1st USAMO 1972

Problem 1
Let (a, b, ... , k) denote the greatest common divisor of the integers a, b, ... k and [a, b, ... , k] denote their least common multiple. Show that for any positive integers a, b, c we have $(a, b, c)^2 [a, b] [b, c] [c, a] = [a, b, c]^2 (a, b) (b, c) (c, a)$.

Solution
If we express a, b, c as a product of primes then the gcd has each prime to the smallest power and the lcm has each prime to the largest power. So the equation given is equivalent to showing that 2 min(r, s, t) + max(r, s) + max(s, t) + max(t, r) = 2 max(r, s, t) + min(r, s) + min(s, t) + min(t, r) for non-negative integers r, s, t. Assume $r \le s \le t$. Then each side is 2r + s + 2t.

Problem 2
A tetrahedron has opposite sides equal. Show that all faces are acute-angled.

Solution
Let the tetrahedron be ABCD. Let M be the midpoint of BC. We have AM + MD > AD (*). Now the triangles ABC and DCB are congruent because AB = DC, BC = CB and AC = DB. Hence AM = DM. Also AD = BC = 2MC. So (*) implies that AM > MC. But that implies that angle BAC is acute. Similarly for all the other angles.

Problem 3
n digits, none of them 0, are randomly (and independently) generated, find the probability that their product is divisible by 10.

Solution
Answer: $1 - (8/9)^n - (5/9)^n + (4/9)^n$.

A number is divisible by 10 iff it has an even number and a 5 amongst its digits. The probability of no 5 is $(8/9)^n$. The probability of no even number is $(5/9)^n$. The probability of no 5 and no even number is $(4/9)^n$. Hence result.

Problem 4
Let k be the real cube root of 2. Find integers A, B, C, a, b, c such that $| (Ax^2 + Bx + C)/(ax^2 + bx + c) - k | < | x - k |$ for all non-negative rational x.

Solution
Taking the limit, we must have $(Ak^2+ Bk + C) = k(ak^2 + bk + c)$, so A = b, B = c, C = 2a. Now notice that $(bx^2 + cx + 2a) - k(ax^2 + bx + c) = (b - ak)x^2 + (c - bk)x + (2a - ck) = (x - k)((b - ak)x + c - ak^2)$. So we require $| (b - ak)x + c - ak^2| < | ax^2 + bx + c |$ for all $x \ge 0$.

There are many ways to satisfy this. For example, take a = 1, b = c = 2. Then (b - ak)x is always positive and less than bx for positive x, and $c - ak^2$ is positive and less than c.

Problem 5
A pentagon is such that each triangle formed by three adjacent vertices has area 1. Find its area, but show that there are infinitely many incongruent pentagons with this property.

Solution

Let the pentagon be ABCDE. Triangles BCD and ECD have the same area, so B and E are the same perpendicular distance from CD, so BE is parallel to CD. The same applies to the other diagonals (each is parallel to the side with which it has no endpoints in common). Let BD and CE meet at X. Then ABXE is a parallelogram, so area BXE = area EAB = 1. Also area CDX + area EDX = area CDX + area BCX = 1. Put area EDX = x. Then DX/XB = area EDX/area BXE = x/1 and also = area CDX/area BCX = (1-x)/x. So $x^2 + x - 1 = 0$, $x = \sqrt{5} - 1)/2$ (we know $x < 0$, so it cannot be the other root). Hence area ABCDE = 3 + x = $(\sqrt{5} + 5)/2$.

Take any triangle XCD of area $(3 - \sqrt{5})/2$ and extend DX to B, so that BCD has area 1, and extend CX to E so that CDE has area 1. Then take BA parallel to CE and EA parallel to BD. It is easy to check that the pentagon has the required property.

2nd USAMO 1973

Problem 1
Show that if two points lie inside a regular tetrahedron the angle they subtend at a vertex is less than $\pi/3$.

Solution
Let the tetrahedron be ABCD and the points be P and Q. Note that we are asked to prove the result for *any* vertex, not just some. So consider angle PAQ. Let the rays AP, AQ meet the plane BCD at P', Q' respectively. So we have to show that angle P'AQ' < 60^o for P' and Q' interior points of the triangle BCD. Extend P'Q' to meet the sides of the triangle at X and Y. Without loss of generality, X lies BC and Y lies on CD. Obviously if is sufficient to show that angle XAY < 60^o.

X and Y cannot both be vertices (or P', Q' would not have been interior points of the triangle and hence P and Q would not have been strictly *inside* the tetrahedron). So suppose X is not a vertex. We show that XY <= XD. Consider triangle XYD. $\angle XDY < \angle BDC = 60^o$, but $\angle XYD = \angle XCD + \angle CXY \geq 60^o$, so $\angle XDY < \angle XYD$. Hence XY $\leq$ XD. But XD = AX (consider, for example, the congruent triangles AXB and DXB). Hence XY < AX. Similarly, XY $\leq$ AY. Hence angle XAY < 60^o.

Problem 2
The sequence a_n is defined by $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + 2a_n$. The sequence b_n is defined by $b_1 = 1$, $b_2 = 7$, $b_{n+2} = 2b_{n+1} + 3b_n$. Show that the only integer in both sequences is 1.

Solution
We can solve the first recurrence relation to give $a_n = A (-1)^n + B 2^n$. Using a_1 and a_2, we get $a_n = (2^n + (-1)^{n+1})/3$. Similarly, for the second recurrence relation we get $b_n = 2.3^{n-1} + (-1)^n$. So if $a_m = b_n$ then $2.3^n + 3 (-1)^n = 2^m + (-1)^{m+1}$ or $2^m = 2.3^n + 3 (-1)^n + (-1)^m$.

If m = 1 or 2, then we find n = 1 is the only solution, corresponding to the fact that the term 1 is in both sequences. If m > 2, then $2^m = 0 \bmod 8$. But $3^n = (-1)^n \bmod 4$, so $2.3^n + 3 (-1)^n + (-1)^m = 5 (-1)^n + (-1)^m \bmod 8$ which cannot be 0 mod 8. So there are no solutions for m > 2.

Problem 3
Three vertices of a regular 2n+1 sided polygon are chosen at random. Find the probability that the center of the polygon lies inside the resulting triangle.

Solution
Answer: (n+1)/(4n-2).

Label the first vertex picked as 1 and the others as 2, 3, ... , 2n+1 (in order). There are 2n(2n-1)/2 ways of choosing the next two vertices. If the second vertex is 2 (or 2n+1), then there is just one way of picking the third vertex so that the center lies in the triangle (vertex n+2). If the second vertex is 3 (or 2n), then there are two (n+2, n+3) and so on. So the total number of favourable triangles is 2(1 + 2 + ... + n) = n(n+1). Thus the required probability is (n+1)/(4n-2).

Problem 4
Find all complex numbers x, y, z which satisfy $x + y + z = x^2 + y^2 + z^2 = x^3 + y^3 + z^3 = 3$.

Solution
Answer: 1, 1, 1.
We have $(x + y + z)^2 = (x^2 + y^2 + z^2) + 2(xy + yz + zx)$, so $xy + yz + zx = 3$.

$(x + y + z)^3 = (x^3 + y^3 + z^3) + 3(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 6xyz$, so $8 = (x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 2xyz$. But $(x + y + z)(xy + yz + zx) = (x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 3xyz$, so $xyz = -1$. Hence x, y, z are the roots of the cubic $w^3 - 3w^2 + 3w - 1 = (w - 1)^3$. Hence $x = y = z = 1$.

Problem 5
Show that the cube roots of three distinct primes cannot be terms in an arithmetic progression (whether consecutive or not).

Solution
Suppose the primes are p, q, r so that $q^{1/3} = p^{1/3} + md$, $r^{1/3} = p^{1/3} + nd$, where m and n (but not necessarily d) are integers. Then $nq^{1/3} - mr^{1/3} = (n - m)p^{1/3}$. Cubing: $n^3q - 3n^2mq^{2/3}r^{1/3} + 3nm^2q^{1/3}r^{2/3} - m^3r = (n-m)^3p$, or $q^{1/3}r^{1/3}(mr^{1/3} - nq^{1/3}) = ((n-m)^3p + m^3r - n^3q)/(3mn)$. But $mr^{1/3} - nq^{1/3} = (m - n)p^{1/3}$, so we have $(pqr)^{1/3} = ((n-m)^3p + m^3r - n^3q)/(3mn(m-n))$ (*).

It is now clear that we do not need p, q, r prime, just that pqr is not a cube, for then by the usual argument it must be irrational so that (*) is impossible.

3rd USAMO 1974

Problem 1
p(x) is a polynomial with integral coefficients. Show that there are no solutions to the equations p(a) = b, p(b) = c, p(c) = a, with a, b, c distinct integers.

Solution
Suppose there a, b, c satisfy the equations. Then p(x) = (x - a)q(x) + b = (x - b)r(x) + c = (x - c)s(x) + a for some polynomials q(x), r(x), s(x) with integer coefficients. Hence (b - a)q(b) + b = p(b) = c, so (b - a) divides (c - b). Similarly, (c - b) divides (a - c), and (a - c) divides (b - a). But (b - a) divides (c - b) divides (a - c) implies that (b - a) divides (a - c). So we have (b - a) and (a - c) dividing each other. Hence (b - a) = ±(a - c).

If b - a = a - c, then b - c = (b - a) + (a - c) = 2(a - c). But (b - a) divides 2(a - c) (and both are non-zero since a, b, c are distinct), so that is impossible. If b - a = -(a - c), the b = c, contradicting the fact that they are distinct. So there are no solutions.

Problem 2
Show that for any positive reals x, y, z we have $x^xy^yz^z \ge (xyz)^a$, where a is the arithmetic mean of x, y, z.

Solution
Without loss of generality $x \ge y \ge z$. We have $x^xy^y \ge x^yy^x$, because that is equivalent to $(x/y)^x \ge (x/y)^y$ which is obviously true. Similarly $y^yz^z \ge y^zz^y$ and $z^zx^x \ge z^xx^z$. Multiplying these three together we get $(x^xy^yz^z)^x \ge x^{y+z}y^{z+x}z^{x+y}$. Multiplying both sides by $x^xy^yz^z$ gives $(x^xy^yz^z)^3 \ge (xyz)^{3a}$. Taking cube roots gives the required result.

Problem 3
Two points in a thin spherical shell are joined by a curve shorter than the diameter of the shell. Show that the curve lies entirely in one hemisphere.

Solution
Suppose the shell has diameter 2. Let M be the midpoint of the curve. Let O be the center of the shell and X the midpoint of MO. Let S be the circle center X radius √3)/2 in the plane normal to OM. Then S lies in the shell and every point of S is a distance (in space) of 1 from M. Hence the curve cannot cross S (because if it crossed at Y, we would have SY ≥ 1 along the curve, but the curve has length < 2, so SY < 1. So we have a stronger result than required.

Problem 4
A, B, C play a series of games. Each game is between two players, The next game is between the winner and the person who was not playing. The series continues until one player has won two games. He wins the series. A is the weakest player, C the strongest. Each player has a fixed probability of winning against a given opponent. A chooses who plays the first game. Show that he should choose to play himself against B.

Solution
It must be wrong to choose B against C, for then after the first game (whatever its outcome) A would be playing one of the other players (X), and that player would already have won a game. That is a worse position than playing that person as the first game, because if he loses the game then X has won the series, whereas if he lost to X on the first game, there is still a chance A could win the series. [If he wins the game as the second game, then he is certainly no better off than he would be after winning the match as the first game.]

Use XbY to denote that X beats Y. If A chooses to play B in the first game, then he wins the series if either (1) AbB, AbC, (2) AbB, CbA, BbC, AbB, or (3) BbA, CbB, AbC, AbB. If A plays C in the first game, then he wins the series if (1') AbC, AbB, (2') AbC, BbA, CbB, AbC, (3') CbA, BbC, AbB, AbC. Evidently the probability of (1) is the same as the probability of (1'). If we compare (2) and (3') they are the same except that in (2) A must beat B and in (3') A must beat C. Similarly, if we compare (3) and (2') they are the same except that in (3) A must beat B and in (2') A must beat C. We assume that, since C is a stronger player than B, A is more likely to beat B than C. Hence prob(2) > prob(3') and prob(3) > prob(2'). Thus A should choose to play B in the first game.

Problem 5
A point inside an equilateral triangle with side 1 is a distance a, b, c from the vertices. The triangle ABC has BC = a, CA = b, AB = c. The sides subtend equal angles at a point inside it. Show that sum of the distances of the point from the vertices is 1.

Solution
Let D be the point inside ABC, so that $\angle$ADB = $\angle$BDC = 120°. The key is to start from ABC and to rotate the triangle BDC through 60° away from the triangle ADB. After that everything is routine.

Suppose D goes to D' and C to C'. Then BD = BD' and $\angle$DBD' = 60°, so BDD' is equilateral. Hence $\angle$D'DB = 60°. $\angle$BDA = 120°, so ADD' is a straight line. Also $\angle$DD'B = 60° and $\angle$C'D'B = 120°, so DD'C' is a straight line. Thus AC' has length DA + DB + DC.

Note that BC = BC' and $\angle$CBC' = 60°, so CBC' is equilateral. Hence $\angle$CC'B = 60°. Now take Y such that AC'Y is equilateral, Y is on the opposite side of AC' to C. Then $\angle$BC'Y = 60° - $\angle$AC'B = $\angle$CC'A. Also BC' = CC' and YC' = AC', so triangles BC'Y and CC'A are congruent. Hence BY = CA = b. Also BC' = BC = a and BA = c. Thus B is a point inside an equilateral triangle and distances a, b, c from the vertices. Hence the triangle must have side 1. So DA + DB + DC = AC' = 1.

4th USAMO 1975

Problem 1
Show that for any non-negative reals x, y, [5x] + [5y] ≥ [3x+y] + [x+3y]. Hence or otherwise show that (5a)! (5b)!/(a! b! (3a+b)! (a+3b)!) is integral for any positive integers a, b.

Solution
If is obviously sufficient to prove that [5x] + [5y] ≥ [3x+y] + [x+3y] for 0 < x < y < 1. If 2x ≥ y, then [5x] ≥ [3x+y] and [5y] ≥ [x+3y], so the result holds. So assume 2x < y. It is now a question of examining a lot of cases.

If y < 2/5, then 3x + y < 5y/2 < 1, so [3x+y] = 0, and [3y+x] <= [5y], so the result holds. If 2/5 ≤ y < 3/5 and x < 1/5, then [5y] = 2, [3x+y] = 0 or 1 and [x+3y] = 1. If 2/5 ≤ y < 3/5 and x ≥ 1/5, the [5x]+[5y] = 3, [3x+y]

$= 1$, $[x+3y] = 1$ or 2. If $3/5 \le y < 4/5$, then $[5y] = 3$, $[3x+y] = 0$ or 1, $[x+3y] = 2$ or 3. If $y \ge 4/5$ and $x < 1/5$, then $[5x] + [5y] = 4$, $[3x+y] = 0$ or 1, $[x+3y]= 2$ or 3. If $y \ge 4/5$ and $1/5 \le x$, then $[5x] + [5y] = 5$, $[3x+y] = 2$, $[x+3y] = 2$ or 3.

The highest power of a prime p dividing n! is $[n/p] + [n/p^2] + [n/p^3] + \dots$. Thus it is sufficient to show that $[5m/p] + [5n/p] \ge [(3m+n)/p] + [(m+3n)/p]$, which follows immediately from the previous result, putting $x = m/p$, $y = n/p$.

Problem 2
Show that for any tetrahedron the sum of the squares of the lengths of two opposite edges is at most the sum of the squares of the other four.

Solution
Let the vertices be A, B, C, D. We will show that $AB^2 + CD^2 \le AC^2 + AD^2 + BC^2 + BD^2$. Use vectors with origin A. Let the vectors from A to B, C, D be **B**, **C**, **D** respectively. We have to show that $\mathbf{B}^2 + (\mathbf{C} - \mathbf{D})^2 \le \mathbf{C}^2 + \mathbf{D}^2 + (\mathbf{B} - \mathbf{C})^2 + (\mathbf{B} - \mathbf{D})^2$. Rearranging, this is equivalent to $(\mathbf{B} - \mathbf{C} - \mathbf{D})^2 \ge 0$.

Problem 3
A polynomial p(x) of degree n satisfies $p(0) = 0$, $p(1) = 1/2$, $p(2) = 2/3$, ... , $p(n) = n/(n+1)$. Find $p(n+1)$.

Solution
The polynomial $(x + 1)p(x) - x$ has degree n+1 and zeros at 0, 1, 2, ... , n, so it must be $k\, x (x - 1)(x - 2) \dots (x - n)$. Also it has value 1 at $x = -1$, so $k (n+1)! (-1)^{n+1}$. Hence $(n+2)\, p(n+1) - (n+1) = (-1)^{n+1}$. So for n odd, $p(n+1) = 1$, and for n even, $p(n+1) = n/(n+2)$.

Problem 4
Two circles intersect at two points, one of them X. Find Y on one circle and Z on the other, so that X, Y and Z are collinear and XY.XZ is as large as possible.

Solution
Let the circle through XY have center O and the other circle have center O'. XY = 2 OX sin XOY, XZ = 2 O'X sin XOZ, so we wish to maximise 2 sin XOY sin XO'Z. But $1/2 \angle XOY = 90^o - \angle OXY$, $1/2 \angle XO'Z = 90^o - \angle OXZ$, so $1/2 \angle XOY + 1/2 \angle XO'Z = \angle OXO'$, which is fixed. Thus $\angle XOY + \angle XO'Z$ is fixed. But 2 sin XOY sin XO'Z = cos(XOY-XO'Z) - cos(XOY+XO'Z), so we maximise by taking XOY = XO'Z.

Note that in this case $\angle OYZ = \angle O'ZY$, so if we extend YO and ZO' to meet at C, then CY = CZ and hence C is the center of a circle containing the two circles and touching them at Y and Z. Also $\angle CZY = \angle OXY$, so O'Z is parallel to OX. Similarly OY is parallel to O'X, which shows how to construct Y and Z.

Problem 5
A pack of n cards, including three aces, is well shuffled. Cards are turned over in turn. Show that the expected number of cards that must be turned over to reach the second ace is $(n+1)/2$.

Solution
For each arrangement A of the cards, let A' be the reflection about the middle of the pack, so that if a card is in position m in A, then it is in position (n+1-m) in A'. Then all possible arrangements can be grouped into pairs (A, A') (note that A cannot equal A'). If the position of the second ace in A is m, then it is n+1-m in A', so the average over A and A' is $(n+1)/2$. Hence that is also the average over all the arrangements.

5th USAMO 1976

Problem 1
The squares of a 4 x 7 chess board are colored red or blue. Show that however the coloring is done, we can find a rectangle with four distinct corner squares all the same color. Find a counter-example to show that this is not true for a 4 x 6 board.

Solution
A counter-example for 4 x 6 is:

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R  B  R  B  R  B
R  B  B  R  B  R
B  R  R  B  B  R
B  R  B  R  R  B
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Every column has two blue and two red squares and no two columns have the red squares in the same two rows or the blue squares in the same two rows, so there can be no rectangles.
Suppose there is a counter-example for a 4 x 7 rectangle. Suppose it has three red squares in the first column. Then in those rows each remaining column can have at most one red square, so four remaining columns each have at least two blue squares in those three columns. Hence two of those columns have blue squares in the same two rows and hence a blue cornered rectangle. Contradiction. Similarly if there are three blue squares in the first column. So each column must have two red and two blue squares. But there are only 6 ways of choosing 2 items from 4, so two columns must have red squares in the same rows. Contradiction.

Problem 2
AB is a fixed chord of a circle, not a diameter. CD is a variable diameter. Find the locus of the intersection of AC and BD.

Solution
Let the lines meet at X and suppose X lies outside the circle. ∠AXB = ∠AXD (same angle) = 180 deg - ∠XAD - ∠XDA = 90^o - ∠XDA (CD is a diameter, so angle CAD = 90^o) = 90^o - ∠BDA (same angle). But ∠BDA is constant, so ∠AXB is constant and hence X lies on a circle through A and B.

Let O be the center of the circle ABC and O' the center of the circle ABX. We have ∠AOB = 2 ∠ADB, ∠AO'B = 2 ∠AXB, so ∠AOB + ∠AO'B = 180^o. Hence ∠OAO' + ∠OBO' = 180^o. But ∠OAO' = ∠OBO', so ∠OAO' = 90^o. In other words, the circles are orthogonal.
If X lies inside the circle center O, then ∠AXB = ∠XAD + ∠XDA = ∠CAD + ∠ADB (same angles) = 90^o + ∠ADB (CD diameter). So X lies on the same circle.

Conversely, suppose X lies on the circle O'. Extend XA, XB to meet the circle center O at C and D respectively. If X lies outside the circle center O, assume C does not lie inside the circle center O' (if not use D). Then ∠CAD = ∠AXD + ∠ADX = ∠AXB + ∠ADB (same angles) = (∠AO'B + ∠AOB)/2 = 90^o. Hence CD is a diameter.
If X lies inside the circle O', then ∠ABX = 90^o + ∠ADB. But ∠AXB = ∠ADB + ∠XBD. So ∠CBD = ∠XBD = 90^o. Hence CD is a diameter.

Problem 3
Find all integral solutions to $a^2 + b^2 + c^2 = a^2b^2$.

Solution
Answer: 0, 0, 0.
Squares must be 0 or 1 mod 4. Since the rhs is a square, each of the squares on the lhs must be 0 mod 4. So a, b, c are even. Put $a = 2a_1$, $b = 2b_1$, $c = 2c_1$. Then $a_1^2 + b_1^2 + c_1^2$ = square. Repeating, we find that a, b, c must each be divisible by an arbitrarily large power of 2. So they must all be zero.

Problem 4
A tetrahedron ABCD has edges of total length 1. The angles at A (BAC etc) are all 90^o. Find the maximum volume of the tetrahedron.

Solution
Answer: $(5\sqrt{2} - 7)/162$.

Let the edges at A have lengths x, y, z. Then the volume is xyz/6 and the perimeter is $x + y + z + \sqrt{x^2 + y^2} + \sqrt{y^2 + z^2} + \sqrt{z^2 + x^2} = 1$. By AM/GM we have $x + y + z \ge 3k$, where $k = (xyz)^{1/3}$. Also $x^2 + y^2 \ge 2xy$, so $\sqrt{x^2 + y^2} + \sqrt{y^2 + z^2} + \sqrt{z^2 + x^2} \ge \sqrt{2xy} + \sqrt{2yx} + \sqrt{2zx}$. By AM/GM that is $\ge 3\sqrt{2}\,k$. So we have $1 \ge 3(1 + \sqrt{2})\,k$. Hence $k^3 \le (5\sqrt{2} - 7)/27$. Hence the volume is at most $(5\sqrt{2} - 7)/162$. This is achieved if $x = y = z = (\sqrt{2} - 1)/3$. Then the other three sides are $(2 - \sqrt{2})/3$ and the perimeter is 1.

Problem 5
The polynomials a(x), b(x), c(x), d(x) satisfy $a(x^5) + x\,b(x^5) + x^2c(x^5) = (1 + x + x^2 + x^3 + x^4)\,d(x)$. Show that a(x) has the factor (x -1).

Solution
Take k to be a complex 5th root of 1, so that $1 + k + k^2 + k^3 + k^4 = 0$. Putting $x = k, k^2, k^3, k^4$ in the given equation we get:

$$a(1) + k\,b(1) + k^2c(1) = 0$$
$$a(1) + k^2b(1) + k^4c(1) = 0$$
$$a(1) + k^3b(1) + k\,c(1) = 0$$
$$a(1) + k^4b(1) + k^3c(1) = 0$$

Multiplying by $-k$, $-k^2$, $-k^3$, $-k^4$ respectively, we get
$$-k\,a(1) - k^2b(1) - k^3c(1) = 0$$
$$-k^2a(1) - k^4b(1) - k\,c(1) = 0$$
$$-k^3a(1) - k\,b(1) - k^4c(1) = 0$$
$$-k^4a(1) - k^3b(1) - k^2c(1) = 0$$

Adding all eight equations gives 5 a(1) = 0. Hence a(x) has the root x = 1 and hence the factor (x - 1).

6th USAMO 1977

Problem 1
For which positive integers a, b does $(x^a + ... + x + 1)$ divide $(x^{ab} + x^{ab-b} + ... + x^{2b} + x^b + 1)$?

Solution
Answer: a+1 and b relatively prime.

The question is when $(x^{a+1} - 1)/(x - 1)$ divides $(x^{b(a+1)} - 1)/(x^b - 1)$ or when $(x^{a+1} - 1)(x^b - 1)$ divides $(x^{b(a+1)} - 1)(x - 1)$. Now both $(x^{a+1} - 1)$ and $(x^b - 1)$ divide $(x^{b(a+1)} - 1)$. They both have a factor (x - 1), so if that is their *only* common factor, then their product divides $(x^{b(a+1)} - 1)(x - 1)$. That is true if a+1 and b are relatively prime, for the roots of $x^k - 1$ are the kth roots of 1. Thus if a+1 and b are relatively prime, then the only (complex) number which is an (a+1)th root of 1 and a bth root of 1 is 1.

But suppose d is a common factor of a+1 and b, then $\exp(2\pi i/d)$ is a root of both $x^{a+1} - 1$ and $x^b - 1$. It is a root of $x^{b(a+1)} - 1$, but only with multiplicity 1, so $(x^{a+1} - 1)(x^b - 1)$ does not divide $(x^{b(a+1)} - 1)(x - 1)$.

Problem 2
The triangles ABC and DEF have AD, BE and CF parallel. Show that [AEF] + [DBF] + [DEC] + [DBC] + [AEC] + [ABF] = 3 [ABC] + 3 [DEF], where [XYZ] denotes the *signed* area of the triangle XYZ. Thus [XYZ] is + area XYZ if the order X, Y, Z is anti-clockwise and - area XYZ if the order X, Y, Z is clockwise. So, in particular, [XYZ] = [YZX] = -[YXZ].

Solution
The starting point is that [ABC] = [XBC] + [AXC] + [ABX] (*) for any point X. If X is inside the triangle, then all the rotations have the same sense. If X is outside, then they do not. But it is easy to check that (*) always holds.

So [ABC] = [DBC] + [ADC] + [ABD], [DEF] = [AEF] + [DAF] + [DEA]. Now, ignoring sign, the triangles ABD and DEA have equal area, because they have a common base AD and the same height (since AD is parallel to BE). But the sign is opposite, so [ABD] + [DEA] = 0. Similarly, [ADC] + [DAF] = 0, so [ABC] + [DEF] = [DBC] + [AEF]. Adding the two similar equations (obtained from E with [ABC] and B with [DEF], and from F with [ABC] and C with [DEF]) gives the required result.

Problem 3
Prove that the product of the two real roots of $x^4 + x^3 - 1 = 0$ is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

Solution
Let the roots of the quartic be a, b, c, d. We show that ab, ac, ad, bc, bd, cd are the roots of the sextic. We have $a + b + c + d = -1$, $ab + ac + ad + bc + bd + cd = 0$, $1/a + 1/b + 1/c + 1/d = 0$, $abcd = -1$.

Let $x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ be the sextic with roots ab, ac, ad, bc, bd, cd. Since their sum is zero, we have $a_5 = 0$. Their product is $(abcd)^3$, so $a_0 = -1$.

Problem 4
ABCD is a tetrahedron. The midpoint of AB is M and the midpoint of CD is N. Show that MN is perpendicular to AB and CD iff AC = BD and AD = BC.

Solution
Use vectors. Take any origin O and write the vector OX as **X**. Then MN perpendicular to AB and CD is equivalent to:
$(\mathbf{A} + \mathbf{B} - \mathbf{C} - \mathbf{D}).(\mathbf{A} - \mathbf{B}) = 0$ and
$(\mathbf{A} + \mathbf{B} - \mathbf{C} - \mathbf{D}).(\mathbf{C} - \mathbf{D}) = 0$.
Expanding and adding the two equations gives $(\mathbf{A} - \mathbf{D})^2 = (\mathbf{B} - \mathbf{C})^2$ or AD = BC. Subtracting gives AC = BD.

Conversely, AD = BC and AC = BD gives:
$(\mathbf{A} - \mathbf{D})^2 = (\mathbf{B} - \mathbf{C})^2$ and
$(\mathbf{A} - \mathbf{C})^2 = (\mathbf{B} - \mathbf{D})^2$.
Adding gives $(\mathbf{A} + \mathbf{B} - \mathbf{C} - \mathbf{D}).(\mathbf{A} - \mathbf{B}) = 0$, so MN is perpendicular to AB. Subtracting gives MN perpendicular to CD.

Problem 5
The positive reals v, w, x, y, z satisfy $0 < h \le v, w, x, y, z \le k$. Show that $(v + w + x + y + z)(1/v + 1/w + 1/x + 1/y + 1/z) \le 25 + 6(\sqrt{(h/k)} - \sqrt{(k/h)})^2$. When do we have equality?

Solution
Fix four of the variables and allow the other to vary. Suppose, for example, we fix all but x. Then the expression on the lhs has the form $(r + x)(s + 1/x) = (rs + 1) + sx + r/x$, where r and s are fixed. But this is convex. That is to say, as x increases if first decreases, then increases. So its maximum must occur at x = h or x = k. This is true for each variable.

Suppose all five are h or all five are k, then the lhs is 25, so the inequality is true and strict unless h = k. If four are h and one is k, then the lhs is $17 + 4(h/k + k/h)$. Similarly if four are k and one is h. If three are h and two are k, then the lhs is $13 + 6(h/k + k/h)$. Similarly if three are k and two are h.

$h/k + k/h \ge 2$ with equality iff h = k, so if h < k, then three of one and two of the other gives a larger lhs than four of one and one of the other. Finally, we note that the rhs is in fact $13 + 6(h/k + k/h)$, so the inequality is true with equality iff either (1) h = k or (2) three of v, w, x, y z are h and two are k or vice versa.

7th USAMO 1978

Problem 1
The sum of 5 real numbers is 8 and the sum of their squares is 16. What is the largest possible value for one of the numbers?

Solution
Answer: 16/5.
Let the numbers be v, w, x, y, z. We have $(v - 6/5)^2 + (w - 6/5)^2 + ... + (z - 6/5)^2 = (v^2 + ... + z^2) - 12/5 (v + ... + z) + 36/5 = 16 - 96/5 + 36/5 = 4$. Hence $|v - 6/5| \le 2$, so $v \le 16/5$. This value can be realized by putting v = 16/5 and setting the other numbers to 6/5.

Problem 2
Two square maps cover exactly the same area of terrain on different scales. The smaller map is placed on top of the larger map and inside its borders. Show that there is a unique point on the top map which lies exactly above the corresponding point on the lower map. How can this point be constructed?

Solution
The point is obviously unique, because the two maps have different scales (but if P and Q where two fixed points the distance between them would be the same on both maps).

Let the small map square be A'B'C'D' and the large be ABCD, where X and X' are corresponding points. We deal first with the special case where A'B' is parallel to AB. In this case let AA' and BB' meet at O. Then triangles OAB and OA'B' are similar, so O must represent the same point. So assume A'B' is not parallel to AB.

Let the lines A'B' and AB meet at W, the lines B'C' and BC meet at X, the lines C'D' and CD meet at Y, and the lines D'A' and DA meet at Z. We claim that the segments WY and XZ meet at a point O inside the smaller square. W cannot lie between A' and B' (or one of the vertices A', B' of the smaller square would lie outside the larger square). If it lies on the opposite side of A' to B', then Y must lie on the opposite side of C' to D'. Thus the segment WY must cut the side A'D' at some point Z' and the side B'C' at some point X'. The same conclusion holds if W lies on the opposite side of B' to A', because then Y must lie on the opposite side of D' to C'. Similarly, the segment XZ must cut the side A'B' at some point W' and the side C'D' at some point Y'. But now the segments X'Z' and W'Y' join pairs of points on opposite sides of the small square and so they must meet at some point O inside the small square.

Now the triangles WOW' and YOY' are similar (WW' and YY' are parallel). Hence OW/OY = OW'/OY'. So if we set up coordinate systems with AB as the x-axis and AD as the y-axis (for the large square) and A'B' as the x'-axis and A'D' as the y'-axis (for the small square) so that corresponding points have the same coordinates, then the y coordinate of O equals the y' coordinate of O. Similarly, XOX' and ZOZ' are similar, so OX/OZ = OX'/OZ', so the x-coordinate of O equals its x'-coordinate. In other words, O represents the same point on both maps.

Problem 3
You are told that all integers from 33 to 73 inclusive can be expressed as a sum of positive integers whose reciprocals sum to 1. Show that the same is true for all integers greater than 73.

Solution
The trick is consider the integers $2a_1, 2a_2, ... , 2a_m$ given that $a_1, a_2, ... , a_m$ is a solution for n. The sum of their reciprocals is 1/2. So if we throw in two 4s, we get a solution for 2n + 8. Similarly, adjoining 3 and 6 gives a solution for 2n + 9. It is now a simple induction. For the starter set gives 74 thru 155, then those give 156 thru 319, and so on. In general, n thru 2n+7 gives 2n+8 thru 4n+23 = 2(2n+8) + 7.

Problem 4
Show that if the angle between each pair of faces of a tetrahedron is equal, then the tetrahedron is regular. Does a tetrahedron have to be regular if five of the angles are equal?

Solution
Answer: no.

Let the tetrahedron be ABCD. Let the insphere have center O and touch the sides at W, X, Y, Z. The OW, OX, OY, OZ are the normals to the faces. But the angle between each pair of normals is equal. So OW, OX, OY and OZ are vectors of equal length at equal angles. Hence each side of WXYZ is equal (eg WX = 2 OW sin(WOX/2)). So WXYZ is a regular tetrahedron. But the faces of ABCD are just the tangent planes at W, X, Y, Z. So if we rotate through an angle 120^o about the line OW, then X goes to Y, Y to Z and Z to X. Hence AB = AC, AC = AD, AD = AB and BC = CD = DB (assuming appropriate labeling). Similarly, for rotation about the other axes. So ABCD has equal edges and hence is regular.

Consider the four normals OW, OX, OY, OZ. We can move X, Y, Z slightly closer together so that XYZ remains an equilateral triangle. Then move W so that WX = WY = XY. So five of the distances are equal, but the sixth is unequal. The reason for *slightly* is that all the angles between pairs of normals must remain less than 180^o. The corresponding tetrahedron will now have the angles between only five pairs of faces equal.

Problem 5
There are 9 delegates at a conference, each speaking at most three languages. Given any three delegates, at least 2 speak a common language. Show that there are three delegates with a common language.

Solution
Suppose not. Then A can shares a language with at most 3 delegates, because if he shared a language with 4, he would have to share the same language with 2 of them (since he can only speak 3 languages). So there are 5 delegates who do not share a language with A. Let one of them be B. By the same argument, there must be at least one of the other 4 (call her C) who does not share a language with B. But now no two of A, B, C share a language. Contradiction.

8th USAMO 1979

Problem 1
Find all sets of 14 or less fourth powers which sum to 1599.

Solution
Answer: none.
The only 4th powers less than 1599 are 1, 16, 81, 256, 625, 1296 ($7^4 = 2401$). Note that 1, 81, 625 = 1 mod 16 and 16, 256, 1296 = 0 mod 16. But 1599 = 15 mod 16, so it cannot be expressed as a sum of 14 or less fourth powers.

Problem 2
N is the north pole. A and B are points on a great circle through N equidistant from N. C is a point on the equator. Show that the great circle through C and N bisects the angle ACB in the spherical triangle ABC (a spherical triangle has great circle arcs as sides).

Solution
Let S_A, S_B, S_N be the great circles through A and C, B and C, and N and C respectively. Let C' be the point directly opposite C on the sphere. Then any great circle through C also goes through C'. So, in particular, S_A, S_B and S_N go through C'.

Two great circles through C meet at the same angle at C and at C', so the spherical angles ACN and AC'N are equal. Now rotate the sphere through an angle 180^o about the diameter through N. Then great circles through N map into themselves, so C and C' change places (C is on the equator). Also A and B change places (they are equidistant from N). S^A must go into another great circle through C and C'. But since A maps to B, it must be S_B. Hence the spherical angle AC'N = angle BCN (since one rotates into the other). Hence ACN and BCN are equal.

Problem 3
$a_1, a_2, \ldots , a_n$ is an arbitrary sequence of positive integers. A member of the sequence is picked at random. Its value is a. Another member is picked at random, independently of the first. Its value is b. Then a third, value c. Show that the probability that a + b + c is divisible by 3 is at least 1/4.

Solution
Let the prob of a value = 0, 1, 2 mod 3 be p, q, r respectively. So $p + q + r = 1$ and p, q, r are non-negative. If $a = b \mod 3$, then to get $a + b + c = 0 \mod 3$, we require $c = a \mod 3$. So the prob of such events is $p^3 + q^3 + r^3$. If the first two values are different mod 3, then the third must be different again, so the prob. is 6pqr. Thus we have to show that $p^3 + q^3 + r^3 + 6pqr >= 1/4$.

$1 = (p + q + r)^3 = p^3 + q^3 + r^3 + 6pqr + 3(p^2q + pq^2 + p^2r + pr^2 + q^2r + qr^2)$. So we have to show that $(p^2q + pq^2 + p^2r + pr^2 + q^2r + qr^2) <= 1/4$. Or $p^2(q + r) + p(q^2 + r^2) + qr(q + r) <= 1/4$, or $p^2(1 - p) + p(q + r)^2 + qr(q + r - 2p) \le 1/4$, or $p^2(1 - p) + p(1 - p)^2 + qr(1 - 3p) = p(1 - p) + qr(1 - 3p) \le 1/4$.

If $p \ge 1/3$, then we maximise $p(1 - p) + qr(1 - 3p)$ by taking $qr = 0$ and $p = 1/2$ to maximise $p(1 - p)$. Thus the maximum is 1/4, achieved at $p = 1/2$, $q = 1/2$, $r = 0$ and $p = 1/2$, $q = 0$, $r = 1/2$. But because of the symmetry, there is no loss of generality in assuming that $p \ge q \ge r$, so p must be $\ge 1/3$. So the maximum value is 1/4.

Problem 4
P lies between the rays OA and OB. Find Q on OA and R on OB collinear with P so that 1/PQ + 1/PR is as large as possible.

Solution
Let the line through P parallel to OA meet OB at C. Note that C is fixed. Let the line through C parallel to QR meet OP at D. D varies as Q varies. Triangles ODC, OPR are similar, so CD/PR = OD/OP. Also triangles OPQ and PDC are similar, so CD/PQ = DP/OP. Adding, CD/PR + CD/PQ = 1. Hence we maximise 1/PQ + 1/PR by making CD as small as possible. That is achieved by making angle CDP = 90^o and hence QR perpendicular to OP.

Problem 5
X has n members. Given n+1 subsets of X, each with 3 members, show that we can always find two which have just one member in common.

Solution
We use induction on n. The result is true for $n < 5$, because there are at most n distinct subsets of 3 elements. Suppose it is true for all sets X with < n members. Let X have n members and consider a collection of n+1 subsets with 3 members. If every member of X was in at most 3 of the subsets, there would be at most n subsets, so some member A is in at least 4 of the subsets. Suppose one of them is {A, B, C}. There are at least three others, so B (say) must be in at least two of the others. Suppose they are {A, B, D} and {A, B, E}. Now any other subset containing A must contain B, because otherwise it would have to contain C, D and E, which is impossible. Similarly, any other subset containing B must contain A. So suppose there are m subsets containing A and B. If {A, B, K} is such a subset, then K cannot belong to any other subsets (because if it also belonged to S, then S and {A, B, K} would have just one member in common). So consider the n-(m+2) people other than those who belong to sets of the type {A, B, K}. There can be at most n-m-2 subsets involving them. Hence at most n-2 in total. So the result is true for n.

Note that we do worse than n subsets if we allow any member to be in 4 subsets. But at least for n = 4m we can achieve n subsets. Just take m groups of 4 and take all subsets with 3 members of each group.

9th USAMO 1980

Problem 1
A balance has unequal arms and pans of unequal weight. It is used to weigh two objects of unequal weight. The first object balances against a weight A, when placed in the left pan and against a weight a, when placed in the right pan. The corresponding weights for the second object are B and b. A third object balances against a weight C, when placed in the left pan. What is its true weight?

Solution
The effect of the unequal arms and pans is that if an object of weight x in the left pan balances an object of weight y in the right pan, then $x = hy + k$ for some constants h and k. Thus if the first object has true weight

x, then $x = hA + k$, $a = hx + k$. So $a = h^2A + (h+1)k$. Similarly, $b = h^2B + (h+1)k$. Subtracting gives $h^2 = (a - b)/(A - B)$. and hence $(h+1)k = a - h^2A = (bA - aB)/(A - B)$.

The true weight of the third object is thus $hC + k = \sqrt{(a-b)/(A-B)}\, C + (bA - aB)/(A - B)\ 1/(\sqrt{(a-b)/(A-B)} + 1)$.

Problem 2
Find the maximum possible number of three term arithmetic progressions in a monotone sequence of n distinct reals.

Solution
Answer: m^2 for $n = 2m+1$, $m(m-1)$ for $n = 2m$.
Let the reals be $a_1, \dots , a_n$. Suppose $n = 2m+1$ and the middle term is a_k. If $k < m+1$, then we are constrained by the shortage of first terms. If $k > m+1$ we are constrained by the shortage of third terms. Thus if $k = 1$, a_k cannot be the middle term. If $k = 2$, there is only one candidate for the first term. If $k = 3$, there are two candidates for the middle terms and so on. Thus the total number of possible progressions certainly cannot exceed: $1 + 2 + \dots + m + m-1 + m-2 + \dots + 1 = m(m+1)/2 + m(m-1)/2 = m^2$. But this bound is achieved by the sequence $1, 2, 3, \dots , n$.

Similarly, if $n = 2m$, then the upper bound is $1 + 2 + \dots + m-1 + m-1 + \dots + 1 = m(m-1)$. Again, this is achieved by the sequence $1, 2, \dots , n$.

Problem 3
$A + B + C$ is an integral multiple of π. x, y, z are real numbers. If $x \sin A + y \sin B + z \sin C = x^2 \sin 2A + y^2 \sin 2B + z^2 \sin 2C = 0$, show that $x^n \sin nA + y^n \sin nB + z^n \sin nC = 0$ for any positive integer n.

Solution
The juxtaposition of x^2 and $\sin 2A$ strongly suggests considering $\cos A + i \sin A$. So put $u = x(\cos A + i \sin A)$, $v = y(\cos B + i \sin B)$, $w = z(\cos C + i \sin C)$. Put $a_n = u^n + v^n + w^n$. So a_1 and a_2 are real. Hence also $uv + vw + wu = (a_1^2 - a_2)/2$ is real. Also $uvw = xyz \exp(i(A+B+C)) = \pm xyz$, since $A+B+C$ is an integral multiple of π. Thus u, v, w are roots of some cubic $p^3 + ap^2 + bp + c = 0$ with real coefficients. Putting $p = u, v, w$ and adding, we get $a_3 + a\, a_2 + b\, a_1 + 3c = 0$, so a_3 is real. Also multiplying through by p^n, then putting $p = u, v, w$ and adding, we get: $a_{n+3} + a\, a_{n+2} + b\, a_{n+1} + c\, a_n = 0$. So by a trivial induction a_n is real for all positive n. Hence result.

Problem 4
The insphere of a tetrahedron touches each face at its centroid. Show that the tetrahedron is regular.

Solution
Let the tetrahedron be ABCD. Let G be the centroid of ABC and H the centroid of ACD. Let AM be a median in ABC and AN a median in ACD. Then AG and AH are tangents to the insphere, so they are equal. CG and CH are also tangents and hence equal. So the triangles ACG and ACH are congruent. Hence $\angle AGC = \angle AHC$ and so $\angle CGM = \angle CHN$. But $GM = AG/2 = AH/2 = GN$, so the triangles CGM and CHN are also congruent. Hence $CM = CN$. Hence $CB = CD$. So every pair of adjacent edges is equal. Hence all the edges are equal and the tetrahedron is regular.

Problem 5
If x, y, z are reals such that $0 \le x, y, z \le 1$, show that $x/(y + z + 1) + y/(z + x + 1) + z/(x + y + 1) \le 1 - (1 - x)(1 - y)(1 - z)$.

Solution
Consider $x/(y+z+1) + y/(z+x+1) + z/(x+y+1) + (1-x)(1-y)(1-z)$ as a function of x, with y and z fixed. Each term is convex, so the whole function is convex. Hence its maximum value occurs at its endpoints. The same is true for x and y, so we need only check the eight possible values $x, y, z = 0$ or 1. In fact, we easily find the expression has value 1 at all eight points. The result follows.

A function is convex if for any three points $a < b < c$, the point $(b, f(b))$ lies on or below the chord joining the points $(a, f(a))$ and $(c, f(c))$. Analytically, this means that if $b = ha + (1-h)c$, where $0 \le h \le 1$, then $f(b) \le h f(a) + (1-h) f(c)$. The linear nature of this relation implies immediately that a sum of convex functions is convex and that a positive multiple of a convex function is convex. Linear functions are obviously convex. It is obvious from the graph that the function $a/(b+x)$ is convex. To prove it analytically we must show that $a/(b+hx+(1-h)y)) \le ha/(b+x) + (1-h)a/(b+y)$ or $a(b^2 + bx + by + xy) \le ha(b+y)(b+hx+(1-h)y) + (1-h)a(b+x)(b+hx+(1-h)y)$. After cancelling some terms, we have to show that $xy \le h(1-h)x^2 + (1-h)^2xy + h^2xy + h(1-h)y^2$. This is obviously true for $h = 1$ or 0. Otherwise we may divide by $1-h$, then h to get $2xy \le x^2 + y^2$, which is true.

To see that the maximum value of a convex function must occur at its endpoints just draw a chord between the endpoints. All other points of the curve must lie below the chord.

10th USAMO 1981

Problem 1
Prove that if n is not a multiple of 3, then the angle π/n can be trisected with ruler and compasses.

Solution
The key is to use $\pi/3$. Since n and 3 are relatively prime we can find integers a and b such that $3a + nb = 1$. Hence $a\pi/n + b\pi/3 = \pi/(3n)$. So take a circle with arcs subtending π/n and $\pi/3$ at the center. Start at a point X on the circumference, mark off $|a|$ arcs subtending π/n in one direction, then mark off $|b|$ arcs subtending $\pi/3$ in the other direction. We end up at a point Y with XY subtending $\pi/3n$ at the center.

Problem 2
What is the largest number of towns that can meet the following criteria. Each pair is directly linked by just one of air, bus or train. At least one pair is linked by air, at least one pair by bus and at least one pair by train. No town has an air link, a bus link and a trian link. No three towns, A, B, C are such that the links between AB, AC and BC are all air, all bus or all train.

Solution
Answer: 4. [eg A and B linked by bus. C and D both linked to A and B by air and to each other by train.]

Suppose A is linked to three other towns by air. Let them be B, C, D. B has at most one other type of link. Suppose it is bus. Then B must be linked to C and D by bus. But now there is a problem with the CD link. If it is air, then A, C, D are all linked by air. If it is bus, then B, C, D are all linked by bus. If it is train, then C has links of all three types. So A cannot be linked to three others by air. Similarly it cannot be linked to three others by train, or to three others by bus. Since it has at most two types of link, it can be linked to at most 4 towns (2 by one type of link and 2 by another). So certainly there cannot be more than 5 towns.

Suppose 5 is possible. A must be linked to two towns by one type of link and two by another. Without loss of generality we may suppose A is linked to B and C by bus and to D and E by train. Now suppose BE is an air link. Then B cannot have a train link (or it will have all three). It cannot be linked to C by bus (or ABC is all bus), so BC must be air. Similarly, ED must be air (it cannot be train or ADE is all train, or bus because then E has air, bus and train). But now there is a problem with the CE link. It cannot be air (or BCE is all air). It cannot be train, because C already has air and bus links. It cannot be bus, because E already has train and air links. So BE cannot be an air link. So it must be a bus or train link. Without loss of generality, we may assume it is a bus link.

E has a bus and a train link, so it cannot have air links. It cannot be linked to D by train (or AED is all train). So ED must be bus. Now DB cannot be air (or D has air, train and bus). It cannot be bus (or DBE is all bus). So it must be train. So BC cannot be air (or B has train, bus and air). It cannot be bus (or ABC is all bus). So BC must be train. CD cannot be air (or C has air, bus and train). It cannot be train (or BCD is all train). So CD must be bus. Finally, CE cannot be air (or C has air, bus and train). It cannot be bus (or CDE is all bus). So CE must be train. So none of the links are air. Contradiction. Hence 5 towns is not possible. But 4 is, as given in the Answer above.

Problem 3
Show that for any triangle, $3(\sqrt{3})/2 \geq \sin 3A + \sin 3B + \sin 3C \geq -2$. When does equality hold?

Solution
Answer: 1st inequality, $A = B = 20^o$, $C = 140^o$. 2nd inequality, $A = 0^o$, $B = C = 90^o$ (which is degenerate).

For x between 0 and 180^o, $\sin 3x$ is negative iff $60^o < x < 120^o$. So at most two angles x in a triangle can have $\sin 3x$ negative. Obviously $\sin 3x \geq -1$, so $\sin 3A + \sin 3B + \sin 3C \geq -2$. We can only get equality in the degenerate case $A = 0^o$, $B = C = 90^o$ (or with the angles relabeled).

We can certainly achieve $3(\sqrt{3})/2$ as shown above. But $3(\sqrt{3})/2 > 2$, so we must have all three sines positive. [If only two are positive, then their sum is at most 2.]. $\sin 3x$ is positive for $x < 60^o$ and $x > 120^o$. So one angle must be $> 120^o$. Assume, for definiteness, that $A \leq B < 60^o$, $C > 120^o$. Put $C' = C - 120^o$. Then $3A + 3B + 3C' = 180^o$. Now $\sin x$ is a concave function for $0 \leq x \leq 180^o$, or $-\sin x$ is a convex function. So by Jensen's inequality $-\sin x - \sin y - \sin z \geq 3 -\sin(x + y + z)/3$. Hence $\sin 3A + \sin 3B + \sin 3C \leq 3 \sin 60^o = 3(\sqrt{3})/2$.

Problem 4
A convex polygon has n sides. Each vertex is joined to a point P not in the same plane. If A, B, C are adjacent vertices of the polygon take the angle between the planes PBA and PBC. The sum of the n such angles equals the sum of the n angles subtended at P by the sides of the polygon (such as the angle APB). Show that n = 3.

Solution
n = 3 is certainly possible. For example, take $\angle APB = \angle APC = \angle BPC = 90^o$ (so that the lines PA, PB, PC are mutually perpendicular). Then the three planes through P are also mutually perpendicular, so the two sums are both 270^o.

We show that n > 3 is not possible.
The sum of the n angles APB etc at P is less than 360^o. This is almost obvious. Take another plane which meets the lines PA, PB, PC etc at A', B', C', ... and so that the foot of the perpendicular from P to the plane lies *inside* the n-gon A'B'C' ... then as we move P down the perpendicular the angles A'PB' etc all increase. But when it reaches the plane their sum is 360^o. However, I do not immediately see how to make that rigorous. Instead, take any point O inside the n-gon ABC... . We have $\angle PBA + \angle PBC > \angle ABC$. Adding the n such equations we get $\sum(180^o - APB) > \sum ABC = (n - 2) 180^o$. So $\sum APB < 360^o$.

The sum of the n angles between the planes is at least $(n - 2) 180^o$. If we take a sphere center P. Then the lines PA, PB intersect it at A", B", ... which form a spherical polygon. The angles of this polygon are the angles between the planes. We can divide the polygon into n - 2 triangles. The angles in a spherical triangle sum to *at least* 180^o. So the angles in the spherical polygon are at least $(n - 2) 180^o$. So we have $(n - 2) 180^o < 360^o$ and hence $n < 4$.

Problem 5
Show that for any positive real x, $[nx] \geq \sum_1^n [kx]/k$.

Solution
If x is an integer, then we have equality. So it is sufficient to prove the result for $0 < x < 1$. The rhs only increases at $x = a/b$, where a, b are coprime positive integers with $1 \leq b \leq n$, and $0 \leq a \leq b$. So it is sufficient to consider x of this form. In fact we can assume $0 < a < b$ since the equality is obvious for x = 0 or 1. We may write:
$a = q_1b + r_1$
$2a = q_2b + r_2$
$3a = q_3b + r_3$
...
$na = q_nb + r_n$
with each $0 \leq r_i < b$.

Thus $kx = q_k$ and we have to prove that $q_1 + q_2/2 + ... + q_n/n \le q_n$.

We claim that $r_1, r_2, ... , r_{b-1}$ is just a permutation of 1, 2, ... , b-1. For if $r_i = r_j$ with $i < j$, then $(j - i)a = (q_j - q_i)b$, but a and b are coprime and $j - i < b$, so that is impossible. So we may use the rearrangement inequality to give $r_1/1 + r_2/2 + ... + r_{b-1}/b-1 \ge 1/1 + 2/2 + ... + (b-1)/(b-1) = b-1$. The inequality remains true if we add some positive terms to the lhs, so we have $r_1/1 + r_2/2 + ... + r_n/n \ge b-1$. Hence $r_1/b + r_2/(2b) + ... + r_n/(nb) \ge (b-1)/b$.

So $(q_1 + q_2/2 + ... + q_n/n) + (b-1)/b \le (q_1 + r_1/b) + (q_2 + r_2/(2b)) + ... (q_n + r_n/(nb)) = a/b + a/b + ... + a/b = na/b = (q_nb + r_n)/b \le q_n + (b-1)/b$. Subtracting $(b-1)/b$ from both sides gives the required result.

11th USAMO 1982

Problem 1
A graph has 1982 points. Given any four points, there is at least one joined to the other three. What is the smallest number of points which are joined to 1981 points?

Solution
Answer: 1979.
Suppose there were 4 points not joined to 1981 points. Let one of them be A. Take a point B not joined to A. Now if X and Y are any two other points, X and Y must be joined, because otherwise none of the points A, B, X, Y could be joined to the other 3. There must be two other points C and D not joined to 1981 points. We have just shown that C must be joined to every point except possibly A and B. So it must be not joined to one of those. Similarly D. But now none of A, B, C, D is joined to the other 3. Contradiction. So there cannot be 4 points not joined to 1981 points. But there can be 3. Just take the graph to have all edges except AB and AC.

Problem 2
Show that if m, n are positive integers such that $(x^{m+n} + y^{m+n} + z^{m+n})/(m+n) = (x^m + y^m + z^m)/m\ (x^n + y^n + z^n)/n$ for all real x, y, z with sum 0, then {m, n} = {2, 3} or {2, 5}.

Solution
Put $z = - x - y$. If m and n are both odd, the lhs has a term in x^{m+n} but the rhs does not. So at least one of m and n is even. Suppose both are even. Then comparing terms in x^{m+n} we get $2/(m+n) = 4/mn$. Put $m = 2M$, $n = 2N$, then $MN = M+N$. So M must divide N and N must divide M. Hence $M = N = 2$. So $m = n = 4$. But put $x = y = 1$, $z = -2$, the lhs is $(1 + 1 + 256)/8 = 129/4$ and the rhs is $((1 + 1 + 16)/4)^2 = 81/4$. So one of m, n must be odd and the other even. Without loss of generality we may take m odd.

Comparing the terms in $x^{m+n-1}y$, the coefficient on the lhs is 1. On the rhs it is 1 x 2/n. So we must have n = 2. Put $x = y = 1$, $z = -2$. We get lhs $= (1 + 1 - 2^{m+2})/(m+2)$ and rhs $= (1 + 1 + 4)/2 \times (1 + 1 - 2^m)/m$. So $(6 - m)2^{m-1}= 2m+6$. We cannot have $m > 6$ or the lhs is negative. Trying m = 1, 3, 5 we find that 3 and 5 are the only solutions.

Arguably, we still have to verify that m=3, n=2 and m=5, n=2 are solutions. That is just tedious algebra. [We do have equality for those values.]

Problem 3
D is a point inside the equilateral triangle ABC. E is a point inside DBC. Show that area DBC/(perimeter DBC)^2 > area EBC/(perimeter EBC)^2.

Solution
Let us find an expression for t = (area DBC)/(perimeter DBC)^2. Let angle B = 2x, angle C = 2y and angle D = 2z and let the inradius be r. Then area DBC = r/2 x perimeter DBC. Also BC = r cot x + r cot y, and similarly for the other sides, so perimeter = 2r(cot x + cot y + cot z). Hence t = 1/(4 (cot x + cot y + cot z)). Putting $z = 90^o - x - y$, we get 1/4t = cot x + cot y + (cot x + cot y)/(cot x cot y - 1). Now EBC has both x and y smaller than DBC, and cot x is a decreasing function of x (certainly for x in the range 0 to 30^o). So writing

$u = \cot x$, $v = \cot y$, it is sufficient to show that $u + v + (u + v)/(uv - 1)$ is an increasing function of u. [It is symmetric, so it follows that it is also an increasing function of v.] We have $x, y < 30^o$, and hence $u, v > \sqrt{3}$, so we need the result at least for $u, v > \sqrt{3}$.

We have $u + v + (u + v)/(uv - 1) = u + v + 1/v\ (uv - 1)/(uv - 1) + (v + 1/v)/(uv - 1)$. In considering the dependence on u, we can ignore the terms that do not depend on u, so we have $u + (1 + 1/v^2)/(u - 1/v)$. Put $k = u - 1/v$, then we have to consider $k + h^2/k$, where $h^2 = 1 + 1/v^2$. But this is an increasing function for $k > h$ (see below). Thus $u + v + (u + v)/(uv - 1)$ is an increasing function of u for $u - 1/v > (1 + 1/v^2)^{1/2}$, or for $u > 1/v + (1 + 1/v^2)^{1/2}$. This bound is highest for the smallest v in other words for $v = \sqrt{3}$, when it is $\sqrt{3}$. So for $v > \sqrt{3}$, $u + v + (u + v)/(uv - 1)$ is an increasing function of u for $u > \sqrt{3}$.

[To see that $k + h^2/k$ is increasing for $k > h$, take $k' > k > h$. Then $k' + h^2/k' - k - h^2/k = (k' - k) - h^2(1/k - 1/k') = (k' - k)(1 - h^2/kk') > 0$.]

Problem 4
Show that there is a positive integer k such that, for every positive integer n, $k\ 2^n + 1$ is composite.

Solution
Answer: k = 542258 (for example).

Suppose p is a prime dividing $2^b - 1$, $0 \le a < b$ and $k = -2^{b-a} \bmod p$. Then if $n = a \bmod b$, we have $k\ 2^n = -2^{b-a}2^{a+hb} = -2^{(h+1)b} = -1 \bmod p$, so $k\ 2^n + 1$ is divisible by p. So we would like to find a collection of pairs (a, b), such that every positive integer n satisfies $n = a \bmod b$ for at least one member of the collection. We need the corresponding p distinct so that we can be sure of finding k by the Chinese Remainder theorem which satisfies $k = -2^{b-a} \bmod p$ for all members of the collection. For the congruences $n = a \bmod b$ to cover all the integers, we need the lcm of the b to be small relative to their size, so we look for an lcm with many factors.

6 = 2.3 does not work because $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^6 - 1 = 3^2 7$, we cannot find distinct primes p. 10 = 2.5 does not work because there are not enough factors to cover all integers. The mod 2 residue covers 1/2, the mod 5 residue covers 1/5 and the mod 10 residue covers 1/10, but that adds up to less than 1. Similarly 12 does not work. We must drop one of 2, 3, 6. But then the rest cover at most 11 residue classes mod 12. So we try 24. Again we drop 6, but we have:

2: $2^2 - 1 = 3$
3: $2^3 - 1 = 7$
4: $2^5 - 1$ has factor 5
8: $2^8 - 1$ has factor 17
12: $2^{12} - 1$ has factor 13
24: $2^{24} - 1$ has factor 241

We now find, for example, the following covering set:

```
0 mod 2 covers the even residues
1 mod 3 covers 1, 7, 13, 19
3 mod 4 covers 3, 11, 15, 23
5 mod 8 covers 5, 21
5 mod 12 covers 17
9 mod 24 covers 9
```

So we now need k which is

-4 mod 3
-4 mod 7
-2 mod 5
-8 mod 17
-128 mod 13
-2^{15} mod 241

The Chinese Remainder Theorem gives 542258.

Problem 5
O is the center of a sphere S. Points A, B, C are inside S, OA is perpendicular to AB and AC, and there are two spheres through A, B, and C which touch S. Show that the sum of their radii equals the radius of S.

Solution
Let D be the circumcenter of ABC. The triangle ABC is in the plane normal to OA. The two spheres both contain through the circumcircle of ABC, so their centers must lie on a line L normal to the plane ABC and hence parallel to OA. Take the plane through OA and the line L. Suppose the centers are P and Q. The sphere center P must touch S at a point X on the line OP. Similarly, the sphere center Q must touch S at a point Y on the line OQ. Since the spheres pass through A, we have PA = PX and hence OP + PA = R, the radius of the sphere S. Similarly OQ + QA = R. Indeed if K is any point on the line L such that OK + KA = R, then the sphere center K will touch S and pass through A. Since it will also have KD perpendicular to DA, it will contain all points on the circumcircle of ABC. But the locus of points such that OK + KA = R is an ellipse with foci O and A. So it meets the line L in just two points, which must therefore be P and Q. Moreover, since the line L is parallel to OA the points must be equidistant from the midpoint of OA (which is the center of the ellipse). Hence OP = AQ and so AQ + AP = R, as required.

12th USAMO 1983

Problem 1
If six points are chosen sequentially at random on the circumference of a circle, what is the probability that the triangle formed by the first three is disjoint from that formed by the second three.

Solution
Answer: 3/10.
Only the order is important. We are interested in permutations of 123456 where the 123 are together (allowing wrapping). wlog the 1 is in first position. So the triangles are disjoint in the cases 123xxx, 132xxx, 1xxx23, 1xxx32, 12xxx3, 13xxx2. So the probability is $6 \cdot 6/5! = 3/10$.

Problem 2
Show that the five roots of the quintic $a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ are not all real if $2a_4^2 < 5a_5a_3$.

Solution
Let the roots be r_i. If the condition holds, then $2 \sum r_i < 5 \sum r_ir_j$. Expanding, $2 \sum r_i^2 + 4 \sum r_ir_j < 5 \sum r_ir_j$, or $2 \sum r_i^2 < \sum r_ir_j$. But if r_i and r_j are real we have we have $2r_ir_j \le r_i^2 + r_j^2$. So if all the roots are real, adding the 10 similar equations gives $2 \sum r_ir_j \le 4 \sum r_i^2$. Contradiction. Hence not all the roots are real.

Problem 3
$S_1, S_2, \dots , S_n$ are subsets of the real line. Each S_i is the union of two closed intervals. Any three S_i have a point in common. Show that there is a point which belongs to at least half the S_i.

Solution
We can write $S_i = [a_i, b_i] \cup [c_i, d_i]$, where $a_i \le b_i \le c_i <= d_i$. Put a = max a_i, d = min d_i. Then a belongs to some S_h, and d belongs to some S_k. Suppose there is some S_i which does not contain a or d. Then $b_i < a$, so any point in S_i and S_h does not belong to $[a_i, b_i]$. Similarly $c_i > b$, so that any point in S_i and S_k does not belong to $[c_i, d_i]$. But that means that S_i, S_h and S_k cannot have a point in common. Contradiction. So every S_i must contain a or d. Hence either a or d belongs to at least half of them.

Problem 4
Show that one can construct (with ruler and compasses) a length equal to the altitude from A of the tetrahedron ABCD, given the lengths of all the sides. [So for each pair of vertices, one is given a pair of points in the plane the appropriate distance apart.]

Solution
Let the altitude from A be AH with H in the plane BCD. The plane normal to BC through A also contains H. Suppose it meets BC at X. Then HX and AX are both perpendicular to BC.

Since we have the side lengths we can construct a cardboard cutout of the tetrahedron: the base BCD and the face BCA next to it, also the face CDA' (and the face BDA", although we do not need it. If we folded along the lines BC, CD and BD, then A, A' and A" would become coincident and we would get the tetrahedron.) We have just shown that in the plane AH is a straight line perpendicular to BC (and meeting it at X). So we draw this line and also the line through A' perpendicular to CD, giving H as their point of intersection. Thus we have AH and HX and we know that in the tetrahedron AH is perpendicular to HX. So draw a circle diameter AX and take a circle center H radius HX meeting the circle at K. Then AK is the required length.

Problem 5
Prove that an open interval of length 1/n in the real line contains at most (n+1)/2 rational points p/q with $1 \le q \le n$.

Solution
This is a variant on the familiar result that m+1 integers from {1, 2, ... , 2m} must include one which divides another. To prove that, take the largest odd divisor of each of the m+1 integers. That gives us m+1 odd numbers from {1, 3, ... , 2m-1}, so by the pigeonhole principle we must have some odd integer b twice. If the corresponding integers are $2^h b$ and $2^k b$, then one must divide the other.

Now if $1 \le q \le n$ and $1 \le kq \le n$, then $|p/q - p'/kq| \ge 1/kq \ge 1/n$. But we cannot have two such points in an *open* interval of length 1/n. Obviously we cannot have two points with the same denominator, so if n = 2m, there are at most m points and if n = 2m+1 there are at most m+1 points.

13th USAMO 1984

Problem 1
Two roots of the real quartic $x^4 - 18x^3 + ax^2 + 200x - 1984 = 0$ have product -32. Find a.

Solution
Let the two roots satisfy the quadratic $x^2 + hx -32 = 0$ (we have not yet shown that h is real). The other two roots have product -1984/-32 = 62. Let them satisfy the quadratic $x^2 + kx + 62$. So $x^4 - 18x^3 + ax^2 + 200x - 1984 = (x^2 + hx -32)(x^2 + kx + 62) = x^4 + (h+k) x^3 + (hk + 30)x^2 + (62h - 32k)x - 1984 = 0$. Equating coefficients: h + k = -18, 62h - 32k = 200. Solving, h = -4, k = -14. Hence a = 86.

Problem 2
Can one find a set of n distinct positive integers such that the geometric mean of any (non-empty, finite) subset is an integer? Can one find an infinite set with this property?

Solution
Answer: yes, no.
Take each member to be an n! power (for example, $1^{n!}, 2^{n!}, \dots , n^{n!}$).

Suppose we could find an infinite set. Take any two members m and n. Then for sufficiently large k, $(m/n)^{1/k}$ must be irrational. But now if we take any other $a_1, a_2, \dots , a_{k-1}$ in the set, $(m\, a_1 \dots a_k)^{1/k}$ and $(n\, a_1 \dots a_k)^{1/k}$ cannot both be integers. Contradiction.

Problem 3
A, B, C, D, X are five points in space, such that AB, BC, CD, DA all subtend the acute angle θ at X. Find the maximum and minimum possible values of $\angle AXC + \angle BXD$ (for all such configurations) in terms of θ.

Solution
Answer: minimum 0 (see below); maximum $2 \cos^{-1}(2 \cos \theta - 1)$, achieved by a square pyramid.

If we take A, B, C, D to lie in a plane so that AC and BD meet at X with the angle AXC = θ, then AB, BC, CD, DA all subtend the angle θ at X, but AC and BD subtend the angle zero. So the minimum is 0.

A little playing around suggests that we should take ABCD to be a square, with X on the normal through the center, so that XABCD is a square pyramid. We calculate the result for this case. Suppose XA = XB = XC = XD = 1, $\angle AXB = \angle BXC = \angle CXD = \angle DXA = \theta$. Then AB = $2 \sin \theta/2$, so AC = $2\sqrt{2} \sin \theta/2$. So $\sin AXC/2 = \sqrt{2} \sin \theta/2$. Hence $\cos AXC = 1 - 2 \sin^2 AXC/2 = 1 - 4 \sin^2\theta/2 = 2 \cos \theta - 1$. Hence $\angle AXC + \angle BXD = 2 \cos^{-1}(2 \cos \theta - 1)$. We note that this increases monotonically from 0 (at $\theta = 0$) to 2π (at $\theta = \pi/2$). For $\theta > 0$, we have $\cos \theta < 1$, hence $2 \cos \theta - 1< \cos \theta$ and hence $2 \cos^{-1}(2 \cos \theta - 1) > 2\theta$. In other words, except for $\theta = 0$, we can certainly do better than 2θ.

Take vectors origin X. Write XA = **A** etc. Since we are only interested in the angles, it is convenient to take **a** to be the unit vector in the direction **A**. Then we have **a.b** = **b.c** = **c.d** = **d.a** = $\cos \theta$. So (**a** - **c**).(**b** - **d**) = 0. So either **a** = **c**, or **b** = **d** or **a** - **c** is perpendicular to **b** - **d**. Suppose **a** = **c**. Then X lies on the line AC. In this case we certainly have AD and CD subtending the same angle at X, and AB and BC subtending the same angle at X. But $\angle AXC = 0$. XABD is a tetrahedron with $\angle AXD = \angle AXB = \theta$. $\angle BXD \le \angle AXB + \angle AXD$ (with equality only if A, B, D, X lie in a plane). So $\angle BXD + \angle AXC \le 2\theta$ and we have just seen that this is not maximal. Similarly if **b** = **d**. So we may assume that **a** - **c** is perpendicular to **b** - **d** and both are non-zero.

We also have (**a** + **c**).(**a** - **c**) = $a^2 - c^2 = 0$ and (**a** + **c**).(**b** - **d**) = 0. So **a** + **c** is normal to the plane containing **a** - **c** and **b** - **d**. Similarly, **b** + **d**. Hence **a** + **c** and **b** + **d** are multiples of each other and (**a** + **c**).(**b** + **d**) = |**a** + **c**| |**b** + **d**|. But lhs = $4 \cos \theta$ and rhs = $(2 \cos AXC/2)(2 \cos BXD/2)$. So $\cos \theta = \cos AXC/2 \cos BXD/2 = 1/2 \cos(AXC/2 + BXD/2) + 1/2 \cos(AXC/2 - BXD/2)$. We want to maximise AXC/2 + BXD/2 and hence to minimise cos(AXC/2 + BXD/2), so we must maximise cos(AXC/2 - BXD/2) and hence take $\angle AXC = \angle BXD$. That then gives $\cos(AXC/2 + BXD/2) = 2 \cos \theta - 1$, so $AXC + BXD = 2 \cos^{-1}(2 \cos \theta - 1)$. So the square pyramid is indeed maximal.

Problem 4
A maths exam has two papers, each with at least one question and 28 questions in total. Each pupil attempted 7 questions. Each pair of questions was attempted by just two pupils. Show that one pupil attempted either nil or at least 4 questions in the first paper.

Solution
Each pupil attempts 7 questions and hence 21 pairs of questions. There are $28\cdot 27/2 = 378$ pairs of questions in total and each is attempted by 2 pupils. So there must be $378\cdot 2/21 = 36$ pupils. Suppose n pupils solved question 1. Each solved 6 pairs involving question 1, so there must be 3n pairs involving question 1. But there are 27 pairs involving question 1, so n = 9. The same applies to any other question. So each question was solved by 9 pupils.

Suppose the result is false. Suppose there are m questions in the first paper, that the number of pupils solving 1, 2, 3 questions in the first paper is a, b, c respectively. So a + b + c = 36, a + 2b + 3c = 9m. Now consider pairs of problems in the first paper. There are m(m-1)/2 such pairs. Pupils solving just 1 solve no pairs, those solving 2 solve 1 pair and those solving 3 solve 3 pairs, so we have b + 3c = m(m-1). Solving for b we get $b = - 2m^2 + 29m - 108 = -2(m - 29/4)^2 - 23/8 < 0$. Contradiction. So the result must be true.

Problem 5
A polynomial of degree 3n has the value 2 at 0, 3, 6, ... , 3n, the value 1 at 1, 4, 7, ... , 3n-2 and the value 0 at 2, 5, 8, ... , 3n-1. Its value at 3n+1 is 730. What is n?

Solution
Answer: n = 4.

The (3n+1)th differences of the polynomial are zero. Call it p(x), so we have p(3n+1) - (3n+1)C1 p(3n) + (3n+1)C2 p(3n-1) - ... + $(-1)^{3n+1}$ p(0) = 0, where rCs is the binomial coefficient. Hence p(3n+1) = 2((3n+1)C1 - (3n+1)C4 + ...) + ((3n+1)C3 - (3n+1)C6 + ...). Putting n = 1, we get: p(4) = 2(4C1 - 4C4) + 4C3 = 6 + 4 = 10. So n is not 1. Putting n = 2, we get: p(7) = 2(7C1 - 7C4 + 7C7) + (7C3 - 7C6) = 2(7 - 35 + 1) + (35 - 7) = -26. So n is not 2. Putting n = 3, we get: p(10) = 2(10C1 - 10C4 + 10C7 - 10C10) + (10C3 - 10C6 + 10C9) = 2(10 - 210 + 120 - 1) + (120 - 210 + 10) = -162 -100 = -262. So n is not 3. Putting n = 4, we get: p(13) = 2(13C1 - 13C4 + 13C7 - 13C10 + 13C13) + (13C3 - 13C6 + 13C9 - 13C12) = 2(13 - 715 + 1716 - 286 + 1) + (286 - 1716 + 715 - 13) = 1458 - 728 = 730. So n = 3 works.

14th USAMO 1985

Problem 1
Do there exist 1985 distinct positive integers such that the sum of their squares is a cube and the sum of their cubes is a square?

Solution
Answer: yes.

Take *any* n integers a_i. Suppose that $\sum a_i^2 = b$ and $\sum a_i^3 = c$. Now multiply each a_i by b^4c^3. The sum of their squares becomes b^9c^6 which is a cube and the sum of their cubes becomes $b^{12}c^{10}$ which is a square.

Problem 2
Find all real roots of the quartic $x^4 - (2N + 1)x^2 - x + N^2 + N - 1 = 0$ correct to 4 decimal places, where $N = 10^{10}$.

Solution
Answer: 99999.9984 and 100000.0016.
We can write the equation as $(x^2 - N - 1/2)^2 = x + 5/4$. For $x < -5/4$ the lhs is positive and the rhs is negative, so there are no roots with $x < -5/4$. If x lies between -5/4 and 0, then the lhs is obviously much larger than the rhs, so again there is no root. Thus there are no negative roots. Descartes' rule of signs (see below) tells us that there are at most 2 positive roots.

If $x = 10^5$, then the lhs is 1/4 and the rhs is much larger (approx 10^5). If $x = 10^5 \pm 1$, then the lhs is $(\pm 2\ 10^5 + 1/2)^2$ which is approximately $4\ 10^{10}$ and much larger than the rhs. So there is a root either side of 10^5. Put $x = 10^5 \pm k$. Then we want $(\pm 2k.10^5 + k^2 - 1/2)^2 = 10^5 + 5/4$, or $(4k^2 10^5 - 1)10^5 \pm 2k(1-2k^2)10^5 - 5/4 = 0$. So evidently we need approximately $4k^2 = 10^{-5}$, or $k = \pm 0.0016$. So it looks as though the roots are $10^5 \pm 0.0016$ = 99999.9984 and 100000.0016.

Put $x = 10^5 \pm 0.00155$. Then $(x^2 - 10^{10} - 1/2)^2 - x - 5/4 = (\pm 310 - 1/2 + 0.00155^2)^2 - 10^5 \pm 0.00155 - 5/4 < 311^2 - 10^5 < 0$. Put $x = 10^5 \pm 0.00165$, then $(x^2 - 10^{10} - 1/2)^2 - x - 5/4 = (\pm 330 -1/2 + 0.00165^2)^2 - 10^5 \pm 0.00165 - 5/4 > 329^2 - 10^5 - 2 > 0$. So indeed one root lies between $10^5 - 0.00165$ and $10^5 - 0.00155$ and the other root lies between $10^5 + 0.00155$ and $10^5 + 0.00165$.

Descartes' rule of signs.

This states that if the number of sign changes in the coefficients of the polynomial is d, then the number of positive roots is d or d less an even number. So, for example, if the polynomial is $x^5 + 14.3 x^4 - 34 x^2 - x + 3.2$, then there are two sign changes (+14.3 to -34 and -1 to 3.2), so there are either 0 or 2 positive roots. Note that we ignore zero coefficients. If r is a positive root of p(-x) = 0, then -r is a negative root of p(x) = 0. So if we substitute -x for x in the polynomial and the number of sign changes is then d', then we can conclude that the number of negative roots of the polynomial is either d' or d' less an even number. With the example above we get $-x^5 + 14.3 x^4 - 34 x^2 + x + 3.2$, which has 3 sign changes. So the polynomial has 1 or 3 negative roots.

The proof is not difficult. The key idea is to show that if k is a positive root, so that $p(x) = (x - k) q(x)$, then (1) $p(x)$ has at least one more sign change than $q(x)$, and (2) the difference between the number of sign changes is odd (note that the signs of the constant coefficients of $p(x)$ and $q(x)$ are different).

Problem 3
A tetrahedron has at most one edge longer than 1. What is the maximum total length of its edges?

Solution
Answer: $5 + \sqrt{3}$.

Suppose AB is the edge which may be longer than 1. Then if we rotate the triangle ACD about CD until A lies in the same plane as BCD and is on the opposite side of CD to B, then we maximise AB without changing any of the other side lengths. So the maximal configuration must be planar.

Now suppose we regard C and D as fixed and the other points as variable. Suppose $CD = 2x$ ($<= 1$). Then A and B must both lie inside the circle center C radius 1 and inside the circle center D radius 1 and hence inside their intersection which is bounded by the two arcs XY (assuming they meet at X and Y). Obviously we maximise $AC + AD + BC + BD$ by taking A at X and B at Y (or vice versa). We claim that that choice also maximises AB. Suppose that is true. Then it also maximises $AC + AD + BC + BD + AB$ at $4 + 2(1 - x^2)^{1/2}$. So we now have to vary CD to maximise $2x + 4 + 2(1 - x^2)^{1/2}$. We show that $x + (1 - x^2)^{1/2}$ is increasing for $x <= 1/2$ and hence that the maximum is at $x = 1/2$. Put $x = \sin t$. Then we have $x + (1 - x^2)^{1/2} = \sqrt{2} \sin(t + \pi/4)$ which is indeed increasing for $x \le \pi/6$.

It remains to prove the claim. Take the circle diameter XY. Then the two arcs both lie inside this circle. [Two circles intersect in at most two points, so each arc must lie entirely inside or entirely outside the circle center O and it obviously does not lie outside.] But AB lies inside a chord of this circle The length of the chord cannot exceed the diameter of the circle (which is XY) and hence $AB \le XY$.

Problem 4
A graph has $n > 2$ points. Show that we can find two points A and B such that at least $[n/2] - 1$ of the remaining points are joined to either both or neither of A and B.

Solution
Consider the number of pairs (X, {Y, Z}), where X, Y, Z are distinct points such that X is joined to just one of Y, Z. If X is joined to just k points, then there are just $k(n - 1 - k) \le (n - 1)^2/4$ such pairs (X, {Y, Z}). Hence in total there are at most $n(n - 1)^2/4$ such pairs. But there are $n(n - 1)/2$ possible {Y, Z}. So we must be able to find one of them {A, B} which belongs to at most $[(n - 1)/2]$ such pairs. Hence there are at least $n - 2 - [(n - 1)/2] = [n/2] - 1$ points X which are joined to both of A and B or to neither of A and B. [If confused by the [ ], consider $n = 2m$ and $n = 2m+1$ separately!]

Problem 5
$0 < a_1 \le a_2 \le a_3 \le \ldots$ is an unbounded sequence of integers. Let $b_n = m$ if a_m is the first member of the sequence to equal or exceed n. Given that $a_{19} = 85$, what is the maximum possible value of $a_1 + a_2 + \ldots + a_{19} + b_1 + b_2 + \ldots + b_{85}$?

Solution
We show that the *only* possible value of the sum is 85.20 = 1700.

That is certainly the value if all $a_i = 85$, for then all $b_j = 1$ and so the sum is $19\cdot85 + 85\cdot1 = 85\cdot20$. Now consider the general case. Suppose that we increase some a_i by 1 from k to k+1 (whilst preserving the property that a_i is increasing, so we must have $a_i < a_{i+1}$ before the increase). The effect of the increase is to change b_{k+1} from i+1 to i, but not to change any other b_j. This is obvious if $a_{i-1} < a_i$ and $a_{i+1} > a_i + 1$. If $a_{i-1} = a_i$, then before the change $b_k = i-1$ (not i) and that is still true after the change. Equally, if $a_i = a_{i+1}$ after the change, then it is still true that b_{k+1} changes from i+1 to i. Thus the overall effect of the increase is not to change the sum of the a_i plus the sum of the b_j. But by a series of such changes we convert any initial sequence to all $a_i = 85$.

15th USAMO 1986

Problem 1
Do there exist 14 consecutive positive integers each divisible by a prime less than 13? What about 21 consecutive positive integers each divisible by a prime less than 17?

Solution
Answer: no, yes.

There are 7 odd numbers. At most one can be a multiple of 7 and one a multiple of 11. It is only possible to have two of them multiples of 5 if either the largest or smallest is a multiple of 5. But it is only possible to have three of them multiples of 3 if both the largest and the smallest are multiples of 3. So at most four numbers are multiples of 3 or 5. That leaves one odd number unaccounted for.

A little juggling shows that we want the odd numbers to be divisible by: 3, 7, 5, 3, 11, 13, 3, 5, 7, 3 (or the 11 and 13 can be interchanged). We need the 6th odd to be 13 mod 143 to get the 11 and 13 correct, and hence the 1st in the sequence to be 2 mod 143. But the 1st must be even, so it must be 2 mod 286. The 2nd must be divisible by 3, so the 1st must be 2 mod 858. But we need the 4th to be divisible by 7, so the 1st must be 3434 mod 6006. Then the 6th must be divisible by 5, so the 1st must be 9440 mod 30030. Thus an example is 9440, 9441, ... , 9460.

Problem 2
Five professors attended a lecture. Each fell asleep just twice. For each pair there was a moment when both were asleep. Show that there was a moment when three of them were asleep.

Solution
This is a slightly tricky application of the pigeonhole principle.

For each pair take the first moment when they are both asleep. There are ten pairs, so ten such moments. If two coincide, then we are done because at that moment at least three professors were asleep. So suppose they are all distinct and form a set S. Each such moment must also be one of the 10 occasions when a professor falls asleep. But consider the earliest member of S. Two professors were asleep at that moment so two fell asleep at or before that moment. Thus each of the remaining 9 members of S must be one of the 8 later occasons when a professor fell asleep. So they cannot all be distinct. Contradiction.

Problem 3
What is the smallest n > 1 for which the average of the first n (non-zero) squares is a square?

Solution
Answer: 337.

We need 1/6 (n+1)(2n+1) a square. We need n = 6m+1 for it to be an integer. So (3m+1)(4m+1) must be a square. But 3m+1 and 4m+1 are relatively prime, so each must be a square. Suppose $3m+1 = a^2$ and $4m+1 = (a+k)^2$, then (subtracting) $m = 2ak + k^2$, so $a^2 - 6ak - (3k^2 + 1) = 0$, so $a = 3k + \sqrt{(12k^2+1)}$. By inspection the smallest solution of this is k = 2, giving a = 13 and hence m = 56 and n = 337.

Problem 4
A T-square allows you to construct a straight line through two points and a line perpendicular to a given line through a given point. Circles C and C' intersect at X and Y. XY is a diameter of C. P is a point on C' inside C. Using only a T-square, find points Q,R on C such that QR is perpendicular to XY and PQ is perpendicular to PR.

Solution
C' seems slightly odd. Why not just say that XY is a diameter of C, P is a point inside C and we want points Q, R on C such that QR is perpendicular etc.? So presumably it is there to help.

Note that PQ perpendicular to PR means that P lies on the circle diameter QR. So take M to be the midpoint of QR. Then MQ = MR = MP. Guided by the hint about C', we extend PM to meet C' at S. Then QM·MR = XM·MY (circle C) = PM·MS (circle C'). So M is also the midpoint of PS. So we want to find S on C' so that XY bisects SP.

Let the line XP meet C at U. Draw the line through U perpendicular to XY to meet C again at V. Draw XV. Draw the line through P perpendicular to XY to meet XV at T. Then XY bisects PT, which is parallel to QR. Now draw the line through T perpendicular to PT (and hence parallel to XY) to meet C' at S. XY must also bisect ST. So we have the required point M. Draw the line through M perpendicular to XY to meet the circle C at Q and R.

Problem 5
A partition of n is an increasing sequence of integers with sum n. For example, the partitions of 5 are: 1, 1, 1, 1, 1; 1, 1, 1, 2; 1, 1, 3; 1, 4; 5; 1, 2, 2; and 2, 3. If p is a partition, f(p) = the number of 1s in p, and g(p) = the number of distinct integers in the partition. Show that ∑ f(p) = ∑ g(p), where the sum is taken over all partitions of n.

Solution
Let π(n) be the number of partitions of n. If p = $a_1, a_2, \ldots , a_m$ is a partition of n, then h(p) = 1, $a_1, \ldots , a_m$ is a partition of n+1 which contains a 1. Moreover, h is obviously a bijection between partitions of n and partitions of n+1 which contain a 1. But h(p) also contains one more 1 than p, so ∑ f(p) for n+1 is ∑ f(p) for n plus π(n). Thus ∑ f(p) for n is 1 + π(1) + π(2) + ... + π(n-1). [Obviously ∑ f(p) for n = 1 is 1.]

Checking, we find π(1) = 1, π(2) = 2, π(3) = 3, π(4) = 5, π(5) = 7. So this formula gives ∑ f(p) for n = 5 is 1 + 1 + 2 + 3 + 5 = 12, which is correct.

Now fix n and consider the number of pairs (p, m), where p is a partition of n containing m. For each p the number of such pairs is g(p). So the total number of such pairs ∑ g(p). But the number of pairs (p, m) for fixed m is just π(n - m) (taking π(0) = 0). So the total is also ∑ π(n - m). So ∑ g(p) = 1 + π(1) + ... + π(n-1) = ∑ f(p).

16th USAMO 1987

Problem 1
Find all solutions to $(m^2 + n)(m + n^2) = (m - n)^3$, where m and n are non-zero integers.

Solution
Answer: (m, n) = (-1, -1), (8, -10), (9, -6), (9, -21).

If m, n > 0, then lhs is certainly positive, so we must have m > n to make the rhs positive. But then $m^2 + n > m^2$ and $n^2 + m > m$, so lhs > m^3 > rhs. Contradiction. So there are no solutions with m and n both positive.

Put M = |m|, N = |n|. Consider next m = M, n = -N. So we have $(M^2 - N)(N^2 + M) = (M + N)^3$. Hence $M^2N^2 + M^3 - N^3 - MN = M^3 + 3M^2N + 3MN^2 + N^3$. N is non-zero, so we can divide to get: $2N^2 + N(3M - M^2) + 3M^2 + M = 0$. Regarding this as a quadratic in N we solve, getting $N = ((M^2 - 3M) \pm \sqrt{(M^4 - 6M^3 + 9M^2 - 24M^2 - 8M)})/4$. Thus $M^4 - 6M^3 - 15M^2 - 8M = M(M - 1)^2(M - 8)$ must be a square. Hence M(M - 8) must be a square. We have $(M - 4)^2 = M(M - 8) + 16$ and for M ≥ 13, we have $(M - 5)^2 = M^2 - 10M + 25 < M^2 - 8M$. So we need only consider M ≤ 12. But obviously we cannot have M < 8, or M(M - 8) is negative. Checking the remaining values, we find M = 8 and 9 are the only solutions. They give the solutions (m, n) = (8, -10), (9, -6), (9, -21).

Next, consider the case m = -M, n = N. That clearly does not work. We get $(M^2 + N)(N^2 - M) = - (M + N)^3$. So $2N^2 + (M^2 + 3M)N + 3M^2 - M = 0$, which has no solutions because $3M^2 > M$.

Finally, consider the case m = -M, n = -N. Then we get $(M^2 - N)(N^2 - M) = (N - M)^3$, so $2N^2 - (M^2 + 3M)N + (3M^2 - M) = 0$. Solving for N, we get $N = ((M^2 + 3M) \pm \sqrt{(M^4 + 6M^3 + 9M^2 - 24M^2 + 8M)})/4$. So we require

$M(M^3 + 6M^2 - 15M + 8) = M(M + 8)(M - 1)^2$ to be a square. Hence $M(M + 8)$ is a square. We have $(M + 4)^2 > M(M + 8)$ and for $M \geq 5$, we have $(M + 3)^2 = M^2 + 6M + 9 < M^2 + 8M$. So we need only check M = 1, 2, 3, 4. We find M = 1 gives the only square, and that gives the solution (m, n) = (-1, -1).

Problem 2
The feet of the angle bisectors of the triangle ABC form a right-angled triangle. If the right-angle is at X, where AX is the bisector of angle A, find all possible values for angle A.

Solution
Answer: 120^o.

Use vectors origin A. Write the vector AB as **B** etc. Using the familiar BX/CX = AB/AC etc, we have **Z** = b**B**/(a+b), **Y** = c**C**/(a+c), **X** = (b**B**+c**C**)/(b+c). Hence **Z-X** = b**B**(c-a)/((a+b)(b+c)) - c**C**/(b+c), **Y-X** = -b**B**/(b+c) + c**C**(b-a)/((a+c)(b+c)).

We have (**Z-X**).(**Y-X**) = 0, so after multiplying through by $(a+b)(a+c)(b+c)^2$, we get $b^2\mathbf{B}^2(a^2-c^2) + c^2\mathbf{C}^2(a^2-b^2) + 2bc\mathbf{B.C}\,(a^2+bc) = 0$. But $\mathbf{B}^2 = c^2$, $\mathbf{C}^2 = b^2$, so $bc(2a^2-b^2-c^2) + 2\mathbf{B.C}\,(a^2+bc) = 0$.

But $a^2 = b^2 + c^2 - 2\mathbf{B.C}$ (cosine rule), so $2a^2-b^2-c^2 = a^2-2\mathbf{B.C}$. Hence $bc(a^2-2\mathbf{B.C}) + 2\mathbf{B.C}(a^2+bc) = 0$, so $\mathbf{B.C} = -bc/2$. Hence cos A = -1/2, so $\angle A = 120^o$.

Problem 3
X is the smallest set of polynomials p(x) such that: (1) p(x) = x belongs to X; and (2) if r(x) belongs to X, then x r(x) and (x + (1 - x) r(x)) both belong to X. Show that if r(x) and s(x) are distinct elements of X, then $r(x) \neq s(x)$ for any $0 < x < 1$.

Solution
If they are never equal, then we must be able to order them, so that r(x) > s(x) for all x in (0, 1) or s(x) > r(x) for all x in (0, 1). Let us use the notation [+ - - + ...]. The operation r(x) to x r(x) is denoted as - , and the operation r(x) to x + (1 - x) r(x) is denoted as + . Then the operations are listed in reverse order with the last carried out put first. So for example [+ -] means we first apply - to get x^2, then + to get $x + (1 - x)x^2 = x + x^2 - x^3$. We claim that we have the ordering + > nothing > - , which we apply starting with the first term. So looking at the first few polynomials we have: [+ +] > [+], because we compare the first terms which are equal, then we compare the second terms. We regard [+] as having nothing for the second term, so + > nothing. Then [+] > [+ -], because they have equal first terms, but unequal second terms (nothing > -). The starting polynomial is represented as [ ]. So we have [+ -] > [ ] because + > nothing. Then [ ] > [- +] > [-] > [- -]. Clearly this ordering, which is a type of lexicographic ordering is a total order, that is, for any two distinct polynomials in X we will find that one is larger than the other.

So suppose r(x) belongs to the set X. We have to establish that + > nothing > - . In other words, that x + (1 - x) r(x) > x and that x > x r(x). But that is obvious, because by a trivial induction we have 0 < r(x) < 1 for all r(x) in X and x in (0, 1). So, if follows that if r(x) and s(x) are in X then x + (1 - x) r(x) > x s(x). It is also obvious that if [a] > [b], where a and b are some strings of + and - , then [+ a] > [+ b] and [- a] > [- b]. But that is sufficient to establish the ordering.

Problem 4
M is the midpoint of XY. The points P and Q lie on a line through Y on opposite sides of Y, such that |XQ| = 2|MP| and |XY|/2 < |MP| < 3|XY|/2. For what value of |PY|/|QY| is |PQ| a minimum?

Solution
Let the angle between the line through Y and XY be θ. Take Y' on the line such that XY' = XY. If P is on the opposite side of Y to Y', then Q is on the opposite side of Y' to Y. As P approaches Y, Q approaches Y' so the minimum value of PQ is YY', corresponding to PY/QY = 0. But it is unrealised, since the problem requires MY < MP. If P is on the same side of Y as Y', then as P approaches the midpoint of YY', Q approaches Y. So the minimum value of PQ is YY'/2 with PY/QY = infinity. Again it is unrealised because

the problem requires MY < MP. P is allowed to be on either side of Y, so the unrealised minimum value of PQ is YY'/2 as PY/QY approaches infinity.

Problem 5

$a_1, a_2, \dots , a_n$ is a sequence of 0s and 1s. T is the number of triples (a_i, a_j, a_k) with $i < j < k$ which are not equal to (0, 1, 0) or (1, 0, 1). For $1 \le i \le n$, f(i) is the number of $j < i$ with $a_j = a_i$ plus the number of $j > i$ with $a_j \ne a_i$. Show that T = f(1) (f(1) - 1)/2 + f(2) (f(2) - 1)/2 + ... + f(n) (f(n) - 1)/2. If n is odd, what is the smallest value of T?

Solution

For n odd, the smallest value of T is n(n-1)(n-3)/8 achieved by 01010... 010.

Suppose a particular $a_i = 0$. Let S_i be the set of a_j with $j < i$ and $a_j = a_i$ and a_j with $j > i$ and $a_j \ne a_i$. Suppose we take any two members of S_i and consider the triple formed with a_i itself. Surprisingly perhaps, they are all of the required form (a_i is bold):

```
... 0 ... 0 ... 0
... 0 ... 0 ... 1
... 0 ... 1 ... 1
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Similarly, if $a_i = 1$:

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... 1 ... 1 ... 1
... 1 ... 1 ... 0
... 1 ... 0 ... 0
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Conversely, any triple of the required form can be considered (uniquely) to be an a_i with members of the triple before it equal to it and members of the triple after it unequal to it. Since f(m) (f(m) - 1) is the number of ways of choosing two items from f(m), we have established that T = ∑ f(m) (f(m) - 1).

We show that ∑ f(m) = n(n-1)/2 (and is hence independent of the details of the particular sequence a_i - it depends only on its length). In the case of a sequence of all 0s, we have f(1) = 0, f(2) = 1, ... , f(n) = n-1, so ∑ f(m) = n(n-1)/2. Now suppose we have a particular sequence and we change a_i from 0 to 1. Suppose there are a 0s before a_i, b 1s before a_i, c 0s after a_i and d 1s after a_i. Then the value of f(i) is changed from a + d to b + c, an increase of (b - a) + (c - d). The value of f(j) with j < i is decreased by 1 if f(j) is 0 and increased by 1 if f(j) is 0. So in total there is an increase of (a - b) for such j. Similarly, for j > i, the value of f(j) is decreased by 1 if f(j) is 0 and increased by 1 if f(j) is 1, a net increase of (d - c). Thus overall ∑ f(m) is increased by (b - a) + (c - d) + (a - b) + (d - c) = 0. But by a sequence of such changes we can get to any sequence a_i, so all sequences (of length n) have the same value for ∑ f(m).

Thus T is minimised when $\sum f(m)^2$ is minimised. But since ∑ f(m) is fixed, that is achieved when the f(m) are as equal as possible. If n =2k+1, then ∑ f(m) = n(n-1)/2, so the average value of f(m) is exactly k, so we look for an arrangement with all f(m) = k. It is not hard to see that 01010...1010 works. So this gives T = ∑ k(k-1)/2 = n(n-1)(n-3)/8.

The question does not ask for it, but the even case is also straightforward. If n = 2k, then the best we can hope for is k values of k and k values of k-1, so that ∑ f(m) = k.k + k.(k-1) = k(2k-1) = n(n-1)/2. That is achieved by 0101...0101 (all the 0s have f(m) = k and all the 1s have f(m) = k-1). Hence T = k.k(k-1)/2 + (k-1).(k-1)(k-2)/2 = $(k-1)(2k^2-3k+2)/2 = (n-2)(n^2-3n+4)/8$.

17th USAMO 1988

Problem 1

The repeating decimal 0.ab ... k $\underline{\text{pq ... u}}$ = m/n, where m and n are relatively prime integers, and there is at least one decimal before the repeating part. Show that n is divisible by 2 or 5 (or both). [For example, $0.011\underline{36}$ = 0.01136363636 ... = 1/88 and 88 is divisible by 2.]

Solution
Note that k and u are not equal (otherwise we should have regarded the repeating part as starting at k). We have $m/n = ab...k/10^r\ pq...u/(10^r(10^s - 1)) = (ab...k\ (10^s - 1) + pq...u)/(10^r(10^s - 1))$. The numerator = u - k mod 10, which is non-zero, so the numerator is not divisible by 10. But the denominator is divisible by 10. Hence after reduction to lowest terms the denominator is divisible by 2 or 5 or both.

Problem 2
The cubic $x^3 + ax^2 + bx + c$ has real coefficients and three real roots $r \ge s \ge t$. Show that $k = a^2 - 3b \ge 0$ and that $\sqrt{k} \le r - t$.

Solution
$a^2 - 3b = (r + s + t)^2 - 3(rs + st + tr) = r^2 + s^2 + t^2 - (rs + st + tr)$. By Cauchy-Schwartz we have $(rs + st + tr)^2 \le (r^2 + s^2 + t^2)^2$, so $r^2 + s^2 + t^2 \ge |rs + st + tr| \ge (rs + st + tr)$. Hence $a^2 - 3b \ge 0$.

$a^2 - 3b \le (r - t)^2$ is the same as $r^2 + s^2 + t^2 - rs - st - tr \le r^2 - 2rt + t^s$ or $s^2 + rt - rs - st \le 0$ or $(r - s)(s - t) \ge 0$, which is true since $r \ge s$ and $s \ge t$. So $a^2 - 3b \le (r - t)^2$. Taking the non-negative square roots, we get the required result.

Problem 3
Let X be the set {1, 2, ... , 20} and let P be the set of all 9-element subsets of X. Show that for any map f: P → X we can find a 10-element subset Y of X, such that f(Y - {k}) ≠ k for any k in Y.

Solution
Consider pairs (S, k) with S in P and k in X such that f(S) = k. There are evidently 20C9 such pairs, since we can choose any S and k is then fixed. Now consider the pairs (Y, k) such that Y is a 10-element subset of X containing k and f(Y - {k}) = k. The map (Y, k) to (Y - {k}, k) is an injection because if (Y - {k}, k) = (Y' - {k'}, k'), then k = k' and hence Y = Y'. It is not necessarily a bijection because if there are any pairs (S, k) with k in S then they do not correspond to any (Y, k). But certainly the number of pairs (Y, k) is at most the number of pairs (S, k). So there are at most 20C9 pairs (Y, k). But there are 20C10 subsets Y with 10 elements, so at least 20C10 - 20C9 of them (more than 16000) do not belong to *any* pairs (Y, k), in other words they are such that f(Y - {k}) is not k for any k in Y.

Problem 4
ABC is a triangle with incenter I. Show that the circumcenters of IAB, IBC, ICA lie on a circle whose center is the circumcenter of ABC.

Solution
In fact they lie on the circumcircle of ABC.

Extend AI to meet the circumcircle again at A'. We show that A' is the circumcenter of BCI. Angle A'AC = angle A'AB, so A' is the midpoint of the arc BC, so A'B = A'C. Also ∠A'CB = ∠A'AB = A/2, so ∠A'CI = A/2 + B/2. But ∠A'IC = ∠IAC + ∠ICA = A/2 + B/2, so A'CI is isosceles, so A'C = A'I.

Problem 5
Let p(x) be the polynomial $(1 - x)^a (1 - x^2)^b (1 - x^3)^c \dots (1 - x^{32})^k$, where a, b, ..., k are integers. When expanded in powers of x, the coefficient of x^1 is -2 and the coefficients of $x^2, x^3, \dots , x^{32}$ are all zero. Find k.

Solution
Answer: $2^{27} - 2^{11}$.

We have $p(x) = 1 - 2x + O(x^{33})$. Hence $p(-x) = 1 + 2x + O(x^{33})$. Multiplying $p(x)p(-x) = 1 - 2^2x^2 + O(x^{33})$. Now p(x) p(-x) cannot have any odd terms, so we can write it as a polynomial in x^2, $q(x^2)$. Hence $q(x^2) = 1 - 2^2x^2 + O(x^{34})$. Similarly, $r(x^4) = q(x^2)\ q(-x^2) = 1 - 2^4x^4 + O(x^{36})$, $s(x^8) = r(x^4)\ r(-x^4) = 1 - 2^8x^8 + O(x^{40})$, and $t(x^{16}) = 1 - 2^{16}x^{16} + O(x^{48})$.

Now go back to the definition of p(x). When we take p(x) p(-x), the term $(1 - x)^a$ becomes $(1 - x^2)^a$. All the even terms just double their exponent, so $(1 - x^2)^b$ becomes $(1 - x^2)^{2b}$, $(1 - x^4)^d$ becomes $(1 - x^4)^{2d}$ and so on. The odd terms all keep the same exponent, so $(1 - x^3)^c$ becomes $(1 - x^6)^c$ and so on. Thus we get $t(x^{16}) = (1 - x^{16})^n(1 - x^{32})^{16k} \dots$. The first exponent is a sum of several exponents from p(x), but the details are unimportant. We know that $t(x^{16}) = 1 - 2^{16}x^{16} + O(x^{48})$. The x^{16} term can only come from $(1 - x^{16})^n$, so $n = 2^{16}$. Now there is no x^{32} term, so putting $N = 2^{16}$ we have NC2 = 16k, were NC2 is the binomial coefficient $N(N - 1)/2 = 2^{31} - 2^{15}$. Hence $k = 2^{27} - 2^{11}$.

18th USAMO 1989

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19th USAMO 1990

Problem 1
A license plate has six digits from 0 to 9 and may have leading zeros. If two plates must always differ in at least two places, what is the largest number of plates that is possible?

Solution
Answer: 10^5.

We show by induction that we can find a set of 10^{n-1} plates for n > 1 digits. It is true for n = 2: take the plates 00, 11, 22, ... , 99. Suppose it is true for n. If d is a digit from 0 to 9 and s is a plate of n digits, let [d, s] be the plate of n+1 digits which has a as its first digit, and the remaining digits the same as those of s, except that the last digit is that for s plus d (reduced mod 10 if necessary). Let S be a set of plates for n digits. We claim that the set S' = { [d, s] : d = 0, 1, ... or 9 and s belongs to S} is a set of plates for n+1 digits. It obviously has 10 times as many members as S, so this claim is sufficient to establish the induction.

We have to show that [a, s] and [b, t] differ in at least two places. If a = b, then s ≠ t, so s and t differ in at least two places. The same change is made to their last digits, so [a, s] and [a, t] differ in at least two places. If a ≠ b and s = t, then [a, s] and [b, s] differ in both their first and last places. If a ≠ b and s ≠ t, then s and t differ in at least two places and so the modified s and t, differ in at least one place. But [a, s] and [b, t] also differ in the first place, so they differ in at least two places.

So we have established that the largest number is at least 10^{n-1} for n digits.
But any two plates which differ only in the last digit cannot both be chosen. So at most 1/10 of the 10^n possible plates can be chosen. That shows that 10^{n-1} is best possible.

Problem 2
Define $f_1(x) = \sqrt{(x^2 + 48)}$ and $f_n(x) = \sqrt{(x^2 + 6f_{n-1}(x))}$. Find all real solutions to $f_n(x) = 2x$.

Solution
Answer: For each n, x = 4 is the only solution.

Obviously x = 4 is a solution. Since $f_n(x) >= 0$, any solution must be non-negative. So we restrict attention to $x \ge 0$.
Suppose x < 4. We show by induction that $f_n(x) > 2x$. For n = 1, the claim is equivalent to $4x^2 < x^2 + 48$, or $x^2 < 16$, which is true. So suppose the result is true for n. Then $x^2 + 6f_n(x) > x^2 + 12x$. But x < 4, so $3x^2 < 12x$, so $4x^2 < x^2 + 12x$. Hence $f_{n+1}(x) > 2x$, as required.
An exactly similar argument shows that $f_n(x) < 2x$ for x > 4. Hence x = 4 is the only solution.

Problem 3
Show that for any odd positive integer we can always divide the set {n, n+1, n+2, ... , n+32} into two parts, one with 14 numbers and one with 19, so that the numbers in each part can be arranged in a circle, with each

number relatively prime to its two neighbours. For example, for n = 1, arranging the numbers as 1, 2, 3, ... , 14 and 15, 16, 17, ... , 33, does not work, because 15 and 33 are not relatively prime.

Solution
Suppose we use n, n+1, ... , n+13 for the first circle. That certainly works for n not divisible by 13, since consecutive numbers are always relatively prime and any common divisor of n and n+13 must also divide their difference 13. We could then take the second circle to be n+15, n+14, n+16, n+17, ... , n+32 for $n \neq 2$ mod 17 or n+14, n+15, ... , n+29, n+30, n+32, n+31 if n = 2 mod 17. Note that any common factor of n+14 and n+16 must divide their difference 2, but n is odd, so they are relatively prime. Similarly, n+30 and n+32.

If n is divisible by 13, then n+19 is not, so we can take n+19, n+20, ... , n+32 for the first circle. Then we can take the second circle to be n+1, n, n+2, n+3, ... , n+18 for $n \neq 16$ mod 17 or n, n+1, ... , n+15, n+16, n+18, n+17 if n = 16 mod 17.

Problem 4
How many positive integers can be written in base n so that (1) the integer has no two digits the same, and (2) each digit after the first differs by one from an earlier digit? For example, in base 3, the possible numbers are 1, 2, 10, 12, 21, 102, 120, 210.

Solution
Answer: $2^{n+1} - 2n - 2$.
We use a more elaborate induction hypothesis. We claim that for base n+1, the following numbers of integers satisfy the two conditions: $2^{n+1}- 2n - 2$ not using the digit n; $2^n - 1$ with the digit n is the last position; 2^{n-1} with the digit n in the last but 1 position; 2^{n-2} with the digit n having 2 following digits; 2^{n-3} with the digit n having 3 following digits; ... ; 1 with the digit n having n following digits.

Thus for n = 2, the claim is that there are 2 numbers only involving 0 and 1 (1, 10), 3 numbers with 2 as the last digit (2, 12, 102), 2 numbers with one digit after 2 (21, 120) and 1 number with two digits after 2 (210). Suppose this holds for base n+1.

If we add up the various possibilities we get $2^{n+1} - 2n - 2 + (2^n - 1 + 2^{n-1} + 2^{n-2} + ... + 1) = 2^{n+1} - 2n - 2 + 2^{n+1} - 2 = 2^{n+2} - 2(n+1) - 2$. The number of base n+2 integers not involving the digit n+1 is the same as the number of base n+1 numbers, which is (by induction and the addition above) $2^{n+2} - 2(n+1) - 2$. If a base n+2 number has the digit n+1 in the last place, then either that is the only digit in the number or the earlier digits must form a base n+1 number with the digit n in it. There are $(2^n - 1 + 2^{n-1} + ... + 1) = 2^{n+1} - 2$ such numbers, so in total we have $2^{n+1} - 1$ base n+2 numbers with n+1 in the last place. If a base n+2 number has the digit n+1 in the penultimate place, then either the number has n+1 as the first digit, in which case it must be n+1 n, or the other digits form a base n+1 number with the digit n in the penultimate place or earlier. There are $2^{n-1} + ... + 1 = 2^n - 1$ such numbers. So 2^n in total. Similarly for the other possibilities.
Finally, we need to check the answer for n = 2 (1, 10, so two numbers and $2^{2+1} - 2\cdot 2 - 2 = 8 - 4 - 2 = 2$).

Problem 5
ABC is acute-angled. The circle diameter AB meets the altitude from C at P and Q. The circle diameter AC meets the altitude from B at R and S. Show that P, Q, R and S lie on a circle.

Solution
Use vectors. Take A as the origin. Let AB = **b**, AC = **c**, AR = **r**. AR is perpendicular to RC, so **r**.(**c** - **r**) = 0. BR is perpendicular to AC, so (**b** - **r**).**c** = 0. Hence **r**.**r** = **r**.**c** = **b**.**c**. Thus |AR| = √(|AB|·|AC| cos A). But the identical argument gives the same value for |AS|. The situtation is symmetrical between B and C, so we get the same result for |AP| and |AQ|. Hence all four points lie on a circle center A.

20th USAMO 1991

Problem 1
An obtuse angled triangle has integral sides and one acute angle is twice the other. Find the smallest possible perimeter.

Solution
Answer: 77 = 16 + 28 + 33.

Let the triangle be ABC with angle A obtuse and angle B = 2 angle C. Let the sides be a, b, c as usual. Note that a > b > c. We have b sin C = c sin 2C and $c^2 = a^2 + b^2 - 2ab \sin C$. Hence, b/2c = sin 2C/sin C = cos C = $(a^2 + b^2 - c^2)/2ab$. So $ab^2 = a^2c + b^2c - c^3$. Hence $b^2(a - c) = c(a^2 - c^2)$. Dividing by a - c we get $b^2 = c(a + c)$.

Now the triangle with smallest perimeter will have a, b, c relatively prime (otherwise we could divide by the common factor). Hence c must be a square. For if c and a+c have a common factor, then so do a and c and hence a, b and c, which means they cannot be the minimal set. Clearly c is not 1 (or the triangle would have nil area). c = 4 gives a = 5, which is too short, or a ≥ 21, which is too long. c = 9 gives a = 7 (too short), 18 (3 divides a, b, c), or ≥ 40 (too long). c = 16 gives a = 20 (too short) or a = 33 which works. Larger c gives a larger perimeter. Eg c = 25 gives a = 56, b = 45 (perimeter 126).

Problem 2
For each non-empty subset of {1, 2, ... , n} take the sum of the elements divided by the product. Show that the sum of the resulting quantities is $n^2 + 2n - (n + 1)s_n$, where $s_n = 1 + 1/2 + 1/3 + ... + 1/n$.

Solution
This is a straightforward induction. For n = 1 the only term in the sum is 1/1 with sum 1. The formula gives $1^2 + 2.1 - 2.1 = 1$. So it is true for n = 1.

Note first that for n the sum of the inverses of the products, including 1 for the empty set, is (1 + 1/1)(1 + 1/2)(1 + 1/3) ... (1 + 1/n) = n+1 (it telescopes). Now the sum for n+1 is the sum for n plus the sum of terms involving n+1. But if a term involves n+1, then the sum is increased by n+1 and the product is increased by a factor n+1. So the sum of all the terms involving n+1 is (1/n+1 x sum for n) + (sum of inverse products for n) = $(n^2 + 2n - (n+1)s_n)/(n+1) + (n+1) = n+1 - 1/(n+1) - s_n + (n+1) = 2(n+1) - s_{n+1}$. Hence the sum for n+1 is $(n^2 + 2n) - (n+1)s_n + 2(n+1) - s_{n+1} = (n+1)^2 + 2(n+1) - 1 - (n+1)s_n - s_{n+1} = (n+1)^2 + 2(n+1) - (n+1)/(n+1) - (n+1)s_n - s_{n+1} = (n+1)^2 + 2(n+1) - (n+2)s_{n+1}.$

Problem 3
Define the function f on the natural numbers by f(1) = 2, $f(n) = 2^{f(n-1)}$. Show that f(n) has the same residue mod m for all sufficiently large n.

Solution
This is the so-called tower of exponents, but HTML is not up showing it! The trick is to use induction on m (which is not at all obvious). The result is trivial for m = 1.

If m is even, then we can write $m = 2^a b$, where b is odd. Then f(n) is eventually constant mod b. Obviously f(n) is eventually 0 mod 2^a, so the residue mod m is eventually constant.

If m is odd, then $2^{\varphi(m)} = 1$ mod m, where $\varphi(m) < m$. So by induction f(n) is eventually constant mod $\varphi(m)$. Hence $f(n+1) = 2^{f(n)}$ is eventually constant mod m.

Problem 4
a and b are positive integers and $c = (a^{a+1} + b^{b+1})/(a^a + b^b)$. By considering $(x^n - n^n)/(x - n)$ or otherwise, show that $c^a + c^b \geq a^a + b^b$.

Solution
$(x^n - n^n)/(x - n) = x^{n-1} + x^{n-2}n + ... + n^{n-1}$. If x > n, then each term is $> n^{n-1}$, so $(x^n - n^n)/(x - n) > n^n$. If x < n, then each term is <= n^n with equality only for the last term, so $(x^n - n^n)/(x - n) < n^n$. So multiplying by x - n, which is positive for x > n and negative for x < n, we get $(x^n - n^n) > (x - n)n^n$ except for x = n, and hence $(x^n - n^n) \geq (x - n)n^n$ for all x ≥ 0.

Putting x = c, n = a, we get $(c^a - a^a) \geq (c - a)a^a$. Similarly, putting x = c, n = b, we get $(c^b - b^b) \geq (c - b)b^b$. Adding $(c^a + c^b) - (a^a + b^b) \geq (c - a)a^a + (c - b)b^b = c(a^a + b^b) - a^{a+1} - b^{b+1} = 0$, which is the required result.

Problem 5
X is a point on the side BC of the triangle ABC. Take the other common tangent (apart from BC) to the incircles of ABX and ACX which intersects the segments AB and AC. Let it meet AX at Y. Show that the locus of Y, as X varies, is the arc of a circle.

Solution
We show that AY = (AB + AC - BC)/2 = constant.

Let the common tangent meet the incircles of ABX, ACX at R, S respectively. Let AX meet them at P, Q respectively and let BC meet them at U, V respectively. Let AB meet the incircle of ABX at K and let AC meet the incircle of ACX at L. We have AY = AQ - QY and AY = AP - PY. So adding 2AY = AP + AQ - (YQ + YP) = AP + AQ - (YS + YR) = AP + AQ - RS. If we reflect about the line of centers of the two incircles R goes to U and S to V. Hence RS = UV. So 2AY = AP + AQ - UV. We have AP = AK = AB - BK = AB - BU and AQ = AL = AC - CL = AC - CV. Hence 2AY = AB + AC - BU - UV - CV = AB + AC - BC.

21st USAMO 1992

Problem 1
Let a_n be the number written with 2^n nines. For example, $a_0 = 9$, $a_1 = 99$, $a_2 = 9999$. Let $b_n = \Pi_0^n a_i$. Find the sum of the digits of b_n.

Solution
Answer: $9 \cdot 2^n$.

Induction on n. We have $b_0 = 9$, digit sum 9, and $b_1 = 891$, digit sum 18, so the result is true for n = 0 and 1. Assume it is true for n-1. Obviously $a_n < 10$ to the power of 2^n, so $b_{n-1} < 10$ to the power of $(1 + 2 + 2^2 + ... + 2^{n-1}) < 10$ to the power of 2^n. Now $b_n = b_{n-1}10^N - b_{n-1}$, where $N = 2^n$. But $b_{n-1} < 10^N$, so $b_n = (b_{n-1} - 1)10^N + (10^N - b_{n-1})$ and the digit sum of b_n is just the digit sum of $(b_{n-1} - 1)10^N$ plus the digit sum of $(10^N - b_{n-1})$.

Now b_{n-1} is odd and so its last digit is non-zero, so the digit sum of $b_{n-1} - 1$ is one less than the digit sum of b_{n-1}, and hence is $9 \cdot 2^{n-1} - 1$. Multiplying by 10^N does not change the digit sum. $(10^N - 1) - b_{n-1}$ has 2^n digits, each 9 minus the corresponding digit of b_{n-1}, so its digit sum is $9 \cdot 2^n - 9 \cdot 2^{n-1}$. b_{n-1} is odd, so its last digit is not 0 and hence the last digit of $(10^N - 1) - b_{n-1}$ is not 9. So the digit sum of $10^N - b_{n-1}$ is $9 \cdot 2^n - 9 \cdot 2^{n-1} + 1$. Hence b_n has digit sum $(9 \cdot 2^{n-1} - 1) + (9 \cdot 2^n - 9 \cdot 2^{n-1} + 1) = 9 \cdot 2^n$.

Problem 2
Let $k = 1^o$. Show that Σ_0^{88} 1/(cos nk cos(n+1)k) = cos k/$\sin^2$k.

Solution
tan(n+1)k - tan nk = (sin(n+1)k cos nk - sin nk cos(n+1)k)/(cos nk cos(n+1)k) = sin k /(cos nk cos(n+1)k). Using this expression the sum telescopes and we get $\sum_0^{88}$ 1/(cos nk cos(n+1)k) = (tan 89k - tan 0)/sin k. But tan 0 = 0 and tan 89k = cot(π/2 - 89k) = cot k.

Problem 3
A set of 11 distinct positive integers has the property that we can find a subset with sum n for any n between 1 and 1500 inclusive. What is the smallest possible value for the second largest element?

Solution
Answer: 248.

By taking the integers to be 1, 2, 4, 8, ... , 1024 we can generate all integers up to 2047. But by taking some integers smaller, we can do better. For example, 1, 2, 4, ... , 128, 247, 248, 750 gives all integers up to 1500. We can obviously use the integers 1, 2, 4, ... , 128 to generate all integers up to 255. Adding 248 gives all integers from 256 up to 503. Then adding 247 gives all integers from 504 to 750. So adding 750 gives all integers up to 1500. We show that we cannot do better than this.

Let the integers be $a_1 < a_2 < ... < a_{11}$. Put $s_n = a_1 + ... + a_n$. Take N such that $s_{N-1} < 1500 \le s_N$. Note that 1500 must be a sum of some of the integers so certainly $1500 \le s_{11}$. Equally we obviously have $a_1 = 1$, $a_2 = 2$, so N is well-defined and we have $1 < N \le 11$. Now if $1 < k \le N$, we have $s_{k-1} < 1500$ and hence $s_{k-1} + 1 \le 1500$. So some sum of distinct a_i must equal $s_{k-1} + 1$ and it must involve an a_i with $i > k-1$ since $s_{k-1} < s_{k-1} + 1$. Hence $a_k \le s_{k-1} + 1$ for $1 < k \le N$.

Now an easy induction gives $s_k \le 2_k - 1$. We must have $a_1 = 1$ and hence $s_1 = 1$, so it is true for $k = 1$. Suppose it is true for k-1. Then $a_k \le s_{k-1} + 1 \le 2^{k-1}$, so $s_k = s_{k-1} + a_k \le 2^{k-1} - 1 + 2^{k-1} = 2^k - 1$, so it is true for k. Hence for all $k \le N$. But $2^{10} -1 = 1023 < 1500$, so if $N < 11$, then $s_N < 1500$. Contradiction. Hence $N = 11$.

We have $s_k = s_{k-1} + a_k \le 2s_{k-1} + 1$. Hence $s_{k-1} \ge (s_k - 1)/2$. So $s_{11} \ge 1500$ implies $s_{10} \ge 750$. But $s_8 \le 255$, so $a_9 + a_{10} \ge 495$, so $a_{10} \ge 248$.

Problem 4
Three chords of a sphere are meet at a point X inside the sphere but are not coplanar. A sphere through an endpoint of each chord and X touches the sphere through the other endpoints and X. Show that the chords have equal length.

Solution
Let two of the chords be AB and CD. Take the plane containing them. In this plane we have a circle through A, B, C, D, and a circle through A, C, X which touches a circle through B, D, X at X. We show that AX = CX. Let the common tangent at X meet the larger circle at E and F. [Assume the points are in the order A, C, F, B, D, E as we go round the circle.] We have $\angle$XAC = $\angle$BAC (same angle) = $\angle$BDC (circle ABCD) = $\angle$BXF (XF tangent to BXD) = $\angle$EXA (opposite angle) = $\angle$XCA (EX tangent to AXC). So XAC is isosceles. So XA = XC. Similarly XB = XD. Hence AB = CD.
Similarly, the other pairs of chords are equal.

Problem 5
A complex polynomial has degree 1992 and distinct zeros. Show that we can find complex numbers z_n, such that if $p_1(z) = z - z_1$ and $p_n(z) = p_{n-1}(z)^2 - z_n$, then the polynomial divides $p_{1992}(z)$.

Solution
Let the polynomial of degree 1992 be q(z). Suppose its roots are $w_1, w_2, ... , w_{1992}$. Let $S_1 = \{w_1, ... , w_{1992}\}$. We now define S_2 as follows. Let $z_1 = (w_1 + w_2)/2$ and take S_2 to be the set of all numbers $(w - z_1)^2$ with w in S_1. Note that $w = w_1$ and $w = w_2$ give the same number. It is possible that other pairs may also give the same number. But certainly $|S_2| <= |S_1| - 1$. We now repeat this process until we get a set with only one member. Thus if S_i has more than one member, then we take we take z_i to be an average of any two *distinct* members. Then we take S_{i+1} to be the set of all $(w - z_i)^2$ with w in S_i. So after picking at most 1991 elements z_i we have a set S_N with only one member. Take z_N to be that one member, so that $S_{N+1} = \{0\}$. Now if $N < 1991$, take the remaining z_i to be 0 until we reach z_{1992}.

Now as we allow z to take the values in S:

$p_1(z) = (z - z_1)$ takes 1992 possible values;
$p_2(z) = (z - z_1)^2 - z_2$ takes at most 1991 possible values;
$p_3(z) = ((z - z_1^2 - z_2)^2 - z_3$ takes at most 1990 possible values;
$p_4(z) = (((z - z_1)^2 - z_2)^2 - z_3)^2 - z_4$ takes at most 1989 possible values;
...
$p_{1992}(z)$ takes only the value 0.

So every root of q(z) is also a root of $p_{1992}(z)$. Hence q(z) must divide $p_{1992}(z)$.

22nd USAMO 1993

Problem 1
$n > 1$, and a and b are positive real numbers such that $a^n - a - 1 = 0$ and $b^{2n} - b - 3a = 0$. Which is larger?

Solution
Answer: $a > b$.

Note that $a^n = a + 1 > 1$ (since a is positive). Hence $a > 1$. So $a^{2n} = (a + 1)^2 = a^2 + 2a + 1$. Put $a = 1 + k$, then $a^2 = 1 + 2k + k^2 > 1 + 2k$, so $a^2 + 1 > 2 + 2k = 2a$. Hence $a^{2n} > 4a$. So $(b/a)^{2n} = (b + 3a)/a^{2n} < (b + 3a)/4a$. If $b/a \geq 1$, then $(b + 3a)/4a \leq (b + 3b)/4a = b/a$, so $(b/a)^{2n} < b/a$. Contradiction. Hence $b/a < 1$.

Problem 2
The diagonals of a convex quadrilateral meet at right angles at X. Show that the four points obtained by reflecting X in each of the sides are cyclic.

Solution
If we shrink the reflected points by 1/2 about the point X, then we get the feet of the perpendiculars from X to the sides. So it is sufficient to show that these four points are cyclic. Let the quadrilateral be ABCD. Let the feet of the perpendiculars from X to AB, BC, CD, DA be P, Q, R, S respectively.

XPBQ is cyclic, so $\angle XQP = \angle XBP$. Similarly, XPAS is cyclic, so $\angle XSP = \angle XAP$. But XBP and XAP are two angles in the triangle XAB and the third is $\angle AXB = 90^o$. Hence $\angle XQP + \angle XSP = 90^o$. Similarly $\angle XQR + \angle XSR = 90^o$. Adding, $\angle PQR + \angle PSR = 180^o$, so PQRS is cyclic.

Problem 3
Let S be the set of functions f defined on reals in the closed interval [0, 1] with non-negative real values such that $f(1) = 1$ and $f(x) + f(y) \leq f(x + y)$ for all x, y such that $x + y \leq 1$. What is the smallest k such that $f(x) \leq kx$ for all f in S and all x?

Solution
Answer: $k = 2$.

Consider the function $f(x) = 0$ for $0 \leq x \leq 1/2$, 1 for $1/2 < x \leq 1$. If $x + y \leq 1$, then at least one of x, y is $\leq 1/2$, so at least one of f(x), f(y) is 0. But f is obviously increasing, so the other of f(x), f(y) is $\leq f(x + y)$. Thus f satisfies the conditions. But $f(1/2 + \varepsilon) = 1$, so k cannot be smaller than 2.

So now let f be any function satisfying the conditions. We wish to show that $f(x) \leq 2x$. Putting $y = 1-x$, we get $f(x) + f(y) <= f(1) = 1$. But f(y) is non-negative, so $f(x) \leq 1$. Put $y = x \leq 1/2$. Then $2f(x) \leq f(2x)$. A simple induction gives $2^n f(x) \leq f(2^n x)$ for $x \leq 1/2^n$. Now take any x in [0, 1]. If $x > 1/2$, then $2x > 1$, so $f(x) < 2x$. If $x \leq 1/2$, choose $n \geq 1$, so that $1/2^{n+1} < x \leq 1/2^n$. Then $2^n f(x) \leq f(2^n x) \leq 1$, so $f(x) \leq 1/2^n \leq 2x$.

Problem 4
The sequence a_n of odd positive integers is defined as follows: $a_1 = r$, $a_2 = s$, and a_n is the greatest odd divisor of $a_{n-1} + a_{n-2}$. Show that, for sufficiently large n, a_n is constant and find this constant (in terms of r and s).

Solution
This is awkward to get started. Note that if $a_{n-1} = a_{n-2}$, then $a_{n-1} + a_{n-2} = 2a_{n-1}$, whose greatest odd divisor is just a_{n-1}, so $a_n = a_{n-1}$. So once two consecutive terms are constant the following terms are constant.

Both a_{n-1} and a_{n-2} are odd, so $a_{n-1} + a_{n-2}$ is even and hence $a_n \leq (a_{n-1} + a_{n-2})/2$. So if a_{n-1} and a_{n-2} are unequal, then $a_n < \max(a_{n-1}, a_{n-2})$. As already noted, if they are equal, then $a_n = \max(a_{n-1}, a_{n-2})$.

Put $b_n = \max(a_n, a_{n-1})$. If $a_n < a_{n-1}$, then $a_{n+1} < \max(a_n, a_{n-1}) = a_{n-1}$. Also $a_{n+2} \leq \max(a_{n+1}, a_n) < a_{n-1}$. Hence $\max(a_{n+2}, a_{n+1}) < \max(a_n, a_{n-1})$ or $b_{n+2} < b_n$. If $a_n > a_{n-1}$, then $a_{n+1} < \max(a_n, a_{n-1}) = a_n$, and $a_{n+2} < \max(a_{n+1}, a_n) =$

a_n. Hence $b_{n+2} < b_n$. Thus if a_n and a_{n-1} are unequal, then $b_{n+2} < b_n$. But obviously $a_n > 0$ for all n, and so $b_n > 0$ for all n. Also it is integral, and $b_1 = \max(r, s)$, so we can only have $b_{2n+1} < b_{2n-1}$ for at most max(r, s) values of n. Hence for some n we must have $a_n = a_{n-1}$ and then a_n is constant from that point on.

We show that the constant is the greatest common divisor of r and s. Use parentheses to denote the greatest common divisor, so the greatest common divisor of r and s is (r, s). We have $(a_{n-1}, a_{n-2}) = (a_{n-1}, a_{n-1} + a_{n-2}) = (a_n, a_{n-1})$. So if a_n is constant for $n \geq N$, we have $(r, s) = (a_1, a_2) = (a_2, a_3) = ... = (a_N, a_{N+1}) = a_N$. We claim that $a_n = d$ for $n > 3$. Clearly d divides r + s and hence d divides a_3. But d divides $a_2 = s$, so d also divides $a_2 + a_3$ and hence d divides a_4. Now suppose some odd k divides r + s, but does not divide s. So k divides a_3, but not a_2. Hence it does not divide a_4.

Problem 5
A sequence x_n of positive reals satisfies $x_{n-1}x_{n+1} \leq x_n^2$. Let a_n be the average of the terms $x_0, x_1, ... , x_n$ and b_n be the average of the terms $x_1, x_2, ... , x_n$. Show that $a_nb_{n-1} \geq a_{n-1}b_n$.

Solution
Put $k = x_1 + x_2 + ... + x_{n-1}$. We have to show that $(x_0 + k + x_n)/(n+1) \quad k/(n-1) \geq (x_0 + k)/n \quad (k + x_n)/n$ or $n^2(k + x_0 + x_n) \geq (n^2 - 1)(k + x_0)(k + x_n)$. Simplifying slightly, this is equivalent to $k(k + x_0 + x_n) \geq (n^2 - 1)x_0x_n$. So it is evidently sufficient to show that $k \geq (n-1)(x_0x_n)^{1/2}$. [Then AM/GM gives $x_0 + x_n \geq 2(x_0x_n)^{1/2}$, so $(k + x_0 + x_n) \geq (n+1)(x_0x_n)^{1/2}$.]

We have $x_0/x_1 \leq x_1/x_2 \leq ... \leq x_{n-1}/x_n$. Hence (apparently weakening) $x_0x_n \leq x_1x_{n-1} \leq x_2x_{n-2} \leq ...$. So $k = (x_1 + x_{n-1}) + (x_2 + x_{n-2}) + ... \geq (n-1) (x_0x_n)^{1/2}$, using first AM/GM, then the relation just established.

23rd USAMO 1994

Problem 1
$a_1, a_2, a_3, ...$ are positive integers such that $a_n > a_{n-1} + 1$. Put $b_n = a_1 + a_2 + ... + a_n$. Show that there is always a square in the range $b_n, b_n+1, b_n+2, ... , b_{n+1}-1$.

Solution
If the result fails then for some m we have $b_n > m^2$ and $b_{n+1} \leq (m+1)^2$. So $b_{n+1}^{1/2} - b_n^{1/2} < 1$. Thus it is sufficient to prove that $b_{n+1}^{1/2} - b_n^{1/2} \geq 1$. Squaring, that is equivalent to $a_{n+1} \geq 2b_n^{1/2} + 1$ or $b_n^{1/2} \leq (a_{n+1} - 1)/2$.

We have $a_{n-1} <= a_n - 2$. So $b_n \leq a_n + (a_n - 2) + (a_n - 4) - ... - (a_n - 2(n-1))$. Adding extra terms to the rhs if necessary we get $b_n = 1 + 3 + 5 + ... + a_n$ for a_n odd, or $2 + 4 + 6 + ... + a_n$ for a_n even. In the odd case there are $(a_n + 1)/2$ terms of average size $(a_n + 1)/2$ (group them in pairs working from the outside in), so the sum is $(a_n + 1)^2/4$. In the even case there are $a_n/2$ terms of average size $(a_n + 2)/2$, so the sum is $a_n(a_n + 2)/4$. So in either case we have $b_n \leq (a_n + 1)^2/4$ and hence $b_n^{1/2} \leq (a_n + 1)/2 \leq (a_{n+1} - 1)/2$.

Problem 2
The sequence $a_1, a_2, ... , a_{99}$ has $a_1 = a_3 = a_5 = ... = a_{97} = 1$, $a_2 = a_4 = a_6 = ... = a_{98} = 2$, and $a_{99} = 3$. We interpret subscripts greater than 99 by subtracting 99, so that a_{100} means a_1 etc. An allowed move is to change the value of any one of the a_n to another member of {1, 2, 3} different from its two neighbors, a_{n-1} and a_{n+1}. Is there a sequence of allowed moves which results in $a_m = a_{m+2} = ... = a_{m+96} = 1$, $a_{m+1} = a_{m+3} = ... = a_{m+95} = 2$, $a_{m+97} = 3$, $a_{n+98} = 2$ for some m? [So if m = 1, we have just interchanged the values of a_{98} and a_{99}.]

Solution
This is a classic invariant problem. We strongly suspect that there is no sequence of moves (otherwise it would be too easy to find), so that we must prove there is no sequence. The standard approach is to look for some invariant which is not changed by the allowed moves, but which is different for the initial and desired final positions.

For each member a_i of the sequence let b_i =
0 if $a_i = a_{i+1}$,

1 if $(a_i, a_{i+1}) = (1, 2)$, $(2, 3)$ or $(3, 1)$
-1 if $(a_i, a_{i+1}) = (1, 3)$, $(2, 1)$ or $(3, 2)$.

We hope that $b_1 + b_2 + ... + b_{99}$will be a suitable invariant. Suppose we make an allowed move by changing a_i. That has the effect of changing just b_{i-1} and b_i. If $a_{i-1} = a_{i+1}$, then $b_{i-1} + b_i = 0$, so the total of the b_j does not change. If a_{i-1} does not equal a_{i+1}, then we cannot change a_i since it must be different from both. Thus allowed moves do not change $b_1 + ... + b_{99}$. So it is an invariant. We now check it is suitable. The initial value is + 3 (there are 49 pairs (1,2), 48 pairs (2,1), 1 pair (2,3) and 1 pair (3,1) total 49 - 48 + 1 + 1 = 3. But the desired final position has value -3 (it has 48 pairs (1, 2), 49 pairs (2, 1), 1 pair (1, 3) and 1 pair (3, 2), total 48 - 49 - 1 - 1 = - 3). So we cannot get from one to the other by allowed moves.

Problem 3
The hexagon ABCDEF has the following properties: (1) its vertices lie on a circle; (2) AB = CD = EF; and (3) the diagonals AD, BE, CF meet at a point. Let X be the intersection of AD and CE. Show that $CX/XE = (AC/CE)^2$.

Solution
Let the diagonals AD, BE, CF meet at Y. We show first that the triangles AEC, YED are similar. $\angle ACE = \angle ADE$ (ACDE circle) = YDE (same angle). $\angle AEB = \angle CED$ (AB = CD), so $\angle AEB + \angle BEC = \angle CED + \angle BEC$ or $\angle AEC = \angle YED$. So AEC and YED are similar, so AC/CE = YD/DE.

We next show that AEC and CDY are similar. $\angle AEC = \angle ADC$ (circle) = $\angle CDY$ (same angle). $\angle EAC = \angle EAD + \angle DAC = \angle ECD$ (circle) + $\angle ECF$ (EF = CD) = $\angle DCY$. So AEC and CDY are similar. So AC/CE = CY/YD.

Hence $(AC/CE)^2 = CY/DE$. Finally we show that CXY and EXD are similar. That is almost obvious because CF is parallel to DE since CD = EF. Hence CY/DE = CX/XE, which gives the required result.

Problem 4
x_i is a infinite sequence of positive reals such that for all n, $x_1 + x_2 + ... + x_n \geq \sqrt{n}$. Show that $x_1^2 + x_2^2 + ... + x_n^2 > (1 + 1/2 + 1/3 + ... + 1/n) / 4$ for all n.

Solution
Note that this is rather a weak inequality. Taking n = 1, we get $x_1 >= 1$, but $(1 + 1/2 + 1/3 + ... + 1/30) < 4$, so it is only for n > 30 that we need to consider x_2! Of course, weak inequalities can be awkward to prove.

For a + b constant, we minimise $a^2 + b^2$ by taking |a - b| as small as possible. So we suspect that the minimum value of $x_1^2 + ... + x_n^2$ is when $x_1 = 1$, $x_2 = \sqrt{2} - 1$, $x_3 = \sqrt{3} - \sqrt{2}$, $x_4 = \sqrt{4} - \sqrt{3}$, ... (*). Note that $\sqrt{n} - \sqrt{(n-1)} = 1/(\sqrt{n} + \sqrt{(n-1)}) > 1/(2\sqrt{n})$. So for the values (*) we have $x_1^2 + ... + x_n^2 > (1 + 1/2 + 1/3 + ... + 1/n)/4$. So it remains to show that the values (*) do indeed give the minimum.

We use Abel's partial summation formula (whose proof is trivial). Put $y_n = \sqrt{n} - \sqrt{(n-1)}$, $s_n = x_1 + x_2 + ... + x_n$, $t_n = y_1 + ... + t_n$. So we assume $s_n \geq t_n$. Note also that $y_1 > y_2 > y_3 > ...$. Then the partial summation formula is: $\sum x_i y_i = \sum s_i(y_i - y_{i+1}) + s_n y_{n+1}$ (where the sum is taken from 1 to n). We also have $\sum y_i^2 = \sum t_i(y_i - y_{i+1}) + t_n y_{n+1}$. But each term is not more than the corresponding term in the first equality, so $\sum y_i^2 \leq \sum x_i y_i$.
Now Cauchy-Schwartz gives $(\sum x_i y_i)^2 \leq \sum x_i^2 \sum y_i^2$. Hence $\sum y_i^2 \leq \sum x_i^2$, as required.

Problem 5
X is a set of n positive integers with sum s and product p. Show for any integer N >= s, ∑(parity(Y) (N - sum(Y))Cs) = p, where aCb is the binomial coefficient a!/(b! (a-b)!), the sum is taken over all subsets Y of X, parity(Y) = 1 if Y is empty or has an even number of elements, -1 if Y has an odd number of elements, and sum(Y) is the sum of the elements in Y.

Solution
($\sum (-1)^n$ binomial) with the sum over all subsets Y of a set X strongly suggests the principle of inclusion and exclusion.

Consider all sequences of length $N \geq s$ of 0s and 1s with a total of s 1s. We can regard positions 1, 2, ... , a_1 as block a_1, positions $a_1 + 1$, $a_2 + 2$, ... , $a_1 + a_2$ as block a_2, positions $a_1+a_2 + 1$, $a_1+a_2 + 2$, ... , $a_1+a_2 + a_3$ as block a_3, and so on up to $a_1 + ... + a_{n-1} + 1$, ... $a_1 + ... + a_n$ as block a_n. We are interested in sequences which have just one 1 in each block. Counting directly there are obviously just p such sequences.

But we can also take N' to be the total number of sequences of 0s and 1s with s 1s and N_Y to be the number with no 1s in block k for k in Y, where Y is a subset of X. Requiring one 1 in each block is equivalent to requiring no block to have no 1s. So the PIE gives that p = N' - N_Y for Y with one element + N_Y for Y with 2 elements and so on. N' = NCs and N_Y = (N - sum(Y))Cs, so we have p = $\sum$(parity(Y) (N - sum(Y))Cs), which is the required relation.

24th USAMO 1995

Problem 1
The sequence a_0a_1, a_2, ... of non-negative integers is defined as follows. The first p-1 terms are 0, 1, 2, 3, ... , p-2. Then a_n is the least positive integer so that there is no arithmetic progression of length p in the first n+1 terms. If p is an odd prime, show that a_n is the number obtained by writing n in base p-1, then treating the result as a number in base p. For example, if p is 5, to get the 5th term one writes 5 as 11 in base 4, then treats this as a base 5 number to get 6.

Solution
Let b_n be the number obtained by writing n in base p-1 and then treating the result as a number in base p. The resulting sequence b_n is all those non-negative integers whose base p representation does not have a digit p-1. We show that b_n cannot contain an arithmetic progression of length p. For suppose there was such a progression with difference d. Suppose the last non-zero digit of d in base p is k. Suppose the first term of the progression has digit h in that position, then the terms of the progression have digit h, h+k, h+2k, ... h+(p-1)k mod p in that position. But these must be a complete set of residues mod p, so one of them must be p-1 mod p. So the corresponding term has a digit p-1 in this position. Contradiction.

Now to show that $a_n = b_n$ we use induction on n. Evidently, it is true for n < p-1. Suppose it is true for all m < n. It is sufficient to show that if $b_n < m < b_{n+1}$, then $\{b_1, b_2, ... , b_n, m\}$ contains an arithmetic progression. m must contain a digit p-1, for otherwise it would be b_k for some k > n. Let m_1 be the number obtained from m by reducing *every* digit p-1 in m by 1. Then m has no digit p-1s, so it must be some b_k and hence one of b_1, ... , b_n. Now take m_2 to be the number obtained by reducing the same digits by another 1. Similarly, define m_3, ... , m_{p-1}. Then each m_i is in $\{b_1, ... , b_n\}$ and m_{p-1}, ... , m_1, m is a progression of length p.

Problem 2
A trigonometric map is any one of sin, cos, tan, arcsin, arccos and arctan. Show that given any positive rational number x, one can find a finite sequence of trigonometric maps which take 0 to x. [So we need to show that we can always find a sequence of trigonometric maps t_i so that: $x_1 = t_0(0)$, $x_2 = t_1(x_1)$, ... , $x_n = t_{n-1}(x_{n-1})$, $x = t_n(x_n)$.]

Solution
We have $\cos^2 t + \sin^2 t = 1$. Hence $\cos t = \cos t/(\sqrt{(\cos^2 t + \sin^2 t)}) = 1/(\sqrt{(1 + \tan^2 t)})$. So if we put $\tan t = \sqrt{x}$, then $\cos \tan^{-1}\sqrt{x} = 1/\sqrt{(1 + x)}$. We also have $\cos(\pi/2 - x) = \sin x$ and $\tan(\pi/2 - x) = 1/\tan x$. So $\tan \cos^{-1} \sin \tan^{-1} x = 1/x$ (1). Hence also $\tan \cos^{-1} \sin \tan^{-1} \cos \tan^{-1} \sqrt{x} = \sqrt{(x + 1)}$ (2). These two relations solve the problem.

Using (2) and iterating we can get $\sqrt{n}$ for any positive integer n. Hence *in particular* we can get n for any positive integer n. Now suppose we want m/n with m and n relatively prime. We show that m/n can be achieved by induction on n. We have just dealt with the case n = 1. Suppose we have dealt with all a/b with b < n. If m > n, then we can write m = qn + r with 0 < r < n and use (2) to reduce the problem to getting r/n. If m < n, then put r = m. Now use (1) to reduce the problem to n/r, which is solved by induction.

Problem 3
The circumcenter O of the triangle ABC does not lie on any side or median. Let the midpoints of BC, CA, AB be L, M, N respectively. Take P, Q, R on the rays OL, OM, ON respectively so that $\angle$OPA = $\angle$OAL, $\angle$OQB = $\angle$OBM and $\angle$ORC = $\angle$OCN. Show that AP, BQ and CR meet at a point.

Solution
We show that the circumcircle ABC is the incircle of PQR. Then (AR/AQ) (CQ/CP) (BP/BR) = 1 since AR = BR, AQ = CQ, BP = CP (equal tangents) and hence PA, QB, RC are concurrent by Ceva's theorem.

So let the tangents to the circumcircle at B and C meet at P'. It is sufficient to show that $\angle$OP'A = $\angle$OAL, for then it follows that P' = P (since there is obviously a unique point on the ray OL at which the segment OA subtends the $\angle$OAL).

This is curiously difficult to prove. Let LP' meet the circle at K. Then $\angle$KCP' = $\angle$KBC (P'C tangent) = $\angle$KCB (KO perpendicular to BC, since L midpoint) = $\angle$KCL (same angle). So KC bisects $\angle$P'CL. Hence KP'/KL = CP'/CL. But obviously CP'/CL = BP'/BL. So K, C and B lie on the circle of Apollonius, the points X such that XP'/XL is constant. But A also lies on the circle of Apollonius. Hence KA bisects $\angle$P'AL. (*See Canada 71/9 if you are not familiar with the circle of Apollonius.)* But $\angle$OP'A + $\angle$KAP' = $\angle$OKA = $\angle$OAK (OA and OK radii) = $\angle$OAL + $\angle$KAL. So $\angle$OP'A = $\angle$OAL.

Problem 4
$a_0, a_1, a_2, \ldots$ is an infinite sequence of integers such that $a_n - a_m$ is divisible by n - m for all (unequal) n and m. For some polynomial p(x) we have $p(n) > |a_n|$ for all n. Show that there is a polynomial q(x) such that $q(n) = a_n$ for all n.

Solution
Clearly for any finite N, we can find a polynomial q(n) of degree N such that $q(n) = a_n$ for n = 0, ... , N. Also, once $a_0, a_1, \ldots, a_N$ are fixed, a_m is somewhat constrained for m > N, because we require a_m to be a_0 mod m, a_1 mod m-1, ... , a_N mod m-N. Put M = lcm(m, m-1, ... , m-N). Then a_m is determined mod M.

We would like to argue that q(m) is known to satisfy the congruences (because m - n divides q(m) - q(n)), that q(m) and a_m are bounded so that they cannot differ by as much as M and hence must be equal. The snag is that q(x) does not necessarily have integer coeffcients, so it is not necessarily true that m - n divides q(m) - q(n). [The argument is that m - n divides $m^k - n^k$ for any integer k and hence divides q(m) - q(n) for any polynomial with integer coefficients.]

However, the coefficients of q(x) are rational. So let Q be the lcm of the denominators. Then Q q(x) does have integer coefficients and m - n does divide Q(q(m) - q(n)). So Q q(m) = Q q(0) = Q a_0 = Q a_m mod m, and similarly Q q(m) = Q a_m mod m-1 and so on. Hence Q q(m) = Q a_m mod M.

But we know that $|q(m)| < c\, m^N$ for some constant c, since q(x) has degree N. Similarly, we know that $|a_m|$ is bounded by a polynomial of degree N and hence $|a_m| < c'\, m^N$ for some constant c'. Now M certainly exceeds the product of the N+1 numbers m, m-1, ... , m-N divided by the greatest common divisor for each of the N(N+1)/2 pairs (m - i, m - j). But each gcd is at most N, so M is more than $c''\, m^{N+1}$ for some (small) constant c'' which does not depend on m. Hence for m $\ge$ some N' we have M > |Q q(m)| + |Q a_m| and hence $q(m) = a_m$.

So we have $q(m) = a_m$ for m $\le$ N and for m $\ge$ N'. Suppose N < m < N'. Then for *any* m' > N' we have that (m' - m) divides Q q(m') - Q q(m) and Q $a_{m'}$ - Q a_m. So it must divide their difference. But $q(m') = a_{m'}$, so it divides Q(q(m) - a_m). But we can choose m' so that (m' - m) is prime to Q and larger than (q(m) - a_m). Hence $q(m) = a_m$. So we have established that $q(m) = a_m$ for all m.

Problem 5
A graph with n points and k edges has no triangles. Show that it has a point P such that there are at most $k(1 - 4k/n^2)$ edges between points not joined to P (by an edge).

Solution
Given a point P, the edges of the graph can be divided into three categories: (1) edges PQ, (2) edges QQ', where Q is joined to P, and (3) edges Q'Q'', where Q' and Q'' are not joined to P. Note that if Q is joined to P and there is an edge QQ', then Q' cannot be joined to P, or PQQ' would be a triangle. So the total number of edges in categories (1) and (2) is $\sum'$ deg Q, where $\sum'$ indicates that the sum is taken over points Q which are joined to P (by an edge). Thus the total number of edges in category (3) is k - $\sum'$ deg Q. So we have to show that for some P, $\sum'$ deg Q $\geq 4k^2/n^2$.

The total for all points P is $\sum_P\sum'$ deg Q. If we invert the order of the summations, this becomes $\sum \deg^2 Q$, where the sum is over all points Q. Now by Cauchy, $(\sum \deg Q)^2 \leq (\sum 1^2)(\sum \deg^2 Q) = n \sum \deg^2 Q$. But $\sum \deg Q = 2k$, so $\sum \deg^2 Q \geq 4k^2/n$ and hence $\sum_P\sum' \deg Q \geq 4k^2/n$. But if a sum of n terms is at least $4k^2/n$, then at least one for the terms must be at least $4k^2/n^2$. In other words, there is a point P such that $\sum' \deg Q \geq 4k^2/n^2$, as required.

25th USAMO 1996

Problem A1
Let $k = 1^o$. Show that 2 sin 2k + 4 sin 4k + 6 sin 6k + ... + 180 sin 180k = 90 cot k.

Solution
Multiply the expression by sin k. We have 2 sin 2nk sin k = cos(2n-1)k - cos(2n+1)k. So 2n sin 2nk sin k = n cos(2n-1)k - n cos(2n+1)k. Adding these equations for n = 1, 2, ... , 90 gives: 2 sin 2k + 4 sin 4k + 6 sin 6k + ... + 180 sin 180k = cos k + (2 - 1) cos 3k + (3 - 2) cos 5k + ... + (90 - 89) cos 179k - 90 cos 181k. Now cos 181k = - cos k, so the last term is + 90 cos k. The other terms sum to zero in pairs: cos k + cos 179k = 0, cos 3k + cos 177k = 0, ... , cos 89k + cos 91k = 0. Hence result.

Problem A2
Let S be a set of n positive integers. Let P be the set of all integers which are the sum of one or more distinct elements of S. Show that we can find n subsets of P whose union is P such that if a, b belong to the same subset, then $a \leq 2b$.

Solution
Let the members of S be $a_1 < a_2 < ... < a_n$. Let $s_m = a_1 + a_2 + ... + a_m$ and put $s_0 = 0$. Let $P_m = \{ s \in P : s_{m-1} < s \leq s_m \}$ for m = 1, 2, ... , n. We claim that this partition works. It is sufficient to show that if $b \in P_m$ then $2b > s_m$ (for then if a also belongs to P_m we have $a \leq s_m <= 2b$).

Now since $b > s_{m-1}$, b must be a sum which includes some a_i with $i \geq m$. So certainly $b \geq a_i \geq a_m = s_m - s_{m-1} > s_m - b$. Hence $2b > s_m$ as required.

Problem A3
Given a triangle, show that we can reflect it in some line so that the area of the intersection of the triangle and its reflection has area greater than 2/3 the area of the triangle.

Solution
Let the triangle be ABC. Assume A is the largest angle. Let AD be the altitude. Assume $AB \leq AC$, so that $BD \leq BC/2$. If BD > BC/3, then reflect in AD. If B' is the reflection of B', then B'D = BD and the intersection of the two triangles is just ABB'. But BB' = 2BD > 2/3 BC, so ABB' has more than 2/3 the area of ABC.

If BD < BC/3, then reflect in the angle bisector of C. The reflection of A' is a point on the segment BD and not D. (It lies on the line BC because we are reflecting in the angle bisector. A'C > DC because $\angle$CAD < $\angle$CDA = 90^o. Finally, $A'C \leq BC$ because we assumed $\angle$B does not exceed $\angle$A). The intersection is just AA'C. But area AA'C/area ABC = CA'/CB > CD/CB $\geq$ 2/3.

Problem B1
A type 1 sequence is a sequence with each term 0 or 1 which does not have 0, 1, 0 as consecutive terms. A

type 2 sequence is a sequence with each term 0 or 1 which does not have 0, 0, 1, 1 or 1, 1, 0, 0 as consecutive terms. Show that there are twice as many type 2 sequences of length n+1 as type 1 sequences of length n.

Solution
Let S be the set of sequences length n and T the set of sequences length n+1 beginning with 0. Define f: S → T as follows. Let the m+1th term of f(s) be the same as the mth term if the mth term of s is 0 and different if the mth term of s is 1. It is clear that f is a bijection. [Define its inverse g by g(t) has 0 as its mth term iff the mth and m+1th terms of t are the same.] Also f(s) includes 0, 0, 1, 1 or 1, 1, 0, 0 iff s includes 0, 1, 0. Hence the number of type 1 sequences in S is the same as the number of type 2 sequences in T. The same result holds if we take T to be sequences which begin with 1.

Problem B2
D lies inside the triangle ABC. $\angle$BAC = 50°. $\angle$DAB = 10°, $\angle$DCA = 30°, $\angle$DBA = 20°. Show that $\angle$DBC = 60°.

Solution
Reflect A in the line BD to get A'. Let Z be the intersection of BD and AA'. Let BA' meet AC at X. Since $\angle$ABX = 2 $\angle$ABD = 40°, and $\angle$BAX = 50°, we have $\angle$BXA = 90°. Now $\angle$DAA' = $\angle$BAA' - $\angle$DAB = $\angle$BAA' - 10°. But $\angle$BAA' = 90° - $\angle$DBA = 70°, so $\angle$DAA' = 60°.

Let BX meet CD at Y. $\angle$DYX = $\angle$YXC + $\angle$DCX = 90° + 30° = 120° = 180° - angle DAA', so DAA'Y is cyclic, so $\angle$A'YA = $\angle$A'DA = 2 $\angle$ZDA = 2($\angle$DBA + $\angle$DAB) = 60°.

But $\angle$XYC = 90 - $\angle$DCA = 60°, so C is the reflection of A in BX. Hence BC = BA, so $\angle$ACB = $\angle$BAC = 50°. Hence $\angle$ABC = 80° and $\angle$DBC = 80° - $\angle$DBA = 60°.

Problem B3
Does there exist a subset S of the integers such that, given any integer n, the equation n = 2s + s' has exactly one solution in S? For example, if T = {-3, 0, 1, 4), then there are unique solutions -3 = 2·0 - 3, -1 = 2·1 - 3, 0 = 2·0 + 0, 1 = 2·0 + 1, 2 = 0 + 2·1, 3 = 2·1 + 1, 4 = 2·0 + 4, 5 = 2·-3 + 1, but not for 6 = 2·1 + 4 = 2·-3 + 0, so T cannot be a subset of S.

Solution
Answer: yes.

We show how to choose S inductively. Suppose we have already chosen $a_1, a_2, \dots , a_n$, but we do not yet have a solution for m ≥ 0. Take N so that all $|a_i| < N$. Now take $a_{n+1} = 5N + m$, $a_{n+2} = -10N - m$. This gives a solution for m: $m = 2a_{n+1} + a_{n+2}$, but it does not duplicate any existing solutions, since $|2a_{n+2} + a_{n+1}|$, $|2a_{n+1} + a_i|$, $|2a_{n+2} + a_i|$, $|a_{n+1} + 2a_i|$, $|a_{n+2} + 2a_i|$ are all ≥ 3N, whereas all existing sums have absolute value < 3N. Similarly for m < 0, we may take $a_{n+1} = - 5N + m$, $a_{n+2} = 10N - m$.

26th USAMO 1997

Problem A1
Let p_n be the nth prime. Let 0 < a < 1 be a real. Define the sequence x_n by $x_0 = a$, x_n = the fractional part of p_n/x_{n-1} if $x_n \neq 0$, or 0 if $x_{n-1} = 0$. Find all a for which the sequence is eventually zero.

Solution
Let {x} denote the fractional part of x. {x} = x minus some integer, so {x} is rational iff x is rational. Hence if x_n is irrational, then p_{n+1}/x_n is irrational and hence x_{n+1} is irrational. So if a is irrational, then the sequence is never zero.

Suppose $x_n = r/s$ with 0 < r < s relatively prime integers. Then $x_{n+1} = u/r$, where u is the remainder on dividing sp_{n+1} by r. So, when expressed as a fraction in lowest terms, the denominator of x_{n+1} is less than that for x_n. So if a is rational and has denominator b, then after at most b iterations we get zero.

Problem A2
ABC is a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent.

Solution
Suppose that the feet of the perpendiculars from A and P to EF are H and K respectively. Then $AF^2 - AE^2 = (AH^2 + FH^2) - (AH^2 + EH^2) = FH^2 - EH^2 = (FH + EH)(FH - EH) = FE(FH - EH)$. Similarly, $PF^2 - PE^2 = FE(FK - EK)$. So H and K coincide iff $AF^2 - AE^2 = PF^2 - PE^2$. In other words, P lies on the line through A perpendicular to EF iff $PF^2 - PE^2 = AF^2 - AE^2$.

Thus if P is the intersection of the line through A perpendicular to EF and the line through B perpendicular to FD, then $PF^2 - PE^2 = AF^2 - AE^2$ and $PD^2 - PF^2 = BD^2 - BF^2$. Hence $PD^2 - PE^2 = AF^2 - BF^2 + BD^2 - AE^2$. But F is equidistant from A and B, so $AF^2 = BF^2$. Similarly, $BD^2 = CD^2$ and $AE^2 = CE^2$. Hence $PD^2 - PE^2 = CD^2 - CE^2$, so P also lies on the perpendicular to DE through C.

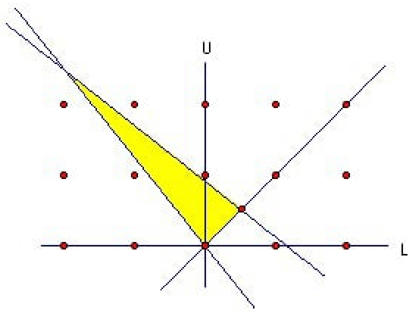
Problem A3
Show that there is a unique polynomial whose coefficients are all single decimal digits which takes the value n at -2 and at -5.

Solution
Call the polynomial $p(x) = p_0 + p_1x + p_2x^2 + ... + p_mx^m$. Since $p(x) - n = 0$ has -2 and -5 as roots, it must have the factor $(x + 2)(x + 5) = x^2 + 7x + 10$. So for some $a_0, a_1, a_2, ...$ we have:

$$\begin{array}{lll}
10a_0 \qquad\qquad +n & = p_0 & \in \{0,1,2,3,4,5,6,7,8,9\} \\
10a_1 + 7a_0 & = p_1 & \in \{0,1,2,3,4,5,6,7,8,9\} \\
10a_2 + 7a_1 + a_0 & = p_2 & \in \{0,1,2,3,4,5,6,7,8,9\} \\
10a_3 + 7a_2 + a_1 & = p_3 & \in \{0,1,2,3,4,5,6,7,8,9\} \\
... & & \\
10a_{r+1} + 7a_r + a_{r-1} & = p_{r+1} & \in \{0,1,2,3,4,5,6,7,8,9\} \\
... & &
\end{array}$$

Now these equations uniquely determine a_i and p_i. For p_0 must be chosen so that $p_0 - n$ is a multiple of 10, which fixes p_0 and a_0 uniquely. Similarly, given p_i and a_i for $0 \le i \le r$, we have $p_{r+1} = 10a_{r+1} + 7a_r + a_{r-1} = 7a_r + a_{r-1} \bmod 10$, so p_{r+1} is uniquely determined and hence also a_{r+1}. Thus any solution is certainly unique, but it is not clear that the process terminates, so that p_i and a_i are zero from some point on.

Evidently the sequence a_i is bounded. For if $a_i, a_{i+1} \le B \ge 9$, then $|a_{i+2}| \le 0.7B + 0.1B + 0.1B \le B$. So if we take $B = \max(9,|a_0|,|a_1|)$, then $|a_i| \le B$ for all i.
So we can define $L_k = \min(a_k, a_{k+1}, a_{k+2}, ...)$, $U_k = \max(a_k, a_{k+1}, a_{k+2}, ...)$. Obviously, we have $L_0 \le L_1 \le ... \le L_k \le L_{k+1} \le ... \le U_{k+1} \le U_k \le ... \le U_1 \le U_0$. So L_i is an increasing integer sequence which is bounded above, so we must have $L_i = L$ for all sufficiently large i. Similarly, $U_i = U$ for all sufficiently large i, and $L \le U$ (1). But if $a_i, a_{i+1} \ge L$, then $a_{i+2} \le -0.7L - 0.1L + 0.9 \le -0.8L + 0.9$. So $U \le -0.8L + 0.9$ (2). Similarly, if $a_i, a_{i+1} \le U$, then $a_{i+2} \ge -0.8U$, so $L \ge -0.8U$ (3).



But as the diagram shows the only lattice point satisfying (1), (2), (3) is (0,0), so $a_i = 0$ for all sufficiently large i, which establishes existence.

Problem B1
A sequence of polygons is derived as follows. The first polygon is a regular hexagon of area 1. Thereafter each polygon is derived from its predecessor by joining to adjacent edge midpoints and cutting off the corner. Show that all the polygons have area greater than 1/3.

Solution
The first point to observe is that each polygon in the sequence is convex. The next point is that we can never completely eliminate the sides of the hexagon, in other words every polygon in the sequence has a vertex on each of the sides of the hexagon.

Let the hexagon be ABCDEF. The diagonals AC, BD, CE, DF, EA, FB meet at the six vertices of a smaller hexagon. Call it UVWXYZ. To be specific, let U be the intersection of FB and AC, V the intersection of AC and BD, W the intersection of BD and CE and so on. Now take any polygon in the sequence. It has a vertex on AB and a vertex on BC. Since it is convex, it must also include the segment joining these points. But any such segment intersects BU and BV. So it has a point on BU and on BV. Similarly for each of the other segments: CV, CW, DW, DX, ... But the convex hull of these 12 points includes the hexagon UVWXYZ. Hence the area of any polygon in the sequence is at least that of UVWXYZ.

The triangle ABF is isosceles with angle ABF = 30 deg, so if AB = k, then BF = $k\sqrt{3}$. But BU = FZ = $k/\sqrt{3}$. Hence UZ = $k\sqrt{3} - 2k/\sqrt{3} = k/\sqrt{3}$. So the area of UVWXYZ is 1/3 the area of ABCDEF.

Problem B2
Show that $xyz/(x^3 + y^3 + xyz) + xyz/(y^3 + z^3 + xyz) + xyz/(z^3 + x^3 + xyz) \le 1$ for all real positive x, y, z.

Solution
For positive x, y, $x \ge y$ iff $x^2 \ge y^2$, so $(x - y)(x^2 - y^2) \ge 0$, or $x^3 + y^3 \ge xy(x + y)$. Hence $x^3 + y^3 + xyz \ge xy(x + y + z)$ and so $xyz/(x^3 + y^3 + xyz) \le z/(x + y + z)$. Adding the two similar equations gives the required inequality.

Problem B3
The sequence of non-negative integers $c_1, c_2, \dots , c_{1997}$ satisfies $c_1 \ge 0$ and $c_m + c_n \le c_{m+n} \le c_m + c_n + 1$ for all m, n > 0 with m + n < 1998. Show that there is a real k such that $c_n = [nk]$ for $1 \le n \le 1997$.

Solution
Any such k must satisfy $c_n/n \le k < c_n/n + 1/n$ for all n. Hence we must have $c_m/m < c_n/n + 1/n$ or $n c_m < m c_n + m$ for all m, n. Conversely, if this inequality holds, then such k exist. For example, we could take $k = \max c_n/n$.

It is tempting to argue that $n c_m < (n-1) c_m + c_{2m} < (n-2) c_m + c_{3m} < \dots < c_{mn} \le c_{mn-n} + c_n + 1 \le \dots \le m c_n + m$. But this does only works for small m, n, because otherwise mn may be out of range. Instead we use induction on m+n. It is obviously true for m = n = 1 and indeed any m = n. Now suppose m < n (and it is true for smaller m + n). Then by induction $(n - m) c_m < m c_{n-m} + m$. But $c_m \le c_n - c_{n-m}$, so $m c_m \le m c_n - m c_{n-m}$. Adding, we get $n c_m < m c_n + m$ as required. Similarly, if m > n (and the result is true for smaller m + n), then by induction, $n c_{m-n} < (m - n) c_n + (m - n)$. But $n c_m <= n c_n + n c_{m-n} + n$, so adding $n c_m < m c_n + m$, as required.

27th USAMO 1998

Problem A1
The sets $\{a_1, a_2, \dots , a_{999}\}$ and $\{b_1, b_2, \dots , b_{999}\}$ together contain all the integers from 1 to 1998. For each i, $|a_i - b_i| = 1$ or 6. For example, we might have $a_1 = 18$, $a_2 = 1$, $b_1 = 17$, $b_2 = 7$. Show that $\sum_1^{999} |a_i - b_i| = 9 \bmod 10$.

Solution

If $|a_i - b_i| = 6$, then a_i and b_i have the same parity, so the set of such a_i and b_i contains an even number of odd numbers. But if $|a_i - b_i| = 1$, then a_i and b_i have opposite parity, so each such pair includes just one odd number. Hence if the number of such pairs is even, then the set of all such a_i and b_i also has an even number of odd numbers. But the total number of a_i and b_i which are odd is 999 which is odd. Hence the number of pairs with $|a_i - b_i| = 1$ must be odd, and hence the number of pairs with $|a_i - b_i| = 6$ must be even. Suppose it is 2k. Then $\sum |a_i - b_i| = (999 - 2k) 1 + 2k 6 = 999 + 10k = 9 \mod 10$.

Problem A2

Two circles are concentric. A chord AC of the outer circle touches the inner circle at Q. P is the midpoint of AQ. A line through A intersects the inner circle at R and S. The perpendicular bisectors of PR and CS meet at T on the line AC. What is the ratio AT/TC?

Solution

We have $AR \cdot AS = AQ^2 = AQ/2\ 2AQ = AP \cdot AC$, so ARP and ACS are similar, so $\angle ACS = \angle ARP$, so PRSC is cyclic. Hence T must be the center of its circumcircle and must also lie on the perpendicular bisector of CP. Hence it must be the midpoint of CP. So CT = 3/8 CA and hence AT/TC = 5/3.

However, that is not quite all. If CS is parallel to PR, then their perpendicular bisectors coincide and both pass through A. So one could also regard A as a possible position for T.

Problem A3

The reals $x_1, x_2, \dots , x_{n+1}$ satisfy $0 < x_i < \pi/2$ and $\sum_1^{n+1} \tan(x_i - \pi/4) \geq n-1$. Show that $\prod_1^{n+1} \tan x_i \geq n^{n+1}$.

Solution

Put $t_i = \tan(x_i - \pi/4)$. Then $(1 + t_i)/(1 - t_i) = \tan(\pi/4 + x_i - \pi/4) = \tan x_i$. So we wish to show that $\prod(1 + t_i)/(1 - t_i) \geq n^{n+1}$.

The given inequality is equivalent to $1 + t_i \geq \sum_{j\neq i} (1 - t_j)$. Using the AM/GM inequality, this implies that $(1 + t_i)/n >= \prod_{j\neq i} (1 - t_j)^{1/n}$. Hence $\prod (1 + t_i)/n^{n+1} \geq \prod_i \prod_{j\neq i} (1 - t_j)^{1/n} = \prod (1 - t_i)$.

Problem B1

A 98 x 98 chess board has the squares colored alternately black and white in the usual way. A move consists of selecting a rectangular subset of the squares (with boundary parallel to the sides of the board) and changing their color. What is the smallest number of moves required to make all the squares black?

Solution

Answer: 98.

There are $4\cdot 97$ adjacent pairs of squares in the border and each pair has one black and one white square. Each move can fix at most 4 pairs, so we need at least 97 moves. However, we start with two corners one color and two another, so at least one rectangle must include a corner square. But such a rectangle can only fix two pairs, so at least 98 moves are needed.

It is easy to see that 98 suffice: take 49 1x98 rectangles (alternate rows), and 49 98x1 rectangles (alternate columns).

Problem B2

Show that one can find a finite set of integers of any size such that for any two members the square of their difference divides their product.

Solution

We find inductively a set with n elements satisfying the slightly stronger condition that if a and b are any two elements, then a - b divides both a and b. For n = 2, we may take {1, 2}. Suppose we have a set S for n. Let m be the lowest common multiple (or any multiple) of all the members of S. Now take the set $\{m + a: a \in S\} \cup \{m\}$ for n+1. A difference (m + a) - (m + b) = a - b divides a and b, hence also m, and hence m + a and m + b. A difference (m + a) - m = a divides a and m and hence also m + a.

Problem B3
What is the largest number of the quadrilaterals formed by four adjacent vertices of an convex n-gon that can have an inscribed circle?

Solution
Answer: [n/2].

Take a regular n-gon and slice off alternate corners (until [n/2] corners have been cut). Specifically, if the vertices are $A_1, A_2, \dots , A_n$, then we slice off the corners at $A_1, A_3, A_5, \dots A_m$, where m = n-1 if n is even, or n-2 if n is odd. At each corner A_i which we slice, we take the inscribed circle to the triangle $A_{i-1}A_iA_{i+1}$, draw a tangent to it parallel to $A_{i-1}A_{i+1}$ and cut along the tangent. This procedure shows that [n/2] can be achieved.

To show that it is optimal it is sufficient to show that if A, B, C, D, E are adjacent vertices of any polygon and ABCD has an inscribed polygon, then BCDE does not. Since ABCD has an inscribed polygon, AD + BC = AB + CD (if the inscribed circle touches at X on AB and Y on AD, then AX = AY. That and the three similar equations give the result). So if BCDE also has an inscribed polygon, then BE + CD = BC + DE. Hence (adding) AD + BE = AB + DE. But the diagonals AD and BE must meet at some point X. Then AD + BE = AX + XD + BX + XE = (AX + XB) + (DX + XE) > AB + DE. Contradiction.

28th USAMO 1999

Problem A1
Certain squares of an n x n board are colored black and the rest white. Every white square shares a side with a black square. Every pair of black squares can be joined by chain of black squares, so that consecutive members of the chain share a side. Show that there are at least $(n^2 - 2)/3$ black squares.

Solution
Concentrate on the chain condition. We show by induction that if k squares are black and satisfy the chain condition, then at most 3k + 2 squares are black or share a side with a black square. This is obvious for k = 1. Suppose it is true for k and that we have k+1 black squares satisfying the chain condition. It must be possible to pick a black square so that the remaining k black squares still satisfy the chain condition. The remaining k black squares give at most 3k+2 black or sharing a side with a black square, and the picked square adds at most 3. That completes the induction. The result follows immediately, since we must have $n^2 \le 3k + 2$, where k is the number of black squares.

Actually, it is not trivial to show that it must be possible to pick a black square so that the remaining k black squares still satisfy the chain condition. It is equivalent to showing that in any connected graph you can find a point such that if you remove the point the graph is still connected. The trick is to take two points A and B which are the maximum distance apart (distance is the minimum number of edges you must traverse to get from one to the other). The claim is that removal of either of these points leaves the graph connected. For suppose removing A left a disconnected graph, then there must be a point C such that B and C are not joined by a path when A is removed. Since they are joined when A is present, all paths joining them must pass through A and hence exceed the length of all paths from A to B. Contradiction.

Problem A2
For each pair of opposite sides of a cyclic quadrilateral take the larger length less the smaller length. Show that the sum of the two resulting differences is at least twice the difference in length of the diagonals.

Solution
We prove the slightly stronger result that the difference between two opposite sides is at least the difference between the diagonals. Suppose the diagonals meet at X. Then AXB, DXC are similar. Suppose AB = kCD with $k \ge 1$. Then BE = kCE and AE = kDE. Suppose $CE \ge DE$. Then CD + DE > CE, so CD > CE - DE, so (k-1) CD > (k-1)(CE - DE) or AB - CD > BE - CE - AE + DE = BD - AC.

Problem A3
p is an odd prime. The integers a, b, c, d are not multiples of p and for any integer n not a multiple of p, we

have {na/p} + {nb/p} + {nc/p} + {nd/p} = 2, where { } denotes the fractional part. Show that we can find two of a, b, c, d whose sum is divisible by p.

Solution

$\underline{n}$ denote the residue of n mod p, so $\underline{n}$ = 0, 1, 2, ... , or p-1. Thus $\{na/p\} = \underline{na}/p$, and we have that $\underline{na} + \underline{nb} + \underline{nc} + \underline{nd} = 2p$ for n not a multiple of p.

Let ω be a complex pth root of 1. We show first that $\omega + 2\omega^2 + 3\omega^3 + ... + (p-1)\omega^{p-1} = p/(\omega - 1)$. Suppose the sum is S. Then $(1 - 2\omega + \omega^2)S = \omega - p\omega^p + (p-1)\omega^{p+1}$ (we need only look at the two lowest and two highest powers - the others all cancel because k - 2(k-1) + (k-2) = 0) = p(ω - 1). Hence S = p/(ω - 1).

Take residues a', b', c', d', so that aa' = bb' = cc' = dd' = 1 mod p. Then for any integers m, n we have mnaa' = mn mod p. Hence $-ma'\,\underline{na} = -mn$ mod p. Hence $\omega^{-ma'\underline{na}} = \omega^{-mn}$. So $\underline{na}\,\omega^{-ma'\underline{na}} = \underline{na}\,\omega^{-mn}$. Similarly for b, c, d. Take n not a multiple of p and add the four equations to get: $\underline{na}\,\omega^{-ma'\underline{na}} + \underline{nb}\,\omega^{-mb'\underline{nb}} + \underline{nc}\,\omega^{-mc'\underline{nc}} + \underline{nd}\,\omega^{-md'\underline{nd}} = 2p\,\omega^{-mn}$.
As n runs through 1, 2, ... , p-1, each of $\underline{na}$, $\underline{nb}$, $\underline{nc}$, $\underline{nd}$ runs through a complete set of non-zero residues. If we take m to be not a multiple of p, then so does -mn, so adding the equations for n = 1, 2, ... , p-1, we get $\underline{na} \sum k\omega^{-a'mk} + \sum k\,\omega^{-b'mk} + \sum k\omega^{-c'mk} + \sum k\omega^{-d'mk} = 2p(\omega + \omega^2 + ... + \omega^{p-1}) = -2p$.

Since $\omega^{-a'm}$ is also a complex pth root of 1, we have $\sum k\omega^{-a'mk} = p/(\omega^{-a'm} - 1)$ and similarly for the other terms. So the equation becomes: $1/(\omega^{-a'm} - 1) + 1/(\omega^{-b'm} - 1) + 1/(\omega^{-c'm} - 1) + 1/(\omega^{-d'm} - 1) = -2$.

Multiplying through by $(\omega^{-a'm} - 1)(\omega^{-b'm} - 1)(\omega^{-c'm} - 1)(\omega^{-d'm} - 1)$, expanding and simplifying, we get $2 + \omega^{-a'm-b'm-c'm} + \omega^{-a'm-b'm-d'm} + \omega^{-a'm-c'm-d'm} + \omega^{-b'm-c'm-d'm} = \omega^{-a'm} + \omega^{-b'm} + \omega^{-c'm} + \omega^{-d'm} + 2\omega^{-a'm-b'm-c'm-d'm}$ (*). Note that this equation is also true for m = 0 (when it is just 5 = 5). Now sum the equations for m = 0, 1, 2, ... , p-1. We have $\sum \omega^k = p$ if k is a multiple of p, and 0 otherwise. Obviously the first term ∑ 2 gives 2p and the other terms on the lhs give non-negative sums, so ∑ lhs is at least 2p. The first four terms on the rhs all have zero sum, so the last term must have sum 2p, so a' + b' + c' + d' must be a multiple of p. Thus (*) becomes $\omega^{a'm} + \omega^{b'm} + \omega^{c'm} + \omega^{d'm} = \omega^{-a'm} + \omega^{-b'm} + \omega^{-c'm} + \omega^{-d'm}$. Multiply through by $\omega^{-a'm}$ and sum for m = 0, 1, 2, ... , p-1. The first term on the lhs has sum p and the others have non-negative sum. The first term on the rhs has zero sum, so one of the others must have positive sum. Hence p divides at least one of (a'+b'), (a'+c'), (a'+d'). Without loss of generality it divides a' + b'. In other words a' + b' = 0 mod p. Multiplying by ab, we get a + b = 0 mod p.

Problem B1

A set of n > 3 real numbers has sum at least n and the sum of the squares of the numbers is at least n^2. Show that the largest positive number is at least 2.

Solution

Let the numbers be $x_1, x_2, ... , x_n$. Notice first that $x_1 = x_2 = ... = x_{n-1} = 2$, $x_n = 2 - n$, gives $\sum x_i = (n - 1)2 + (2 - n) = n$, $\sum x_i^2 = (n - 1)4 + (4 - 4n + n^2) = n^2$, so the inequality is best possible.

Suppose the result is false. So we have a set of numbers with $\Sigma x_i \ge n$, $\sum x_i^2 \ge n^2$ and max $x_i < 2$. At least one of the numbers must be negative, since otherwise we have $n \ge 4$, so $n^2 \ge 4n > \sum x_i^2$. Contradiction. This allows us to assume that $\sum x_i = n$, for if it is greater, we may just decrease a negative x_i until it becomes true ($\sum x_i^2$ will be increased, so it will remain at least n^2).

Now suppose two of the x_i, namely x and y, are less than 2. Then if we replace them by 2 and x + y - 2, the sum is unaffected and the sum of squares is increased by 2(2 - x)(2 - y). Since we start with all the x_i less than 2, we may do this repeatedly until we reach a set with all the numbers 2 except one. Since the sum is unchanged, the other number must be 2 - n, and, as shown above, that makes the sum of the squares n^2. But we have increased the sum of the squares at each step. Contradiction.

Problem B2

Two players play a game on a line of 2000 squares. Each player in turn puts either S or O into an empty square. The game stops when three adjacent squares contain S, O, S in that order and the last player wins. If all the squares are filled without getting S, O, S, then the game is drawn. Show that the second player can always win.

Solution
Suppose a square is such that if you play there then that allows your opponent to win on the following move. If you play an O, then your opponent must win by playing an adjacent S. So we must have S 1 2 3, where 1 and 2 are empty and you play O on square 1. But you also lose if you play S, so your opponent must then win by playing O on 2, which means that 3 must already contain an S. But now the situation is symmetrical, so that 2 is also a losing square. Thus, until someone plays on one of them, losing squares always occur in pairs.

The board has an even number of squares, so the first player always faces a board with an even number of squares not yet occupied, whereas the second player always faces a board with an odd number of squares not yet occupied. Thus provided (1) there is at least one pair of losing squares, (2) he never plays on a losing square, and (3) he makes the obvious winning move if the first player ever creates the opportunity, then the second player is sure to win, because the first player will eventually face a board with only losing squares available for play.

To make sure there is at least one pair of losing squares the second player must create it. He can always do this by placing an S on his first move well away from the first player's move and from the edges of the board. Then on his second move (assuming the first player has not been stupid enough to allow him an immediate win) he can always play another S three away from it, creating a pair of losing squares. Thereafter, he must simply take care to win if there is a winning move and otherwise to avoid losing plays.

Problem B3
I is the incenter of the triangle ABC. The point D outside the triangle is such DA is parallel to BC and DB = AC, but ABCD is not a parallelogram. The angle bisector of BDC meets the line through I perpendicular to BC at X. The circumcircle of CDX meets the line BC again at Y. Show that DXY is isosceles.

Solution
Let IX meet BC at Z. Then using equal tangents, (BC - CZ) + (AC - CZ) = AB, so CZ = (AC + BC - AB)/2. Suppose the excircle opposite D of DBC touches BC at Z'. Then, again considering equal tangents, DB + (BC - CZ') = DC + CZ', so CZ' = (BD + BC - DC)/2 = (AC + BC - AB)/2 = CZ, so Z' and Z coincide. Since X lies on the perpendicular to BC at Z and on the bisector of $\angle$BDC, it must also be the center of the excircle. Hence XC is the exterior bisector of $\angle$BCD. So $\angle$XCB = 90 - $\angle$BCD/2.

By construction, YDCX is cyclic, so $\angle$YDX = $\angle$YCX = $\angle$XCB. Also $\angle$BCD = $\angle$YCD = $\angle$YXD. Hence $\angle$YDX = 90 - $\angle$YXD/2. Hence YX = DX.

29th USAMO 2000

Problem A1
Show that there is no real-valued function f on the reals such that (f(x) + f(y))/2 ≥ f((x+y)/2) + |x - y| for all x, y.

Solution
Put x = a + b, y = a - b with b > 0. Then we have f(a) ≤ 1/2 f(a+b) + 1/2 f(a-b) - 2b. Also f(a + b/2) ≤ 1/2 f(a) + 1/2 f(a+b) - b, f(a - b/2) ≤ 1/2 f(a-b) + 1/2 f(a) - b, and f(a) ≤ 1/2 f(a - b/2) + 1/2 f(a + b/2) - b ≤ 1/4 f(a-b) + 1/2 f(a) + 1/4 f(a+b) - 2b. Hence f(a) ≤ 1/2 f(a-b) + 1/2 f(a+b) - 4b. But a and b are arbitrary (apart from b > 0) so this argument can now be repeated to show that f(a) ≤ 1/2 f(a-b) + 1/2 f(a+b) + 2^nb for any positive integer n. Contradiction.

Problem A2
The incircle of the triangle ABC touches BC, CA, AB at D, E, F respectively. We have AF ≤ BD ≤ CE, the

inradius is r and we have 2/AF + 5/BD + 5/CE = 6/r. Show that ABC is isosceles and find the lengths of its sides if r = 4.

Solution
Answer: sides 24, 15, 15. AF = 3, BD = CE = 12.

Let the incenter be I. The triangle AFI has $\angle AFI = 90^o$, $\angle FAI = A/2$, and FI = r. So r/AF = tan A/2. Similarly, r/BD = tan B/2, r/CE = tan C/2. So the given relation is 2 tan A/2 + 5 tan B/2 + 5 tan C/2 = 6. We have A/2 = 90^o - (B/2 + C/2), so we can eliminate A/2, using tan A/2 = cot(B/2 + C/2) = (1 - tan B/2 tan C/2)/(tan B/2 + tan C/2). Hence 5 $\tan^2$B/2 + 5 $\tan^2$C/2 + 8 tan B/2 tan C/2 - 6 tan B/2 - 6 tan C/2 + 2 = 0 (*).

It is not immediately clear where we go from here. But we are asked to prove that ABC is isosceles. Since the given relation is symmetrical in B and C, presumably AB = AC and angle B = angle C, in which case (*) reduces to $(3 \tan B/2 - 1)^2 = 0$. So our goal must be to show that 3 tan B/2 - 1 = 3 tan C/2 - 1 = 0. If we use 3 tan B/2 - 1 and 3 tan C/2 - 1 as variables, we have $(3 \tan B/2 - 1)^2 = 9 \tan^2 B/2 - 6 \tan B/2 + 1$, $(3 \tan C/2 - 1)^2 = 9 \tan^2 C/2 - 6 \tan C/2 + 1$, (3 tan B/2 - 1)(3 tan C/2 - 1) = 9 tan B/2 tan C/2 - 3 tan B/2 - 3 tan C/2 + 1. Comparing to (*), we see that it can be written as $5 (3 \tan B/2 - 1)^2 + 5 (3 \tan C/2 - 1)^2 + 8(3 \tan B/2 - 1)(3 \tan C/2 - 1) = 0$. But $8^2 < 4 \cdot 5 \cdot 5$, so this implies 3 tan B/2 - 1 = 3 tan C/2 - 1 = 0. So tan A/2 = 4/3 and we have found all the angles in the triangle. We have AF = r cot A/2 = 3, BD = CE = r cot B/2 = 12. So the triangle has sides 3 + 12 = 15, 3 + 12 = 15 and 12 + 12 = 24.

Problem A3
A player starts with A blue cards, B red cards and C white cards. He scores points as he plays each card. If he plays a blue card, his score is the number of white cards remaining in his hand. If he plays a red card it is three times the number of blue cards remaining in his hand. If he plays a white card, it is twice the number of red cards remaining in his hand. What is the lowest possible score as a function of A, B and C and how many different ways can it be achieved?

Solution
Answer: the lowest score is min(AC, 2BC, 3AB). If the maximum of B, A/2, C/3 is unique, then there is only one way to achieve the lowest score. If B = A/2 > C/3, there are C+1 ways; if B = C/3 > A/2, there are A+1 ways; if A/2 = C/3 > B, there are B+1 ways. If B = A/2 = C/3, then there are A+B+C ways.

If A = 0, then the unique solution is to play all the red cards followed by all the white cards (total score nil). Similarly, if B = 0, the unique solution is to play all the white cards followed by all the blue cards, and if C = 0, the unique solution is to play all the blue cards followed by all the red cards. So assume A, B, C are all non-zero.

It is never correct to play a red card immediately before a blue card, because the score would be reduced by 3 if the order was reversed. Similarly, it is never correct to play a white card immediately before a red card, or a blue card immediately before a white card. Hence the optimum play must be either (1) BRWBRWB ... or (2) RWBRWB ... or (3) WBRWB ... , where B denotes the play of one or more blue cards, R denotes the play of one or more red cards and W denotes the play of one or more white cards.

Suppose the optimum involves two or more separate plays of blue cards, so we have ... b, r, w, b' , ... meaning that the sequence includes the plays b blue cards, followed by r red cards, followed by w white cards, followed by b' blue cards. Then the score for ... (b-1), r, w, (b'+1), ... is (w-3r) lower. That is independent of b. So if w is not equal to 3r, then the sequence cannot be optimal, because either ... (b+b'), r, w, ... or ... r, w, (b+b'), ... gives a lower score. If w = 3r, then both ... (b+b'), r, w, ... and ... r, w, (b+b'), ... are also optimal. But that implies that the original sequence cannot have had any more terms, it must have been simply b, r, w, b', otherwise one of ... (b+b'), r, w, ... or ... r, w, (b+b'), ... would involve playing a white card immediately before a red, which is never optimal.

A similar argument applies to two or more separate plays of red cards and to two or more separate plays of white cards. So one of BRW, RWB, WBR is always optimal. They give scores of AC, 3AB, 2BC respectively, so the minimum score is min(AC, 3AB, 2BC).

The argument above also shows that if a play sequence is optimal, then it must be one of the three above or BRWB, RWBR or WBRW. Also BRWB can only be optimal if C = 3B. Similarly, RWBR can only be optimal if 3A = 2C and WBRW can only be optimal if 2B = A.

If BRW, RWB and WBR are all optimal, then A = B/2 = C/3. In this case, BRWB, RWBR and WBRW are also optimal. There are A possibilities for BRW or BRWB (start with 1, 2, 3, ... or A blue cards, then play all the red, then all the white, then any remaining blue). Similarly, there are B possibilities for RWB and RWBR and C for WBR and WBRW, so A+B+C possibilities in total. So assume BRW, RWB and WBR are not all optimal.

If BRWB is optimal, then BRW and RWB must also be optimal, so A/2 < B = C/3. So there are A+1 possibilities (start with 0, 1, 2, ... or A blue cards, then all the red, then all the white, then any remaining blue).

Similarly, if RWBR is optimal, then B < A/2 = C/3. There are B+1 possibilities (start with 0, 1, 2, ... or B red cards, then all the white, then all the blue, then any remaining red). Similarly, if WBRW is optimal, then C/3 < B = A/2 and there are C+1 possibilities (start with 0, 1, 2, ... or C white cards, then all the blue, then all the red, then any remaining white).

If none of BRWB, RWBR, WBRW are optimal, then B, A/2 and C/3 are all unequal and the solution is unique.

Problem B1
How many squares of a 1000 x 1000 chessboard can be chosen, so that we cannot find three chosen squares with two in the same row and two in the same column?

Solution
Answer: 1998. Choose every square in the first row or column but not both.

We prove the slightly more general result that the maximum number for an m x n rectangle is n if m = 1, or m + n - 2 for m, n > 1.

We may assume m ≤ n. We use induction on m. The result for m = 1 is obvious. For m = 2, if we choose two in the same row, then we cannot choose any more, so it is better (or no worse for n = 2) to choose all the squares in a column giving n = m + n - 2 in total. That establishes the result for m = 1 and 2.

Now suppose n ≥ m > 2 and that the result is true for smaller m. We can certainly do at least m + n - 2 by choosing all the squares in the first row or column but not both. Assume we have m columns. If no two are in the same row, then there are at most n which we know is not optimal. So assume there are two in the same row. Now there cannot be any more in the two corresponding columns. So consider the remaining m-2 columns. If m > 3, then by induction we cannot choose more than m-2 + n - 2 in those columns and with the two already chosen that gives m + n - 2. If m = 3, then we cannot choose more than n in the remaining column. But we can combine at most n-1 of those with the existing two, since if we pick the square in the same row as the two squares already chosen, then we cannot choose any others. So for this case also we can do at most n+1 = m + n - 2. Hence the result is true for all m.

Problem B2
ABC is a triangle. C_1 is a circle through A and B. We can find circle C_2 through B and C touching C_1, circle C_3 through C and A touching C_2, circle C_4 through A and B touching C_3 and so on. Show that C_7 is the same as C_1.

Solution
Let O_i be the center of C_i. Evidently O_i lies on the perpendicular bisector of the relevant side. Since C_1 and C_2 touch, O_1, B and O_2 must be collinear. Let M be the midpoint of AB. Let $\angle MO_1A = x_1$. Define x_i similarly. Since O_1, B and O_2 are collinear, we have $(90^\circ - x_1) + B + (90^\circ - x_2) = 180^\circ$. So $B = x_1 + x_2$.

Similarly, $x_2 + x_3 = C$, $x_3 + x_4 + A$, $x_4 + x_5 = B$, $x_5 + x_6 = C$, $x_6 + x_7 = A$. Hence $(x_1 + x_2) + (x_3 + x_4) + (x_5 + x_6) = A + B + C = (x_2 + x_3) + (x_4 + x_5) + (x_6 + x_7)$. So $x_1 = x_7$. Hence $O_1 = O_7$.

Problem B3

$x_1, x_2, \dots , x_n$, and $y_1, y_2, \dots , y_n$ are non-negative reals. Show that $\sum \min(x_ix_j, y_iy_j) \le \sum \min(x_iy_j, x_jy_i)$, where each sum is taken over all n^2 pairs (i, j).

Solution

Let $f(x_1, y_1, x_2, y_2, \dots , x_n, y_n) = \Sigma (\min(x_iy_j, x_jy_i) - \min(x_ix_j, y_iy_j))$. So we have to show that $f(x_1, y_1, \dots , x_n, y_n) \ge 0$. We use induction on n. There is nothing to prove for $n = 1$. Suppose the result is true for all positive integers $< n$.

If any x_i or y_i is zero, or if any $x_i = y_i$, then the result follows immediately from that for n-1. Suppose $x_1/y_1 = x_2/y_2$. We claim that $f(x_1, \dots , y_n) = f(x_1+x_2, y_1+y_2, x_3, y_3, \dots , x_n, y_n)$. Note that the rhs has one less pair of terms. For convenience we write $x_1 = k y_1$, so $x_2 = k y_2$. The sum of the (1, i) and (2, i) terms on the lhs is $\min(x_1y_i, y_1x_i) + \min(x_2y_i, y_2x_i) - \min(x_1x_i, y_1y_i) - \min(x_2x_i, y_2y_i) = (y_1 + y_2) \min(ky_i, x_i) - (y_1 + y_2) \min(kx_i, y_i)$. The corresponding (1+2, i) term on the rhs is $\min((x_1+x_2)y_i, (y_1+y_2)x_i) - \min((x_1+x_2)x_i, (y_1+y_2)y_i) = (y_1+y_2) \min(ky_i, x_i) - (y_1+y_2) \min(kx_i, y_i)$, which is the same. Similarly for the (i, 1) + (i, 2) versus (i, 1+2) terms. The (i, j) terms (with $i, j > 2$) are obviously unchanged. So we just have to consider the various 1, 2 terms. On the lhs there are four of them: the (1, 1), (1, 2) = (2, 1) and the (2, 2) terms. Their sum is $x_1y_1 - \min(x_1^2, y_1^2) + x_2y_2 - \min(x_2^2, y_2^2) + 2\min(x_1y_2, x_2y_1) - 2\min(x_1x_2, y_1y_2) = ky_1^2 - y_1^2\min(k^2, 1) + ky_2^2 - y_2^2\min(k^2, 1) + 2ky_1y_2 - 2y_1y_2\min(1,k^2) = (y_1+y_2)^2(k - \min(1,k^2))$. On the rhs there is just the one term $(x_1+x_2)(y_1+y_2) - \min((x_1+x_2)^2, (y_1+y_2)^2) = k(y_1+y_2)^2 - (y_1+y_2)^2\min(k^2, 1)$, which is the same. So we have established the claim. Thus if we have any distinct i, j such that $x_i/y_i = x_j/y_j$, then the result for n follows from that for n-1.

We now show that the same is true if we have $x_i/y_i = y_j/x_j$. Assume for convenience that $x_1/y_1 = y_2/x_2$. We can also assume without loss of generality that $x_1 \le y_2$. So we can take $x_2 = ky_1$, $y_2 = k x_1$ with $k \ge 1$. Then we claim that $f(x_1, \dots , y_n) = f(x_2 - y_1, y_2 - x_1, x_3, y_3, \dots , x_n, y_n)$. Again, the rhs has one less pair of terms than the lhs. On the lhs the sum of the (1, i) and (2, i) terms on the lhs is $\min(x_1y_i, y_1x_i) + \min(x_2y_i, y_2x_i) - \min(x_1x_i, y_1y_i) - \min(x_2x_i, y_2y_i) = (k-1) \min(x_1x_i, y_1y_i) - (k-1) \min(x_1y_i, y_1x_i)$. The corresponding (1+2, i) term on the rhs is $\min((x_2-y_1)y_i, (y_2-x_1)x_i) - \min((x_2-y_1)x_i, (y_2-x_1)y_i) = (k-1) \min(y_1y_i, x_1x_i) - (k-1) \min(y_1x_i, x_1y_i)$, which is the same. Similarly, the 1, 2 terms on the lhs are $x_1y_1 - \min(x_1^2, y_1^2) + x_2y_2 - \min(x_2^2, y_2^2) + 2\min(x_1y_2, x_2y_1) - 2\min(x_1x_2, y_1y_2) = x_1y_1 - \min(x_1^2, y_1^2) + k^2x_1y_1 - k^2\min(x_1^2, y_1^2) + 2k \min(x_1^2, y_1^2) - 2kx_1y_1 = (1 - k)^2x_1y_1 - (1 - k)^2\min(x_1^2, y_1^2)$. On the rhs we have $(x_2 - y_1)(y_2 - x_1) - \min((x_2 - y_1)^2, (y_2 - x_1)^2) = (k - 1)^2x_1y_1 - (k - 1)^2\min(y_1^2, x_1^2)$, which is the same. Again the terms not involving 1 or 2 are the same on both sides. So we have shown that if we have any distinct i, j such that $x_i/y_i = y_j/x_j$ then the result for n follows from that for n-1.

Put $r_i = \max(x_i/y_i, y_i/x_i)$. Reordering the pairs, if necessary, we can take $1 \le r_1 \le r_2 \le \dots \le r_n$. Now let us consider all the x_i, y_i as fixed except y_1. For convenience, let us write $t = y_1$. We have $1 \le r_1 \le r_2$, so t can take any value in the interval $[x_1, x_1r_2]$ or any value in the interval $[x_1/r_2, x_1]$. We examine f on each of these intervals. Since only t is varying, the only terms in f which vary are the (1, 1), (1, i) and (i, 1) terms. Writing their sum as g(t), we have $g(t) = x_1t - \min(x_1^2, t^2) + 2 \sum \min(x_1y_i, t x_i) - 2 \sum \min(x_1x_i, t y_i)$. Now if $x_i \ge y_i$, then $x_i/y_i = r_i \ge r_2 \ge t/x_1$, so $x_1x_i \ge t y_i$. Also $t x_i \ge x_1x_i \ge x_1y_i$, so $\min(x_1y_i, t x_i) = x_1y_i$, and $\min(x_1x_i, t y_i) = t y_i$. So if we put $z_i = - y_i$, then these two terms in g(t) give $2(t - x_1) z_i$. Similarly, if $x_i < y_i$, then $x_1y_i \ge t x_i$ and $t y_i \ge x_1x_i$, so if we put $z_i = x_i$, then the two terms in g(t) still give $2(t - x_1) z_i$. Thus $g(t) = x_1t - x_1^2 + 2(t - x_1) \sum z_i$, which is linear in t. But a linear function takes its minimum value in an interval at one of the endpoints, so the minimum value of g(t) (and hence of f as t varies) must occur at $t = x_1$ or x_1r_2. But then we have $r_1 = 1$ or r_2 and in both those cases we have established that the value of f equals the value of f for n-1 pairs and is therefore non-negative.

Now suppose t is in the other interval $[x_1/r_2, x_1]$. Again, we put g(t) equal to the sum of the variable terms. So $g(t) = x_1t - \min(x_1^2, t^2) + 2 \sum \min(x_1y_i, t x_i) - 2 \sum \min(x_1x_i, t y_i)$. Again, we consider separately the case $x_i \ge y_i$ which gives a pair of terms with sum $2(x_1 - t)z_i$ if we put $z_i = y_i$, whilst if $x_i < y_i$, then the pair of terms has the sum $2(x_1 - t)z_i$ if we put $z_i = -x_i$. So we get $g(t) = x_1t - t_2 + 2(x_1 - t) \sum z_i$. This time we have a quadratic.

But the leading term $-t^2$ has a negative coefficient, so g(t) has a single *maximum* as t varies over all real values. Thus it is again true that the *minimum* value over the interval $[x_1/r_2, x_1]$ must occur at one of the endpoints. So again the minimum value of f as t varies over the allowed range is at $r_1 = r_2$ or 1 and is hence non-negative by induction.

So the induction is complete and the result established.

30th USAMO 2001

Problem A1
What is the smallest number of colors needed to color 8 boxes of 6 balls (one color for each ball), so that the balls in each box are all different colors and any pair of colors occurs in at most one box.

Solution
If each color occurs only twice, then we need at least $8\cdot 6/2 = 24$ colors. But we can do better, so at least one color occurs more than twice.

If a color occurs 4 times or more, let the first 4 boxes each include it. Then those 4 boxes use $1 + 4\cdot 5 = 21$ colors. Now the 5th box can use at most one color from each of the first 4 boxes, so it must use another 2 colors as well. Now the 6th box can use at most one color from each of the first 5 boxes, so it must use another color as well. We are now up to 24 colors. But we can do better.

So assume a color occurs 3 times. Let the first 3 boxes each include it. Then those 3 boxes use $1 + 3\cdot 5 = 16$ colors. The 4th box can use at most one color from each of the first 3 boxes, so it needs at least another 3 colors as well. The 5th box can use at most one color from each of the first 4 boxes, so it needs at least another 2 colors as well. Similarly the 6th box needs at least 1 more color. We are now up to 22.

It can be done with 23 colors, as we show later. The question therefore is whether 22 suffice. If so, then we cannot exceed the lower limits given above, so the boxes must be as shown below, where Ax denotes one of A1, A2, A3, A4, A5. Similarly Bx etc. But the three occurrences of Ex F1 must include a repetition, because there are only two Es to choose from. So 22 does not work.

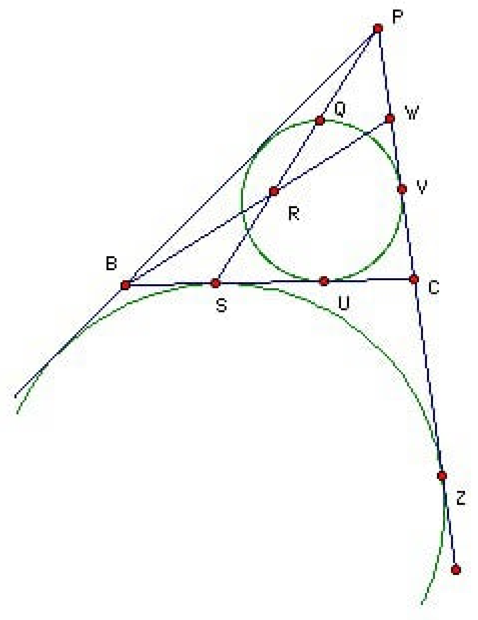
```
 1 A1 A2 A3 A4 A5
 1 B1 B2 B3 B4 B5
 1 C1 C2 C3 C4 C5
Ax Bx Cx D1 D2 D3
Ax Bx Cx Dx E1 E2
Ax Bx Cx Dx Ex F1
Ax Bx Cx Dx Ex F1
Ax Bx Cx Dx Ex F1
```

Finally, 23 does work:

```
1  2  3  4  5  6
1  7  8  9 10 11
1 12 13 14 15 16
2  7 12 17 18 19
3  8 13 17 20 21
4  9 14 17 21 22
5 10 15 18 20 22
6 11 16 19 21 23
```

Problem A2
The incircle of the triangle PBC touches BC at U and PC at V. The point S on BC is such that BS = CU. PS meets the incircle at two points. The nearer to P is Q. Take W on PC such that PW = CV. Let BW and PS meet at R. Show that PQ = RS.

Solution



The excircle opposite P touches BC at S (consider tangents - the tangents from P have length PC + CS etc). Contract about P so that the excircle becomes the incircle. The point S goes to a point on PS at which the incircle touches a line parallel to BC. This point must be Q. Let PC touch the excircle at Z. Then Z goes to V, so PQ/QS = PV/VZ = CW/(VC + CZ) = CW/(CU + CS) = CW/(CU + BU) = CW/BC.

Now consider the triangle PSC. The line BRW cuts PS at R, CP at W and SC at B, so by Menelaus' theorem, we have (SR/RP) (PW/WC) (CB/SB) = 1. But SB = CU = CV = PW, so this gives SR/RP = CW/BC = PQ/QS. Hence PQ = RS.

Problem A3

Non-negative reals x, y, z satisfy $x^2 + y^2 + z^2 + xyz = 4$. Show that $xyz \le xy + yz + zx \le xyz + 2$.

Solution

Assume $x \ge y \ge z$. If $z > 1$, then $x^2 + y^2 + z^2 + xyz > 1 + 1 + 1 + 1 = 4$. Contradiction. So $z \le 1$. Hence $xy + yz + zx \ge xy \ge xyz$.

Put $x = u + v$, $y = u - v$, so that $u, v \ge 0$. Then the equation given becomes $u^2(2 + z) + (2 - z)v^2 + z^2 = 4$. So we we keep z fixed and reduce v to nil, then we must increase u. But $xy + yz + zx - xyz = (u^2 - v^2)(1 - z) + 2zu$, so decreasing v and increasing u has the effect of increasing $xy + yz + zx - xyz$. Hence $xy + yz + zx - xyz$ takes its maximum value when $x = y$. But if $x = y$, then the equation gives $x = y = \sqrt{(2 - z)}$. So to establish that $xy + yz + zx - xyz \le 2$ it is sufficient to show that $2 - z + 2z\sqrt{(2 - z)} \le 2 + z(2 - z)$. Evidently we have equality if $z = 0$. If z is non-zero, then the relation is equivalent to $2\sqrt{(2 - z)} \le 3 - z$ or $(z - 1)^2 \ge 0$. Hence the relation is true and we have equality only for $z = 0$ or 1.

Problem B1

ABC is a triangle and X is a point in the same plane. The three lengths XA, XB, XC can be used to form an obtuse-angled triangle. Show that if XA is the longest length, then angle BAC is acute.

Solution

Suppose first that $\angle BAC = 90^o$. We may choose coordinates so that A is (-a, -b), B is (-a, b) and C is (a, -b). Let X be (x, y). We have that $XB^2 + XC^2 - XA^2 = (x + a)^2 + (y - b)^2 + (x - a)^2 + (y + b)^2 - (x + a)^2 - (y + b)^2 = (x - a)^2 + (y - b)^2 \ge 0$. So in this case the lengths XA, XB, XC cannot form an obtuse-angled triangle.

Now suppose BAC is obtuse. Let A' be the foot of the perpendicular from C to the line AB. We show that if X is situated so that XA is the longest length, then XA < XA'. But since BA'C is right-angled, we have just shown that $XB^2 + XC^2 \ge XA^2$ and hence $XB^2 + XC^2 \ge XA'^2$, showing that the lengths XA, XB, XC form an acute-angled triangle.

This is almost obvious from a diagram. Let L, M, N be the midpoints of BC, CA, AB. Let N' be the midpoint of A'B. Let the perpendiculars through M, N meet at O. Take P on the line MO on the opposite side of O to M, and Q on the line NO on the opposite side of O to N. So for XA to be the longest length, X must lie on the same side of the line MP as C (for XA ≥ XC) and on the same side of the line NQ as B (for XA ≥ XB). So it must lie in the region bounded by the rays OP and OQ.

We now find the similar region for A'BC. The perpendicular bisector of A'C is just the line ML. Take P' on this line on the opposite side of L to M. The perpendicular bisector of AB meets AB at a point N' (say) which must lie on the segment AN. It also passes through L. Take a point Q' on this line on the opposite side of L to

N'. Then for $XA' \geq XB, XC$ we require X to lie in the sector bounded by the lines LP' and LQ'. But the sector bounded by OP and OQ lies entirely inside this sector. [This is obvious from a diagram, but LQ' is parallel to OQ and lies between it and C. OP and LP' both pass through M and OP cuts BC at a point between L and C.]

Also the region between LP' and LQ' lies on the same side of the perpendicular bisector of AA' (which is parallel to LQ') as A. So any point in the region is closer to A than A'. This gives us all we need. If $XA \geq XB$ and XC, then it lies in the region bounded by OP and OQ. Hence it also lies in the region bounded by LP' and LQ', so $XA'^2 \leq XB^2 + XC^2$. Since also $XA < XA'$, we have $XA^2 < XB^2 + XC^2$ and hence XA, XB, XC form an obtuse-angled triangle.

Problem B2
A set of integers is such that if a and b belong to it, then so do $a^2 - a$, and $a^2 - b$. Also, there are two members a, b whose greatest common divisor is 1 and such that a - 2 and b - 2 also have greatest common divisor 1. Show that the set contains all the integers.

Solution
Suppose 1 belongs to the set. Then so does $0 = 1^2 - 1$.We have $0^2 - 1 = -1$, $1^2 - (-1) = 2$, $2^2 - 1 = 3$, $2^2 - 0 = 4$, $2^2 - (-1) = 5$. Now given that we have every integer up to k^2 we can get the integers from $k^2 + 1$ to $(k + 1)^2$ using $(k + 1)^2 - h$ for h = 0, 1, ... , 2k (assuming that $k \geq 2$). Hence we can get all positive integers. Now to get any negative integer -k just take $0^2 - k$.

Now if a, b, c belong to the set, then so do $(a^2 - b^2) + c = a^2 - (b^2 - c)$ and $-(a^2 - b^2) + c$. So by induction $n(a^2 - b^2) + c$ belongs for any integer n. Put $A = a^2 - b^2$. Now if a' and b' are two other numbers in the set, put $B = a'^2 - b'^2$. Then mA + nB + c belongs to the set for all integers m and n. If we could find A and B which were relatively prime, then we would be home, because we could find m, n for which mA + nB = 1 and hence we could find m, n for which mA + nB = -(c - 1), so that mA + nB + c = 1.

It is not obvious how to find such A, B. But we can find A, B, C such that the greatest common divisor is 1 and that is sufficient. Let $A = a^2 - b^2$, where a, b are the two given numbers. Let $B = a^3(a - 2)$ and let $C = b^3(b - 2)$. Since a is in the set so is $a^2 - a$ and $B = (a^2 - a)^2 - a^2$. Similarly C. So certainly all numbers mA + nB + rC + a are in the set for any integers m, n, r. If a prime p divides B, then it must divide a or a - 2. If it also divides A, then it cannot divide a, for then it would also divide b^2 and hence b, but we are told that a and b have no common factor. So any prime dividing all of A, B and C must divide a - 2. Similarly, it must divide b - 2. But we are told that a - 2 and b - 2 have no common factor. Hence A, B, C have no common factor. So we can find m, n, r such that mA + nB + rC = 1, and hence m, n, r such that mA + nB + rC = -(a - 1) and hence such that mA + nB + rC + a = 1.

Problem B3
Every point in the plane is assigned a real number, so that for any three points which are not collinear, the number assigned to the incenter is the mean of the numbers assigned to the three points. Show that the same number is assigned to every point.

Solution
Let f(P) be the number assigned to any point P. Let P, Q be any points. Take R on the segment PQ. Its position will be determined later. Take AA' perpendicular to PQ with R its midpoint. Take a rectangle ACFA' on the opposite side of AA' to P and AA'/AC = 3/2. Let B be the midpoint of AC and B' the midpoint of A'F. Take equally spaced points D, E on CF, so that AB = BC = CD = DE = EF = FB' = B'A'. Finally take X on the ray RP. We will choose the lengths XP and RA so that P is the incenter of XAA' and Q is the incenter of XBB'.

The incircles of ACD and BCE coincide, so f(A) + f(D) = f(B) + f(E). Hence f(D) - f(E) = f(B) - f(A). Similarly, the incircles of A'EF and B'DF coincide, so f(D) - f(E) = f(A') - f(B'). Hence f(A) + f(A') = f(B) + f(B'). Hence f(P) = (f(A) + f(A') + f(X))/3 = (f(B) + f(B') + f(X))/3 = f(Q). That is all we need, since it shows that the same number is assigned to two arbitrary points.

So it remains to show that XP and RA can be chosen as claimed. The incenter of an isosceles triangle base 2a and height h is $ah/(a + \sqrt{(a^2+h^2)})$ above the base (if the distance is x, then by similar triangles $x/(h-x) = a/\sqrt{(a^2+h^2)}$). So if we take $XR/RA = 3/2$ and $RA/PR = (1 + \sqrt{5})/2$ then P is the incenter of XAA'.

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Problem A1
Let S be a set with 2002 elements and P the set of all its subsets. Prove that for any n (in the range from zero to |P|) we can color n elements of P white, and the rest black, so that the union of any two elements of P with the same color has the same color.

Solution
Let S have m elements and P be the set of its subsets. We show by induction on m that a coloring is possible for any $n \le |P|$. If m = 1, we color both subsets black for n = 0, the empty set white (and the other subset black) for n = 1, and both subsets white for n = 2. Suppose now that a coloring is possible for m (and any n). Consider a set S with m+1 elements. Let b be any element of S. For $n \le 2^m$, use induction to color just n subsets of S - {b} white and color black all subsets of S which include b. Then the union of two white subsets is still a subset of S - {b} and hence (by assumption) white. The union of two black subsets of S - {b} is black for the same reason. If one black subset includes b, then so does the union, which must therefore be black. For $n > 2^m$, we have $2^{m+1} - n < 2^m$, so we can find a coloring for $2^{m+1} - n$ and then swap the colors.

Problem A2
The triangle ABC satisfies the relation $\cot^2 A/2 + 4 \cot^2 B/2 + 9 \cot^2 C/2 = 9(a+b+c)^2/(49r^2)$, where r is the radius of the incircle (and a = |BC| etc, as usual). Show that ABC is similar to a triangle whose sides are integers and find the smallest set of such integers.

Solution
Answer: a =13, b = 40, c = 45.
Let the incenter be I. Consider the triangle IBC. It has angle IBC = B/2, angle ICB = C/2 and height r. Hence $a = r \cot B/2 + r \cot C/2$. With the two similar relations for the other sides, that gives $2r \cot A/2 = (b + c - a)$, $2r \cot B/2 = (c + a - b)$, $2r \cot C/2 = (a + b - c)$. So the given relation becomes: $49((b + c - a)^2 + 4(c + a - b)^2 + 9(a + b - c)^2) = 36(a + b + c)^2$.

Multiplying out is a mistake. It leads nowhere. It is more helpful to change variable to d = b + c - a, e = c + a - b, f = a + b - c giving $49(d^2 + 4e^2 + 9f^2) = 36(d + e + f)^2$, or $13d^2 + 160e^2 + 405f^2 - 72(de + ef + fd) = 0$. We would like to express this as $(hd + ke)^2 + (h'e + k'f)^2 + (h''f + k''d)^2 = 0$. Presumably $13 = 3^2 + 2^2$. Then $72 = 2\cdot 3\cdot$something and $2\cdot 2\cdot$something, giving 12 and 18. Squares 144, 324. Fortunately, we see that $160 = 12^2+ 4^2$, $405 = 18^2 + 9^2$ and $2\cdot 4\cdot 9 = 72$. So putting that together we get: $(2d - 18f)^2 + (3d - 12e)^2 + (4e - 9f)^2 = 0$.

So we conclude that b + c - a = 9(a + b - c) = 4(c + a - b), or 5a + 4b = 5c, 5a + 3c = 5b, or a = 13k, b = 40k, c = 45k. We get the smallest triangle with integer sides by taking k = 1.

Problem A3
p(x) is a polynomial of degree n with real coefficients and leading coefficient 1. Show that we can find two polynomials q(x) and r(x) which both have degree n, all roots real and leading coefficient 1, such that p(x) = q(x)/2 + r(x)/2.

Solution
The easiest way to show that a polynomial has a root between a and b is to show that it changes sign. So the idea is to take some polynomial that obviously changes sign n times. Then if we take k s(x) and -k s(x) + 2p(x), for sufficiently large k the sign of -k s(x) + 2p(x) should be dominated by s(x). That does not quite deal with the leading coefficient. But we know that ultimately the leading term dominates, so something like $k\, s(x) + x^n$ and $-k\, s(x) - x^n + 2p(x)$ ought to work.

Specifically, put s(x) = (1 - x)(2 - x)(3 - x) ... (n-1 - x). It is zero at x = 1, 2, 3, ... , n-1. It is alternately positive and negative at x = 1/2, 1 1/2, ... , n - 1/2. Suppose n is even. Let $M = n^n$ so that $x^n < M$ on the interval [0, n]. Clearly, if we take k sufficiently large (in relation to M), then k s(x) + x^n has the same sign as s(x) at x = 1/2, 1 1/2, ... , n - 1/2. In particular, it is negative at x = n - 1/2, but, whatever k, if x is sufficiently large k s(x) + x^n is positive. So k s(x) + x^n changes sign at least n times and hence has n real roots.

Similarly, for k sufficiently large (in relation to M and the max value of 2p(x) over the interval [0, n]), -k s(x) - x^n + 2p(x) will have the opposite sign to s(x) at x = 1/2, 1 1/2, ... , n - 1/2 and in particular will be negative at x = 1/2. But the leading term in - k s(x) - x^n + 2p(x) is x^n and n is even, so for x sufficiently negative, the sign will be positive. Thus - k s(x) - x^n + 2p(x) also changes sign at least n times and hence has n real roots.

Exactly similar arguments work for n odd. We get n-1 sign changes from the k s(x) term and one extra for x large and positive or large and negative (this time k s(x) has the same sign at x = 1/2 and x = n - 1/2, but x^n has different signs for large positive and large negative).

Problem B1
Find all real-valued functions f on the reals such that $f(x^2 - y^2) = x f(x) - y f(y)$ for all x, y.

Solution
Answer: f(x) = kx for some real k.
Putting y = 0, $f(x^2) = x f(x)$. Hence $f(x^2 - y^2) = f(x^2) - f(y^2)$. So for any non-negative x, y, we have f(x - y) = f(x) - f(y). Hence also f(x) = f(x + y - y) = f(x + y) - f(y), so f(x + y) = f(x) + f(y) for non-negative x, y. Also $f(0) = f(0^2) = 0 f(0) = 0$, and for non-negative y, f(-y) = f(0 - y) = f(0) - f(y) = -f(y). Hence also f(-y) = -f(y) for negative y. So we have f(x + y) = f(x) + f(y) for non-negative x and any y. But now if x is negative, f(x + y) = -f(-x - y) = - (f(-x) - f(y)) = f(x) + f(y). So f(x + y) = f(x) + f(y) for all x and y.

Now for any x we have $f(x) + f(x - 1) = f(2x - 1) = f(x^2 - (x - 1)^2) = x f(x) - (x - 1) f(x - 1) = x f(x - 1) + x f(1) - (x - 1) f(x - 1) = x f(1) + f(x - 1)$, so f(x) = x f(1). So if f(1) = k, then f(x) = kx. It is trivial to check that this does indeed satsify the equation given for any k.

Problem B2
Show that we can link any two integers m, n greater than 2 by a chain of positive integers m = a_1, a_2, ... , a_{k+1} = n, so that the product of any two consecutive members of the chain is divisible by their sum. [For example, 7, 42, 21, 28, 70, 30, 6, 3 links 7 and 3.]

Solution
We write a ↔ b if (a + b) divides ab. The starting point is that for n > 1 we have n ↔ n(n - 1). As slight variants we also have 2n ↔ n(n - 2) for n > 2, and in any case where a ↔ b, then also ma ↔ mb (for m > 0). That allows us to link n > 2 and 2n, thus: n ↔ n(n - 1) ↔ n(n - 1)(n - 2) = n(n - 2) ↔ 2n.

To go much further we need some inspiration. Note that n(n - 3) + 2 = (n - 1)(n - 2). So 2(n - 1)(n - 2) ↔ n(n - 3)(n - 1)(n - 2). That is critical, for it is a general way of allowing us to reduce the largest factor. Thus for n > 3, n ↔ n(n - 1) ↔ n(n - 1)(n - 2) ↔ n(n - 1)(n - 2)(n - 3) ↔ 2(n - 1)(n - 2) ↔ (n - 1)(n - 2) ↔ n - 1. But linking n and n-1 obviously allows us to link any two integers > 3. That leaves 3 itself, but the question already shows how to link that to at least one integer > 3, which is all we need.

Problem B3
A tromino is a 1 x 3 rectangle. Trominoes are placed on an n x n board. Each tromino must line up with the squares on the board, so that it covers exactly three squares. Let f(n) be the smallest number of trominoes required to stop any more being placed. Show that for all n > 0, $n^2/7 + hn \le f(n) \le n^2/5 + kn$ for some reals h and k.

Solution
A tromino may be placed in n - 2 positions in each row and column, so there are $2n^2 - 4n$ possible positions in total. Placing a tromino occupies or blocks at most 14 of these positions (5 parallel and 9 perpendicular). Hence any placement of $(2n^2 - 4n)/14 = n^2/7 - 2n/7$ trominoes will block further trominoes. So f(n) >= $n^2/7$ - 2n/7.

If we place trominoes roughly like this:

```
x x x o o x x x o o x x x o o x x x o o x
o x x x o o x x x o o x x x o o x x x o o
o o x x x o o x x x o o x x x o o x x x o
x o o x x x o o x x x o o x x x o o x x x
x x o o x x x o o x x x o o x x x o o x x
x x x o o x x x o o x x x o o x x x o o x
```

it is obvious that no further trominoes are possible and the number of occupied squares is about $3n^2/5$. Hence the number of trominoes is about $n^2/5$. But we need to do some tidying up in relation to edge effects.

The safe way to deal with partial trominoes at the beginning or end of rows is to pull them completely onto the board. Each complete group of five cells in a row needs a tromino, but we may need one extra at the start and one extra at the end. So $[n/5] + 2$ will always suffice for the row. Thus $n^2/5 + 2n$ will suffice for the board and so $f(n) \le n^2/5 + 2n$.

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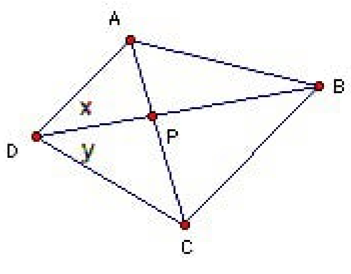
Problem A1
Show that for each n we can find an n-digit number with all its digits odd which is divisible by 5^n.

Solution
Induction on n. For n = 1 we have 5. So suppose N works for n. Consider the five n+1 digit numbers $10^n + N$, $3\cdot 10^n + N$, $5\cdot 10^n + N$, $7\cdot 10^n$, $9\cdot 10^n$. We may take out the common factor 5^n to get the five numbers k, k + h, k + 2h, k + 3h, k + 4h, for some k and $h = 2^{n+1}$. Since h is relatively prime to 5, the five numbers are all incongruent mod 5 and so one must be a multiple of 5.

Problem A2
A convex polygon has all its sides and diagonals with rational length. It is dissected into smaller polygons by drawing all its diagonals. Show that the small polygons have all sides rational.

Solution
It is not hard to see that it is sufficient to prove the result for convex quadrilaterals. For in the case of an n-gon any side of a small polygon is either a side of the n-gon (in which case there is nothing to prove) or a segment of a diagonal. Suppose the diagonal is A_iA_j. Suppose the points of intersection along this diagonal are (in order) $P_0 = A_i, P_1, P_2, \dots , P_m = A_j$. Suppose P_k is the intersection of A_iA_j with A_rA_s. Then using the quadrilateral $A_iA_rA_jA_s$ we deduce that P_0P_k (= A_iP_k) is rational. Hence $P_hP_k = P_0P_k - P_0P_h$ is rational. So all the segments of the diagonal are rational.



It is immediate from the cosine rule that the angles in a triangle with rational sides have rational cosines. So cos x, cos y and cos(x+y) are rational (using triangles ABD, BCD, ADC). Using the formula for cos(x+y) it follows that sin x sin y is rational. Now $\sin^2 y = 1 - \cos^2 y$ is rational, so sin x/sin y is rational.

Now area APD = (AD·PD sin x)/2 and area CPD = CD·PD sin y)/2, so AP/PC = area APD/area CPD = (sin x/sin y)(AD/CD) = rational. But AP + PC is rational, so AP is rational. Similarly for the other segments.

Problem A3
Given a sequence S_1 of n+1 non-negative integers, $a_0, a_1, \dots , a_n$ we derive another sequence S_2 with terms b_0, $b_1, \dots , b_n$, where b_i is the number of terms preceding a_i in S_1 which are different from a_i (so $b_0 = 0$). Similarly, we derive S_2 from S_1 and so on. Show that if $a_i \le i$ for each i, then $S_n = S_{n+1}$.

Solution

Note that the derived sequence b_i also satisfies $b_i \le i$ (since there are only i terms preceding b_i). We show that $b_i \ge a_i$ for each i. That is obvious if $a_i = 0$. If $a_i = k > 0$, then since each of the first k terms $(a_0, a_1, \dots , a_{k-1})$ must be $< k$, we certainly have $b_i \ge k$.

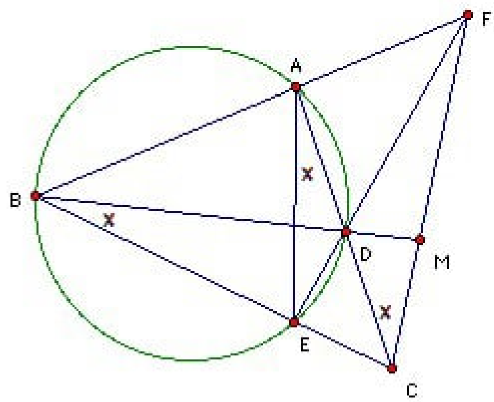
Next we show that if $b_i = a_i$, then further iterations do not change term i. If $b_i = a_i = 0$, then none of the terms before a_i differ from 0, so all the terms before b_i are also 0. But that means the corresponding terms of the next iteration must also all be 0. If $b_i = a_i = k > 0$, then since $a_0, a_1, \dots , a_{k-1}$ all differ from a_i, the remaining terms (if any) $a_k, a_{k+1}, \dots , a_{i-1}$ must all be the same as a_i. But that implies that each of $b_k, b_{k+1}, \dots , b_{i-1}$ must also be k. Hence if the next iteration is $c_0, c_1, \dots$ then $c_i = k$ also.

Now we use induction on k. Clearly term 0 is always 0. Considering the two cases, we see that term 1 does not change at iteration 1. So suppose that term i does not change at iteration i. If term i+1 does change at iteration i+1, then it must have changed at all previous iterations. So it must have started at 0 and increased by 1 at each iteration.

Problem B1

ABC is a triangle. A circle through A and B meets the sides AC, BC at D, E respectively. The lines AB and DE meet at F. The lines BD and CF meet at M. Show that M is the midpoint of CF iff $MB\cdot MD = MC^2$.

Solution



If $MB\cdot MD = MC^2$, then $BM/MC = CM/MD$, so triangles CMD and BMC are similar, so $\angle MCD = \angle MBC$. But ABED is cyclic, so $\angle MBC = \angle DAE$, so AE is parallel to CF. But now we can reverse the argument, but "reflecting" about BM so that we interchange A and E, and C and F, to conclude that $MB\cdot MD = MF^2$.

Suppose conversely that MC = MF. Applying Ceva's theorem to triangle BCF, we have that $(BA/AF)(1)(CE/EB) = 1$, so $BA/AF = BE/EC$ so AE is parallel to CF. We can now use the argument above to show that $MB\cdot MD = MC^2$.

Problem B2

Prove that for any positive reals x, y, z we have $(2x+y+z)^2(2x^2 + (y+z)^2) + (2y+z+x)^2(2y^2 + (z+x)^2) + (2z+x+y)^2(2z^2 + (x+y)^2) \le 8$.

Solution

If the inequality holds for x, y, z, then it also holds for kx, ky, kz, so it is sufficient to prove the result for x+y+z=3. The first term becomes $(x+3)^2/(2x^2+(3-x)^2) = (1/3)\ (x^2+6x+9)/(x^2-2x+3) = (1/3)\ (1 + (8x+6)/(2+(x-1)^2) \le (1/3)\ (1 + (8x+6)/2) = 4/3 + 4x/3$. Similarly for the other terms, so the whole expression $\le (4/3 + 4x/3) + (4/3 + 4y/3) + (4/3 + 4z/3) = 8$.

Problem B3

A positive integer is written at each vertex of a hexagon. A move is to replace a number by the (non-negative) difference between the two numbers at the adjacent vertices. If the starting numbers sum to 2003^{2003}, show that it is always possible to make a sequence of moves ending with zeros at every vertex.

Solution

It is possible to get stuck, so the result is not trivial. For example:

```
   1  1
0       0
   1  1
```

We show that provided the sum of the numbers is odd, we can always find a sequence of moves which give either (1) a lower maximum number, and odd sum, or (2) all zeros. Since the starting sum is odd, that is sufficient.

Note that no move *increases* the maximum. The first step is to show that if the sum is odd, we can find moves which give just one number odd. For convenience we refer to the numbers as

```
  A B
F     C
  E D
```

We also use the letters to refer to moves. So, for example, B means the move replacing B. Since the sum is odd, either A+C+E is odd or B+D+F is odd. wlog A+C+E is odd. Suppose just one of A,B,C is odd. wlog it is A. Making the moves B, F, A, F, D and working mod 2 we get successively:

```
  1 B       1 1       1 1       0 1       0 1       0 1
F     0   F     0   1     0   1     0   0     0   0     0
  0 D       0 D       0 D       0 D       0 D       0 0
```

Similarly, if all of A, B, C are odd, then B, F, D, E, C we get mod 2:

```
  1 B       1 0       1 0       1 0       1 0       1 0
F     1   F     1   0     1   0     1   0     1   0     0
  1 D       1 D       1 D       1 0       0 0       0 0
```

So wlog A is odd and all the other numbers are even. Suppose that the maximum is M. We show how to reduce M. There are two cases. Suppose first that M is even so that A < M. Then we make the moves B, C, D, E, F giving (mod 2):

```
  1 0       1 1       1 1       1 1       1 1       1 1
0     0   0     0   0     1   0     1   0     1   0     1
  0 0       0 0       0 0       0 1       1 1       1 1
```

The first four moves do not change A, which is neither 1 nor M, so the last move must reduce F, so the new maximum must be an odd number. But it must be ≤M, which is even, so the new maximum is <M and the sum is still odd.
The second case is M odd, so that A = M. If C > 0 we make the moves B, F, A, F giving (mod 2):

```
  1 0       1 1       1 1       0 1       0 1
0     0   0     0   1     0   1     0   0     0
  0 0       0 0       0 0       0 0       0 0
```

Since C is not 0 or M, the new B must be <M and the other new numbers are all even and hence <M. So the new maximum is <M and the sum is still odd. The same argument works if E > 0 (just reflect about AD). So finally suppose C = E = 0. Then we get (no reduction mod 2):

```
  A B      A A      A A      A A      A A      0 A      0 A      0 0
F     0  F     0  A     0  A     0  A     0  A     0  0     0  0     0
  0 D      0 D      0 D      0 D      0 0      0 0      0 0      0 0
```