British Mathematical Olympiad

Round 1 : Wednesday 13th January 1993

Time allowed *Three and a half hours.*

Instructions
- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find, showing your method, a six-digit integer n with the following properties: (i) n is a perfect square, (ii) the number formed by the last three digits of n is exactly one greater than the number formed by the first three digits of n. (Thus n might look like 123124, although this is not a square.)

2. A square piece of toast $ABCD$ of side length 1 and centre O is cut in half to form two equal pieces ABC and CDA. If the triangle ABC has to be cut into two parts of equal area, one would usually cut along the line of symmetry BO. However, there are other ways of doing this. Find, with justification, the length and location of the shortest straight cut which divides the triangle ABC into two parts of equal area.

3. For each positive integer c, the sequence u_n of integers is defined by
$$u_1 = 1,\ u_2 = c, \quad u_n = (2n+1)u_{n-1} - (n^2-1)u_{n-2},\ (n \ge 3).$$
For which values of c does this sequence have the property that u_i divides u_j whenever $i \le j$?
(Note: If x and y are integers, then x *divides* y if and only if there exists an integer z such that $y = xz$. For example, $x = 4$ divides $y = -12$, since we can take $z = -3$.)

4. Two circles touch internally at M. A straight line touches the inner circle at P and cuts the outer circle at Q and R. Prove that $\angle QMP = \angle RMP$.

5. Let x, y, z be positive real numbers satisfying
$$\frac{1}{3} \le xy + yz + zx \le 3.$$
Determine the range of values for (i) xyz, and (ii) $x+y+z$.

British Mathematical Olympiad

Round 2 : Thursday, 11 February 1993

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than trying all four problems.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

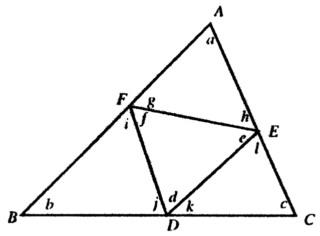
Before March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (on 15-18 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team for this summer's International Mathematical Olympiad (to be held in Istanbul, Turkey, July 13–24) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for Istanbul.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. We usually measure angles in degrees, but we can use any other unit we choose. For example, if we use $30°$ as a new unit, then the angles of a $30°$, $60°$, $90°$ triangle would be equal to 1, 2, 3 new units respectively.

   The diagram shows a triangle ABC with a second triangle DEF inscribed in it. All the angles in the diagram are whole number multiples of some new (unknown unit); their sizes $a, b, c, d, e, f, g, h, i, j, k, \ell$ with respect to this new angle unit are all distinct.



   Find the smallest possible value of $a+b+c$ for which such an angle unit can be chosen, and mark the corresponding values of the angles a to ℓ in the diagram.

2. Let $m = (4^p - 1)/3$, where p is a prime number exceeding 3. Prove that 2^{m-1} has remainder 1 when divided by m.

3. Let P be an internal point of triangle ABC and let α, β, γ be defined by
$$\alpha = \angle BPC - \angle BAC,\ \beta = \angle CPA - \angle CBA,\ \gamma = \angle APB - \angle ACB.$$
   Prove that
$$PA\,\frac{\sin\angle BAC}{\sin\alpha} = PB\,\frac{\sin\angle CBA}{\sin\beta} = PC\,\frac{\sin\angle ACB}{\sin\gamma}.$$

4. The set $Z(m,n)$ consists of all integers N with mn digits which have precisely n ones, n twos, n threes, $\ldots$, n ms. For each integer $N \in Z(m,n)$, define $d(N)$ to be the sum of the absolute values of the differences of all pairs of consecutive digits. For example, $122313 \in Z(3,2)$ with $d(122313) = 1+0+1+2+2 = 6$. Find the average value of $d(N)$ as N ranges over all possible elements of $Z(m,n)$.

British Mathematical Olympiad

Round 1 : Wednesday 19th January 1994

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Starting with any three digit number n (such as $n = 625$) we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs, and the product of all three digits.
   (i) Find the value of $n/f(n)$ when $n = 625$. (The answer is an integer!)
   (ii) Find all three digit numbers such that the ratio $n/f(n)$=1.

2. In triangle ABC the point X lies on BC.
   (i) Suppose that $\angle BAC = 90°$, that X is the midpoint of BC, and that $\angle BAX$ is one third of $\angle BAC$. What can you say (and prove!) about triangle ACX?
   (ii) Suppose that $\angle BAC = 60°$, that X lies one third of the way from B to C, and that AX bisects $\angle BAC$. What can you say (and prove!) about triangle ACX?

3. The sequence of integers $u_0, u_1, u_2, u_3, \ldots$ satisfies $u_0 = 1$ and
$$u_{n+1}u_{n-1} = ku_n \quad \text{for each} \quad n \geq 1,$$
where k is some fixed positive integer. If $u_{2000} = 2000$, determine all possible values of k.

4. The points Q, R lie on the circle γ, and P is a point such that PQ, PR are tangents to γ. A is a point on the extension of PQ, and γ' is the circumcircle of triangle PAR. The circle γ' cuts γ again at B, and AR cuts γ at the point C. Prove that $\angle PAR = \angle ABC$.

5. An *increasing* sequence of integers is said to be **alternating** if it *starts* with an *odd* term, the second term is even, the third term is odd, the fourth is even, and so on. The empty sequence (with no term at all!) is considered to be alternating. Let $A(n)$ denote the number of alternating sequences which only involve integers from the set $\{1, 2, \ldots, n\}$. Show that $A(1) = 2$ and $A(2) = 3$. Find the value of $A(20)$, and prove that your value is correct.

British Mathematical Olympiad

Round 2 : Thursday, 24 February 1994

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than trying all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (on 7-10 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Hong Kong, 8-20 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for Hong Kong.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find the first integer $n > 1$ such that the average of
$$1^2, 2^2, 3^2, \ldots, n^2$$
is itself a perfect square.

2. How many different (i.e. pairwise non-congruent) triangles are there with integer sides and with perimeter 1994?

3. AP, AQ, AR, AS are chords of a given circle with the property that
$$\angle PAQ = \angle QAR = \angle RAS.$$
Prove that
$$AR(AP + AR) = AQ(AQ + AS).$$

4. How many perfect squares are there (mod 2^n)?

British Mathematical Olympiad

Round 1 : Wednesday 18th January 1995

Time allowed *Three and a half hours.*

Instructions

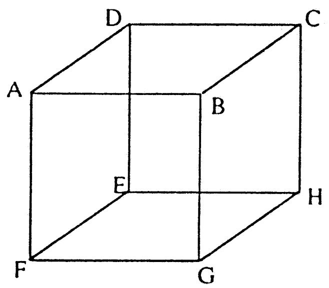
- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find the first positive integer whose square ends in three 4's.
   Find all positive integers whose squares end in three 4's.
   Show that no perfect square ends with four 4's.

2. $ABCDEFGH$ is a cube of side 2.
   (a) Find the area of the quadrilateral $AMHN$, where M is the midpoint of BC, and N is the midpoint of EF.
   (b) Let P be the midpoint of AB, and Q the midpoint of HE. Let AM meet CP at X, and HN meet FQ at Y. Find the length of XY.



3. (a) Find the maximum value of the expression $x^2y - y^2x$ when $0 \le x \le 1$ and $0 \le y \le 1$.
   (b) Find the maximum value of the expression
   $$x^2y + y^2z + z^2x - x^2z - y^2x - z^2y$$
   when $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

4. ABC is a triangle, right-angled at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q, respectively. M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

5. The seven dwarfs walk to work each morning in single file. As they go, they sing their famous song, "*High - low - high -low, it's off to work we go* ...". Each day they line up so that no three successive dwarfs are either increasing or decreasing in height. Thus, the line-up must go *up-down-up-down-* $\cdots$ or *down-up-down-up-* $\cdots$. If they all have different heights, for how many days they go to work like this if they insist on using a different order each day?
   What if Snow White always came along too?

British Mathematical Olympiad

Round 2 : Thursday, 16 February 1995

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (30 March – 2 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Toronto, Canada, 13–23 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session 2–6 July before leaving for Canada.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Find all triples of positive integers (a, b, c) such that
$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2.$$

2. Let ABC be a triangle, and D, E, F be the midpoints of BC, CA, AB, respectively.

   Prove that $\angle DAC = \angle ABE$ if, and only if, $\angle AFC = \angle ADB$.

3. Let a, b, c be real numbers satisfying $a < b < c$, $a + b + c = 6$ and $ab + bc + ca = 9$.

   Prove that $0 < a < 1 < b < 3 < c < 4$.

4. (a) Determine, with careful explanation, how many ways $2n$ people can be paired off to form n teams of 2.

   (b) Prove that $\{(mn)!\}^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$ for all positive integers m, n.

British Mathematical Olympiad

Round 1 : Wednesday, 17th January 1996

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
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British Mathematical Olympiad

1. Consider the pair of four-digit positive integers
$$(M, N) = (3600, 2500).$$
Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four-digit positive integers (M, N) with these properties.

2. A function f is defined over the set of all positive integers and satisfies
$$f(1) = 1996$$
and
$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \quad \text{for all } n > 1.$$
Calculate the exact value of $f(1996)$.

3. Let ABC be an acute-angled triangle, and let O be its circumcentre. The circle through A, O and B is called S. The lines CA and CB meet the circle S again at P and Q respectively. Prove that the lines CO and PQ are perpendicular.
(*Given any triangle XYZ, its* **circumcentre** *is the centre of the circle which passes through the three vertices X, Y and Z.*)

4. For any real number x, let $[x]$ denote the greatest integer which is less than or equal to x. Define
$$q(n) = \left[\frac{n}{[\sqrt{n}]}\right] \quad \text{for} \quad n = 1, 2, 3, \ldots .$$
Determine all positive integers n for which $q(n) > q(n+1)$.

5. Let a, b and c be positive real numbers.
(i) Prove that $4(a^3 + b^3) \geq (a + b)^3$.
(ii) Prove that $9(a^3 + b^3 + c^3) \geq (a + b + c)^3$.

British Mathematical Olympiad

Round 2 : Thursday, 15 February 1996

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (28–31 March). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in New Delhi, India, 7–17 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session 30 June–4 July before leaving for India.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Determine all sets of non-negative integers x, y and z which satisfy the equation
$$2^x + 3^y = z^2.$$

2. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations
$$u(v + w - u) = a^2,$$
$$v(w + u - v) = b^2,$$
$$w(u + v - w) = c^2.$$
Prove that triangle ABC is acute-angled and express the angles U, V, W in terms of A, B, C.

3. Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2 respectively. Let P be one point of intersection of S with the common tangent to S_1 and S_2 at K. The line PA_1 meets S_1 again at B_1, and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 and S_2.

4. Let a, b, c and d be positive real numbers such that
$$a + b + c + d = 12$$
and
$$abcd = 27 + ab + ac + ad + bc + bd + cd.$$
Find all possible values of a, b, c, d satisfying these equations.

British Mathematical Olympiad

Round 1 : Wednesday, 15 January 1997

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
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British Mathematical Olympiad

1. N is a four-digit integer, not ending in zero, and $R(N)$ is the four-digit integer obtained by reversing the digits of N; for example, $R(3275) = 5723$.
   Determine all such integers N for which $R(N) = 4N + 3$.

2. For positive integers n, the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is defined by
   $$a_1 = 1; \quad a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + a_3 + \cdots + a_{n-1}), \quad n > 1.$$
   Determine the value of a_{1997}.

3. The Dwarfs in the Land-under-the-Mountain have just adopted a completely decimal currency system based on the *Pippin*, with gold coins to the value of 1 *Pippin*, 10 *Pippins*, 100 *Pippins* and 1000 *Pippins*.
   In how many ways is it possible for a Dwarf to pay, in exact coinage, a bill of 1997 *Pippins*?

4. Let $ABCD$ be a convex quadrilateral. The midpoints of AB, BC, CD and DA are P, Q, R and S, respectively. Given that the quadrilateral $PQRS$ has area 1, prove that the area of the quadrilateral $ABCD$ is 2.

5. Let x, y and z be positive real numbers.
   (i) If $x + y + z \geq 3$, is it necessarily true that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \leq 3$?
   (ii) If $x + y + z \leq 3$, is it necessarily true that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 3$?

British Mathematical Olympiad

Round 2 : Thursday, 27 February 1997

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (10-13 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Mar del Plata, Argentina, 21-31 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session in late June or early July before leaving for Argentina.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Let M and N be two 9-digit positive integers with the property that if **any** one digit of M is replaced by the digit of N in the corresponding place (e.g., the 'tens' digit of M replaced by the 'tens' digit of N) then the resulting integer is a multiple of 7.

   Prove that any number obtained by replacing a digit of N by the corresponding digit of M is also a multiple of 7.

   Find an integer $d > 9$ such that the above result concerning divisibility by 7 remains true when M and N are two d-digit positive integers.

2. In the acute-angled triangle ABC, CF is an altitude, with F on AB, and BM is a median, with M on CA. Given that $BM = CF$ and $\angle MBC = \angle FCA$, prove that the triangle ABC is equilateral.

3. Find the number of polynomials of degree 5 with **distinct** coefficients from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are divisible by $x^2 - x + 1$.

4. The set $S = \{1/r : r = 1, 2, 3, \ldots\}$ of reciprocals of the positive integers contains arithmetic progressions of various lengths. For instance, $1/20, 1/8, 1/5$ is such a progression, of length 3 (and common difference $3/40$). Moreover, this is a *maximal progression* in S of length 3 since it cannot be extended to the left or right within S ($-1/40$ and $11/40$ not being members of S).

   (i) Find a maximal progression in S of length 1996.

   (ii) Is there a maximal progression in S of length 1997?

British Mathematical Olympiad

Round 1 : Wednesday, 14 January 1998

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*.
   Show that it is impossible for the score to be 20.
   What is the highest possible score?

2. Let $a_1 = 19$, $a_2 = 98$. For $n \geq 1$, define a_{n+2} to be the remainder of $a_n + a_{n+1}$ when it is divided by 100. What is the remainder when
   $$a_1^2 + a_2^2 + \cdots + a_{1998}^2$$
   is divided by 8?

3. ABP is an isosceles triangle with $AB = AP$ and $\angle PAB$ acute. PC is the line through P perpendicular to BP, and C is a point on this line on the same side of BP as A. (You may assume that C is not on the line AB.) D completes the parallelogram $ABCD$. PC meets DA at M.
   Prove that M is the midpoint of DA.

4. Show that there is a unique sequence of positive integers (a_n) satisfying the following conditions:
   $$a_1 = 1, \quad a_2 = 2, \quad a_4 = 12,$$
   $$a_{n+1}a_{n-1} = a_n^2 \pm 1 \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

5. In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.

British Mathematical Olympiad

Round 2 : Thursday, 26 February 1998

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (2-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Taiwan, 13-21 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session in early July before leaving for Taiwan.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. A booking office at a railway station sells tickets to 200 destinations. One day, tickets were issued to 3800 passengers. Show that

   (i) there are (at least) 6 destinations at which the passenger arrival numbers are the same;

   (ii) the statement in (i) becomes false if '6' is replaced by '7'.

2. A triangle ABC has $\angle BAC > \angle BCA$. A line AP is drawn so that $\angle PAC = \angle BCA$ where P is inside the triangle. A point Q outside the triangle is constructed so that PQ is parallel to AB, and BQ is parallel to AC. R is the point on BC (separated from Q by the line AP) such that $\angle PRQ = \angle BCA$.

   Prove that the circumcircle of ABC touches the circumcircle of PQR.

3. Suppose x, y, z are positive integers satisfying the equation
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{z},$$
   and let h be the highest common factor of x, y, z.

   Prove that $hxyz$ is a perfect square.

   Prove also that $h(y - x)$ is a perfect square.

4. Find a solution of the simultaneous equations
$$xy + yz + zx = 12$$
$$xyz = 2 + x + y + z$$
   in which all of x, y, z are positive, and prove that it is the only such solution.

   Show that a solution exists in which x, y, z are real and distinct.

British Mathematical Olympiad

Round 1 : Wednesday, 13 January 1999

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. I have four children. The age in years of each child is a positive integer between 2 and 16 inclusive and all four ages are distinct. A year ago the square of the age of the oldest child was equal to the sum of the squares of the ages of the other three. In one year's time the sum of the squares of the ages of the oldest and the youngest will be equal to the sum of the squares of the other two children.
Decide whether this information is sufficient to determine their ages uniquely, and find all possibilities for their ages.

2. A circle has diameter AB and X is a fixed point of AB lying between A and B. A point P, distinct from A and B, lies on the circumference of the circle. Prove that, for all possible positions of P,
$$\frac{\tan \angle APX}{\tan \angle PAX}$$
remains constant.

3. Determine a positive constant c such that the equation
$$xy^2 - y^2 - x + y = c$$
has precisely three solutions (x, y) in positive integers.

4. Any positive integer m can be written uniquely in base 3 form as a string of 0's, 1's and 2's (not beginning with a zero). For example,
$$98 = (1\times81) + (0\times27) + (1\times9) + (2\times3) + (2\times1) = (10122)_3.$$
Let $c(m)$ denote the sum of the cubes of the digits of the base 3 form of m; thus, for instance
$$c(98) = 1^3 + 0^3 + 1^3 + 2^3 + 2^3 = 18.$$
Let n be any fixed positive integer. Define the sequence (u_r) by
$$u_1 = n \quad \text{and} \quad u_r = c(u_{r-1}) \quad \text{for} \quad r \geq 2.$$
Show that there is a positive integer r for which $u_r = 1, 2$ or 17.

5. Consider all functions f from the positive integers to the positive integers such that
(i) for each positive integer m, there is a unique positive integer n such that $f(n) = m$;
(ii) for each positive integer n, we have
$f(n+1)$ is **either** $4f(n) - 1$ **or** $f(n) - 1$.
Find the set of positive integers p such that $f(1999) = p$ for some function f with properties (i) and (ii).

British Mathematical Olympiad

Round 2 : Thursday, 25 February 1999

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (8-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in Bucharest, Romania, 13-22 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session (3-7 July) in Birmingham before leaving for Bucharest.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. For each positive integer n, let S_n denote the set consisting of the first n natural numbers, that is
$$S_n = \{1, 2, 3, 4, \ldots, n-1, n\}.$$
   (i) For which values of n is it possible to express S_n as the union of two non-empty disjoint subsets so that the elements in the two subsets have equal sums?

   (ii) For which values of n is it possible to express S_n as the union of three non-empty disjoint subsets so that the elements in the three subsets have equal sums?

2. Let $ABCDEF$ be a hexagon (which may not be regular), which circumscribes a circle S. (That is, S is tangent to each of the six sides of the hexagon.) The circle S touches AB, CD, EF at their midpoints P, Q, R respectively. Let X, Y, Z be the points of contact of S with BC, DE, FA respectively. Prove that PY, QZ, RX are concurrent.

3. Non-negative real numbers p, q and r satisfy $p + q + r = 1$. Prove that
$$7(pq + qr + rp) \leq 2 + 9pqr.$$

4. Consider all numbers of the form $3n^2 + n + 1$, where n is a positive integer.

   (i) How small can the sum of the digits (in base 10) of such a number be?

   (ii) Can such a number have the sum of its digits (in base 10) equal to 1999?

British Mathematical Olympiad

Round 1 : Wednesday, 12 January 2000

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

2. Show that, for every positive integer n,
$$121^n - 25^n + 1900^n - (-4)^n$$
is divisible by 2000.

3. Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that
$$AP + BP + CP$$
is minimized.

4. For each positive integer $k > 1$, define the sequence $\{a_n\}$ by
$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n a_{n-1} \qquad \text{for each} \quad n \geq 1.$$
Determine all values of k for which 2000 is a term of the sequence.

5. The seven dwarfs decide to form four teams to compete in the Millennium Quiz. Of course, the sizes of the teams will not all be equal. For instance, one team might consist of Doc alone, one of Dopey alone, one of Sleepy, Happy & Grumpy, and one of Bashful & Sneezy. In how many ways can the four teams be made up? (The order of the teams or of the dwarfs within the teams does not matter, but each dwarf must be in exactly one of the teams.)
Suppose Snow-White agreed to take part as well. In how many ways could the four teams then be formed?

British Mathematical Olympiad

Round 2 : Wednesday, 23 February 2000

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (6-9 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in South Korea, 13-24 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session before leaving for South Korea.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. Prove that the triangles MNP and MNQ have equal areas.

2. Given that x, y, z are positive real numbers satisfying $xyz = 32$, find the minimum value of
$$x^2 + 4xy + 4y^2 + 2z^2.$$

3. Find positive integers a and b such that
$$(\sqrt[3]{a} + \sqrt[3]{b} - 1)^2 = 49 + 20\sqrt[3]{6}.$$

4. (a) Find a set A of ten positive integers such that no six distinct elements of A have a sum which is divisible by 6.

   (b) Is it possible to find such a set if "ten" is replaced by "eleven"?

British Mathematical Olympiad

Round 1 : Wednesday, 17 January 2001

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
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2001 British Mathematical Olympiad
Round 1

1. Find all two-digit integers N for which the sum of the digits of $10^N - N$ is divisible by 170.

2. Circle S lies inside circle T and touches it at A. From a point P (distinct from A) on T, chords PQ and PR of T are drawn touching S at X and Y respectively. Show that $\angle QAR = 2\angle XAY$.

3. A *tetromino* is a figure made up of four unit squares connected by common edges.
   (i) If we do not distinguish between the possible rotations of a tetromino within its plane, prove that there are seven distinct tetrominoes.
   (ii) Prove or disprove the statement: It is possible to pack all seven distinct tetrominoes into a 4×7 rectangle without overlapping.

4. Define the sequence (a_n) by
$$a_n = n + \{\sqrt{n}\},$$
where n is a positive integer and $\{x\}$ denotes the nearest integer to x, where halves are rounded up if necessary. Determine the smallest integer k for which the terms $a_k, a_{k+1}, \ldots, a_{k+2000}$ form a sequence of 2001 consecutive integers.

5. A triangle has sides of length a, b, c and its circumcircle has radius R. Prove that the triangle is right-angled if and only if $a^2 + b^2 + c^2 = 8R^2$.

British Mathematical Olympiad

Round 2 : Tuesday, 27 February 2001

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
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- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (8-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend another meeting in Cambridge (probably 26-29 May). The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Washington DC, USA, 3-14 July) will then be chosen.

Do not turn over until **told to do so.**

2001 British Mathematical Olympiad
Round 2

1. Ahmed and Beth have respectively p and q marbles, with $p > q$.

   Starting with Ahmed, each in turn gives to the other as many marbles as the other already possesses. It is found that after $2n$ such transfers, Ahmed has q marbles and Beth has p marbles.
   Find $\frac{p}{q}$ in terms of n.

2. Find all pairs of integers (x, y) satisfying
$$1 + x^2y = x^2 + 2xy + 2x + y.$$

3. A triangle ABC has $\angle ACB > \angle ABC$.

   The internal bisector of $\angle BAC$ meets BC at D.

   The point E on AB is such that $\angle EDB = 90°$.

   The point F on AC is such that $\angle BED = \angle DEF$.

   Show that $\angle BAD = \angle FDC$.

4. N dwarfs of heights $1, 2, 3, \ldots, N$ are arranged in a circle. For each pair of neighbouring dwarfs the positive difference between the heights is calculated; the sum of these N differences is called the "total variance" V of the arrangement.

   Find (with proof) the maximum and minimum possible values of V.

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British Mathematical Olympiad

Round 1 : Wednesday, 5 December 2001

Time allowed *Three and a half hours.*

Instructions

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- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
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2001 British Mathematical Olympiad
Round 1

1. Find all positive integers m, n, where n is odd, that satisfy
$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}.$$

2. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC, BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P.
Given that $CD = CP = DQ$, prove that $\angle CAD = 60°$.

3. Find all positive real solutions to the equation
$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor,$$
where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

4. Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross?
(Nobody is allowed to shake hands with more than one person at once.)

5. f is a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers, which has the following properties:-
a) $f(n+1) > f(n)$ for each $n \in \mathbb{Z}^+$,
b) $f(n + f(m)) = f(n) + m + 1$ for all $m, n \in \mathbb{Z}^+$.
Find all possible values of $f(2001)$.

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British Mathematical Olympiad

Round 2 : Tuesday, 26 February 2002

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

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  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (4 – 7 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend another meeting in Cambridge. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Glasgow, 22 –31 July) will then be chosen.

Do not turn over until **told to do so.**

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2002 British Mathematical Olympiad
Round 2

1. The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

2. A conference hall has a round table wth n chairs. There are n delegates to the conference. The first delegate chooses his or her seat arbitrarily. Thereafter the $(k+1)$th delegate sits k places to the right of the kth delegate, for $1 \le k \le n-1$. (In particular, the second delegate sits next to the first.) No chair can be occupied by more than one delegate.

   Find the set of values n for which this is possible.

3. Prove that the sequence defined by

   $$y_0 = 1, \qquad y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right), \quad (n \ge 0)$$

   consists only of integers.

4. Suppose that $B_1, \ldots, B_N$ are N spheres of unit radius arranged in space so that each sphere touches exactly two others externally. Let P be a point outside all these spheres, and let the N points of contact be $C_1, \ldots, C_N$. The length of the tangent from P to the sphere B_i $(1 \le i \le N)$ is denoted by t_i. Prove the product of the quantities t_i is not more than the product of the distances PC_i.

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British Mathematical Olympiad

Round 1 : Wednesday, 11 December 2002

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
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- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
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- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
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2002/3 British Mathematical Olympiad
Round 1

1. Given that

$$34! = 295\,232\,799\,cd9\,604\,140\,847\,618\,609\,643\,5ab\,000\,000,$$

determine the digits a, b, c, d.

2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.
Show that $BX = CX$ if and only if PQ is a diameter of S.

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$.
Prove that

$$x^2yz + xy^2z + xyz^2 \le \frac{1}{3}.$$

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid. Determine the greatest possible value of k.

5. Find all solutions in positive integers a, b, c to the equation

$$a!\,b! = a! + b! + c!$$

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British Mathematical Olympiad

Round 2 : Tuesday, 25 February 2003

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

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- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (3-6 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Japan, 7-19 July) will then be chosen.

Do not turn over until **told to do so.**

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2003 British Mathematical Olympiad
Round 2

1. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples x, y, z of distinct positive integers satisfying
   (i) x, y, z are in arithmetic progression, and
   (ii) $p(xyz) \leq 3$.

2. Let ABC be a triangle and let D be a point on AB such that $4AD = AB$. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of θ with DA where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line ℓ at P, show that $PB = 2PD$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.
   (i) Show that there is an arithmetic progression of positive integers $a, a+d, a+2d$, where $d > 0$, such that
   $$f(a) < f(a+d) < f(a+2d).$$
   (ii) Must there be an arithmetic progression $a, a+d, \ldots, a+2003d$, where $d > 0$, such that
   $$f(a) < f(a+d) < \ldots < f(a+2003d)?$$
   [*A permutation of* $\mathbb{N}$ *is a one-to-one function whose image is the whole of* $\mathbb{N}$*; that is, a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *there exists a unique* $n \in \mathbb{N}$ *such that* $f(n) = m$.]

4. Let f be a function from the set of non-negative integers into itself such that for all $n \geq 0$
   (i) $\left(f(2n+1)\right)^2 - \left(f(2n)\right)^2 = 6f(n) + 1$, and
   (ii) $f(2n) \geq f(n)$.
   How many numbers less than 2003 are there in the image of f?

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British Mathematical Olympiad

Round 1 : Wednesday, 3 December 2003

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
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- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
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2003/4 British Mathematical Olympiad
Round 1

1. Solve the simultaneous equations
$$ab + c + d = 3, \quad bc + d + a = 5, \quad cd + a + b = 2, \quad da + b + c = 6,$$
where a, b, c, d are real numbers.

2. $ABCD$ is a rectangle, P is the midpoint of AB, and Q is the point on PD such that CQ is perpendicular to PD.
Prove that the triangle BQC is isosceles.

3. Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game.
Prove that Alice can always obtain a score at least as great as Barbara's.

4. A set of positive integers is defined to be *wicked* if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set.
Find the number of wicked subsets of the set
$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

5. Let p, q and r be prime numbers. It is given that p divides $qr - 1$, q divides $rp - 1$, and r divides $pq - 1$.
Determine all possible values of pqr.

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British Mathematical Olympiad

Round 2 : Tuesday, 24 February 2004

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (1-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Athens, 9-18 July) will then be chosen.

Do not turn over until **told to do so.**

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2004 British Mathematical Olympiad
Round 2

1. Let ABC be an equilateral triangle and D an internal point of the side BC. A circle, tangent to BC at D, cuts AB internally at M and N, and AC internally at P and Q.

   Show that $BD + AM + AN = CD + AP + AQ$.

2. Show that there is an integer n with the following properties:

   (i) the binary expansion of n has precisely 2004 0s and 2004 1s;

   (ii) 2004 divides n.

3. (a) Given real numbers a, b, c, with $a + b + c = 0$, prove that
$$a^3 + b^3 + c^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 > 0.$$
   (b) Given real numbers a, b, c, d, with $a + b + c + d = 0$, prove that
$$a^3 + b^3 + c^3 + d^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 + d^5 > 0.$$

4. The real number x between 0 and 1 has decimal representation
$$0{\cdot}a_1a_2a_3a_4\ldots$$
   with the following property: the number of *distinct* blocks of the form
$$a_ka_{k+1}a_{k+2}\ldots a_{k+2003},$$
   as k ranges through all positive integers, is less than or equal to 2004.

   Prove that x is rational.

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British Mathematical Olympiad

Round 1 : Wednesday, 1 December 2004

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

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2004/5 British Mathematical Olympiad
Round 1

1. Each of Paul and Jenny has a whole number of pounds.
He says to her: "If you give me £3, I will have n times as much as you".
She says to him: "If you give me £n, I will have 3 times as much as you".
Given that all these statements are true and that n is a positive integer, what are the possible values for n?

2. Let ABC be an acute-angled triangle, and let D, E be the feet of the perpendiculars from A, B to BC, CA respectively. Let P be the point where the line AD meets the semicircle constructed outwardly on BC, and Q be the point where the line BE meets the semicircle constructed outwardly on AC. Prove that $CP = CQ$.

3. Determine the least natural number n for which the following result holds:
No matter how the elements of the set $\{1, 2, \ldots, n\}$ are coloured red or blue, there are integers x, y, z, w in the set (not necessarily distinct) of the same colour such that $x + y + z = w$.

4. Determine the least possible value of the largest term in an arithmetic progression of seven distinct primes.

5. Let S be a set of rational numbers with the following properties:
i) $\frac{1}{2} \in S$;
ii) If $x \in S$, then both $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.
Prove that S contains all rational numbers in the interval $0 < x < 1$.

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British Mathematical Olympiad

Round 2 : Tuesday, 1 February 2005

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
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In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (7-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Merida, Mexico, 8 - 19 July) will then be chosen.

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2005 British Mathematical Olympiad
Round 2

1. The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$
Prove that N is a perfect square.

2. In triangle ABC, $\angle BAC = 120°$. Let the angle bisectors of angles A, B and C meet the opposite sides in D, E and F respectively.

   Prove that the circle on diameter EF passes through D.

3. Let a, b, c be positive real numbers. Prove that
$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 \geq (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

4. Let $X = \{A_1, A_2, \ldots, A_n\}$ be a set of distinct 3-element subsets of $\{1, 2, \ldots, 36\}$ such that

   i) A_i and A_j have non-empty intersection for every i, j.

   ii) The intersection of all the elements of X is the empty set.

   Show that $n \leq 100$. How many such sets X are there when $n = 100$?