 + bz + c)/(z^3 - 3z + 1)$ looks a good candidate. Differentiating this, we find that it does in fact work for a = -1, b = 1, c = 0. So the value is K(-10) - K(-100), where $K(z) = (z^2 - z)/(z^3 - 3z + 1)$. This is obviously rational. [In fact it evaluates to the answer given above with a little more effort.]

Problem A6
Let $a_0, a_1, a_2, \ldots$ be a sequence such that: $a_0 = 2$; each $a_n = 2$ or 3; a_n = the number of 3s between the nth and n+1th 2 in the sequence. So the sequence starts: 233233233233 2 ... . Show that we can find α such that $a_n = 2$ iff $n = [\alpha m]$ for some integer $m \geq 0$.

Solution
Let b_n be the index of the nth 2, so $b_0 = 0$, $b_1 = 3$, $b_2 = 7$, $b_3 = 11$, $b_4 = 14$, ... Then we are told that $b_n = (a_0 + 1) + (a_1 + 1) + \ldots + (a_{n-1} + 1)$. Now each $a_i = 2$ or 3, so $b_n = 4n$ - (number of b_ms in 0, 1, 2, ... , n-1).

Suppose that $b_n = [n\alpha]$ as claimed. Then the number of b_ms in 0, 1, ... , n-1 is the number of $[m\alpha] < n$, which is the same as the number of $(m\alpha) < n$ (since α is irrational). And that is just $1 + [n/\alpha]$. So we have $[n\alpha] = 4n - 1 - [n/\alpha]$. But $4n - [n/\alpha] = [4n - n/\alpha] + 1$ (since n/α is irrational), so $[n\alpha] = [4n - n/\alpha] = [n\beta]$, where $\beta = 4 - 1/\alpha$. But this holds for all n, so we must have $\beta = \alpha$. Solving the quadratic gives $\alpha = 2 \pm \sqrt{3}$. But clearly $\alpha > 3$, so $\alpha = 2 + \sqrt{3}$.

We are not done yet. All we have done is shown that *if* it is true that $b_n = [n\alpha]$ for some α, then α must be $2 + \sqrt{3}$. But we can work the previous argument in reverse. Suppose we define $d_n = [n(2 + \sqrt{3})]$. Then the argument shows that $d_n = 4n$ - (number of d_ms in 0, 1, 2, ... , n-1) and that is sufficient to identify d_n and b_n.

Problem B1
What is the smallest integer n > 0 such that for any integer m in the range 1, 2, 3, ... , 1992 we can always find an integral multiple of 1/n in the open interval (m/1993, (m + 1)/1994)?

Solution
Answer: 1993 + 1994 = 3987.

For positive a, b, a', b', with a/b < a'/b', we have that a/b < (a+a')/(b+b') < a'/b'. So certainly m/1993 < (2m+1)/3987 < (m+1)/1994. The question is whether a smaller number also works.

The simplest idea is to try the extreme value m = 1992. That gives: 1992/1993 < k/n < 1993/1994. Subtracting from 1 gives: 1/1993 > k'/n > 1/1994, where k' = n - k, or 1993k' < n < 1994k'. That is clearly not possible for k' = 1, so the smallest possible n corresponds to k' = 2 and is 3987.

Problem B2
A deck of 2n cards numbered from 1 to 2n is shuffled and n cards are dealt to A and B. A and B alternately discard a card face up, starting with A. The game when the sum of the discards is first divisible by 2n + 1, and the last person to discard wins. What is the probability that A wins if neither player makes a mistake?

Solution
Answer: 0.

B always wins irrespective of the deal. The stuff about random shuffling and probabilities is a red herring. Note first that both players have perfect information (because each can see his own cards and the discards and hence deduce the other player's hand).

A cannot win on his first play. Now suppose it is B's turn. Each card B might play will give a different total and so requires a different card for A to win on the following move. But B has one more card than A, so he can choose a card which does not allow A to win on the next move. Thus B can always play so that A does not win on the following move. But one player must win after 2n plays or less, so it must be B.

Problem B3
x and y are chosen at random (with uniform density) from the interval (0, 1). What is the probability that the closest integer to x/y is even?

Solution
The closest integer to x/y is 0 if $x < 2y$. It is 2n (for $n > 0$) if $2x/(4n+1) < y < 2x/(4n-1)$. [We can ignore $y/x = 2/(2m+1)$ since it has probability zero.]

Hence the required probability $p = 1/4 + (1/3 - 1/5) + (1/7 - 1/9) + \ldots$. But $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$, so $p = 5/4 - \pi/4$.
[It makes things easier to draw a picture of the unit square with the lines $y = 2x$, $y = 2/3\ x$, $y = 2/5x$ etc.]

Problem B4
K(x, y), f(x) and g(x) are positive and continuous for $x, y \in [0, 1]$. $\int_0^1 f(y)\, K(x, y)\, dy = g(x)$ and $\int_0^1 g(y)\, K(x, y)\, dy = f(x)$ for all $x \in [0, 1]$. Show that f = g on [0, 1].

Solution
Let $h = \min \{ f(x)/g(x): x \in [0, 1] \}$, $k = \min\{ g(x)/f(x): x \in [0, 1] \}$. Assume $h < k$. Then for all x, $g(x)/f(x) \geq k \geq h$, or $g(x) - h\, f(x) \geq 0$. Since [0, 1] is compact, the value h must be attained. So take x_0 such that $f(x_0) = h\, g(x_0)$. Then we have $0 = f(x_0) - h\, g(x_0) = \int K(x_0, y)\, (g(y) - h\, f(y))\, dy$. But K is positive, and $g(y) - h\, f(y)$ is continuous and non-negative, so if we had $g(y) - h\, f(y) > 0$ for any y, then there would be an interval on which $g(y) - h\, f(y) > 0$ and the integral would be non-zero. Hence $g(y) - h\, f(y) = 0$ for all y. But now we have $f(x) = \int K(x, y)\, g(y)\, dy = \int K(x, y)\, h\, f(y)\, dy = h\, g(x) = h^2 f(x)$. Hence h = 1, and f(x) = g(x).

Problem B5
Show that given any 4 points in the plane we can find two whose distance apart is not an odd integer.

Solution
This is a curious question. It is easy if you happen to know the formula for the volume of a

tetrahedron in terms of the lengths of its sides (which almost no one does). Otherwise it harder. Some alternative solutions are below.

The formula is $V = 1/12\ \sqrt{(\sum_1 a^2b^2c^2 - \sum_2 a^2b^2c^2 - \sum_3 a^4b^2)}$. Here $\sum_1$ is taken over the 12 cases where a, b, c form an open path of length 3. $\sum_2$ is taken over the 4 cases where a, b, c form the sides of a face. $\sum_3$ is taken over the 6 cases where a, b are opposite sides (without a vertex in common). This is usually written as a 5 x 5 symmetric determinant D with zeros on the diagonal (the volume is $12/\sqrt{2}$ times the square root of D). If we label the vertices as 1, 2, 3, 4 so that the lengths of the sides become d_{ij}, then the elements below the diagonal are given by $D_{21} = d_{21}^2$, $D_{31} = d_{31}^2$, $D_{32} = d_{32}^2$, $D_{41} = d_{41}^2$, $D_{42} = d_{42}^2$, $D_{43} = d_{43}^2$, and $D_{51} = D_{52} = D_{53} = D_{54} = 1$.

In either case, we use a parity argument. If all the lengths are odd integers, then their squares are all congruent to 1 mod 4. So the sum is obviously congruent to 2 mod 4 and hence non-zero. With a little more work the determinant is congruent to 4 mod 8 and hence non-zero.

I do not know an elegant proof of the formula for the volume of a tetrahedron. It can be derived in a straightforward way, but the algebra is fairly horrendous (and probably takes half an hour, if you are careful enough not to make any mistakes). Choose coordinates so that the points are (0, 0, 0), (a, 0, 0), (d, b, 0), (e, f, c). Take the lengths as: P from (0, 0, 0) to (a, 0, 0); Q from (0, 0, 0) to (d, b, 0); R from (a, 0, 0) to (d, b, 0); L from (0, 0, 0) to (e, f, c); M from (a, 0, 0) to (e, f, c); and N from (d, b, 0) to (e, f, c). Then the volume is 1/6 abc. We have the following equations:
$P^2 = a^2$ (1); $Q^2 = b^2 + d^2$ (2);
$R^2 = (a - d)^2 + b^2$ (3); $L^2 = e^2 + f^2 + c^2$ (4);
$M^2 = (a - e)^2 + f^2 + c^2$ (5); $N^2 = (d - e)^2 + (b - f)^2 + c^2$ (6).
(1) gives: a = P.
(1) + (2) - (3) gives: $2ad = P^2 + Q^2 - R^2$, or $d = (P^2 + Q^2 - R^2)/2P$;
subs in (2) gives: $b^2 = Q^2 - d^2 = (2P^2Q^2 + 2Q^2R^2 + 2R^2P^2 - P^4 - Q^4 - R^4)/4P^2$;
(1) + (4) - (5) gives: $2ae = P^2 + L^2 - M^2$, or $e = (P^2 + L^2 - M^2)/2P$;
(2) + (4) - (6) gives: $Q^2 + L^2 - N^2 = 2de + 2bf$, so $2bf = Q^2 + L^2 - N^2 - (P^2 + Q^2 - R^2)(P^2 + L^2 - M^2)/2P^2$;

Finally, $a^4b^2c^2 = a^4b^2(L^2 - e^2 - f^2) = a^2(a^2b^2)L^2 - (a^2b^2)(a^2e^2) - a^4(bf)^2$. Substituting the results already obtained and multiplying out, gives $16a^4b^2c^2$ as a sum of almost 100 terms. Most of these cancel and we are left with the result quoted.

I subsequently discovered that there are several better ways of doing this problem. First, we can get (a variant of) the tetrahedral formula more easily using vectors.

Take one of the points at the origin and the vectors to the other three points as **a**, **b**, **c**. The volume $V = |\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$. Expressing as a matrix and multiplying by its transpose gives $V^2 = \det X$, where X is the 3 x 3 matrix with first row **a.a**, **a.b**, **a.c**, second row **b.a**, **b.b**, **b.c**, third row **c.a**, **c.b**, **c.c**. Now using $2\mathbf{a}.\mathbf{b} = a^2 + b^2 - |\mathbf{a} - \mathbf{b}|^2$ etc, we get $8V^2 = \det Y$, where Y is the 3 x 3 matrix with first row $2a^2$, $a^2 + b^2 - z^2$, $a^2 - y^2 + c^2$, second row $a^2 + b^2 - z^2$, $2b^2$, $-x^2 + b^2 + c^2$, third row $a^2 - y^2 + c^2$, $-x^2 + b^2 + c^2$, $2c^2$ (and x is the distance from the tip of vector **c** to the tip of vector **b** etc). Now since odd squares are 1 mod 8, we get immediately that $8V^2 = 4$ mod 8, which is impossible.

There are many variants on this, some using explicitly the angles in the figure.

Problem B6
Given a triple of positive integers $x \le y \le z$, derive a new triple as follows. Replace x and y by 2x and y - x (and reorder), or replace x and z by 2x and z - x (and reorder), or replace y and z by 2y and

z - y (and reorder). Show that by finitely many tranformations of this type we can always derive a triple with smallest element zero.

Solution
Let the three integers in S be $0 < a \leq b \leq c$. We show that we can make a finite series of transformations which give a set S' with an element a' satisfying $0 \leq a' < a$. A finite number of applications of such a series must yield a set with zero.

Let $[b/a] = b_n b_{n-1} \ldots b_0$ in base 2. Let $c_i = 1 - b_i$ and k = the binary number $c_n c_{n-1} \ldots c_0$. Then since $[b/a] \geq 2^n$ and $[b/a] + k = 2^{n+1} - 1$, $k < [b/a]$.

We now carry out n+1 transformations, for i = 0, 1, ... , n. Step i gives $2^{i+1}a$, $b - (b_i b_{i-1} \ldots b_0)\, a$, $c - (c_i c_{i-1} \ldots c_0)\, a$. So we always double the first element and we subtract it from the second or third according as $b_i = 1$ or 0. Note that this works because after step n, the second element is $b - [b/a]\, a \geq 0$, and the third element is $c - k\, a \geq b - k\, a > b - [b/a]\, a \geq 0$. After the final step the second element is $b - [b/a]\, a < a$, as required.

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Problem A1
a_n is a sequence of positive reals satisfying $a_n \le a_{2n} + a_{2n+1}$ for all n. Prove that $\sum a_n$ diverges.

Solution
$\sum a_n = a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + ...$ and each bracket has sum at least $a_1 > 0$.

Problem A2
Let R be the region in the first quadrant bounded by the x-axis, the line 2y = x, and the ellipse $x^2/9 + y^2 = 1$. Let R' be the region in the first quadrant bounded by the y-axis, the line y = mx, and the ellipse. Find m such that R and R' have the same area.

Solution
Stretch the figure by a factor 3 along the y-axis, so that the point (x, y) goes to (x, 3y). Then the ellipse becomes a circle and the line 2y = x, becomes the line 2y = 3x. Obviously, the required line in the stretched figure is 3y = 2x. Shrinking back to the original preserves the ratio of the areas and gives m = 2/9.

Problem A3
X is the set of points on one or more sides of a triangle with sides length 1, 1 and √2. Show that if X is 4-colored, then there must be two points of the same color a distance 2 - √2 or more apart.

Solution
Let the vertices be A, B, C with angle B = 90 degrees. Let the point on AB a distance k = 2 - √2 from A be D. Let the point on AC a distance k from A be E. Let the point on AC a distance k from C be F, and let the point on BC a distance k from C be G. It is easy to check that DF = EG = DG = k.

We show that these 7 points cannot be colored with 4 colors so that no two points the same color are a distance k or more apart. A and C are each a distance k or more from all the other points, so they use up two colors. D and G are a distance k apart so they must be different colors. E is a distance k from G and F is a distance k from D, so E and F must be different colors, but now there is no color left for B, which is a distance greater than k from E and F.

[This result is best possible. Color [A, D) and [A, E) color 1, color [C, F) and [C, G) color 2, color [E, F] color 3, and color [B, D] and [B, G] color 4. Then the only two points of the same color a distance k or more apart are D and G.]

Problem A4
A and B are 2 x 2 matrices with integral values. A, A + B, A + 2B, A + 3B, and A + 4B all have inverses with integral values. Show that A + 5B does also.

Solution
Let $A = (a_{ij})$, $B = (b_{ij})$. Given any matrix $C = (c_{ij})$, C is invertible iff the determinant $d = c_{11}c_{22} - c_{12}c_{21}$ is non-zero and the inverse is then (d_{ij}) where $d_{11} = c_{22}/d$, $d_{12} = -c_{12}/d$, $d_{21} = -c_{21}/d$, $d_{22} = c_{11}/d$. Thus if the inverse has integer entries, then d divides all the entries in the original matrix and hence d^2 divides d, so d = 1 or -1.

Multiplying out, we find that $\det(A + nB) = a + nh + n^2k$, where a = det A = 1 or -1, $h = (a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11})$, and k = det B. Since A + B has integral entries, a + h + k = 1 or -1 (1).

Similarly, A + 2B has integral entries, so a + 2h + 4k = 1 or -1 (2). We find that the only solutions to (1) and (2) are a, h, k = 1,0,0; 1,1,-1; 1,-3,1; 1,-4,2; -1,0,0; -1,-1,1; -1,3,-1; -1,4,-2 (*).

But A + 3B also has integral entries, so in the same way we conclude that a + 3h + 9k = 1 or -1 and hence h + 5k = 0, 2, or -2 (3). The only solutions (*) which also satisfy (3) are 1,0,0 and -1,0,0. Hence det(A + nB) = det A for all n, and so A + nB is invertible and has integer entries for all integers n.

Note that we do not need to use A + 4B. Note also that the fact of integer entries is essential. It is easy to find counter-examples with non-integer entries, eg take $a_{11} = 4$, $a_{12} = a_{21} = 0$, $a_{22} = 1$, and $b_{11} = -1$, $b_{12} = b_{21} = b_{22} = 0$.

On reflection, the reason for including A + 4B was probably to allow the following simpler solution: $a + hn + kn^2$ takes the values 1 or -1 for the 5 values n = 0, 1, 2, 3, 4. Hence it must take one value at least 3 times. But that is impossible for a quadratic unless it is constant. Hence it is constant. Then as before.

Problem A5
Given a sequence of positive real numbers which tends to zero, show that every non-empty interval (a, b) contains a non-empty subinterval (c, d) that does not contain any numbers equal to a sum of 1994 distinct elements of the sequence.

Solution
Let the sequence be r_n. Let S_k be the set of all sums of k distinct elements of the sequence. We prove the result by induction on k.

Consider first k = 1, so we are given an interval (a, b). If b <= 0, there is nothing to prove. If b > 0, take a subinterval (c, b) with c > 0. Then for all n > some N, we have $r_n < c$ and hence $r_n \notin (c, b)$. So there are only finitely many r_i in (c, b). Hence we can pick an interval that lies between two of them.

Now assume the result for k, so given (a, b), take (c', d) which does not intersect S_k. Let h > 0 be half the width of (c', d), so that c = c'+h is the midpoint of (c', d). Then we can find m such that $r_i < h$ for i > m. Now take a sum s in S_{k+1}. If it falls in (c, d), then the last term must be one of $r_1, \ldots, r_m$, because if it was a later term, then it would be smaller than h and hence the sum of the first k terms in s would fall in (c', d) [contradicting the choice of (c', d).]. Now by induction on m we can find a subinterval of (c, d) which also excludes sums with last term $r_1, \ldots$, or r_m. Consider the interval $(c-r_1, d-r_1)$. We can find a subinterval (e, f) of this which does not intersect S_k. Now (c_1, d_1), where $c_1 = e + r_1$, $d_1 = f + r_1$, is a subinterval of (c, d) and does not contain any sum s in S_{k+1} with last term r_1 (or any r_i with i > m). In the same way we find a subinterval (c_2, d_2) of (c_1, d_1) which does not contain any sum s with last term r_2 (or r_1 or r_i with i > m, since it is a subinterval). And so on.

Problem A6
Let Z be the integers. Let $f_1, f_2, \ldots, f_{10} : Z \to Z$ be bijections. Given any $n \in Z$ we can find some composition of the f_i (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions $S = \{ g_1 g_2 \ldots g_{10} \}$, where g_i = the identity or f_i. Show that at most half the functions in S map a finite (non-empty) subset of Z onto itself.

Solution
Let A be a finite subset of Z. Suppose all f_i map A to itself. Then, since all the f_i are bijections, all compositions of the f_i also map A to itself. If $0 \in A$, then since A is finite we can find $n \notin A$, and hence no composition of the f_i could take 0 to n, contradicting the hypothesis. If $0 \notin A$, then take

any $n \in A$. In this case no composition of the f_i could take 0 to n, contradicting the hypothesis. Thus some f_k does not map A to itself.

Let g be any composition $g_1...g_{k-1}$ and g' any composition $g_{k+1}...g_{10}$, where g_i = the identity or f_i. [If k = 1, then we take g to be the identity, and if k = 10, then we take g' to be the identity.]. Now if both gg' and gf_kg' map A to itself, then f_k must map A to itself (since all the functions are bijections), contradiction. Hence at least one of the maps gg' and gf_kg' does not map A to itself. But the 1024 maps in S form 512 disjoint pairs gg', gf_kg', so at least 512 maps in S (one from each pair) do not map A to itself.

Problem B1
For a positive integer n, let f(n) be the number of perfect squares d such that $|n - d| \le 250$. Find all n such that f(n) = 15. [The perfect squares are 0, 1, 4, 9, 16, ...]

Solution
Answer: 315 - 325 and 332 - 350.

We may readily compile the following table:

0 -250	≥ 16	$0^2 - 15^2$
251-299	≥ 16	$7^2 - 22^2$
300-314	16	$8^2 - 23^2$
315-325	15	$9^2 - 23^2$
326-331	16	$9^2 - 24^2$
332-350	15	$10^2 - 24^2$
351-371	14	$11^2 - 24^2$

$25^2 - 11^2 = 504 > 500$ and hence the gap $(n+15)^2 - (n+1)^2 > 500$ for all n > 10. So no numbers $n > 11^2 + 250 = 371$ have f(n) > 14.

Problem B2
For which real α does the curve $y = x^4 + 9x^3 + \alpha x^2 + 9x + 4$ contain four collinear points?

Solution
Answer: α < 30.375.

We look at the second derivative $y'' = 12x^2 + 54x + 2\alpha = 12(x + 2.25)^2 + 2(\alpha - 30.375)$. This has two zeros iff α < 30.375.

If α > 30.375, then y'' has no zeros and hence y' is strictly monotonically increasing. Hence no line cuts the curve in more than two points (if a line gradient k cut the curve at x = a, b and c, with a < b < c, then there would be a point between a and b with gradient k and another between b and c, contradicting the strict monotonicity of y'.].

If α < 30.375, the y'' has two zeros either side of x = 2.25. Hence the curve has two points of inflection either side of x = 2.25. The line through these two inflection points will cut the curve in four points. [Suppose the two points are at x = a and x = b with a < b. The line has a smaller gradient than the tangent at the point of inflection at x = a and hence cuts the curve again for x < a. Similarly it has larger gradient than the tangent at the point of inflection at x = b and hence cuts the curve again for x > b.]

It remains to consider the case $\alpha = 30.375$. In this case y'' has a single zero at $x = 2.25$. In this case there is no point of inflection. The gradient is monotonic and hence a line can only cut the curve in two points. [We cannot have degeneracy with infinitely many solutions because we are dealing with a polynomial which has at most 4 solutions.]

Problem B3
Let R be the reals and R^+ the positive reals. $f : R \to R^+$ is differentiable and $f'(x) > f(x)$ for all x. For what k must f(x) exceed e^{kx} for all sufficiently large k?

Solution
Answer: $k \leq 1$.

Let $g(x) = e^{-x}f(x)$, then $g'(x) = e^{-x}(f'(x) - f(x)) > 0$ for all x, so g is a strictly increasing function. Hence for any $h < 0$, $g(x) > e^{hx}$ for all sufficiently large x. Hence if $k < 1$, then $f(x) > e^{kx}$ for all sufficiently large x.

The tricky case is obviously $k = 1$. The argument above suggests that if g(x) keeps below 1, then we will have $f(x) < e^x$ for all x. For example, we can take $f(x) = e^{x - e^{-x}}$. Then $f'(x) = f(x)(1 + e^{-x}) > f(x)$, but $f(x) < e^{kx}$ for all x, provided $k \geq 1$.

Problem B4
A is the 2 x 2 matrix (a_{ij}) with $a_{11} = a_{22} = 3$, $a_{12} = 2$, $a_{21} = 4$ and I is the 2 x 2 unit matrix. Show that the greatest common divisor of the entries of $A^n - I$ tends to infinity.

Solution
It is a trivial induction that A^n has the form (c_{ij}), where $c_{11} = r$, $c_{12} = s$, $c_{21} = 2s$, $c_{22} = r$, with r an odd integer which tends to infinity with n (for example, $r > 2^n$), and s an even integer. Hence the gcd of the entries of $A^n - I$ is the gcd of $r - 1$ and s. Also $\det A = 1$, hence $\det(A^n) = 1$ and so $r^2 - 2s^2 = 1$, or $(r - 1)(r + 1) = 2s^2$. It follows that the gcd of $r - 1$ and s is either $\sqrt{(r - 1)}$ or $\sqrt{(2r - 2)}$. [If p is an odd prime and h is the highest power of p that divides $r - 1$, then p does not divide $r + 1$, and so h must be even and $h/2$ the highest power of p dividing n. Both $r - 1$ and $r + 1$ are even, but one is not divisible by 4. If h is the highest power of 2 dividing the other, then h must be even and $h/2$ is the highest power of 2 dividing n.] But $r - 1$ tends to infinity with n, so the gcd does also.

Problem B5
For any real α define $f_\alpha(x) = [\alpha x]$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$, $f_\alpha^k(n^2) = n^2 - k = f_\alpha^k(n^2)$, where f_α^k denotes the k-fold composition of f_α.

Solution
Answer: $\alpha = e^{-1/n2}$.

We require $\alpha(n^2 - k) < n^2 - k$, so $\alpha < 1$. We also require $\alpha(n^2 - k) \geq n^2 - (k + 1)$, so $\alpha \geq 1 - 1/(n^2 - k)$. This suggests trying a value near $1 - 1/n^2$. However, we also need $[\alpha^k n^2] = n^2 - k$, so we would like an α such that α^k is easy to work with. This suggests trying $e^{-1/n2}$. Certainly $1 > e^{-1/n2} > 1 - 1/n^2$, so it follows immediately that $[\alpha(n^2 - k)] = n^2 - (k+1)$ and hence that $f_\alpha^k(n^2) = n^2 - k$ for $1 \leq k \leq n$.

For $0 < x < 1$, the Taylor series for e^{-x} has decreasing terms of alternating sign, so $1 - x < e^{-x} < 1 - x + x^2/2$. Hence, in particular: $1 - k/n^2 < \alpha^k < 1 - k/n^2 + k^2/(2n^4)$. Hence α^k lies between $n^2 - k$ and $n^2 - k + k^2/(2n^2)$, so $[\alpha^k n^2] = n^2 - k$, as required.

Problem B6
a, b, c, d are integers in the range 0 - 99. Show that if $101a - 100\cdot 2^a + 101b - 100\cdot 2^b = 101c - 100\cdot 2^c + 101d - 100\cdot 2^d \pmod{10100}$ then $\{a, b\} = \{c, d\}$.

Solution
We have immediately that $a + b = c + d \pmod{100}$ and $2^a + 2^b = 2^c + 2^d \pmod{101}$. But by Fermat's little theorem $2^{100} = 1 \pmod{101}$, so $2^{a+b} = 2^{c+d} \pmod{101}$.

Now $(2^c - 2^a)(2^c - 2^b) = 2^{2c} - (2^a + 2^b)2^c + 2^{a+b} = 2^{2c} - (2^c + 2^d)2^c + 2^{c+d} = 0 \pmod{101}$. But 101 is prime, so either $2^c = 2^a \pmod{101}$, or $2^c = 2^b \pmod{101}$. If $2^c = 2^a$, then $2^d = 2^b$, and so $\{2^a, 2^b\} = \{2^c, 2^d\} \pmod{101}$. Similarly if $2^c = 2^b \pmod{101}$.

It remains to show that 2 is a primitive root of 101. $2^{10} = 1024 = 14 \pmod{101}$, so $2^{20} = 196 = -6 \pmod{101}$, and $2^{50} = 36\cdot 14 = 504 = -1 \pmod{101}$. Hence 2 is a primitive root and so $2^m = 2^n \pmod{101}$ implies $m = n \pmod{100}$. We are told that a, b, c, d all lie in the range 0 to 99, so $\{2^a, 2^b\} = \{2^c, 2^d\} \pmod{101}$ implies $\{a, b\} = \{c, d\}$.

56th Putnam 1995

Problem A1
S is a set of real numbers which is closed under multiplication. S = A ∪ B, and A ∩ B = ∅. If a, b, c ∈ A, then abc ∈ A. Similarly, if a, b, c ∈ B, then abc ∈ B. Show that at least one of the A, B is closed under multiplication.

Solution
Suppose the result is false, then we can find a, b, c, d such that a, b ∈ A, but ab ∈ B, and c, d ∈ B, but cd ∈ A. But now a.b.cd is a product of three elements of A and hence in A, whereas ab.c.d is a product of three elements of B and hence in B. Contradiction.

Problem A2
For what positive reals α, β does $\int_\beta^\infty \sqrt{(\sqrt{(x+\alpha)} - \sqrt{x})} - \sqrt{(\sqrt{x} - \sqrt{(x-\beta)})}\, dx$ converge?

Solution
Answer: it converges for α = β.

Multiplying up shows that given positive k < 1, $\alpha k/(2\sqrt{x}) < (\sqrt{(x+\alpha)} - \sqrt{x}) < \alpha/(2\sqrt{x})$ for all sufficiently large x.

So if α > β, take k such that 0 < k < 1 and αk > β. Then the integrand is greater than $(\sqrt{(\alpha k)} - \sqrt{\beta})/(x^{1/4}\sqrt{2})$. But the integral of $x^{-1/4}$ diverges (because $x^{3/4}$ tends to infinity as x tends to infinity), do the original integral does also.

Similarly, if α < β, take k such that 0 < k < 1 and βk > α. Then the integrand is less than $(\sqrt{\alpha} - \sqrt{(\beta k)})/(x^{1/4}\sqrt{2})$ and so the integral again diverges.

If α = β, then $(\sqrt{(x+\alpha)} - \sqrt{x})$ takes the same values on the interval [nα, nα + α] as $(\sqrt{x} - \sqrt{(x-\beta)})$ does on the interval [nα+α, nα+2α]. Thus the integral from α to nα equals the K + L where - K is the integral from 0 to α of $\sqrt{(\sqrt{(x+\alpha)} - \sqrt{x})}$ and L is the integral from nα-α to nα of $\sqrt{(\sqrt{(x+\alpha)} - \sqrt{x})}$. But the integrand $\sqrt{(\sqrt{(x+\alpha)} - \sqrt{x})}$ tends to zero as x tends to infinity, so L tends to zero as n tends to infinity and the integral converges to K.

Problem A3
d, e and f each have nine digits when written in base 10. Each of the nine numbers formed from d by replacing one of its digits by the corresponding digit of e is divisible by 7. Similarly, each of the nine numbers formed from e by replacing one of its digits by the corresponding digit of f is divisible by 7. Show that each of the nine differences between corresponding digits of d and f is divisible by 7.

Solution
Let $d = d_1d_2... d_9$ and similarly for e and f. We have are given that $(d + e_9 - d_9)$, $(d + 10e_8 - 10d_8)$, ... , $(d + 10^8e_1 - 10^8d_1)$ are all multiples of 7. Adding gives that (9d + e - d) is a multiple of 7 and hence also (d + e).

$(d + e_9 - d_9)$ and $(e + f_9 - e_9)$ are multiples of 7. So, adding, $(d + e + f_9 - d_9)$ is a multiple of 7 and hence also $(f_9 - d_9)$. Similarly, $(d + 10e_8 - 10d_8) + (d + 10f_8 - 10e_8) - (d + e) = 10(f_8 - d_8)$ is a multiple of 7, and hence also $(f_8 - d_8)$. Similarly, for the other terms.

Problem A4
n integers totalling n - 1 are arranged in a circle. Prove that we choose one of the integers x_1, so that

the other integers going around the circle are, in order, x_2, ... , x_n and we have $\sum_1^k x_i \le k - 1$ for k = 1, 2, ... , n.

Solution
Let the integers be y_i (starting from an arbitary position). We use the convention that subscripts are reduced mod n when necessary. Let $z_j = \sum_1^j (y_i - (n-1)/n)$. The amount (n-1)/n is chosen so that $z_n = 0$ and hence the sum of any group of s consecutive y_i: $y_{r+1} + ... + y_{r+s}$ is simply $z_{r+s} - z_r + s(n-1)/n$.

Let z_h be the largest z_j. Take $x_1 = z_{h+1}$. Then the sum of the first k terms is $z_{h+k} - z_h + k(n-1)/n <= k(n-1)/n$, since $z_h >= z_{h+k}$. Hence the sum of the first k terms is < k, but it is integral, so it is ≤ k-1.

Problem A5
R is the reals. x_i **:** R → R are differentiable for i = 1, 2, ... , n and satisfy $x_i' = a_{i1}x_1 + ... + a_{in}x_n$ for some constants $a_{ij} \ge 0$. Also $\lim_{t\to\infty} x_i(t) = 0$. Can the functions x_i be linearly independent?

Solution
Answer: no.

Let A be the n x n matrix (a_{ij}). Let A^T be its transpose. We know that the trace of A^T is non-negative (since all its elements are non-negative). So it must have an eigenvalue λ with $Re(\lambda) \ge 0$. Let **v** = (v_i) be a corresponding eigenvector and consider the linear combination $y = \sum v_i x_i$. We have $y' = \sum v_i a_{ij} x_j = \lambda \sum v_j x_j = \lambda y$. So, integrating, $y = k e^{\lambda t}$ for some constant k. But $|e^{\lambda t}|$ does not tend to zero as t → ∞, since $Re(\lambda) \ge 0$. We are given that each $x_i(t)$ and hence y(t) → 0, so k = 0. In other words we have found a (non-trivial) linear combination of the x_i which is zero, so they are not linearly independent.

Problem A6
Each of the n triples (r_i, s_i, t_i) is a randomly chosen permutation of (1, 2, 3) and each triple is chosen independently. Let p be the probability that each of the three sums $\sum r_i$, $\sum s_i$, $\sum t_i$ equals 2n, and let q be the probability that they are 2n - 1, 2n, 2n + 1 in some order. Show that for some n ≥ 1995, 4p ≤ q.

Solution
Let a_n be the probability that the sums are 2n, 2n, 2n (type 1); b_n that they are 2n - 1, 2n, 2n + 1 in some order (type 2); c_n that they are 2n - 2, 2n + 1, 2n + 1 or 2n - 1, 2n - 1, 2n + 2 in some order (type 3). We show that for any n, either $a_n < b_n/4$ or $a_{n+1} < b_{n+1}/4$. It is easy to verify that:

$a_{n+1} = b_n/6$;
$b_{n+1} >= a_n + b_n/3 + c_n/3$;
$c_{n+1} >= b_n/3$

Suppose that $a_n > b_n/4$. Then $b_{n+1} \ge b_n/4 + b_n/3 + b_{n-1}/9 = 7b_n/12 + 2a_n/3 > 7b_n/12 + b_n/6 = 9b_n/12 = 9a_{n+1}/2 > 4\ a_{n+1}$.

Thus we have the required $a_n < b_n/4$ for either n = 1995 or n = 1996.

[This looks deceptively easy. But how did we know that the inequality is alternately true and false? How do we know how many types of patterns to look at? The second question is easier. We might start with just type 1 and type 2. Then we get $a_{n+1} = b_n/6$, and $b_{n+1} \ge a_n + b_n/3$. But that is not enough, because $a_n > b_n/4$ only implies $a_{n+1} < b_{n+1}/(3\ 1/2)$. So we need at least one more type.]

Problem B1
Let X be a set with 9 elements. Given a partition π of X, let $\pi(h)$ be the number of elements in the part containing h. Given any two partitions π_1 and π_2 of X, show that we can find $h \neq k$ such that $\pi_1(h) = \pi_1(k)$ and $\pi_2(h) = \pi_2(k)$.

Solution
At least 4 elements x must have the same value of $\pi_1(x)$. For if not, then certainly every part (in the partition by π_1) has size 3 or less. Moreover, there can only be one part size 3 and one part size 2 (since $2 \cdot 3$ and $2 \cdot 2 \geq 4$), and three parts size 1, but that only covers $3 + 2 + 3 = 8$ elements. Contradiction.

These 4 elements cannot all have different values of π_2, because $1 + 2 + 3 + 4 > 9$, so π_2 cannot have 4 or more different sized parts. So at least two of the 4 elements have the same value of π_2.

Problem B2
An ellipse with semi-axes a and b rolls without slipping on the curve $y = c \sin (x/a)$ and completes one revolution in one period of the sine curve. What conditions do a, b, c satisfy?

Solution
Answer: (1) $b > a$; (2) $c = \pm\sqrt{(b^2 - a^2)}$; (3) $c\, b^2 < a^3$.

The first point to consider is which semi-axis is longer. A complete period of the curve is from $x = 0$ to $x = 2\pi a$, so the curve length is $\geq 2\pi a$. If the ellipse has $a > b$, then its length $< 2\pi a$. Thus we can assume that $b \geq a$.

The obvious necessary condition is that the arc-lengths should be the same. This is a routine calculation. We find that the length for the curve is $\int_0^{2\pi} (a^2 + c^2\cos^2 t)^{1/2}\, dt$ and the length for the ellipse is $\int_0^{2\pi} (a^2 + (b^2 - a^2)\cos^t)^{1/2}\, dt$. Since the curve length is strictly monotonic increasing with $|c|$, we must have $c = \pm \sqrt{(b^2 - a^2)}$.

However, there is another necessary condition. The ellipse must be able to roll, in other words its maximum radius of curvature must be less than the minimum radius of curvature for the curve.

This is another routine calculation, using the expression $|(1 + y'^2)^{3/2}/y''|$ for the radius of curvature. For the curve we find that the minimum value is a^2/c, and for the ellipse we find that the maximum value is b^2/a. Hence we require $c\, b^2 < a^3$, or $(b^2 - a^2)b^4 < a^6$, giving $(b/a)^6 - (b/a)^4 < 1$, or $b/a < 1.211$.

Problem B3
For each positive integer k with n^2 decimal digits (and leading digit non-zero), let d(k) be the determinant of the matrix formed by writing the digits in order across the rows (so if k has decimal form $a_1 a_2 \ldots a_n$, then the matrix has elements $b_{ij} = a_{n(i-1)+j}$). Find $f(n) = \sum d(k)$, where the sum is taken over all $9 \cdot 10^{n2-1}$ such integers.

Solution
Answer: $f(1) = 45$; $f(2) = 10.45^2$; for $n \geq 3$, $f(n) = 0$.

$f(1) = 1 + 2 + \ldots + 9 = 45$.
$f(2) = \sum (a_{11}a_{22} - a_{12}a_{21})$. Summing $a_{11}a_{22}$ over a term that is present, such as a_{22} gives a factor 45. Summing over a term that is not present gives a factor 10, unless the sum is over a_{11} in which case it gives a factor 9. Hence the sum for $a_{11}a_{22}$ is $45^2 10^2$. Similarly, the sum for $-a_{12}a_{21}$ is $-45^2 90$. Hence the total is 10.45^2.

The same approach gives zero for $n \geq 3$, because there are an even number of terms involving a_{11}, half with a plus sign and half with a minus sign. Thus the sum over these terms is zero. Similarly, there are an even number of terms not involving a_{11}, half with a plus sign and half with a minus sign. Again, the sum over these terms is zero.

Problem B4
Express $(2207-1/(2207-1/(2207-1/(2207- \ldots))))^{1/8}$ in the form $(a + b\sqrt{c})/d$, where a, b, c, d are integers.

Solution
Answer: $(3 + \sqrt{5})/2$.

Let $x = (2207-1/(2207-1/(2207-1/(2207- \ldots))))$. Then $x = 1/(2207 - x)$, so $x^2 - 2207x + 1 = 0$. Evidently x is only just less than 2207, so the root is $2207/2 + \sqrt{(2207^2 - 4)}/2 = 2207/2 + 987/2\ \sqrt{5}$. It is easy to verify that if $y = (3 + \sqrt{5})/2$, then $y^8 = x$.

[How do we come up with y? Well, we are effectively told to look for something of the form $(a + b\sqrt{5})/c$. $2^8 = 256$, $3^8 = 6561$, so we are looking for y between 2 and 3. We end up with a factor 2 in the denominator, so the simplest choice for c is 2. a, b = 2, 1 looks as though y will be too close to 2, so we try a, b = 3, 1.]

[Or you can take the square root three times. $(a + b\sqrt{5})^2 = (a^2 + 5b^2) + 2ab\sqrt{5}$. The simplest thing is to look for a and b both half odd integers. So we look for factors of 987. We have $987 = 3.7.47$. So we can try each of the pairs (3, 329), (7, 141), (21, 47). The first two obviously do not work (the squares are too large), but the third does. Similarly, for the next two square roots.]

[Or, better, use $y = x^{1/8}$. Note that $2207 + 2 = 47^2$. Start from $2207 = y^8 + 1/y^8$. Iterate twice more to get $3 = y + 1/y$. An elegant approach, for which I thank *Henry Pan* from Toronto, email 29 May 2001.]

[Strictly we should also establish that the continued fraction converges and that we take the larger root for x. So, define $a_1 = 2207$, $a_{n+1} = 2207 - 1/a_n$, $b_1 = 2206$, $b_{n+1} = 2207 - 1/b_n$. We show that $a_n > a_{n+1} > b_{n+1} > b_n$ and that $a_n - b_n$ tends to zero. It follows that the continued fraction converges to a value larger than 2206, which suffices.

If $y = 2207 - 1/z$, then $y^2 - 2207y + 1 = (z^2 - 2207z + 1)/z^2$. But $a_1^2 - 2207a_1 + 1 > 0$, so $a_n^2 - 2207a_n + 1 > 0$ for all n. Hence $a_n > 2207 - 1/a_n = a_{n+1}$. Similarly, $b_1^2 - 2207b_1 + 1 < 0$, so $b_n^2 - 2207b_n + 1 < 0$ for all n, and hence $b_n < b_{n+1}$.

$a_{n+1} - b_{n+1} = 1/b_n - 1/a_n = (a_n - b_n)/(a_nb_n$. But $a_1 - b_1 > 0$ and $b_1 > 0$, so $a_n - b_n > 0$ for all n. Also $a_n > b_n \geq b_1 = 2206$, so $(a_{n+1} - b_{n+1}) < (a_n - b_n)/10^6$, and hence $(a_n - b_n)$ tends to zero.]

Problem B5
A game starts with four heaps, containing 3, 4, 5 and 6 items respectively. The two players move alternately. A player may take a complete heap of two or three items or take one item from a heap provided that leaves more than one item in that heap. The player who takes the last item wins. Give a winning strategy for the first or second player.

Solution
Answer: first player wins.

The first player leaves 2, 4, 5, 6. The second player now has a choice of 4 moves: (A) leave 4, 5, 6, (B) 2, 3, 5, 6, (C) 2, 4, 4, 6, (D) 2, 4, 5, 5.

In case (A), the first player leaves 4, 5, 5. The second player now has two choices (A1) 3, 5, 5, and (A2) 4, 4, 5. In case (A1), the first player leaves 5, 5. The first player now wins, because whatever his opponent does to one pile, he does to the other. In case (A2), the first player leaves 4, 4, 4, so the second player is forced to leave 3, 4, 4. The first player now leaves 4, 4 and wins because he copies his opponents moves on the other pile.

In case (B), the first player leaves 2, 5, 6. If the second player leaves 5, 6 or 2, 5, 5, then the first player leaves 5, 5 and wins as before. If the second player leaves 2, 4, 6, then the first player leaves 4, 6. If the second player leaves 4, 5, the first player leaves 4, 4 and wins as before; if the second player leaves 3, 6, then the first player leaves 6 and wins (after some forced moves the second player leaves 3 and the first player takes it to win).

In cases (C) and (D), the first player leaves 2, 4, 4, 5. The second player now has 3 moves (C1) leave 4, 4, 5, (C2) 2, 3, 4, 5, (C3) 2, 4, 4, 4. Case (C1) is just a repeat of (A2). In case (C2) the first player leaves 2, 4, 5. The second player has three choices: if he leaves 4, 5 or 2, 4, 4, then the first player leaves 4, 4 and wins as before; if he leaves 2, 3, 5, then the first player leaves 2, 5. This is a win for the first player, because after three moves (one by the first player, two by the second) the first player is left with 3, which he takes to win. Finally, in case (C3) the first player leaves 4, 4, 4 and wins as in case (A2) above.

The general solution is as follows.

Classify piles into three types: *3-piles* (piles with just 3 beans); *odd-piles* (piles with 2 beans or with an odd number of beans ≥ 5); *even-piles* (piles with an even number of beans ≥ 4).

The winning strategy is to leave an even number of 3-piles and an even number of odd-piles. We show that this strategy can always be followed unless you are facing an even number of 3-piles and an even number of odd-piles.

Let the number of piles of each type be T, O2 and E respectively. There are 4 cases to consider.

(1) T odd, O2 odd. Then $T \geq 1$, so remove a single bean from it, which reduces T by 1 and increases O2 by 1.

(2) T odd, O2 even. Then $T \geq 1$, so remove a 3-pile, thus leaving T and O2 even.

(3) T even, O2 odd. Then $O2 \geq 1$. Remove a bean from an odd-pile (or remove the pile entirely if it has only 2 beans). This reduces O2 by 1 and hence leaves T and O2 even.

(4) T even, O2 even. If there are no piles, then you have already lost! Removing a bean from an even-pile increases O2 or T by 1. Removing a bean from an O2 pile (or removing a pile of 2 beans entirely) reduces O2 by 1. Removing a bean from a T pile (or removing the whole pile), reduces T by 1. In any case at least one of T and O2 becomes odd.

Problem B6

Let N be the positive integers. For any $\alpha > 0$, define $S_\alpha = \{ [n\alpha]: n \in N\}$. Prove that we cannot find α, β, γ such that $N = S_\alpha \cup S_\beta \cup S_\gamma$ and $S_\alpha, S_\beta, S_\gamma$ are (pairwise) disjoint.

Solution
We prove the stronger result that the three sets cannot be disjoint (whether or not their union is N).

Take any integer $k > \alpha, \beta, \gamma$. Let $\{\ \}$ denote the fractional part. Then at least two of the $k^3 + 1$ points $(\{i/\alpha\}, \{i/\beta\}, \{i/\gamma\})$, where $i = 0, 1, 2, \ldots, k^3$, must lie in a cube side $1/k$. Suppose they are $i > j$. Let $h = i - j$, then for some positive integers a, b, c we have $|h/\alpha - a|, |h/\beta - b|, |h/\gamma - c| \leq 1/k$. Hence $|h - a\alpha|, |h - b\beta|, |h - c\gamma| < 1$ (since $k > \alpha, \beta, \gamma$). So each of $[a\alpha], [b\beta], [c\gamma]$ is h or h - 1. Hence two are equal.

57th Putnam 1996

Problem A1
What is the smallest α such that two squares with total area 1 can always be placed inside a rectangle area α with sides parallel to those of the rectangle and with no overlap (of their interiors)?

Solution
Answer: 1/2 (1 + √2).

Let the squares, have sides $x \geq 1/\sqrt{2} \geq \sqrt{(1 - x^2)}$. Then the smallest rectangle containing them has sides x, $x + \sqrt{(1 - x^2)}$, and hence area $f(x) = x^2 + x\sqrt{(1 - x^2)}$. Evidently α is the maximum value of f(x) in the range $1/\sqrt{2} \leq x \leq 1$.

Evidently x^2 is an increasing function, and $x\sqrt{(1 - x^2)}$ a decreasing function, so f(x) has a single maximum where f '(x) = 0. $f'(x) = 2x + \sqrt{(1 - x^2)} - x^2/\sqrt{(1 - x^2)}$, so f '(x) = 0 for $8x^4 - 8x^2 + 1 = 0$, or $x^2 = 1/2 + \sqrt{2}/4$ (we want the larger root so that $x \geq 1/\sqrt{2}$). Substituting in f(x) gives f(x) = 1/2 (1 + √2).

Problem A2
Two circles have radii 1 and 3 and centers a distance 10 apart. Find the locus of all points which are the midpoint of a segment with one end on each circle.

Solution
Answer: Let O_1, O_2 be the centers of C_1, C_2 and O the midpoint of O_1O_2. The locus is the annulus radii 1 and 2, center O.

Fix Y on the C_2. Then the locus of the midpoint is a circle radius 1/2, center N, the midpoint of XO_1. Now vary X. The locus of N is a circle radius 1 1/2 center O. Hence the locus of M is the area swept out by the circle radius 1/2 as its center moves around the circle radius 1 1/2. This is an annulus radii 1 and 2.

Problem A3
There are six courses on offer. Each of 20 students chooses some all or none of the courses. Is it true that we can find two courses C and C' and five students S_1, S_2, S_3, S_4, S_5 such that each S_i has chosen C and C', or such that each S_i has chosen neither C nor C'?

Solution
Answer: false.

A little experimentation suggests that if the result is false the choice of courses taken and not taken must be fairly balanced. This suggests taking the set S of all distinct subsets of 3 courses. |S| = 20, so we allocate one member of S to each student. Now there are 4 members of S containing any two given courses and 4 members of S not containing either of two given courses, so the assertion in the question is false.

Problem A4
A is a finite set. S is a set of ordered triples (a, b, c) of distinct elements of A, such that:
(a, b, c) ∈ S iff (b, c, a) ∈ S;
(a, b, c) ∈ S iff (c, b, a) ∉ S;
(a, b, c) and (c, d, a) ∈ S iff (b, c, d) and (d, a, b) ∈ S.
Prove that there exists an injection g from A to the reals, such that g(a) < g(b) < g(c) implies (a, b, c) ∈ S.

Solution
Fix any element a in A. Now define $\prec$ on the other elements of A by $b \prec c$ iff (a, b, c) is in S. We check that $\prec$ is a total order on the other elements. If $b \prec c$ and $c \prec d$, then (a, b, c) and (a, c, d) are in S, so (a, b, d) is in S and hence $b \prec d$, so the relation is transitive. Also for distinct b, c just one of (a, b, c) and (a, c, b) is in S, so just one of $b \prec c$, $c \prec b$ is true.

Now extend this ordering to A, by taking $a \prec b$ for all $b \neq a$. Finally, arrange the elements of A in order $a_1 \prec a_2 \prec a_3 \prec ... \prec a_n$ and define $g(a_i) = i$. Then g has the required property.

Problem A5
Let p be a prime ≥ 5. Prove that p^2 divides $\sum_0^{[2p/3]}$ pCr.

Solution
Obviously each pCi is divisible by p. So we have to show that S = pC1/p + pC2/p + ... + pCk/p is divisible by p. We work in the field mod p. pCi/p = (p-1)...(p-i)/(1.2. ... i) = $(-1)^{i-1}$1/i. Hence S = 1 - 1/2 + ... +/- 1/k = 1 + 1/2 + ... + 1/k - 2(1/2 + 1/4 + ... + 1/2h) = 1 + 1/2 + ... + 1/k - (1 + 1/2 + ... + 1/h) = 1 + 1/2 + ... + 1/k + (1/(p-1) + 1/(p-2) + ... + 1/(p-h)). By considering p = 1, 2 (mod 3) separately, we can easily check that p-h = k+1 and hence S = 1 + 1/2 + ... + 1/(p-1), which is a complete set of reduced residues and hence sums to zero.

Problem A6
Let R be the reals and k a non-negative real. Find all continuous functions f **:** R → R such that f(x) = $f(x^2 + k)$ for all x.

Solution
We note that first that f(-x) = f(x), so it is sufficient to define f for non-negative x.

We note that $(x^2 + k) \geq k$ for all x, with equality iff x = 0, so it is tempting to say that on [0, k] f is arbitary except that it must be continuous and f(0) = f(k). We then extend f to the rest of the real line using the relation given.

The trap is that this does not work for $k \leq 1/4$. The extension process just generates smaller and smaller intervals which converge on a root of the equation $x^2 + k = x$. If we try starting with a larger interval above the roots, then we can extend the function towards infinity without difficulty, but extending back to smaller values gives the same problem as before.

In fact, it is not too difficult to see that for $k \leq 1/4$ the function must be constant.

Actually, spelling this out in detail is fairly painful! It is most easily seen from the graph. Let $a_1 < a_2$ be the two roots of $x^2 - x + k = 0$. If we define $x_1 = a$, $x_{n+1} = x_n^2 + k$ for $0 < a < a_2$ (and also $a \neq a_1$) then x_n tends to a_1. By the continuity of f, $f(x_n)$ must tend to $f(a_1)$, but f is constant on the sequence, so $f(a) = f(a_1)$. Similarly, for any $a > a_2$, we define $x_1 = a$, $x_{n+1} = \sqrt{(x_n - k)}$. This time x_n tends to a_2 and by the same argument as before we must have $f(a) = f(a_2)$. But we have $f(x) = f(a_1)$ for values of x arbitarily close to a_2 (and below it) and $f(x) = f(a_2)$ for values of x arbitarily close to a_2 and above it, so $f(a_1) = f(a_2)$ and hence f is constant.

If we want a more rigorous treatment of the fact that x_n tends to the root, there is further pain! Write $x_n = a_1 + k_n$, $x_{n+1} = a_1 + k_{n+1}$, then $k_{n+1} = 2a_1k_n + k_n^2$. But $0 < 2a_1 < 1$, so for $k_n < 0$, we have $0 > k_{n+1} > 2a_1k_n$, and hence $k_n$tends to zero. For $k_n > 0$, we observe first that $x_{n+1} < x_n$, so we have $k_n < k_1 = a_2 - a_1 - \varepsilon$ for some $\varepsilon > 0$, so $k_{n+1} = (2a_1 + k_n)k_n < (2a_1 + k_1)k_n = (2a_1 + a_2 - a_1 - \varepsilon)k_n = (1 - \varepsilon)k_n$. Hence k_n tends to zero. A similar argument deals with the case $a > a_2$. We have skated over the case k = 1/4, but I leave that as an exercise. [f is still constant. In graphical terms the curves $y = x^2 + 1/4$

and $y = \sqrt{(x - 1/4)}$ touches the line $y = x$ at $x = 1/2$ and it is clear from the graph that the same sequences work as before.]

Problem B1
Let N be the set $\{1, 2, 3, \ldots , n\}$. X is *selfish* if $|X| \in X$. How many subsets of N are selfish and have no proper selfish subsets.

Solution
Answer: Fibonacci sequence.

Observe first that X is a minimal selfish set iff $|X|$ is the smallest element of X. Let be the collection of selfish subsets of $\{1, 2, \ldots , n\}$ and let $u_n = |S_n|$. Clearly $S_n \subseteq S_{n+1}$. Indeed if $n+1 \notin A \in S_{n+1}$, then $A \in S_{n+1}$. Suppose now that $n+1 \in A \in S_{n+1}$. Define f(A) as follows: remove n+1 and reduce the other elements by 1. No element of f(A) can exceed n-1, so f is a map to S_{n-1}, and it is clearly (1, 1). We show that is also onto. Suppose $B \in S_{n-1}$. Form A by increasing each element by 1 and adjoining n+1. A is a minimal selfish set because $|B|$ is the smallest element of B, so $|A| = |B|+1$ is the smallest element of A. Hence $A \in S_{n+1}$ and $f(A) = B$, so f is onto. So $S_{n+1} = S_n \cup S_{n-1}$, and hence $u_{n+1} = u_n + u_{n-1}$. Clearly $u_2 = 1$, $u_3 = 2$, so u_n is the Fibonacci sequence.

Problem B2
Let $f(n) = ((2n+1)/e)^{(2n+1)/2}$. Show that for $n > 0$: $f(n-1) < 1.3.5 \ldots (2n-1) < f(n)$.

Solution
We estimate the integral of ln x, which is convex and hence easy to estimate. Take the integral from 1 to 2n-1. This is less than $2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. But the integral of ln x is x ln x - x, so the integral evaluates to $(2n-1) \ln(2n-1) - 2n + 2$. Hence $(2n-1) \ln(2n-1) - (2n-1) < (2n-1) \ln(2n-1) - 2n + 2 < 2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. Exponentiating gives the left hand inequality.

Similarly, the integral from 1 to 2n+1 is greater than $2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. However, this is not quite good enough, because the integral evaluates to $(2n+1) \ln(2n+1) - 2n$, and this time the odd 1 works against us. However, we can slightly improve our estimate. Because $\ln 1 = 0$, our estimate treated the integral from 1 to 3 as nil, so we can raise the lower end of the integration range: the integral from 3 to 2n+1 is greater than $2(\ln 3 + \ln 5 + \ldots + \ln(2n-1))$. Actually, it is more convenient to take the lower bound as $e < 3$, because then x ln x - x evaluates to 0 at the lower bound, thus getting rid of the odd 1. Exponentiating now gives the right hand inequality.

Problem B3
$(x_1, x_2, \ldots , x_n)$ is a permutation of $(1, 2, \ldots , n)$. What is the maximum of $x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1$?

Solution
Answer: $(2n^3 + 3n^2 - 11n + 18)/6$.

For n = 2, the maximum is simply $1.2 + 2.1 = 4$. If we take an arrangement for $\{1, 2, \ldots , n-1\}$ and insert n between i and j then the net increase in the sum is $n(i + j) - ij$. This is an increasing function of i [$= (n - j)i + nj$] and of j, so its maximum value is $n^2 - 2$, achieved if n is inserted between $(n - 1)$ and $(n - 2)$. Of course, this might not be possible for a particular arrangement of $\{1, 2, \ldots , n-1\}$, but it shows that the maximum value for n elements is at most $4 + (3^2 - 2) + (4^2 - 2) + \ldots + (n^2 - 2) = n(n + 1)(n + 2)/6 - 2(n - 1) + 1 = (2n^3 + 3n^2 - 11n + 18)/6$. This can actually be achieved by repeatedly inserting n between the two maximum elements. That puts the two new maximum elements together ready for the next insertion.

[Definitely no extra credit for checking this explicitly! You find the arrangement can be written as {2, 4, 6, ... , 2[n/2], 2[(n-1]/2]+1, 2[(n-1]/2]-1, ... , 5, 3, 1 - the even elements in increasing sequence followed by the odd elements in decreasing sequence. For n = 2m, this gives $4(1.2 + 2.3 + ... + (m-1)m) + 2m(2m-1) + (1.3 + 3.5 + ... + (2m-3)(2m-1)) + 2.1 = 4/3\ (m-1)m(m+1) + 2m(2m-1) + 1/3\ (m-1)(4n^2 - 2m - 3) + 2 = 1/3\ (8m^3 + 6m^2 - 11m + 9)$, as required, and similarly for n odd. But why go through all that rigmarole?]

Problem B4
Let B be the 2 x 2 matrix (b_{ij}) with $b_{11} = b_{22} = 1$, $b_{12} = 1996$, $b_{21} = 0$. Can we find a 2 x 2 matrix A such that sin A = B? [We define sin A by the usual power series: $A - A^3/3! + A^5/5! - ... .$]

Solution
Answer: no.

If sin A has the value given, then $1 - \sin^2 A = (c_{ij})$ with $c_{11} = c_{21} = c_{22} = 0$, $c_{21} \neq 0$. But we should have $\cos^2 A = (c_{ij})$.

This is not possible. Suppose $\cos A = (d_{ij})$, then we need $d_{11}d_{12} + d_{12}d_{22} \neq 0$, and hence $(d_{11} + d_{22}) \neq 0$. But we also need $0 = c_{21} = d_{21}(d_{11} + d_{22})$ and so $d_{21} = 0$. Now $0 = c_{11} = d_{11}^2 + d_{12}d_{21} = d_{11}^2$, so $d_{11} = 0$. Similarly, $0 = c_{22} = d_{22}^2 + d_{12}d_{21} = d_{22}^2$, so $d_{22} = 0$. So $d_{11} + d_{22} = 0$. Contradiction.
Hence sin A cannot have the value given.

The only problem with this argument is that it is too easy! One worries about justifying the various steps. The relation $\sin^2 A + \cos^2 A = 1$ is an easy calculation directly from the power series definitions, once one has established absolute convergence.

Problem B5
We call a finite string of the symbols X and O *balanced* iff every substring of consecutive symbols has a difference of at most 2 between the number of Xs and the number of Os. For example, XOOXOOX is not balanced, because the substring OOXOO has a difference of 3. Find the number of balanced strings of length n.

Solution
Answer: $3.2^{n/2} - 2$ for n even, $2^{[n/2]+2} - 2$ for n odd.

The insight is to change focus from the identity of the symbol to the number of consecutive symbols of a given type. We cannot have XXX or OOO, so we have blocks of length 1 or 2, alternately X and O. If we code the string as 11122112... , then each such code corresponds to two possible strings depending on whether we start with X or O.

The balance requirement translates to requiring an even number of 1s between each pair of adjacent 2s. This is clearly a necessary condition because if there are an odd number 2m+1 of 1s then the flanking 2s represent the same symbol and so the substring corresponding to 21...12 has $|\Delta| = 4 + m - (m+1) = 3$, which is inadmissible. On the other hand, it is not hard to see that it is also a sufficient condition. We want $|\Delta|$ for a substring. In calculating this we can ignore any complete runs of 1s and their flanking 2s, because of the evenness property, so the only possibilities are:

```
some 1s                    |Δ| = 0 or 1;
some 1s, 2                 |Δ| = 1 or 2;
some 1s, 2, some 1s        |Δ| = 1 or 2;
2                          |Δ| = 2;
2, some 1s                 |Δ| = 1 or 2;
```

Now take n even = 2m. The balance requirement implies that each sequence of 1s and 2s can be obtained by starting with 22...2 or 122...21 and replacing zero or more 2s by a pair of 1s. Moreover this process generates each acceptable sequence just once, except for 11...1 which is generated twice. Thus there are $2^m + 2^{m-1} - 1$ sequences and hence $2(2^m + 2^{m-1} - 1) = 3 \cdot 2^m - 2$ balanced strings. For n odd = 2m+1, we start with 122...2 or 22...21 and get $2(2^m + 2^m - 1) = 2^{m+2} - 1$ balanced strings.

For a less elegant solution, we try to get a recurrence relation. It quickly becomes clear that we need to distinguish strings which start with a repeated symbol from those which do not. The main task turns out to be showing that the number of balanced strings length n which begin with XX or OO is $2^{[n/2]}$. Let b be a balanced string length 2n+1 beginning with XX. Since 2n+1 is odd $\Delta(b) = 1$ or -1. But if $\Delta = -1$, then the substring formed by deleting the initial XX would have $\Delta = -3$. Hence $\Delta = 1$. Similarly, if a balanced string length 2n beginning with XX must have $\Delta = 0$ or 2.

Now we claim that if we add either X or O add the end of a balanced string length 2n+1 beginning with XX to get a new string b, then b is balanced. It is sufficient to show that $|\Delta(s)| \le 2$ for a substring ending with the final symbol. Note first that $\Delta(b) = 0$ or 2, depending on whether we added a O or an X. Let t be the complementary string b \ s. Then $\Delta(s) = \Delta(b) - \Delta(t) = 0$ or 2 less 0, 1, or 2 = something in the range -2 to 2. Thus the number of balanced strings beginning XX and length 2n is twice the number length 2n-1.

On the other hand, if have a balanced string length 2n, the symbol we add must be chosen to fix Δ for the new string at 1, so at most one of X and O will work. In fact a similar argument shows that if we pick X or O to make $\Delta = 1$, then the new string is balanced. Taking b, s and t as before we have $\Delta(s) = \Delta(b) - \Delta(t) = 1$ less 0, 1 or 2 = 1, 0 or -1. Thus the number of balanced strings beginning XX and length 2n equals the number length 2n+1. A trivial induction now gives that the number length n is $2^{[n/2]-1}$. But the numbers of balanced strings beginning XX and OO are obviously the same, so we get the result stated at the start.

Now we claim that the number of balanced strings length n+1 beginning XO is the same as the number of balanced strings length n beginning O. For clearly if b is a balanced string length n+1 beginning XO, then the substring formed by deleting the initial X is balanced, so it is sufficient to show that if we add X at the start of a balanced string b beginning with O, then the new string b' is balanced. So suppose s is a substring of b' beginning with the first symbol X. s has the same Δ as the string formed by deleting its first two symbols, so we must have $|\Delta| \le 2$, since b is balanced.

So let u_n be the number of balanced strings length n. Then we have established that $u_{2n} = u_{2n-1} + 2^n$, $u_{2n+1} = u_{2n} + 2^n$. So we get immediately that $u_{2n} = (2^n + 2^{n-1}) + u_{2n-2} = ... = ({}^n + 2^{n-1}) + (2^{n-1} + 2^{n-2}) + ... + (2^1 + 2^0) + 1 = 2^n + (2^n + 2^{n-1} + ... + 2^1) = 2^n + 2^{n+1} - 2 = 3 \cdot 2^n - 2$ (checking that $u_2 = 4$). Hence $u_{2n+1} = 4 \cdot 2^n - 2 = 2^{n+2} - 2$.

Problem B6
The origin lies inside a convex polygon whose vertices have coordinates (a_i, b_i) for i = 1, 2, ... , n. Show that we can find x, y > 0 such that $a_1x^{a_1}y^{b_1} + a_2x^{a_2}y^{b_2} + ... + a_nx^{a_n}y^{b_n} = 0$ and $b_1x^{a_1}y^{b_1} + b_2x^{a_2}y^{b_2} + ... + b_nx^{a_n}y^{b_n} = 0$.

Solution
Convex may trigger thoughts of λ_i such that $\lambda_1(a_1, b_1) + ... + \lambda_n(a_n, b_n) = 0$, but that is a dead-end. The problem is that there are n λs, whereas we only want 2 parameters x and y. But if we look at the requirement for the first coordinate $a_1x^{a_1}y^{b_1} + a_2x^{a_2}y^{b_2} + ... + a_nx^{a_n}y^{b_n} = 0$, the $a_1x^{a_1}$ is strongly reminiscent of $k\,x^{k-1}$, and indeed we could make it more so by dividing through by x. In fact the requirement can be written as grad f(x,y) = 0, where $f(x,y) = x^{a_1}y^{b_1} + x^{a_2}y^{b_2} + ... + x^{a_n}y^{b_n}$. So it looks as though all we have to do is show that grad f = 0 in the first quadrant.

That is true. But care is needed. For example, suppose we take the three points as (1, -1), (-1, 1), (-1, -1), which puts the origin on the boundary of the polygon, rather than in its interior. Then $f(x, y) = x/y + y/x + 1/xy$. This is clearly > 2 for x, y > 0, but it *decreases* monotonically to 2 as we go to infinity along the line y = x, so there is no minimum (and in fact no stationary point at all) in the first quadrant. So simple-minded arguments to the effect that the a_i and b_i include both positive and negative values and hence the function tends to infinity as x, y tend to 0 or infinity are not enough. The point of the problem is precisely to establish that grad f (or grad of something) really does vanish in the first quadrant.

A more instructive case is $f(x, y) = x/y + y/x + xy$ (which again puts the origin on the boundary of the polygon). If we take a ray y = kx, then $f(x, kx) = 1/k + k + k\,x^2$ which clearly does tend to infinity as we go to infinity along the ray. Also as if we fix x or y non-zero and let the other tend to zero, we clearly get f tending to infinity. So f tends to infinity as we approach the boundary of the first quadrant in any direction except towards the origin, where the function is evidently discontinuous. Unfortunately, this is enough to invalidate the argument that grad f must be zero in the first quadrant (as we easily see by setting grad f to zero and solving - there are no solutions). This strongly suggests that f is the wrong function to use, since the problems with the discontinuity of f at the origin will still be there even if we take the origin in the interior of the polygon.

A better behaved variant is $f(u, v) = e^{a_1 u+b_1 v} + ... + e^{a_n u+b_n v}$, which is defined on the plane with no problems at the origin. Now if we take any ray through the origin it passes between two vertices. The two vertices must subtend an angle less than π at the origin (since the origin is in the interior of the polygon), so the line to one of the two vertices makes an angle $\theta < \pi/2$ with the ray. Take this vertex (a_k, b_k) [or in the degenerate case where the ray passes through a vertex, we simply take that vertex]. Now if (u, v) is a point on the ray, we have $a_k u + b_k v = |(u, v)|\, h \cos \theta$, where $|(u, v)|$ is the distance from the origin to the point (u, v) and h is the distance from the origin to the vertex. Hence $a_k u + b_k v$ tends to infinity as we go to infinity along the ray and hence g(u, v) tends to infinity. Thus g tends to infinity in *all* directions. Hence grad g = 0 at some point in the plane. Putting $x = e^u$, $y = e^v$ gives the result.

58th Putnam 1997

Problem A1
ROMN is a rectangle with vertices in that order and RO = 11, OM = 5. The triangle ABC has circumcenter O and its altitudes intersect at R. M is the midpoint of BC, and AN is the altitude from A to BC. What is the length of BC?

Solution
Answer: 28.
The orthocenter, median and circumcenter are collinear (the Euler line), so X, the intersection of AM and RO is the median. But AX = 2XM, so AR = 2RN = 10. Hence $AO^2 = AR^2 + RO^2 = 221$. So $MC^2 = OC^2 - OM^2 = 221 - 25 = 196$, so MC = 14 and BC = 28.

Problem A2
Players 1, 2, ... , n are seated in a circle. Each has one penny. Starting with player 1 passing one penny to player 2, players alternately pass one or two pennies to the next player still seated. A player leaves as soon as he runs out of pennies. So player 1 leaves after move 1, and player 2 leaves after move 2. Find an infinite set of numbers n for which some player ends up with all n pennies.

Solution
Answer: for $2^n + 1$ or $2^n + 2$ players (where $n \geq 0$) one player ends up with all the coins. In all other cases the game does not terminate.

We give a complete solution for the game, although to solve the problem one need only consider one of the cases $2^n + 1$ or $2^n + 2$.
It is convenient to deal separately with odd and even numbers of players. Suppose first that the number is even, so we may write it as $2^m(2r + 1) + 2$, where $m \geq 1$ and $r \geq 0$.
Call a *round* a set of moves in which each remaining player plays once, starting with the lowest numbered remaining player (the *low player*). Thus the first round always leaves the (new) low player with 3 or 4 coins and the other remaining players with 2 each. Call an *eliminating round* a round in which one or more players drop out.

The kth eliminating round, for $k \leq m$, leaves $2^{m-k}(2r + 1)$ players, the low player with $2^k + 2$ coins, and the others with 2^k each, and the low player about to pass 1 coin. This is an easy induction. It is easily verified for k = 1. Assume it is true for k-1. The number of players is even, so until the next elimination any given player always passes and receives the same number of coins each round. The players in even positions receive 1 coin and pass 2, so they lose 1 coin in each round. They are thus all eliminated together after 2^k rounds. Similarly, the players in odd positions (including the low player) gain a coin each round, a total gain of 2^k. This establishes the result for k.
So after m eliminating rounds there are (2r + 1) players, the low player with $2^m + 2$ coins, the others with 2^m each, and the low player about to pass 1 coin. Of course, if r = 0, then a single player has all the coins. If not, then we show that play continues indefinitely. After the next round the sequence of coins is $2^m + 2$, $2^m - 1$, $2^m + 1$, $2^m - 1$, ... , $2^m + 1$, and the low player is about to pass 1. After the next round the situation is just as it was after the mth eliminating round, so the game now cycles indefinitely.

For an odd number of players, it is more convenient to regard the round as ending one play earlier with the highest numbered remaining player (the *high player*) about to play. Thus in the first round the high player does not play. In subsequent rounds each player plays once. We can write the initial number of players as $2^m(2r + 1) + 1$, with $m \geq 1$ and $r \geq 0$.

Arguing as before, the kth eliminating round for $k \leq m$ leaves $2^{m-k}(2r + 1)$ players, the high player with $2^k + 1$ coins and about to pass 1, and the others with 2^k each. So after the mth eliminating round, there are $(2r + 1)$ players, the high player with $2^m + 1$, the others with 2^m each. Again if $r = 0$, then one player has all the coins and the game terminates. If not, then after the next round the sequence of coins is $2^m - 1, 2^m + 1, 2^m - 1, \ldots , 2^m + 1, 2^m + 1$, with the high player about to pass 2. After the next round the situtation is just as it was after the mth eliminating round, so the game now cycles indefinitely.

Problem A3
Let $f(x) = (x - x^3/2 + x^5/(2.4) - x^7/(2.4.6) + \ldots)$, and $g(x) = (1 + x^2/2^2 + x^4/(2^2 4^2) + x^6/(2^2 4^2 6^2) + \ldots)$. Find $\int_0^\infty f(x)\, g(x)\, dx$.

Solution
Answer: $\sqrt{e}$.

We notice that the first series is $x\, e^{-x2/2}$. The second series is clearly absolutely convergent and so we can multiply out and integrate each term separately. Thus we have to integrate $x^{2n+1}e^{-x2/2}$. This is easily seen to be:
$-n!\, e^{-x2/2}(x^{2n}/n! + 2\, x^{2n-2}/(n-1)! + 2^2 x^{2n-4}/(n-2)! + \ldots + 2^n)$.
Integrating, only the last term contributes, and we get $n!\, 2^n$. Thus the x^{2n+1} term integrates to give $n!\, 2^n/(n!\, n!\, 2^{2n}) = 1/(n!\, 2^n)$. Hence the answer to the question is the sum from zero to infinity of $1/(n!\, 2^n)$ which is $\sqrt{e}$.

Problem A4
G is a group, not necessarily abelian. We write the operation as juxtapostion and the identity as 1. There is a function $\varphi : G \rightarrow G$ such that if $abc = def = 1$, then $\varphi(a)\varphi(b)\varphi(c) = \varphi(d)\varphi(e)\varphi(f)$. Prove that there exists an element $k \in G$ such that $k\varphi(x)$ is a homomorphism.

Solution
$x^{-1}1\, x = 1 = x^{-1}\, x\, 1$, so $\varphi(x^{-1})\varphi(1)\varphi(x) = \varphi(x^{-1})\varphi(x)\varphi(1)$. Hence $\varphi(1)\varphi(x) = \varphi(x)\varphi(1)$, so $\varphi(1)$ commutes with all $\varphi(x)$, and hence $k = \varphi(1)^{-1}$ also commutes with all $\varphi(x)$.

$x\, y\, (y^{-1}x^{-1}) = 1 = (x\, y)(y^{-1}x^{-1})\, 1$, so $\varphi(x)\varphi(y)\varphi(y^{-1}x^{-1}) = \varphi(xy)\varphi(y^{-1}x^{-1})\varphi(1)$. But $\varphi(1)$ commutes, so $\varphi(xy)\varphi(y^{-1}x^{-1})\varphi(1) = \varphi(1)\varphi(xy)\varphi(y^{-1}x^{-1})$ and hence $\varphi(x)\varphi(y)\varphi(y^{-1}x^{-1}) = \varphi(1)\varphi(xy)\varphi(y^{-1}x^{-1})$. So $\varphi(x)\varphi(y) = \varphi(1)\varphi(xy)$, and hence $k\, \varphi(xy) = k\varphi(x)k\varphi(y)$, as required.

Problem A5
Is the number of ordered 10-tuples of positive integers $(a_1, a_2, \ldots , a_{10})$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_{10} = 1$ even or odd?

Solution
Answer: odd.
Suppose that there are s unequal values for the a_i, with m_1 equal to the first value, m_2 equal to the second and so on. Then there are $n!/(m_1! \ldots m_s!)$ distinct arrangements of the a_i. But $n!/(m_1! \ldots m_s!) = nCm_i$ x integer, so it is only odd if all the nCm_i are odd [where nCm is the binomial coefficient $n!/(m!(n-m)!)$.]

But the only values m in the range 1, 2, ... , 10 for which 10Cm is odd are 2, 8 and 10. Thus we need only consider solutions involving these values (because all the other solutions come in even sized groups). It is easy to see that the only such solutions are:

(1) 1/4 + 1/4 + 1/16 + ... + 1/16,
(2) 1/3 + 1/3 + 1/24 + ... + 1/24,
(3) 1/6 + 1/6 + 1/12 + ... + 1/12,
(4) 1/18 + 1/18 + 1/9 + ... + 1/9,
(5) 1/10 + ... + 1/10.

[With only 10 terms available, the possibilities are 1 = 10/a, 1 = 2/a + 8/b, 1 = 2/a + 2/b + 2/c + 2/d + 2/e, but the latter does not work because 10!/(2!2!2!2!2!) is even. The interesting case is 1 = 2/a + 8/b, which is equivalent to (a - 2)(b - 8) = 16, which gives the solutions shown.] There are an odd number of solutions of each of these (45 for each of (1) - (4) and 1 for (5)), so the total number of solutions is odd.

Problem A6
Let N be a fixed positive integer. For real α, define the sequence x_k by: $x_0 = 0$, $x_1 = 1$, $x_{k+2} = (\alpha x_{k+1} - (N - k)x_k)/(k + 1)$. Find the largest α such that $x_{N+1} = 0$ and the resulting x_k.

Solution
Answer: $\alpha = N - 1$, $x_k = (N-1)C(k-1)$ for $k \leq N$ and 0 for $k > N$.

Playing around with low values of N quickly generates the binomial coefficients and the value $\alpha = N - 1$. It is then straightforward to verify that these values satisfy the recurrence relation. But that still leaves open the possibility that a larger value of α might work.
So we have to use a little more machinery. Notice first that if $x_{N+1} = 0$, then $x_{N+2} = 0$ and hence $x_i = 0$ for all $i > N$. Now define the generating function $f(z) = \sum_0^\infty x_{k+1}z^k$. We have no convergence worries, because f(z) is in fact a polynomial. We have $f'(z) = \sum_0^\infty (k+1)x_{k+2}z^k$, $zf(z) = \sum_1^\infty x_k z^k$, and $z^2f'(z) + zf(z) = \sum_1^\infty k\, x_k z^k$, and so the recurrence relation is equivalent to:

$$f'(z)/f(z) = (\alpha - (N-1)z)/(1 - z^2) = (\alpha - (N-1))/(2(1 - z)) + (\alpha + N-1)/(2(1+z)).$$

Integrating (and using $x_1 = 0$): $f(z) = (1-z)^{(N-1-\alpha)/2}(1+z)^{(\alpha+N-1)/2}$.
Now we see immediately that the largest value of α which gives a polynomial is $\alpha = N - 1$. The generating function is then $(1 + z)^{N-1}$, and so $x_k = (N-1)C(k-1)$.

Problem B1
Find $\sum_1^{6N-1} \min(\{r/3N\}, \{r/3N\})$, where $\{\alpha\} = \min(\alpha - [\alpha], [\alpha] + 1 - \alpha)$, the distance to the nearest integer.

Solution
Answer: N.

Notice first that $\{(6N-r)/3N\} = \{r/3N\}$ and $\{(6N-r)/6N\} = \{r/6N\}$. Also $\{3N/3N\} = 0$, so the sum is twice the sum from 1 to 3N-1. Next notice that the second term is lower for $r \leq 2N$ and in fact evaluates to r/6N. The first term is lower for the remaining r and evaluates to (3N-r)/N. So the answer is:

$$2/6N\,(1 + 2 + ... + 2N) + 2/3N\,(N-1 + N-2 + ... + 1) = (2N+1)/3 + (N-1)/3 = N.$$

Problem B2
Let R be the reals. f **:** R → R is twice-differentiable and we can find g **:** R → R such that $g(x) >= 0$ and $f(x) + f''(x) = -xg(x)f'(x)$ for all x. Prove that f(x) is bounded.

Solution
This needs a trick. Multiply through by f '(x). Then the lhs is (half) the derivative of $f^2 + f'^2$. The

rhs is non-positive for x > 0 and non-negative for x < 0. Hence $f^2 + f'^2$ has a maximum at x = 0. A fortiori, it is bounded. Hence f is bounded.

Problem B3
Let $(1 + 1/2 + 1/3 + ... + 1/n) = p_n/q_n$, where p_n and q_n are relatively prime positive integers. For which n is q_n not divisible by 5?

Solution
Answer: n = 1 - 4, 20 - 24, 100 - 104, and 120 - 124.

Let S(n) = 1 + 1/2 + 1/3 + ... + 1/n, and let S'(n) be the same sum excluding terms 1/m with m a multiple of 5. For x non-integral it is convenient to use S(x) to mean S([x]). We find that S(1) = 1/1, S(2) = 3/2, S(3) = 11/6, S(4) = 25/12.
Let N, N' etc denote fractions p/q where 5 does not divide q, let F, F' etc denote fractions p/q where 5 divides q, but not p, and let X, X' etc denote fractions p/q where 5 does not divide p or q. For example, we obviously have S'(n) = N. Notice that 1/(5r+1) + 1/(5r+2) + 1/(5r+3) + 1/(5r+4) = (1/(5r+1) + 1/(5r+4)) + (1/(5r+2) + 1/(5r+3)) = (10r+5)(1/(5r'+4) + 1/(5r''+6)) = 25 N. So for n = 4 or 5 (mod 5) we have 1/5 S'(n) = 5N.

For n = 5, ... , 19, S(n) = S'(n) + 1/5 S(n/5) = N + 1/5 X = F.
For n = 20, ... ,24, S(n) = S'(n) + 1/5 25/12 = N + N' = N''.
We need also that S(21) = S'(20) + 1/21 + 5/12 = 5N + 1/21 + 5N' = X, and similarly that S(22) = 5N + 1/21 + 1/22 = X, S(23) = 5N + 1/21 + 1/22 + 1/23 = X.
For n = 25, ... , 99, S(n) = S'(n) + 1/5 S'(n/5) + 1/25 S(n/25) = N + 1/5 N' + 1/25 X = F.
For n = 100, ... , 104, S(n) = S'(n) + 1/5 S'(20) + 1/25 25/12 = N + 5N' + 1/12 = N.
But S'(100) = 5N, so we have the stronger result that S(100) = 5N'' + 1/12 = X. Similarly for S'(104). S(101) = S(100) + 1/101 = 5N'' + 1/12 + 1/101 = X (just check that the numerator of 1/12 + 1/101 is not a multiple of 5) and, similarly, S(102) = X, S(103) = X.
For n = 105, ... , 119, S(n) = S'(n) + 1/5 S'(n/5) + 1/12 = N + 1/5 X + 1/12 = F (since S'(n/5) = S'(21) or S'(22) or S'(23) = X).
For n = 120, ... , 124, S(n) = S'(n) + 1/5 S'(24) + 1/12 = N + 5N' + 1/12 = N.
Exactly as for 100-104, we also have the stronger result that S(n) = X.

We have established that S(n) has the denominator divisible by 5 for n = 1-4, 20-24, 100-104 and 120-124. It remains to show that the denominator is not divisible by 5 for $n \geq 125$.
But we have also established that the numerator is not divisible by 5 for $25 \leq n \leq 124$. So let $n \geq 125$, and let h be the highest power of 5 not exceeding n, so $h \geq 3$. We now divide S(n) into A, the sum of terms which are multiples of 5^{h-2} and B, the sum of the remaining terms. Clearly $B = N/5^{h-3}$. But $A = S(r)/5^{h-2}$, where $25 \leq r \leq 124$. But the numerator of S(r) is not divisible by 5, so the denominator of A is divisible by at least 5^{h-2} and hence S(n) = A + B has denominator (in lowest terms) divisible by at least 5^{h-2} and hence by 5.

Problem B4
Let $(1 + x + x^2)^m = \sum_0^{2m} a_{m,n}x^n$. Prove that for all $k \geq 0$, $\sum_0^{[2k/3]} (-1)^i a_{k-i,i} \in [0,1]$.

Solution
Notice first that, if we let $a_{m,n} = 0$ where there is no term x^n in the expansion, then the required sum is simply taken over all i.

There are two ways to do this. The straightforward way is induction, using the fact that $a_{m+1,n} = a_{m,n} + a_{m,n-1} + a_{m,n-2}$. If S(k) denotes the sum for k, then we find that S(k) = S(k-1) - S(k-2) + S(k-3). This is a routine computation.

The slick way is to use a generating function in two variables $f(x,y) = \sum a_{m,n} y^m x^n$. Then if we set $x = - y$, we find that the coefficient of y^k is the required sum. But $f(x, y) = 1/(1 - y(1+x+x^2))$, so $f(-y, y) = (1 + y)/(1 - y^4) = (1 + y)(1 + y^4 + y^8 + ...)$ and the coefficient of $y^k = 1$ if $k = 0$ or 1 (mod 4), 0 otherwise.

Problem B5
Define the sequence a_n by $a_1 = 2$, $a_{n+1} = 2^{a_n}$. Prove that $a_n = a_{n-1} \pmod n$ for $n \geq 2$.

Solution
We use induction. The result is obviously true for $n = 2$. Assume it is true for $< n$, then we must prove that $2^{a_{n-1}} = 2^{a_{n-2}} \pmod m$, where $m \leq n$. If let $m = 2^r(2s+1)$. Then $2^{\varphi(2r+1)} = 1 \pmod{2r+1}$, and since $\varphi(2r+1) < 2r+1 \leq m \leq n$, we have by induction that $a_{n-1} = a_{n-2} \pmod{\varphi(2r+1)}$, and hence $2^{a_{n-1}} = 2^{a_{n-2}} \pmod{2r+1}$. Obviously 2^s divides both $2^{a_{n-1}}$ and $2^{a_{n-2}}$, so $2^{a_{n-1}} = 2^{a_{n-2}} \pmod{2^s}$. But $(2r+1)$ and 2^s are coprime, so $2^{a_{n-1}} = 2^{a_{n-2}} \pmod m$, as required.

Problem B6
For a plane set S define d(S), the diameter of S, to be $\sup \{PQ : P, Q \in S\}$. Let K be a triangle with sides 3, 4, 5 and its interior. If $K = H_1 \cup H_2 \cup H_3 \cup H_4$, what is the smallest possible value of $\max(d(H_1), d(H_2), d(H_3), d(H_4))$?

Solution
Let the vertices be ABC with AB = 5, AB = 4 and BC = 3. Take a point D on AC with AD = 25/13, and a point E on AB with BE = 25/13. We show that DE = 25/13. Take X on AC so that AX = 32/13, then EX is parallel to BC and so EX = 24/13. Also DX = 7/13, and EDX is right-angled, so DE = 25/13. Now the distance between any two of the 5 points A, B, C, D, E is at least 25/13, so we cannot have a dissection into 4 parts of diameter less than 25/13. We now have to show that a dissection into parts with diameter ≤ 25/13 is possible.

Such a dissection is not unique. We just have to exhibit one. The trick is to choose one for which it is reasonably easy to demonstrate that the sides and diagonals (and hence diameter) are less than 25/13.
Since AB, AC > 2 x 25/13, we must divide them each into three parts. We can divide BC into just two parts, so the first idea is to slice off the three vertices, giving three triangles and a pentagon. However, it is soon apparent that the pentagon necessarily has a diagonal longer than 25/13, so we need to convert the two triangles including B or C into quadrilaterals.

So, take F on AB with AF = 25/13. We thus have AF = 25/13, FE = 15/13 and EB = 25/13. Now take a point H in the interior of the triangle so that BH = FH = 25/13. Take G on DC with DG = 14/13 (and hence GC = 1), and I on BC with IC = 20/13. So the four parts are an isosceles triangle AFD with AF = AD = 25/13; a pentagon EFDGH; and two quadrilaterals, BEHI and CGHI.

It is clear from the construction (at least if you draw a figure!) that DH is parallel to AB. The projections of AD and BH onto AB have length 20/13 (3-4-5 again), so DH = 5 - 20/13 - 20/13 = 25/13. Also X is the midpoint of DG and so EGX is congruent to EDX and hence EG = 25/13. Thus 4 of the pentagons five diagonals have length 25/13. The fifth diagonal is FG. If we take a point E' on GC with GE' = 1/13, then AE' = AE and AF = AD, so FE' = ED = 25/13. So FG < FE' = 25/13. Clearly the sides of the pentagon are less than 25/13, so the pentagon has diameter 25/13. So does the triangle. For the quadrilateral BEIH, we need to check that EI < 25/13 (the other diagonal BH = 25/13). The perpendicular from E to BC has length 20/13 and its foot is 15/13 from B (3-4-5 again). So EI < (BI - 15/13) + (20/13) = 24/13. CGHI obviously has diameter < 25/13.

59th Putnam 1998

Problem A1
A cone has circular base radius 1, and vertex a height 3 directly above the center of the circle. A cube has four vertices in the base and four on the sloping sides. What length is a side of the cube?

Solution
Answer: $3\sqrt{2}/(3 + \sqrt{2}) = 0.9611$.

Let O be the apex of the cone. Let C be the center of the base. Let V be one of the vertices of the cube not on the base. V must lie in the sloping side of the cone. Continue OV to meet the base at X. Let OC meet the top side of the cube at K. Now consider the triangle OXC. XC = 1, OC = 3. OVK is similar and VK = $x/\sqrt{2}$, where x is the side length of the cube. KC = x, so OK = 3 - x. Since the triangles are similar, OK/OC = VK/XC leading to $x = 3\sqrt{2}/(3 + \sqrt{2}) = 0.9611$.

Problem A2
Let C be the circle center (0, 0), radius 1. Let X, Y be two points on C with positive x and y coordinates. Let X_1, Y_1 be the points on the x-axis with the same x-coordinates as X and Y respectively, and let X_2, Y_2 be the points on the y-axis with the same y-coordinates. Show that the area of the region XYY_1X_1 plus the area of the region XYY_2X_2 depends only on the length of the arc XY, and not on its position.

Solution
Let O the origin. Area XYY_1X_1 = area sector OXY + area OYY_1 - area OXX_1, and area XYY_2X_2 = area sector OXY + area OXX_2 - area OYY_2. But area OXX_1 = area OXX_2 and area OYY_1 = area OYY_2. Hence area XYY_1X_1 + area XYY_2X_2 = 2 area sector OXY, which is independent of the position of the arc.

Problem A3
Let R be the reals. Let f **:** R → R have a continuous third derivative. Show that there is a point a with $f(a)\, f'(a)\, f''(a)\, f'''(a) \geq 0$.

Solution
If any of f, f ', f '', f ''' change sign, then we are done. We show that if f, f ' and f '' do not change sign, then f and f '' have the same sign. Suppose f '' is positive. Then $f(x) = f(0) + f'(0)\, x + f''(c)\, x^2/2$ for some c, so for all x, $f(x) > f(0) + f'(0)\, x$. That is certainly positive for x sufficiently large and of the same sign as f '(0). Hence it is positive for all x. Similarly, if f '' is negative, then f is negative. Hence $f(x)f''(x) \geq 0$ for all x. Similarly $f'(x)f'''(x) \geq 0$ for all x.

Problem A4
Define the sequence of decimal integers a_n as follows: $a_1 = 0$; $a_2 = 1$; a_{n+2} is obtained by writing the digits of a_{n+1} immediately followed by those of a_n. When is a_n a multiple of 11?

Solution
Answer: n = 1 (mod 6).

Recall that if $n = d_m d_{m-1} \ldots d_1 d_0$, where the d_i are decimal digits, then the residue of n mod 11 is $f(n) = (d_0 + d_2 + \ldots) - (d_1 + d_3 + \ldots)$. A trivial induction shows that a_n has an odd number of digits unless n is a multiple of 3. Hence $f(a_n) = -f(a_{n-1}) + f(a_{n-2})$ unless n = 2 (mod 3) when $f(a_n) = f(a_{n-1}) + f(a_{n-2})$. Another simple induction shows that $f(a_n) = 1$ for n = 0, 2, 5 (mod 6), -1 for n = 3 (mod 6), 2 for n = 4 (mod 6), and 0 for n = 1 (mod 6).

Problem A5
A finite collection of disks covers a subset X of the plane. Show that we can find a pairwise disjoint subcollection S, such that $X \subseteq \cup \{3D : D \in S\}$, where 3D denotes the disk with the same center as D and 3 times the radius.

Solution
The key observation is that if disk D has larger radius than any disk intersecting it, then 3D covers D and any disks intersecting it.
So take D_1 to be the largest disk. Remove D_1 and those intersecting it. Now take D_2 to be the largest disk remaining. And so on.

Problem A6
P, Q, R are three (distinct) lattice points in the plane. Prove that if $(PQ + QR)^2 < 8$ area PQR + 1, then P, Q, R are vertices of a square.

Solution
The integer coordinates imply that PQ^2, QR^2, and 2 area PQR are all integral. Hence $(PQ + QR)^2 \leq$ 8 area PQR. Hence $(PQ - QR)^2 \leq 8$ area PQR - 4 PQ·QR. But 2 area PQR ≤ PQ·QR with equality if and only if PQR is a right-angle. Hence $(PQ - QR)^2 \leq 0$ with equality only if PQR is a right-angle. Hence PQ = QR and PQR is a right angle.

Problem B1
Find the minimum of $\{ (x + 1/x)^6 - (x^6 + 1/x^6) - 2 \}/\{ (x + 1/x)^3 + (x^3 + 1/x^3) \}$, for x > 0.

Solution
Simplifying, we find that the expression is 3(x + 1/x). This has minimum at x = 1, value 6.

Problem B2
Let P be the point (a, b) with 0 < b < a. Find Q on the x-axis and R on y = x, so that PQ + QR + RP is as small as possible.

Solution
Reflect P in the x-axis to get A (a, -b) and in the line y = x to get B (b, a). Then PQ = AQ and RP = RB, so PQ + QR + RP = AQ + QR + RB ≥ AB, with equality iff Q and R lie on AB. So the minimum is $\sqrt{(2a^2 + 2b^2)}$.

Problem B3
Let S be the sphere center the origin and radius 1. Let P be a regular pentagon in the plane z = 0 with vertices on S. Find the surface area of the part of the sphere which lies above (z > 0) P or its interior.

Solution
Answer: $\pi(5 \cos \pi/5 - 3) = 3.2832$.

Call the area on a sphere bounded by a circle a *cap*. The *depth* of the cap is the distance from its central point to the plane through its boundary. A simple integration gives that the area of a cap depth d is $2\pi rd$, where r is the radius of the sphere. In this case the vertical plane through a side of the pentagon defines a cap. The five caps do not overlap and so the area required is (area S - 5 area cap)/2. We have that $d = 1 - \cos \pi/5$. Hence area required = $\pi(5 \cos \pi/5 - 3)$.

Problem B4
For what m, n > 0 is $\sum_0^{mn-1} (-1)^{[i/m] + [i/n]} = 0$?

Solution
Answer: The highest powers of 2 dividing m and n are different.

If m and n are both odd, then there are an odd number of terms, so the sum must be non-zero. If just one of m and n is odd, then m+n is odd and hence [i/m] + [i/n] has opposite parity to [(mn-1-i)/m] + [(mn-1-i)/n] = n - [i/m] -1 + m - [i/n] - 1= (m + n) + ([i/m] + [i/n]) - 2([i/m] + [i/n] + 1). So in this case the sum is zero.

Finally, suppose m=2m' and n=2n'. Then for any i satisfying 0 <= i < m'n', we have [i/m'] + [i/n'] = [2i/m] + [2i/n] = [(2i+1)/m] + [(2i+1)/n]. Moreover, [(j+m'n')/m] + [(j+m'n')m] = m'+n' + [j/m] + [j/n]. So for m'+n' both even or both odd, [j/m] + [j/n] = [i/m'] + [i/n'] for j = 2i, 2i+1, m'n'+2i, m'n'+2i+1 and hence the sum for m, n has 4 times the value for m', n'. For one of m', n' even and the other odd, m'+n' is odd and hence the two terms j = 2i, 2i+1 cancel with the two terms j = m'n'+2i, m'n'+2i+1, so the sum is zero.

Problem B5
Let n be the decimal integer 11...1 (with 1998 digits). What is the 1000th digit after the decimal point of √n?

Solution
Answer: 1.

A little experimentation suggests that if n is the integer with 2m decimal digits, all of them 1, then √n has m 3s, followed by a decimal point, followed by m 3s, followed by a 1, followed by some 6s. If true, that would imply that the required digit was 1.
$n = (10^{2m} - 1)/9$. Let $x = (10^m - 10^{-m})/3$, then x has m 3s followed by a decimal point, followed by m 3s. So we have to show that $(x + 10^{-m-1})^2 < n$ and $(x + 2.10^{-m-1})^2 > n$. That is a straightforward computation.

Problem B6
Show that for any integers a, b, c we can find a positive integer n such that $n^3 + a\,n^2 + b\,n + c$ is not a perfect square.

Solution
In fact, $f(n) = n^3 + a\,n^2 + b\,n + c$ cannot be a perfect square for all of n = 1, 2, 3 and 4. Perfect squares must be = 0 or 1 (mod 4). But f(2) -f(4) = 2b (mod 4), so if f(2) and f(4) are perfect squares then 2b = 0 (mod 4), and f(3) - f(1) = 2b + 2 (mod 4), so if f(1) and f(3) are perfect squares then 2b + 2 also = 0 (mod 4), which is impossible.

60th Putnam 1999

Problem A1
Find polynomials a(x), b(x), c(x) such that $|a(x)| - |b(x)| + c(x) = -1$ for $x < -1$, $3x + 2$ for $-1 \le x \le 0$, $-2x + 2$ for $x > 0$.

Solution
Note that we do not have to find *all* solutions, just a solution.

We look first for linear polynomials. The points of discontinuity in the derivative are evidently where a(x) and b(x) change sign. At -1 the function jumps by 3x + 3, so the function changing sign must be (3x + 3)/2. Moreover, it must be a(x) that changes sign, since the jump is +(3x + 3) not -(3x + 3). So a(x) = (3x + 3)/2. Similarly, at 0, the function jumps by -5x, so b(x) = 5x/2. Hence $c(x) = -1 + |b(x)| - |a(x)| = -1 - 5x/2 + (3x + 3)/2 = -x + 1/2$ (for $x < -1$ and hence for all x)
Finally, notice that the sign of a(x) and b(x) is not determined uniquely.

Problem A2
Show that for some fixed positive n we can always express a polynomial with real coefficients which is nowhere negative as a sum of the squares of n polynomials.

Solution
We consider what polynomials are nowhere negative. If p(x) has a root of odd order, then it changes sign at the root. So any real roots of the polynomial must be of even order. If $x = \alpha + i\,\beta$ is a complex root, then so is $x = \alpha - i\,\beta$, since the coefficients of the polynomial are real.

Collecting all the real roots together gives us a square factor $q(x)^2$. If the complex roots are $\alpha_1 \pm i\,\beta_1$, $\alpha_2 \pm i\,\beta_2$, $\alpha_3 \pm i\,\beta_3$, ... , $\alpha_m \pm i\,\beta_m$, then let $a(x) = \prod (x - \alpha_i - i\,\beta_i)$ and $b(x) = \prod (x - \alpha_i + i\,\beta_i)$, so that $p(x) = q(x)^2 a(x) b(x)$. Then we can find real polynomials, r(x) and s(x), such that $a(x) = r(x) + i\,s(x)$. Substituting -i for i, we must have $b(x) = r(x) - i\,s(x)$ and hence $a(x)\,b(x) = r(x)^2 + s(x)^2$. So the original polynomial $p(x) = q(x)^2(r(x)^2 + s(x)^2)$, which is the sum of two squares.

Problem A3
Let $1/(1 - 2x - x^2) = s_0 + s_1 x + s_2 x^2 + \ldots$. Prove that for some f(n) we have $s_n^2 + s_{n+1}^2 = s_{f(n)}$.

Solution
Multiplying across, we have $1 = (1 - 2x - x^2)(s_0 + s_1 x + s_2 x^2 + \ldots) = s_0 + (s_1 - 2s_0)x + (s_2 - 2s_1 - s_0)x^2 + \ldots + (s_n - 2s_{n-1} - s_{n-2})x^n + \ldots$. Hence, the first few terms are: $s_0 = 1$, $s_1 = 2$, $s_2 = 5$, $s_3 = 12$, $s_4 = 29$, $s_5 = 70$, $s_6 = 169$. Also we have the recurrence relation $s_n = 2s_{n-1} + s_{n-2}$ (*).

For the early terms, we find: $s_0^2 + s_1^2 = s_2$, $s_1^2 + s_2^2 = s_4$, $s_2^2 + s_3^2 = s_6$. So it is natural to conjecture that $s_n^2 + s_{n+1}^2 = s_{2n+2}$. It might seem that the obvious way to prove this is by induction using the recurrence relation (*), but after some fruitless fiddling around, I decided it would be easier to solve (*) and use the solution.
The associated quadratic has roots $1 \pm \sqrt{2}$, so the general solution of the recurrence relation is $A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n$. Using the values for s_0 and s_1 gives $s_n = ((1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1})/(2\sqrt{2})$. Hence $s_n^2 + s_{n+1}^2 = 1/8 ((1 + \sqrt{2})^{2n+2} - 2 (-1)^{n+1} + (1 - \sqrt{2})^{2n+2}) + 1/8 ((1 + \sqrt{2})^{2n+4} - 2 (-1)^{n+2} + (1 - \sqrt{2})^{2n+4})$ $= 1/8 (1 + \sqrt{2})^{2n+2}(1 + 3 + 2\sqrt{2}) + 1/8 (1 - \sqrt{2})^{2n+2}(1 + 3 - 2\sqrt{2}) = (1 + \sqrt{2})^{2n+3}/(2\sqrt{2}) - (1 - \sqrt{2})^{2n+3}/(2\sqrt{2}) = s_{2n+2}$.

Problem A4
Let $a_{ij} = i^2 j/(3^i(j\,3^i + i\,3^j))$. Find $\sum a_{ij}$ where the sum is taken over all pairs of integers (i, j) with $i, j > 0$.

Solution
Let $b_i = i/3^i$, then $a_{ij} = b_i^2 b_j (b_i + b_j)$. Let the sum be k. Then $k = \sum a_{ij} = 1/2 \sum (a_{ij} + a_{ji}) = 1/2 \sum b_i b_j$. We have the familiar $1 + 2x + 3x^2 + ... = 1/(1 - x)^2$, so $x + 2x^2 + 3x^3 + ... = x/(1 - x)^2$, and hence $\sum b_i = 1/3\ (2/3)^{-2} = 3/4$. So $k = 1/2\ (3/4)^2 = 9/32$.

Problem A5
Find a constant k such that for any polynomial f(x) of degree 1999, we have $|f(0)| \le k \int_{-1}^{1} |f(x)|\, dx$.

Solution
The basic idea is simple. Divide the interval [-1, 1] into 2000 subintervals of length 1/1000, then there must be at least one subinterval K such that there is no root of f(x) whose real part falls in K. Take K' to be the central 1/3 of K. Then every point of K' is a distance at least 1/3000 from any root of f(x). Now $|f(x)| = \prod |x - \alpha| \ge (1/3000)^{1999}$. Hence the integral is at least $(1/3000)^{2000}$.

That would work if we were looking for a lower bound for the ratio of the integral to the leading coefficient. But we were asked for the ratio of the integral to the modulus of constant term (which is $\prod |\alpha|$), so we need to estimate $\prod |x - \alpha|/|\alpha|$. If f(0) = 0, then any constant k works, so we can assume that all the roots $\alpha \ne 0$.

If $|\alpha| \ge 1$, then $|x| < 1 - 1/3000$ (K' is the central third of a subinterval length 1/1000 of [-1, 1], so its endpoint is at least 1/3000 from ±1). Hence $|x - \alpha| \ge |\alpha| - |x| \ge |\alpha| - 1 + 1/3000$, and $|x - \alpha|/|\alpha| > 1/3000$. If $|\alpha| < 1$, then $|x - \alpha|/|\alpha| > |x - \alpha| > 1/3000$.
So we have established that $|f(x)|/|f(0)| > (1/3000)^{1999}$ on the interval K'. Hence the integral is at least $(1/3000)^{2000}\ |f(0)|$ and we have established the result with $k = (1/3000)^{2000}$.

Problem A6
$u_1 = 1$, $u_2 = 2$, $u_3 = 24$, $u_n = (6\ u_{n-1}^2 u_{n-3} - 8\ u_{n-1} u_{n-2}^2)/(u_{n-2} u_{n-3})$. Show that u_n is always a multiple of n.

Solution
Put $v_n = u_n/u_{n-1}$. Then the relation simplifies to: $v_n = 6v_{n-1} - 8v_{n-2}$. This has general solution: $v_n = A\ 2^n + B\ 4^n$. But $v_2 = u_2/u_1 = 2$, $v_3 = u_3/u_2 = 12$, so $v_{n+1} = 4^n - 2^n$. Hence $u_n = (4^{n-1} - 2^{n-1})(4^{n-2} - 2^{n-2}) ... (4 - 2)$.
For any prime p, we have $4^{p-1} = 2^{p-1} = 1 \pmod p$, so p divides $4^{p-1} - 2^{p-1}$ and also $4^s - 2^s$ for s a multiple of p - 1. But if p^r divides n, then there are at least r multiples of (p - 1) less than n, so p^r divides u_n. Hence n divides u_n.

Problem B1
The triangle ABC has AC = 1, angle ACB = 90^o, and angle BAC = φ. D is the point between A and B such that AD = 1. E is the point between B and C such that angle EDC = φ. The perpendicular to BC at E meets AB at F. Find $\lim_{\varphi \to 0}$ EF.

Solution
Considering ACD, angle ACD = angle ADC = $90^o - \varphi/2$. Hence angle DCE = φ/2. So, using the sine rule on triangle CDE: CE/DE = sin φ/sin(φ/2) which tends to 2. Angle EBD = $90^o - \varphi$, angle EDB = $90^o - \varphi/2$. So DE/EB = cos φ/cos(φ/2) which tends to 1. Hence EB/CE tends to 1/2 or EB/BC tends to 1/3. But EF/AC = EB/BC, so EF tends to AC/3 = 1/3.

Problem B2
p(x) is a polynomial of degree n. q(x) is a polynomial of degree 2. p(x) = p''(x)q(x) and the roots of p(x) are not all equal. Show that the roots of p(x) are all distinct.

Solution
Suppose not, so for some α and n > 1, we have $p(x) = (x - \alpha)^n r(x)$ with $r(\alpha) \neq 0$. Then $(x - \alpha)^n r(x) = (n(n - 1)(x - \alpha)^{n-2} r(x) + 2n(x - \alpha)^{n-1} r'(x) + (x - \alpha)^n r''(x)) q(x)$. Dividing by $(x - \alpha)^{n-2}$, we get $(x - \alpha)^2 r(x) = (n(n - 1) r(x) + 2n(x - \alpha) r'(x) + (x - \alpha)^2 r''(x)) q(x)$. Setting $x = \alpha$, we conclude that $q(\alpha) = 0$ and hence $(x - \alpha)$ divides $q(x)$. Dividing through by $(x - \alpha)$ and repeating the argument, we find that $q(x)$ has a second factor $(x - \alpha)$, in other words $q(x) = c(x - \alpha)^2$. Dividing that out too, we get: $r(x) = c\, n(n - 1)\, r(x) + c\, 2\, n(x - \alpha) r'(x) + c(x - \alpha)^2 r''(x)$. Setting $x = \alpha$ gives $c = 1/(n(n - 1))$. Hence $r''(x)/r'(x) = -2n/(x - \alpha)$. Integrating: $r'(x) = A/(x - \alpha)^{2n}$, which is impossible since $r(x)$ is supposed to be a polynomial.

Problem B3
Let R be the reals. Define f **:**$[0, 1) \times [0, 1) \to R$ by $f(x, y) = \sum x^m y^n$, where the sum is taken over all pairs of positive integers (m, n) satisfying $m \geq n/2$, $n \geq m/2$. Find $\lim_{(x, y) \to (1, 1)} (1 - xy^2)(1 - x^2 y) f(x, y)$.

Solution
The straight slog approach is to sum the series. This is not too bad. If m = 2r, the y terms are $y^r(1 + y + ... + y^{3r}) = y^r(1 - y^{3r+1})/(1 - y)$. Now summing over r we have $\sum x^{2r} y^r/(1 - y) - \sum x^{2r} y^{4r+1}/(1 - y) = x^2 y/((1 - x^2 y)(1 - y)) - x^2 y^5/((1 - x^2 y^4)(1 - y))$. We deal with odd m in the same way. Let m = 2r - 1, then the y terms are $y^r(1 + y + ... + y^{3r-2}) = y^r(1 - y^{3r-1})/(1 - y)$. So summing over r, we get $xy/((1 - x^2 y)(1 - y)) - xy^3/((1 - x^2 y^4)(1 - y))$. After some algebra, we get that the complete sum is $xy(1 + x + y + xy - x^2 y^2)/((1 - x^2 y)(1 - xy^2))$. Hence the required limit is 3.

A slightly more elegant approach is to note that the sum over pairs (m, n) with $m > n/2$ is $\sum x^{2n+m} y^n$ where the sum is over all m, $n \geq 1$. That is just $x/(1 - x)\, x^2 y/(1 - x^2 y)$. Similarly, the sum over pairs (m, n) with $n > m/2$ is $y/(1 - y)\, xy^2/(1 - xy^2)$. Putting the three sums together gives $x/(1 - x)\, y/(1 - y)$. Hence the required sum is $x/(1 - x)\, y/(1 - y) - x/(1 - x)\, x^2 y/(1 - x^2 y) - y/(1 - y)\, xy^2/(1 - xy^2)$. Unfortunately, things deteriorate from this point on. Getting the expression into the required form $xy(1 + x + y + xy - x^2 y^2)/((1 - x^2 y)(1 - xy^2))$ requires, if anything, rather more slogging than the straight slog approach.

Problem B4
Let R be the reals. f **:**$R \to R$ is three times differentiable, and f(x), f '(x), f ''(x), f '''(x) are all positive for all x. Also $f(x) \geq f'''(x)$ for all x. Show that $f'(x) < 2 f(x)$ for all x.

Solution
Take an arbitrary point x = a. Let f(a) = b, f '(a) = c, f ''(a) = d. We wish to show that c < 2b.

f ''(x) has positive derivative, so $f''(x) \leq f''(a) = d$ for x <= a. Hence $f'(x) \geq f'(a) + d(x - a) = c + d(x - a)$, and $f(x) \leq f(a) + c(x - a) + d(x - a)^2/2 = b + c(x - a) + d(x - a)^2/2$ for $x \leq a$. But f(x) is positive for $x \leq a$, so the quadratic $b + c(x - a) + d(x - a)^2/2$ has no roots for (x - a) negative. But it has no roots for positive (x - a) since all its coefficients are positive. Hence it has no real roots and so $c^2 < 2bd$.
We are given that $f'''(x) \leq f(x)$, so $f'''(x) < f(x) < f(a) = b$ for x < a. Hence $f''(x) \geq f''(a) + b(x - a) = d + b(x - a)$ for $x \leq a$. Hence $f'(x) \leq f'(a) + d(x - a) + b(x - a)^2/2 = c + d(x - a) + b(x - a)^2/2$ for x <= a. But f '(x) is everywhere positive, so the quadratic $c + d(x - a) + b(x - a)^2/2$ has no roots for negative (x - a). It also has none for positive (x - a), so it has no real roots and $d^2 < 2bc$.
So $c^4 < 4b^2 d^2 < 8b^3 c$, and hence c < 2b, which is the required result.

Problem B5
n is an integer greater than 2 and $\varphi = 2\pi/n$. A is the n x n matrix (a_{ij}), where $a_{ij} = \cos(i + j)\varphi$ for $i \neq j$, $1 + \cos 2j\varphi$ for i = j. Find det A.

Solution
A direct approach does not work (at least, I could not get it to work). However, if we let B be the n x n matrix (b_{ij}) with $b_{ij} = \cos(i + j)\varphi$. Then the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of B satisfy the characteristic polynomial $\det(B - \lambda I) = 0$. Hence $\det A = \Pi(\lambda_i + 1)$.

For r = 1, 2, 3, ... , n, let v_r be the column vector $(e^{i\varphi r}, e^{2i\varphi r}, e^{3i\varphi r}, \ldots, e^{ni\varphi r})$. Using $\cos s\varphi = (e^{is\varphi} + e^{-is\varphi})/2$, we have that $2Bv_r$ is the column vector with jth element $(e^{i\varphi(j+1+r)} + e^{i\varphi(j+2+2r)} + e^{i\varphi(j+3+3r)} + \ldots + e^{i\varphi(j+n+nr)}) + (e^{i\varphi(-j-1+r)} + e^{i\varphi(-j-2+2r)} + e^{i\varphi(-j-3+3r)} + \ldots + e^{i\varphi(-j-n+nr)}) = e^{i\varphi j}(\alpha + \alpha^2 + \ldots + \alpha^n) + e^{-i\varphi j}(\beta + \beta^2 + \ldots + \beta^n)$, where $\alpha = e^{i\varphi(r+1)}$ and $\beta = e^{i\varphi(r-1)}$. If r is not 1 or n - 1, then α and β are nth roots of 1 other than 1 and hence $\alpha + \alpha^2 + \ldots + \alpha^n = \beta + \beta^2 + \ldots + \beta^n = 0$, so $2Bv_r = 0$, and so v_r is an eigenvector with eigenvalue 0. If r = 1, then $\beta = 1$, and so $\alpha + \alpha^2 + \ldots + \alpha^n = 0$, $\beta + \beta^2 + \ldots + \beta^n = n$. Hence $Bv_1 = n/2\ v_{n-1}$. Similarly, if r = n - 1, then $\alpha = 1$, and so $\alpha + \alpha^2 + \ldots + \alpha^n = n$, $\beta + \beta^2 + \ldots + \beta^n = 0$. Hence $Bv_{n-1} = n/2\ v_1$. So $(v_1 + v_{n-1})/2$ [which is the column vector $(\cos\varphi, \cos 2\varphi, \cos 3\varphi, \ldots, \cos n\varphi)$] is an eigenvector with eigenvalue n/2, and $(v_1 - v_{n-1})/2i$ [which is the column vector $(\sin\varphi, \sin 2\varphi, \ldots, \sin n\varphi)$] is an eigenvector with eigenvalue -n/2.

Finally, notice that the determinant with columns v_r is a Vandermonde determinant, whose value (using the well-known formula) is $\prod_{r>s}(e^{ir\varphi} - e^{is\varphi}) \neq 0$. So we have found n - 2 linearly independent eigenvectors for the eigenvalue 0, and hence 0 is an eigenvalue of multiplicity n - 2 (and there are no other eigenvalues).
Thus $\det A = \prod(\lambda_i + 1) = (1 + n/2)(1 - n/2) = -n^2/4 + 1$.

Problem B6
X is a finite set of integers greater than 1 such that for any positive integer n, we can find $m \in X$ such that m divides n or is relatively prime to n. Show that X contains a prime or two elements whose greatest common divisor is prime.

Solution
There are elements not relatively prime to any element of X (for example, the product of all elements of X). Take the smallest such, n. Then there must be $m \in X$ which divides n. If m is prime we are home. If not, take a prime p which divides m. p must also divide n (a multiple of m). Set n' = n/p, then $n' < n$, so we can find $m' \in X$ which is relatively prime to n'. n = n'p is not relatively prime to m', so p must divide m'. So p divides both m and m'. Moreover, it must be the greatest common divisor. For if pd divides m and m', then it divides n and m' and hence d divides n' and m' which are relatively prime. So we have found two elements of X (m and m') whose greatest common divisor is prime.

61st Putnam 2000

Problem A1
k is a positive constant. The sequence x_i of positive reals has sum k. What are the possible values for the sum of x_i^2 ?

Solution
Answer: any value in the open interval $(0, k^2)$.

Since the terms are positive we have $0 < \sum x_n^2 < (\sum x_n)^2 = k^2$. So certainly all sums must lie in the interval $(0, k^2)$.
We show how to get any value in the interval by explicit construction. We may represent the value as hk^2 where $0 < h < 1$. Take $c = (1 - h)/(1 + h)$. Take the geometric sequence with first term $k(1 - c)$ and ratio c ($k(1 - c), kc(1 - c), kc^2(1 - c), \ldots$). We have $\sum x_n = k$ as required, and $\sum x_n^2 = k^2(1 - c)/(1 + c) = hk^2$ as required.

Problem A2
Show that we can find infinitely many triples N, N + 1, N + 2 such that each member of the triple is a sum of one or two squares.

Solution
Take $N = 2n^2(n + 1)^2$. Let $m = n(n + 1)$. So $N = m^2 + m^2$, $N + 2 = (m + 1)^2 + (m - 1)^2$.

Slightly less obviously, $N + 1 = (n^2 - 1)^2 + (n^2 + 2n)^2$.
That is clean, but a little unobvious. Fumbling our way to a solution, we note that n^2, $n^2 + 1$ are easy. So we are home if we can do $n^2 - 1$. We need to remember that if A and B are the sum of two squares, then so is AB: $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$. Thus if we can find a solution for n - 1, n, n + 1, then we can also find one for $(n + 1)(n - 1) = n^2 - 1$, n^2, $n^2 + 1$. A starting solution is $8 = 2^2 + 2^2$, $9 = 3^2$, $10 = 3^2 + 1$.

Problem A3
An octagon is incribed in a circle. One set of alternate vertices forms a square area 5. The other set forms a rectangle area 4. What is the maximum possible area for the octagon?

Solution
Answer: $3\sqrt{5}$.

Evidently the circle has radius $\sqrt{5}/\sqrt{2}$, and the rectangle has sides $\sqrt{2}$ and $2\sqrt{2}$.
The trick is to focus on the rectangle. The area of the octagon is the area of the rectangle plus the area of the four triangles on its sides. The area of each of these triangles is maximised by taking the vertex midway between the two vertices of the rectangle. But these four midpoints happen to form a square.
The area of the two triangles with common base on a side of the rectangle, one with third vertex at the centre of the circle and the other at the midpoint of the arc is (radius x side)/2. So the total area of the octagon is (radius x sum of two adjacent sides of the rectangle) = $3\sqrt{5}$.

Problem A4
Show that $\lim_{k\to\infty} \int_0^k \sin x \sin x^2 \, dx$ converges.

Solution
We integrate by parts to get $\int_0^k \sin x \sin x^2 \, dx = -\int_0^k (\sin x)/2x \, d(\cos x^2) = (\cos x^2 \sin x)/2x \,|_k^0 + \int_0^k \cos x^2 \, ((\cos x)/2x - (\sin x)/2x^2) \, dx$ = term 1 + term 2 - term 3.

term 1 obviously converges to 1/2.

For term 3 the integral of the absolute value is less than $\int_0^k 1/2x^2\,dx$, which converges.
For term 2, we integrate by parts again to get $\int_0^k (\cos x)/4x^2\, d(\sin x^2)\, dx = (\sin x^2 \cos x)/4x^2 \,|_0^k + \int_0^k (\sin x^2 \sin x)/4x^2\, dx + \int_0^k (\sin x^2 \cos x)/8x^3\, dx$.
The first term obviously tends to zero and the second two converge, by comparison of the absolute value of the integrands to $1/x^2$ and $1/x^3$.

Problem A5
A, B, C each have integral coordinates and lie on a circle radius R. Show that at least one of the distances AB, BC, CA exceeds $R^{1/3}$.

Solution
Let the sides have lengths a, b, c as usual. The question suggests that we use some relationship of the form abc = constant x R. That would give the power of 1/3. If you happen to know the relation abc = 4ΔR, where Δ is the area of the triangle, then the rest is easy. If not, you will have a rough time.

To prove the relation, let O be the centre of the circumcircle. Project AO to meet the circle again at K. Let AH be the altitude. Then angle ABC = angle AKC, so triangles ABH and AKC are similar. Hence AB/AH = AK/AC or c/AH = 2R/b. Hence $abc = 2R \cdot a \cdot AH = 4\Delta R$.
So far we have not used the fact that the triangle has integral coordinates. Take the vectors along two sides. The area is half the magnitude of the cross product of these vectors. But the vectors have integral components, so the cross product has also. Hence its magnitude is at least 1 and the area $\Delta \geq 1/2$, so $abc \geq 2R > R$, hence at least one of a, b, $c > R^{1/3}$.

Problem A6
p(x) is a polynomial with integer coefficients. A sequence $x_0, x_1, x_2, \ldots$ is defined by $x_0 = 0$, $x_{n+1} = p(x_n)$. Prove that if $x_n = 0$ for some $n > 0$, then $x_1 = 0$ or $x_2 = 0$.

Solution
If $x_1 = 0$, then obviously all $x_n = 0$. So that is one possibility. Assume, then, that x_1 is non-zero. Assume also that $x_N = 0$ for some $N > 1$. Then $x_{N+1} = p(0) = x_1$.

a - b always divides $a^s - b^s$ and hence also p(a) - p(b). So $x_{n+1} - x_n$ divides $x_{n+2} - x_{n+1}$. But $x_{N+1} - x_N = x_1 - x_0$. So for all $0 <= n <= N$, we have $x_{n+1} - x_n = \pm x_1$. Moreover, some signs must be positive and some negative, since $(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_N - x_{N-1}) = x_N - x_0 = 0$. So we must be able to find an adjacent pair of opposite signs: $x_{n+1} - x_n = -(x_n - x_{n-1})$, with $1 \leq n \leq N$. Hence $x_{n-1} = x_{n+1}$. But now $p(x_{n-1}) = p(x_{n+1})$, or $x_n = x_{n+2}$. So by a simple induction $x_N = x_{N+2}$. But $x_N = 0$, and $x_{N+2} = x_2$, so $x_2 = 0$, as required.

Problem B1
We are given N triples of integers $(a_1, b_1, c_1), (a_2, b_2, c_2), \ldots (a_N, b_N, c_N)$. At least one member of each triple is odd. Show that we can find integers A, B, C such that at least 4N/7 of the N values $A\, a_i + B\, b_i + C\, c_i$ are odd.

Solution
Evidently we are only interested in the residues of the integers mod 2. So there are 7 possibilities for the each triple: (1, 1, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1). These are also the possibilities for (A, B, C).

It is easy to verify that if we take any fixed triple (a, b, c), then just 4 of the 7 possibilities for Aa + Bb + Cc are odd. So given any set of N triples, just 4N of the 7N sums $Aa_i + Bb_i + Cc_i$ are odd. These are divided into 7 sets (corresponding to the 7 possibilities for (A, B, C)), so at least one of the these sets must contain at least 4N/7 odds.

Problem B2
m and n are positive integers with $m \leq n$. d is their greatest common divisor. nCm is the binomial coefficient. Show that d/n nCm is integral.

Solution
Recall that d is a linear combination of m and n. n/n nCm is obviously integral, whilst m/n nCm = (n-1)C(m-1), which is also integral.

Problem B3
$a_1, a_2, \ldots, a_N$ are real and a_N is non-zero. $f(x) = a_1 \sin 2\pi x + a_2 \sin 4\pi x + a_3 \sin 6\pi x + \ldots + a_N \sin 2N\pi x$. Show that the number of zeros of $f^{(i)}(x) = 0$ in the interval [0, 1) is a non-decreasing function of i and tends to 2N (as i tends to infinity).

Solution
By Rolle's theorem there is a root of f ' between two adjacent roots of f. Also the function is periodic with period 1, so there number of zeros is non-decreasing.

If we write $z = e^{2\pi i x}$, then the nth derivative is $1/z^N$ times a polynomial degree 2N in z, so it has at most 2N complex roots. Hence there are at most 2N real roots for x in the interval [0, 1).

Finally, we need to show that for sufficiently high derivatives the term $a_N \sin 2N\pi x$ dominates and hence gives exactly 2N roots. We use Rolle's theorem again. At the values x = 0, 1/2N, 2/2N, ... , 2N/2N, cos 2Nπx is alternately ±1, so if we choose k sufficiently large, then the odd derivative $f^{(k)}(x)$ is alternately positive and negative at these values (because $(2N)^k |a_N| > 2^k |a_1| + 4^k |a_2| + \ldots + (2N-1)^k |a_{N-1}|$ and hence $f^{(k)}(x)$ has at least 2N zeros in the interval (0, 1).

Problem B4
f(x) is a continuous real function satisfying $f(2x^2 - 1) = 2 x f(x)$. Show that f(x) is zero on the interval [-1, 1].

Solution
Set F(t) = f(cos t)/sin t. So F(t) is defined and continuous except possibly at multiples of π. Then F(t + 2π) = F(t). Also $F(2^m t) = F(t)$. Hence $F(1) = F(2^{m+1}) = F(2^{m+1} + 2n\pi) = F(1 + n\pi/2^m)$ for all integers n, m. But the set $\{1 + n\pi/2^m\}$ is dense in the reals, so F must be constant on each open interval (nπ, (n+1)π).

But we see directly from the relation $f(2x^2 - 1) = 2 x f(x)$, that f(-x) = f(x). Hence F(t + π) = f(-cos t)/(- sin t) = F(t). So the constant value on each open interval is the same. In other words F is constant except at integral multiples of π.
But F(-t) = - F(t). Hence F must be zero, except at integral multiples of π. So f(x) must be zero on the interval on the open interval (-1, 1). But f is continuous, so f(1) and f(-1) are also zero. (Or, directly, from $f(2x^2 - 1) = 2 x f(x)$ putting x = 0 and then x = 1).

Problem B5
S_0 is an arbitrary finite set of positive integers. Define S_{n+1} as the set of integers k such that just one of k - 1, k is in S_n. Show that for infinitely many n, S_n is the union of S_0 and a translate of S_0.

Solution
We show that S_n has the required form for all sufficiently large powers of 2.

We need to establish first that if $n = 2^k$, then all the binomial coefficients in the expansion of $(a + b)^n$ except the first and last are even. This is a simple induction on k - the result for k = 1 also allows us to deduce the result for k+1 from that for k.

Define the polynomial $p_0(x)$ as $\equiv a_i x^i$, where $a_i = 1$ if i belongs to S_0 and 0 otherwise. We may consider $p_0(x)$ as a polynomial over F_2, the finite field with 2 elements. Similarly, we may define $p_m(x)$ by reference to S_m. Evidently, $p_{m+1}(x) = (1 + x)\, p_m(x)$. Hence $p_m(x) = (1 + x)^m\, p_0(x)$. But since we are in F_2 we have $(1 + x)^n = 1 + x^n$ for $n = 2^k$. So S_n has the required form if k is sufficiently large that all elements of S_0 are less than 2^k.

Problem B6
Let X be the set of 2^n points $(\pm 1, \pm 1, \ldots, \pm 1)$ in Euclidean n-space. Show that any subset of X with at least $2^{n+1}/n$ points contains an equilateral triangle.

Solution
Let the subset be Y. For each point P in X let S_P be the set of n points a distance 2 from P (in other words the points which differ from P in just one coordinate). The points in S_P are all equidistant from each other (each pair is a distance $2\sqrt{2}$ apart), so it is sufficient to show that at least one of the sets S_P contains more than 2 points of the subset Y.

Each point of Y (indeed of X) is in just n sets S_P, so $\sum_P |S_P \cap Y| = |Y|\, n > 2^{n+1}$. But there are 2^n sets $S_P \cap Y$, so at least one of them must have more than 2 points.

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