1st Vietnam 1962 problems

Problem
Prove that $1/(1/a + 1/b) + 1/(1/c + 1/d) \le 1/(1/(a+c) + 1/(b+d))$ for positive reals a, b, c, d.

Solution
A straightforward, if inelegant, approach is to multiply out and expand everything. All terms cancel except four and we are left with $2abcd \le a^2d^2 + b^2c^2$, which is obviously true since $(ad - bc)^2 \ge 0$.

Problem 2
$f(x) = (1 + x)(2 + x^2)^{1/2}(3 + x^3)^{1/3}$. Find $f'(-1)$.

Solution
Differentiating gives $f'(x) = (2 + x^2)^{1/2}(3 + x^3)^{1/3}$ + terms with factor $(1 + x)$. Hence $f'(-1) = 3^{1/2}2^{1/3}$.

Problem 3
ABCD is a tetrahedron. A' is the foot of the perpendicular from A to the opposite face, and B' is the foot of the perpendicular from B to the opposite face. Show that AA' and BB' intersect iff AB is perpendicular to CD. Do they intersect if AC = AD = BC = BD?

Solution
Let the ray AB' meet CD at X and the ray BA' meet CD at Y. If AB' and A'B intersect, then X = Y. Let L be the line through A' parallel to CD. Then L is perpendicular to AA'. Hence CD is perpendicular to AA'. Similarly, let L' be the line through B' parallel to CD. Then L' is perpendicular to BB', and hence CD is perpendicular to BB'. So CD is perpendicular to two non-parallel lines in the plane ABX. Hence it is perpendicular to all lines in the plane ABX and, in particular, to AB.

Suppose conversely that AB is perpendicular to CD. Consider the plane ABY. CD is perpendicular to AB and to AA', so CD is perpendicular to the plane. Similarly CD is perpendicular to the plane ABX. But it can only be perpendicular to a single plane through AB. Hence X = Y and so AA' and BB' belong to the same plane and therefore meet.

Problem 4
The tetrahedron ABCD has BCD equilateral and AB = AC = AD. The height is h and the angle between ABC and BCD is α. The point X is taken on AB such that the plane XCD is perpendicular to AB. Find the volume of the tetrahedron XBCD.

Problem 5
Solve the equation $\sin^6 x + \cos^6 x = 1/4$.

Solution
Put $a = \sin^2 x$, $b = \cos^2 x$. Then a and b are non-negative with sum 1, so we may put $a = 1/2 + h$, $b = 1/2 - h$. Then $a^3 + b^3 = 1/4 + 3h^2 \ge 1/4$ with equality iff $h = 0$. Hence x is a solution of the equation given iff $\sin^2 x = \cos^2 x = 1/2$ or x is an odd multiple of $\pi/4$.

2nd VMO 1963

Problem 1
A conference has 47 people attending. One woman knows 16 of the men who are attending, another knows 17, and so on up to the last woman who knows all the men who are attending. Find the number of men and women attending the conference.

Solution
Suppose there are m women. Then the last woman knows 15+m men, so 15+2m = 47, so m = 16. Hence there are 31 men and 16 women.

Problem 2
For what values of m does the equation $x^2 + (2m + 6)x + 4m + 12 = 0$ has two real roots, both of them greater than -2.

Answer
$m \le -3$

Solution
For real roots we must have $(m+3)^2 \ge 4m+12$ or $(m-1)(m+3) \ge 0$, so $m \ge 1$ or $m \le -3$. If $m \ge 1$, then $-(2m+6) \le -8$, so at least one of the roots is < -2. So we must have $m \le -3$.

The roots are $-(m+3) \pm\sqrt{(m^2+2m-3)}$. Now $-(m+3) \ge 0$, so $-(m+3) + \sqrt{(m^2+2m-3)} \ge 0 > -2$. So we need $-(m+3) - \sqrt{(m^2+2m-3)} > -2$, or $\sqrt{(m^2+2m-3)} < -m-1 = \sqrt{(m^2+2m+1)}$, which is always true.

Problem 3
Solve the equation $\sin^3x \cos 3x + \cos^3x \sin 3x = 3/8$.

Answer
$7½^o + k90^o$ or $37½^o + k90^o$

Solution
We have $\sin 3x = 3 \sin x - 4 \sin^3x$, $\cos 3x = 4 \cos^3x - 3 \cos x$. So we need $4 \sin^3x \cos^3x - 3 \sin^3x \cos x + 3 \sin x \cos^3x - 4 \sin^3x \cos^3x = 3/8$ or $8 \sin x \cos x(\cos^2x - \sin^2x) = 1$, or $4 \sin 2x \cos 2x = 1$ or $\sin 4x = 1/2$. Hence $4x = 30^o + k360^o$ or $150^o + k360^o$. So $x = 7½^o + k90^o$ or $37½^o + k90^o$.

Problem 4
The tetrahedron SABC has the faces SBC and ABC perpendicular. The three angles at S are all 60^o and SB = SC = 1. Find its volume.

Problem 5
The triangle ABC has perimeter p. Find the side length AB and the area S in terms of $\angle A$, $\angle B$ and p. In particular, find S if p = 23.6, A = 52.7 deg, B = 46 4/15 deg.

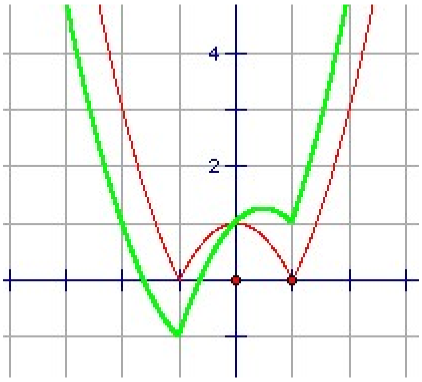
3rd VMO 1964

Problem A1
Find cos x + cos(x + 2π/3) + cos(x + 4π/3) and sin x + sin(x + 2π/3) + sin(x + 4π/3).

Solution
Using cos(A+B) = cos A cos B - sin A sin B, we have cos(x + 2π/3) = -(1/2) cos x + (√3)/2 sin x, cos(x + 4π/3) = -(1/2) cos x - (√3)/2 sin x. Hence cos x + cos(x + 2π/3) + cos(x + 4π/3) = 0. Similarly, sin(x + 2π/3) = -1/2 sin x + (√3)/2 cos x, sin(x + 4π/3) = -1/2 sin x - (√3)/2 cos x, so sin x + sin(x + 2π/3) + sin(x + 4π/3) = 0.

Problem A2
Draw the graph of the functions y = | x^2 - 1 | and y = x + | x^2 - 1 |. Find the number of roots of the equation x + | x^2 - 1 | = k, where k is a real constant.

Answer
0 for k < -1 , 1 for k = -1 , 2 for -1 < k < 1 , 3 for k = 1 , 4 for 1 < k < 5/4 , 3 for k = 5/4 , 2 for k > 5/4



Solution

It is clear from the graph that there are no roots for k < -1, and one root for k = -1 (namely x = -1). Then for k > -1 there are two roots except for a small interval [1, 1+h]. At k = 1, there are 3 roots (x = -2, 0, 1). The upper bound is at the local maximum between 0 and 1. For such x, y = x + 1 - x2 = 5/4 - (x - 1/2)2, so the local maximum is at 5/4. Thus there are 3 roots at k = 5/4 and 4 roots for k ∈ (1, 5/4).

Problem A3
Let O be a point not in the plane p and A a point in p. For each line in p through A, let H be the foot of the perpendicular from O to the line. Find the locus of H.

Answer: circle diameter AB, where OB is the normal to p

Solution
Let B be the foot of the perpendicular from O to p. We claim that the locus is the circle diameter AB. Any line in p through A meets this circle at one other point K (except for the tangent to the circle at A, but in that case A is obviously the foot of the perpendicular from O to the line). Now BK is perpendicular to AK, so OK is also perpendicular to AK, and hence K must be the foot of the perpendicular from O to the line.

Problem A4
Define the sequence of positive integers f_n by $f_0 = 1$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$. Show that $f_n = (a^{n+1} - b^{n+1})/\sqrt{5}$, where a, b are real numbers such that a + b = 1, ab = -1 and a > b.

Solution
Put a = (1+√5)/2, b = (1-√5)/2. Then a, b are the roots of $x^2 - x - 1 = 0$ and satisfy a + b = 1, ab = -1. We show by induction that $f_n = (a^{n+1} - b^{n+1})/\sqrt{5}$. We have $f_0 = (a-b)/\sqrt{5} = 1$, $f_1 = (a^2-b^2)/\sqrt{5} = (a+1 - b-1)/\sqrt{5} = 1$, so the result is true for n = 0, 1. Finally, suppose $f_n = (a^{n+1} - b^{n+1})/\sqrt{5}$ and $f_{n+1} = (a^{n+2} - b^{n+2})/\sqrt{5}$. Then $f_{n+2} = f_{n+1} + f_n = (1/\sqrt{5})(a^{n+1}(a+1) - b^{n+1}(b+1)\ ) = (a^{n+1}a^2 - b^{n+1}b^2)/\sqrt{5}$, so the result is true for n+1.

4th Vietnam 1965 problems

Problem 1
At time $t = 0$, a lion L is standing at point O and a horse H is at point A running with speed v perpendicular to OA. The speed and direction of the horse does not change. The lion's strategy is to run with constant speed u at an angle $0 < \varphi < \pi/2$ to the line LH. What is the condition on u and v for this strategy to result in the lion catching the horse? If the lion does not catch the horse, how close does he get? What is the choice of φ required to minimise this distance?

Problem 2
AB and CD are two fixed parallel chords of the circle S. M is a variable point on the circle. Q is the intersection of the lines MD and AB. X is the circumcenter of the triangle MCQ. Find the locus of X. What happens to X as M tends to (1) D, (2) C? Find a point E outside the plane of S such that the circumcenter of the tetrahedron MCQE has the same locus as X.

Problem 3
m an n are fixed positive integers and k is a fixed positive real. Show that the minimum value of $x_1^m + x_2^m + x_3^m + \dots + x_n^m$ for real x_i satisfying $x_1 + x_2 + \dots + x_n = k$ occurs at $x_1 = x_2 = \dots = x_n$.

5th Vietnam 1966 problems

Problem 1
[missing]

Problem 2
a, b are two fixed lines through O. Variable lines x, y are parallel. x intersects a at A and b at C, y intersects a at B and b at D. The lines AD and BC meet at M. The line through M parallel to x meets a at L and b at N. What can you say about L, M, N? Find the locus M.

Problem 3
(1) ABCD is a rhombus. A tangent to the inscribed circle meets AB, DA, BC, CD at M, N, P, Q respectively. Find a relationship between BM and DN.
(2) ABCD is a rhombus and P a point inside. The circles through P with centers A, B, C, D meet the four sides AB, BC, CD, DA in eight points. Find a property of the resulting octagon. Use it to construct a regular octagon.
(3) [unclear].

6th Vietnam 1967 problems

Problem 1
Draw the graph of the function $y = | x^3 - x^2 - 2x | /3 - | x + 1 |$.

Problem 2
A river flows at speed u. A boat has speed v relative to the water. If its velocity is at an angle α relative the direction of the river, what is its speed relative to the river bank? What α minimises the time taken to cross the river?

Problem 3
(1) ABCD is a rhombus. A tangent to the inscribed circle meets AB, DA, BC, CD at M, N, P, Q respectively. Find a relationship between BM and DN.
(2) ABCD is a rhombus and P a point inside. The circles through P with centers A, B, C, D meet the four sides AB, BC, CD, DA in eight points. Find a property of the resulting octagon. Use it to construct an equiangular octagon.
(3) Rotate the figure about the line AC to form a solid. State a similar result.

7th Vietnam 1968 problems

Problem 1
The real numbers a and b satisfy $a \geq b > 0$, $a + b = 1$. Show that $a^m - a^n \geq b^m - b^n > 0$ for any positive integers $m < n$. Show that the quadratic $x^2 - b^n x - a^n$ has two real roots in the interval $(-1, 1)$.

Problem 2
L and M are two parallel lines a distance d apart. Given r and x, construct a triangle ABC, with A on L, and B and C on M, such that the inradius is r, and angle A = x. Calculate angles B and C in terms of d, r and x. If the incircle touches the side BC at D, find a relation between BD and DC.

8th VMO 1969

Problem A1
A graph G has n + k points. A is a subset of n points and B is the subset of the other k points. Each point of A is joined to at least k - m points of B where $nm < k$. Show that there is a point in B which is joined every point in A.

Solution
This is just the pigeonhole principle. There are at least $n(k-m) > (n-1)k$ edges from points of A to points of B and there are k points in B. So some point of B must have at least n edges (otherwise there would be $\leq (n-1)k$ edges). That point is joined to every point of A.

Problem A2
Find all real x such that $0 < x < \pi$ and $8/(3 \sin x - \sin 3x) + 3 \sin^2 x \leq 5$.

Answer: $\pi/2$

Solution

We have $3 \sin x - \sin 3x = 4 \sin^3 x$. Put $s = \sin x$. Then we want $2/s^3 + 3s^2 \leq 5$. Note that since $0 < x < \pi$ we have s positive. But by AM/GM we have $1/s^3 + 1/s^3 + s^2 + s^2 + s^2 > 5$ with equality iff $s = 1$, so we must have $\sin x = 1$ and hence $x = \pi/2$.

Problem A3
The real numbers x_1, x_4, y_1, y_2 are positive and the real numbers x_2, x_3, y_3, y_4 are negative. We have $(x_i - a)^2 + (y_i - b)^2 \leq c^2$ for i = 1, 2, 3, 4. Show that $a^2 + b^2 \leq c^2$. State the result in geometric language.

Solution
Stated geometrically, the result is: if a disk includes a point in each quadrant, then it must also include the origin. We use the fact that a disk is convex. Let P_i be the point (x_i,y_i). The segment P_1P_2 must intersect the positive x-axis. By convexity, the point of intersection, call it X, must lie in the disk. Similarly, P_3P_4 must intersect the negative x-axis at some point Y, which must be in the disk. Then all points of the segment XY are in the disk and hence, in particular, the origin.

Problem 4
Two circles centers O and O', radii R and R', meet at two points. A variable line L meets the circles at A, C, B, D in that order and AC/AD = CB/BD. The perpendiculars from O and O' to L have feet H and H'. Find the locus of H and H'. If $OO'^2 < R^2 + R'^2$, find a point P on L such that PO + PO' has the smallest possible value. Show that this value does not depend on the position of L. Comment on the case $OO'^2 > R^2 + R'^2$.

9th VMO 1970

Problem A1
ABC is a triangle. Show that sin A/2 sin B/2 sin C/2 < 1/4.

Solution
Put x = A/2, y = B/2. We have sin C/2 = sin(90^o-x-y) = cos(x+y). So we need to show that sin x sin y cos(x+y) < 1/4, or (cos(x-y) - cos(x+y))cos(x+y) < 1/2, or 2 cos(x-y) cos(x+y) < 1 + 2 $\cos^2$(x+y). But 2 cos(x-y) cos(x+y) ≤ $\cos^2$(x+y) + $\cos^2$(x-y) ≤ 1 + $\cos^2$(x+y) < 1 + 2 $\cos^2$(x+y) (since 0 < x,y < 90^o.

Problem A2
Find all positive integers which divide 1890·1930·1970 and are not divisible by 45.

Answer
$k\cdot 2^a 7^b 193^c 197^d$, where k = 1, 3, 3^2, 3^3, 5, 3·5, a = 0, 1, 2, or 3, b = 0 or 1, c = 0 or 1, d = 0 or 1 (192 solutions in all)

Solution
1890 = $2\cdot 3^3 5\cdot 7$, 1930 = 2·5·193, 1970 = 2·5·197 (and 193 and 197 are prime). So 1890·1930·1970 = $2^3 3^3 5^3 7\cdot 193\cdot 197$.

Problem A3.
The function f(x, y) is defined for all real numbers x, y. It satisfies f(x,0) = ax (where a is a non-zero constant) and if (c, d) and (h, k) are distinct points such that f(c, d) = f(h, k), then f(x, y) is constant on the line through (c, d) and (h, k). Show that for any real b, the set of points such that f(x, y) = b is a straight line and that all such lines are parallel. Show that f(x, y) = ax + by, for some constant b.

Problem B1
AB and CD are perpendicular diameters of a circle. L is the tangent to the circle at A. M is a variable point on the minor arc AC. The ray BM, DM meet the line L at P and Q respectively. Show that AP·AQ = AB·PQ. Show how to construct the point M which gives BQ parallel to DP. If the lines OP and BQ meet at N find the locus of N. The lines BP and BQ meet the tangent at D at P' and Q' respectively. Find the relation between P' and Q'. The lines DP and DQ meet the line BC at P" and Q" respectively. Find the relation between P" and Q".

Problem B2
A plane p passes through a vertex of a cube so that the three edges at the vertex make equal angles with p. Find the cosine of this angle. Find the positions of the feet of the perpendiculars from the vertices of the cube onto p. There are 28 lines through two vertices of the cube and 20 planes through three vertices of the cube. Find some relationship between these lines and planes and the plane p.

10th Vietnam 1971 problems

Problem A1
m, n, r, s are positive integers such that: (1) $m < n$ and $r < s$; (2) m and n are relatively prime, and r and s are relatively prime; and (3) $\tan^{-1} m/n + \tan^{-1} r/s = \pi/4$. Given m and n, find r and s. Given n and s, find m and r. Given m and s, find n and r.

Solution
Put $k = \tan^{-1} m/n$, $h = \tan^{-1} r/s$. Then $h + k = \pi/4$, so $1 = \tan(h+k) = (\tan h + \tan k)/(1 - \tan h \tan k) = (r/s + m/n)/(1 - mr/ns) = (nr + ms)/(ns - mr)$. Hence $(m+n)(r+s) = 2ns$.

Suppose we are given m and n. We have $(m+n)r = (n-m)s$. If m and n have opposite parity, then m+n and m-n are coprime, so $r = n-m$, $s = m+n$. If m and n have the same parity, then m+n and m-n are both even, so $r = (n-m)/2$, $s = (m+n)/2$.

Suppose we are given n and s. wlog $n \geq s$. If $d > 1$ divides n and d divides m+n, then d divides m and n. Contradiction. So n must divide r+s. But $r < s \leq n$, so $r+s < 2n$. Hence $r+s = n$, so $m+n = 2s$. Hence n and s must be relatively prime and $n < 2s$.

Suppose we are given m and s. Then $(m+n)(s-r) = 2ms$. If $d > 1$ divides m and m+n, then d divides m and n. Contradiction. So m must divide s-r. So if $d > 1$ divides m and s, then d divides r, so r and s are not coprime. Contradiction. Hence m and s must be coprime. Also m must be $< s$. But now we can take $r = s-m$, $n = 2s-m$.

Problem B1
ABCDA'B'C'D' is a cube (with ABCD and A'B'C'D' faces, and AA', BB', CC', DD' edges). L is a line which intersects or is parallel to the lines AA', BC and DB'. L meets the line BC at M (which may be the point at infinity). Let $m = |BM|$. The plane MAA' meets the line B'C' at E. Show that $|B'E| = m$. The plane MDB' meets the line A'D' at F. Show that $|D'F| = m$. Hence or otherwise show how to construct the point P at the intersection of L and the plane A'B'C'D'. Find the distance between P and the line A'B' and the distance between P and the line A'D' in terms of m. Find a relation between these two distances that does not depend on m. Find the locus of M. Let S be the envelope of the line L as M varies. Find the intersection of S with the faces of the cube.

11th Vietnam 1972 problems

Problem A1
Let x = cos α, y = cos nα, where n is a positive integer. Show that for each x in the range [-1, 1], there is only one corresponding y. So consider y as a function of x and put $y = T_n(x)$. Find $T_1(x)$ and $T_2(x)$ and show that $T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$. Show that $T_n(x)$ is a polynomial of degree n with n roots in [-1, 1].

Problem A2
For any positive integer n, let $f(n) = \sum (-1)^{(d-1)/2}$ where the sum is taken over all odd d dividing n. Show that:

$f(2^n) = 1$
f(p) = 2 for p a prime congruent to 1 mod 4
f(p) = 0 for p a prime congruent to 3 mod 4
$f(p^n) = n+1$ for p a prime congruent to 1 mod 4
$f(p^n) = 1$ for p a prime congruent to 3 mod 4, and n even
$f(p^n) = 0$ for p a prime congruent to 3 mod 4, and n odd

Show that f(mn) = f(m) f(n) for m and n relatively prime. Find $f(5^4 11^{28} 17^{19})$ and f(1980). Show how to calculate f(n).

Problem B1
ABC is a triangle. U is a point on the line BC. I is the midpoint of BC. The line through C parallel to AI meets the line AU at E. The line through E parallel to BC meets the line AB at F. The line through E parallel to AB meets the line BC at H. The line through H parallel to AU meets the line AB at K. The lines HK and FG meet at T. V is the point on the line AU such that A is the midpoint of UV. Show that V, T and I are collinear. [Next part unclear.]
B2. ABCD is a regular tetrahedron with side a. Take E, E' on the edge AB such that AE = a/6, AE' = 5a/6. Take F, F' on the edge AC such that AF = a/4, AF' = 3a/4. Take G, G' on the edge AD such that AG = a/3, AG' = 2a/3. Find the intersection of the planes BCD, EFG and E'F'G' and its position in the triangle BCD. Calculate the volume of EFGE'F'G' and find the angles between the lines AB, AC, AD and the plane EFG.

12th VMO 1974

Problem A1

Find all positive integers n and b with $0 < b < 10$ such that if a_n is the positive integer with n digits, all of them 1, then $a_{2n} - b\, a_n$ is a square.

Answer

b = 2 works for any n
b = 7 works for n = 1

Solution

$a_n = (10^n-1)/9$, so we need $10^{2n} - b10^n + (b-1)$ to be a square. For b = 2, this is true for all n. Note that $(10^n - c)^2 = 10^{2n} - 2c10^n + c^2$, so if b = 2c, we need $c^2 = b-1 = 2c - 1$ and hence c = 1. So b = 4, 6, 8 do not work. Equally $10^{2n} > 10^{2n} - 10^n > (10^n - 1)^2 = 10^{2n} - 2\cdot 10^n + 1$, so it cannot be a square for b = 1. Similarly, $(10^n - 1)^2 = 10^{2n} - 2\cdot 10^n + 1 > 10^{2n} - 3\cdot 10^n + 2 > 10^{2n} - 4\cdot 10^n + 4 = (10^n - 2)^2$, so b = 3 does not work. Similarly, $(10^n - 2)^2 = 10^{2n} - 4\cdot 10^n + 4 > 10^{2n} - 5\cdot 10^n + 4 > 10^{2n} - 6\cdot 10^n + 9 = (10^n - 3)^2$, so b = 5 does not work. Similarly, $(10^n - 3)^2 = 10^{2n} - 6\cdot 10^n + 9 > 10^{2n} - 7\cdot 10^n + 6 > 10^{2n} - 8\cdot 10^n + 16 = (10^n - 4)^2$ for n > 1, so b = 7 does not work, except possibly for n = 1. Since $11 - 7 = 2^2$, b = 1 does work for n = 1. Finally, $(10^n - 4)^2 = 10^{2n} - 8\cdot 10^n + 16 > 10^{2n} - 9\cdot 10^n + 8 > 10^{2n} - 10\cdot 10^n + 25 = (10^n - 4)^2$ for n > 1, so b = 9 does not work, except possibly for n = 1. It is easy to check it does not work for n = 1.

Problem A2

(1) How many positive integers n are such that n is divisible by 8 and n+1 is divisible by 25?
(2) How many positive integers n are such that n is divisible by 21 and n+1 is divisible by 165?
(3) Find all integers n such that n is divisible by 9, n+1 is divisible by 25 and n+2 is divisible by 4.

Answer

infinitely many, none, n = 774 mod 900

Solution

(1) We need n = 0 mod 8 and -1 mod 25. Hence n = 24 mod 200.
(2) We need n = 0 mod 21, n = -1 mod 165. But 3 divides 165, so we require n = 0 mod 3 and 2 mod 3, which is impossible.
(3) We need n = 0 mod 9, -1 mod 25, 2 mod 4. Hence n = 99 mod 225, and 2 mod 4, so n = 774 mod 900

Problem B1

ABC is a triangle. AH is the altitude. P, Q are the feet of the perpendiculars from P to AB, AC respectively. M is a variable point on PQ. The line through M perpendicular to MH meets the lines AB, AC at R, S respectively. Show that ARHS is cyclic. If M' is another position of M with corresponding points R', S', show that the ratio RR'/SS' is constant. Find the conditions on ABC such that if M moves at constant speed along PQ, then the speeds of R along AB and S along AC are the same. The point K on the line HM is on the other side of M to H and satisfies KM = HM. The line through K perpendicular to PQ meets the line RS at D. Show that if $\angle A = 90^o$, then $\angle BHR = \angle DHR$.

Problem B2

C is a cube side 1. The 12 lines containing the sides of the cube meet at plane p in 12 points. What can you say about the 12 points?

13th Vietnam 1975 problems

A1.
The roots of the equation $x^3 - x + 1 = 0$ are a, b, c. Find $a^8 + b^8 + c^8$.

Answer
10

Solution
We have $a+b+c = 0$, $ab+bc+ca = -1$, $abc = -1$.
$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 2$. Similarly, $a^2b^2+b^2c^2+c^2a^2 = (ab+bc+ca)^2 - 2abc(a+b+c) = 1$. Hence $a^4+b^4+c^4 = (a^2+b^2+c^2)^2 - 2(a^2b^2+b^2c^2+c^2a^2) = 2$. Similarly, $a^4b^4+b^4c^4+c^4a^4 = (a^2b^2+b^2c^2+c^2a^2)^2 - 2a^2b^2c^2(a^2+b^2+c^2) = 1 - 4 = -3$. Finally $a^8+b^8+c^8 = (a^4+b^4+c^4)^2 - 2(a^4b^4+b^4c^4+c^4a^4) = 4 + 6 = 10$.

A2.
Find all real x which satisfy $(x^3 + a^3)/(x + a)^3 + (x^3 + b^3)/(x + b)^3 + (x^3 + c^3)/(x + c)^3 + 3(x - a)(x - b)(x - c)/(2(x + a)(x + b)(x + c)) = 3/2$.

A3.
ABCD is a tetrahedron. The three edges at B are mutually perpendicular. O is the midpoint of AB and K is the foot of the perpendicular from O to CD. Show that vol KOAC/vol KOBD = AC/BD iff $2\cdot AC\cdot BD = AB^2$.

B1.
Find all terms of the arithmetic progression -1, 18, 37, 56, ... whose only digit is 5.

Answer
5...5 with 18k+5 digits for k = 0, 1, 2, 3, ...

Solution
The general term is $19k - 1$ and this must equal $5(10^n - 1)/9$ for some n. Hence $5\cdot 10^n = 171k - 4 = -4 \bmod 19$. So we certainly require $10^n = 3 \bmod 19$ and hence $n = 5 \bmod 18$. Conversely if $n = 5 \bmod 18$, then $10^n = 3 \bmod 19$, so $5\cdot 10^n = -4 \bmod 19$, so $5\cdot 10^n = 19h - 4$ for some h. So $5(10^n - 1) = 19h - 9$. Now lhs and 9 are divisible by 9, so 19h must be divisible by 9, so h must be divisible by 9. Put $h = 9k$ and we get $19k - 1 = 5(10^n - 1)/9$.

B2.
Show that the sum of the maximum and minimum values of the function $\tan(3x)/\tan^3 x$ on the interval $(0, \pi/2)$ is rational.

B3.
L is a fixed line and A a fixed point not on L. L' is a variable line (in space) through A. Let M be the point on L and N the point on L' such that MN is perpendicular to L and L'. Find the locus of M and the locus of the midpoint of MN.

14th Vietnam 1976 problems

A1. Find all integer solutions to $m^{m+n} = n^{12}$, $n^{m+n} = m^3$.

Answer
(m,n) = (4,2)

Solution
Suppose a prime p divides n. Then from the first equation it must also divide m. So suppose pa is the highest power of p dividing m, and pb is the highest power of p dividing n. Then the first equation gives 12b = a(m+n), and the second gives 3a = b(m+n). Hence $36b = 3a(m+n) = b(m+n)^2$, so m+n = 6 and a = 2b. Hence m = 4, n = 2.

A2.
Find all triangles ABC such that (a cos A + b cos B + c cos C)/(a sin A + b sin B + c sin C) = (a + b + c)/9R, where, as usual, a, b, c are the lengths of sides BC, CA, AB and R is the circumradius.

A3.
P is a point inside the triangle ABC. The perpendicular distances from P to the three sides have product p. Show that $p \le 8\ S^3/(27abc)$, where S = area ABC and a, b, c are the sides. Prove a similar result for a tetrahedron.

B1.
Find all three digit integers abc = n, such that 2n/3 = a! b! c!

Answer
432

Solution
2n/3 < 1000 < 7!, so a, b, c ≤ 6. Hence n ≤ 666, so 2n/3 ≤ 444, so a ≤ 4. Also 6! > 444, so b, c ≤ 5.

Consider first the case a = 4. Then (2n/3)/a! ≤ 2·455/(3·24) < 13. So b!c! ≤ 12, so b, c ≤ 3. Also 2n is divisible by a!3, so n is divisible by 9. So {b,c} = {2,3}, so n = (3/2)4!3!2! = 432.
Now suppose a = 3. Then n = (3/2) 6 b! c! = 9 b! c!, so we need 300 < 9 b! c! < 355, so 33 < b!c! < 40. Hence b, c ≤ 4. But 4! < 33 and 2·4! > 40, so b, c ≤ 3. Hence b = c = 3. But (3/2)3!3!3! = 324 ≠ 333, so there are no solutions with a = 3.
Suppose a = 2. Then n = 3b!c!, so 200 < 3b!c! < 255. But 3·5! > 255 and 3·3!3! < 200, so at least one of b, c must be 4. If the other is x, then 200/72 ≤ x! ≤ 255/72, so x! = 3. Contradiction, so there are no solutions with a = 2.

Finally, suppose a = 1. Then n = (3/2)b!c!, so (2/3)100 ≤ b!c! ≤ (2/3)155, and so 67 ≤ b!c! ≤ 103. But 5! > 103 and 3!3! < 67, so one of b, c must be 4. If the other is x, then 2.7 < x! < 4.3, so x! = 3 or 4. Contradiction.

B2.
L, L' are two skew lines in space and p is a plane not containing either line. M is a variable line parallel to p which meets L at X and L' at Y. Find the position of M which minimises the distance XY. L" is another fixed line. Find the line M which is also perpendicular to L".

B3.
Show that $1/x_1^n + 1/x_2^n + ... + 1/x_k^n \ge k^{n+1}$ for real numbers x_i with sum 1.

15th Vietnam 1977 problems

A1.
Find all real x such that $\sqrt{(x - 1/x)} + \sqrt{(1 - 1/x)} > (x - 1)/x$.

A2.
Show that there are 1977 non-similar triangles such that the angles A, B, C satisfy $(\sin A + \sin B + \sin C)/(\cos A + \cos B + \cos C) = 12/7$ and $\sin A \sin B \sin C = 12/25$.

A3.
Into how many regions do n circles divide the plane, if each pair of circles intersects in two points and no point lies on three circles?

B1.
p(x) is a real polynomial of degree 3. Find necessary and sufficient conditions on its coefficients in order that p(n) is integral for every integer n.

B2.
The real numbers $a_0, a_1, \dots , a_{n+1}$ satisfy $a_0 = a_{n+1} = 0$ and $| a_{k-1} - 2a_k + a_{k+1}| \le 1$ for $k = 1, 2, \dots , n$. Show that $|a_k| \le k(n + 1 - k)/2$ for all k.

B3.
The planes p and p' are parallel. A polygon P on p has m sides and a polygon P' on p' has n sides. Find the largest and smallest distances between a vertex of P and a vertex of P'.

16th Vietnam 1978 problems

A1. Find all three digit numbers abc such that 2(abc) = bca + cab.

Answer
111, 222, 333, 370, 407, 444, 481, 518, 555, 592, 629, 666, 777, 888, 999.

Solution
We have 200a + 20b + 2c = 100b + 10c + a + 100c + 10a + b, so 7a = 3b + 4c (*). There are the obvious solutions a = b = c. If any two of a, b, c are equal, then (*) implies that they are all equal. So we can assume they are all distinct. Note that we must have a = b mod 4. It is now a question of looking in turn at a = 1, 2, ... , 9. For example a = 1, so b = 5 or 9. But in both cases 3b > 7a.

A2.
Find all values of the parameter m such that the equations $x^2 = 2^{|x|} + |x| - y - m = 1 - y^2$ have only one root.

Answer
0, 2

Solution
If x is a non-zero solution, then -x is another, so we must have x = 0. Hence $0 = 1 - y - m = 1 - y^2$. So y = ±1, so m = 0 or 2, and each of these gives just one root.

A3.
The triangle ABC has angle A = 30^o and AB = 3/4 AC. Find the point P inside the triangle which minimises 5 PA + 4 PB + 3 PC.

B1.
Find three rational numbers a/d, b/d, c/d in their lowest terms such that they form an arithmetic progression and b/a = (a + 1)/(d + 1), c/b = (b + 1)/(d + 1).

B2.
A river has a right-angle bend. Except at the bend, its banks are parallel lines a distance a apart. At the bend the river forms a square with the river flowing in across one side and out across an adjacent side. What is the longest boat of length c and negligible width which can pass through the bend?

B3.
ABCDA'B'C'D' is a rectangular parallelepiped (so that ABCD and A'B'C'D' are faces and AA', BB', CC', DD' are edges). We have AB = a, AD = b, AA' = c. The perpendicular distances of A, A', D from the line BD' are p, q, r. Show that there is a triangle with sides p, q, r. Find a relation between a, b, c and p, q, r.

17th Vietnam 1979 problems

A1.
Show that for all $x > 1$ there is a triangle with sides, $x^4 + x^3 + 2x^2 + x + 1$, $2x^3 + x^2 + 2x + 1$, $x^4 - 1$.

Solution
Put $a = x^4 + x^3 + 2x^2 + x + 1$, $b = 2x^3 + x^2 + 2x + 1$, $c = x^4 - 1$. Then obviously $a > c$. Also $a - b = (x^4 - x^3) + (x^2 - x) > 0$. But $a - b = x^4 - (x^3 - x^2) - x < c$. So a is the longest side but shorter than $b + c$. That is sufficient to ensure that there is a triangle with sides a, b, c.

A2.
Find all real numbers a, b, c such that $x^3 + ax^2 + bx + c$ has three real roots α,β,γ (not necessarily all distinct) and the equation $x^3 + a^3x^2 + b^3x + c^3$ has roots α^3, β^3, γ^3.

Answer
$(a,b,c) = (h,k,hk)$, where $k \le 0$

Solution
We have $-a = \alpha + \beta + \gamma$, $b = \alpha\beta + \beta\gamma + \gamma\alpha$, $-c = \alpha\beta\gamma$. We need $-a^3 = \alpha^3 + \beta^3 + \gamma^3$. Since $-a^3 = (\alpha + \beta + \gamma)^3 = \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha\beta^2 + \dots) + 6\alpha\beta\gamma = -a^3 - 3ab + 3c$, we require $c = ab$.

Now if $c = ab$, the cubic factorises as $(x+a)(x^2+b)$, so we get three real roots iff $b \le 0$. But if $b \le 0$ and $c = ab$, then the roots are $-a$, $\pm\sqrt{-b}$. So the equation with roots α^3, β^3, γ^3 is $(x + a^3)(x^2 + b^3) = x^3 + a^3x^2 + b^3x + a^3b^3$. Thus $c = ab$, $b \le 0$ is both a necessary and a sufficient condition.

A3.
ABC is a triangle. Find a point X on BC such that area ABX/area ACX = perimeter ABX/perimeter ACX.

B1.
For each integer $n > 0$ show that there is a polynomial $p(x)$ such that $p(2 \cos x) = 2 \cos nx$.

Solution
Induction on n. Obvious for $n = 1$. For $n = 2$, we have $2 \cos 2x = 4 \cos^2 x - 2 = (2 \cos x)^2 - 2$, so it is true for $n = 2$. Now suppose it is true for n and n+1. Then $2 \cos(n+1) \cos x = \cos(n+2)x + \cos nx$, so $2 \cos(n+2) x = (2 \cos(n+1)x)(2 \cos x) - (2 \cos nx)$, and so it is true for n+2.

B2.
Find all real numbers k such that $x^2 - 2 x [x] + x - k = 0$ has at least two non-negative roots.

B3.
ABCD is a rectangle with $BC/AB = \sqrt{2}$. ABEF is a congruent rectangle in a different plane. Find the angle DAF such that the lines CA and BF are perpendicular. In this configuration, find two points on the line CA and two points on the line BF so that the four points form a regular tetrahedron.

18th Vietnam 1980 problems

A1.
Let $x_1, x_2, \dots, x_n$ be real numbers in the interval $[0, \pi]$ such that $(1 + \cos x_1) + (1 + \cos x_2) + \dots + (1 + \cos x_n)$ is an odd integer. Show that $\sin x_1 + \sin x_2 + \dots + \sin x_n \ge 1$.

A2.
Let $x_1, x_2, \dots, x_n$ be positive reals with sum s. Show that $(x_1 + 1/x_1)^2 + (x_2 + 1/x_2)^2 + \dots + (x_n + 1/x_n)^2 \ge n(n/s + s/n)^2$.

A3.
P is a point inside the triangle $A_1A_2A_3$. The ray A_iP meets the opposite side at B_i. C_i is the midpoint of A_iB_i and D_i is the midpoint of PB_i. Show that area $C_1C_2C_3$ = area $D_1D_2D_3$.

B1.
Show that for any tetrahedron it is possible to find two perpendicular planes such that if the projection of the tetrahedron onto the two planes has areas A and A', then $A'/A > \sqrt{2}$.

B2.
Does there exist real m such that the equation $x^3 - 2x^2 - 2x + m$ has three different rational roots?

B3.
Given $n > 1$ and real $s > 0$, find the maximum of $x_1x_2 + x_2x_3 + x_3x_4 + \dots + x_{n-1}x_n$ for non-negative reals x_i such that $x_1 + x_2 + \dots + x_n = s$.

19th Vietnam 1981 problems

A1.
Show that the triangle ABC is right-angled iff $\sin A + \sin B + \sin C = \cos A + \cos B + \cos C + 1$.

A2.
Find all integral values of m such that $x^3 + 2x + m$ divides $x^{12} - x^{11} + 3x^{10} + 11x^3 - x^2 + 23x + 30$.

A3.
Given two points A, B not in the plane p, find the point X in the plane such that XA/XB has the smallest possible value.

B1.
Find all real solutions to:
$w^2 + x^2 + y^2 + z^2 = 50$
$w^2 - x^2 + y^2 - z^2 = -24$
$wx = yz$
$w - x + y - z = 0$.

B2.
$x_1, x_2, x_3, \dots, x_n$ are reals in the interval $[a, b]$. $M = (x_1 + x_2 + \dots + x_n)/n$, $V = (x_1^2 + x_2^2 + \dots + x_n^2)/n$. Show that $M^2 \ge 4Vab/(a + b)^2$.

B3.
Two circles touch externally at A. P is a point inside one of the circles, not on the line of centers. A variable line L through P meets one circle at B (and possibly another point) and the other circle at C (and possibly another point). Find L such that the circumcircle of ABC touches the line of centers at A.

20th Vietnam 1982 problems

A1.
Find a quadratic with integer coefficients whose roots are cos 72^o and cos 144^o.

A2.
Find all real solutions to $x(x + 1)(x + 2)(x + 3) = m - 1$.

Answer
$m < 0$, no solutions
$m = 0$, two solutions, $x = -3\pm\sqrt{5})/2$
$0 < m < 25/16$, four solutions, $x = -3/2 \pm\sqrt{(5/4 \pm\sqrt{m})}$
$m = 25/16$, three solutions, $x = -3/2, -3/2 \pm\sqrt{(5/2)}$
$m > 25/16$, two solutions, $x = -3/2 \pm\sqrt{(5/4 + \sqrt{m})}$

Solution
We have $m = (x^2+3x+1)^2 = (x + 3/2)^2 - 5/4$. So if m is negative there are no solutions. For non-negative m we have solutions as above

A3.
ABC is a triangle. A' is on the same side of BC as A, and A" is on the opposite side of BC. A'BC and A"BC are equilateral. B', B", C', C" are defined similarly. Show that area ABC + area A'B'C' = area A"B"C".

B1.
Find all positive integer solutions to $2^a + 2^b + 2^c = 2336$.

Answer
$\{a,b,c\} = \{5,8,11\}$

Solution
Assume $a \geq b \geq c$ (then we get other solutions by permuting a, b, c). $3\cdot 512 < 2336 < 4096$, so a = 10 or 11. If a = 10, then since $512 + 512 < 2336 - 1024 = 1312$, we must have b = 10. But that leaves 288, which is not a power of 2. Hence a = 11. So $2^b + 2^c = 288$. But $128 + 128 < 288 < 512$, so b = 8. Then c = 5.

B2.
n is a positive integer. x and y are reals such that $0 \leq x \leq 1$ and $x^{n+1} \leq y \leq 1$. Show that the absolute value of $(y - x)(y - x^2)(y - x^3) \dots (y - x^n)(1 + x)(1 + x^2) \dots (1 + x^n)$ is at most $(y + x)(y + x^2) \dots (y + x^n)(1 - x)(1 - x^2) \dots (1 - x^n)$.

B3.
ABCDA'B'C'D' is a cube (ABCD and A'B'C'D' are faces and AA', BB', CC', DD' are edges). L is the line joining the midpoints of BB' and DD'. Show that there is no line which meets L and the lines AA', BC and C'D'.

21st Vietnam 1983 problems

A1.
For which positive integers m, n with n > 1 does 2^n - 1 divides 2^m + 1?

Answer
(m,n) = (k,1) or (2k+1,2)

Solution
Suppose n > 2. The residues 1, 2, 4, 8, ... , 2^{n-1} are obviously distinct mod 2^n-1. But 2^n = 1 mod 2^n-1, so all power of 2 must equal one of these residues mod 2^n-1. Hence for any m we have 2^m+1 = one of 2, 3, 5, 9, ... , 2^{n-1}+1 mod 2^n-1. Since 2^{n-1}+1 < 2^n-1, these are all non-zero and so there are no solutions.

If n = 1, then obviously any value of m works. If n = 2, then we need 2^m+1 = $(3-1)^m$+1 to be a multiple of 3 and hence m odd.

A2.
(1) Show that (sin x + cos x) √2 ≥ 2 $\sin(2x)^{1/4}$ for all 0 ≤ x ≤ π/2.
(2) Find all x such that 0 < x < π and 1 + 2 cot(2x)/cot x ≥ tan(2x)/tan x.

A3.
P is a variable point inside the triangle ABC. D, E, F are the feet of the perpendiculars from P to the sides of the triangles. FInd the locus of P such that the area of DEF is constant.

B1.
For which n can we find n different odd positive integers such that the sum of their reciprocals is 1?

B2.
Let s_n = 1/((2n-1)2n) + 2/((2n-3)(2n-1)) + 3/((2n-5)(2n-2)) + 4/((2n-7)(2n-3) + ... + n/(1(n+1)) and t_n = 1/1 + 1/2 + 1/3 + ... + 1/n. Which is larger?

B3.
ABCD is a tetrahedron with AB = CD. A variable plane intersects the tetrahedron in a quadrilateral. Find the positions of the plane which minimise the perimeter of the quadrilateral. Find the locus of the centroid for those quadrilaterals with minimum perimeter.

22nd Vietnam 1984 problems

A1.
(1) Find a polynomial with integral coefficients which has the real number $2^{1/2} + 3^{1/3}$ as a root and the smallest possible degree.
(2) Find all real solutions to $1 + \sqrt{(1 + x^2)}\,(\sqrt{(1 + x)^3} - \sqrt{(1 - x)^3}\,) = 2 + \sqrt{(1 - x^2)}$.

A2.
The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 3a_{n+1} - a_n$. Find $\cot^{-1}a_1 + \cot^{-1}a_2 + \cot^{-1}a_3 + \ldots$.

A3.
A cube side 2a has ABCD as one face. S is the other vertex (apart from B and D) adjacent to A. M, N are variable points on the lines BC, CD respectively.
(1) Find the positions of M and N such that the planes SMA and SMN are perpendicular, $BM + DN \geq 3a/2$, and $BM \cdot DN$ has the smallest value possible.
(2) Find the positions of M and N such that angle NAM = 45 deg, and the volume of SAMN is (a) a maximum, (b) a minimum, and find the maximum and minimum.
(3) Q is a variable point (in space) such that $\angle AQB = \angle AQD = 90^o$. p is the plane ABS. Q' is the intersection of DQ and p. Find the locus K of Q'. Let CQ meet K again at R. Let R' be the intersection of DR and p. Show that $\sin^2 Q'DB + \sin^2 R'DB$ is constant.

B1.
(1) m, n are integers not both zero. Find the minimum value of $| 5m^2 + 11mn - 5n^2 |$.
(2) Find all positive reals x such that $9x/10 = [x]/(x - [x])$.

B2.
a, b are unequal reals. Find all polynomials p(x) which satisfy $x\, p(x - a) = (x - b)\, p(x)$ for all x.

B3.
(1) ABCD is a tetrahedron. $\angle CAD = z$, $\angle BAC = y$, $\angle BAD = x$, the angle between the planes ACB and ACD is X, the angle between the planes ABC and ABD is Z, the angle between the planes ADB and ADC is Y. Show that $\sin x/\sin X = \sin y/\sin Y = \sin z/\sin Z$ and that $x + y = 180^o$ iff $X + Y = 180^o$.

(2) ABCD is a tetrahedron with $\angle BAC = \angle CAD = \angle DAB = 90^o$. Points A and B are fixed. C and D are variable. Show that $\angle CBD + \angle ABD + \angle ABC$ is constant. Find the locus of the center of the insphere of ABCD.

23rd Vietnam 1985 problems

A1.
Find all integer solutions to $m^3 - n^3 = 2mn + 8$.

Answer
(m,n) = (2,0), (0,-2)

Solution
Put m = n+k. Then $3n^2k+3nk^2+k^3 = 2n^2+2nk+8$, so $(3k-2)n^2+(3k^2-2k)n+k^3-8 = 0$. For real solutions we require $(3k^2-2k)^2 \ge 4(3k-2)(k^3-8)$ or $(3k-2)(32-2k^2-k^3) \ge 0$. The first bracket is +ve for $k \ge 1$, -ve for $k \le 0$, the second is +ve for $k \le 2$, -ve for $k \ge 3$. Hence k = 1 or 2.

If k = 1, then $n^2+n-7 = 0$, which has no integer solutions. If k = 2, then $4n^2+8n = 0$, so n = 0 or -2.

A2.
Find all real-valued functions f(n) on the integers such that f(1) = 5/2, f(0) is not 0, and f(m) f(n) = f(m+n) + f(m-n) for all m, n.

Answer
$f(n) = 2^n + 1/2^n$

Solution
Putting m = n = 0, we get $f(0)^2 = 2f(0)$, so f(0) = 2. Putting n = 1, we get f(m+1) = 5/2 f(m) + f(m-1). That is a standard linear recurrence relation. The associated quadratic has roots 2, 1/2, so the general solution is $f(n) = A\ 2^n + B/2^n$. f(0) = 1 gives A + B = 2, f(1) = 5/2 gives 2A + B/2 = 5/2, so A = B = 1. It is now easy to check that this solutions satisfies the conditions.

A3.
A parallelepiped has side lengths a, b, c. Its center is O. The plane p passes through O and is perpendicular to one of the diagonals. Find the area of its intersection with the parallelepiped.

B1.
a, b, m are positive integers. Show that there is a positive integer n such that $(a^n - 1)b$ is divisible by m iff the greatest common divisor of ab and m is also the greatest common divisor of b and m.

B2.
Find all real values a such that the roots of $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16$ are all real and form an arithmetic progression.

B3.
ABCD is a tetrahedron. The base BCD has area S. The altitude from B is at least (AC + AD)/2, the altitude from C is at least (AD + AB)/2, and the altitude from D is at least (AB + AC)/2. Find the volume of the tetrahedron.

24th Vietnam 1986 problems

A1.
$a_1, a_2, \dots , a_n$ are real numbers such that $1/2 \le a_i \le 5$ for each i. The real numbers $x_1, x_2, \dots , x_n$ satisfy $4x_i^2 - 4a_ix_i + (a_i - 1)^2 = 0$. Let $S = (x_1 + x_2 + \dots + x_n)/n$, $S_2 = (x_1^2 + x_2^2 + \dots + x_n^2)/n$. Show that $\sqrt{S_2} \le S + 1$.

A2.
P is a pyramid whose base is a regular 1986-gon, and whose sloping sides are all equal. Its inradius is r and its circumradius is R. Show that $R/r \ge 1 + 1/\cos(\pi/1986)$. Find the total area of the pyramid's faces when equality occurs.

A3.
The polynomial p(x) has degree n and $p(1) = 2$, $p(2) = 4$, $p(3) = 8, \dots , p(n+1) = 2^{n+1}$. Find $p(n+2)$.

B1.
ABCD is a square. ABM is an equilateral triangle in the plane perpendicular to ABCD. E is the midpoint of AB. O is the midpoint of CM. The variable point X on the side AB is a distance x from B. P is the foot of the perpendicular from M to the line CX. Find the locus of P. Find the maximum and minimum values of XO.

B2.
Find all $n > 1$ such that $(x_1^2 + x_2^2 + \dots + x_n^2) \ge x_n(x_1 + x_2 + \dots + x_{n-1})$ for all real x_i.

B3.
A sequence of positive integers is constructed as follows. The first term is 1. Then we take the next two even numbers: 2, 4. Then we take the next three odd numbers: 5, 7, 9. Then we take the next four even numbers: 10, 12, 14, 16. And so on. Find the nth term of the sequence.

25th Vietnam 1987 problems

A1.
Let $x_n = (n+1)\pi/3974$. Find the sum of all $\cos(\pm x_1 \pm x_2 \pm \dots \pm x_{1987})$.

A2.
The sequences $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ are defined as follows. $a_0 = 365$, $a_{n+1} = a_n(a_n^{1986} + 1) + 1622$, $b_0 = 16$, $b_{n+1} = b_n(b_n^3 + 1) - 1952$. Show that there is no number in both sequences.

A3.
There are $n > 2$ lines in the plane, no two parallel. The lines are not all concurrent. Show that there is a point on just two lines.

B1.
$x_1, x_2, \dots , x_n$ are positive reals with sum X and $n > 1$. $h \le k$ are two positive integers. $H = 2^h$ and $K = 2^k$. Show that $x_1^K/(X - x_1)^{H-1} + x_2^K/(X - x_2)^{H-1} + x_3^K/(X - x_3)^{H-1} + \dots + x_n^K/(X - x_n)^{H-1} \ge X^{K-H+1}/((n-1)^{2H-1}n^{K-H})$. When does equality hold?

B2.
The function f(x) is defined and differentiable on the non-negative reals. It satisfies $| f(x) | \le 5$, $f(x) f'(x) \ge \sin x$ for all x. Show that it tends to a limit as x tends to infinity.

B3.
Given 5 rays in space from the same point, show that we can always find two with an angle between them of at most 90^o.

26th Vietnam 1988 problems

A1.
994 cages each contain 2 chickens. Each day we rearrange the chickens so that the same pair of chickens are never together twice. What is the maximum number of days we can do this?

A2.
The real polynomial $p(x) = x^n - nx^{n-1} + (n^2-n)/2\ x^{n-2} + a_{n-3}x^{n-3} + \dots + a_1x + a_0$ (where $n > 2$) has n real roots. Find the values of $a_0, a_1, \dots, a_{n-3}$.

A3.
The plane is dissected into equilateral triangles of side 1 by three sets of equally spaced parallel lines. Does there exist a circle such that just 1988 vertices lie inside it?

B1.
The sequence of reals $x_1, x_2, x_3, \dots$ satisfies $x_{n+2} \le (x_n + x_{n+1})/2$. Show that it converges to a finite limit.

B2.
ABC is an acute-angled triangle. Tan A, tan B, tan C are the roots of the equation $x^3 + px^2 + qx + p = 0$, where q is not 1. Show that $p \le \sqrt{27}$ and $q > 1$.

B3.
For a line L in space let R(L) be the operation of rotation through 180 deg about L. Show that three skew lines L, M, N have a common perpendicular iff R(L) R(M) R(N) has the form R(K) for some line K.

27th Vietnam 1989 problems

A1.
Show that the absolute value of $\sin(kx)/N + \sin(kx + x)/(N+1) + \sin(kx + 2x)/(N+2) + \dots + \sin(kx + nx)/(N+n)$ does not exceed the smaller of $(n+1)|x|$ and $1/(N \sin(x/2))$, where N is a positive integer and k is real and satisfies $0 \le k \le N$.

A2.
Let $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ be the Fibonacci sequence. Show that there are infinitely many terms of the sequence such that $1985a_n^2 + 1956a_n + 1960$ is divisible by 1989. Does there exist a term such that $1985a_n^2 + 1956a_n + 1960 + 2$ is divisible by 1989?

A3.
ABCD is a square side 2. The segment AB is moved continuously until it coincides with CD (note that A is brought into coincidence with the *opposite* corner). Show that this can be done in such a way that the region passed over by AB during the motion has area $< 5\pi/6$.

B1.
Do there exist integers m, n not both divisible by 5 such that $m^2 + 19n^2 = 198\cdot10^{1989}$?

B2.
Define the sequence of polynomials $p_0(x), p_1(x), p_2(x), \dots$ by $p_0(x) = 0$, $p_{n+1}(x) = p_n(x) + (x - p_n(x)^2)/2$. Show that for any $0 \le x \le 1$, $0 < \sqrt{x} - p_n(x) \le 2/(n+1)$.

B3.
ABCDA'B'C'D' is a parallelepiped (with edges AB, BC, CD, DA, AA', BB', CC', DD', A'B', B'C', C'D', D'A'). Show that if a line intersects three of the lines AB', BC', CA', AD', then it also intersects the fourth.

28th Vietnam 1990 problems

A1.
$-1 < a < 1$. The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = a$, $x_{n+1} = (\sqrt{3 - 3x_n^2} - x_n)/2$. Find necessary and sufficient conditions on a for all members of the sequence to be positive. Show that the sequence is always periodic.

Answer
$0 < a < (\sqrt{3})/2$ period 2

Solution
Put $x_1 = \sin k$. Then $x_2 = (\sqrt{3})/2 \cos k - (1/2) \sin k = \sin 60^\circ \cos k - \cos 60^\circ \sin k = \sin(60^\circ-k)$. Hence $x_3 = \sin(60^\circ - 60^\circ + k) = x_1$.
For $\sin k$ and $\sin(60^\circ-k)$ to be positive we need $0 < k < 60^\circ$ and hence $0 < a < (\sqrt{3})/2$.

A2.
n-1 or more numbers are removed from $\{1, 2, \ldots , 2n-1\}$ so that if a is removed, so is 2a and if a and b are removed, so is a + b. What is the largest possible sum for the remaining numbers?

A3.
ABCD is a tetrahedron with volume V. We wish to make three plane cuts to give a parallelepiped three of whose faces and all of whose vertices belong to the surface of the tetrahedron. Find the intersection of the three planes if the volume of the parallelepiped is 11V/50. Can it be done so that the volume of the parallelepiped is 9V/40?

B1.
ABC is a triangle. P is a variable point. The feet of the perpendiculars from P to the lines BC, CA, AB are A', B', C' respectively. Find the locus of P such that PA PA' = PB PB' = PC PC'.

B2.
The polynomial $p(x)$ with degree at least 1 satisfies $p(x) p(2x^2) = p(3x^3 + x)$. Show that it does not have any real roots.

Solution
Suppose it has a positive root α, then 3α3+α is another root, which is larger than α. Proceeding, we get infinitely many roots. Contradiction. So there are no positive roots. Similarly, there are no negative roots. So the only possibility is 0. Suppose bx^m is the lowest power of x in $p(x)$. Then the lowest power of x in $p(x)p(2x^2)$ is x^{3m}, but the lowest power of x in $p(3x^3 + x)$ is x^m, so $m = 0$ and hence $p(0) = b \neq 0$, so 0 is not a root.

B3.
Some children are sitting in a circle. Each has an even number of tokens (possibly zero). A child gives half his tokens to the child on his right. Then the child on his right does the same and so on. If a child about to give tokens has an odd number, then the teacher gives him an extra token. Show that after several steps, all the children will have the same number of tokens, except one who has twice the number.

29th Vietnam 1991 problems

A1.
Find all real-valued functions f(x) on the reals such that $f(xy)/2 + f(xz)/2 - f(x) f(yz) \geq 1/4$ for all x, y, z.

Answer
$f(x) = 1/2$ for all x

Solution
Put $x = y = z = 0$, then $(1/2 - f(0))^2 \leq 0$, so $f(0) = 1/2$. Put $z = 0$, then $f(xy) \geq f(x)$ for all x,y. Taking $x = 1$ we get $f(x) \geq f(1)$ for all x. Taking $y = 1/x$ we get $f(1) \geq f(x)$ for all x except possibly $x = 0$, so $f(x) = f(1)$ for all x except possibly $x = 0$. But putting $x = y = z = 1$ we get $(1/2 - f(1))^2 \leq 0$, so $f(1) = 1/2$. Hence $f(x) = 1/2$ for all x.

A2.
For each positive integer n and odd $k > 1$, find the largest number N such that 2^N divides $k^n - 1$.

A3.
The lines L, M, N in space are mutually perpendicular. A variable sphere passes through three fixed points A on L, B on M, C on N and meets the lines again at A', B', C'. Find the locus of the midpoint of the line joining the centroids of ABC and A'B'C'.

B1.
1991 students sit in a circle. Starting from student A and counting clockwise round the remaining students, every second and third student is asked to leave the circle until only one remains. (So if the students clockwise from A are A, B, C, D, E, F, ... , then B, C, E, F are the first students to leave.) Where was the surviving student originally sitting relative to A?

B2.
The triangle ABC has centroid G. The lines GA, GB, GC meet the circumcircle again at D, E, F. Show that $3/R \leq 1/GD + 1/GE + 1/GF \leq \sqrt{3} (1/AB + 1/BC + 1/CA)$, where R is the circumradius.

B3.
Show that $x^2y/z + y^2z/x + z^2x/y \geq x^2 + y^2 + z^2$ for any non-negative reals x, y, z.
[This is false, (1,2,3), (1,1,1), (1,2,8) give >, =, < . Does anyone know the correct question?]

30th Vietnam 1992 problems

A1.
ABCD is a tetrahedron. The three face angles at A sum to 180^o, and the three face angles at B sum to 180^o. Two of the face angles at C, $\angle ACD$ and $\angle BCD$, sum to 180^o. Find the sum of the areas of the four faces in terms of AC + CB = k and $\angle ACB = x$.

A2.
For any positive integer n, let f(n) be the number of positive divisors of n which equal ±1 mod 10, and let g(n) be the number of positive divisors of n which equal ±3 mod 10. Show that $f(n) \geq g(n)$.

A3.
Given a > 0, b > 0, c > 0, define the sequences a, b_n, c_n by $a_0 = a$, $b_0 = b$, $c_0 = c$, $a_{n+1} = a_n + 2/(b_n + c_n)$, $b_{n+1} = 2/(c_n + a_n)$, $c_{n+1} = c_n + 2/(a_n + b_n)$. Show that a_n tends to infinity.

B1.
Label the squares of a 1991 x 1992 rectangle (m, n) with $1 \leq m \leq 1991$ and $1 \leq n \leq 1992$. We wish to color all the squares red. The first move is to color red the squares (m, n), (m+1, n+1), (m+2, n+1) for some m < 1990, n < 1992. Subsequent moves are to color any three (uncolored) squares in the same row, or to color any three (uncolored) squares in the same column. Can we color all the squares in this way?

B2.
ABCD is a rectangle with center O and angle AOB $\leq 45^o$. Rotate the rectangle about O through an angle $0 < x < 360^o$. Find x such that the intersection of the old and new rectangles has the smallest possible area.

B3.
Let p(x) be a polynomial with constant term 1 and every coefficient 0 or 1. Show that p(x) does not have any real roots $> (1 - \sqrt{5})/2$.

31st Vietnam 1993 problems

A1.
$f : [\sqrt{1995}, \infty) \rightarrow R$ is defined by $f(x) = x(1993 + \sqrt{(1995 - x^2)}\ )$. Find its maximum and minimum values.

A2.
ABCD is a quadrilateral such that AB is not parallel to CD, and BC is not parallel to AD. Variable points P, Q, R, S are taken on AB, BC, CD, DA respectively so that PQRS is a parallelogram. Find the locus of its center.

A3.
Find a function f(n) on the positive integers with positive integer values such that $f(\ f(n)\ ) = 1993\ n^{1945}$ for all n.

B1.
The tetrahedron ABCD has its vertices on the fixed sphere S. Find all configurations which minimise $AB^2 + AC^2 + AD^2 - BC^2 - BD^2 - CD^2$.

B2.
1993 points are arranged in a circle. At time 0 each point is arbitrarily labeled +1 or -1. At times n = 1, 2, 3, ... the vertices are relabeled. At time n a vertex is given the label +1 if its two neighbours had the same label at time n-1, and it is given the label -1 if its two neighbours had different labels at time n-1. Show that for some time n > 1 the labeling will be the same as at time 1.

B3.
Define the sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ by $a_0 = 2$, $b_0 = 1$, $a_{n+1} = 2a_nb_n/(a_n + b_n)$, $b_{n+1} = \sqrt{(a_{n+1}b_n)}$. Show that the two sequences converge to the same limit, and find the limit.

32nd Vietnam 1994 problems

A1.
Find all real solutions to:
$x^3 + 3x - 3 + \ln(x^2 - x + 1) = y$
$y^3 + 3y - 3 + \ln(y^2 - y + 1) = z$
$z^3 + 3z - 3 + \ln(z^2 - z + 1) = x$.

A2.
ABC is a triangle. Reflect each vertex in the opposite side to get the triangle A'B'C'. Find a necessary and sufficient condition on ABC for A'B'C' to be equilateral.

A3.
Define the sequence $x_0, x_1, x_2, \ldots$ by $x_0 = a$, where $0 < a < 1$, $x_{n+1} = 4/\pi^2 (\cos^{-1}x_n + \pi/2) \sin^{-1}x_n$. Show that the sequence converges and find its limit.

B1.
There are n+1 containers arranged in a circle. One container has n stones, the others are empty. A *move* is to choose two containers A and B, take a stone from A and put it in one of the containers adjacent to B, and to take a stone from B and put it in one of the containers adjacent to A. We can take A = B. For which n is it possible by series of moves to end up with one stone in each container except that which originally held n stones.

B2.
S is a sphere center O. G and G' are two perpendicular great circles on S. Take A, B, C on G and D on G' such that the altitudes of the tetrahedron ABCD intersect at a point. Find the locus of the intersection.

B3.
Do there exist polynomials p(x), q(x), r(x) whose coefficients are positive integers such that $p(x) = (x^2 - 3x + 3) q(x)$ and $q(x) = (x^2/20 - x/15 + 1/12) r(x)$?

33rd Vietnam 1995 problems

A1.
Find all real solutions to $x^3 - 3x^2 - 8x + 40 = 8(4x + 4)^{1/4} = 0$.

A2.
The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 1$, $a_1 = 3$, $a_{n+2} = a_{n+1} + 9a_n$ for n even, $9a_{n+1} + 5a_n$ for n odd. Show that $a_{1995}^2 + a_{1996}^2 + a_{1997}^2 + a_{1998}^2 + a_{1999}^2 + a_{2000}^2$ is divisible by 20, and that no a_{2n+1} is a square.

A3.
ABC is a triangle with altitudes AD, BE, CF. A', B', C' are points on AD, BE, CF such that AA'/AD = BB'/BE = CC'/CF = k. Find all k such that A'B'C' is similar to ABC for all triangles ABC.

B1.
ABCD is a tetrahedron. A' is the circumcenter of the face opposite A. B', C', D' are defined similarly. p_A is the plane through A perpendicular to C'D', p_B is the plane through B perpendicular to D'A', p_C is the plane through C perpendicular to A'B', and p_D is the plane through D perpendicular to B'C'. Show that the four planes have a common point. If this point is the circumcenter of ABCD, must ABCD be regular?

B2.
Find all real polynomials p(x) such that p(x) = a has more than 1995 real roots, all greater than 1995, for any a > 1995. Multiple roots are counted according to their multiplicities.

B3.
How many ways are there of coloring the vertices of a regular 2n-gon with n colors, such that each vertex is given one color, and every color is used for two non-adjacent vertices? Colorings are regarded as the same if one is obtained from the other by rotation.

34th Vietnam 1996 problems

A1.
Find all real x, y such that $\sqrt{(3x)}\ (1 + 1/(x + y)) = 2$ and $\sqrt{(7y)}\ (1 - 1/(x + y)) = 4\sqrt{2}$.

A2.
SABC is a tetrahedron. DAB, EBC, FCA are triangles in the plane of ABC congruent to SAB, SBC, SCA respectively. O is the circumcenter of DEF. Let K be the exsphere of SABC opposite O (which touches the planes SAB, SBC, SCA, ABC, lies on the opposite side of ABC to S, but on the same side of SAB as C, the same side of SBC as A, and the same side of SCA as B). Show that K touches the plane ABC at O.

A3.
Let n be a positive integer and k a positive integer not greater than n. Find the number of ordered k-tuples $(a_1, a_2, \dots , a_n)$ such that: (1) all a_i are different, but all belong to $\{1, 2, \dots , n\}$; (2) $a_r > a_s$ for some $r < s$; (3) a_r has the opposite parity to r for some r.

B1.
Find all functions f(n) on the positive integers with positive integer values, such that $f(n) + f(n+1) = f(n+2) f(n+3) - 1996$ for all n.

B2.
The triangle ABC has BC = 1 and angle A = x. Let f(x) be the shortest possible distance between its incenter and its centroid. Find f(x). What is the largest value of f(x) for $60^o < x < 180^o$?

B3.
Let w, x, y, z be non-negative reals such that $2(wx + wy + wz + xy + xz + yz) + wxy + xyz + yzw + zwx = 16$. Show that $3(w + x + y + z) \ge 2(wx + wy + wz + xy + xz + yz)$.

35th Vietnam 1997 problems

A1.
S is a fixed circle with radius R. P is a fixed point inside the circle with OP = d < R. ABCD is a variable quadrilateral, such that A, B, C, D lie on S, AC intersects BD at P, and AC is perpendicular to BD. Find the maximum and minimum values of the perimeter of ABCD in terms of R and d.

A2.
$n > 1$ is any integer not divisible by 1997. Put $a_m = m + mn/1997$ for $m = 1, 2, \dots , 1996$ and $b_m = m + 1997m/n$ for $m = 1, 2, \dots , n-1$. Arrange all the terms a_i, b_j in a single sequence in ascending order. Show that the difference between any two consecutive terms is less than 2.

A3.
How many functions f(n) defined on the positive integers with positive integer values satisfy f(1) = 1 and $f(n) f(n+2) = f(n+1)^2 + 1997$ for all n?

B1.
Let $k = 3^{1/3}$. Find a polynomial p(x) with rational coefficients and degree as small as possible such that $p(k + k^2) = 3 + k$. Does there exist a polynomial q(x) with integer coefficients such that $q(k + k^2) = 3 + k$?

B2.
Show that for any positive integer n, we can find a positive integer f(n) such that $19^{f(n)} - 97$ is divisible by 2^n.

B3.
Given 75 points in a unit cube, no three collinear, show that we can choose three points which form a triangle with area at most 7/72.

36th Vietnam 1998 problems

A1.
Define the sequence $x_1, x_2, x_3, \ldots$ by $x_1 = a \geq 1$, $x_{n+1} = 1 + \ln(x_n(x_n^2+3)/(1 + 3x_n^2))$. Show that the sequence converges and find the limit.

Solution
$(x-1)^3 \geq 0$, so $x(x^2+3)/(1+3x^2) \geq 1$. So the ln term is well-defined and non-negative and $1 + \ln(x(x^2+3)/(1+3x^2)) \geq 1$. So by a trivial induction all $x_n \geq 1$.

Also $x^2 \geq 1$ implies $(x^2+3)/(1+3x^2) \leq 1$, so $x(x^2+3)/(1+3x^2) \leq x$ and hence $1 + \ln(x(x^2+3)/(1+3x^2)) \leq 1 + \ln x \leq x$ (*). So the sequence is monotonically decreasing and bounded below by 1. So it must converge. Suppose the limit is L. Then $L = 1 + \ln(L(L^2+3)/(1+Lx^2))$. But we only have equality in (*) at 1. Hence L = 1.

A2.
Let O be the circumcenter of the tetrahedron ABCD. Let A', B', C', D' be points on the circumsphere such that AA', BB', CC' and DD' are diameters. Let A" be the centroid of the triangle BCD. Define B", C", D" similarly. Show that the lines A'A", B'B", C'C", D'D" are concurrent. Suppose they meet at X. Show that the line through X and the midpoint of AB is perpendicular to CD.

A3.
The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0= 20$, $a_1 = 100$, $a_{n+2} = 4a_{n+1} + 5a_n + 20$. Find the smallest m such that $a_m - a_0$, $a_{m+1} - a_1$, $a_{m+2} - a_2, \ldots$ are all divisible by 1998.

B1.
Does there exist an infinite real sequence $x_1, x_2, x_3, \ldots$ such that $|x_n| \leq 0.666$, and $|x_m - x_n| \geq 1/(n^2 + n + m^2 + m)$ for all distinct m, n?

B2.
What is the minimum value of $\sqrt{(x+1)^2 + (y-1)^2} + \sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+2)^2}$?

B3.
Find all positive integers n for which there is a polynomial p(x) with real coefficients such that $p(x^{1998} - x^{-1998}) = (x^n - x^{-n})$ for all x.

37th Vietnam 1999 problems

A1.
Find all real solutions to $(1 + 4^{2x-y})(5^{y-2x+1}) = 2^{2x-y+1} + 1$, $y^3 + 4x + \ln(y^2 + 2x) + 1 = 0$.

A2.
ABC is a triangle. A' is the midpoint of the arc BC of the circumcircle not containing A. B' and C' are defined similarly. The segments A'B', B'C', C'A' intersect the sides of the triangle in six points, two on each side. These points divide each side of the triangle into three parts. Show that the three middle parts are equal iff ABC is equilateral.

A3.
The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 3a_{n+1} - a_n$. The sequence $b_1, b_2, b_3, \ldots$ is defined by $b_1 = 1$, $b_2 = 4$, $b_{n+2} = 3b_{n+1} - b_n$. Show that the positive integers a, b satisfy $5a^2 - b^2 = 4$ iff $a = a_n$, $b = b_n$ for some n.

B1.
Find the maximum value of $2/(x^2 + 1) - 2/(y^2 + 1) + 3/(z^2 + 1)$ for positive reals x, y, z which satisfy $xyz + x + z = y$.

B2.
OA, OB, OC, OD are 4 rays in space such that the angle between any two is the same. Show that for a variable ray OX, the sum of the cosines of the angles XOA, XOB, XOC, XOD is constant and the sum of the squares of the cosines is also constant.

B3.
Find all functions f(n) defined on the non-negative integers with values in the set {0, 1, 2, ... , 2000} such that: (1) $f(n) = n$ for $0 \le n \le 2000$; and (2) $f(f(m) + f(n)) = f(m + n)$ for all m, n.

38th Vietnam 2000 problems

A1.
Define a sequence of positive reals $x_0, x_1, x_2, \ldots$ by $x_0 = b$, $x_{n+1} = \sqrt{c - \sqrt{c + x_n}}$. Find all values of c such that for all b in the interval (0, c), such a sequence exists and converges to a finite limit as n tends to infinity.

A2.
C and C' are circles centers O and O' respectively. X and X' are points on C and C' respectively such that the lines OX and O'X' intersect. M and M' are variable points on C and C' respectively, such that $\angle XOM = \angle X'O'M'$ (both measured clockwise). Find the locus of the midpoint of MM'. Let OM and O'M' meet at Q. Show that the circumcircle of QMM' passes through a fixed point.

A3.
Let $p(x) = x^3 + 153x^2 - 111x + 38$. Show that p(n) is divisible by 3^{2000} for at least nine positive integers n less than 3^{2000}. For how many such n is it divisible?

B1.
Given an angle α such that $0 < \alpha < \pi$, show that there is a unique real monic quadratic $x^2 + ax + b$ which is a factor of $p_n(x) = \sin\alpha\, x^n - \sin(n\alpha)\, x + \sin(n\alpha-\alpha)$ for all $n > 2$. Show that there is no linear polynomial $x + c$ which divides $p_n(x)$ for all $n > 2$.

B2.
Find all $n > 3$ such that we can find n points in space, no three collinear and no four on the same circle, such that the circles through any three points all have the same radius.

B3.
p(x) is a polynomial with real coefficients such that $p(x^2 - 1) = p(x)\, p(-x)$. What is the largest number of real roots that p(x) can have?

39th Vietnam 2001 problems

A1.
A circle center O meets a circle center O' at A and B. The line TT' touches the first circle at T and the second at T'. The perpendiculars from T and T' meet the line OO' at S and S'. The ray AS meets the first circle again at R, and the ray AS' meets the second circle again at R'. Show that R, B and R' are collinear.

A2.
Let $N = 6^n$, where n is a positive integer, and let $M = a^N + b^N$, where a and b are relatively prime integers greater than 1. M has at least two odd divisors greater than 1. Find the residue of M mod $6\ 12^n$.

A3.
For real a, b define the sequence $x_0, x_1, x_2, \ldots$ by $x_0 = a$, $x_{n+1} = x_n + b \sin x_n$. If b = 1, show that the sequence converges to a finite limit for all a. If b > 2, show that the sequence diverges for some a.

B1.
Find the maximum value of $1/\sqrt{x} + 2/\sqrt{y} + 3/\sqrt{z}$, where x, y, z are positive reals satisfying $1/\sqrt{2} \le z \le \min(x\sqrt{2}, y\sqrt{3})$, $x + z\sqrt{3} \ge \sqrt{6}$, $y\sqrt{3} + z\sqrt{10} \ge= 2\sqrt{5}$.

B2.
Find all real-valued continuous functions defined on the interval (-1, 1) such that $(1 - x^2) f(2x/(1 + x^2)) = (1 + x^2)^2 f(x)$ for all x.

B3.
$a_1, a_2, \ldots, a_{2n}$ is a permutation of 1, 2, ... , 2n such that $|a_i - a_{i+1}| \ne |a_j - a_{j+1}|$ for $i \ne j$. Show that $a_1 = a_{2n} + n$ iff $1 \le a_{2i} \le n$ for i = 1, 2, ... n.

40th Vietnam 2002 problems

A1. Solve the following equation: $\sqrt{(4 - 3\sqrt{(10 - 3x)})} = x - 2$.

Answer: $x = 3$.
Squaring twice we get $x^4 - 8x^3 + 16x^2 + 9x - 90 = 0$. Factorising, we get $(x + 2)(x - 3)(x^2 - 7x + 15)$, so the only real roots are $x = -2$ and 3. Checking, we find that 3 is indeed a solution of the original equation, but $x = -2$ is not because we get $\sqrt{(-8)}$ on the lhs.

A2.
ABC is an isosceles triangle with AB = AC. O is a variable point on the line BC such that the circle center O radius OA does not have the lines AB or AC as tangents. The lines AB, AC meet the circle again at M, N respectively. Find the locus of the orthocenter of the triangle AMN.

Solution:
Let A' be the reflection of A in BC. Then OA = OA', so the circle also passes through A'. $\angle$MAA' = $\angle$NAA', so A' is the midpoint of the arc MN. Hence OA' is perpendicular to MN. Let X be midpoint of MN, so X lies on OA'.

Now $\angle OA'M = \angle MA'N/2 = (180^\circ - \angle A)/2 = 90^\circ - \angle A/2$, so $\angle MOX = 2(90^\circ - \angle OA'M) = \angle A$. Hence OX/OA' = OM cos A/OA' = cos A, which is fixed. So the locus of X is line parallel to BC. Let G be centroid of AMN, then AG/AX = 2/3, so G also lies on a line parallel to BC. But H lies on ray OG with GH = 2OG (Euler line), so H also lies on a line parallel to BC.

Given any point H on the line take a line through A' parallel to AH. It meets the line BC at a point O, which is the required point to generate H. (Arguably, we do not generate the two points on the line corresponding to OA perpendicular to AB and AC, because then one of M, N coincides with A and AMN is degenerate.)

A3.
$m < 2001$ and $n < 2002$ are fixed positive integers. A set of distinct real numbers are arranged in an array with 2001 rows and 2002 columns. A number in the array is *bad* if it is smaller than at least m numbers in the same column and at least n numbers in the same row. What is the smallest possible number of bad numbers in the array?

B1.
If all the roots of the polynomial $x^3 + a x^2 + bx + c$ are real, show that $12ab + 27c \le 6a^3 + 10(a^2 - 2b)^{3/2}$. When does equality hold?

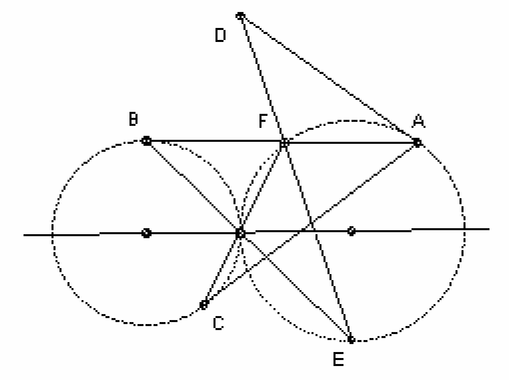
B2.
Find all positive integers n for which the equation $a + b + c + d = n\sqrt{(abcd)}$ has a solution in positive integers.

B3.
n is a positive integer. Show that the equation $1/(x - 1) + 1/(2^2x - 1) + ... + 1/(n^2x - 1) = 1/2$ has a unique solution $x_n > 1$. Show that as n tends to infinity, x_n tends to 4.

41st Vietnam 2003 problems

A1.
Let R be the reals and f: R → R a function such that f(cot x) = cos 2x + sin 2x for all $0 < x < \pi$. Define g(x) = f(x) f(1-x) for $-1 \le x \le 1$. Find the maximum and minimum values of g on the closed interval [-1, 1].

A2.
The circles C_1 and C_2 touch externally at M and the radius of C_2 is larger than that of C_1. A is any point on C_2 which does not lie on the line joining the centers of the circles. B and C are points on C_1 such that AB and AC are tangent to C_1. The lines BM, CM intersect C_2 again at E, F respectively. D is the intersection of the tangent at A and the line EF. Show that the locus of D as A varies is a straight line.



A3.
Let S_n be the number of permutations $(a_1, a_2, \dots, a_n)$ of $(1, 2, \dots, n)$ such that $1 \le |a_k - k| \le 2$ for all k. Show that $(7/4)\ S_{n-1} < S_n < 2\ S_{n-1}$ for $n > 6$.

B1.
Find the largest positive integer n such that the following equations have integer solutions in $x, y_1, y_2, \dots, y_n$:
$(x + 1)^2 + y_1^2 = (x + 2)^2 + y_2^2 = \dots = (x + n)^2 + y_n^2$.

B2.
Define $p(x) = 4x^3 - 2x^2 - 15x + 9$, $q(x) = 12x^3 + 6x^2 - 7x + 1$. Show that each polynomial has just three distinct real roots. Let A be the largest root of p(x) and B the largest root of q(x). Show that $A^2 + 3 B^2 = 4$.

B3.
Let R^+ be the set of positive reals and let F be the set of all functions $f : R^+ \to R^+$ such that $f(3x) \ge f(f(2x)) + x$ for all x. Find the largest A such that $f(x) \ge A x$ for all f in F and all x in R^+.