Problems of the 1st International Physics Olympiad[1] (Warsaw, 1967)

Waldemar Gorzkowski

Institute of Physics, Polish Academy of Sciences, Warsaw, Poland[2]

Abstract

The article contains the competition problems given at he 1st International Physics Olympiad (Warsaw, 1967) and their solutions. Additionally it contains comments of historical character.

Introduction

One of the most important points when preparing the students to the International Physics Olympiads is solving and analysis of the competition problems given in the past. Unfortunately, it is very difficult to find appropriate materials. The proceedings of the subsequent Olympiads are published starting from the XV IPhO in Sigtuna (Sweden, 1984). It is true that some of very old problems were published (not always in English) in different books or articles, but they are practically unavailable. Moreover, sometimes they are more or less substantially changed.

The original English versions of the problems of the 1st IPhO have not been conserved. The permanent Secretariat of the IPhOs was created in 1983. Until this year the Olympic materials were collected by different persons in their private archives. These archives as a rule were of amateur character and practically no one of them was complete. This article is based on the books by R. Kunfalvi [1], Tadeusz Pniewski [2] and Waldemar Gorzkowski [3]. Tadeusz Pniewski was one of the members of the Organizing Committee of the Polish Physics Olympiad when the 1st IPhO took place, while R. Kunfalvi was one of the members of the International Board at the 1st IPhO. For that it seems that credibility of these materials is very high. The differences between versions presented by R. Kunfalvi and T. Pniewski are rather very small (although the book by Pniewski is richer, especially with respect to the solution to the experimental problem).

As regards the competition problems given in Sigtuna (1984) or later, they are available, in principle, in appropriate proceedings. “In principle” as the proceedings usually were published in a small number of copies, not enough to satisfy present needs of people interested in our competition. It is true that every year the organizers provide the permanent Secretariat with a number of copies of the proceedings for free dissemination. But the needs are continually growing up and we have disseminated practically all what we had.

The competition problems were commonly available (at least for some time) just only from the XXVI IPhO in Canberra (Australia) as from that time the organizers started putting the problems on their home pages. The Olympic home page www.jyu.fi/ipho contains the problems starting from the XXVIII IPhO in Sudbury (Canada). Unfortunately, the problems given in Canberra (XXVI IPhO) and in Oslo (XXVII IPhO) are not present there.

The net result is such that finding the competition problems of the Olympiads organized prior to Sudbury is very difficult. It seems that the best way of improving the situation is publishing the competition problems of the older Olympiads in our journal. The

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[1] This is somewhat extended version of the article sent for publication in *Physics Competitions* in July 2003.

[2] e-mail: gorzk@ifpan.edu.pl

question arises, however, who should do it. According to the Statutes the problems are created by the local organizing committees. It is true that the texts are improved and accepted by the International Board, but always the organizers bear the main responsibility for the topics of the problems, their structure and quality. On the other hand, the glory resulting of high level problems goes to them. For the above it is absolutely clear to me that they should have an absolute priority with respect to any form of publication. So, the best way would be to publish the problems of the older Olympiads by representatives of the organizers from different countries.

Poland organized the IPhOs for thee times: I IPhO (1967), VII IPhO (1974) and XX IPhO (1989). So, I have decided to give a good example and present the competition problems of these Olympiads in three subsequent articles. At the same time I ask our Colleagues and Friends from other countries for doing the same with respect to the Olympiads organized in their countries prior to the XXVIII IPhO (Sudbury).

I IPhO (Warsaw 1967)

The problems were created by the Organizing Committee. At present we are not able to recover the names of the authors of the problems.

Theoretical problems

Problem 1

A small ball with mass M = 0.2 kg rests on a vertical column with height h = 5m. A bullet with mass m = 0.01 kg, moving with velocity v_0 = 500 m/s, passes horizontally through the center of the ball (Fig. 1). The ball reaches the ground at a distance s = 20 m. Where does the bullet reach the ground? What part of the kinetic energy of the bullet was converted into heat when the bullet passed trough the ball? Neglect resistance of the air. Assume that g = 10 m/s^2.

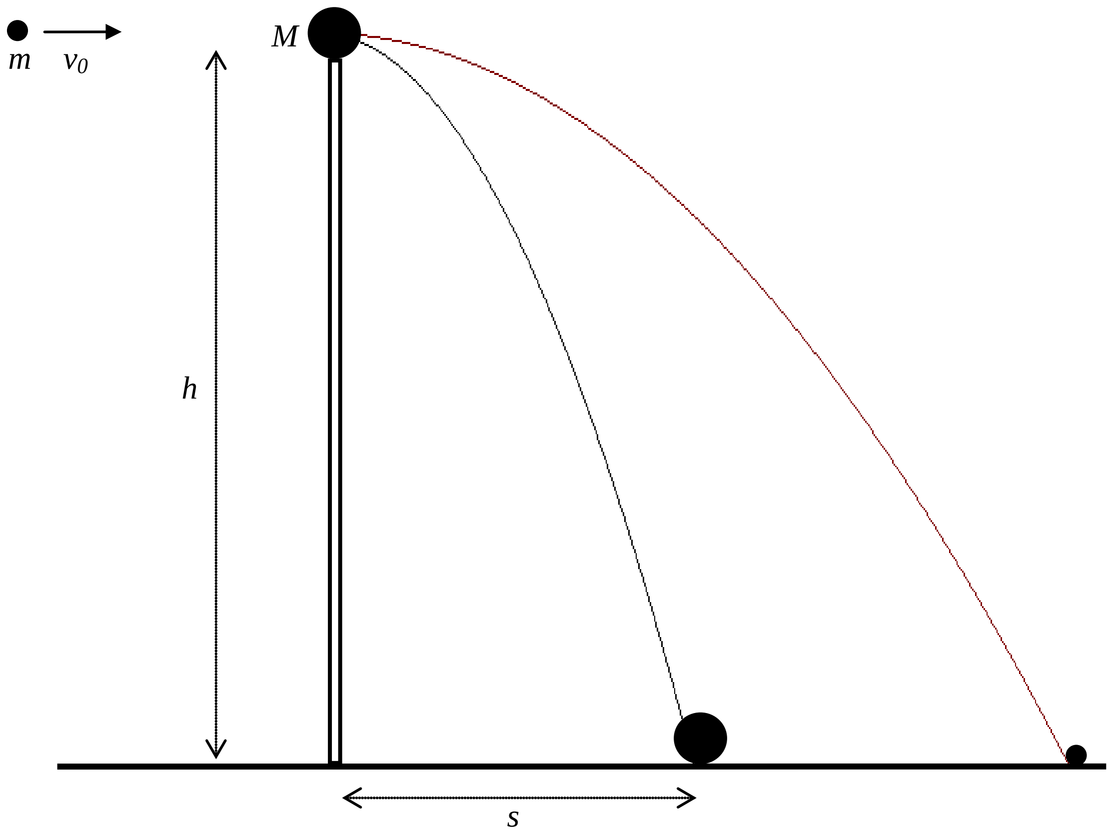

Fig. 1

Solution

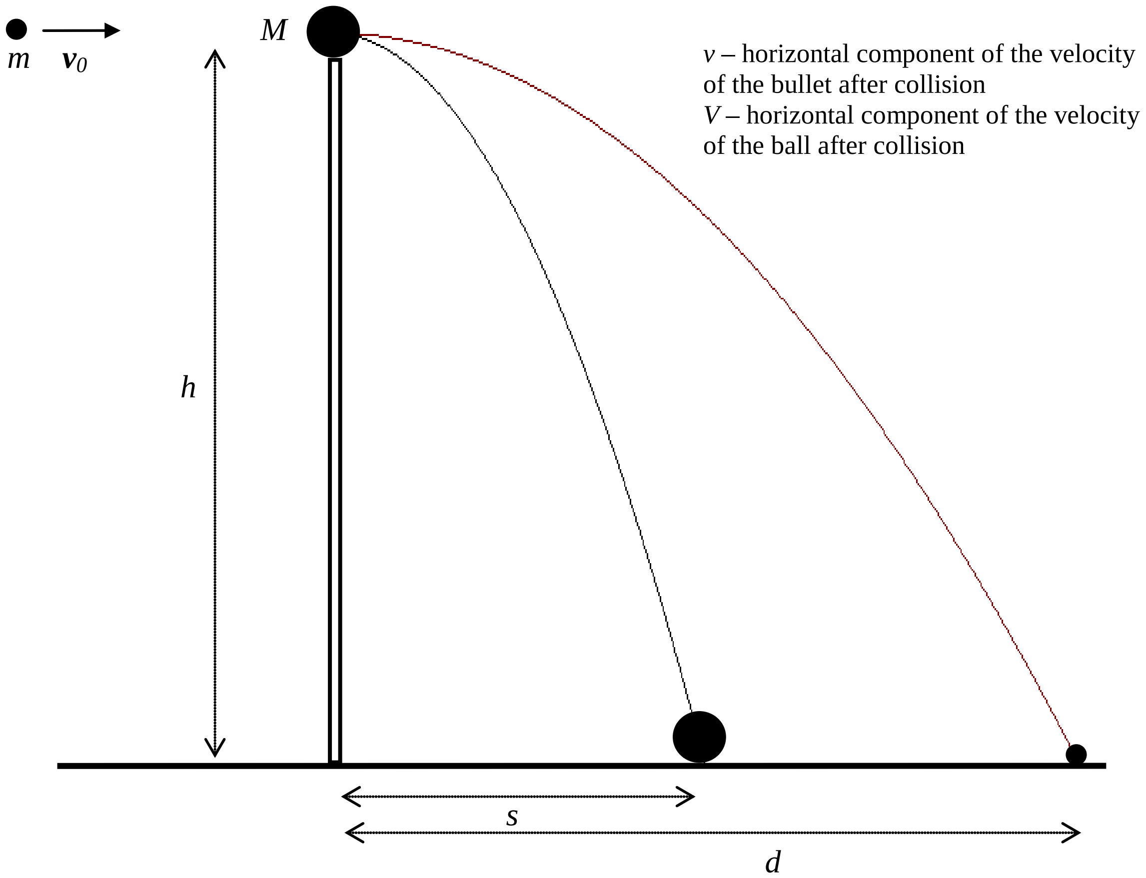

Fig. 2

We will use notation shown in Fig. 2.

As no horizontal force acts on the system ball + bullet, the horizontal component of momentum of this system before collision and after collision must be the same:

$$mv_0 = mv + MV.$$

So,

$$v = v_0 - \frac{M}{m}V.$$

From conditions described in the text of the problem it follows that

$$v > V.$$

After collision both the ball and the bullet continue a free motion in the gravitational field with initial horizontal velocities v and V, respectively. Motion of the ball and motion of the bullet are continued for the same time:

$$t = \sqrt{\frac{2h}{g}}.$$

It is time of free fall from height *h*.

The distances passed by the ball and bullet during time *t* are:

$$s = Vt \quad \text{and} \quad d = vt\,,$$

respectively. Thus

$$V = s\sqrt{\frac{g}{2h}}.$$

Therefore

$$v = v_0 - \frac{M}{m}s\sqrt{\frac{g}{2h}}\,.$$

Finally:

$$d = v_0\sqrt{\frac{2h}{g}} - \frac{M}{m}s\,.$$

Numerically:

$$d = 100 \text{ m.}$$

The total kinetic energy of the system was equal to the initial kinetic energy of the bullet:

$$E_0 = \frac{mv_0^2}{2}\,.$$

Immediately after the collision the total kinetic energy of the system is equal to the sum of the kinetic energy of the bullet and the ball:

$$E_m = \frac{mv^2}{2}\,, \qquad E_M = \frac{MV^2}{2}\,.$$

Their difference, converted into heat, was

$$\Delta E = E_0 - (E_m + E_M)\,.$$

It is the following part of the initial kinetic energy of the bullet:

$$p = \frac{\Delta E}{E_0} = 1 - \frac{E_m + E_M}{E_0}.$$

By using expressions for energies and velocities (quoted earlier) we get

$$p = \frac{M}{m}\frac{s^2}{v_0^2}\frac{g}{2h}\left(2\frac{v_0}{s}\sqrt{\frac{2h}{g}} - \frac{M+m}{m}\right).$$

Numerically:

$$p = 92{,}8\%.$$

Problem 2

Consider an infinite network consisting of resistors (resistance of each of them is r) shown in Fig. 3. Find the resultant resistance R_{AB} between points A and B.

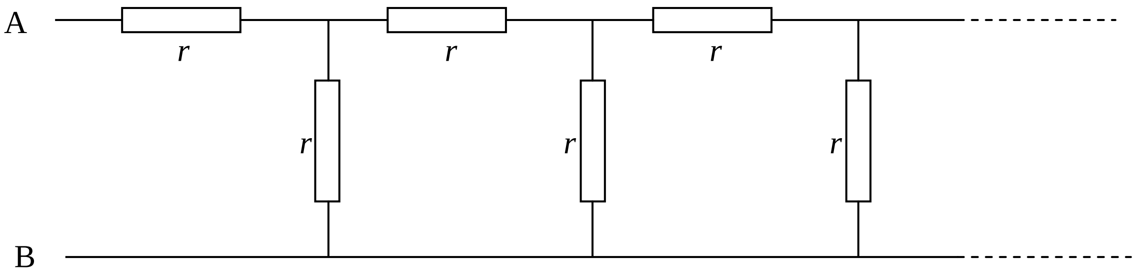

Fig. 3

Solution

It is easy to remark that after removing the left part of the network, shown in Fig. 4 with the dotted square, then we receive a network that is identical with the initial network (it is result of the fact that the network is infinite).

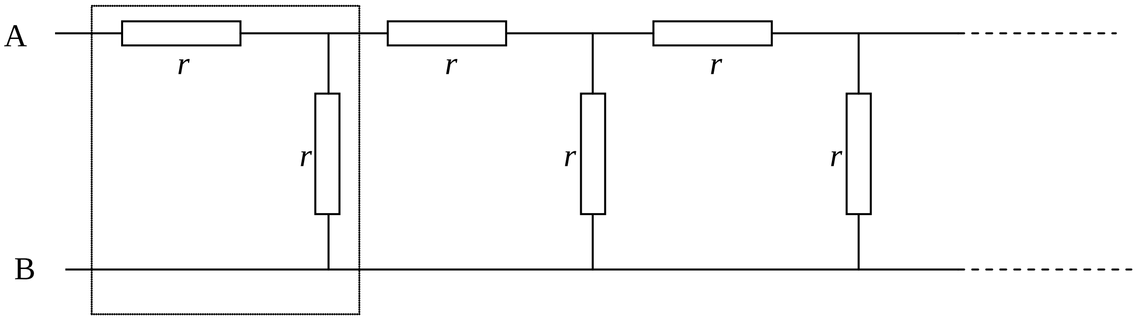

Fig. 4

Thus, we may use the equivalence shown graphically in Fig. 5.

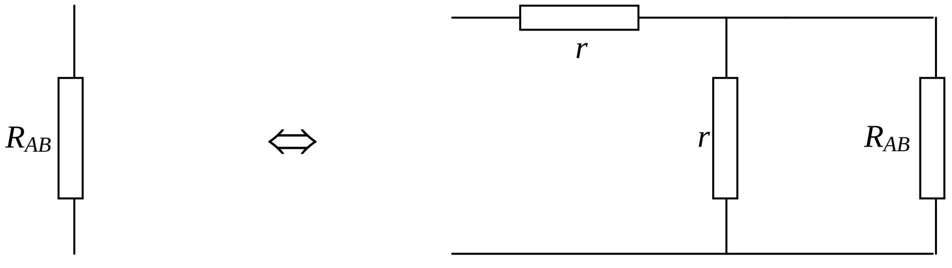

Fig. 5

Algebraically this equivalence can be written as

$$R_{AB} = r + \frac{1}{\frac{1}{r} + \frac{1}{R_{AB}}}.$$

Thus

$$R_{AB}^2 - rR_{AB} - r^2 = 0.$$

This equation has two solutions:

$$R_{AB} = \tfrac{1}{2}(1 \pm \sqrt{5})r.$$

The solution corresponding to "-" in the above formula is negative, while resistance must be positive. So, we reject it. Finally we receive

$$R_{AB} = \tfrac{1}{2}(1 + \sqrt{5})r.$$

Problem 3

Consider two identical homogeneous balls, A and B, with the same initial temperatures. One of them is at rest on a horizontal plane, while the second one hangs on a thread (Fig. 6). The same quantities of heat have been supplied to both balls. Are the final temperatures of the balls the same or not? Justify your answer. (All kinds of heat losses are negligible.)

Fig. 6

Solution

Fig. 7

As regards the text of the problem, the sentence "The same quantities of heat have been supplied to both balls." is not too clear. We will follow intuitive understanding of this sentence, i.e. we will assume that both systems (A – the hanging ball and B – the ball resting on the plane) received the same portion of energy from outside. One should realize, however, that it is not the only possible interpretation.

When the balls are warmed up, their mass centers are moving as the radii of the balls are changing. The mass center of the ball A goes down, while the mass center of the ball B goes up. It is shown in Fig. 7 (scale is not conserved).

Displacement of the mass center corresponds to a change of the potential energy of the ball in the gravitational field.

In case of the ball A the potential energy decreases. From the 1^{st} principle of thermodynamics it corresponds to additional heating of the ball.

In case of the ball B the potential energy increases. From the 1^{st} principle of thermodynamics it corresponds to some "losses of the heat provided" for performing a mechanical work necessary to rise the ball. The net result is that the final temperature of the ball B should be lower than the final temperature of the ball A.

The above effect is very small. For example, one may find (see later) that for balls made of lead, with radius 10 cm, and portion of heat equal to 50 kcal, the difference of the final temperatures of the balls is of order 10^{-5} K. For spatial and time fluctuations such small quantity practically cannot be measured.

Calculation of the difference of the final temperatures was not required from the participants. Nevertheless, we present it here as an element of discussion.

We may assume that the work against the atmospheric pressure can be neglected. It is obvious that this work is small. Moreover, it is almost the same for both balls. So, it should not affect the difference of the temperatures substantially. We will assume that such quantities as specific heat of lead and coefficient of thermal expansion of lead are constant (i.e. do not depend on temperature).

The heat used for changing the temperatures of balls may be written as

$$Q_i = mc\Delta t_i, \quad \text{where } i = A \text{ or } B,$$

Here: m denotes the mass of ball, c - the specific heat of lead and Δt_i - the change of the temperature of ball.

The changes of the potential energy of the balls are (neglecting signs):

$$\Delta E_i = mgr\alpha\Delta t_i, \quad \text{where } \ i = A \text{ or } B.$$

Here: g denotes the gravitational acceleration, r - initial radius of the ball, α - coefficient of thermal expansion of lead. We assume here that the thread does not change its length.

Taking into account conditions described in the text of the problem and the interpretation mentioned at the beginning of the solution, we may write:

$$Q = Q_A - A\Delta E_A, \quad \text{for the ball } A,$$
$$Q = Q_B + A\Delta E_B, \quad \text{for the ball } B.$$

A denotes the thermal equivalent of work: $A \approx 0.24\frac{\text{cal}}{\text{J}}$. In fact, A is only a conversion ratio between calories and joules. If you use a system of units in which calories are not present, you may omit A at all.

Thus

$$Q = (mc - Amgr\alpha)\Delta t_A, \quad \text{for the ball } A,$$
$$Q = (mc + Amgr\alpha)\Delta t_B, \quad \text{for the ball } B$$

and

$$\Delta t_A = \frac{Q}{mc - Amgr\alpha}, \qquad \Delta t_B = \frac{Q}{mc + Amgr\alpha}.$$

Finally we get

$$\Delta t = \Delta t_A - \Delta t_B = \frac{2Agr\alpha}{c^2 - (Agr\alpha)^2}\frac{Q}{m} \approx \frac{2AQgr\alpha}{mc^2}.$$

(We neglected the term with α^2 as the coefficient α is very small.)

Now we may put the numerical values: $Q = 50$ kcal, $A \approx 0.24$ cal/J, $g \approx 9.8$ m/s^2, $m \approx 47$ kg (mass of the lead ball with radius equal to 10 cm), $r = 0.1$ m, $c \approx 0.031$ cal/(g·K), $\alpha \approx 29 \cdot 10^{-6}$ K^{-1}. After calculations we get $\Delta t \approx 1.5 \cdot 10^{-5}$ K.

Problem 4

Comment: *The Organizing Committee prepared three theoretical problems. Unfortunately, at the time of the 1st Olympiad the Romanian students from the last class had the entrance examinations at the universities. For that Romania sent a team consisting of students from younger classes. They were not familiar with electricity. To give them a chance the Organizers (under agreement of the International Board) added the fourth problem presented here. The students (not only from Romania) were allowed to chose three problems. The maximum possible scores for the problems were: 1st problem – 10 points, 2nd problem – 10 points, 3rd problem – 10 points and 4th problem – 6 points. The fourth problem was solved by 8 students. Only four of them solved the problem for 6 points.*

A closed vessel with volume V_0 = 10 l contains dry air in the normal conditions (t_0 = 0°C, p_0 = 1 atm). In some moment 3 g of water were added to the vessel and the system was warmed up to t = 100°C. Find the pressure in the vessel. Discuss assumption you made to solve the problem.

Solution

The water added to the vessel evaporates. Assume that the whole portion of water evaporated. Then the density of water vapor in 100°C should be 0.300 g/l. It is less than the density of saturated vapor at 100°C equal to 0.597 g/l. (The students were allowed to use physical tables.) So, at 100°C the vessel contains air and unsaturated water vapor only (without any liquid phase).

Now we assume that both air and unsaturated water vapor behave as ideal gases. In view of Dalton law, the total pressure p in the vessel at 100°C is equal to the sum of partial pressures of the air p_a and unsaturated water vapor p_v:

$$p = p_a + p_v .$$

As the volume of the vessel is constant, we may apply the Gay-Lussac law to the air. We obtain:

$$p_a = p_0\left(\frac{273+t}{273}\right).$$

The pressure of the water vapor may be found from the equation of state of the ideal gas:

$$\frac{p_v V_0}{273+t} = \frac{m}{\mu} R ,$$

where m denotes the mass of the vapor, μ - the molecular mass of the water and R – the universal gas constant. Thus,

$$p_v = \frac{m}{\mu} R \frac{273+t}{V_0}$$

and finally

$$p = p_0 \frac{273+t}{273} + \frac{m}{\mu} R \frac{273+t}{V_0} .$$

Numerically:

$$p = (1.366 + 0.516)\,\text{atm} \approx 1.88\,\text{atm}.$$

Experimental problem

The following devices and materials are given:

1. Balance (without weights)
2. Calorimeter
3. Thermometer
4. Source of voltage
5. Switches
6. Wires
7. Electric heater
8. Stop-watch
9. Beakers
10. Water
11. Petroleum
12. Sand (for balancing)

Determine specific heat of petroleum. The specific heat of water is 1 cal/(g·°C). The specific heat of the calorimeter is 0.092 cal/(g·°C).

Discuss assumptions made in the solution.

Solution

The devices given to the students allowed using several methods. The students used the following three methods:

1. Comparison of velocity of warming up water and petroleum;
2. Comparison of cooling down water and petroleum;
3. Traditional heat balance.

As no weights were given, the students had to use the sand to find portions of petroleum and water with masses equal to the mass of calorimeter.

First method: comparison of velocity of warming up

If the heater is inside water then both water and calorimeter are warming up. The heat taken by water and calorimeter is:

$$Q_1 = m_w c_w \Delta t_1 + m_c c_c \Delta t_1,$$

where: m_w denotes mass of water, m_c - mass of calorimeter, c_w - specific heat of water, c_c - specific heat of calorimeter, Δt_1 - change of temperature of the system water + calorimeter.

On the other hand, the heat provided by the heater is equal:

$$Q_2 = A\frac{U^2}{R}\tau_1,$$

where: A – denotes the thermal equivalent of work, U – voltage, R – resistance of the heater, τ_1 – time of work of the heater in the water.

Of course,

$$Q_1 = Q_2.$$

Thus

$$A\frac{U^2}{R}\tau_1 = m_w c_w \Delta t_1 + m_c c_c \Delta t_1.$$

For petroleum in the calorimeter we get a similar formula:

$$A\frac{U^2}{R}\tau_2 = m_p c_p \Delta t_2 + m_c c_c \Delta t_2.$$

where: m_p denotes mass of petroleum, c_p - specific heat of petroleum, Δt_2 - change of temperature of the system water + petroleum, τ_2 – time of work of the heater in the petroleum.

By dividing the last equations we get

$$\frac{\tau_1}{\tau_2} = \frac{m_w c_w \Delta t_1 + m_c c_c \Delta t}{m_p c_p \Delta t_2 + m_c c_c \Delta t_2}.$$

It is convenient to perform the experiment by taking masses of water and petroleum equal to the mass of the calorimeter (for that we use the balance and the sand). For

$$m_w = m_p = m_c$$

the last formula can be written in a very simple form:

$$\frac{\tau_1}{\tau_2} = \frac{c_w \Delta t_1 + c_c \Delta t_1}{c_p \Delta t_2 + c_c \Delta t_2}.$$

Thus

$$c_c = \frac{\Delta t_1}{\tau_1}\frac{\tau_2}{\Delta t_2} c_w - \left(1 - \frac{\Delta t_1}{\tau_1}\frac{\tau_2}{\Delta t_2}\right) c_c$$

or

$$c_c = \frac{k_1}{k_2} c_w - \left(1 - \frac{k_1}{k_2}\right) c_c,$$

where

$$k_1 = \frac{\Delta t_1}{\tau_1} \quad \text{and} \quad k_2 = \frac{\Delta t_2}{\tau_2}$$

denote "velocities of heating" water and petroleum, respectively. These quantities can be determined experimentally by drawing graphs representing dependence Δt_1 and Δt_2 on time (τ). The experiment shows that these dependences are linear. Thus, it is enough to take slopes of appropriate straight lines. The experimental setup given to the students allowed measurements of the specific heat of petroleum, equal to 0.53 cal/(g°·C), with accuracy about 1%.

Some students used certain mutations of this method by performing measurements at $\Delta t_1 = \Delta t_2$ or at $\tau_1 = \tau_2$. Then, of course, the error of the final result is greater (it is additionally affected by accuracy of establishing the conditions $\Delta t_1 = \Delta t_2$ or at $\tau_1 = \tau_2$).

Second method: comparison of velocity of cooling down

Some students initially heated the liquids in the calorimeter and later observed their cooling down. This method is based on the Newton's law of cooling. It says that the heat Q transferred during cooling in time τ is given by the formula:

$$Q = h(t - \vartheta) s \tau,$$

where: t denotes the temperature of the body, ϑ - the temperature of surrounding, s – area of the body, and h – certain coefficient characterizing properties of the surface. This formula is correct for small differences of temperatures $t - \vartheta$ only (small compared to t and ϑ in the absolute scale).

This method, like the previous one, can be applied in different versions. We will consider only one of them.

Consider the situation when cooling of water and petroleum is observed in the same calorimeter (containing initially water and later petroleum). The heat lost by the system water + calorimeter is

$$\Delta Q_1 = (m_w c_w + m_c c_c)\Delta t \, ,$$

where Δt denotes a change of the temperature of the system during certain period τ_1. For the system petroleum + calorimeter, under assumption that the change in the temperature Δt is the same, we have

$$\Delta Q_2 = (m_p c_p + m_c c_c)\Delta t \, .$$

Of course, the time corresponding to Δt in the second case will be different. Let it be τ_2.

From the Newton's law we get

$$\frac{\Delta Q_1}{\Delta Q_2} = \frac{\tau_1}{\tau_2} \, .$$

Thus

$$\frac{\tau_1}{\tau_2} = \frac{m_w c_w + m_c c_c}{m_p c_p + m_c c_c} \, .$$

If we conduct the experiment at

$$m_w = m_p = m_c \, ,$$

then we get

$$c_p = \frac{T_2}{T_1} c_w - \left(1 - \frac{T_2}{T_1}\right) c_c \, .$$

As cooling is rather a very slow process, this method gives the result with definitely greater error.

Third method: heat balance

This method is rather typical. The students heated the water in the calorimeter to certain temperature t_1 and added the petroleum with the temperature t_2. After reaching the thermal equilibrium the final temperature was t. From the thermal balance (neglecting the heat losses) we have

$$(m_w c_w + m_c c_c)(t_1 - t) = m_p c_p (t - t_2).$$

If, like previously, the experiment is conducted at

$$m_w = m_p = m_c,$$

then

$$c_p = (c_w + c_c)\frac{t_1 - t}{t - t_2}.$$

In this methods the heat losses (when adding the petroleum to the water) always played a substantial role.

The accuracy of the result equal or better than 5% can be reached by using any of the methods described above. However, one should remark that in the first method it was easiest. The most common mistake was neglecting the heat capacity of the calorimeter. This mistake increased the error additionally by about 8%.

Marks

No marking schemes are present in my archive materials. Only the mean scores are available. They are:

Problem # 1	7.6 points
Problem # 2	7.8 points (without the Romanian students)
Problem # 3	5.9 points
Experimental problem	7.7 points

Thanks

The author would like to express deep thanks to Prof. Jan Mostowski and Dr. Yohanes Surya for reviewing the text and for valuable comments and remarks.

Literature

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Problems of the 2^{nd} International Physics Olympiads (Budapest, Hungary, 1968)

Péter Vankó
Institute of Physics, Budapest University of Technical Engineering, Budapest, Hungary

Abstract

After a short introduction the problems of the 2^{nd} and the 9^{th} International Physics Olympiad, organized in Budapest, Hungary, 1968 and 1976, and their solutions are presented.

Introduction

Following the initiative of Dr. Waldemar Gorzkowski [1] I present the problems and solutions of the 2^{nd} and the 9^{th} International Physics Olympiad, organized by Hungary. I have used Prof. Rezső Kunfalvi's problem collection [2], its Hungarian version [3] and in the case of the 9^{th} Olympiad the original Hungarian problem sheet given to the students (my own copy). Besides the digitalization of the text, the equations and the figures it has been made only small corrections where it was needed (type mistakes, small grammatical changes). I omitted old units, where both old and SI units were given, and converted them into SI units, where it was necessary.

If we compare the problem sheets of the early Olympiads with the last ones, we can realize at once the difference in length. It is not so easy to judge the difficulty of the problems, but the solutions are surely much shorter.

The problems of the 2^{nd} Olympiad followed the more than hundred years tradition of physics competitions in Hungary. The tasks of the most important Hungarian theoretical physics competition (Eötvös Competition), for example, are always very short. Sometimes the solution is only a few lines, too, but to find the idea for this solution is rather difficult.

Of the 9^{th} Olympiad I have personal memories; I was the youngest member of the Hungarian team. The problems of this Olympiad were collected and partly invented by Miklós Vermes, a legendary and famous Hungarian secondary school physics teacher. In the first problem only the detailed investigation of the stability was unusual, in the second problem one could forget to subtract the work of the atmospheric pressure, but the fully "open" third problem was really unexpected for us.

The experimental problem was difficult in the same way: in contrast to the Olympiads of today we got no instructions how to measure. (In the last years the only similarly open experimental problem was the investigation of "The magnetic puck" in Leicester, 2000, a really nice problem by Cyril Isenberg.) The challenge was not to perform many-many measurements in a short time, but to find out what to measure and how to do it.

Of course, the evaluating of such open problems is very difficult, especially for several hundred students. But in the 9^{th} Olympiad, for example, only ten countries participated and the same person could read, compare, grade and mark all of the solutions.

2^{nd} IPhO (Budapest, 1968)

Theoretical problems

Problem 1

On an inclined plane of 30° a block, mass $m_2 = 4$ kg, is joined by a light cord to a solid cylinder, mass $m_1 = 8$ kg, radius $r = 5$ cm (*Fig. 1*). Find the acceleration if the bodies are released. The coefficient of friction between the block and the inclined plane $\mu = 0.2$. Friction at the bearing and rolling friction are negligible.

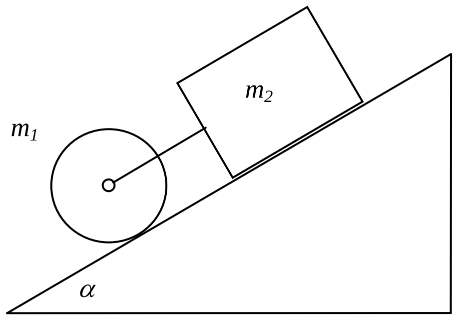

Figure 1

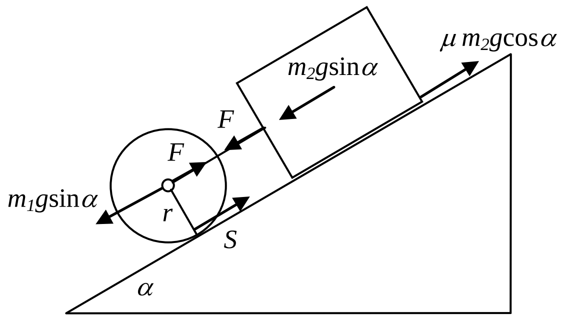

Figure 2

Solution

If the cord is stressed the cylinder and the block are moving with the same acceleration a. Let F be the tension in the cord, S the frictional force between the cylinder and the inclined plane (*Fig. 2*). The angular acceleration of the cylinder is a/r. The net force causing the acceleration of the block:

$$m_2 a = m_2 g \sin\alpha - \mu m_2 g \cos\alpha + F \,,$$

and the net force causing the acceleration of the cylinder:

$$m_1 a = m_1 g \sin\alpha - S - F \;.$$

The equation of motion for the rotation of the cylinder:

$$S\,r = \frac{a}{r}\cdot I \,.$$

(I is the moment of inertia of the cylinder, $S\cdot r$ is the torque of the frictional force.)
Solving the system of equations we get:

$$a = g\cdot\frac{(m_1+m_2)\sin\alpha - \mu m_2\cos\alpha}{m_1+m_2+\dfrac{I}{r^2}}\,, \qquad (1)$$

$$S = \frac{I}{r^2}\cdot g\cdot\frac{(m_1+m_2)\sin\alpha - \mu m_2\cos\alpha}{m_1+m_2+\dfrac{I}{r^2}}\,, \qquad (2)$$

$$F = m_2 g \cdot \frac{\mu\left(m_1 + \frac{I}{r^2}\right)\cos\alpha - \frac{I\sin\alpha}{r^2}}{m_1 + m_2 + \frac{I}{r^2}}. \qquad (3)$$

The moment of inertia of a solid cylinder is $I = \frac{m_1 r^2}{2}$. Using the given numerical values:

$$a = g \cdot \frac{(m_1 + m_2)\sin\alpha - \mu m_2 \cos\alpha}{1.5m_1 + m_2} = 0.3317g = \mathbf{3.25\ m/s^2},$$

$$S = \frac{m_1 g}{2} \cdot \frac{(m_1 + m_2)\sin\alpha - \mu m_2 \cos\alpha}{1.5m_1 + m_2} = \mathbf{13.01\ N},$$

$$F = m_2 g \cdot \frac{(1.5\mu\cos\alpha - 0.5\sin\alpha)m_1}{1.5m_1 + m_2} = \mathbf{0.192\ N}.$$

Discussion (See *Fig. 3*.)

The condition for the system to start moving is $a > 0$. Inserting $a = 0$ into (1) we obtain the limit for angle α_1:

$$\tan\alpha_1 = \mu \cdot \frac{m_2}{m_1 + m_2} = \frac{\mu}{3} = 0.0667, \quad \alpha_1 = 3.81°.$$

For the cylinder separately $\alpha_1 = 0$, and for the block separately $\alpha_1 = \tan^{-1}\mu = 11.31°$.

If the cord is not stretched the bodies move separately. We obtain the limit by inserting $F = 0$ into (3):

$$\tan\alpha_2 = \mu \cdot \left(1 + \frac{m_1 r^2}{I}\right) = 3\mu = 0.6, \quad \alpha_2 = 30.96°.$$

The condition for the cylinder to slip is that the value of S (calculated from (2) taking the same coefficient of friction) exceeds the value of $\mu m_1 g\cos\alpha$. This gives the same value for α_3 as we had for α_2. The acceleration of the centers of the cylinder and the block is the same: $g(\sin\alpha - \mu\cos\alpha)$, the frictional force at the bottom of the cylinder is $\mu m_1 g\cos\alpha$, the peripheral acceleration of the cylinder is $\mu \cdot \frac{m_1 r^2}{I} \cdot g\cos\alpha$.

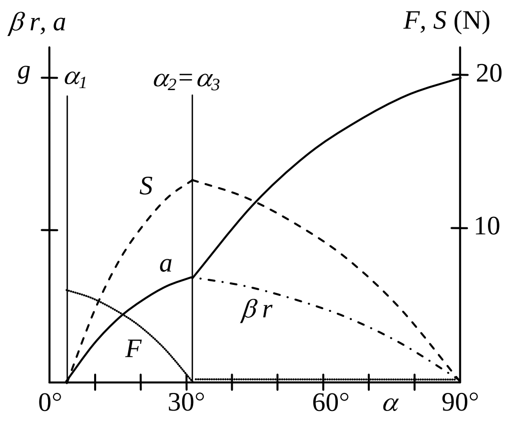

Figure 3

Problem 2

There are 300 cm^3 toluene of 0°C temperature in a glass and 110 cm^3 toluene of 100°C temperature in another glass. (The sum of the volumes is 410 cm^3.) Find the final volume after the two liquids are mixed. The coefficient of volume expansion of toluene $\beta = 0.001(°C)^{-1}$. Neglect the loss of heat.

Solution

If the volume at temperature t_1 is V_1, then the volume at temperature 0°C is $V_{10} = V_1 / (1 + \beta t_1)$. In the same way if the volume at t_2 temperature is V_2, at 0°C we have $V_{20} = V_2 / (1 + \beta t_2)$. Furthermore if the density of the liquid at 0°C is d, then the masses are $m_1 = V_{10} d$ and $m_2 = V_{20} d$, respectively. After mixing the liquids the temperature is

$$t = \frac{m_1 t_1 + m_2 t_2}{m_1 + m_2}.$$

The volumes at this temperature are $V_{10}(1 + \beta t)$ and $V_{20}(1 + \beta t)$.
The sum of the volumes after mixing:

$$V_{10}(1+\beta t) + V_{20}(1+\beta t) = V_{10} + V_{20} + \beta (V_{10} + V_{20}) t =$$
$$= V_{10} + V_{20} + \beta \cdot \frac{m_1 + m_2}{d} \cdot \frac{m_1 t_1 + m_2 t_2}{m_1 + m_2} =$$
$$= V_{10} + V_{20} + \beta \left(\frac{m_1 t_1}{d} + \frac{m_2 t_2}{d} \right) = V_{10} + \beta V_{10} t_1 + V_{20} + \beta V_{20} t_2 =$$
$$= V_{10}(1 + \beta t_1) + V_{20}(1 + \beta t_2) = V_1 + V_2$$

The sum of the volumes is constant. In our case it is 410 cm^3. The result is valid for any number of quantities of toluene, as the mixing can be done successively adding always one more glass of liquid to the mixture.

Problem 3

Parallel light rays are falling on the plane surface of a semi-cylinder made of glass, at an angle of 45°, in such a plane which is perpendicular to the axis of the semi-cylinder (*Fig. 4*). (Index of refraction is $\sqrt{2}$.) Where are the rays emerging out of the cylindrical surface?

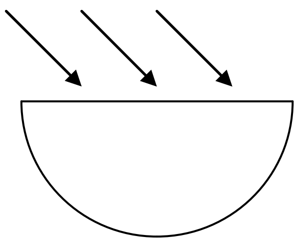

Figure 4

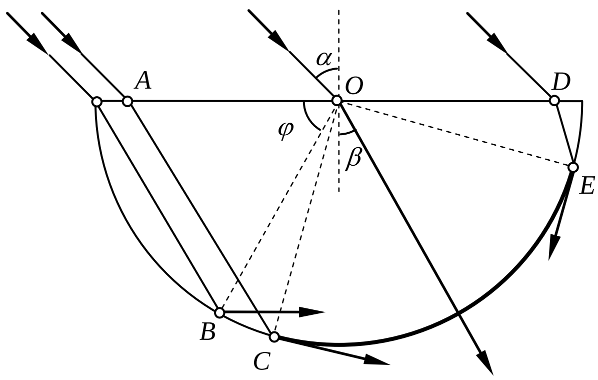

Figure 5

Solution

Let us use angle φ to describe the position of the rays in the glass (*Fig. 5*). According to the law of refraction $\sin 45° / \sin \beta = \sqrt{2}$, $\sin \beta = 0.5$, $\beta = 30°$. The refracted angle is 30° for all of the incoming rays. We have to investigate what happens if φ changes from 0° to 180°.

It is easy to see that φ can not be less than 60° ($AOB\angle = 60°$). The critical angle is given by $\sin\beta_{crit} = 1/n = \sqrt{2}/2$; hence $\beta_{crit} = 45°$. In the case of total internal reflection $ACO\angle = 45°$, hence $\varphi = 180° - 60° - 45° = 75°$. If φ is more than 75° the rays can emerge the cylinder. Increasing the angle we reach the critical angle again if $OED\angle = 45°$. Thus the rays are leaving the glass cylinder if:

$$75° < \varphi < 165°,$$

CE, arc of the emerging rays, subtends a central angle of 90°.

Experimental problem

Three closed boxes (black boxes) with two plug sockets on each are present for investigation. The participants have to find out, without opening the boxes, what kind of elements are in them and measure their characteristic properties. AC and DC meters (their internal resistance and accuracy are given) and AC (5O Hz) and DC sources are put at the participants' disposal.

Solution

No voltage is observed at any of the plug sockets therefore none of the boxes contains a source.

Measuring the resistances using first AC then DC, one of the boxes gives the same result. Conclusion: the box contains a simple resistor. Its resistance is determined by measurement.

One of the boxes has a very great resistance for DC but conducts AC well. It contains a capacitor, the value can be computed as $C = \dfrac{1}{\omega X_C}$.

The third box conducts both AC and DC, its resistance for AC is greater. It contains a resistor and an inductor connected in series. The values of the resistance and the inductance can be computed from the measurements.

3^{rd} International Physics Olympiad
1969, Brno, Czechoslovakia

Problem 1. Figure 1 shows a mechanical system consisting of three carts A, B and C of masses $m_1 = 0.3$ kg, $m_2 = 0.2$ kg and $m_3 = 1.5$ kg respectively. Carts B and A are connected by a light taut inelastic string which passes over a light smooth pulley attaches to the cart C as shown. For this problem, all resistive and frictional forces may be ignored as may the moments of inertia of the pulley and of the wheels of all three carts. Take the acceleration due to gravity g to be 9.81 m s^{-2}.

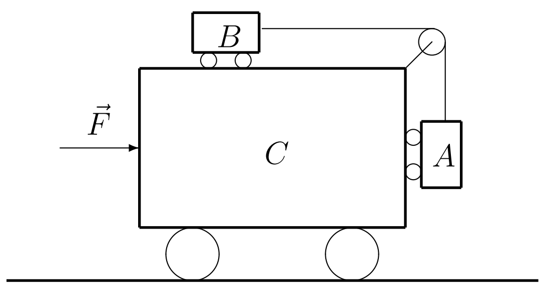

Figure 1:

1. A horizontal force $\vec{F}$ is now applied to cart C as shown. The size of $\vec{F}$ is such that carts A and B remain at rest relative to cart C.
   a) Find the tension in the string connecting carts A and B.
   b) Determine the magnitude of $\vec{F}$.
2. Later cart C is held stationary, while carts A and B are released from rest.
   a) Determine the accelerations of carts A and B.
   b) Calculate also the tension in the string.

Solution:
Case 1. The force $\vec{F}$ has so big magnitude that the carts A and B remain at the rest with respect to the cart C, *i.e.* they are moving with the same acceleration as the cart C is. Let $\vec{G_1}$, $\vec{T_1}$ and $\vec{T_2}$ denote forces acting on particular carts as shown in the Figure 2 and let us write the equations of motion for the carts A and B and also for whole mechanical system. Note that certain internal forces (viz. normal reactions) are not shown.

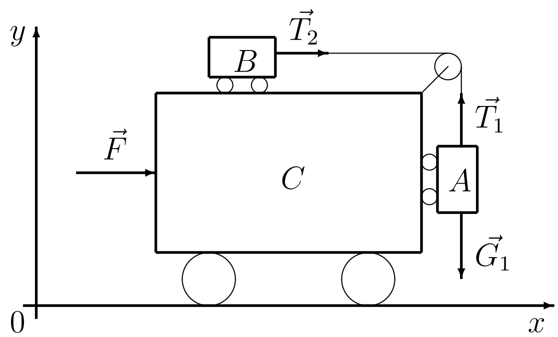

Figure 2:

The cart B is moving in the coordinate system Oxy with an acceleration a_x. The only force acting on the cart B is the force $\vec{T_2}$, thus

$$T_2 = m_2\, a_x\,. \tag{1}$$

Since $\vec{T_1}$ and $\vec{T_2}$ denote tensions in the same cord, their magnitudes satisfy

$$T_1 = T_2\,.$$

The forces $\vec{T_1}$ and $\vec{G_1}$ act on the cart A in the direction of the y-axis. Since, according to condition 1, the carts A and B are at rest with respect to the cart C, the acceleration in the direction of the y-axis equals to zero, $a_y = 0$, which yields

$$T_1 - m_1\, g = 0\,.$$

Consequently

$$T_2 = m_1\, g\,. \tag{2}$$

So the motion of the whole mechanical system is described by the equation

$$F = (m_1 + m_2 + m_3)\, a_x\,, \tag{3}$$

because forces between the carts A and C and also between the carts B and C are internal forces with respect to the system of all three bodies. Let us remark here that also the tension $\vec{T_2}$ is the internal force with respect to the system of all bodies, as can be easily seen from the analysis of forces acting on the pulley. From equations (1) and (2) we obtain

$$a_x = \frac{m_1}{m_2} g \, .$$

Substituting the last result to (3) we arrive at

$$F = (m_1 + m_2 + m_3) \frac{m_1}{m_2} g \, .$$

Numerical solution:

$$T_2 = T_1 = 0.3 \cdot 9.81 \text{ N} = 2.94 \text{ N} \, ,$$
$$F = 2 \cdot \frac{3}{2} \cdot 9.81 \text{ N} = 29.4 \text{ N} \, .$$

Case 2. If the cart C is immovable then the cart A moves with an acceleration a_y and the cart B with an acceleration a_x. Since the cord is inextensible (*i.e.* it cannot lengthen), the equality

$$a_x = -a_y = a$$

holds true. Then the equations of motion for the carts A, respectively B, can be written in following form

$$T_1 = G_1 - m_1 \, a \, , \qquad (4)$$
$$T_2 = m_2 \, a \, . \qquad (5)$$

The magnitudes of the tensions in the cord again satisfy

$$T_1 = T_2 \, . \qquad (6)$$

The equalities (4), (5) and (6) immediately yield

$$(m_1 + m_2) \, a = m_1 \, g \, .$$

Using the last result we can calculate

$$a = a_x = -a_y = \frac{m_1}{m_1 + m_2}\, g\,,$$
$$T_2 = T_1 = \frac{m_2 m_1}{m_1 + m_2}\, g\,.$$

Numerical results:

$$a = a_x = \frac{3}{5} \cdot 9.81 \text{ m s}^{-2} = 5.89 \text{ m s}^{-2}\,,$$
$$T_1 = T_2 = 1.18 \text{ N}\,.$$

Problem 2. Water of mass m_2 is contained in a copper calorimeter of mass m_1. Their common temperature is t_2. A piece of ice of mass m_3 and temperature $t_3 < 0\,^\circ$C is dropped into the calorimeter.

a) Determine the temperature and masses of water and ice in the equilibrium state for general values of m_1, m_2, m_3, t_2 and t_3. Write equilibrium equations for all possible processes which have to be considered.

b) Find the final temperature and final masses of water and ice for $m_1 = 1.00$ kg, $m_2 = 1.00$ kg, $m_3 = 2.00$ kg, $t_2 = 10\,^\circ$C, $t_3 = -20\,^\circ$C.

Neglect the energy losses, assume the normal barometric pressure. Specific heat of copper is $c_1 = 0.1$ kcal/kg·°C, specific heat of water $c_2 = 1$ kcal/kg·°C, specific heat of ice $c_3 = 0.492$ kcal/kg·°C, latent heat of fusion of ice $l = 78,7$ kcal/kg. Take 1 cal = 4.2 J.

Solution:
We use the following notation:

t	temperature of the final equilibrium state,
$t_0 = 0\,^\circ\text{C}$	the melting point of ice under normal pressure conditions,
M_2	final mass of water,
M_3	final mass of ice,
$m_2' \le m_2$	mass of water, which freezes to ice,
$m_3' \le m_3$	mass of ice, which melts to water.

a) Generally, four possible processes and corresponding equilibrium states can occur:

1. $t_0 < t < t_2$, $m'_2 = 0$, $m'_3 = m_3$, $M_2 = m_2 + m_3$, $M_3 = 0$.
   Unknown final temperature t can be determined from the equation

$$(m_1c_1 + m_2c_2)(t_2 - t) = m_3c_3(t_0 - t_3) + m_3l + m_3c_2(t - t_0)\,. \qquad (7)$$

   However, only the solution satisfying the condition $t_0 < t < t_2$ does make physical sense.

2. $t_3 < t < t_0$, $m'_2 = m_2$, $m'_3 = 0$, $M_2 = 0$, $M_3 = m_2 + m_3$.
   Unknown final temperature t can be determined from the equation

$$m_1c_1(t_2 - t) + m_2c_2(t_2 - t_0) + m_2l + m_2c_3(t_0 - t) = m_3c_3(t - t_3)\,. \quad (8)$$

   However, only the solution satisfying the condition $t_3 < t < t_0$ does make physical sense.

3. $t = t_0$, $m'_2 = 0$, $0 \leq m'_3 \leq m_3$, $M_2 = m_2 + m'_3$, $M_3 = m_3 - m'_3$.
   Unknown mass m'_3 can be calculated from the equation

$$(m_1c_1 + m_2c_2)(t_2 - t_0) = m_3c_3(t - t_3) + m'_3l\,. \qquad (9)$$

   However, only the solution satisfying the condition $0 \leq m'_3 \leq m_3$ does make physical sense.

4. $t = t_0$, $0 \leq m'_2 \leq m_2$, $m'_3 = 0$, $M_2 = m_2 - m'_2$, $M_3 = m_3 + m'_2$.
   Unknown mass m'_2 can be calculated from the equation

$$(m_1c_1 + m_2c_2)(t_2 - t_0) + m'_2l = m_3c_3(t_0 - t_3)\,. \qquad (10)$$

   However, only the solution satisfying the condition $0 \leq m'_2 \leq m_2$ does make physical sense.

b) Substituting the particular values of m_1, m_2, m_3, t_2 and t_3 to equations (7), (8) and (9) one obtains solutions not making the physical sense (not satisfying the above conditions for t, respectively m'_3). The real physical process under given conditions is given by the equation (10) which yields

$$m'_2 = \frac{m_3c_3(t_0 - t_3) - (m_1c_1 + m_2c_2)(t_2 - t_0)}{l}\,.$$

Substituting given numerical values one gets $m'_2 = 0.11$ kg. Hence, $t = 0\,^{\circ}\mathrm{C}$, $M_2 = m_2 - m'_2 = 0.89$ kg, $M_3 = m_3 + m'_2 = 2.11$ kg.

Problem 3. A small charged ball of mass m and charge q is suspended from the highest point of a ring of radius R by means of an insulating cord of negligible mass. The ring is made of a rigid wire of negligible cross section and lies in a vertical plane. On the ring there is uniformly distributed charge Q of the same sign as q. Determine the length l of the cord so as the equilibrium position of the ball lies on the symmetry axis perpendicular to the plane of the ring.

Find first the general solution a then for particular values $Q = q = 9.0 \cdot 10^{-8}$ C, $R = 5$ cm, $m = 1.0$ g, $\varepsilon_0 = 8.9 \cdot 10^{-12}$ F/m.

Solution:
In equilibrium, the cord is stretched in the direction of resultant force of $\vec{G} = m\vec{g}$ and $\vec{F} = q\vec{E}$, where $\vec{E}$ stands for the electric field strength of the ring on the axis in distance x from the plane of the ring, see Figure 3. Using the triangle similarity, one can write

$$\frac{x}{R} = \frac{Eq}{mg}\,. \tag{11}$$

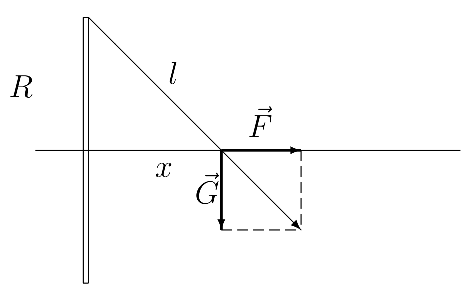

Figure 3:

For the calculation of the electric field strength let us divide the ring to n identical parts, so as every part carries the charge Q/n. The electric field strength magnitude of one part of the ring is given by

$$\Delta E = \frac{Q}{4\pi\varepsilon_0 l^2 n}\,.$$

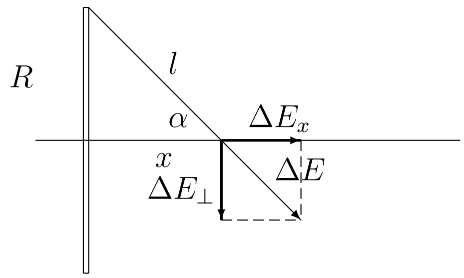

Figure 4:

This electric field strength can be decomposed into the component in the direction of the x-axis and the one perpendicular to the x-axis, see Figure 4. Magnitudes of both components obey

$$\Delta E_x = \Delta E \, \cos\alpha = \frac{\Delta E \, x}{l} \,,$$
$$\Delta E_\perp = \Delta E \, \sin\alpha \,.$$

It follows from the symmetry, that for every part of the ring there exists another one having the component $\Delta\vec{E}_\perp$ of the same magnitude, but however oppositely oriented. Hence, components perpendicular to the axis cancel each other and resultant electric field strength has the magnitude

$$E = E_x = n\Delta E_x = \frac{Q\,x}{4\pi\varepsilon_0\, l^3} \,. \qquad (12)$$

Substituting (12) into (11) we obtain for the cord length

$$l = \sqrt[3]{\frac{Q\,q\,R}{4\pi\varepsilon_0\, m\, g}} \,.$$

Numerically

$$l = \sqrt[3]{\frac{9.0\cdot 10^{-8}\cdot 9.0\cdot 10^{-8}\cdot 5.0\cdot 10^{-2}}{4\pi\cdot 8.9\cdot 10^{-12}\cdot 10^{-3}\cdot 9.8}}\ \mathrm{m} = 7.2\cdot 10^{-2}\ \mathrm{m}\,.$$

Problem 4. A glass plate is placed above a glass cube of 2 cm edges in such a way that there remains a thin air layer between them, see Figure 5.

Electromagnetic radiation of wavelength between 400 nm and 1150 nm (for which the plate is penetrable) incident perpendicular to the plate from above is reflected from both air surfaces and interferes. In this range only two wavelengths give maximum reinforcements, one of them is $\lambda = 400$ nm. Find the second wavelength. Determine how it is necessary to warm up the cube so as it would touch the plate. The coefficient of linear thermal expansion is $\alpha = 8.0 \cdot 10^{-6}\,^{\circ}\mathrm{C}^{-1}$, the refractive index of the air $n = 1$. The distance of the bottom of the cube from the plate does not change during warming up.

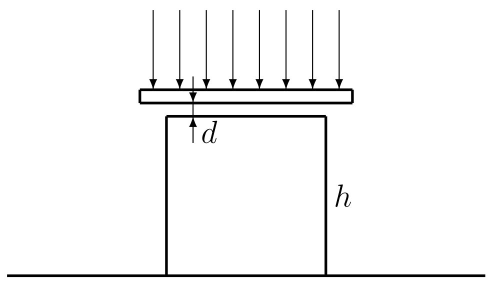

Figure 5:

Solution:
Condition for the maximum reinforcement can be written as

$$2dn - \frac{\lambda_k}{2} = k\lambda_k\,, \text{ for } k = 0, 1, 2, \dots\,,$$

i.e.

$$2dn = (2k+1)\frac{\lambda_k}{2}\,, \tag{13}$$

with d being thickness of the layer, n the refractive index and k maximum order. Let us denote $\lambda' = 1150$ nm. Since for $\lambda = 400$ nm the condition for maximum is satisfied by the assumption, let us denote $\lambda_p = 400$ nm, where p is an unknown integer identifying the maximum order, for which

$$\lambda_p(2p+1) = 4dn \tag{14}$$

holds true. The equation (13) yields that for fixed d the wavelength λ_k increases with decreasing maximum order k and vise versa. According to the assumption,

$$\lambda_{p-1} < \lambda' < \lambda_{p-2}\,,$$

i.e.

$$\frac{4dn}{2(p-1)+1} < \lambda' < \frac{4dn}{2(p-2)+1}\,.$$

Substituting to the last inequalities for $4dn$ using (14) one gets

$$\frac{\lambda_p(2p+1)}{2(p-1)+1} < \lambda' < \frac{\lambda_p(2p+1)}{2(p-2)+1}\,.$$

Let us first investigate the first inequality, straightforward calculations give us gradually

$$\lambda_p(2p+1) < \lambda'(2p-1)\,, \quad 2p(\lambda'-\lambda_p) > \lambda'+\lambda_p\,,$$

i.e.

$$p > \frac{1}{2}\frac{\lambda'+\lambda_p}{\lambda'-\lambda_p} = \frac{1}{2}\frac{1150+400}{1150-400} = 1.\ \ldots \tag{15}$$

Similarly, from the second inequality we have

$$\lambda_p(2p+1) > \lambda'(2p-3)\,, \quad 2p(\lambda'-\lambda_p) < 3\lambda'+\lambda_p\,,$$

i.e.

$$p < \frac{1}{2}\frac{3\lambda'+\lambda_p}{\lambda'-\lambda_p} = \frac{1}{2}\frac{3\cdot 1150+400}{1150-400} = 2.\ \ldots \tag{16}$$

The only integer p satisfying both (15) and (16) is $p = 2$.

Let us now find the thickness d of the air layer:

$$d = \frac{\lambda_p}{4}(2p+1) = \frac{400}{4}(2\cdot 2+1)\ \text{nm} = 500\ \text{nm}\,.$$

Substituting d to the equation (13) we can calculate λ_{p-1}, *i.e.* λ_1:

$$\lambda_1 = \frac{4dn}{2(p-1)+1} = \frac{4dn}{2p-1}.$$

Introducing the particular values we obtain

$$\lambda_1 = \frac{4\cdot 500\cdot 1}{2\cdot 2-1}\ \text{nm} = 666.7\ \text{nm}\,.$$

Finally, let us determine temperature growth Δt. Generally, $\Delta l = \alpha l \Delta t$ holds true. Denoting the cube edge by h we arrive at $d = \alpha h \Delta t$. Hence

$$\Delta t = \frac{d}{\alpha h} = \frac{5\cdot 10^{-7}}{8\cdot 10^{-6}\cdot 2\cdot 10^{-2}}\ {}^\circ\text{C} = 3.1\ {}^\circ\text{C}\,.$$

Problems of the IV International Olympiad, Moscow, 1970

The publication is prepared by Prof. S. Kozel & Prof. V.Orlov

(Moscow Institute of Physics and Technology)

The IV International Olympiad in Physics for schoolchildren took place in Moscow (USSR) in July 1970 on the basis of Moscow State University. Teams from 8 countries participated in the competition, namely Bulgaria, Hungary, Poland, Romania, Czechoslovakia, the DDR, the SFR Yugoslavia, the USSR. The problems for the theoretical competition have been prepared by the group from Moscow University stuff headed by professor V.Zubov. The problem for the experimental competition has been worked out by B. Zvorikin from the Academy of Pedagogical Sciences.

It is pity that marking schemes were not preserved.

Theoretical Problems

Problem 1.

A long bar with the mass $M = 1$ kg is placed on a smooth horizontal surface of a table where it can move frictionless. A carriage equipped with a motor can slide along the upper horizontal panel of the bar, the mass of the carriage is $m = 0.1$ kg. The friction coefficient of the carriage is $\mu = 0.02$. The motor is winding a thread around a shaft at a constant speed $v_0 = 0.1$ m/s. The other end of the thread is tied up to a rather distant stationary support in one case (Fig.1, a), whereas in the other case it is attached to a picket at the edge of the bar (Fig.1, b). While holding the bar fixed one allows the carriage to start moving at the velocity V_0 then the bar is let loose.

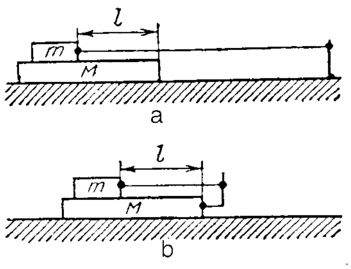

Fig. 1

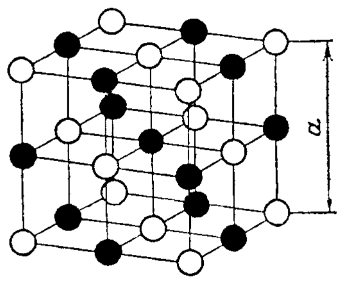

Fig. 2

By the moment the bar is released the front edge of the carriage is at the distance $l = 0.5$ m from the front edge of the bar. For both cases find the laws of movement of both the bar and the carriage and the time during which the carriage will reach the front edge of the bar.

Problem 2.

A unit cell of a crystal of natrium chloride (common salt- NaCl) is a cube with the edge length a = $5.6\cdot10^{-10}$ m (Fig.2). The black circles in the figure stand for the position of natrium atoms whereas the white ones are chlorine atoms. The entire crystal of common salt turns out to be a repetition of such unit cells. The relative atomic mass of natrium is 23 and that of chlorine is 35,5. The density of the common salt $\rho = 2.22\cdot10^3$ kg/m^3 . Find the mass of a hydrogen atom.

Problem 3.

Inside a thin-walled metal sphere with radius R=20 cm there is a metal ball with the radius r = 10 cm which has a common centre with the sphere. The ball is connected with a very long wire to the Earth via an opening in the sphere (Fig. 3). A charge $Q = 10^{-8}$ C is placed onto the outside sphere. Calculate the potential of this sphere, electrical capacity of the obtained system of conducting bodies and draw out an equivalent electric scheme.

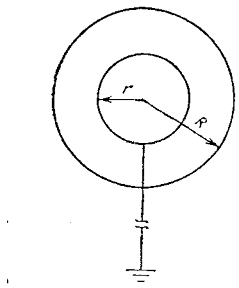

Fig. 3

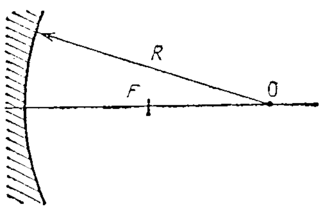

Fig. 4

Problem 4.

A spherical mirror is installed into a telescope. Its lateral diameter is D=0,5 m and the radius of the curvature R=2 m. In the main focus of the mirror there is an emission receiver in the form of a round disk. The disk is placed perpendicular to the optical axis of the mirror (Fig.7). What should the radius r of the receiver be so that it could receive the entire flux of the emission reflected by the mirror? How would the received flux of the emission decrease if the detector's dimensions decreased by 8 times?

Directions: 1) When calculating small values α (α<<1) one may perform a substitution $\sqrt{1-\alpha} \approx 1-\frac{\alpha}{2}$; 2) diffraction should not be taken into account.

Experimental Problem

Determine the focal distances of lenses.

List of instruments: three different lenses installed on posts, a screen bearing an image of a geometric figure, some vertical wiring also fixed on the posts and a ruler.

Solutions of the problems of the IV International Olympiad, Moscow, 1970

Theoretical Competition

Problem 1.

a) By the moment of releasing the bar the carriage has a velocity v_0 relative to the table and continues to move at the same velocity.

The bar, influenced by the friction force $F_{fr} = \mu mg$ from the carriage, gets an acceleration $a = F_{fr}/ M = \mu mg/M$; $a = 0.02$ m/s , while the velocity of the bar changes with time according to the law $v_b = at$.

Since the bar can not move faster than the carriage then at a moment of time $t = t_0$ its sliding will stop, that is $v_b = v_0$. Let us determine this moment of time:

$$t_0 = \frac{v_0}{a} = \frac{v_0 M}{\mu mg} = 5\text{s}$$

By that moment the displacement of the *Sb* bar and the carriage *Sc* relative to the table will be equal to

$$S_c = v_0 t_0 = \frac{v_0^2 M}{\mu mg}\ , \quad S_b = \frac{a t_0^2}{2} = \frac{v_0^2 M}{2\mu mg}\ .$$

The displacement of the carriage relative to the bar is equal to

$$S = S_c - S_b = \frac{v_0^2 M}{2\mu mg} = 0.25\text{m}$$

Since $S<l$, the carriage will not reach the edge of the bar until the bar is stopped by an immovable support. The distance to the support is not indicated in the problem condition so we can not calculate this time. Thus, the carriage is moving evenly at the velocity $v_0 = 0.1$ m/s, whereas the bar is moving for the first 5 sec uniformly accelerated with an acceleration $a = 0.02$ m/s and then the bar is moving with constant velocity together with the carriage.

b) Since there is no friction between the bar and the table surface the system of the bodies “bar-carriage” is a closed one. For this system one can apply the law of conservation of momentum:

$$mv + Mu = mv_0 \qquad (1)$$

where v and u are projections of velocities of the carriage and the bar relative to the table onto the horizontal axis directed along the vector of the velocity v_0. The velocity of the thread winding v_0 is equal to the velocity of the carriage relative to the bar (v-u), that is

$$v_0 = v - u \qquad (2)$$

Solving the system of equations (1) and (2) we obtain:

$$u = 0\,, \quad v = v_0\,.$$

Thus, being released the bar remains fixed relative to the table, whereas the carriage will be moving with the same velocity v_0 and will reach the edge of the bar within the time t equal to

$$t = l/v_0 = 5 \text{ s}.$$

Problem 2.

Let's calculate the quantities of natrium atoms (n_1) and chlorine atoms (n_2) embedded in a single NaCl unit crystal cell (Fig.2).

One atom of natrium occupies the middle of the cell and it entirely belongs to the cell. 12 atoms of natrium hold the edges of a large cube and they belong to three more cells so as 1/4 part of each belongs to the first cell. Thus we have

$n_1 = 1+12\cdot1/4 = 4$ atoms of natrium per unit cell.

In one cell there are 6 atoms of chlorine placed on the side of the cube and 8 placed in the vertices. Each atom from a side belongs to another cell and the atom in the vertex - to seven others. Then for one cell we have

$n_2 = 6\cdot1/2 + 8\cdot 1/8 = 4$ atoms of chlorine.

Thus 4 atoms of natriun and 4 atoms of chlorine belong to one unit cell of NaCl crystal.

The mass m of such a cell is equal

$$m = 4(m_{rNa} + m_{rCl}) \text{ (amu)},$$

where m_{rNa} and m_{rCl} are relative atomic masses of natrium and clorine. Since the mass of hydrogen atom m_H is approximately equal to one atomic mass unit: $m_H = 1.008$ amu $\approx$ 1 amu then the mass of an unit cell of NaCl is

$$m = 4(m_{rNa} + m_{rCl})\, m_H\,.$$

On the other hand, it is equal $m = \rho a^3$, hence

$$m_H = \frac{\rho a^3}{4(m_{rNa} + m_{rCl})} \approx 1.67\cdot10^{-27}\,\text{kg}\,.$$

Problem 3.

Having no charge on the ball the sphere has the potential

$$\varphi_{0s} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R} = 450\text{V}\,.$$

When connected with the Earth the ball inside the sphere has the potential equal to zero so there is an electric field between the ball and the sphere. This field moves a certain charge *q* from the Earth to the ball. Charge *Q*`, uniformly distributed on the sphere, doesn't create any field inside thus the electric field inside the sphere is defined by the ball's charge *q*. The potential difference between the balls and the sphere is equal

$$\Delta\varphi = \varphi_b - \varphi_s = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} - \frac{q}{R}\right), \qquad (1)$$

Outside the sphere the field is the same as in the case when all the charges were placed in its center. When the ball was connected with the Earth the potential of the sphere φ_s is equal

$$\varphi_s = \frac{1}{4\pi\varepsilon_0}\frac{q+Q}{R}. \qquad (2)$$

Then the potential of the ball

$$\varphi_b = \varphi_s + \Delta\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{q+Q}{R} + \frac{q}{r} - \frac{q}{R}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R} + \frac{q}{r}\right) = 0 \qquad (3)$$

Which leads to

$$q = -Q\frac{r}{R}. \qquad (4)$$

Substituting (4) into (2) we obtain for potential of the sphere to be found:

$$\varphi_s = \frac{1}{4\pi\varepsilon_0}\frac{Q - Q\frac{r}{R}}{R} = \frac{1}{4\pi\varepsilon_0}\frac{Q(R-r)}{R^2} = 225\text{V}.$$

The electric capacity of whole system of conductors is

$$C = \frac{Q}{\varphi_s} = \frac{4\pi\varepsilon_0 R^2}{R-r} = 4.4\cdot 10^{-11}\,\text{F} = 44\text{pF}$$

The equivalent electric scheme consists of two parallel capacitors: 1) a spherical one with charges +*q* and –*q* at the plates and 2) a capacitor "sphere – Earth" with charges +(*Q*-*q*) and –(*Q*–*q*) at the plates (Fig.5).

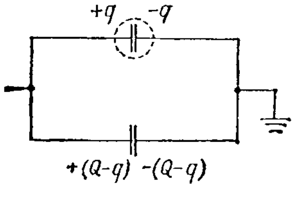

Fig. 5

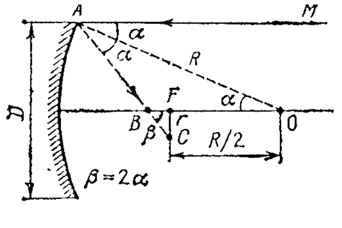

Fig. 6

Problem 4.

As known, rays parallel to the main optical axis of a spherical mirror, passing at little distances from it after having been reflected, join at the main focus of the mirror *F* which is at the distance *R*/2 from the centre *O* of the spherical surface. Let us consider now the movement of the ray reflected near the edge of the spherical mirror of large diameter *D* (Fig. 6). The angle of incidence α of the ray onto the surface is equal to the angle of reflection. That is why the angle *OAB* within the triangle, formed by the radius *OA* of the sphere, traced to the incidence point of the ray by the reflected ray *AB* and an intercept BO of the main optical axis, is equal to α. The angles BOA and MAO are equal, that is the angle BOA is equal to α.

Thus, the triangle AOB is isosceles with its side AB being equal to the side BO. Since the sum of the lengths of its two other sides exceeds the length of its third side, $AB+BO>OA=R$, *hence* $BO>R/2$. This means that a ray parallel to the main optical axis of the spherical mirror and passing not too close to it, after having been reflected, crosses the main optical axis at the point *B* lying between the focus *F* and the mirror. The focal surface is crossed by this ray at the point *C* which is at a certain distance *CF* = r from the main focus.

Thus, when reflecting a parallel beam of rays by a spherical mirror finite in size it does not join at the focus of the mirror but forms a beam with radius *r* on the focal plane.

From $\Delta\, BFC$ we can write :

$$r = BF \operatorname{tg} \beta = BF \operatorname{tg} 2\alpha ,$$

where α is the maximum angle of incidence of the extreme ray onto the mirror, while $\sin \alpha = D/2R$:

$$BF = BO - OF = \frac{R}{2\cos\alpha} - \frac{R}{2} = \frac{R}{2}\frac{1-\cos\alpha}{\cos\alpha}.$$

Thus, $r = \frac{R}{2}\frac{1-\cos\alpha}{\cos\alpha}\frac{\sin 2\alpha}{\cos 2\alpha}$. Let us express the values of $\cos\alpha$, $\sin 2\alpha$, $\cos 2\alpha$ via $\sin\alpha$ taking into account the small value of the angle α:

$$\cos\alpha = \sqrt{1-\sin^2\alpha} \approx 1 - \frac{\sin^2\alpha}{2},$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha ,$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha .$$

Then

$$r = \frac{R}{2}\frac{\sin^3\alpha}{1-2\sin^2\alpha} \approx \frac{R}{2}\sin^3\alpha \approx \frac{D^3}{16R^2}.$$

Substituting numerical data we will obtain: $r \approx 1.95\cdot 10^{-3}$ m $\approx$ 2mm .

From the expression $D = \sqrt[3]{16R^2 r}$ one can see that if the radius of the receiver is decreased 8 times the transversal diameter *D'* of the mirror, from which the light comes to the receiver, will be decreased 2 times and thus the "effective" area of the mirror will be decreased 4 times.

The radiation flux Φ reflected by the mirror and received by the receiver will also be decreased twice since $\Phi \sim S$.

Solution of the Experimental Problem

While looking at objects through lenses it is easy to establish that there were given two converging lenses and a diverging one.

The peculiarity of the given problem is the absence of a white screen on the list of the equipment that is used to observe real images. The competitors were supposed to determine the position of the images by the parallance method observing the images with their eyes.

The focal distance of the converging lens may be determined by the following method.

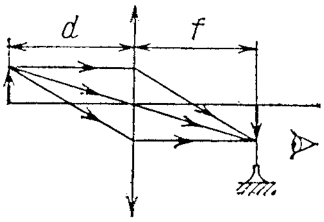

Fig. 7

Using a lens one can obtain a real image of a geometrical figure shown on the screen. The position of the real image is registered by the parallax method: if one places a vertical wire (Fig.7) to the point, in which the image is located, then at small displacements of the eye from the main optical axis of the lens the image of this object and the wire will not diverge.

We obtain the value of focal distance *F* from the formula of thin lens by the measured distances *d* and *f* :

$$\frac{1}{F_{1,2}} = \frac{1}{d} + \frac{1}{f}; \qquad F_{1,2} = \frac{df}{d+f}\ .$$

In this method the best accuracy is achieved in the case of

$$f = d.$$

The competitors were not asked to make a conclusion.

The error of measuring the focal distance for each of the two converging lenses can be determined by multiple repeated measurements. The total number of points was given to those competitors who carried out not less fewer than n=5 measurements of the focal distance and estimated the mean value of the focal distance Fav:

$$F_{\text{av}} = \frac{1}{n}\sum_{1}^{n} F_i$$

and the absolute error ΔF

$$\Delta F = \frac{1}{n}\sum_{1}^{n} \Delta F_i\,, \qquad \Delta F_i = \left|F_i - F_{\text{av}}\right|$$

or root mean square error ΔF_{rms}

$$\Delta F_{\text{rms}} = \frac{1}{n}\sqrt{\sum\left(\Delta F_i\right)^2}\,.$$

One could calculate the error by graphic method.

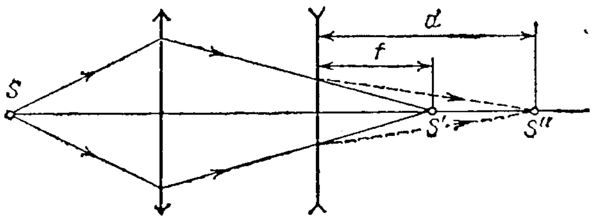

Fig. 8

Determination of the focal distance of the diverging lens can be carried out by the method of compensation. With this goal one has to obtain a real image *S*' of the object *S* using a converging lens. The position of the image can be registered using the parallax method.

If one places a diverging lens between the image and the converging lens the image will be displaced. Let us find a new position of the image *S*". Using the reversibility property of the light rays, one can admit that the light rays leave the point *S*". Then point *S*' is a virtual image of the point *S*", whereas the distances from the optical centre of the concave lens to the points *S*' and *S*" are, respectively, the distances *f* to the image and *d* to the object (Fig.8). Using the formula of a thin lens we obtain

$$\frac{1}{F_3} = -\frac{1}{f} + \frac{1}{d}; \qquad F_3 = -\frac{fd}{d-f} < 0\,.$$

Here F < 0 is the focal distance of the diverging lens. In this case the error of measuring the focal distance can also be estimated by the method of repeated measurements similar to the case of the converging lens.

Typical results are:

$F_1 = (22{,}0 \pm 0{,}4)cm$, $F_2 = (12{,}3 \pm 0{,}3)cm$, $F_3 = (-8{,}4 \pm 0{,}4)cm$.

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V International Physics Olympiad, 1971

Sofia, Bulgaria

The problems and the solutions are adapted by
Victor Ivanov
University of Sofia, Faculty of Physics, 5 James Bourchier Blvd., 1164 Sofia, Bulgaria

Reference: O. F. Kabardin, V. A. Orlov, in "International Physics Olympiads for High School Students", eds. V. G. Razumovski, Moscow, Nauka, 1985. (In Russian).

Theoretical problems

Question 1.

A triangular prism of mass M is placed one side on a frictionless horizontal plane as shown in Fig. 1. The other two sides are inclined with respect to the plane at angles α_1 and α_2 respectively. Two blocks of masses m_1 and m_2, connected by an inextensible thread, can slide without friction on the surface of the prism. The mass of the pulley, which supports the thread, is negligible.

- Express the acceleration a of the blocks relative to the prism in terms of the acceleration a_0 of the prism.
- Find the acceleration a_0 of the prism in terms of quantities given and the acceleration g due to gravity.
- At what ratio m_1/m_2 the prism will be in equilibrium?

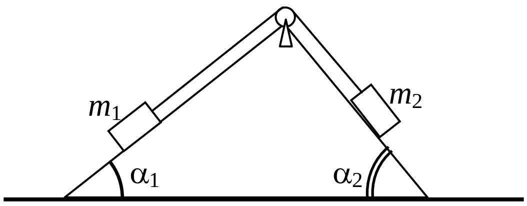


Fig. 1

Question 2.

A vertical glass tube of cross section S = 1.0 cm^2 contains unknown amount of hydrogen. The upper end of the tube is closed. The other end is opened and is immersed in a pan filled with mercury. The tube and the pan are placed in a sealed chamber containing air at temperature T_0 = 273 K and pressure P_0 = 1.334×10^5 Pa. Under these conditions the height of mercury column in the tube above the mercury level in the pan is h_0 = 0.70 m.

One of the walls of the chamber is a piston, which expands the air isothermally to a pressure of P_1 = 8.00×10^4 Pa. As a result the height of the mercury column in the tube decreases to h_1 = 0.40 m. Then the chamber is heated up at a constant volume to some temperature T_2 until the mercury column rises to h_2 = 0.50 m. Finally, the air in the chamber is expanded at constant pressure and the mercury level in the tube settles at h_3 = 0.45 m above the mercury level in the pan.

Provided that the system is in mechanical and thermal equilibrium during all the processes calculate the mass m of the hydrogen, the intermediate temperature T_2, and the pressure P in the final state.

The density of mercury at temperature T_0 is $\rho_0 = 1.36\times10^4$ kg/m^3, the coefficient of expansion for mercury $\beta = 1.84\times10^{-4}$ K^{-1}, and the gas constant $R = 8.314$ J/(mol×K). The thermal expansion of the glass tube and the variations of the mercury level in the pan are not considered.

Hint. If ΔT is the interval of temperature variations of the system then $\beta\Delta T = x << 1$ In that case you can use the approximation: $\frac{1}{1+x} \approx 1 - x$.

Question 3.

Four batteries of EMF $E_1 = 4$ V, $E_2 = 8$ V, $E_3 = 12$ V, and $E_4 = 16$ V, four capacitors with the same capacitance $C_1 = C_2 = C_3 = C_4 = 1$ µF, and four equivalent resistors are connected in the circuit shown in Fig. 3. The internal resistance of the batteries is negligible.

- Calculate the total energy W accumulated on the capacitors when a steady state of the system is established.
- The points H and B are short connected. Find the charge on the capacitor C_2 in the new steady state.

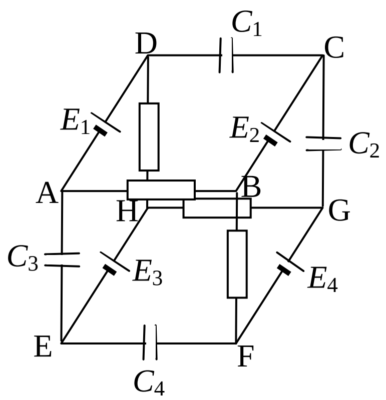


Fig. 3

Question 4.

A spherical aquarium, filled with water, is placed in front of a flat vertical mirror. The radius of the aquarium is R, and the distance between its center and the mirror is $3R$. A small fish, which is initially at the point nearest to the mirror, starts to move with velocity v along the wall. An observer looks at the fish from a very large distance along a horizontal line passing trough the center of the aquarium.

What is the relative velocity v_{rel} at which the two images of the fish seen by the observer will move apart? Express your answer in terms of v. Assume that:

- The wall of the aquarium is made of a very thin glass.
- The index of refraction of water is $n = 4/3$.

Experimental Problem

Apparatus: dc source, ammeter, voltmeter, rheostat (coil of high resistance wire with sliding contact), and connecting wires.

Problem: Construct appropriate circuit and establish the dependence of the electric power P dissipated in the rheostat as a function of the current I supplied by the dc source.

1. Make a plot of P versus I.
2. Find the internal resistance of the dc source.
3. Determine the electromotive force E of the source.
4. Make a graph of the electric power P versus resistance R of the rheostat.
5. Make a graph of the total power P_{tot} dissipated in the circuit as a function of R.
6. Make a graph of the efficiency η of the dc source versus R.

Solutions to the problems of the 5-th International Physics Olympiad, 1971, Sofia, Bulgaria

The problems and the solutions are adapted by
Victor Ivanov
Sofia State University, Faculty of Physics, 5 James Bourcier Blvd., 1164 Sofia, Bulgaria

Reference: O. F. Kabardin, V. A. Orlov, in "International Physics Olympiads for High School Students", eds. V. G. Razumovski, Moscow, Nauka, 1985. (In Russian).

Theoretical problems

Question 1.

The blocks slide relative to the prism with accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$, which are parallel to its sides and have the same magnitude a (see Fig. 1.1). The blocks move relative to the earth with accelerations:

(1.1) $\mathbf{w}_1 = \mathbf{a}_1 + \mathbf{a}_0$;

(1.2) $\mathbf{w}_2 = \mathbf{a}_2 + \mathbf{a}_0$.

Now we project $\mathbf{w}_1$ and $\mathbf{w}_2$ along the x- and y-axes:

(1.3) $w_{1x} = a\cos\alpha_1 - a_0$;

(1.4) $w_{1y} = a\sin\alpha_1$;

(1.5) $w_{2x} = a\cos\alpha_2 - a_0$;

(1.6) $w_{2y} = -a\sin\alpha_2$.

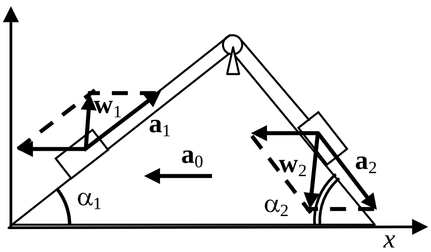


Fig. 1.1

The equations of motion for the blocks and for the prism have the following vector forms (see Fig. 1.2):

(1.7) $m_1\mathbf{w}_1 = m_1\mathbf{g} + \mathbf{R}_1 + \mathbf{T}_1$;

(1.8) $m_2\mathbf{w}_2 = m_2\mathbf{g} + \mathbf{R}_2 + \mathbf{T}_2$;

(1.9) $M\mathbf{a}_0 = M\mathbf{g} - \mathbf{R}_1 - \mathbf{R}_2 + \mathbf{R} - \mathbf{T}_1 - \mathbf{T}_2$.

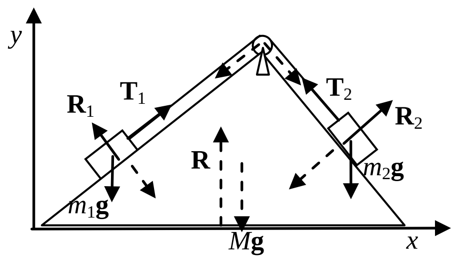


Fig. 1.2

The forces of tension $\mathbf{T}_1$ and $\mathbf{T}_2$ at the ends of the thread are of the same magnitude T since the masses of the thread and that of the pulley are negligible. Note that in equation (1.9) we account for the net force $-(\mathbf{T}_1 + \mathbf{T}_2)$, which the bended thread exerts on the prism through the pulley. The equations of motion result in a system of six scalar equations when projected along x and y:

(1.10) $$m_1 a\cos\alpha_1 - m_1 a_0 = T\cos\alpha_1 - R_1\sin\alpha_1;$$

(1.11) $$m_1 a\sin\alpha_1 = T\sin\alpha_1 + R_1\cos\alpha_1 - m_1 g;$$

(1.12) $$m_2 a\cos\alpha_2 - m_2 a_0 = -T\cos\alpha_2 + R_2\sin\alpha_2;$$

(1.13) $$m_2 a\sin\alpha_2 = T\sin\alpha_2 + R_2\sin\alpha_2 - m_2 g;$$

(1.14) $$-Ma_0 = R_1\sin\alpha_1 - R_2\sin\alpha_2 - T\cos\alpha_1 + T\cos\alpha_2;$$

(1.15) $$0 = R - R_1\cos\alpha_1 - R_2\cos\alpha_2 - Mg.$$

By adding up equations (1.10), (1.12), and (1.14) all forces internal to the system cancel each other. In this way we obtain the required relation between accelerations a and a_0:

(1.16) $$a = a_0\frac{M + m_1 + m_2}{m_1\cos\alpha_1 + m_2\cos\alpha_2}.$$

The straightforward elimination of the unknown forces gives the final answer for a_0:

(1.17) $$a_0 = \frac{(m_1\sin\alpha_1 - m_2\sin\alpha_2)(m_1\cos\alpha_1 + m_2\cos\alpha_2)}{(m_1 + m_2 + M)(m_1 + m_2) - (m_1\cos\alpha_1 + m_2\cos\alpha_2)^2}.$$

It follows from equation (1.17) that the prism will be in equilibrium ($a_0 = 0$) if:

(1.18) $$\frac{m_1}{m_2} = \frac{\sin\alpha_2}{\sin\alpha_1}.$$

Question 2.

We will denote by H (H = const) the height of the tube above the mercury level in the pan, and the height of the mercury column in the tube by h_i. Under conditions of mechanical equilibrium the hydrogen pressure in the tube is:

(2.1) $$P_{H_2} = P_{air} - \rho g h_i,$$

where ρ is the density of mercury at temperature t_i:

(2.2) $$\rho = \rho_0(1 - \beta t)$$

The index i enumerates different stages undergone by the system, ρ_0 is the density of mercury at t_0 = 0 °C, or T_0 = 273 K, and β its coefficient of expansion. The volume of the hydrogen is given by:

(2.3) $$V_i = S(H - h_i).$$

Now we can write down the equations of state for hydrogen at points 0, 1, 2, and 3 of the PV diagram (see Fig. 2):

(2.4) $$(P_0 - \rho_0 g h_0)S(H - h_0) = \frac{m}{M}RT_0;$$

(2.5) $$(P_1 - \rho_0 g h_1)S(H - h_1) = \frac{m}{M}RT_0;$$

(2.6) $$(P_2 - \rho_1 g h_2)S(H - h_2) = \frac{m}{M}RT_2,$$

where $P_2 = \frac{P_1 T_2}{T_0}$, $\rho_1 = \frac{\rho_0}{1 + \beta(T_2 - T_0)} \approx \rho_0[1 - \beta(T_2 - T_0)]$ since the process 1–3 is isochoric, and:

(2.7) $$(P_2 - \rho_2 g h_3) S (H - h_3) = \frac{m}{M} R T_3$$

where $\rho_2 \approx \rho_0 \left[1 - \beta(T_3 - T_0)\right]$, $T_3 = T_2 \dfrac{V_3}{V_2} = T_2 \dfrac{H - h_3}{H - h_2}$ for the isobaric process 2–3.

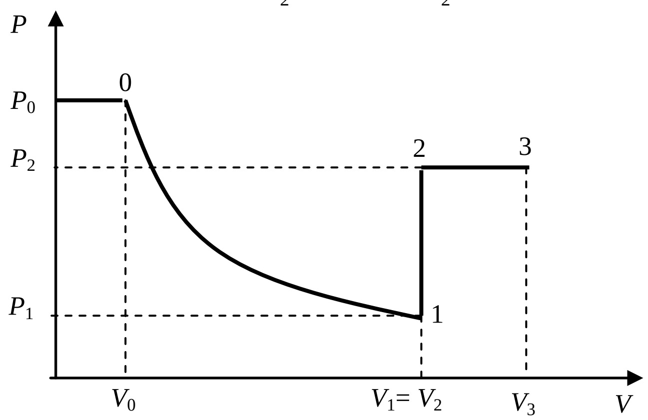


Fig. 2

After a good deal of algebra the above system of equations can be solved for the unknown quantities, an exercise, which is left to the reader. The numerical answers, however, will be given for reference:

$H \approx 1.3$ m;
$m \approx 2.11\times10^{-6}$ kg;
$T_2 \approx 364$ K;
$P_2 \approx 1.067\times10^{5}$ Pa;
$T_3 \approx 546$ K;
$P_2 \approx 4.8\times10^{4}$ Pa.

Question 3.

A circuit equivalent to the given one is shown in Fig. 3. In a steady state (the capacitors are completely charged already) the same current I flows through all the resistors in the closed circuit ABFGHDA. From the Kirchhoff's second rule we obtain:

(3.1) $$I = \frac{E_4 - E_1}{4R}.$$

Next we apply this rule for the circuit ABCDA:

(3.2) $$V_1 + IR = E_2 - E_1,$$

where V_1 is the potential difference across the capacitor C_1. By using the expression (3.1) for I, and the equation (3.2) we obtain:

(3.3) $$V_1 = E_2 - E_1 - \frac{E_4 - E_1}{4} = 1\,\mathrm{V}.$$

Similarly, we obtain the potential differences V_2 and V_4 across the capacitors C_2 and C_4 by considering circuits BFGCB and FGHEF:

(3.4) $$V_2 = E_4 - E_2 - \frac{E_4 - E_1}{4} = 5\,\mathrm{V},$$

(3.5) $$V_4 = E_4 - E_3 - \frac{E_4 - E_1}{4} = 1\,\text{V}.$$

Finally, the voltage V_3 across C_3 is found by applying the Kirchhoff's rule for the outermost circuit EHDAH:

(3.6) $$V_3 = E_3 - E_1 - \frac{E_4 - E_1}{4} = 5\,\text{V}.$$

The total energy of the capacitors is expressed by the formula:

(3.7) $$W = \frac{C}{2}\left(V_1^2 + V_2^2 + V_3^2 + V_4^2\right) = 26\,\mu\text{J}.$$

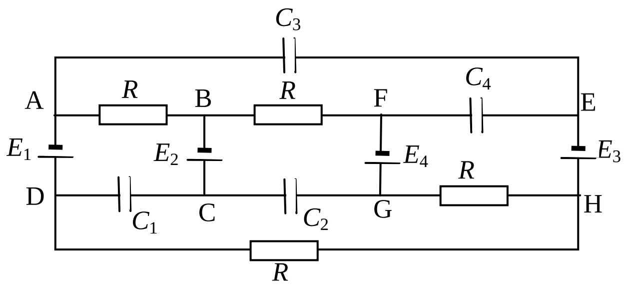


Fig. 3

When points B and H are short connected the same electric current *I'* flows through the resistors in the BFGH circuit. It can be calculated, again by means of the Kirchhoff's rule, that:

(3.8) $$I' = \frac{E_4}{2R}.$$

The new steady-state voltage on C_2 is found by considering the BFGCB circuit:

(3.9) $$V_2' + I'R = E_4 - E_2$$

or finally:

(3.10) $$V_2' = \frac{E_4}{2} - E_2 = 0\,\text{V}.$$

Therefore the charge q_2' on C_2 in the new steady state is zero.

Question 4.

In a small time interval Δt the fish moves upward, from point *A* to point *B*, at a small distance $d = v\Delta t$. Since the glass wall is very thin we can assume that the rays leaving the aquarium refract as if there was water – air interface. The divergent rays undergoing one single refraction, as show in Fig. 4.1, form the first, virtual, image of the fish. The corresponding vertical displacement A_1B_1 of that image is equal to the distance d_1 between the optical axis *a* and the ray b_1, which leaves the aquarium parallel to *a*. Since distances *d* and d_1 are small compared to *R* we can use the small-angle approximation: $\sin\alpha \approx \tan\alpha \approx \alpha$ (rad). Thus we obtain:

(4.1) $d_1 \approx R\,\alpha;$

(4.2) $d \approx R\,\gamma;$

(4.3) $\alpha + \gamma = 2\beta;$

(4.4) $\alpha \approx n\beta.$

From equations (4.1) - (4.4) we find the vertical displacement of the first image in terms of d:

(4.5) $$d_1 = \frac{n}{2-n} d \,,$$

and respectively its velocity v_1 in terms of v:

(4.6) $$v_1 = \frac{n}{2-n} = 2v \,.$$

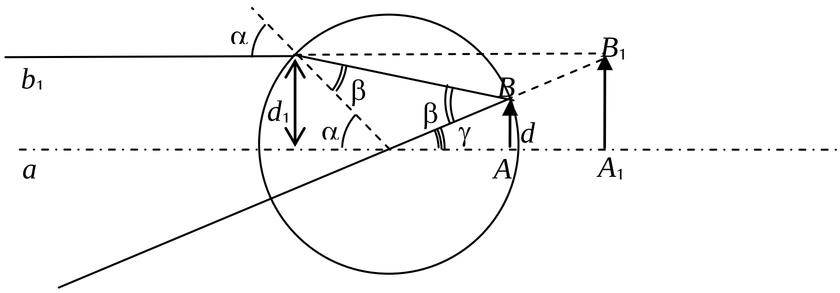


Fig. 4.1

The rays, which are first reflected by the mirror, and then are refracted twice at the walls of the aquarium form the second, real image (see Fig. 4.2). It can be considered as originating from the mirror image of the fish, which move along the line *A'B'* at exactly the same distance d as the fish do.

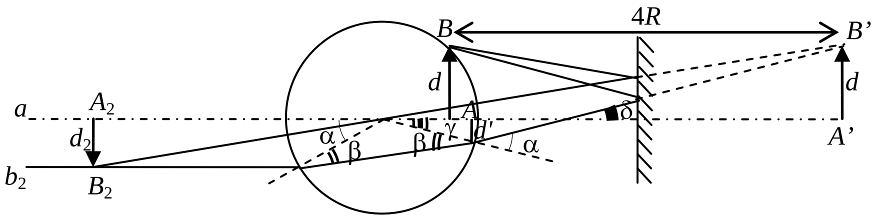


Fig. 4.2

The vertical displacement A_2B_2 of the second image is equal to the distance d_2 between the optical axis a and the ray b_2, which is parallel to a. Again, using the small-angle approximation we have:

(4.7) $d' \approx 4R\delta - d$,

(4.8) $d_2 \approx R\alpha$

Following the derivation of equation (4.5) we obtain:

(4.9) $$d_2 = \frac{n}{2-n} d' \,.$$

Now using the exact geometric relations:

(4.10) $\delta = 2\alpha - 2\beta$

and the Snell's law (4.4) in a small-angle limit, we finally express d_2 in terms of d:

(4.11) $$d_2 = \frac{n}{9n-10} d ,$$

and the velocity v_2 of the second image in terms of v:

(4.12) $$v_2 = \frac{n}{9n-10} v = \frac{2}{3} v .$$

The relative velocity of the two images is:

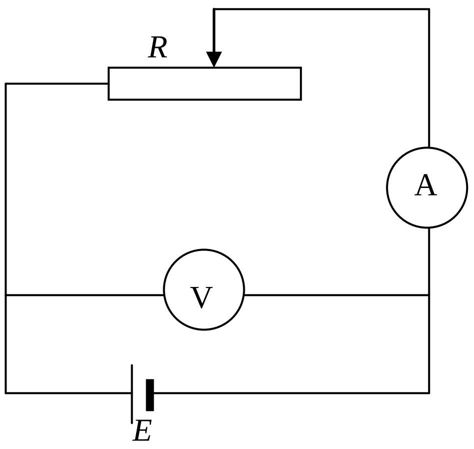
(4.13) $$\mathbf{v}_{rel} = \mathbf{v}_1 - \mathbf{v}_2$$

in a vector form. Since vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are oppositely directed (one of the images moves upward, the other, downward) the magnitude of the relative velocity is:

(4.14) $$v_{rel} = v_1 + v_2 = \frac{8}{3} v .$$

Experimental problem

The circuit is given in the figure below:

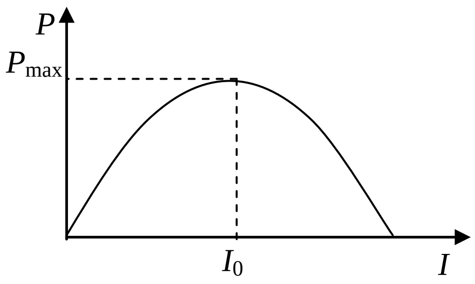


Sliding the contact along the rheostat sets the current I supplied by the source. For each value of I the voltage U across the source terminals is recorded by the voltmeter. The power dissipated in the rheostat is:

$$P = UI$$

provided that the heat losses in the internal resistance of the ammeter are negligible.

1. A typical P–I curve is shown below:



If the current varies in a sufficiently large interval a maximum power P_{max} can be detected at a certain value, I_0, of I. Theoretically, the $P(I)$ dependence is given by:

(5.1) $$P = EI - I^2 r,$$

where E and r are the EMF and the internal resistance of the dc source respectively. The maxim value of P therefore is:

(5.2) $$P_{max} = \frac{E^2}{4r},$$

and corresponds to a current:

(5.3) $$I_0 = \frac{E}{2r}.$$

2. The internal resistance is determined trough (5.2) and (5.3) by recording P_{max} and I_0 from the experimental plot:

$$r = \frac{P_{max}}{I_0^2} \quad .$$

3. Similarly, EMF is calculated as:

$$E = \frac{2P_{max}}{I_0}.$$

4. The current depends on the resistance of the rheostat as:

$$I = \frac{E}{R+r}.$$

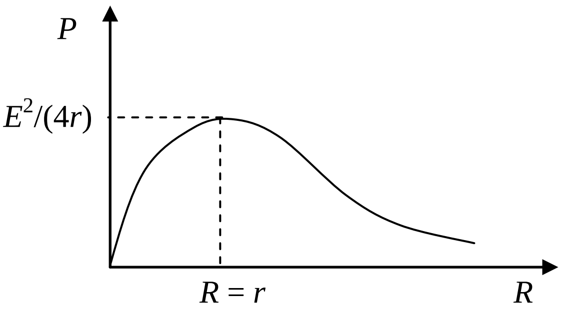
Therefore a value of R can be calculated for each value of I:

(5.4) $$R = \frac{E}{I} - r.$$

The power dissipated in the rheostat is given in terms of R respectively by:

(5.5) $$P = \frac{E^2 R}{(R+r)^2}.$$

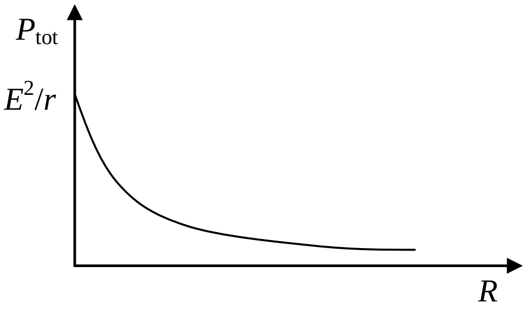
The P–R plot is given below:



Its maximum is obtained at $R = r$.

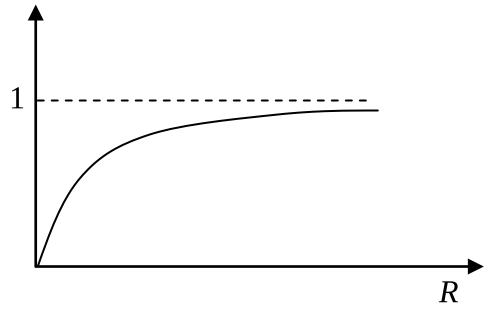
5. The total power supplied by the dc source is:

(5.6) $$P_{tot} = \frac{E^2}{R+r}.$$



6. The efficiency respectively is:

(5.7) $$\eta = \frac{P}{P_{tot}} = \frac{R}{R + r}.$$



Problems of the 6th International Physics Olympiad
(Bucharest, 1972)

Romulus Pop
Civil Engineering University, Physics Department[1]
Bucharest, Romania

The sixth IPhO was held in Bucharest and the participants were: Bulgaria, Czechoslovakia, Cuba, France, German Democratic Republic, Hungary, Poland, Romania, and Soviet Union. It was an important event because it was the first time when a non-European country and a western country participated (Cuba), and Sweden sent one observer.

The International Board selected four theoretical problems and an experimental problem. Each theoretical problem was scored from 0 to 10 and the maximum score for the experimental problem was 20. The highest score corresponding to actual marking system was 47,5 points. Each team consisted in six students. Four students obtained the first prize, seven students obtained the second prize, ten students obtained the third prize, thirteen students had got honorable mentions, and two special prizes were awarded too.

The article contains the competition problems given at the 6th International Physics Olympiad (Bucharest, 1972) and their solutions. The problems were translated from the book published in Romania concerning the first nine International Physics Olympiads[2], because I couldn't find the original English version.

Theoretical problems

Problem 1 (Mechanics)

Three cylinders with the same mass, the same length and the same external radius are initially resting on an inclined plane. The coefficient of sliding friction on the inclined plane, μ, is known and has the same value for all the cylinders. The first cylinder is empty (tube) , the second is homogeneous filled, and the third has a cavity exactly like the first, but closed with two negligible mass lids and filled with a liquid with the same density like the cylinder's walls. The friction between the liquid and the cylinder wall is considered negligible. The density of the material of the first cylinder is n times greater than that of the second or of the third cylinder.

Determine:

a) The linear acceleration of the cylinders in the non-sliding case. Compare all the accelerations.

b) Condition for angle α of the inclined plane so that no cylinders is sliding.

c) The reciprocal ratios of the angular accelerations in the case of roll over with sliding of all the three cylinders. Make a comparison between these accelerations.

d) The interaction force between the liquid and the walls of the cylinder in the case of sliding of this cylinder, knowing that the liquid mass is m_l.

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[1] E-mail: popr@tvet.ro

[2] Marius Gall and Anatolie Hristev, Probleme date la Olimpiadele de Fizica, Editura Didactica si Pedagogica – Bucuresti, 1978

Solution Problem 1

The inertia moments of the three cylinders are:

$$I_1 = \frac{1}{2}\rho_1\pi\left(R^4 - r^4\right)h, \quad I_2 = \frac{1}{2}\rho_2\pi R^4 h = \frac{1}{2}mR^2 \quad , \quad I3 = \frac{1}{2}\rho_2\pi\left(R^4 - r^4\right)h, \qquad (1)$$

Because the three cylinders have the same mass :

$$m = \rho_1\pi\left(R^2 - r^2\right)h = \rho_2\pi R^2 h \qquad (2)$$

it results:

$$r^2 = R^2\left(1 - \frac{\rho_2}{\rho_1}\right) = R^2\left(1 - \frac{1}{n}\right), n = \frac{\rho_1}{\rho_2} \qquad (3)$$

The inertia moments can be written:

$$I_1 = I_2\left(2 - \frac{1}{n}\right)\rangle I_2 \ , \quad I_3 = I_2\left(2 - \frac{1}{n}\right)\cdot\frac{1}{n} = \frac{I_1}{n} \qquad (4)$$

In the expression of the inertia momentum I_3 the sum of the two factors is constant:

$$\left(2 - \frac{1}{n}\right) + \frac{1}{n} = 2$$

independent of n, so that their products are maximum when these factors are equal: $2 - \frac{1}{n} = \frac{1}{n}$; it results n = 1, and the products $\left(2 - \frac{1}{n}\right)\cdot\frac{1}{n} = 1$. In fact n > 1, so that the products is les than 1. It results:

$$I_1 > I_2 > I_3 \qquad (5)$$

For a cylinder rolling over freely on the inclined plane (fig. 1.1) we can write the equations:

$$mg\sin\alpha - F_f = ma \qquad (6)$$

$$N - mg\cos\alpha = 0$$

$$F_f R = I\varepsilon \qquad (7)$$

where ε is the angular acceleration. If the cylinder doesn't slide we have the condition:

$$a = \varepsilon R \qquad (8)$$

Solving the equation system (6-8) we find:

$$a = \frac{g\sin\alpha}{1 + \frac{I}{mR^2}} \quad , \quad F_f = \frac{mg\sin\alpha}{1 + \frac{mR^2}{I}} \qquad (9)$$

The condition of non-sliding is:

$$F_f < \mu N = \mu mg\sin\alpha$$

$$\mathrm{tg}\alpha < \mu\left(1 + \frac{mR^2}{I_1}\right) \qquad (10)$$

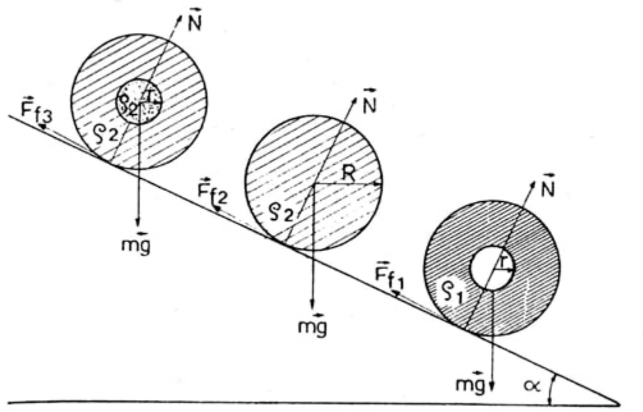

Fig. 1.1

In the case of the cylinders from this problem, the condition necessary so that none of them slides is obtained for maximum I:

$$tg\alpha \langle \mu\left(1 + \frac{mR^2}{I_1}\right) = \mu\frac{4n-1}{2n-1} \qquad (11)$$

The accelerations of the cylinders are:

$$a_1 = \frac{2g\sin\alpha}{3 + (1 - \frac{1}{n})} \quad , \quad a_2 = \frac{2g\sin\alpha}{3} \quad , \quad a_3 = \frac{2g\sin\alpha}{3 - (1 - \frac{1}{n})^2}. \qquad (12)$$

The relation between accelerations:

$$a_1 < a_2 < a_3 \qquad (13)$$

In the case than all the three cylinders slide:

$$F_f = \mu N = \mu mg\cos\alpha \qquad (14)$$

and from (7) results:

$$\varepsilon = \frac{R}{I}\mu mg\cos\alpha \qquad (15)$$

for the cylinders of the problem:

$$\varepsilon_1 : \varepsilon_2 : \varepsilon_3 = \frac{1}{I_1} : \frac{1}{I_2} : \frac{1}{I_3} = 1 : \left(1 - \frac{1}{n}\right) : n$$

$$\varepsilon_1 < \varepsilon_2 < \varepsilon_3 \qquad (16)$$

In the case that one of the cylinders is sliding:

$$mg\sin\alpha - F_f = ma, \quad F_f = \mu mg\cos\alpha, \qquad (17)$$

$$a = g(\sin\alpha - \mu\cos\alpha) \qquad (18)$$

Let $\vec{F}$ be the total force acting on the liquid mass m_l inside the cylinder (fig.1.2), we can write:

$$F_x + m_l g\sin\alpha = m_l a = m_l g(\sin\alpha - \mu\cos\alpha), \quad F_y - m_l g\cos\alpha = 0 \quad (19)$$

$$F = \sqrt{F_x^2 + F_y^2} = m_l g\cos\alpha \cdot \sqrt{1+\mu^2} = m_l g\frac{\cos\alpha}{\cos\phi} \qquad (20)$$

where ϕ is the friction angle $(tg\phi = \mu)$.

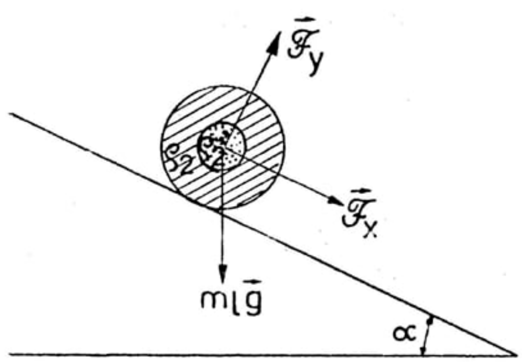

Fig. 1.2

Problem 2 (Molecular Physics)

Two cylinders A and B, with equal diameters have inside two pistons with negligible mass connected by a rigid rod. The pistons can move freely. The rod is a short tube with a valve. The valve is initially closed (fig. 2.1).

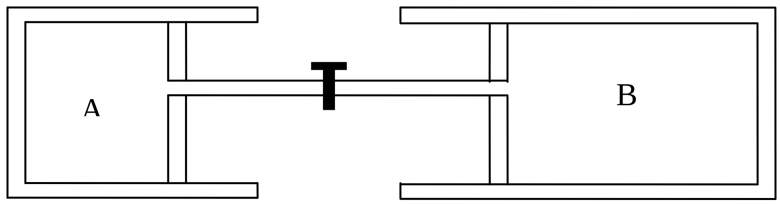

Fig. 2.1

The cylinder A and his piston is adiabatically insulated and the cylinder B is in thermal contact with a thermostat which has the temperature $\theta = 27^oC$.
Initially the piston of the cylinder A is fixed and inside there is a mass m= 32 kg of argon at a pressure higher than the atmospheric pressure. Inside the cylinder B there is a mass of oxygen at the normal atmospheric pressure.
Liberating the piston of the cylinder A, it moves slowly enough (quasi-static) and at equilibrium the volume of the gas is eight times higher, and in the cylinder B de oxygen's density increased two times. Knowing that the thermostat received the heat $Q^{'}=747{,}9.10^4$J, determine:

a) Establish on the base of the kinetic theory of the gases, studying the elastic collisions of the molecules with the piston, that the thermal equation of the process taking place in the cylinder A is $TV^{2/3}$ = constant.

b) Calculate the parameters p, V, and T of argon in the initial and final states.

c) Opening the valve which separates the two cylinders, calculate the final pressure of the mixture of the gases.

The kilo-molar mass of argon is μ = 40 kg/kmol.

Solution Problem 2

a) We consider argon an ideal mono-atomic gas and the collisions of the atoms with the piston perfect elastic. In such a collision with a fix wall the speed $\vec{v}$ of the particle changes only the direction so that the speed $\vec{v}$ and the speed $\vec{v}'$ after collision there are in the same plane with the normal and the incident and reflection angle are equal.

$$v_n^{'} = -v_n\,,\; v_t^{'} = v_t \qquad (1)$$

In the problem the wall moves with the speed $\vec{u}$ perpendicular on the wall. The relative speed of the particle with respect the wall is $\vec{v}-\vec{u}$. Choosing the Oz axis perpendicular on the wall in the sense of $\vec{u}$, the conditions of the elastic collision give:

$$(\vec{v}-\vec{u})_z = -(\vec{v}^{'}-\vec{u})_z\;,\;(\vec{v}-\vec{u})_{x,y} = (\vec{v}^{'}-\vec{u})_{x,y}\,;$$

$$v_z - u = -(v_z^{'}-u)\;,\; v_z^{'} = 2u - v_z\,,\; v_{x,y}^{'} = v_{x,y} \qquad (2)$$

The increase of the kinetic energy of the particle with mass m_o after collision is:

$$\frac{1}{2}m_o v^{'2} - \frac{1}{2}m_o v^2 = \frac{1}{2}m_o\left(v_z^{'2}-v_z^2\right) = 2m_o u(u - v_z) \cong -2m_o u v_z \qquad (3)$$

because u is much smaller than v_z.
If n_k is the number of molecules from unit volume with the speed component v_{zk}, then the number of molecules with this component which collide in the time dt the area dS of the piston is:

$$\frac{1}{2}n_k v_{zk} dtdS \quad (4)$$

These molecules will have a change of the kinetic energy:

$$\frac{1}{2}n_k v_{zk} dtdS(-2m_o u v_{zk}) = -m_o n_k v_{zk}^2 dV \quad (5)$$

where $dV = udtdS$ is the increase of the volume of gas.

The change of the kinetic energy of the gas corresponding to the increase of volume dV is:

$$dE_c = -m_o dV \sum_k n_k v_{zk}^2 = -\frac{1}{3}nm_o \bar{v}^2 dV \quad (6)$$

and:

$$dU = -\frac{2}{3}N\frac{m_o\bar{v}^2}{2}\cdot\frac{dV}{V} = -\frac{2}{3}U\frac{dV}{V} \quad (7)$$

Integrating equation (7) results:

$$UV^{2/3} = const. \quad (8)$$

The internal energy of the ideal mono-atomic gas is proportional with the absolute temperature T and the equation (8) can be written:

$$TV^{2/3} = const. \quad (9)$$

b) The oxygen is in contact with a thermostat and will suffer an isothermal process. The internal energy will be modified only by the adiabatic process suffered by argon gas:

$\Delta U = \nu C_V \Delta T = mc_V \Delta T$ (10)

where ν is the number of kilomoles. For argon $C_V = \frac{3}{2}R$.

For the entire system L=0 and $\Delta U = Q$.

We will use indices 1, respectively 2, for the measures corresponding to argon from cylinder A, respectively oxygen from the cylinder B:

$$\Delta U = \frac{m_1}{\mu_1}\cdot\frac{3}{2}\cdot R(T_1^{'} - T_1) = Q = \frac{m_1}{\mu_1}\cdot\frac{3}{2}RT_1\left[\left(\frac{V_1}{V_1^{'}}\right)^{2/3} - 1\right] \quad (11)$$

From equation (11) results:

$$T_1 = \frac{2}{3}\cdot\frac{\mu_1}{m_1}\cdot\frac{Q}{R}\cdot\frac{1}{\left(\frac{V_1}{V_1^{'}}\right)^{2/3} - 1} = 1000K \quad (12)$$

$$T_1^{'} = \frac{T_1}{4} = 250K \quad (13)$$

For the isothermal process suffered by oxygen:

$$\frac{\rho_2^{'}}{\rho_2} = \frac{p_2^{'}}{p_2} \quad (14)$$

$p_2^{'} = 2{,}00atm = 2{,}026\cdot 10^5 N/m^2$

From the equilibrium condition:

$$p_1^{'} = p_2^{'} = 2atm \qquad (15)$$

For argon:

$$p_1 = p_1^{'} \cdot \frac{V_1^{'}}{V_1} \cdot \frac{T_1}{T_1^{'}} = 64atm = 64{,}9 \cdot 10^5\, N/m^2 \quad (16)$$

$$V_1 = \frac{m_1}{\mu_1} \cdot \frac{RT_1}{p_1} = 1{,}02m^3, V_1^{'} = 8V_1 = 8{,}16m^3 \quad (17)$$

c) When the valve is opened the gases intermix and at thermal equilibrium the final pressure will be $p^{'}$ and the temperature T. The total number of kilomoles is constant:

$$\nu_1 + \nu_2 = \nu^{'},\ \frac{p_1^{'}V_1^{'}}{RT_1^{'}} + \frac{p_2^{'}V_2^{'}}{RT} = \frac{p\left(V_1^{'} + V_2^{'}\right)}{RT} \qquad (18)$$

$$p_1^{'} + p_2^{'} = 2atm, T_2 = T_2^{'} = T = 300K$$

The total volume of the system is constant:

$$V_1 + V_2 = V_1^{'} + V_2^{'},\quad \frac{V_2^{'}}{V_2} = \frac{\rho_2}{\rho_2^{'}},\quad V_2^{'} = \frac{V_2}{2} = 7{,}14m^3 \quad (19)$$

From equation (18) results the final pressure:

$$p = p_1^{'} \cdot \frac{1}{V_1 + V_2} \cdot \left(V_1^{'} \cdot \frac{T}{T_1^{'}} + V_2^{'}\right) = 2{,}2atm = 2{,}23 \cdot 10^5\, N/m^2 \quad (20)$$

Problem 3 (Electricity)

A plane capacitor with rectangular plates is fixed in a vertical position having the lower part in contact with a dielectric liquid (fig. 3.1)

Determine the height, h, of the liquid between the plates and explain the phenomenon.

The capillarity effects are neglected.

It is supposed that the distance between the plates is much smaller than the linear dimensions of the plates.

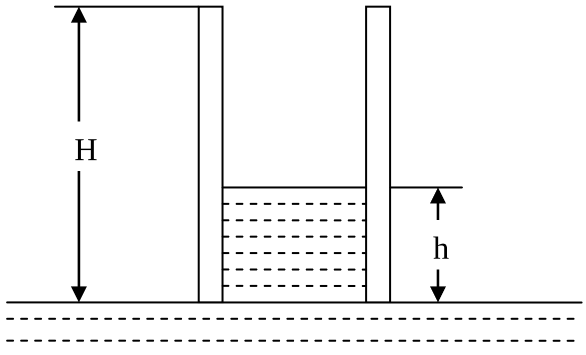

Fig. 3.1

It is known: the initial intensity of the electric field of the charged capacitor, E, the density ρ, the relative electric permittivity ε_r of the liquid, and the height H of the plates of the capacitor. Discussion.

Solution Problem 3

The initial energy on the capacitor is:

$$W_o = \frac{1}{2}\cdot C_o U_o^2 = \frac{1}{2}\cdot\frac{Q_o^2}{C_o}, \text{ where } \quad C_o = \frac{\varepsilon_o H l}{d} \quad (1)$$

H is the height of the plates, l is the width of the capacitor's plates, and d is the distance between the plates.
When the plates contact the liquid's surface on the dielectric liquid is exerted a vertical force. The total electric charge remains constant and there is no energy transferred to the system from outside. The increase of the gravitational energy is compensated by the decrease of the electrical energy on the capacitor:

$$W_o = W_1 + W_2 \qquad (2)$$

$$W_1 = \frac{1}{2}\cdot\frac{Q_o^2}{C}, \; W_2 = \frac{1}{2}\rho g h^2 l d \qquad (3)$$

$$C = C_1 + C_2 = \frac{\varepsilon_o \varepsilon_r h l}{d} + \frac{\varepsilon_o (H-h) l}{d} \qquad (4)$$

Introducing (3) and (4) in equation (2) it results:

$$(\varepsilon_r - 1)h^2 + Hh - \frac{E_o^2 \varepsilon_o H(\varepsilon_r - 1)}{\rho g} = 0$$

The solution is:

$$h_{1,2} = \frac{H}{2(\varepsilon_r - 1)}\cdot\left[-1 \pm \sqrt{1 \pm \frac{4E_o^2 \varepsilon_o (\varepsilon_r - 1)^2}{\rho g H}}\right] \qquad (8)$$

Discussion: Only the positive solution has sense. Taking in account that H is much more grater than h we obtain the final result:

$$h \approx \frac{\varepsilon_o(\varepsilon_r - 1)}{\rho g}\cdot E_o^2$$

Problem 4 (Optics)

A thin lens plane-convex with the diameter 2r, the curvature radius R and the refractive index n_o is positioned so that on its left side is air ($n_1 = 1$), and on its right side there is a transparent medium with the refractive index $n_2 \neq 1$. The convex face of the lens is directed towards air. In the air, at the distance $s_1$from the lens, measured on the principal optic ax, there is a punctual source of monochromatic light.
a) Demonstrate, using Gauss approximation, that between the position of the image, given by the distance s_2 from the lens, and the position of the light source, exists the relation:

$$\frac{f_1}{s_1} + \frac{f_2}{s_2} = 1$$

where f_1 and f_2 are the focal distances of the lens, in air, respectively in the medium with the refractive index n_2.
Observation: All the refractive indexes are absolute indexes.
b) The lens is cut perpendicular on its plane face in two equal parts. These parts are moved away at a distance $\delta << r$ (Billet lens). On the symmetry axis of the system obtained is led a punctual source of light at the distance s_1 ($s_1 > f_1$) (fig. 4.1). On the right side of the lens there is a screen E at the distance d. The screen is parallel with the plane face of the lens. On this screen there are N interference fringes, if on the right side of the lens is air.
Determine N function of the wave length.

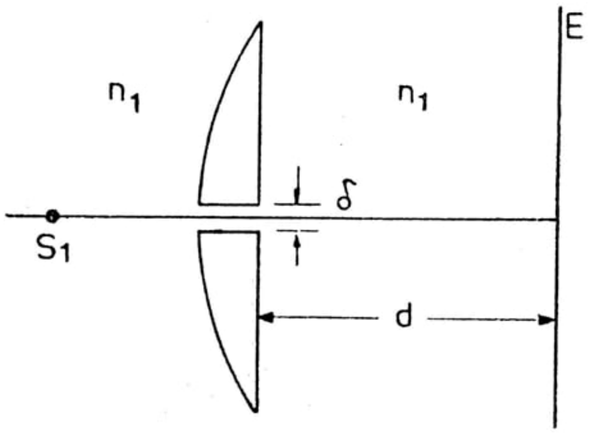

Fig. 4.1

Solution problem 4

a) From the Fermat principle it results that the time the light arrives from P_1 to P_2 is not dependent of the way, in gauss approximation (P_1 and P_2 are conjugated points).

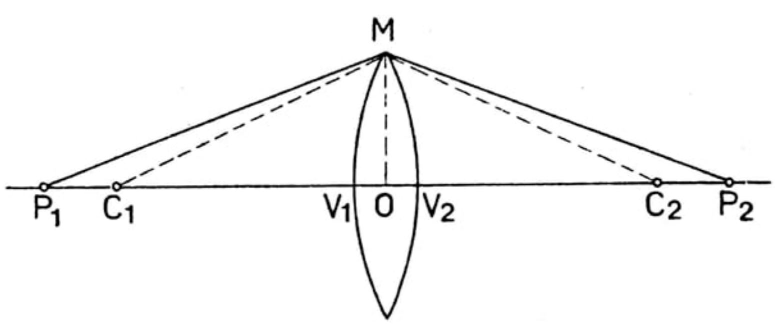

Fig. 4.2

T_1 is the time the light roams the optical way $P_1V_1OV_2P_2$ (fig. 4.2):

$$T_1 = \frac{P_1M}{v_1} + \frac{P_2M}{v_2}, \text{ where } P_1M = \sqrt{P_1O^2 + MO^2} \approx P_1O + \frac{h^2}{2P_1O}, \text{ and } P_2M \approx P_2O + \frac{h^2}{2P_2O}$$

because $h = OM$ is much more smaller than P_1O or P_2O.

$$T_1 = \frac{P_1O}{v_1} + \frac{P_2O}{v_2} + \frac{h^2}{2} \cdot \left(\frac{1}{v_1P_1O} + \frac{1}{v_2P_2O} \right); \; T_2 = \frac{P_1V_1}{v_1} + \frac{V_2P_2}{v_2} + \frac{V_1V_2}{v} \qquad (1)$$

$$V_1V_2 \cong \frac{h^2}{2}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \quad (2)$$

From condition $T_1 = T_2$**,** it results:

$$\frac{1}{v_1P_1O}+\frac{1}{v_2P_2O}=\frac{1}{v}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-\frac{1}{v_1R_1}-\frac{1}{v_2R_2} \quad (3)$$

Taking in account the relation $v=\frac{c}{n}$, and using $P_1O = s_1, OP_2 = s_2$, the relation (3) can be written:

$$\frac{n_1}{s_1}+\frac{n_2}{s_2}=n_o\left(\frac{1}{R_1}+\frac{1}{R_2}\right)-\frac{1}{v_1R_1}-\frac{1}{v_2R_2} \quad (4)$$

If the point P_1 is at infinite, s_2 becomes the focal distance; the same for P_2.

$$\frac{1}{f_2}=\frac{1}{n_2}\cdot\left(\frac{n_o-n_1}{R_1}+\frac{n_o-n_2}{R_2}\right);\quad \frac{1}{f_1}=\frac{1}{n_1}\cdot\left(\frac{n_o-n_1}{R_1}+\frac{n_o-n_2}{R_2}\right) \quad (5)$$

From the equations (30 and (4) it results:

$$\frac{f_1}{s_1}+\frac{f_2}{s_2}=1 \quad (6)$$

The lens is plane-convex (fig. 4.3) and its focal distances are:

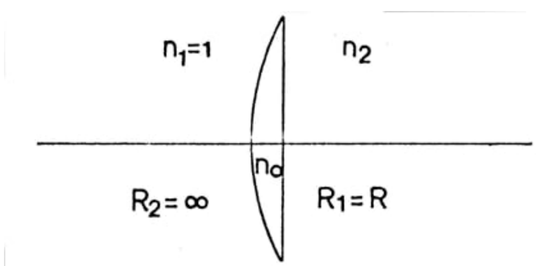

Fig. 4.3

$$f_1=\frac{n_1R}{n_o-n_1}=\frac{R}{n_o-1} \quad ; \quad f_2=\frac{n_2R}{n_o-n_1}=\frac{n_2R}{n_0-1} \quad (7)$$

b) In the case of Billet lenses, S_1 and S_2 are the real images of the object S and can be considered like coherent light sources (fig. 4.4).

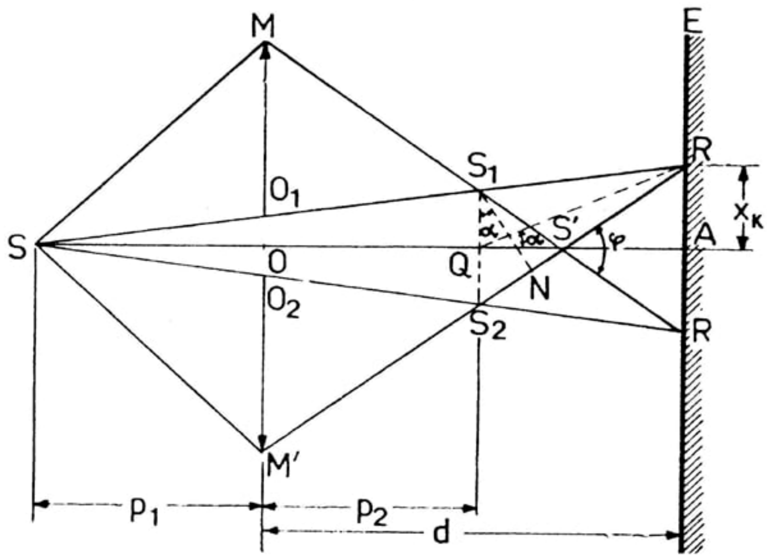

Fig. 4.4

$O_1O_2 = \Delta$ is much more smaller than r:

$$OM = \Delta + r \approx r\,,\;\; SO \approx SO_1 \approx SO_2 = p_1\,,\; S_1O_1 = S_2O_2 \approx S^{'}O = p_2\,,\; S_1S_2 = \Delta \cdot \left(1 + \frac{p_1}{p_2}\right)$$

We calculate the width of the interference field $RR^{'}$ (fig. 4.4).

$$RR^{'} = 2 \cdot RA = 2 \cdot S^{'}A \cdot tg\frac{\varphi}{2}\,,\;\; S^{'}A \cong d - p_2\,,\; tg\frac{\varphi}{2} = \frac{r}{p_2}\,,\;\;\; RR^{'} = 2(d - p_2)\cdot\frac{r}{p_2}$$

Maximum interference condition is:

$$S_2N = k \cdot \lambda$$

The fringe of k order is located at a distance x_k from A:

$$x_k = k \cdot \frac{\lambda(d - p_2)}{\Delta\left(1 + \frac{p_2}{p_1}\right)} \qquad (8)$$

The expression of the inter-fringes distance is:

$$i = \frac{\lambda(d - p_2)}{\Delta\left(1 + \frac{p}{p_1}\right)} \qquad (9)$$

The number of observed fringes on the screen is:

$$N = \frac{RR^{'}}{i} = 2r\Delta \cdot \frac{1 + \frac{p_2}{p_1}}{\lambda p_2} \qquad (10)$$

p_2 can be expressed from the lenses' formula:

$$p_2 = \frac{p_1 f}{p_1 - f}$$

Experimental part (Mechanics)

There are given two cylindrical bodies (having identical external shapes and from the same material), two measuring rules, one graduated and other un-graduated, and a vessel with water.

It is known that one of the bodies is homogenous and the other has an internal cavity with the following characteristics:

- the cavity is cylindrical
- has the axis parallel with the axis of the body
- its length is practically equal with that of the body

Determine experimentally and justify theoretically:

a) The density of the material the two bodies consist of.
b) The radius of the internal cavity.
c) The distance between the axis of the cavity and the axis of the cylinder.
d) Indicate the sources of errors and appreciate which of them influences more the final results.

Write all the variants you have found.

Solution of the experimental problem

a) Determination of the density of the material

The average density of the two bodies was chosen so that the bodies float on the water. Using the mass of the liquid crowded out it is determined the mass of the first body (the homogenous body):

$$m = m_a = V_a \rho_a = S_a H \rho_a \qquad (1)$$

where S_a is the area of the base immersed in water, H the length of the cylinder and ρ_a is the density of water.

The mass of the cylinder is:

$$m = V \cdot \rho = \pi R^2 H \rho \qquad (2)$$

It results the density of the body:

$$\rho = \rho_a \frac{S_a}{\pi R^2} \qquad (3)$$

To calculate the area S_a it is measured the distance h above the water surface (fig. 5.1). Area is composed by the area of the triangle OAB plus the area of the circular sector with the angle $2\pi - 2\theta$.

The triangle area:

$$\frac{1}{2} \cdot 2\sqrt{R^2 - (R-h)} \cdot (R-h) = (R-h)\sqrt{h(2R-h)} \qquad (4)$$

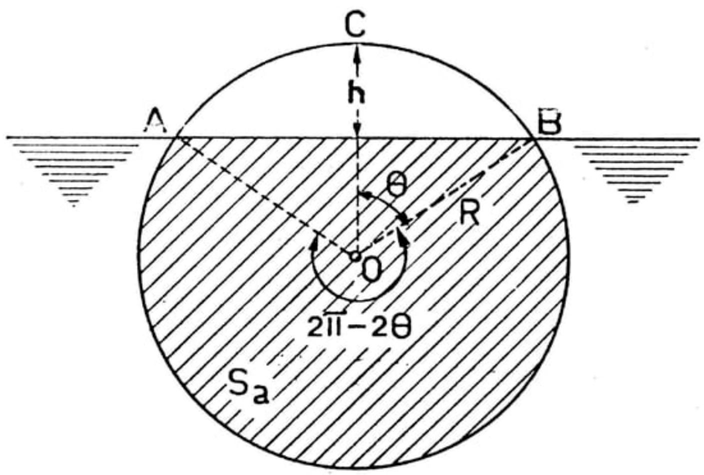

Fig. 5.1

The circular sector area is:

$$\frac{2(\pi-\theta)}{2\pi}\pi R^2 = R^2\left(\pi - \arccos\frac{R-h}{R}\right) \quad (5)$$

The immersed area is:

$$S_a = (R-h)\sqrt{h(2R-h)} + R^2\left(\pi - \arccos\frac{R-h}{R}\right) \quad (6)$$

where R and h are measured by the graduated rule.

b) The radius of the cylindrical cavity

The second body (with cavity) is dislocating a water mass:

$$m' = m_a' = S_a' H\rho_a \quad (7)$$

where S_a' is area immersed in water.

The mass of the body having the cavity inside is:

$$m' = (V-v)\rho = \pi(R^2 - r^2)H\rho \quad (8)$$

The cavity radius is:

$$r = \sqrt{R^2 - \frac{\rho_a}{\pi\rho}\cdot S_a'} \quad (9)$$

S_a' is determined like S_a.

c) The distance between the cylinder's axis and the cavity axis

We put the second body on the horizontal table (or let it to float in water) and we trace the vertical symmetry axis AB (fig. 5.2).

Using the rule we make an inclined plane. We put the body on this plane and we determine the maximum angle of the inclined plane for the situation the body remains in rest (the body doesn't roll). Taking in account that the weight centre is located on the axis AB on the left side of the cylinder axis (point G in fig. 5.2) and that at equilibrium the weight centre is on the same vertical with the contact point between the cylinder and the inclined plane, we obtain the situation corresponding to the maximum angle of the inclined plane (the diameter AB is horizontal).

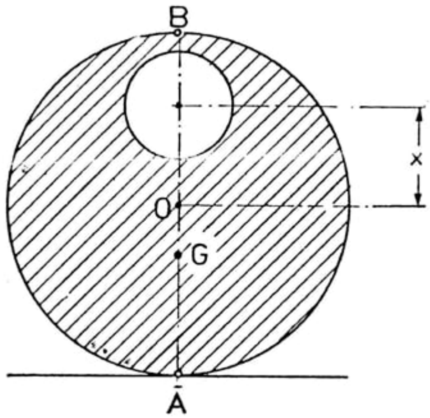

Fig. 5.2

The distance OG is calculated from the equilibrium condition:

$$m' \cdot OG = m_c \cdot x$$, (m_c = the mass dislocated by the cavity) (10)

OG = Rsinα (11)

$$x = OG \cdot \frac{m'}{m_c} = R \cdot \sin\alpha \cdot \frac{R^2 - r^2}{r^2} \quad (12)$$

d) At every measurement it must be estimated the reading error. Taking in account the expressions for ρ, r and x it is evaluated the maximum error for the determination of these measures.

Problems of the 7th International Physics Olympiad[1]
(Warsaw, 1974)

Waldemar Gorzkowski

Institute of Physics, Polish Academy of Sciences, Warsaw, Poland [2]

Abstract

The article contains the competition problems given at the 7th International Physics Olympiad (Warsaw, 1974) and their solutions.

Introduction

The 7th International Physics Olympiad (Warsaw, 1974) was the second one organized in Poland. It took place after a one-year organizational gap, as no country was able to organize the competition in 1973.

The original English version of the problems of the 7th IPhO has not been preserved. We would like to remind that the permanent Secretariat of the IPhOs was established only in 1983; previously the Olympic materials had been collected by individual people in their private archives and, in general, are not complete. English texts of the problems and simplified solutions are available in the book by R. Kunfalvi [1]. Unfortunately, they are somewhat deformed as compared to the originals. Fortunately, we have very precise Polish texts. Also the full solutions in Polish are available. This article is based on the books [2, 3] and article [4].

The competition problems were prepared especially for the 7th IPhO by Andrzej Szymacha (theoretical problems) and Jerzy Langer (experimental problem).

THEORETICAL PROBLEMS

Problem 1

A hydrogen atom in the ground state, moving with velocity v, collides with another hydrogen atom in the ground state at rest. Using the Bohr model find the smallest velocity v_0 of the atom below which the collision must be elastic.

At velocity v_0 the collision may be inelastic and the colliding atoms may emit electromagnetic radiation. Estimate the difference of frequencies of the radiation emitted in the direction of the initial velocity of the hydrogen atom and in the opposite direction as a fraction (expressed in percents) of their arithmetic mean value.

Data:

$E_i = \frac{me^4}{2\hbar^2} = 13.6\,\text{eV} = 2.18\cdot 18^{-18}$ J; (ionization energy of hydrogen atom)

$m_H = 1.67\cdot 10^{-27}$ kg; (mass of hydrogen atom)

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[1] This article has been sent for publication in *Physics Competitions* in September 2003

[2] e-mail: gorzk@ifpan.edu.pl

(m - mass of electron; e - electric charge of electron; $\hbar$ - Planck constant; numerical values of these quantities are not necessary.)

Solution

According to the Bohr model the energy levels of the hydrogen atom are given by the formula:

$$E_n = -\frac{E_i}{n^2},$$

where n = 1, 2, 3, … The ground state corresponds to $n = 1$, while the lowest excited state corresponds to $n = 2$. Thus, the smallest energy necessary for excitation of the hydrogen atom is:

$$\Delta E = E_2 - E_1 = E_i(1 - \tfrac{1}{4}) = \tfrac{3}{4}E_i.$$

During an inelastic collision a part of kinetic energy of the colliding particles is converted into their internal energy. The internal energy of the system of two hydrogen atoms considered in the problem cannot be changed by less than ΔE. It means that if the kinetic energy of the colliding atoms with respect to their center of mass is less than ΔE, then the collision must be an elastic one. The value of v_0 can be found by considering the critical case, when the kinetic energy of the colliding atoms is equal to the smallest energy of excitation. With respect to the center of mass the atoms move in opposite direction with velocities $\tfrac{1}{2}v_0$. Thus

$$\frac{1}{2}m_H\left(\tfrac{1}{2}v_0\right)^2 + \frac{1}{2}m_H\left(\tfrac{1}{2}v_0\right)^2 = \tfrac{3}{4}E_i$$

and

$$v_0 = \sqrt{\frac{3E_i}{m_H}} \qquad (\approx 6.26 \cdot 10^4 \text{ m/s}).$$

Consider the case when $v = v_0$. The collision may be elastic or inelastic. When the collision is elastic the atoms remain in their ground states and do not emit radiation. Radiation is possible only when the collision is inelastic. Of course, only the atom excited in the collision can emit the radiation. In principle, the radiation can be emitted in any direction, but according to the text of the problem we have to consider radiation emitted in the direction of the initial velocity and in the opposite direction only. After the inelastic collision both atom are moving (in the laboratory system) with the same velocities equal to $\tfrac{1}{2}v_0$. Let f denotes the frequency of radiation emitted by the hydrogen atom in the mass center (i.e. at rest). Because of the Doppler effect, in the laboratory system this frequency is observed as (c denotes the velocity of light):

a) $f_1 = \left(1 + \frac{\frac{1}{2}v_0}{c}\right) f$ - for radiation emitted in the direction of the initial velocity of the hydrogen atom,

b) $f_2 = \left(1 - \frac{\frac{1}{2}v_0}{c}\right) f$ - for radiation emitted in opposite direction.

The arithmetic mean value of these frequencies is equal to f . Thus the required ratio is

$$\frac{\Delta f}{f} = \frac{f_1 - f_2}{f} = \frac{v_0}{c} \qquad (\approx 2 \cdot 10^{-2}\%).$$

In the above solution we took into account that $v_0 << c$. Otherwise it would be necessary to use relativistic formulae for the Doppler effect. Also we neglected the recoil of atom(s) in the emission process. One should notice that for the visible radiation or radiation not too far from the visible range the recoil cannot change significantly the numerical results for the critical velocity v_0 and the ratio $\frac{\Delta f}{f}$. The recoil is important for high-energy quanta, but it is not this case.

The solutions were marked according to the following scheme (draft):

1. Energy of excitation — up to 3 points
2. Correct description of the physical processes — up to 4 points
3. Doppler effect — up to 3 points

Problem 2

Consider a parallel, transparent plate of thickness d – Fig. 1. Its refraction index varies as

$$n = \frac{n_0}{1 - \frac{x}{R}}.$$

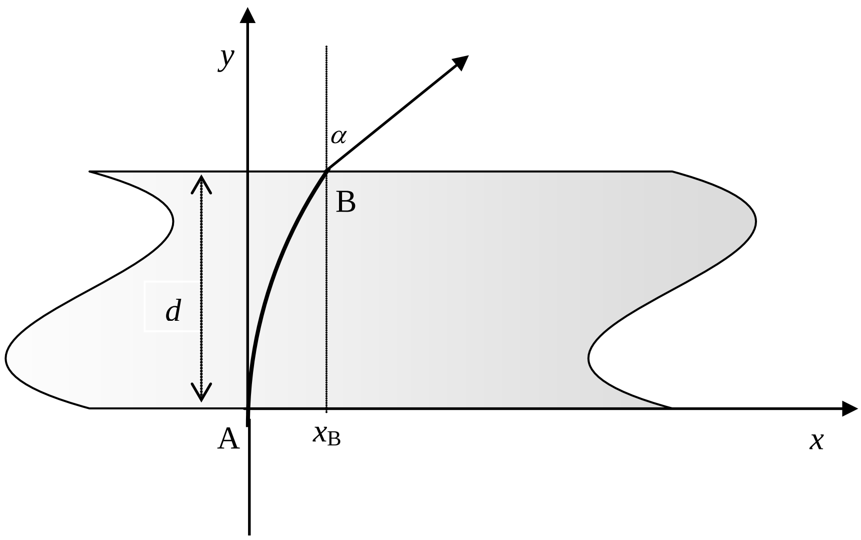


Fig. 1

A light beam enters from the air perpendicularly to the plate at the point A ($x_A = 0$) and emerges from it at the point B at an angle α.

1. Find the refraction index n_B at the point B.
2. Find x_B (i.e. value of x at the point B)
3. Find the thickness d of the plate.

Data:

$n_0 = 1.2$; $R = 13\,\text{cm}$; $\alpha = 30°$.

Solution

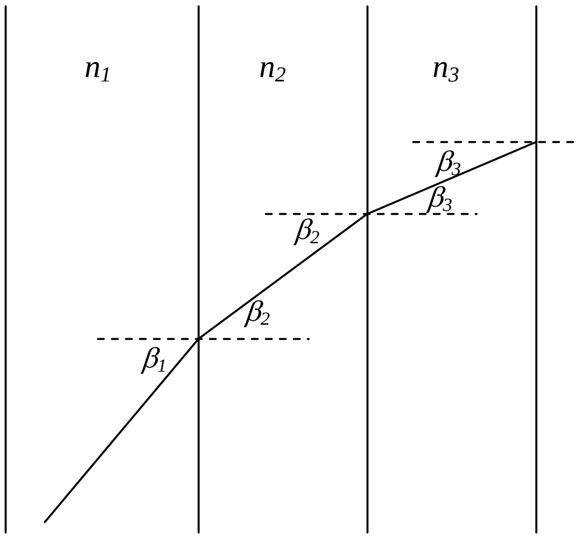


Fig. 2

Consider a light ray passing through a system of parallel plates with different refractive indexes – Fig. 2. From the Snell law we have

$$\frac{\sin\beta_2}{\sin\beta_1} = \frac{n_1}{n_2}$$

i.e.

$$n_2 \sin\beta_2 = n_1 \sin\beta_1 .$$

In the same way we get

$$n_3 \sin\beta_3 = n_2 \sin\beta_2 , \text{ etc.}$$

Thus, in general:

$$n_i \sin\beta_i = \text{const.}$$

This relation does not involve plates thickness nor their number. So, we may make use of it also in case of continuous dependence of the refractive index in one direction (in our case in the *x* direction).

Consider the situation shown in Fig. 3.

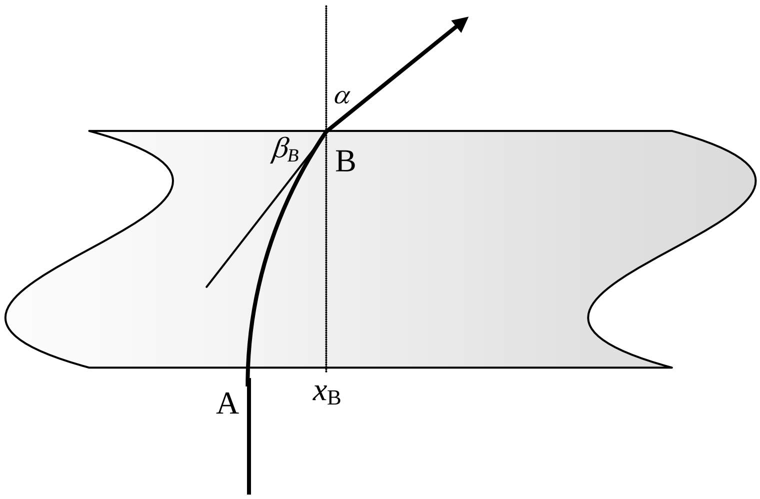


Fig. 3

At the point A the angle $\beta_A = 90°$. The refractive index at this point is n_0. Thus, we have

$$n_A \sin\beta_A = n_B \sin\beta_B ,$$
$$n_0 = n_B \sin\beta_B .$$

Additionally, from the Snell law applied to the refraction at the point B, we have

$$\frac{\sin\alpha}{\sin(90° - \beta_B)} = n_B .$$

Therefore

$$\sin\alpha = n_B \cos\beta_B = n_B\sqrt{1 - \sin^2\beta_B} = \sqrt{n_B^2 - (n_B \sin\beta_B)^2} = \sqrt{n_B^2 - n_0^2}$$

and finally

$$n_B = \sqrt{n_0^2 + \sin^2\alpha} .$$

Numerically

$$n_B = \sqrt{\left(\frac{12}{10}\right)^2 + \left(\frac{5}{10}\right)^2} = 1.3$$

The value of x_B can be found from the dependence $n(x)$ given in the text of the problem. We have

$$n_B = n(x_B) = \frac{n_0}{1-\frac{x_B}{R}},$$

$$x_B = R\left(1-\frac{n_0}{n_B}\right),$$

Numerically

$$x_B = 1 \text{ cm}.$$

The answer to the third question requires determination of the trajectory of the light ray. According to considerations described at the beginning of the solution we may write (see Fig. 4):

$$n(x)\sin\beta(x) = n_0 .$$

Thus

$$\sin\beta(x) = \frac{n_0}{n(x)} = \frac{R-x}{R} .$$

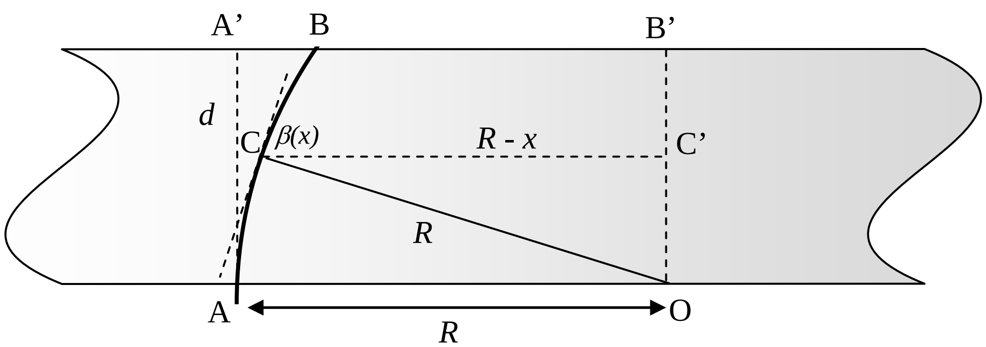


Fig. 4

Consider the direction of the ray crossing a point C on the circle with radius R and center in point O as shown in Fig. 4. We see that

$$\sin\angle\text{COC}' = \frac{R-x}{R} = \sin\beta(x) .$$

Therefore, the angle $\angle\text{COC}'$ must be equal to the angle $\beta(x)$ formed at the point C by the light ray and CC'. It means that at the point C the ray must be tangent to the circle. Moreover, the ray that is tangent to the circle at some point must be tangent also at farther points. Therefore, the ray cannot leave the circle (as long as it is inside the plate)! But at the beginning the ray (at the point A) is tangent to the circle. Thus, the ray must propagate along the circle shown in Fig. 4 until reaching point B where it leaves the plate.

Already we know that A'B = 1 cm. Thus, B'B = 12 cm and from the rectangular triangle BB'O we get

$$d = \mathrm{B'O} = \sqrt{13^2 - 12^2}\ \mathrm{cm} = 5\ \mathrm{cm}.$$

The shape of the trajectory $y(x)$ can be determined also by using more sophisticated calculations. Knowing $\beta(x)$ we find $\mathrm{tg}\,\beta(x)$:

$$\mathrm{tg}\,\beta(x) = \frac{R-x}{\sqrt{R^2-(R-x)^2}}.$$

But $\mathrm{tg}\,\beta(x)$ is the derivative of $y(x)$. So, we have

$$\frac{dy}{dx} = \frac{R-x}{\sqrt{R^2-(r-x)^2}} = \frac{d}{dx}\left(\sqrt{R^2-(R-x)^2}\right).$$

Thus

$$y = \sqrt{R^2-(R-x)^2} + const$$

Value of *const* can be found from the condition

$$y(0) = 0.$$

Finally:

$$y = \sqrt{R^2-(R-x)^2}.$$

It means that the ray moves in the plate along to the circle as found previously.

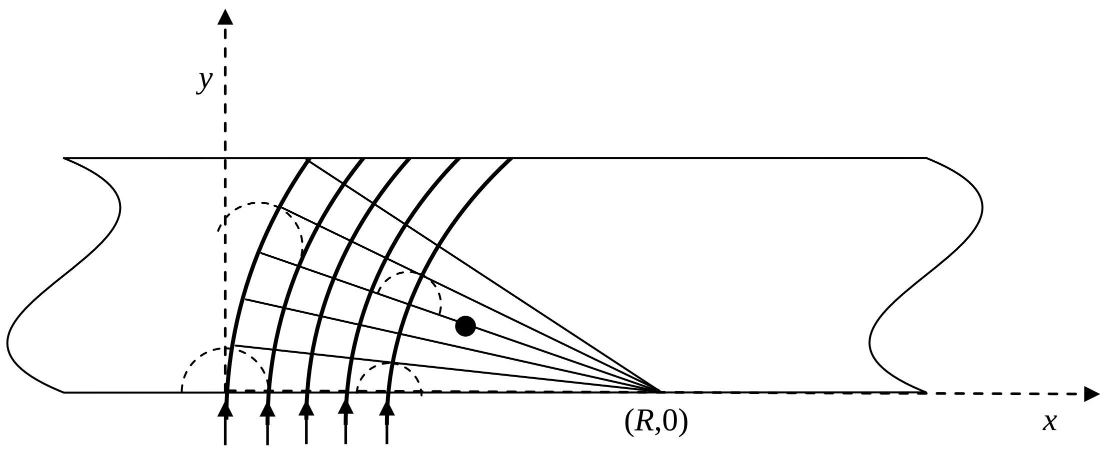


Fig. 5

Now we will present yet another, already the third, method of proving that the light in the plate must move along the circle.

We draw a number of straight lines (inside the plate) close to each other and passing trough the point (R,0) - Fig. 5. From the formula given in the text of the problem it follows that the refraction index on each of these lines is inversely proportional to the distance to the point (R,0). Now we draw several arcs with the center at (R,0). It is obvious that the geometric length of each arc between two lines is proportional to the distance to the point (R,0).

It follows from the above that the optical path (a product of geometric length and refractive index) along each arc between the two lines (close to each other) is the same for all the arcs.

Assume that at +-certain moment t the wave front reached one of the lines, e.g. the line marked with a black dot in Fig. 5. According to the Huygens principle, the secondary sources on this line emit secondary waves. Their envelope forms the wave front of the real wave at some time $t + \Delta t$. The wave fronts of secondary waves, shown in Fig. 5, have different geometric radii, but - in view of our previous considerations - their optical radii are exactly the same. It means that at the time $t + \Delta t$ the new wave front will correspond to one of the lines passing trough (R,0). At the beginning the wave front of the light coincided with the x axis, it means that inside the plate the light will move along the circle with center at the point (R,0).

The solutions were marked according to the following scheme (draft):

1.	Proof of the relation $n \sin \beta =$ const	up to 2 points
2.	Correct description of refraction at points A and B	up to 2 points
3.	Calculation of x_B	up to 1 point
4.	Calculation of d	up to 5 points

Problem 3

A scientific expedition stayed on an uninhabited island. The members of the expedition had had some sources of energy, but after some time these sources exhausted. Then they decided to construct an alternative energy source. Unfortunately, the island was very quiet: there were no winds, clouds uniformly covered the sky, the air pressure was constant and the temperatures of air and water in the sea were the same during day and night. Fortunately, they found a source of chemically neutral gas outgoing very slowly from a cavity. The pressure and temperature of the gas are exactly the same as the pressure and temperature of the atmosphere.

The expedition had, however, certain membranes in its equipment. One of them was ideally transparent for gas and ideally non-transparent for air. Another one had an opposite property: it was ideally transparent for air and ideally non-transparent for gas. The members of the expedition had materials and tools that allowed them to make different mechanical devices such as cylinders with pistons, valves etc. They decided to construct an engine by using the gas from the cavity.

Show that there is no theoretical limit on the power of an ideal engine that uses the gas and the membranes considered above.

Solution

Let us construct the device shown in Fig. 6. B_1 denotes the membrane transparent for the gas from the cavity, but non-transparent for the air, while B_2 denotes the membrane with opposite property: it is transparent for the air but non-transparent for the gas.

Initially the valve Z_1 is open and the valve Z_2 is closed. In the initial situation, when we keep the piston at rest, the pressure under the piston is equal to $p_0 + p_0$ due to the Dalton law. Let V_0 denotes an initial volume of the gas (at pressure p_0).

Now we close the valve Z_1 and allow the gas in the cylinder to expand. During movement of the piston in the downwards direction we obtain certain work performed by excess pressure inside the cylinder with respect to the atmospheric pressure p_0. The partial pressure of the gas in the cylinder will be reduced according to the formula $p = p_0 V_0 / V$, where V denotes volume closed by the piston (isothermal process). Due to the membrane B_2 the partial pressure of the air in the cylinder all the time is p_0 and balances the air pressure outside the cylinder. It means that only the gas from the cavity effectively performs the work.

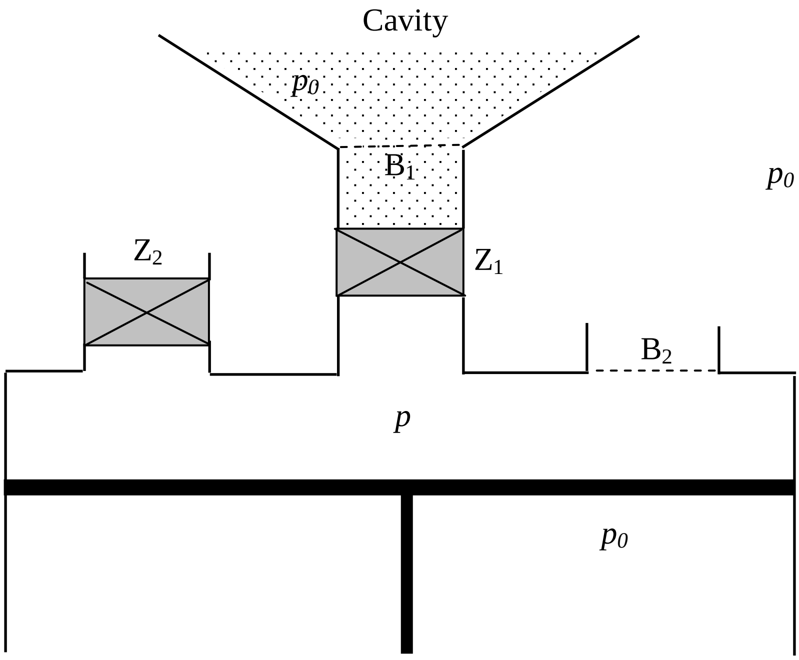


Fig. 6

Consider the problem of limits for the work that can be performed during isothermal expansion of an initial portion of the gas. Let us analyze the graph of the function $p_0 V_0 / V$ versus V shown in Fig. 7.

It is obvious that the amount of work performed by the gas during isothermal expansion from V_0 to V_k is represented by the area under the curve (shown in the graph) from V_0 to V_k. Of course, the work is proportional to V_0. We shall prove that for large enough V_k the work can be arbitrarily large.

Consider $V = V_0, 2V_0, 4V_0, 8V_0, 16V_0, ...$ It is clear that the rectangles I, II, III, … (see Fig. 7) have the same area and that one may draw arbitrarily large number of such rectangles under the considered curve. It means that during isothermal expansion of a given portion of the gas we may obtain arbitrarily large work (at the cost of the heat taken from sthe urrounding) – it is enough to take V_k large enough.

After reaching V_k we open the valve Z_2 and move the piston to its initial position without performing any work. The cycle can be repeated as many times as we want.

In the above considerations we focused our attention on the work obtained during one cycle only. We entirely neglected dynamics of the process, while each cycle lasts some time. One may think that - in principle - the length of the cycle increases very rapidly with the effective work we obtain. This would limit the power of the device we consider.

Take, however, into account that, by proper choice of various parameters of the device, the time taken by one cycle can be made small and the initial volume of the gas V_0 can be made arbitrarily large (we consider only theoretical possibilities – we neglect practical difficulties entirely). E.g. by taking large size of the membrane B_1 and large size of the piston we may minimize the time of taking the initial portion of the gas V_0 from the cavity and make this portion very great.

In our analysis we neglected all losses, friction, etc. One should remark that there are no theoretical limits for them. These losses, friction etc. can be made negligibly small.

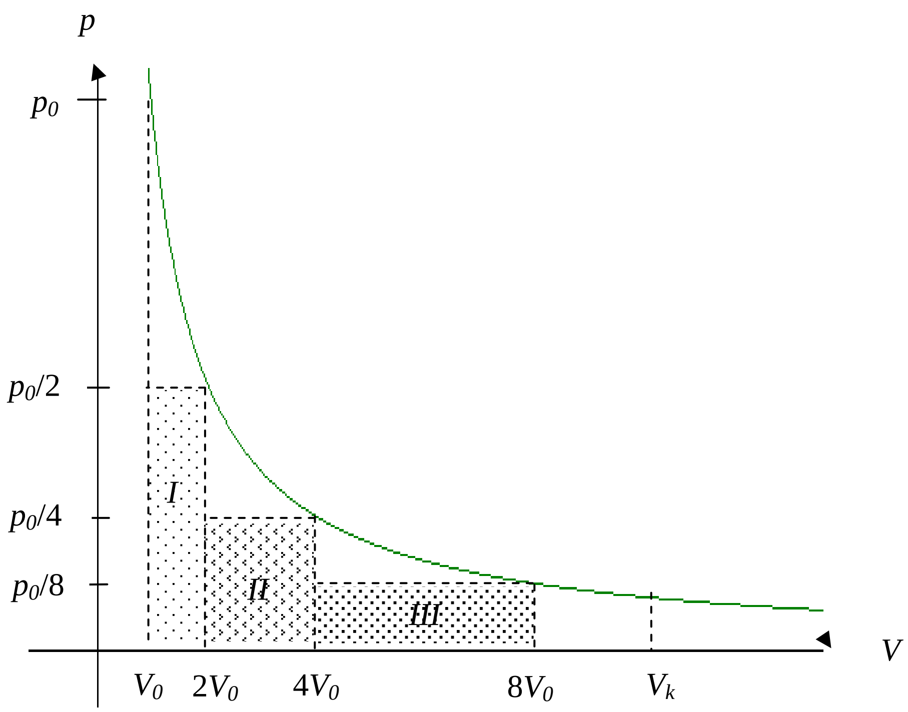


Fig. 7

The device we analyzed is very interesting: it produces work at cost of heat taken from surrounding without any difference in temperatures. Does this contradict the second law of thermodynamics? No! It is true that there is no temperature difference in the system, but the work of the device makes irreversible changes in the system (mixing of the gas from the cavity and the air).

The solutions were marked according to the following scheme (draft):

1. Model of an engine and its description — up to 4 points
2. Proof that there is no theoretical limit for power — up to 4 points
3. Remark on II law of thermodynamics — up to 2 points

EXPERIMENTAL PROBLEM

In a "black box" there are two identical semiconducting diodes and one resistor connected in some unknown way. By using instruments provided by the organizers find the resistance of the resistor.

Remark: One may assume that the diode conducts current in one direction only.

List of instruments: two universal volt-ammeters (without ohmmeters), battery, wires with endings, graph paper, resistor with regulated resistance.

Solution

At the beginning we perform preliminary measurements by using the circuit shown in Fig. 8. For two values of voltage U_1 and U_2, applied to the black box in both directions, we measure four values of current: $I(U_1)$, $I(U_2)$, $I(-U_1)$ and $I(-U_2)$. In this way we find that:

1. The black box conducts current in both directions;
2. There is an asymmetry with respect to the sign of the voltage;
3. In both directions current is a nonlinear function of voltage.

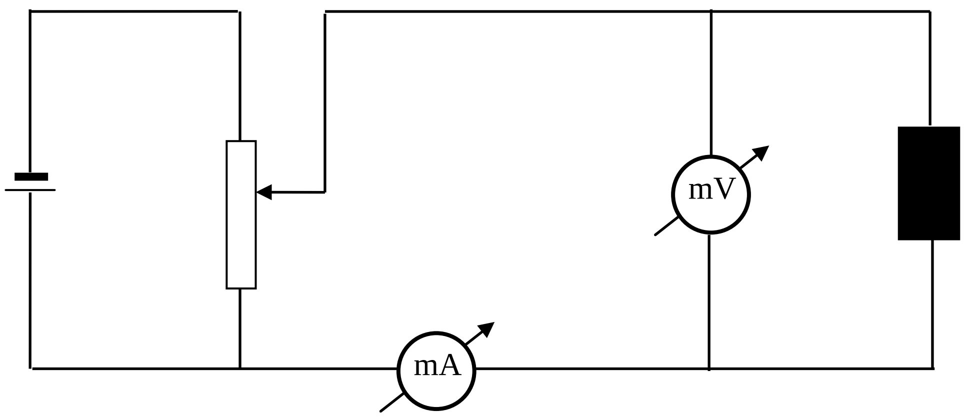
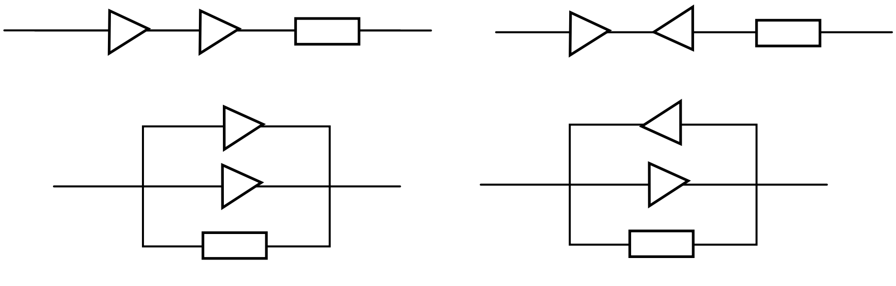


Fig. 8

The diodes and resistor can be connected in a limited number of ways shown in Fig. 9 (connections that differ from each other in a trivial way have been omitted).



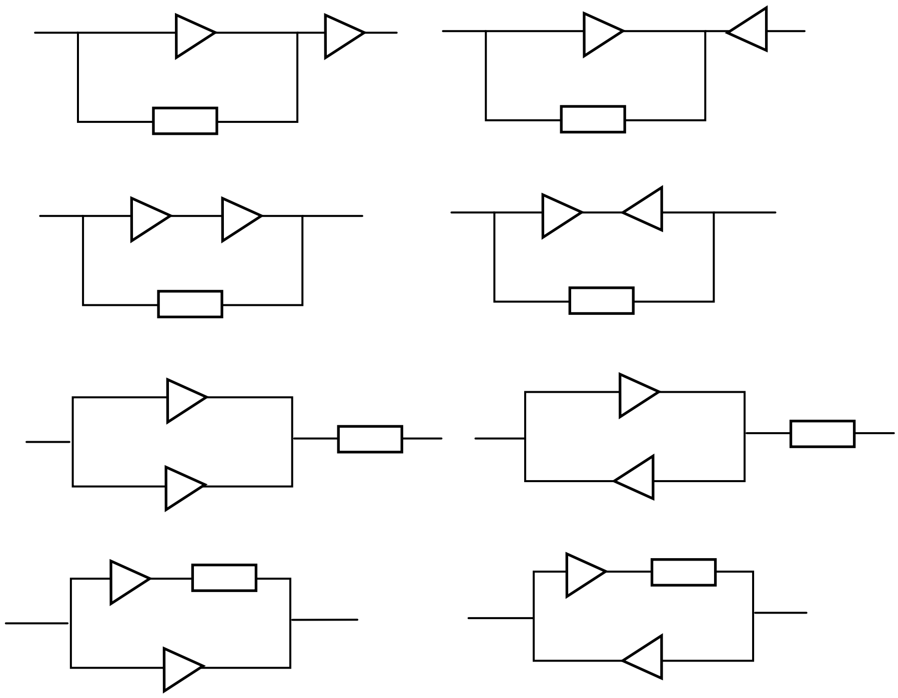


Fig. 9

Only one of these connections has the properties mentioned at the beginning. It is:

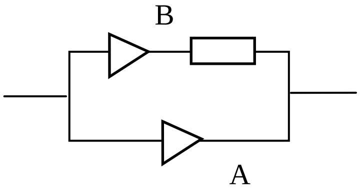


Fig. 10

For absolute values of voltages we have

$$U_R = U_B - U_A = \Delta U ,$$

where U_R denotes voltage on the resistor when a current I flows trough the branch B, U_A - voltage on the black box when the current I flows through the branch A, and U_B - voltage on the black box when the current I flows through the branch B.

Therefore

$$R = \frac{U_R(I)}{I} = \frac{U_B(I) - U_A(I)}{I} = \frac{\Delta U}{I} .$$

It follows from the above that it is enough to take characteristics of the black box in both directions: by subtraction of the corresponding points (graphically) we obtain a straight line (example is shown in Fig. 11) whose slope allows to determine the value of R .

The solutions were marked according to the following scheme (draft):

Theoretical part:

1. Proper circuit and method allowing determination of connections the elements in the black box — up to 6 points
2. Determination of R (principle) — up to 2 points
3. Remark that measurements at the same voltage in both directions make the error smaller — up to 1 point
4. Role of number of measurements (affect on errors) — up to 1 point

Experimental part:

1. Proper use of regulated resistor as potentiometer — up to 2 points
2. Practical determination of R (including error) — up to 4 points
3. Proper use of measuring instruments — up to 2 points
4. Taking into account that temperature of diodes increases during measurements — up to 1 point
5. Taking class of measuring instruments into account — up to 1 point

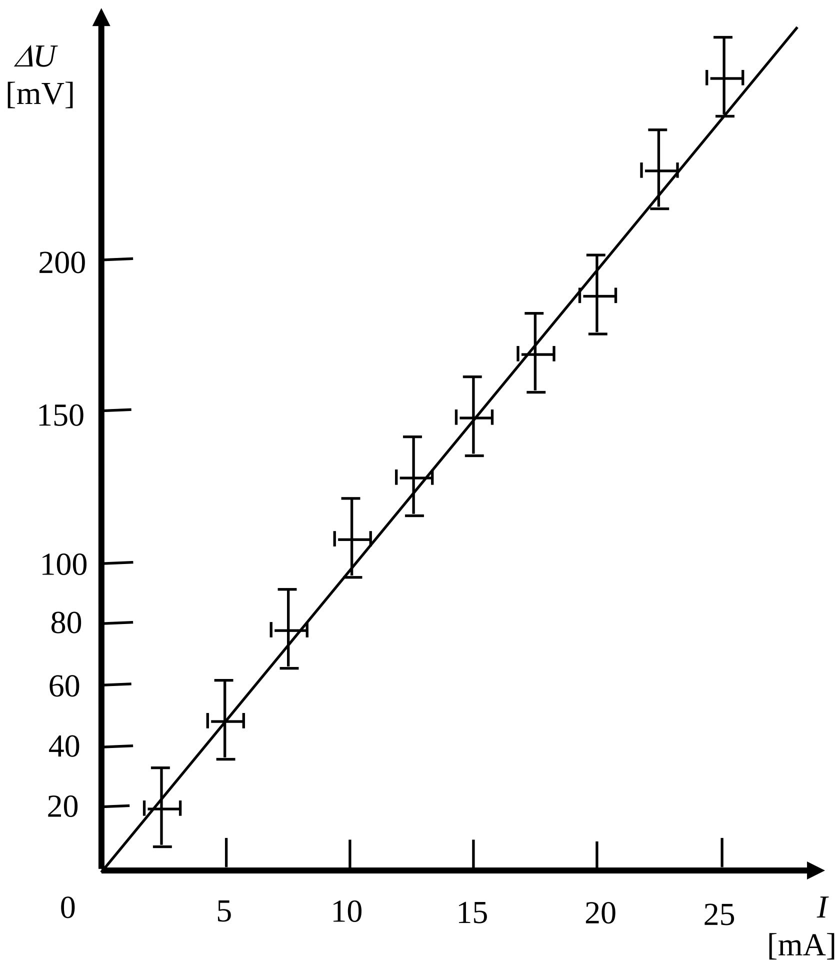


Fig. 11

Acknowledgement

Author wishes to express many thanks to Prof. Jan Mostowski for reading the text and for many valuable comments and remarks that allowed improving the final version.

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Problems of the 8th International Physics Olympiad (Güstrow, 1975)

Gunnar Friege[1] & Gunter Lind

Introduction

The 8th International Physics Olympiad took place from the 7.7. to the 12.7. 1975 in Güstrow, in the German Democratic Republic (GDR). Altogether, 9 countries with 45 pupils participated. The teams came from Bulgaria, the German Democratic Republic, the Federal Republic of Germany (FRG), France, Poland, Rumania, Tchechoslowakia, Hungary and the USSR. The entire event took place in the pedagogic academy of Güstrow. Pupils and leaders were accommodated inside the university academy complex. On the schedule there was the competition and receptions as well as excursions to Schwerin, Rostock, and Berlin were offered. The delegation of the FRG reported of a very good organisation of the olympiad.

The problems and solutions of the 8th International Physics Olympiad were created by a commission of university physics professors and lecturers. The same commission set marking schemes and conducted the correction of the tests. The correction was carried out very quickly and was considered as righteous and, in cases of doubt, as very generous.

The main competition consisted of a 5 hour test in theory and a 4.5 hour experimental test. The time for the theoretical part was rather short and for the experimental part rather long. The problems originated from central areas of classical physics. The theoretical problems were relatively difficult, although solvable with good physics knowledge taught at school. The level of difficulty of the experimental problem was adequate. There were no additional devices necessary for the solution of the problems. Only basic formula knowledge was requested, and could be demanded from all pupils. Critics were only uttered concerning the second theoretical problem (thick lens). This problem requested relatively little physical understanding, but tested the mathematical skills and the routine in approaching problems (e.g. correct distinction of cases). However, it is also difficult to find substantial physics problems in the area of geometrical optics.

[1] Remark: This article was written due to the special request to us by Dr. W. Gorzkowski, in order to close one of the last few gaps in the IPhO-report collection.

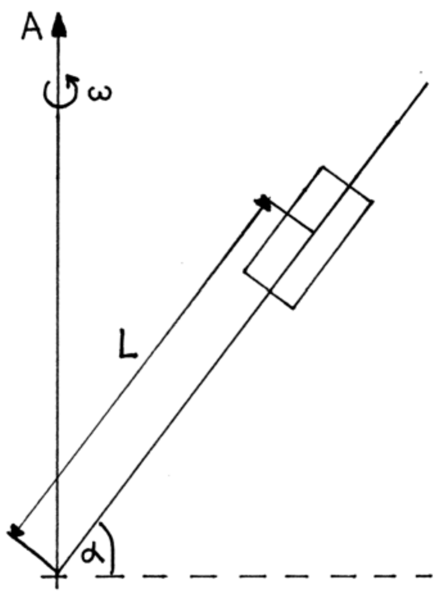
Contact: Dr. Gunnar Friege, Leibniz-Institute for Science Education (IPN) at the University of Kiel, Olshausenstr. 62, 24098 Kiel, Germany, friege@ipn.uni-kiel.de

Altogether 50 points were the maximum to achieve; 30 in the theoretical test and 20 in the experimental test. The best contestant came from the USSR and had 43 points. The first prize (gold medal) was awarded with 39 points, the second prize (silver medal) with 34 points, the third prize (bronze medal) with 28 points and the fourth prize (honourable mention) with 22 points. Among the 45 contestants, 7 I. prizes, 9 II. prizes, 12 III. prizes and 8 IV. prizes were awarded, meaning that 80 % of all contestants were awarded.

The following problem descriptions and solution are based mainly on a translation of the original German version from 1975. Because the original drafts are not well preserved, some new sketches were drawn. We also gave the problems headlines and the solutions are in more detail.

Theoretical problem 1: "Rotating rod"

A rod revolves with a constant angular velocity ω around a vertical axis A. The rod includes a fixed angle of $\pi/2-\alpha$ with the axis. A body of mass m can glide along the rod. The coefficient of friction is $\mu = \tan\beta$. The angle β is called „friction angle".



a) Determine the angles α under which the body remains at rest and under which the body is in motion if the rod is not rotating (i.e. $\omega = 0$).

b) The rod rotates with constant angular velocity $\omega > 0$. The angle α does not change during rotation. Find the condition for the body to remain at rest relative to the rod.

You can use the following relations:

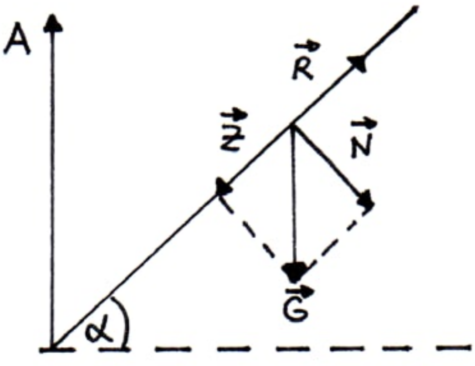
$\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta$

Solution of problem 1:

a) $\omega = 0$:

The forces in this case are (see figure):



$\vec{G} = \vec{Z} + \vec{N} = m \cdot \vec{g}$ (1),

$|\vec{Z}| = m \cdot g \cdot \sin\alpha = Z$ (2),

$|\vec{N}| = m \cdot g \cdot \cos\alpha = N$ (3),

$|\vec{R}| = \mu \cdot N = \mu \cdot m \cdot g \cdot \cos\alpha = R$ (4).

[$\vec{R}$: force of friction]

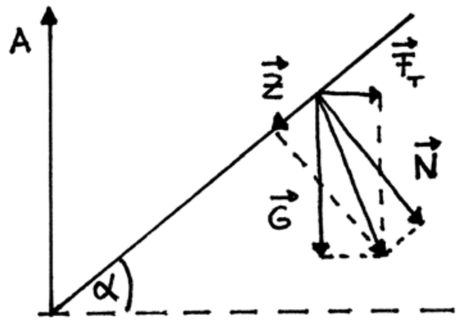
The body is at rest relative to the rod, if $Z \leq R$. According to equations (2) and (4) this is equivalent to $\tan\alpha \leq \tan\beta$. That means, the body is at rest relative to the rod for $\alpha \leq \beta$ and the body moves along the rod for $\alpha > \beta$.

b) $\omega > 0$:

Two different situations have to be considered: 1. $\alpha > \beta$ and 2. $\alpha \leq \beta$.

If the rod is moving ($\omega \neq 0$) the forces are $\vec{G} = m \cdot \vec{g}$ and $|\vec{F_r}| = m \cdot r \cdot \vec{\omega}^2$.

From the parallelogramm of forces (see figure):



$\vec{Z} + \vec{N} = \vec{G} + \vec{F_r}$ (5).

The condition of equilibrium is:

$|\vec{Z}| = \mu |\vec{N}|$ (6).

Case 1: $\vec{Z}$ is oriented downwards, i.e. $g \cdot \sin\alpha > r \cdot \omega^2 \cdot \cos\alpha$.

$|\vec{Z}| = m \cdot g \cdot \sin\alpha - m \cdot r \cdot \omega^2 \cdot \cos\alpha$ and $|\vec{N}| = m \cdot g \cdot \cos\alpha + m \cdot r \cdot \omega^2 \cdot \sin\alpha$

Case 2: $\vec{Z}$ is oriented upwards, i.e. $g \cdot \sin\alpha < r \cdot \omega^2 \cdot \cos\alpha$.

$|\vec{Z}| = -m \cdot g \cdot \sin\alpha + m \cdot r \cdot \omega^2 \cdot \cos\alpha$ and $|\vec{N}| = m \cdot g \cdot \cos\alpha + m \cdot r \cdot \omega^2 \cdot \sin\alpha$

It follows from the condition of equilibrium equation (6) that

$$\pm\left(g \cdot \sin\alpha - r \cdot \omega^2 \cdot \cos\alpha\right) = \tan\beta \cdot \left(g \cdot \cos\alpha + r \cdot \omega^2 \cdot \sin\alpha\right) \quad (7).$$

Algebraic manipulation of equation (7) leads to:

$$g \cdot \sin(\alpha - \beta) = r \cdot \omega^2 \cdot \cos(\alpha - \beta) \qquad (8),$$

$$g \cdot \sin(\alpha + \beta) = r \cdot \omega^2 \cdot \cos(\alpha + \beta) \qquad (9).$$

That means,

$$r_{1,2} = \frac{g}{\omega^2} \cdot \tan(\alpha \mp \beta) \qquad (10).$$

The body is at rest relative to the rotating rod in the case $\alpha > \beta$ if the following inequalities hold:

$$r_1 \le r \le r_2 \qquad \text{with } r_1, r_2 > 0 \qquad (11)$$

or

$$L_1 \le L \le L_2 \qquad \text{with } L_1 = r_1 / \cos\alpha \text{ and } L_2 = r_2 / \cos\alpha \qquad (12).$$

The body is at rest relative to the rotating rod in the case $\alpha \le \beta$ if the following inequalities hold:

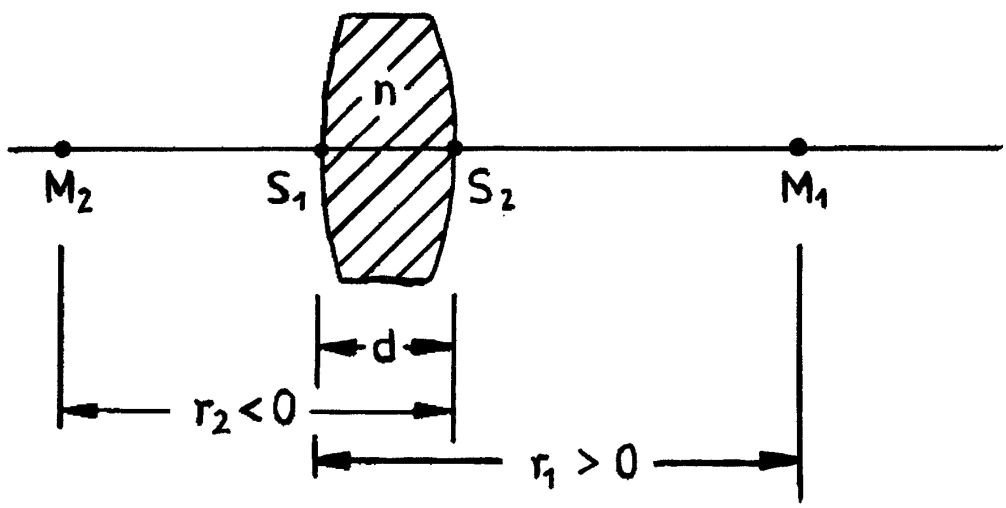
$$0 \le r \le r_2 \qquad \text{with } r_1 = 0 \text{ (since } r_1 < 0 \text{ is not a physical solution), } r_2 > 0 \qquad (13).$$

Inequality (13) is equivalent to

$$0 \le L \le L_2 \qquad \text{with } L_2 = r_2 / \cos\alpha > 0 \qquad (14).$$

Theoretical problem 2: “Thick lens”

The focal length f of a thick glass lens in air with refractive index n, radius curvatures r_1, r_2 and vertex distance d (see figure) is given by: $f = \frac{n r_1 r_2}{(n-1)\left[n(r_2 - r_1) + d(n-1)\right]}$



Remark: $r_i > 0$ means that the central curvature point M_i is on the right side of the aerial vertex S_i, $r_i < 0$ means that the central curvature point M_i is on the left side of the aerial vertex S_i (i = 1,2).

For some special applications it is required, that the focal length is independent from the wavelength.

a) For how many different wavelengths can the same focal length be achieved?

b) Describe a relation between r_i (i = 1,2), d and the refractive index n for which the required wavelength independence can be fulfilled and discuss this relation.
Sketch possible shapes of lenses and mark the central curvature points M_1 and M_2.

c) Prove that for a given planconvex lens a specific focal length can be achieved by only one wavelength.

d) State possible parameters of the thick lens for two further cases in which a certain focal length can be realized for one wavelength only. Take into account the physical and the geometrical circumstances.

Solution of problem 2:

a) The refractive index n is a function of the wavelength λ, i.e. $n = n(\lambda)$. According to the given formula for the focal length f (see above) which for a given f yields to an equation quadratic in n there are at most two different wavelengths (indices of refraction) for the same focal length.

b) If the focal length is the same for two different wavelengths, then the equation

$$f(\lambda_1) = f(\lambda_2) \quad or \quad f(n_1) = f(n_2) \tag{1}$$

holds. Using the given equation for the focal length it follows from equation (1):

$$\frac{n_1 r_1 r_2}{(n_1 - 1)\left[n_1(r_2 - r_1) + d(n_1 - 1)\right]} = \frac{n_2 r_1 r_2}{(n_2 - 1)\left[n_2(r_2 - r_1) + d(n_2 - 1)\right]}$$

Algebraic calculations lead to:

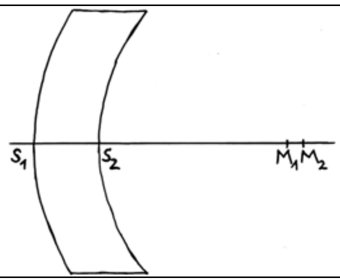
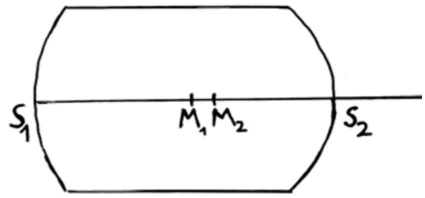
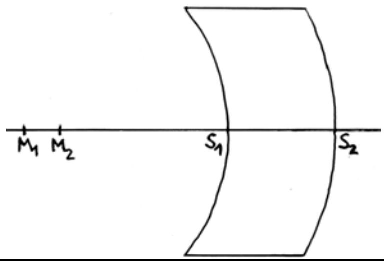
$$r_1 - r_2 = d \cdot \left(1 - \frac{1}{n_1 n_2}\right) \tag{2}.$$

If the values of the radii r_1, r_2 and the thickness satisfy this condition the focal length will be the same for two wavelengths (indices of refraction). The parameters in this equation are subject to some physical restrictions: The indices of refraction are greater than 1 and the thickness of the lens is greater than 0 m. Therefore, from equation (2) the relation

$$d > r_1 - r_2 > 0 \tag{3}$$

is obtained.

The following table shows a discussion of different cases:

r_1	r_2	***condition***	***shape of the lens***	***centre of curvature***
$r_1 > 0$	$r_2 > 0$	$0 < r_1 - r_2 < d$ or $r_2 < r_1 < d + r_2$		*M_2 is always right of M_1.* $\overline{M_1M_2} < \overline{S_1S_2}$
$r_1 > 0$	$r_2 < 0$	$r_1 + \lvert r_2 \rvert < \mathrm{d}$		*Order of points:* $S_1M_1M_2S_2$
$r_1 < 0$	$r_2 > 0$	*never fulfilled*		
$r_1 < 0$	$r_2 < 0$	$0 < \lvert r_2 \rvert - \lvert r_1 \rvert < d$ or $\lvert r_1 \rvert < \lvert r_2 \rvert < d + \lvert r_1 \rvert$		*M_2 is always right of M_1.* $\overline{M_1M_2} < \overline{S_1S_2}$

c) The radius r_1 or the radius r_2 is infinite in the case of the planconvex lens. In the following it is assumed that r_1 is infinite and r_2 is finite.

$$\lim_{r_1\to\infty} f = \lim_{r_1\to\infty} \frac{n\,r_2}{(n-1)\left[n\left(\frac{r_2}{r_1}-1\right)+(n-1)\frac{d}{r_1}\right]} = \frac{r_2}{1-n} \tag{4}$$

Equation (4) means, that for each wavelength (refractive index) there exists a different value of the focal length.

d) From the given formula for the focal length (see problem formulation) one obtains the following quadratic equation in n:

$$A\cdot n^2 + B\cdot n + C = 0 \tag{5}$$

with $\quad A = (r_2 - r_1 + d)\cdot f$, $B = -\left[f\cdot(r_2 - r_1) + 2\cdot f\cdot d + r_1\cdot r_2\right]$ and $C = f\cdot d$.

Solutions of equation (5) are:

$$n_{1,2} = -\frac{B}{2\cdot A} \pm \sqrt{\frac{B^2}{4\cdot A^2} - \frac{C}{A}} \qquad (6).$$

Equation (5) has only one physical correct solution, if...

I) A = 0 (i.e., the coefficient of n^2 in equation (5) vanishes)

In this case the following relationships exists:

$$r_1 - r_2 = d \qquad (7),$$

$$n = \frac{f\cdot d}{f\cdot d + r_1\cdot r_2} > 1 \qquad (8).$$

II) B = 0 (i.e. the coefficient of n in equation (5) vanishes)

In this case the equation has a positive and a negative solution. Only the positve solution makes sense from the physical point of view. It is:

$$f\cdot(r_2 - r_1) + 2\cdot f\cdot d + r_1\cdot r_2 = 0 \qquad (9),$$

$$n^2 = -\frac{C}{A} = -\frac{d}{(r_2 - r_1 + d)} > 1 \qquad (10),$$

III) $B^2 = 4$ AC

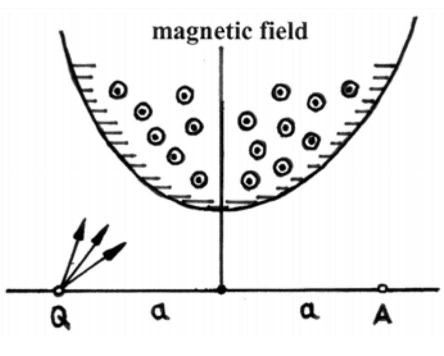
In this case two identical real solutions exist. It is:

$$\left[f\cdot(r_2 - r_1) + 2\cdot f\cdot d + r_1\cdot r_2\right]^2 = 4\cdot(r_2 - r_1 + d)\cdot f^2\cdot d \qquad (11),$$

$$n = -\frac{B}{2\cdot A} = \frac{f\cdot(r_2 - r_1) + 2\cdot f\cdot d + r_1\cdot r_2}{2\cdot f(r_2 - r_1 + d)} > 1 \qquad (12).$$

Theoretical problem 3: "Ions in a magnetic field"

A beam of positive ions (charge +e) of the same and constant mass *m* spread from point Q in different directions in the plane of paper (see figure[2]). The ions were accelerated by a voltage *U*. They are deflected in a uniform magnetic field *B* that is perpendicular to the plane of paper. The boundaries of the magnetic field are made in a way that the initially diverging ions are focussed in point A ($\overline{QA} = 2\cdot a$). The trajectories of the ions are symmetric to the middle perpendicular on $\overline{QA}$.



[2] Remark: This illustrative figure was <u>not</u> part of the original problem formulation.

Among different possible boundaries of magnetic fields a specific type shall be considered in which a contiguous magnetic field acts around the middle perpendicular and in which the points Q and A are in the field free area.

a) Describe the radius curvature R of the particle path in the magnetic field as a function of the voltage U and the induction B.

b) Describe the characteristic properties of the particle paths in the setup mentioned above.

c) Obtain the boundaries of the magnetic field boundaries by geometrical constructions for the cases $R < a$, $R = a$ and $R > 0$.

d) Describe the general equation for the boundaries of the magnetic field.

Solution of problem 3:

a) The kinetic energy of the ion after acceleration by a voltage U is:

$$\tfrac{1}{2} mv^2 = eU \qquad (1).$$

From equation (1) the velocity of the ions is calculated:

$$v = \sqrt{\frac{2 \cdot e \cdot U}{m}} \qquad (2).$$

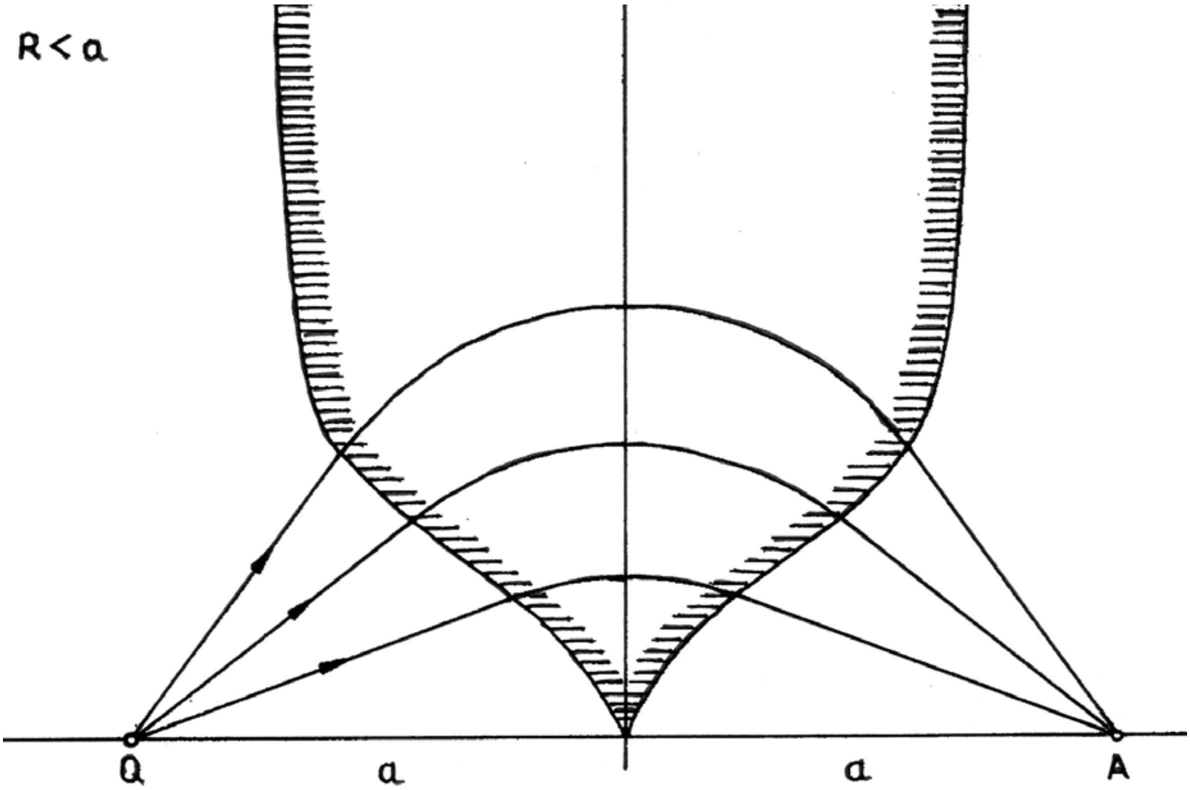
On a moving ion (charge e and velocity v) in a homogenous magnetic field B acts a Lorentz force F. Under the given conditions the velocity is always perpendicular to the magnetic field. Therefore, the paths of the ions are circular with Radius R. Lorentz force and centrifugal force are of the same amount:

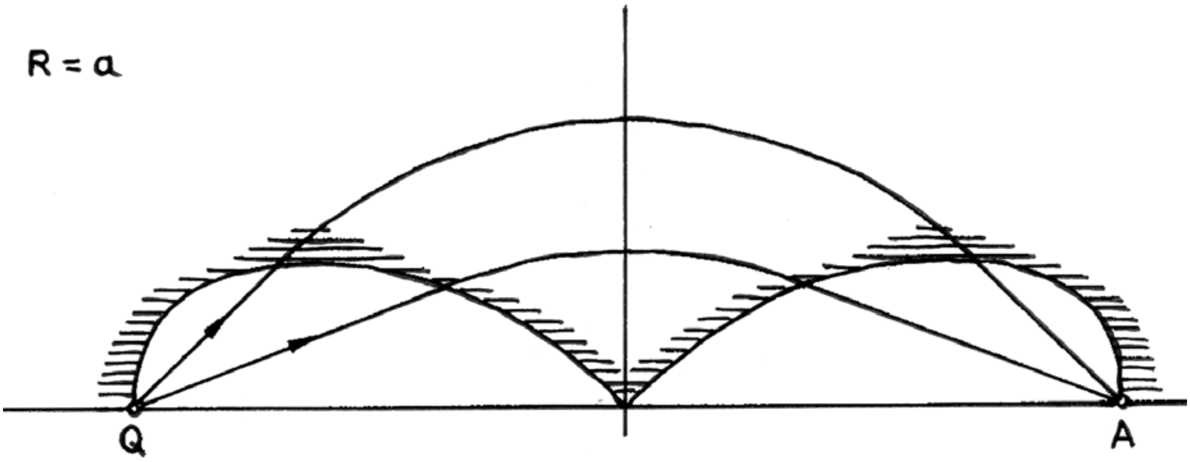
$$e \cdot v \cdot B = \frac{m \cdot v^2}{R} \qquad (3).$$

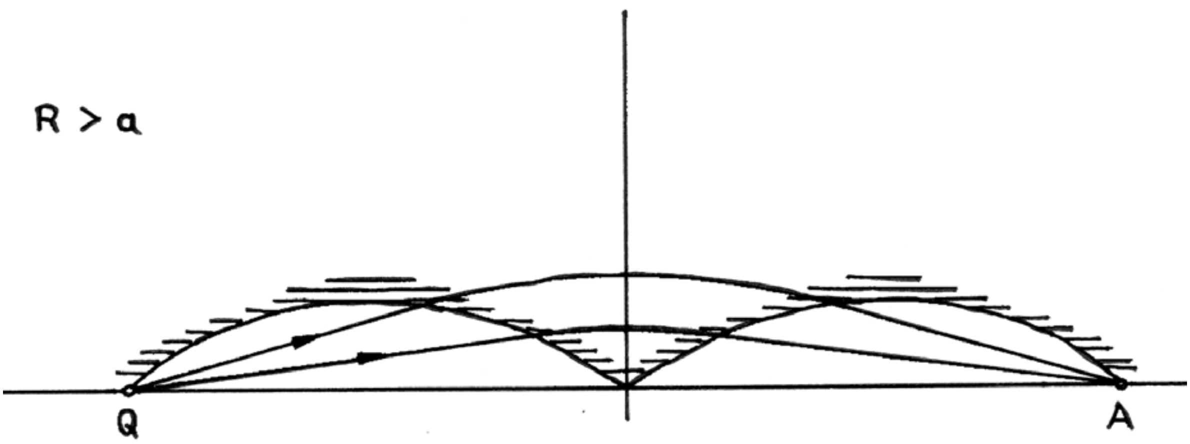
From equation (3) the radius of the ion path is calculated:

$$R = \frac{1}{B}\sqrt{\frac{2 \cdot m \cdot U}{e}} \qquad (4).$$

b) All ions of mass m travel on circular paths of radius $R = v{\cdot}m / e{\cdot}B$ inside the magnetic field. Leaving the magnetic field they fly in a straight line along the last tangent. The centres of curvature of the ion paths lie on the middle perpendicular on $\overline{QA}$ since the magnetic field is assumed to be symmetric to the middle perpendicular on $\overline{QA}$. The paths of the focussed ions are above $\overline{QA}$ due to the direction of the magnetic field.


R < a
Q
a
a
A

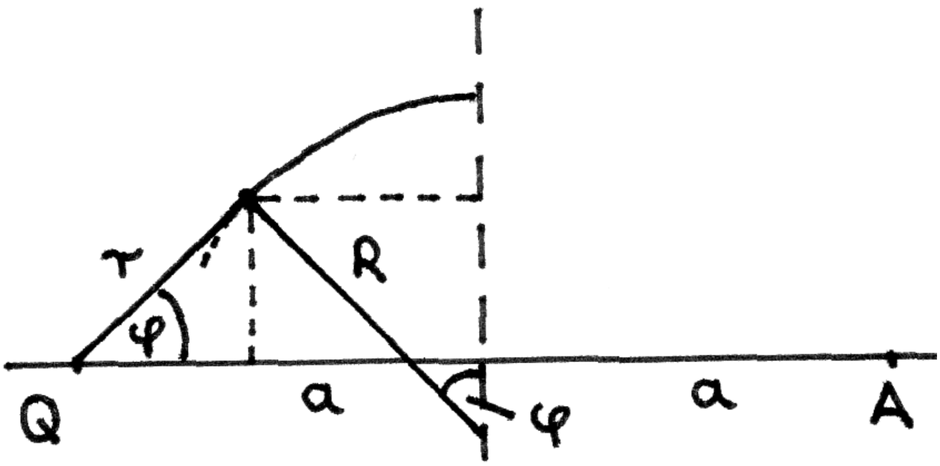

R = a
Q
A


R > a
Q
A

c) The construction method of the boundaries of the magnetic fields is based on the considerations in part b:

- Sketch circles of radius R and different centres of curvature on the middle perpendicular on $\overline{\mathrm{QA}}$.
- Sketch tangents on the circle with either point Q or point A on these straight lines.
- The points of tangency make up the boundaries of the magnetic field. If $R > a$ then not all ions will reach point A. Ions starting at an angle steeper than the tangent at Q, do not arrive in A. The figure on the last page shows the boundaries of the magnetic field for the three cases $R < a$, $R = a$ and $R > a$.

d) It is convenient to deduce a general equation for the boundaries of the magnetic field in polar coordinates (r, φ) instead of using cartesian coordinates (x, y).



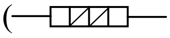
The following relation is obtained from the figure:

$$r \cdot \cos\varphi + R \sin\varphi = a \qquad (7).$$

The boundaries of the magnetic field are given by:

$$r = \frac{a}{\cos\varphi}\left(1 - \frac{R}{a}\sin\varphi\right) \qquad (8).$$

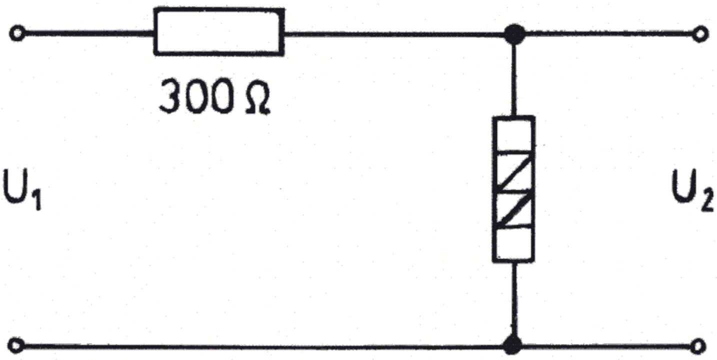
Experimental problem: “Semiconductor element”

In this experiment a semiconductor element (), an adjustable resistor (up to 140 Ω), a fixed resistor (300 Ω), a 9-V-direct voltage source, cables and two multimeters are at disposal. It is not allowed to use the multimeters as ohmmeters.

a) Determine the current-voltage-characteristics of the semiconductor element taking into account the fact that the maximum load permitted is 250 mW. Write down your data in tabular form and plot your data. Before your measurements consider how an overload of the semiconductor element can surely be avoided and note down your thoughts. Sketch the circuit diagram of the chosen setup and discuss the systematic errors of the circuit.

b) Calculate the resistance (dynamic resistance) of the semiconductor element for a current of 25 mA.

c) Determine the dependence of output voltage U_2 from the input voltage U_1 by using the circuit described below. Write down your data in tabular form and plot your data.



The input voltage U_1 varies between 0 V and 9 V. The semiconductor element is to be placed in the circuit in such a manner, that U_2 is as high as possible. Describe the entire circuit diagram in the protocol and discuss the results of the measurements.

d) How does the output voltage U_2 change, when the input voltage is raised from 7 V to 9 V? Explain qualitatively the ratio $\Delta U_1 / \Delta U_2$.

e) What type of semiconductor element is used in the experiment? What is a practical application of the circuit shown above?

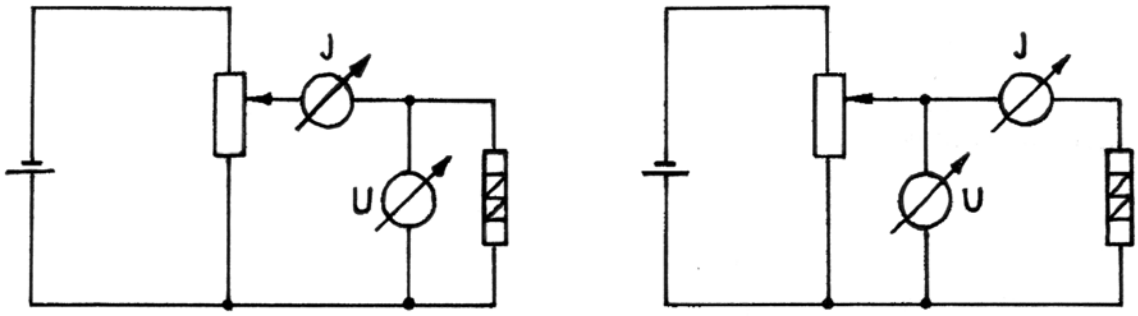
Hints: The multimeters can be used as voltmeter or as ammeter. The precision class of these instruments is 2.5% and they have the following features:

measuring range	50 μA	300 μA	3 mA	30 mA	300 mA	0,3 V	1 V	3 V	10 V
internal resistance	2 kΩ	1 kΩ	100 Ω	10 Ω	1 Ω	6 kΩ	20 kΩ	60 kΩ	200 kΩ

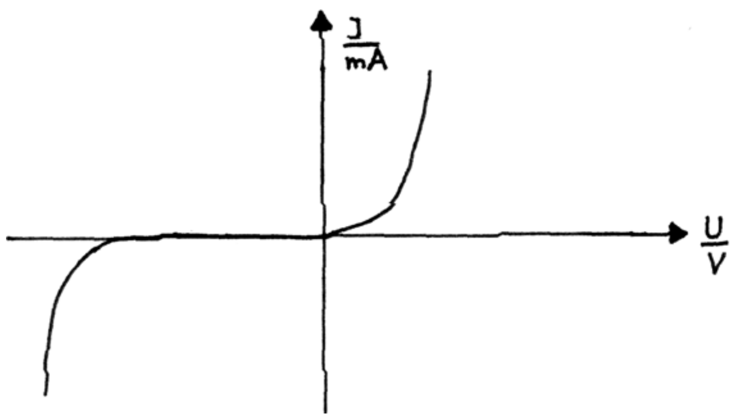
Solution of the experimental problem:

a) Some considerations: the product of the voltage across the semiconductor element U and current I through this element is not allowed to be larger than the maximum permitted load of 250 mW. Therefore the measurements have to be processed in a way, that the product $U \cdot I$ is always smaller than 250 mW.

The figure shows two different circuit diagram that can be used in this experiment:



The complete current-voltage-characteristics look like this:



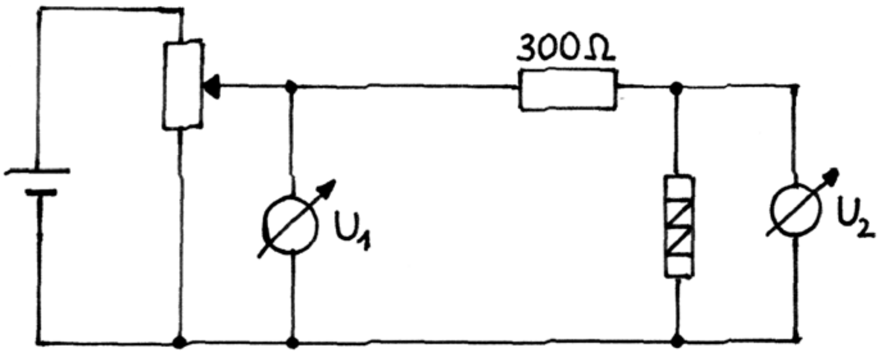
The systematic error is produced by the measuring instruments. Concerning the circuit diagram on the left ("Stromfehlerschaltung"), the ammeter also measures the current running through the voltmeter. The current must therefore be corrected. Concerning the circuit diagram on the right ("Spannungsfehlerschaltung") the voltmeter also measures the voltage across the ammeter. This error must also be corrected. To this end, the given internal resistances of the measuring instruments can be used. Another systematic error is produced by the uncontrolled temperature increase of the semiconductor element, whereby the electric conductivity rises.

b) The dynamic resistance is obtained as ratio of small differences by

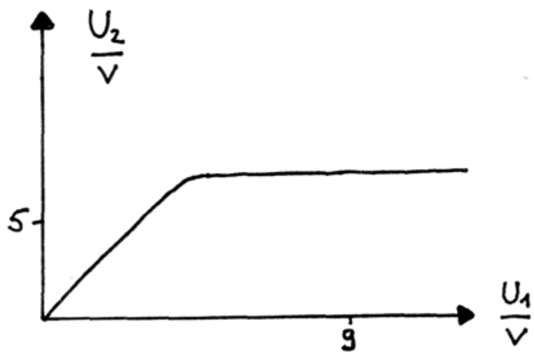
$$R_i = \frac{\Delta U}{\Delta I} \qquad (1).$$

The dynamic resistance is different for the two directions of the current. The order of magnitude in one direction (backward direction) is 10 Ω ± 50% and the order of magnitude in the other direction (flux direction) is 1 Ω ± 50%.

c) The complete circuit diagram contains a potentiometer and two voltmeters.



The graph of the function $U_2 = f(U_1)$ has generally the same form for both directions of the current, but the absolute values are different. By requesting that the semiconductor element has to be placed in such a way, that the output voltage U_2 is as high as possible, a backward direction should be used.



Comment: After exceeding a specific input voltage U_1 the output voltage increases only a little, because with the alteration of U_1 the current I increases (breakdown of the diode) and therefore also the voltage drop at the resistance.

d) The output voltages belonging to U_1 = 7 V and U_1 = 9 V are measured and their difference ΔU_2 is calculated:

$\Delta U_2 = 0.1$ V ± 50% (2).

Comment: The circuit is a voltage divider circuit. Its special behaviour results from the different resistances. The resistance of the semiconductor element is much smaller than the resistance. It changes nonlinear with the voltage across the element. From $R_i << R_V$ follows $\Delta U_2 < \Delta U_1$ in the case of $U_1 > U_2$.

e) The semiconductor element is a Z-diode (Zener diode); also correct: diode and rectifier. The circuit diagram can be used for stabilisation of voltages.

Marking scheme

Problem 1: “Rotating rod” (10 points)

Part a	1 point
Part b – cases 1. and 2.	1 point
– forces and condition of equilibrium	1 point
– case *Z* downwards	2 points
– case *Z* upwards	2 points
– calculation of $r_{1,2}$	1 point
– case $\alpha > \beta$	1 point
– case $\alpha \leq \beta$	1 point

Problem 2: “Thick lens” (10 points)

Part a	1 point
Part b – equation (1), equation (2)	2 points
– physical restrictions, equation (3)	1 point
– discussion of different cases	2 points
– shapes of lenses	1 point
Part c – discussion and equation (4)	1 point
Part d	2 point

Problem 3: “Ions in a magnetic field” (10 points)

Part a – derivation of equations (1) and (2)	1 point
– derivation of equation (4)	1 point
Part b – characteristics properties of the particle paths	3 points
Part c – boundaries of the magnetic field for the three cases	3 points
Part d	2 points

Experimental problem: "Semiconductor element" (20 points)

Part a – considerations concerning overload, circuit diagram, experiment and measurements, complete current-voltage- -characteristics discussion of the systematic errors	6 points
Part b – equation (1) dynamic resistance for both directions correct results within ±50%	3 points
Part c – complete circuit diagram, measurements, graph of the function $U_2 = f(U_1)$, correct comment	5 points
Part d – correct ΔU_2 within ±50%, correct comment	3 points
Part e – Zener-diode (diode, rectifier) and stabilisation of voltages	3 points

Remarks: If the diode is destroyed two points are deducted.

If a multimeter is destroyed five points are deducted.

Problems of the 9th International Physics Olympiads
(Budapest, Hungary, 1976)

Theoretical problems

Problem 1

A hollow sphere of radius R = 0.5 m rotates about a vertical axis through its centre with an angular velocity of ω = 5 s^{-1}. Inside the sphere a small block is moving together with the sphere at the height of R/2 (*Fig. 6*). (g = 10 m/s^2.)

a) What should be at least the coefficient of friction to fulfill this condition?

b) Find the minimal coefficient of friction also for the case of ω = 8 s^{-1}.

c) Investigate the problem of stability in both cases,

α) for a small change of the position of the block,

β) for a small change of the angular velocity of the sphere.

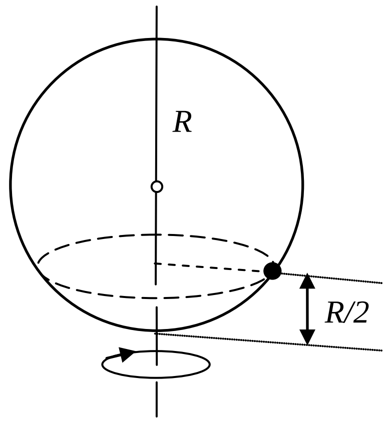


Figure 6

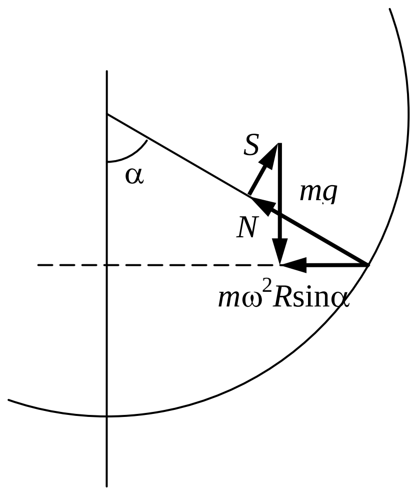


Figure 7

Solution

a) The block moves along a horizontal circle of radius $R\sin\alpha$. The net force acting on the block is pointed to the centre of this circle (*Fig. 7*). The vector sum of the normal force exerted by the wall N, the frictional force S and the weight mg is equal to the resultant: $m\omega^2 R\sin\alpha$.

The connections between the horizontal and vertical components:

$$m\omega^2 R\sin\alpha = N\sin\alpha - S\cos\alpha\ ,$$

$$mg = N\cos\alpha + S\sin\alpha\ .$$

The solution of the system of equations:

$$S = mg\sin\alpha\left(1 - \frac{\omega^2 R\cos\alpha}{g}\right),$$

$$N = mg\left(\cos\alpha + \frac{\omega^2 R\sin^2\alpha}{g}\right).$$

The block does not slip down if

$$\mu_a \geq \frac{S}{N} = \sin\alpha \cdot \frac{1 - \dfrac{\omega^2 R\cos\alpha}{g}}{\cos\alpha + \dfrac{\omega^2 R\sin^2\alpha}{g}} = \frac{3\sqrt{3}}{23} = \mathbf{0.2259}.$$

In this case there must be at least this friction to prevent slipping, i.e. sliding down.

b) If on the other hand $\dfrac{\omega^2 R\cos\alpha}{g} > 1$ some friction is necessary to prevent the block to slip upwards. $m\omega^2 R\sin\alpha$ must be equal to the resultant of forces *S*, *N* and *mg*. Condition for the minimal coefficient of friction is (*Fig. 8*):

$$\mu_b \geq \frac{S}{N} = \sin\alpha \cdot \frac{\dfrac{\omega^2 R\cos\alpha}{g} - 1}{\cos\alpha + \dfrac{\omega^2 R\sin^2\alpha}{g}} =$$

$$= \frac{3\sqrt{3}}{29} = \mathbf{0.1792}.$$

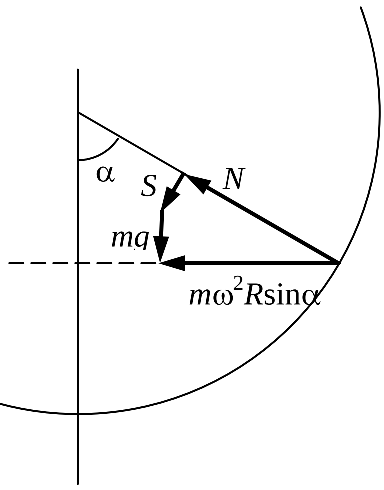
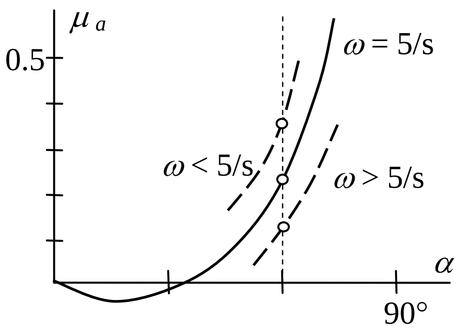
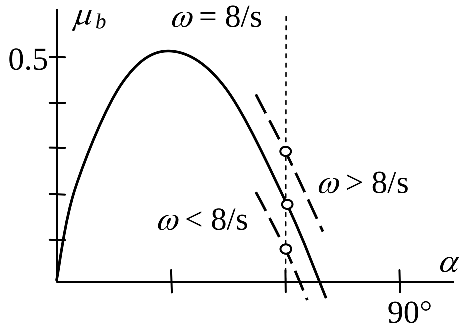


Figure 8

c) We have to investigate μ_a and μ_b as functions of α and ω in the cases a) and b) (see *Fig. 9/a* and *9/b*):



Figure



Figure

In case a): if the block slips upwards, it comes back; if it slips down it does not return. If ω increases, the block remains in equilibrium, if ω decreases it slips downwards.

In case b): if the block slips upwards it stays there; if the block slips downwards it returns. If ω increases the block climbs upwards, if ω decreases the block remains in equilibrium.

Problem 2

The walls of a cylinder of base 1 dm^2, the piston and the inner dividing wall are perfect heat insulators (*Fig. 10*). The valve in the dividing wall opens if the pressure on the right side is greater than on the left side. Initially there is 12 g helium in the left side and 2 g helium in the right side. The lengths of both sides are 11.2 dm each and the temperature is $0°C$. Outside we have a pressure of 100 kPa. The specific heat at constant volume is $c_v = 3.15$ J/gK, at constant pressure it is $c_p = 5.25$ J/gK. The piston is pushed slowly towards the dividing wall. When the valve opens we stop then continue pushing slowly until the wall is reached. Find the work done on the piston by us.

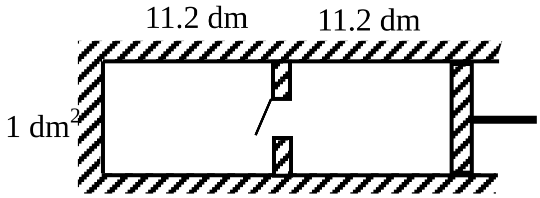


Figure 10

Solution

The volume of 4 g helium at $0°C$ temperature and a pressure of 100 kPa is 22.4 dm^3 (molar volume). It follows that initially the pressure on the left hand side is 600 kPa, on the right hand side 100 kPa. Therefore the valve is closed.

An adiabatic compression happens until the pressure in the right side reaches 600 kPa ($\kappa = 5/3$).

$$100 \cdot 11.2^{5/3} = 600 \cdot V^{5/3},$$

hence the volume on the right side (when the valve opens):

$$V = 3.82 \text{ dm}^3.$$

From the ideal gas equation the temperature is on the right side at this point

$$T_1 = \frac{pV}{nR} = 552\text{K}.$$

During this phase the whole work performed increases the internal energy of the gas:

$$W_1 = (3.15 \text{ J/gK}) \cdot (2 \text{ g}) \cdot (552 \text{ K} - 273 \text{ K}) = 1760 \text{ J}.$$

Next the valve opens, the piston is arrested. The temperature after the mixing has been completed:

$$T_2 = \frac{12 \cdot 273 + 2 \cdot 552}{14} = 313\text{K}.$$

During this phase there is no change in the energy, no work done on the piston.

An adiabatic compression follows from 11.2 + 3.82 = 15.02 dm^3 to 11.2 dm^3:

$$313 \cdot 15.02^{2/3} = T_3 \cdot 11.2^{2/3},$$

hence

$$T_3 = 381 \text{ K}.$$

The whole work done increases the energy of the gas:

$$W_3 = (3.15 \text{ J/gK}) \cdot (14 \text{ g}) \cdot (381 \text{ K} - 313 \text{ K}) = 3000 \text{ J}.$$

The total work done:

$$W_{total} = W_1 + W_3 = 4760 \text{ J}.$$

The work done by the outside atmospheric pressure should be subtracted:

$$W_{atm} = 100 \text{ kPa} \cdot 11.2 \text{ dm}^3 = 1120 \text{ J}.$$

The work done on the piston by us:

$$W = W_{\text{total}} - W_{\text{atm}} = \mathbf{3640\ J}.$$

Problem 3

Somewhere in a glass sphere there is an air bubble. Describe methods how to determine the diameter of the bubble without damaging the sphere.

Solution

We can not rely on any value about the density of the glass. It is quite uncertain. The index of refraction can be determined using a light beam which does not touch the bubble. Another method consists of immersing the sphere into a liquid of same refraction index: its surface becomes invisible.

A great number of methods can be found.

We can start by determining the axis, the line which joins the centers of the sphere and the bubble. The easiest way is to use the "tumbler-over" method. If the sphere is placed on a horizontal plane the axis takes up a vertical position. The image of the bubble, seen from both directions along the axis, is a circle.

If the sphere is immersed in a liquid of same index of refraction the spherical bubble is practically inside a parallel plate (*Fig. 11*). Its boundaries can be determined either by a micrometer or using parallel light beams.

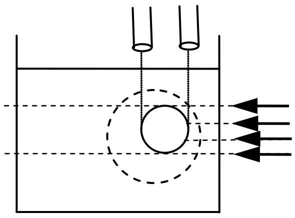
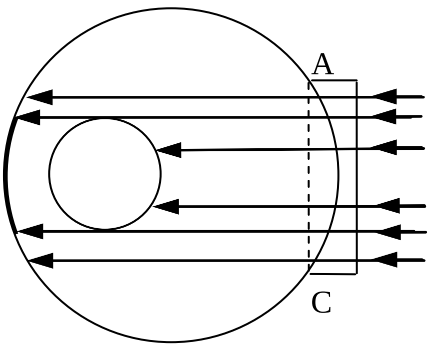


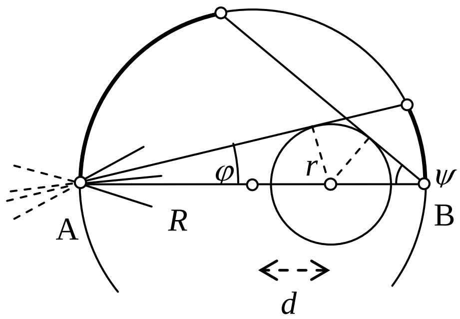
Figure11

Along the axis we have a lens system consisting of two thick negative lenses. The diameter of the bubble can be determined by several measurements and complicated calculations.

If the index of refraction of the glass is known we can fit a plano-concave lens of same index of refraction to the sphere at the end of the axis (*Fig. 12*). As ABCD forms a parallel plate the diameter of the bubble can be measured using parallel light beams.



Fiaure12



Fiaure13

Focusing a light beam on point A of the surface of the sphere (*Fig. 13*) we get a diverging beam from point A inside the sphere. The rays strike the surface at the other side and illuminate a cap. Measuring the spherical cap we get angle φ. Angle ψ can be obtained in a similar way at point B. From

$$\sin\varphi = \frac{r}{R+d} \quad \text{and} \quad \sin\psi = \frac{r}{R-d}$$

we have

$$r = 2R \cdot \frac{\sin\psi \sin\varphi}{\sin\psi + \sin\varphi}, \qquad d = R \cdot \frac{\sin\psi - \sin\varphi}{\sin\psi + \sin\varphi}.$$

The diameter of the bubble can be determined also by the help of X-rays. X-rays are not refracted by glass. They will cast shadows indicating the structure of the body, in our case the position and diameter of the bubble.

We can also determine the moment of inertia with respect to the axis and thus the diameter of the bubble.

Experimental problem

The whole text given to the students:

At the workplace there are beyond other devices a test tube with 12 V electrical heating, a liquid with known specific heat (c_0 = 2.1 J/g°C) and an X material with unknown thermal properties. The X material is insoluble in the liquid.

Examine the thermal properties of the X crystal material between room temperature and 70 °C. Determine the thermal data of the X material. Tabulate and plot the measured data.

(You can use only the devices and materials prepared on the table. The damaged devices and the used up materials are not replaceable.)

Solution

Heating first the liquid then the liquid and the crystalline substance together two time-temperature graphs can be plotted. From the graphs specific heat, melting point and heat of fusion can be easily obtained.

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Problem 1. The compression ratio of a four-stroke internal combustion engine is $\varepsilon = 9.5$. The engine draws in air and gaseous fuel at a temperature 27 °C at a pressure 1 atm = 100 kPa. Compression follows an adiabatic process from point 1 to point 2, see Fig. 1. The pressure in the cylinder is doubled during the mixture ignition (2–3). The hot exhaust gas expands adiabatically to the volume V_2 pushing the piston downwards (3–4). Then the exhaust valve opens and the pressure gets back to the initial value of 1 atm. All processes in the cylinder are supposed to be ideal. The Poisson constant (i.e. the ratio of specific heats C_p/C_V) for the mixture and exhaust gas is $\kappa = 1.40$. (The compression ratio is the ratio of the volume of the cylinder when the piston is at the bottom to the volume when the piston is at the top.)

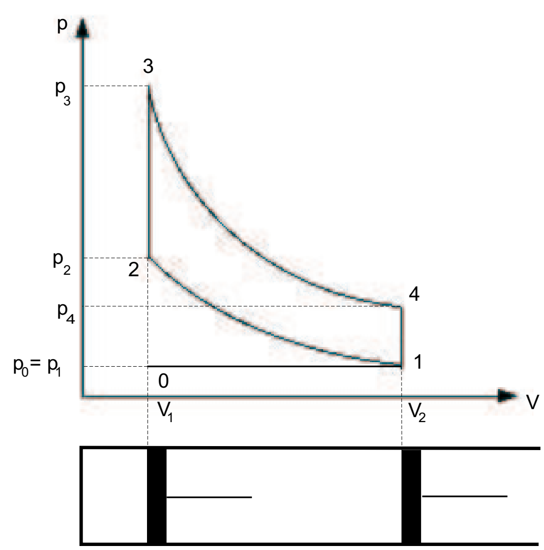


Figure 1:

a) Which processes run between the points 0–1, 2–3, 4–1, 1–0?

b) Determine the pressure and the temperature in the states 1, 2, 3 and 4.

c) Find the thermal efficiency of the cycle.

d) Discuss obtained results. Are they realistic?

Solution: a) The description of the processes between particular points is the following:

0–1 :	intake stroke	isobaric and isothermal process
1–2 :	compression of the mixture	adiabatic process
2–3 :	mixture ignition	isochoric process
3–4 :	expansion of the exhaust gas	adiabatic process
4–1 :	exhaust	isochoric process
1–0 :	exhaust	isobaric process

Let us denote the initial volume of the cylinder before induction at the point 0 by V_1, after induction at the point 1 by V_2 and the temperatures at the particular points by T_0, T_1, T_2, T_3 and T_4.

b) The equations for particular processes are as follows.

0–1 : The fuel-air mixture is drawn into the cylinder at the temperature of $T_0 = T_1 = 300$ K and a pressure of $p_0 = p_1 = 0.10$ MPa.

1–2 : Since the compression is very fast, one can suppose the process to be adiabatic. Hence:

$$p_1 V_2^\kappa = p_2 V_1^\kappa \quad \text{and} \quad \frac{p_1 V_2}{T_1} = \frac{p_2 V_1}{T_2}\,.$$

From the first equation one obtains

$$p_2 = p_1 \left(\frac{V_2}{V_1}\right)^\kappa = p_1 \varepsilon^\kappa$$

and by the dividing of both equations we arrive after a straightforward calculation at

$$T_1 V_2^{\kappa-1} = T_2 V_1^{\kappa-1}\,, \quad T_2 = T_1 \left(\frac{V_2}{V_1}\right)^{\kappa-1} = T_1 \varepsilon^{\kappa-1}\,.$$

For given values $\kappa = 1.40$, $\varepsilon = 9.5$, $p_1 = 0.10$ MPa, $T_1 = 300$ K we have $p_2 = 2.34$ MPa and $T_2 = 738$ K ($t_2 = 465\,°\mathrm{C}$).

2–3 : Because the process is isochoric and $p_3 = 2p_2$ holds true, we can write

$$\frac{p_3}{p_2} = \frac{T_3}{T_2}\,, \quad \text{which implies} \quad T_3 = T_2\frac{p_3}{p_2} = 2T_2\,.$$

Numerically, $p_3 = 4.68$ MPa, $T_3 = 1476$ K ($t_3 = 1203\,^\circ$C).

3–4 : The expansion is adiabatic, therefore

$$p_3V_1^\kappa = p_4V_2^\kappa\,, \quad \frac{p_3V_1}{T_3} = \frac{p_4V_2}{T_4}\,.$$

The first equation gives

$$p_4 = p_3\left(\frac{V_1}{V_2}\right)^\kappa = 2p_2\varepsilon^{-\kappa} = 2p_1$$

and by dividing we get

$$T_3V_1^{\kappa-1} = T_4V_2^{\kappa-1}\,.$$

Consequently,

$$T_4 = T_3\varepsilon^{1-\kappa} = 2T_2\varepsilon^{1-\kappa} = 2T_1\,.$$

Numerical results: $p_4 = 0.20$ MPa, $T_3 = 600$ K ($t_3 = 327\,^\circ$C).

4–1 : The process is isochoric. Denoting the temperature by T_1' we can write

$$\frac{p_4}{p_1} = \frac{T_4}{T_1'}\,,$$

which yields

$$T_1' = T_4\frac{p_1}{p_4} = \frac{T_4}{2} = T_1\,.$$

We have thus obtained the correct result $T_1' = T_1$. Numerically, $p_1 =$ 0.10 MPa, $T_1' = 300$ K.

c) Thermal efficiency of the engine is defined as the proportion of the heat supplied that is converted to net work. The exhaust gas does work on the piston during the expansion 3–4, on the other hand, the work is done on the mixture during the compression 1–2. No work is done by/on the gas during the processes 2–3 and 4–1. The heat is supplied to the gas during the process 2–3.

The net work done by 1 mol of the gas is

$$W = \frac{R}{\kappa - 1}(T_1 - T_2) + \frac{R}{\kappa - 1}(T_3 - T_4) = \frac{R}{\kappa - 1}(T_1 - T_2 + T_3 - T_4)$$

and the heat supplied to the gas is

$$Q_{23} = C_V(T_3 - T_2)\,.$$

Hence, we have for thermal efficiency

$$\eta = \frac{W}{Q_{23}} = \frac{R}{(\kappa - 1)C_V}\,\frac{T_1 - T_2 + T_3 - T_4}{T_3 - T_2}\,.$$

Since

$$\frac{R}{(\kappa - 1)C_V} = \frac{C_p - C_V}{(\kappa - 1)C_V} = \frac{\kappa - 1}{\kappa - 1} = 1\,,$$

we obtain

$$\eta = 1 - \frac{T_4 - T_1}{T_3 - T_2} = 1 - \frac{T_1}{T_2} = 1 - \varepsilon^{1-\kappa}\,.$$

Numerically, $\eta = 1 - 300/738 = 1 - 0.407$, $\eta = 59,3\%$.

d) Actually, the real pV-diagram of the cycle is smooth, without the sharp angles. Since the gas is not ideal, the real efficiency would be lower than the calculated one.

Problem 2. Dipping the frame in a soap solution, the soap forms a rectangle film of length b and height h. White light falls on the film at an angle α (measured with respect to the normal direction). The reflected light displays a green color of wavelength λ_0.

a) Find out if it is possible to determine the mass of the soap film using the laboratory scales which has calibration accuracy of 0.1 mg.

b) What color does the thinnest possible soap film display being seen from the perpendicular direction? Derive the related equations.

Constants and given data: relative refractive index $n = 1.33$, the wavelength of the reflected green light $\lambda_0 = 500$ nm, $\alpha = 30^o$, $b = 0.020$ m, $h = 0.030$ m, density $\varrho = 1000$ kg m^{-3}.

Solution: The thin layer reflects the monochromatic light of the wavelength λ in the best way, if the following equation holds true

$$2nd\cos\beta = (2k+1)\frac{\lambda}{2}\,, \quad k = 0, 1, 2, \dots\,, \tag{1}$$

where k denotes an integer and β is the angle of refraction satisfying

$$\frac{\sin\alpha}{\sin\beta} = n\,.$$

Hence,

$$\cos\beta = \sqrt{1-\sin^2\beta} = \frac{1}{n}\sqrt{n^2-\sin^2\alpha}\,.$$

Substituting to (1) we obtain

$$2d\sqrt{n^2-\sin^2\alpha} = (2k+1)\frac{\lambda}{2}\,. \tag{2}$$

If the white light falls on a layer, the colors of wavelengths obeying (2) are reinforced in the reflected light. If the wavelength of the reflected light is λ_0, the thickness of the layer satisfies for the kth order interference

$$d_k = \frac{(2k+1)\lambda_0}{4\sqrt{n^2-\sin^2\alpha}} = (2k+1)d_0\,.$$

For given values and $k = 0$ we obtain $d_0 = 1.01 \cdot 10^{-7}$ m.

a) The mass of the soap film is $m_k = \varrho_k b\,h\,d_k$. Substituting the given values, we get $m_0 = 6.06 \cdot 10^{-2}$ mg, $m_1 = 18.2 \cdot 10^{-2}$ mg, $m_2 = 30.3 \cdot 10^{-8}$ mg, etc. The mass of the thinnest film thus cannot be determined by given laboratory scales.

b) If the light falls at the angle of 30° then the film seen from the perpendicular direction cannot be colored. It would appear dark.

Problem 3. An electron gun T emits electrons accelerated by a potential difference U in a vacuum in the direction of the line a as shown in Fig. 2. The target M is placed at a distance d from the electron gun in such a way that the line segment connecting the points T and M and the line a subtend the angle α as shown in Fig. 2. Find the magnetic induction B of the uniform magnetic field

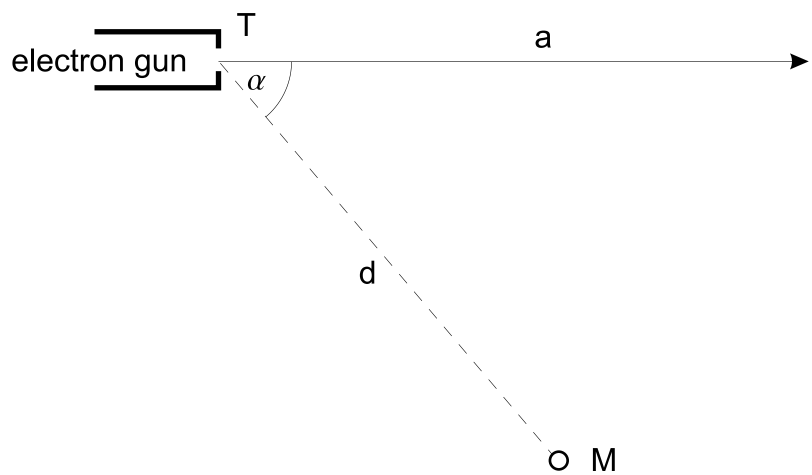


Figure 2:

a) perpendicular to the plane determined by the line a and the point M

b) parallel to the segment TM

in order that the electrons hit the target M. Find first the general solution and then substitute the following values: $U = 1000$ V, $e = 1.60 \cdot 10^{-19}$ C, $m_e = 9.11 \cdot 10^{-31}$ kg, $\alpha = 60^{\circ}$, $d = 5.0$ cm, $B < 0.030$ T.

Solution: a) If a uniform magnetic field is perpendicular to the initial direction of motion of an electron beam, the electrons will be deflected by a force that is always perpendicular to their velocity and to the magnetic field. Consequently, the beam will be deflected into a circular trajectory. The origin of the centripetal force is the Lorentz force, so

$$Bev = \frac{m_e v^2}{r} \,. \tag{3}$$

Geometrical considerations yield that the radius of the trajectory obeys (cf. Fig. 3).

$$r = \frac{d}{2\sin\alpha} \,. \tag{4}$$

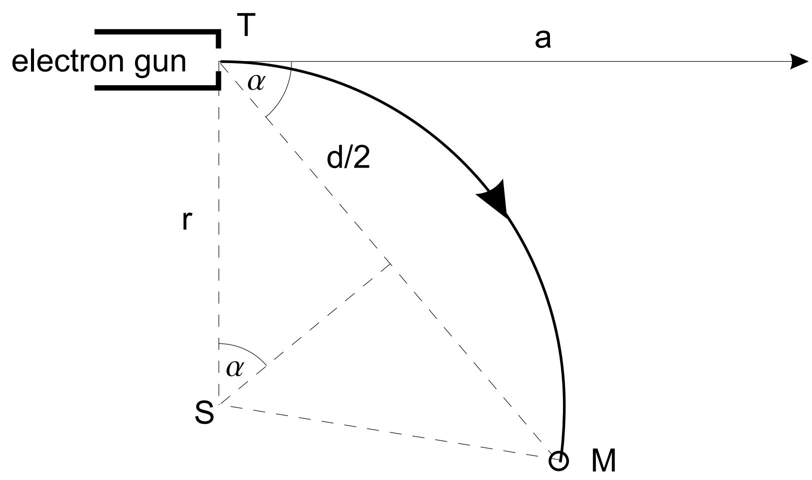


Figure 3:

The velocity of electrons can be determined from the relation between the kinetic energy of an electron and the work done on this electron by the electric field of the voltage U inside the gun,

$$\frac{1}{2}m_e v^2 = eU\,. \tag{5}$$

Using (3), (4) and (5) one obtains

$$B = m_e\sqrt{\frac{2eU}{m_e}}\frac{2\sin\alpha}{ed} = 2\sqrt{\frac{2Um_e}{e}}\frac{\sin\alpha}{d}\,.$$

Substituting the given values we have $B = 3.70\cdot 10^{-3}$ T.

b) If a uniform magnetic field is neither perpendicular nor parallel to the initial direction of motion of an electron beam, the electrons will be deflected into a helical trajectory. Namely, the motion of electrons will be composed of an uniform motion on a circle in the plane perpendicular to the magnetic field and of an uniform rectilinear motion in the direction of the magnetic field. The component $\vec{v}_1$ of the initial velocity $\vec{v}$, which is perpendicular to the magnetic field (see Fig. 4), will manifest itself at the Lorentz force and during the motion will rotate uniformly around the line parallel to the magnetic field. The component $\vec{v}_2$ parallel to the magnetic field will remain

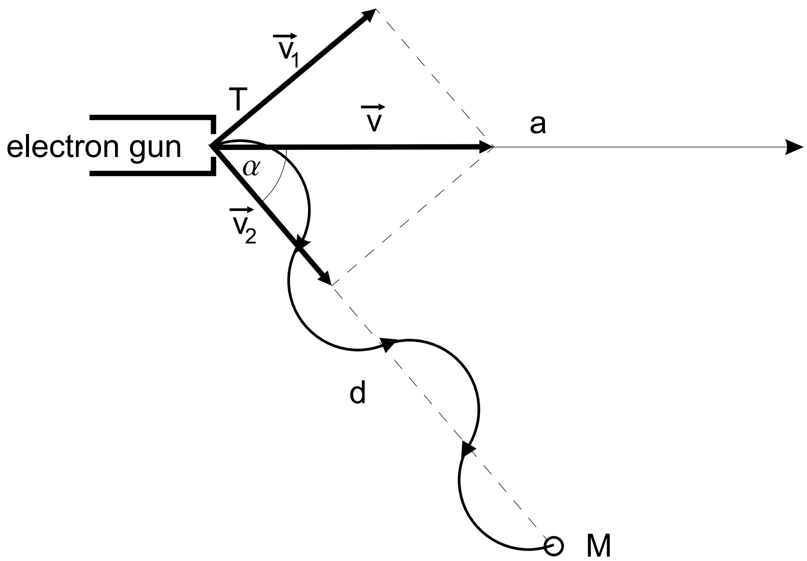


Figure 4:

constant during the motion, it will be the velocity of the uniform rectilinear motion. Magnitudes of the components of the velocity can be expressed as

$$v_1 = v \sin\alpha \quad v_2 = v \cos\alpha \,.$$

Denoting by N the number of screws of the helix we can write for the time of motion of the electron

$$t = \frac{d}{v_2} = \frac{d}{v\cos\alpha} = \frac{2\pi r N}{v_1} = \frac{2\pi r N}{v \sin\alpha} \,.$$

Hence we can calculate the radius of the circular trajectory

$$r = \frac{d \sin\alpha}{2\pi N \cos\alpha} \,.$$

However, the Lorentz force must be equated to the centripetal force

$$B e v \sin\alpha = \frac{m_e v^2 \sin^2\alpha}{r} = \frac{m_e v^2 \sin^2\alpha}{\frac{d\sin\alpha}{2\pi N \cos\alpha}} \,. \qquad (6)$$

Consequently,

$$B = \frac{m_e v^2 \sin^2 \alpha \, 2\pi N \cos \alpha}{d \sin \alpha \, ev \sin \alpha} = \frac{2\pi N m_e v \cos \alpha}{de} \, .$$

The magnitude of velocity v again satisfies (5), so

$$v = \sqrt{\frac{2Ue}{m_e}} \, .$$

Substituting into (6) one obtains

$$B = \frac{2\pi N \cos \alpha}{d} \sqrt{\frac{2Um_e}{e}} \, .$$

Numerically we get $B = N \cdot 6.70 \cdot 10^{-3}$ T. If $B < 0.030$ T should hold true, we have four possibilities ($N \leq 4$). Namely,

$$B_1 = 6.70 \cdot 10^{-3} \text{ T} \, ,$$
$$B_2 = 13.4 \cdot 10^{-3} \text{ T} \, ,$$
$$B_3 = 20.1 \cdot 10^{-3} \text{ T} \, ,$$
$$B_4 = 26.8 \cdot 10^{-3} \text{ T} \, .$$

Problems of the XI International Olympiad, Moscow, 1979

The publication has been prepared by Prof. S. Kozel and Prof. V.Orlov

(Moscow Institute of Physics and Technology)

The XI International Olympiad in Physics for students took place in Moscow, USSR, in July 1979 on the basis of Moscow Institute of Physics and Technology (MIPT). Teams from 11 countries participated in the competition, namely Bulgaria, Finland, Germany, Hungary, Poland, Romania, Sweden, Czechoslovakia, the DDR, the SFR Yugoslavia, the USSR. The problems for the theoretical competition have been prepared by professors of MIPT (V.Belonuchkin, I.Slobodetsky, S.Kozel). The problem for the experimental competition has been worked out by O.Kabardin from the Academy of Pedagogical Sciences.

It is pity that marking schemes were not preserved.

Theoretical Problems

Problem 1.

A space rocket with mass M=12t is moving around the Moon along the circular orbit at the height of h =100 km. The engine is activated for a short time to pass at the lunar landing orbit. The velocity of the ejected gases u = 10^4 m/s. The Moon radius R_M = $1{,}7\cdot 10^3$ km, the acceleration of gravity near the Moon surface g_M = 1.7 m/s^2

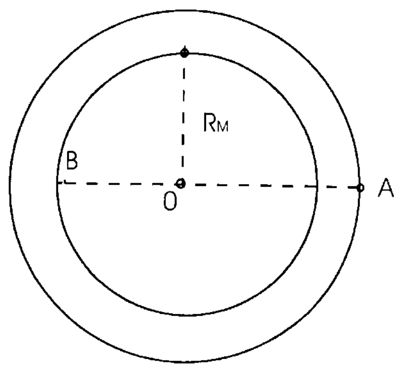


Fig.1

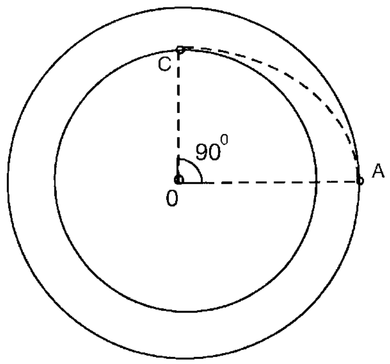


Fig.2

1). What amount of fuel should be spent so that when activating the braking engine at point A of the trajectory, the rocket would land on the Moon at point B (Fig.1)?

2). In the second scenario of landing, at point A the rocket is given an impulse directed towards the center of the Moon, to put the rocket to the orbit meeting the Moon surface at point C (Fig.2). What amount of fuel is needed in this case?

Problem 2.

Brass weights are used to weigh an aluminum-made sample on an analytical balance. The weighing is ones in dry air and another time in humid air with the water vapor pressure $P_h = 2 \cdot 10^3$ Pa. The total atmospheric pressure ($P = 10^5$ Pa) and the temperature ($t = 20°$ C) are the same in both cases.

What should the mass of the sample be to be able to tell the difference in the balance readings provided their sensitivity is $m_0 = 0.1$ mg ?

Aluminum density $\rho_1 = 2700$ kg/m^3, brass density $\rho_2 = .8500$ kg/m^3.

Problem 3

.During the Soviet-French experiment on the optical location of the Moon the light pulse of a ruby laser ($\lambda = 0,69 \mu m$) was directed to the Moon's surface by the telescope with a diameter of the mirror D = 2,6 m. The reflector on the Moon's surface reflected the light backward as an ideal mirror with the diameter d = 20 cm. The reflected light was then collected by the same telescope and focused at the photodetector.

1) What must the accuracy to direct the telescope optical axis be in this experiment?
2) What part of emitted laser energy can be detected after reflection on the Moon, if we neglect the light loses in the Earth's atmosphere?
3) Can we see a reflected light pulse with naked eye if the energy of single laser pulse E = 1 J and the threshold sensitivity of eye is equal n =100 light quantum?
4) Suppose the Moon's surface reflects α = 10% of the incident light in the spatial angle 2π steradian, estimate the advantage of a using reflector.

The distance from the Earth to the Moon is L = 380000 km. The diameter of pupil of the eye is d_p = 5mm. Plank constant is $h = 6.6 \cdot 10^{-34}$ J·s.

Experimental Problem

Define the electrical circuit scheme in a "black box" and determine the parameters of its elements.

List of instruments: A DC source with tension 4.5 V, an AC source with 50 Hz frequency and output voltage up to 30 V, two multimeters for measuring AC/DC current and voltage, variable resistor, connection wires.

Solution of Problems of the XI International Olympiad, Moscow, 1979

Solution of Theoretical Problems

Problem 1.

1) During the rocket moving along the circular orbit its centripetal acceleration is created by moon gravity force:

$$G\frac{MM_M}{R^2} = \frac{Mv_0^2}{R},$$

where $R = R_M + h$ is the primary orbit radius, v_0 -the rocket velocity on the circular orbit:

$$v_0 = \sqrt{G\frac{M_M}{R}}$$

Since $g_M = G\frac{M_M}{R_M^2}$ it yields

$$v_0 = \sqrt{\frac{g_M R_M^2}{R}} = R_M\sqrt{\frac{g_M}{R_M + h}} \qquad (1)$$

The rocket velocity will remain perpendicular to the radius-vector OA after the braking engine sends tangential momentum to the rocket (Fig.1). The rocket should then move along the elliptical trajectory with the focus in the Moon's center.

Denoting the rocket velocity at points A and B as v_A and v_B we can write the equations for energy and momentum conservation as follows:

$$\frac{Mv_A^2}{2} - G\frac{MM_M}{R} = \frac{Mv_B^2}{2} - G\frac{MM_M}{R_M} \qquad (2)$$

$$Mv_AR = Mv_BR_M \qquad (3)$$

Solving equations (2) and (3) jointly we find

$$v_A = \sqrt{2G\frac{M_M R_M}{R(R + R_M)}}$$

Taking (1) into account, we get

$$v_A = v_0\sqrt{\frac{2R_M}{R+R_M}}\ .$$

Thus the rocket velocity change Δv at point A must be

$$\Delta v = v_0 - v_A = v_0\left(1-\sqrt{\frac{2R_M}{R+R_M}}\right) = v_0\left(1-\sqrt{\frac{2R_M}{2R_M+h}}\right) = 24 m/s.$$

Since the engine switches on for a short time the momentum conservation low in the system "rocket-fuel" can be written in the form

$$(M - m_1)\Delta v = m_1 u$$

where m_1 is the burnt fuel mass.

This yields

$$m_1 = \frac{\Delta v}{u+\Delta v}$$

Allow for $\Delta v << u$ we find

$$m_1 \approx \frac{\Delta v}{u}M = 29\text{kg}$$

2) In the second case the vector $\vec{v}_2$ is directed perpendicular to the vector $\vec{v}_0$ thus giving

$$\vec{v}_A = \vec{v}_0 + \Delta\vec{v}_2, \qquad v_A = \sqrt{v_0^2 + \Delta v_2^2}\ .$$

Based on the energy conservation law in this case the equation can be written as

$$\frac{M\left(v_0^2+\Delta v_2^2\right)}{2} - \frac{GMM_M}{R} = \frac{Mv_C^2}{2} - \frac{GMM_M}{R_M} \qquad (4)$$

and from the momentum conservation law

$$Mv_0 R = Mv_C R_M\ . \qquad (5)$$

Solving equations (4) and (5) jointly and taking into account (1) we find

$$\Delta v_2 = \sqrt{g_M\frac{(R-R_M)^2}{R}} = h\sqrt{\frac{g_M}{R_M+h}} \approx 97\text{m/s}\ .$$

Using the momentum conservation law we obtain

$$m_2 = \frac{\Delta v_2}{u}M \approx 116\text{kg}\ .$$

Problem 2.

A sample and weights are affected by the Archimede's buoyancy force of either dry or humid air in the first and second cases, respectively. The difference in the scale indication ΔF is determined by the change of difference of these forces.

The difference of Archimede's buoyancy forces in dry air:

$$\Delta F_1 = \Delta V \rho_a^{'} g$$

Whereas in humid air it is:

$$\Delta F_2 = \Delta V \rho_a^{''} g$$

where ΔV - the difference in volumes between the sample and the weights, and $\rho_a^{'}$ and $\rho_a^{''}$ - densities of dry and humid air, respectively.

Then the difference in the scale indications ΔF could be written as follows:

$$\Delta F = \Delta F_1 - \Delta F_2 = \Delta V g \left(\rho_a^{'} - \rho_a^{''}\right) \qquad (1)$$

According to the problem conditions this difference should be distinguished, i.e.
$\Delta F \geq m_0 g$ or $\Delta V g \left(\rho_a^{'} - \rho_a^{''}\right) \geq m_0$, wherefrom

$$\Delta V \geq \frac{m_0}{\rho_a^{'} - \rho_a^{''}} . \qquad (2)$$

The difference in volumes between the aluminum sample and brass weights can be found from the equation

$$\Delta V = \frac{m}{\rho_1} - \frac{m}{\rho_2} = m\left(\frac{\rho_2 - \rho_1}{\rho_1 \rho_2}\right) , \qquad (3)$$

where m is the sought mass of the sample. From expressions (2) and (3) we obtain

$$m = \Delta V\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right) \geq \frac{m_0}{\rho_a^{'} - \rho_a^{''}}\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right) . \qquad (4)$$

To find the mass m of the sample one has to determine the difference $\left(\rho_a^{'} - \rho_a^{''}\right)$.

With the general pressure being equal, in the second case, some part of dry air is replaced by vapor:

$$\rho_a^{'} - \rho_a^{''} = \frac{\Delta m_a}{V} - \frac{\Delta m_v}{V} .$$

Changes of mass of air Δm_a and vapor Δm_v can be found from the ideal-gas equation of state

$$\Delta m_a = \frac{P_a V M_a}{RT} , \quad \Delta m_v = \frac{P_v V M_v}{RT} ,$$

wherefrom we obtain

$$\rho_a^{'} - \rho_a^{''} = \frac{P_a\left(M_a - M_v\right)}{RT} . \qquad (5)$$

From equations (4) and (5) we obtain

$$m \geq \frac{m_0 RT}{P_a\left(M_a - M_v\right)}\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right) . \qquad (6)$$

The substitution of numerical values gives the answer: $m \geq 0.0432$ kg ≈ 43 g.

Note. When we wrote down expression (3), we considered the sample mass be equal to the weights' mass, at the same time allowing for a small error.

One may choose another way of solving this problem. Let us calculate the change of Archimede's force by the change of the air average molar mass.

In dry air the condition of the balance between the sample and weights could be written down in the form of

$$\left(\rho_1 - \frac{M_a P}{RT}\right)V_1 = \left(\rho_2 - \frac{M_a P}{RT}\right)V_2 \quad . \tag{7}$$

In humid air its molar mass is equal to

$$M = M_a \frac{P_a}{P} + M_v \frac{P - P_a}{P}, \tag{8}$$

whereas the condition of finding the scale error could be written in the form

$$\left(\rho_1 - \frac{M_a P}{RT}\right)V_1 - \left(\rho_2 - \frac{M_a P}{RT}\right)V_2 \geq m_0 . \tag{9}$$

From expressions (7) –(9) one can get a more precise answer

$$m \geq \frac{m_0 RT \rho_1 \rho_2 - M_a P_a}{(M_a - M_v)(\rho_2 - \rho_1)P_a} \quad . \tag{10}$$

Since $M_a P_a << m_0 \rho_1 \rho_2 RT$, then both expressions (6) and (10) lead practically to the same quantitative result, i.e. m $\geq$ 43 g.

Problem 3.

1) The beam divergence angle $\delta\varphi$ caused by diffraction defines the accuracy of the telescope optical axis installation:

$$\delta\varphi \approx \lambda/D \approx 2.6 \cdot 10^{-7} \text{ rad.} \approx 0.05'' .$$

2) The part K_1 of the light energy of a laser, directed to a reflector, may be found by the ratio of the area of S_1 reflector ($S_1 = \pi d^2/4$) versus the area S_2 of the light spot on the Moon ($S_2 = \pi r^2$, where $r = L\,\delta\varphi \approx L\lambda/D$, L – the distance from the Earth to the Moon)

$$K_1 = \frac{S_1}{S_2} = \frac{d^2}{(2r)^2} = \frac{d^2 D^2}{4\lambda^2 L^2}$$

The reflected light beam diverges as well and forms a light spot with the radius R on the Earth's surface:

$$R = \lambda L/d, \quad \text{as} \quad r << R$$

That's why the part K_2 of the reflected energy, which got into the telescope, makes

$$K_2 = \frac{D^2}{(2R)^2} = \frac{D^2 d^2}{4\lambda^2 L^2}$$

The part K_0 of the laser energy, that got into the telescope after having been reflected by the reflector on the Moon, equals

$$K_0 = K_1 K_2 = \left(\frac{dD}{2\lambda L}\right)^4 \approx 10^{-12}$$

3) The pupil of a naked eye receives as less a part of the light flux compared to a telescope, as the area of the pupil S_e is less than the area of the telescope mirror S_t:

$$K_e = K_0 \frac{S_e}{S_t} = K_0 \frac{d_e^2}{D^2} \approx 3.7 \cdot 10^{-18}.$$

So the number of photons N getting into the pupil of the eye is equal

$$N = \frac{E}{h\nu} K_e = 12.$$

Since $N<n$, one can not perceive the reflected pulse with a naked eye.

4) In the absence of a reflector α =10% of the laser energy, that got onto the Moon, are dispersed by the lunar surface within a solid angle $\Omega_1 = 2\pi$ steradian.

The solid angle in which one can see the telescope mirror from the Moon, constitutes

$$\Omega_2 = S_t/L^2 = \pi D^2/4L^2$$

That is why the part K of the energy gets into the telescope and it is equal

$$K = \alpha \frac{\Omega_2}{\Omega_1} = \alpha \frac{D^2}{8L^2} \approx 0.5 \cdot 10^{-18}$$

Thus, the gain β, which is obtained through the use of the reflector is equal

$$\beta = K_0/K \approx 2 \cdot 10^6$$

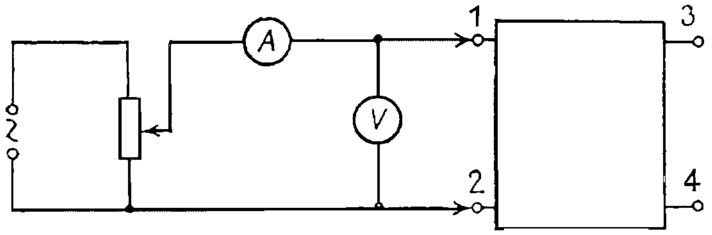
Note. The result obtained is only evaluative as the light flux is unevenly distributed inside the angle of diffraction.

Solution of Experimental Problem.

A transformer is built-in in a "black box". The black box has 4 terminals. To be able to determine the equivalent circuit and the parameters of its elements one may first carry out measurements of the direct current. The most expedient is to mount the circuit according to the layout in Fig.3 and to build volt-ampere characteristics for various terminals of the "box". This enables one to make sure rightway that there were no e.m.f. sources in the "box" (the plot *I=f(U)* goes through the origin of the coordinates), no diodes (the current strength does not depend on the polarity of the current's external source), by the inclination angle of the plot one may define the resistances between different terminals of the "box". The tests allowed for some estimations of values $R_{1\text{-}2}$ and $R_{3\text{-}4}$. The ammeter did not register any current between the other terminals. This means that between these terminals there might be some other resistors with resistances larger than R_L :

$$R_L = \frac{U_{max}}{I_{min}} = \frac{4{,}5\text{V}}{2\cdot 10^{-6}\,\text{A}} = 2.25\cdot 10^{6}\,\text{ohm}$$

where I_{min} - the minimum value of the strength of the current which the instrument would have registered.



"Black box"

Fig.3

Probably there might be some capacitors between terminals 1-3, 1-4, 2-3, 2-4 (Fig.4).

Then, one can carry out analogous measurements of an alternative current. The taken volt-ampere characteristics enabled one to find full resistances on the alternative current of sections 1-2 and 3-4: Z_1 and Z_2 and to compare them to the values R_1 and R_2. It turned out, that $Z_1>R_2$ and $Z_2>R_2$.

Fig.4 Fig.5

This fact allows one to conclude that in the “black box” the coils are connected to terminals 1-2 and 3-4 (Fig.5). Inductances of coils L_1 and L_2 can be determined by the formulas

$$L_1 = \frac{\sqrt{Z_1^2 - R_1^2}}{2\pi\nu}, \qquad L_2 = \frac{\sqrt{Z_2^2 - R_2^2}}{2\pi\nu}.$$

After that the dependences $Z = f(I)$, $L=f(I)$ are to be investigated. The character of the found dependences enabled one to draw a conclusion about the presence of ferromagnetic cores in the coils. Judging by the results of the measurements on the alternative current one could identify the upper limit of capacitance of the capacitors which could be placed between terminals 1-3, 1-4, 2-3, 2-4:

$$C_{\max} = \frac{I_{\min}}{2\pi\nu U_{\max}} = \frac{5\cdot 10^{-6}\,\mathrm{A}}{2\cdot 3.14\cdot 50\mathrm{s}^{-1}\cdot 3\mathrm{V}} = 5\cdot 10^{-9}\,\mathrm{F} = 5\mathrm{nF}$$

Then one could check the availability of inductive coupling between circuits 1-2 and 3-4. The plot of dependence of voltage $U_{3\text{-}4}$ versus voltage $U_{1\text{-}2}$ (Fig. 6) allows one to find both the transformation coefficient

$$K = \frac{U_{1-2}}{U_{3-4}} = \frac{1}{2}$$

and the maximum operational voltages on coils L_1 and L_2, when the transformation

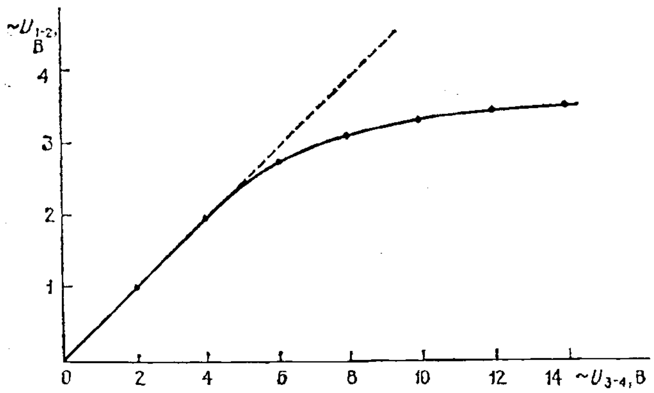


Fig.6

coefficient has not changed yet, i.e. before saturation of the core.

$U_{1\text{-}2(\max)}$ =2.5 V, $U_{3\text{-}4(\max)}$ = 5 V.

One could build either plot $K(U_{1\text{-}2})$ or $K(U_{3\text{-}4})$ (Fig. 7).

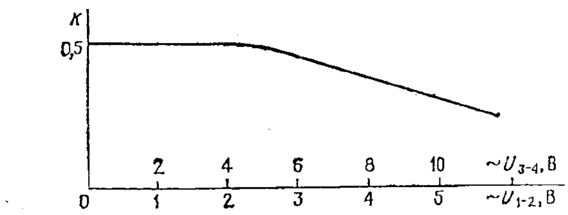


Fig.7

Note: It was also possible to define the “box” circuit after tests of the direct current. To do that one had to find the presence of induction coupling between terminals 1-2 and 3-4, that is the appearance of e.m.f. of induction in circuit 3-4, when closing and breaking circuits 1-2 and vice-versa. When comparing the direction of the pointer’s rejection of the voltmeters connected to terminals 1-2 and 3-4 one could identify directions of the transformer’s windings.

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XII International Physics Olympiad

Varna, Bulgaria, July 1981

The problems and the solutions are adapted by Miroslav Abrashev
University of Sofia, Faculty of Physics, 5 James Bourchier Blvd., 1164 Sofia, Bulgaria

Reference: O. F. Kabardin, V. A. Orlov, in “International Physics Olympiads for High School Students”, ed. V. G. Razumovski, Moscow, Nauka, 1985. (In Russian).

The Experimental Problem

Materials and Instruments: elastic rubber cord (the length of free cord is l_0 = 150 mm), vertically hanged up to a stand, set of weights from 10 g to 100 g, pan for the weights with mass 5 g, chronometer, ruler, millimeter (scaled) paper.

Note: The Earth Acceleration is g = 10 m/s^2. The mass of the rubber cord can be neglected.

Make the following study:

1. Load the rubber cord with weights in the range 15 g to 105 g. Put the data obtained into a table. Make a graph (using suitable scale) with the experimentally obtained dependence of the prolongation of the cord on the stress force F.

2. Using the experimental results, obtained in p.1, calculate and put into a table the volume of the cord as a function of the loading in the range 35 g to 95 g. Do the calculations consequently for each two adjacent values of the loading in this range. Write down the formulas you have used for the calculations. Make an analytical proposition about the dependence of the volume on the loading.

Assume that Young’s modulus is constant: E = 2.10^6 Pa. Take in mind that the Hooke’s law is only approximately valid and the deviations from it can be up to 10%.

3. Determine the volume of the rubber cord, using the chronometer, at mass of the weight equal to 60 g. Write the formulas used.

Solution of the Experimental Problem

1. The measurements of the cord length l_n at different loadings m_n must be at least 10. The results are shown in Table I.

Table 1.

m_n, kg	$F_n = m_n.g$, N	l_n, mm	$\Delta l_n = l_n - l_0$, mm
0.005	0.05	153	3
0.015	0.15	158	8
0.025	0.25	164	14
0.035	0.35	172	22
0.045	0.45	181	31
0.055	0.55	191	41
0.065	0.65	202	53
0.075	0.75	215	65
0.085	0.85	228	78
0.095	0.95	243	93

0.105	10.5	261	111

The obtained dependence of the prolongation of the cord on the stress force F can be drawn on graph. It is shown in Fig. 1.

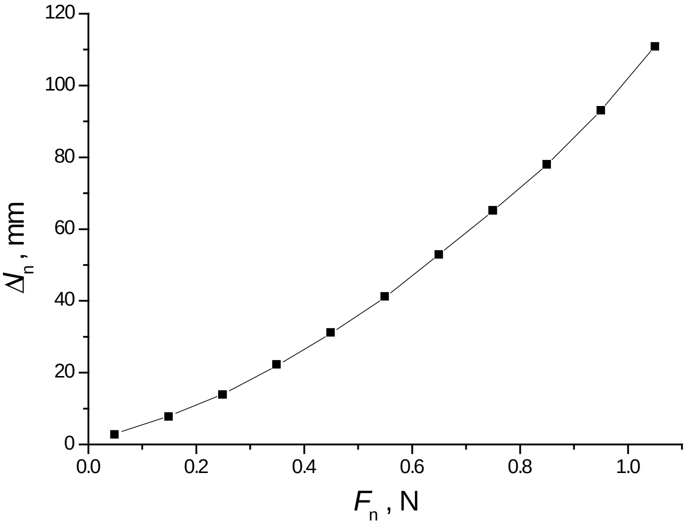


Fig.1

2.For the calculations of the volume the Hooke's law can be used for each measurement:

$$\frac{\Delta l_n'}{l_n} = \frac{1}{E}\frac{\Delta F_n}{S_n},$$

therefore

$$S_n = \frac{l_n \Delta F_n}{E \Delta l_n'},$$

where $\Delta l_n' = l_n - l_{n-1}$, $\Delta F_n = \Delta mg$. (Using the Hooke's law in the form $\frac{\Delta l_n}{l_n} = \frac{1}{E}\frac{F_n}{S_n}$ leads to larger error, because the value of the Δl_n is of the same order as l_n).

As the value of the S_n is determined, it is easy to calculate the volume V_n at each value of F_n:

$$V_n = S_n l_n = \frac{l_n^2 \Delta F_n}{E \Delta l_n'}.$$

Using the data from Table 1, all calculations can be presented in Table 2:

$\Delta m_n = m_n - m_{n-1}$, kg	$\Delta F_n = \Delta m_n g$, N	l_n, m	$\Delta l_n = l_n - l_{n-1}$, m	$S_n = \frac{l_n \Delta F_n}{E \Delta l_n'}$, m^2	$V_n = l_n S_n$, m^3
0.035 – 0.025	0.1	0.172	0.008	$1{,}07.10^{-6}$	184.10^{-9}
0.045 – 0.035	0.1	0.181	0.009	$1{,}01.10^{-6}$	183.10^{-9}
0.055 – 0.045	0.1	0.191	0.010	$0{,}95.10^{-6}$	182.10^{-9}
0.065 – 0.055	0.1	0.203	0.012	$0{,}92.10^{-6}$	187.10^{-9}
0.075 – 0.065	0.1	0.215	0.012	$0{,}89.10^{-6}$	191.10^{-9}
0.085 – 0.075	0.1	0.228	0.013	$0{,}88.10^{-6}$	200.10^{-9}

0.095 – 0.085	0.1	0.243	0.015	$0{,}81.10^{-6}$	196.10^{-9}
0.105 – 0.095	0.1	0.261	0.018	$0{,}72.10^{-6}$	188.10^{-9}

The results show that the relative deviation from the averaged value of the calculated values of the volume is:

$$\varepsilon = \frac{\Delta V_{n,aver.}.100\%}{V_{aver.}} \approx \frac{5{,}3.10^{-9}}{189.10^{-9}}.100\% \approx 2.8\%$$

Therefore, the conclusion is that the volume of the rubber cord upon stretching is constant:

$$V_n = const.$$

3. The volume of the rubber cord at fixed loading can be determined investigating the small vibrations of the cord. The reason for these vibrations is the elastic force:

$$F = ES\frac{\Delta l}{l}$$

Using the second law of Newton:

$$-ES\frac{\Delta l}{l} = m\frac{d^2(\Delta l)}{dt^2},$$

the period of the vibrations can be determined:

$$T = 2\pi\sqrt{\frac{ml}{ES}}.$$

Then

$$S = \frac{(2\pi)^2 ml}{ET^2},$$

and the volume of the cord is equal to:

$$V = Sl = \frac{4\pi^2 ml^2}{ET^2}$$

The measurement of the period gives: $T = t/n$ = 5.25s /10 = 0.52 s at used mass m = 0.065 kg. The result for the volume $V \approx 195.10^{-9}$ m^3, in agreement with the results obtained in part 2.

XII International Physics Olympiad

Varna, Bulgaria, July 1981

The problems and the solutions are adapted by Miroslav Abrashev
University of Sofia, Faculty of Physics, 5 James Bourchier Blvd., 1164 Sofia, Bulgaria

Reference: O. F. Kabardin, V. A. Orlov, in "International Physics Olympiads for High School Students", ed. V. G. Razumovski, Moscow, Nauka, 1985. (In Russian).

Theoretical Problem 1

A static container of mass M and cylindrical shape is placed in vacuum. One of its ends is closed. A fixed piston of mass m and negligible width separates the volume of the container into two equal parts. The closed part contains n moles of monoatomic perfect gas with molar mass M_0 and temperature T. After releasing of the piston, it leaves the container without friction. After that the gas also leaves the container. What is the final velocity of the container?

The gas constant is R. The momentum of the gas up to the leaving of the piston can be neglected. There is no heat exchange between the gas, container and the piston. The change of the temperature of the gas, when it leaves the container, can be neglected. Do not account for the gravitation of the Earth.

Theoretical Problem 2

An electric lamp of resistance $R_0 = 2\ \Omega$ working at nominal voltage $U_0 = 4.5$ V is connected to accumulator of electromotive force E = 6 V and negligible internal resistance.

1. The nominal voltage of the lamp is ensured as the lamp is connected potentiometrically to the accumulator using a rheostat with resistance R. What should be the resistance R and what is the maximal electric current I_{max}, flowing in the rheostat, if the efficiency of the system must not be smaller than $\eta_0 = 0.6$?

2. What is the maximal possible efficiency η of the system and how the lamp can be connected to the rheostat in this case?

Theoretical Problem 3

A detector of radiowaves in a radioastronomical observatory is placed on the sea beach at height $h = 2$ m above the sea level. After the rise of a star, radiating electromagnetic waves of wavelength $\lambda = 21$ cm, above the horizont the detector registers series of alternating maxima and minima. The registered signal is proportional to the intensity of the detected waves. The detector registers waves with electric vector, vibrating in a direction parallel to the sea surface.

1. Determine the angle between the star and the horizont in the moment when the detector registers maxima and minima (in general form).

2. Does the signal decrease or increase just after the rise of the star?

3. Determine the signal ratio of the first maximum to the next minimum. At reflection of the electromagnetic wave on the water surface, the ratio of the intensities of the electric field of the reflected (E_r) and incident (E_i) wave follows the low:

$$\frac{E_r}{E_i} = \frac{n - \cos\varphi}{n + \cos\varphi},$$

where n is the refraction index and φ is the incident angle of the wave. For the surface "air-water" for λ = 21 cm, the refraction index n = 9.

4. Does the ratio of the intensities of consecutive maxima and minima increase or decrease with rising of the star?

Assume that the sea surface is flat.

Solution of the Theoretical Problem 1

Up to the moment when the piston leaves the container, the system can be considered as a closed one. It follows from the laws of the conservation of the momentum and the energy:

$$(M + nM_0)v_1 - mu = 0 \qquad (1)$$

$$\frac{(M + nM_0)v_1^2}{2} + \frac{mu^2}{2} = \Delta U \,, \qquad (2)$$

where v_1 – velocity of the container when the piston leaves it, u – velocity of the piston in the same moment, ΔU – the change of the internal energy of the gas. The gas is perfect and monoatomic, therefore

$$\Delta U = \frac{3}{2} nR\Delta T = \frac{3}{2} nR(T - T_f); \qquad (3)$$

T_f - the temperature of the gas in the moment when the piston leaves the container. This temperature can be determined by the law of the adiabatic process:

$$pV^{\gamma} = const.$$

Using the perfect gas equation $pV = nRT$, one obtains

$$TV^{\gamma-1} = const., \qquad TV^{\gamma-1} = T_f V_f^{\gamma-1}$$

Using the relation $V_f = 2V$, and the fact that the adiabatic coefficient for one-atomic gas is $\gamma = \frac{c_p}{c_v} = \frac{\frac{5}{2}R}{\frac{3}{2}R} = \frac{5}{3}$, the result for final temperature is:

$$T_f = T\left(\frac{V}{V_f}\right)^{\gamma-1} = \frac{T}{2^{2/3}} = T\,2^{-2/3} \qquad (4)$$

Solving the equations (1) – (4) we obtain

$$v_1 = \sqrt{3(1 - 2^{-2/3})\frac{mnRT}{(nM_0 + M)(m + nM_0 + M)}} \qquad (5)$$

If the gas mass nM_0 is much smaller than the masses of the container M and the piston m, then the equation (5) is simplified to:

$$v_1 = \sqrt{3(1 - 2^{-2/3})\frac{mnRT}{M(m + M)}} \qquad (5')$$

When the piston leaves the container, the velocity of the container additionally increases to value v_2 due to the hits of the atoms in the bottom of the container. Each atom gives the container momentum:

$$p = 2m_A \Delta \overline{v_x},$$

where m_A – mass of the atom; $m_A = \frac{M_0}{N_A}$, and $\overline{v_x}$ can be obtained by the averaged quadratic velocity of the atoms $\overline{v^2}$ as follows:

$\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} = \overline{v^2}$, and $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$, therefore $\overline{v_x} = \sqrt{\frac{\overline{v^2}}{3}}$. It appears that due to the elastic impact of one atom the container receives averaged momentum

$$p = 2\frac{M_0}{N_A}\sqrt{\frac{\overline{v^2}}{3}}$$

All calculations are done assuming that the thermal velocities of the atoms are much larger than the velocity of the container and that the movement is described using system connected with the container.

Have in mind that only half of the atoms hit the bottom of the container, the total momentum received by the container is

$$p_t = \frac{1}{2} n N_A p = n M_0 \sqrt{\frac{\overline{v^2}}{3}} \qquad (6)$$

and additional increase of the velocity of the container is

$$v_2 = \frac{p_t}{M} = n\frac{M_0}{M}\sqrt{\frac{\overline{v^2}}{3}}. \qquad (7)$$

Using the formula for the averaged quadratic velocity

$$\sqrt{\overline{v^2}} = \sqrt{\frac{3RT_f}{M_0}}$$

as well eq. (4) for the temperature T_f, the final result for v_2 is

$$v_2 = 2^{-1/3}\frac{n\sqrt{M_0 RT}}{M}. \qquad (8)$$

Therefore the final velocity of the container is

$$v = v_1 + v_2 = \sqrt{3(1-2^{-2/3})\frac{mnRT}{(nM_0 + M)(m + nM_0 + M)}} + 2^{-1/3}\frac{n\sqrt{M_0 RT}}{M} \approx$$

$$\approx \sqrt{3(1-2^{-2/3})\frac{mnRT}{M(m+M)}} + 2^{-1/3}\frac{n\sqrt{M_0 RT}}{M}. \qquad (9)$$

Solution of the Theoretical Problem 2

1) The voltage U_0 of the lamp of resistance R_0 is adjusted using the rheostat of resistance R. Using the Kirchhoff laws one obtains:

$$I = \frac{U_0}{R} + \frac{U_0}{R - R_x}, \qquad (1)$$

where $R-R_x$ is the resistance of the part of the rheostat, parallel connected to the lamp, R_x is the resistance of the rest part,

$$U_0 = E - IR_x \tag{2}$$

The efficiency η of such a circuit is

$$\eta = \frac{P_{lamp}}{P_{accum.}} = \frac{U_0^2 / r}{IE} = \frac{U_0^2}{RIE}. \tag{3}$$

From eq. (3) it is seen that the maximal current, flowing in the rheostat, is determined by the minimal value of the efficiency:

$$I_{\max} = \frac{U_0^2}{RE\eta_{\min}} = \frac{U_0^2}{RE\eta_0}. \tag{4}$$

The dependence of the resistance of the rheostat R on the efficiency η can determined replacing the value for the current I, obtained by the eq. (3), $I = \frac{U_0^2}{RE\eta}$, in the eqs. (1) and (2):

$$\frac{U_0}{RE\eta} = \frac{1}{R_0} + \frac{1}{R-R_x}, \tag{5}$$

$$R_x = (E - U_0)\frac{RE\eta}{U_0^2}. \tag{6}$$

Then

$$R = R_0\eta\frac{E^2}{U_0^2}\frac{1+\eta(1-\frac{E}{U_0})}{1-\frac{E}{U_0}\eta}. \tag{7}$$

To answer the questions, the dependence $R(\eta)$ must be investigated. By this reason we find the first derivative R'_η:

$$R'_\eta \propto \left(\frac{\eta + \eta^2(1-\frac{E}{U_0})}{1-\frac{E}{U_0}\eta}\right)' \propto$$

$$\propto 1 + 2\eta(1-\frac{E}{U_0})(1-\frac{E}{U_0}\eta) + \left[\eta + \eta^2(1-\frac{E}{U_0})\right]\frac{E}{U_0} = \eta(2-\frac{E}{U_0}\eta)(1-\frac{E}{U_0}) + 1.$$

$\eta < 1$, therefore the above obtained derivative is positive and the function $R(\eta)$ is increasing. It means that the efficiency will be minimal when the rheostat resistance is minimal. Then

$$R \geq R_{\min} = R_0\eta_0\frac{E^2}{U_0^2}\frac{1+\eta_0(1-\frac{E}{U_0})}{1-\frac{E}{U_0}\eta_0} \approx 8.53\Omega.$$

The maximal current I_{max} can be calculated using eq. (4). The result is: $I_{max} \approx 660$ mA.

2) As the function $R(\eta)$ is increasing one, $\eta \to \eta_{\max}$, when $R \to \infty$. In this case the total current I will be minimal and equal to $\frac{U_0}{R}$. Therefore the maximal efficiency is

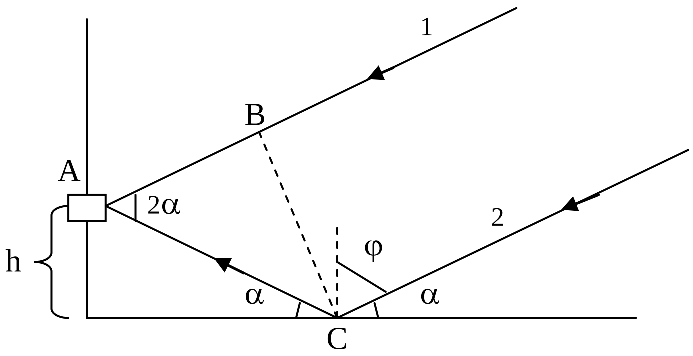
$$\eta_{\max} = \frac{U_0}{E} = 0.75$$

This case can be realized connecting the rheostat in the circuit using only two of its three plugs. The used part of the rheostat is R_1:

$$R_1 = \frac{E - U_0}{I_0} = \frac{E - U_0}{U_0} R_0 \approx 0.67\Omega .$$

Solution of the Theoretical Problem 3

1) The signal, registered by the detector A, is result of the interference of two rays: the ray 1, incident directly from the star and the ray 2, reflected from the sea surface (see the figure).



The phase of the second ray is shifted by π due to the reflection by a medium of larger refractive index. Therefore, the phase difference between the two rays is:

$$\Delta = AC + \frac{\lambda}{2} - AB = \frac{h}{\sin\alpha} + \frac{\lambda}{2} - \left(\frac{h}{\sin\alpha}\right)\cos(2\alpha) =$$

$$= \frac{\lambda}{2} + \frac{h}{\sin\alpha}[1 - \cos(2\alpha)] = \frac{\lambda}{2} + 2h\sin\alpha \qquad (1)$$

The condition for an interference maximum is:

$$\frac{\lambda}{2} + 2h\sin\alpha_{\max} = k\lambda \text{, or}$$

$$\sin\alpha_{\max} = (k - \frac{1}{2})\frac{\lambda}{2h} = (2k-1)\frac{\lambda}{4h}, \qquad (2)$$

where k = 1,2,3,…,19. (the difference of the optical paths cannot exceed $2h$, therefore k cannot exceed 19).

The condition for an interference minimum is:

$$\frac{\lambda}{2} + 2h\sin\alpha_{\max} = (2k+1)\frac{\lambda}{2} \text{, or}$$

$$\sin\alpha_{\min} = \frac{k\lambda}{2h} \qquad (3)$$

where k = 1,2,3,…,19.

2) Just after the rise of the star the angular height α is zero, therefore the condition for an interference minimum is satisfied. By this reason just after the rise of the star, the signal will increase.

3) If the condition for an interference maximum is satisfied, the intensity of the electric field is a sum of the intensities of the direct ray E_i and the reflected ray E_r , respectively: $E_{\max} = E_i + E_r$.

Because $E_r = E_i \dfrac{n-\cos\varphi}{n+\cos\varphi}$, then $E_{\max} = E_i\left(1+\dfrac{n-\cos\varphi_{\max}}{n+\cos\varphi_{\max}}\right)$.

From the figure it is seen that $\varphi_{\max} = \dfrac{\pi}{2} - \alpha_{\max}$, we obtain

$$E_{\max} = E_i\left(1+\frac{n-\sin\alpha_{\max}}{n+\sin\alpha_{\max}}\right) = E_i\frac{2n}{n+\sin(2\alpha_{\max})}. \qquad (4)$$

At the interference minimum, the resulting intensity is:

$$E_{\min} = E_i - E_r = E_i\frac{2\sin\alpha_{\min}}{n+\sin\alpha_{\min}}. \qquad (5)$$

The intensity I of the signal is proportional to the square of the intensity of the electric field E, therefore the ratio of the intensities of the consecutive maxima and minima is:

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{E_{\max}}{E_{\min}}\right)^2 = \frac{n^2}{\sin^2\alpha_{\min}}\frac{(n+\sin\alpha_{\min})^2}{(n+\sin\alpha_{\max})^2}. \qquad (6)$$

Using the eqs. (2) and (3), the eq. (6) can be transformed into the following form:

$$\frac{I_{\max}}{I_{\min}} = \frac{4n^2h^2}{k^2\lambda^2}\left[\frac{n+k\dfrac{\lambda}{2h}}{n+(2k-1)\dfrac{\lambda}{4h}}\right]^2. \qquad (7)$$

Using this general formula, we can determine the ratio for the first maximum (k =1) and the next minimum:

$$\frac{I_{\max}}{I_{\min}} = \frac{4n^2h^2}{\lambda^2}\left(\frac{n+\dfrac{\lambda}{2h}}{n+\dfrac{\lambda}{4h}}\right)^2 = 3.10^4$$

4) Using that $n \gg \dfrac{\lambda}{2h}$, from the eq. (7) follows :

$$\frac{I_{\max}}{I_{\min}} \approx \frac{4n^2h^2}{k^2\lambda^2}.$$

So, with the rising of the star the ratio of the intensities of the consecutive maxima and minima decreases.

Problems of the 13th International Physics Olympiad

(Malente, 1982)

Gunter Lind & Gunnar Friege [1]

Leibniz-Institute for Science Education (IPN) at the University of Kiel, Germany

Abstract

The 13th International Physics Olympiad took place in 1982 in the Federal Republic of Germany. This article contains the competition problems, their solutions and a grading scheme.

Introduction

In 1982 the Federal Republic of Germany was the first host of the Physics Olympiad outside the so-called Eastern bloc. The 13th International Physics Olympiad took place in Malente, Schleswig-Holstein. The competition was funded by the German Federal Ministry of Science and Education. The organisational guidelines were laid down by the work group "Olympiads for pupils" of the conference of ministers of education of the German federal states. The Institute for Science Education (IPN) at the University of Kiel was responsible for the realisation of the event. A commission of professors, whose chairman was appointed by the German Physical Society, were concerned with the formulation of the competition problems. All other members of the commission came from physics department of the university of Kiel or from the college of education at Kiel.

The problems as usual covered different fields of classical physics. In 1982 the pupils had to deal with three theoretical and two experimental problems, whereas at the previous Olympiads only one experimental task was given. However, it seemed to be reasonable to put more stress on experimental work. The degree of difficulty was well balanced. One of the theoretical problems could be considered as quite simple (problem 3: "hot-air balloon"). Another theoretical problem (problem 1: "fluorescent lamp") had a mean degree of difficulty and the distribution of the points was a normal distribution with only a few

[1] Contact: Leibniz-Institute for Science Education (IPN) at the University of Kiel
Olshausenstrasse 62, 24098 Kiel, Germany
ipho@ipn.uni-kiel.de

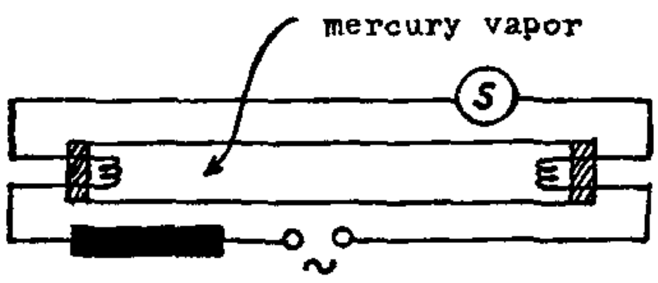
excellent and only a few unsatisfying solutions. The third problem (problem 2: “oscillation coat hanger”) turned out to be the most difficult problem. This problem was generally considered as quite interesting because different ways of solving were possible. About one third of the pupils did not find an adequate start to the problem, but nearly one third of the pupils was able to solve the substantial part of the problem. That means, this problem polarized between the pupils. The two experimental tasks were quite different in respect of the input for the experimental setup and the time required for dealing with the problems, whereas they were quite similar in the degree of difficulty. Both required demandingly theoretical considerations and experimental skills. Both experimental problems turned out to be rather difficult. The tasks were composed in a way that on the one hand almost every pupil had the possibility to come to certain partial results and that there were some difficulties on the other hand which could only be solved by very few pupils. The difficulty in the second experimental problem (problem5: “motion of a rolling cylinder”) was the explanation of the experimental results, which were initially quite surprising. The difficulty in the other task (problem 4: “lens experiment") was the revealing of an observation method with a high accuracy (parallax). The five hours provided for solving the two experimental problems were slightly too short. According to that, in both experiments only a few pupils came up with excellent solutions. In problem 5 nobody got the full points.

The problems presented here are based on the original German and English versions of the competition problems. The solutions are complete but in some parts condensed to the essentials. Almost all of the original hand-made figures are published here.

Theoretical Problems

Problem 1: Fluorescent lamp

An alternating voltage of 50 Hz frequency is applied to the fluorescent lamp shown in the accompanying circuit diagram.



The following quantities are measured:

overall voltage (main voltage)	$U = 228.5$ V
electric current	$I = 0.6$ A
partial voltage across the fluorescent lamp	$U' = 84$ V
ohmic resistance of the series reactor	$R_d = 26.3\Omega$

The fluorescent lamp itself may be considered as an ohmic resistor in the calculations.

a) What is the inductance L of the series reactor?

b) What is the phase shift φ between voltage and current?

c) What is the active power P_w transformed by the apparatus?

d) Apart from limiting the current the series reactor has another important function. Name and explain this function!

Hint: The starter —(s)— includes a contact which closes shortly after switching on the lamp, opens up again and stays open.

e) In a diagram with a quantitative time scale sketch the time sequence of the luminous flux emitted by the lamp.

f) Why has the lamp to be ignited only once although the applied alternating voltage goes through zero in regular intervals?

g) According to the statement of the manufacturer, for a fluorescent lamp of the described type a capacitor of about 4.7 μF can be switched in series with the series reactor. How does this affect the operation of the lamp and to what intent is this possibility provided for?

h) Examine both halves of the displayed demonstrator lamp with the added spectroscope. Explain the differences between the two spectra. You may walk up to the lamp and you may keep the spectroscope as a souvenir.

Solution of problem 1:

a) The total resistance of the apparatus is $Z = \frac{228.5\,V}{0.6\,A} = 380.8\ \Omega$,

the ohmic resistance of the tube is $R_R = \frac{84\,V}{0.6\,A} = 140\ \Omega$.

Hence the total ohmic resistance is $R = 140\,\Omega + 26.3\,\Omega = 166.3\,\Omega$.

Therefore the inductance of the series reactor is: $\omega \cdot L = \sqrt{Z^2 - R^2} = 342.6\,\Omega$.

This yields $L = \frac{342.6\,\Omega}{100\,\pi\,s^{-1}} = 1.09\ H$.

b) The impedance angle is obtained from $\tan\varphi = \frac{\omega \cdot L}{R} = \frac{342.6\,\Omega}{166.3\,\Omega} = 2.06$.

Thus $\varphi = 64.1^{\circ}$.

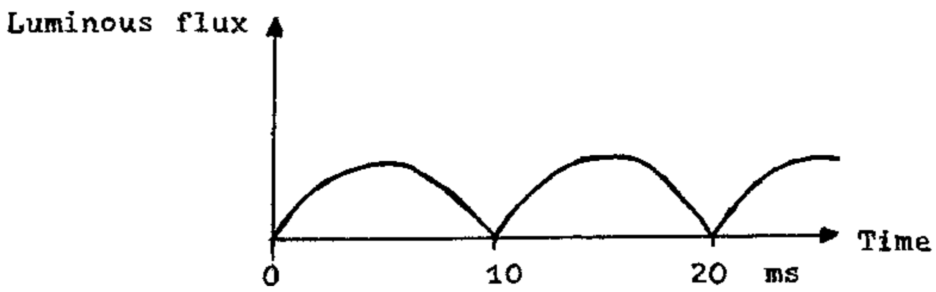
c) The active power can be calculated in different ways:

1) $P_w = U \cdot I \cdot \cos\varphi = 228.5\,V \cdot 0.6\ A \cdot \cos 64.1^{\circ} = 59.88\ W$

2) $P_w = R \cdot I^2 = 166.3\,\Omega \cdot (0.6\ A)^2 = 59.87\ W$

d) By opening the contact in the starter a high induction voltage is produced across the series reactor (provided the contact does not open exactly the same moment, when the current goes through zero). This voltage is sufficient to ignite the lamp. The main voltage itself, however, is smaller than the ignition voltage of the fluorescent tube.

e)



f) The recombination time of the ions and electrons in the gaseous discharge is sufficiently large.

g) The capacitive resistance of a capacitor of 4.7 μF is

$$\frac{1}{\omega \cdot C} = (100 \cdot \pi \cdot 4.7 \cdot 10^{-6})^{-1}\ \Omega = 677.3\ \Omega .$$

The two reactances subtract and there remains a reactance of 334.7 Ω acting as a capacitor.

The total resistance of the arrangement is now

$$Z' = \sqrt{(334.7)^2 + (166.3)^2}\ \Omega = 373.7\ \Omega ,$$

which is very close to the total resistance without capacitor, if you assume the capacitor to be loss-free (cf. a)). Thus the lamp has the same operating qualities, ignites the same way, and a difference is found only in the impedance angle φ', which is opposite to the angle φ calculated in b):

$$\tan \varphi' = \frac{\omega \cdot L - (\omega \cdot C)^{-1}}{R} = -\frac{334.7}{166.3} = -2.01$$

$$\varphi' = -63.6^{\circ} .$$

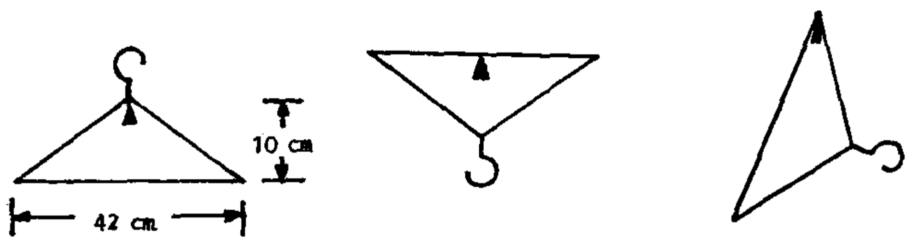
Such additional capacitors are used for compensation of reactive currents in buildings with a high number of fluorescent lamps, frequently they are prescribed by the electricity supply companies. That is, a high portion of reactive current is unwelcome, because the power generators have to be layed out much bigger than would be really necessary and transport losses also have to be added which are not payed for by the customer, if pure active current meters are used.

h) The uncoated part of the demonstrator lamp reveals the line spectrum of mercury, the coated part shows the same line spectrum over a continuous background. The continuous spectrum results from the ultraviolet part of the mercury light, which is absorbed by the fluorescence and re-emitted with smaller frequency (energy loss of the photons) or larger wavelength respectively.

Problem 2: Oscillating coat hanger

A (suitably made) wire coat hanger can perform small amplitude oscillations in the plane of the figure around the equilibrium positions shown. In positions a) and b) the long side is horizontal. The other two sides have equal length. The period of oscillation is the same in all cases.

What is the location of the center of mass, and how long is the period?

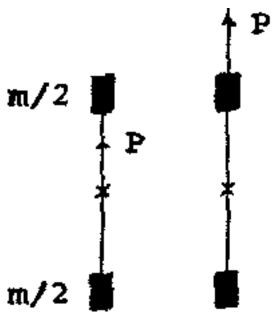


The figure does not contain any information beyond the dimensions given. Nothing is known, e.g., concerning the detailed distribution of mass.

Solution of problem 2

First method:

The motions of a rigid body in a plane correspond to the motion of two equal point masses connected by a rigid massless rod. The moment of inertia then determines their distance. Because of the equilibrium position a) the center of mass is on the perpendicular bipartition of the long side of the coat hanger. If one imagines the equivalent masses and the supporting point P being arranged in a straight line in each case, only two positions of P yield the same period of oscillation (see sketch). One can understand this by considering the limiting cases: 1. both supporting points in the upper mass and 2. one point in the center of mass and the other infinitely high above. Between these extremes the period of oscillation grows continuously. The supporting point placed in the corner of the long side c) has the largest distance from the center of mass, and therefore this point lies outside the two point masses. The two other supporting points a), b) then have to be placed symmetrically to the center of mass between the two point masses, i.e., the center of mass bisects the perpendicular bipartition. One knows of the reversible pendulum that for every supporting point of the physical pendulum it generally has a second supporting point of the pendulum rotated by 180^o, with the same period of oscillation but at a different distance from the center of mass. The



section between the two supporting points equals the length of the corresponding mathematical pendulum. Therefore the period of oscillation is obtained through the corresponding length of the pendulum $s_b + s_c$, where $s_b = 5$ cm and $s_c = \sqrt{5^2 + 21^2}$ cm, to be $T = 1.03$ s .

Second method:

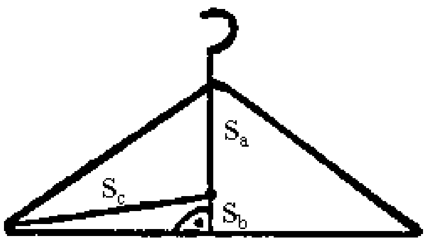
Let s denote the distance between the supporting point and the center of mass, m the mass itself and θ the moment of inertia referring to the supporting point. Then we have the period of oscillation T :

$$T = 2\pi\sqrt{\frac{\theta}{m \cdot g \cdot s}} \quad , \qquad (1)$$

where g is the acceleration of gravity, $g = 9.81$ m/s^2. Here θ can be obtained from the moment of inertia θ_o related to the center of mass:

$$\theta = \theta_0 + m \cdot s^2 \qquad (2)$$

Because of the symmetrical position in case a) the center of mass is to be found on the perpendicular bisection above the long side. Now (1) and (2) yield



$$\theta_0 + m \cdot s^2 = \left(\frac{T}{2 \cdot \pi}\right)^2 \cdot m \cdot g \cdot s \quad \text{for } s = s_a,\ s_b \text{ and } s_c. \qquad (3)$$

because all periods of oscillation are the same. This quadratic equation has only two different solutions at most. Therefore at least two of the three distances are equal. Because of $s_c > 21$ cm $> s_a + s_b$, only s_a and s_b can equal each other. Thus we have

$$s_a = 5 \text{ cm} \qquad (4)$$

The moment of inertia θ_0 is eliminated through (3):

$$m \cdot (s_c^2 - s_a^2) = \left(\frac{T}{2 \cdot \pi}\right)^2 \cdot m \cdot g \cdot (s_c - s_a)$$

and we have $$T = 2 \cdot \pi\sqrt{\frac{s_c + s_a}{g}} \qquad (5)$$

with the numerical value $T = 1.03\,s$,

which has been rounded off after two decimals because of the accuracy of g.

Third method:

This solution is identical to the previous one up to equation (2).

From (1) and (2) we generally have for equal periods of oscillation $T_1 = T_2$:

$$\frac{\theta_0 + m \cdot s_1^{\,2}}{m \cdot g \cdot s_1} = \frac{\theta_0 + m \cdot s_2^{\,2}}{m \cdot g \cdot s_2}$$

and therefore $s_2 \cdot \left(\theta_0 + m \cdot s_1^{\,2}\right) = s_1 \cdot \left(\theta_0 + m \cdot s_2^{\,2}\right)$

or $$\left(s_2 - s_1\right) \cdot \left(\theta_0 - m \cdot s_1\, s_2\right) = 0 \qquad (6)$$

The solution of (6) includes two possibilities: $s_1 = s_2$ or $s_1 \cdot s_2 = \dfrac{\theta_0}{m}$

Let $2 \cdot a$ be the length of the long side and b the height of the coat hanger. Because of $T_b = T_c$ we then have either $s_b = s_c$ or $s_b \cdot s_c = \dfrac{\theta_0}{m}$, where $s_c = \sqrt{s_b^{\,2} + a^2}$,

which excludes the first possibility. Thus $$s_b \cdot s_c = \frac{\theta_0}{m} . \qquad (7)$$

For $T_a = T_b$ the case $s_a \cdot s_b = \dfrac{\theta_0}{m}$ is excluded because of eq. (7), for we have

$$s_a \cdot s_b < s_c \cdot s_b = \frac{\theta_0}{m} .$$

Hence $s_a = s_b = \dfrac{1}{2} b$, $s_c = \sqrt{\dfrac{1}{4} b^2 + a^2}$

and $$T = 2 \cdot \pi \sqrt{\frac{\frac{\theta_0}{m} + s_b^{\,2}}{g \cdot s_b}} = 2\pi \sqrt{\frac{s_b \cdot s_c + s_b^{\,2}}{g \cdot s_b}}$$

The numerical calculation yields the value $T = 1.03\,s$.

Problem 3: Hot-air-balloon

Consider a hot-air balloon with fixed volume $V_B = 1.1\ m^3$. The mass of the balloon-envelope, whose volume is to be neglected in comparison to V_B, is $m_H = 0.187$ kg.

The balloon shall be started, where the external air temperature is $\vartheta_1 = 20\ ^oC$ and the normal external air pressure is $p_o = 1.013 \cdot 10^5$ Pa. Under these conditions the density of air is $\rho_1 = 1.2\ kg/m^3$.

a) What temperature ϑ_2 must the warmed air inside the balloon have to make the balloon just float?

b) First the balloon is held fast to the ground and the internal air is heated to a steady-state temperature of $\vartheta_3 = 110\ ^oC$. The balloon is fastened with a rope.

   Calculate the force on the rope.

c) Consider the balloon being tied up at the bottom (the density of the internal air stays constant). With a steady-state temperature $\vartheta_3 = 110\ ^oC$ of the internal air the balloon rises in an isothermal atmosphere of 20 ^oC and a ground pressure of $p_0 = 1.013 \cdot 10^5$ Pa. Which height h can be gained by the balloon under these conditions?

d) At the height h the balloon (question c)) is pulled out of its equilibrium position by 10 m and then is released again.
   Find out by qualitative reasoning what kind of motion it is going to perform!

Solution of problem 3:

a) Floating condition:

   The total mass of the balloon, consisting of the mass of the envelope m_H and the mass of the air quantity of temperature ϑ_2 must equal the mass of the displaced air quantity with temperature $\vartheta_1 = 20\ ^oC$.

$$V_B \cdot \rho_2 + m_H = V_B \cdot \rho_1$$

$$\rho_2 = \rho_1 - \frac{m_H}{V_B} \qquad (1)$$

Then the temperature may by obtained from

$$\frac{\rho_1}{\rho_2} = \frac{T_2}{T_1},$$

$$T_2 = \frac{\rho_1}{\rho_2} \cdot T_1 = 341.53\ \mathrm{K} = 68.38\ °\mathrm{C} \qquad (2)$$

b) The force F_B acting on the rope is the difference between the buoyant force F_A and the weight force F_G:

$$F_B = V_B \cdot \rho_1 \cdot g \ - \ (V_B \cdot \rho_3 \ + m_H) \cdot g \qquad (3)$$

It follows with $\rho_3 \cdot T_3 \ = \ \rho_1 \cdot T_1$

$$F_B = V_B \cdot \rho_1 \cdot g \cdot \left(1 - \frac{T_1}{T_3}\right) - m_H \cdot g = 1{,}21\ \mathrm{N} \qquad (4)$$

c) The balloon rises to the height h, where the density of the external air ρ_h has the same value as the effective density ρ_{eff}, which is evaluated from the mass of the air of temperature $\vartheta_3 = 110\ ^o\mathrm{C}$ (inside the balloon) and the mass of the envelope m_H:

$$\rho_{eff} = \frac{m_2}{V_B} = \frac{\rho_3 \cdot V_B + m_H}{V_B} = \rho_h = \rho_1 \cdot e^{-\frac{\rho_1 \cdot g \cdot h}{\rho_0}} \qquad (5)$$

Resolving eq. (5) for h gives: $\quad h \ = \ \dfrac{p_o}{\rho_1 \cdot g} \cdot 1n \dfrac{\rho_1}{\rho_{eff}} \ = \ 843\,\mathrm{m}$ (6).

d) For *small* height differences (10 m in comparison to 843 m) the exponential pressure drop (or density drop respectively) with height can be approximated by a linear function of height. Therefore the driving force is proportional to the elongation out of the equilibrium position.

This is the condition in which harmonic oscillations result, which of course are damped by the air resistance.

Experimental Problems

Problem 4: Lens experiment

The apparatus consists of a symmetric biconvex lens, a plane mirror, water, a meter stick, an optical object (pencil), a supporting base and a right angle clamp. Only these parts may be used in the experiment.

a) Determine the focal length of the lens with a maximum error of $\pm$ 1 %.

b) Determine the index of refraction of the glass from which the lens is made.

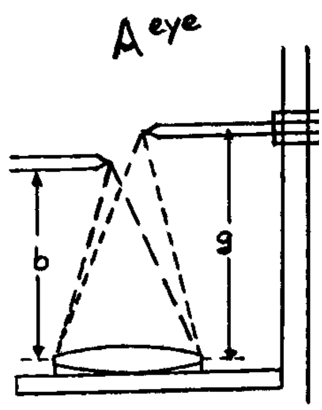
The index of refraction of water is $n_w = 1.33$. The focal length of a thin lens is given by

$$\frac{1}{f} = (n-1) \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right),$$

where n is the index of refraction of the lens material and r_1 and r_2 are the curvature radii of the refracting surfaces. For a symmetric biconvex lens we have $r_1 = - r_2 = r$, for a symmetric biconcave lens $r_1 = - r_2 = - r$.

Solution of problem 4:

a) For the determination of f_L , place the lens on the mirror and with the clamp fix the pencil to the supporting base. Lens and mirror are then moved around until the vertically downward looking eye sees the pencil and its image side by side.



In order to have object and image in focus at the same time, they must be placed at an equal distance to the eye. In this case object distance and image distance are the same and the magnification factor is 1 .

It may be proved quite accurately, whether magnification 1 has in fact been obtained, if one concentrates on parallatical shifts between object and image when moving the eye: only when the distances are equal do the pencil-tips point at each other all the time.

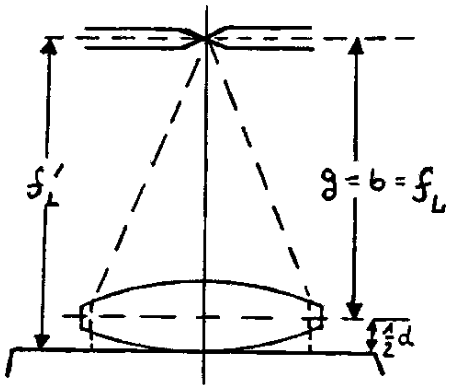
The light rays pass the lens twice because they are reflected by the mirror. Therefore the optical mapping under consideration corresponds to a mapping with two lenses placed directly one after another:

$$\frac{1}{g}+\frac{1}{b}=\frac{1}{f}, \quad \text{where} \quad \frac{1}{f}=\frac{1}{f_L}+\frac{1}{f_L}$$

i.e. the effective focal length f is just half the focal length of the lens. Thus we find for magnification 1:

$$g=b \quad \text{and} \quad \frac{2}{g}=\frac{2}{f_L} \quad \text{i.e.} \quad f_L=g.$$

A different derivation of $f_L = g = b$: For a mapping of magnification 1 the light rays emerging from a point on the optical axis are reflected into themselves. Therefore these rays have to hit the mirror at right angle and so the object distance g equals the focal length f_L of the lens in this case.



The distance between pencil point and mirror has to be determined with an accuracy, which enables one to state f_L with a maximum error of $\pm 1\,\%$. This is accomplished either by averaging several measurements or by stating an uncertainty interval, which is found through the appearance of parallaxe.

Half the thickness of the lens has to be subtracted from the distance between pencil-point and mirror.

$$f_L = f_L' - \frac{1}{2}d, \quad d = 3.0 \pm 0.5\,\text{mm}$$

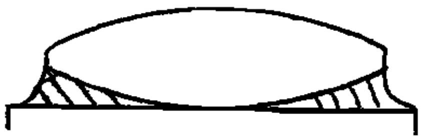
The nominal value of the focal length of the lens is f_L = 30 cm. However, the actual focal length of the single lenses spread considerably. Each lens was measured separately, so the individual result of the student can be compared with the exact value.

b) The refractive index n of the lens material can be evaluated from the equation

$$\frac{1}{f_L} = (n-1)\cdot\frac{2}{r}$$

if the focal length f_L and the curvature radius r of the symmetric biconvex lens are known. f_L was determined in part a) of this problem.

The still unknown curvature radius r of the symmetric biconvex lens is found in the following way: If one pours some water onto the mirror and places the lens in the water, one gets a plane-concave water lens, which has one curvature radius equalling the glass lens' radius and the other radius is ∞.



Because the refractive index of water is known in this case, one can evaluate the curvature radius through the formula above, where $r_1 = -r$ and $r_2 = \infty$:

$$-\frac{1}{f_w} = (n_w - 1)\cdot\frac{1}{r}.$$

Only the focal length f' of the combination of lenses is directly measured, for which we have

$$\frac{1}{f'} = \frac{1}{f_L} + \frac{1}{f_w}.$$

This focal length is to be determined by a mapping of magnification 1 as above.

Then the focal length of the water lens is $\frac{1}{f_w} = \frac{1}{f'} - \frac{1}{f_L}$

and one has the curvature radius $r = -(n_w - 1)\cdot f_w$.

Now the refractive index of the lens is determined by $n = \frac{r}{2\cdot f_L} + 1$

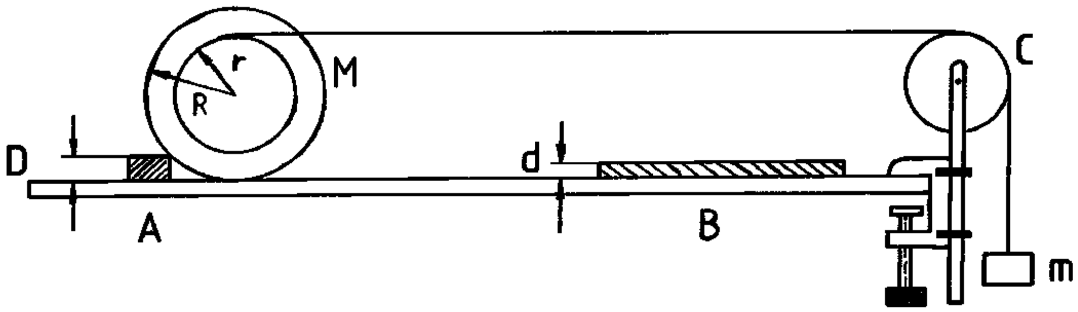
with the known values of f_L and r, or, if one wants to express n explicitly through the measured quantities: $n = \frac{f'\cdot(n_w - 1)}{2\cdot(f' - f_L)} + 1.$

The nominal values are: $f' = 43.9$ cm, $f_w = -94.5$ cm, $r = 31.2$ cm, $n = 1.52$.

Problem 5: Motion of a rolling cylinder

The rolling motion of a cylinder may be decomposed into rotation about its axis and horizontal translation of the center of gravity. In the present experiment only the translatory acceleration and the forces causing it are determined directly.

Given a cylinder of mass M, radius R, which is placed on a horizontal plane board. At a distance r_i ($i = 1 \ldots 6$) from the cylinder axis a force acts on it (see sketch). After letting the cylinder go, it rolls with constant acceleration.



a) Determine the linear accelerations a_i ($i = 1 \ldots 6$) of the cylinder axis experimentally for several distances r_i ($i = 1 \ldots 6$).

b) From the accelerations a_i and given quantities, compute the forces F_i which act in horizontal direction between cylinder and plane board.

c) Plot the experimental values F_i as functions of r_i. Discuss the results.

Before starting the measurements, adjust the plane board horizontally. For present purposes it suffices to realize the horizontal position with an uncertainty of ± 1 mm of height difference on 1 m of length; this corresponds to the distance between adjacent markings on the level. What would be the result of a not horizontal position of the plane board?

Describe the determination of auxiliary quantities and possible further adjustments; indicate the extent to which misadjustments would influence the results.

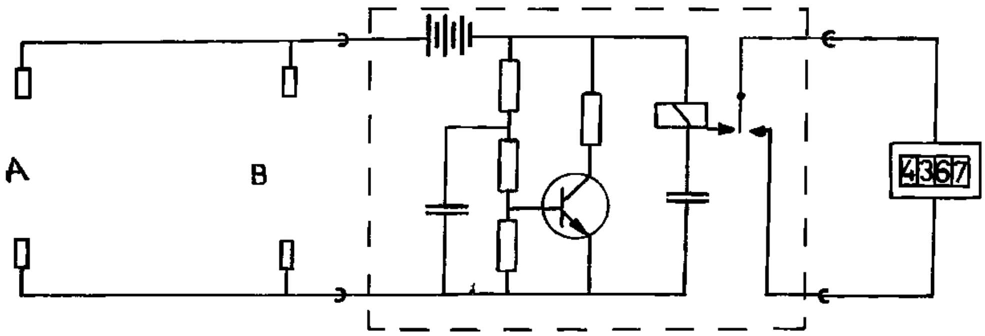
The following quantities are given:

R	=	5 cm	r_1	=	0.75	cm
M	=	3.275 kg	r_2	=	1.50	cm
m	=	2 x 50 g	r_3	=	2.25	cm
D	=	1.50 cm	r_4	=	3.00	cm
d	=	0.1 mm	r_5	=	3.75	cm
			r_6	=	4.50	cm

Mass and friction of the pulleys c may be neglected in the evaluation of the data.

By means of knots, the strings are put into slots at the cylinder. They should be inserted as deeply as possible. You may use the attached paper clip to help in this job.

The stop watch should be connected, as shown in the sketch, with electrical contacts at A and B via an electronic circuit box. The stop watch starts running as soon as the contact at A is opened, and it stops when the contact at B is closed.



The purpose of the transistor circuit is to keep the relay position after closing of the contact at B, even if this contact is opened afterwards for a few milliseconds by a jump or chatter of the cylinder.

Solution of problem 5:

Theoretical considerations:

a) The acceleration of the center of mass of the cylinder is $a = \frac{2 \cdot s}{t^2}$ (1)

b) Let a_m be the acceleration of the masses m and T the sum of the tensions in the two strings, then

$$T = m \cdot g - m \cdot a_m \tag{2}$$

The acceleration a of the center of mass of the cylinder is determined by the resultant force of the string-tension T and the force of interaction F between cylinder and the horizontal plane.

$$M \cdot a = T - F \tag{3}$$

If the cylinder rotates through an angle θ the mass m moves a distance x_m.
It holds

$$x_m = (R + r) \cdot \theta$$

$$a_m = (R + r) \cdot \frac{a}{R} \tag{4}$$

From (2), (3) and (4) follows $F = mg - \left[M + m \cdot \left(1 + \frac{r}{R} \right) \right] \cdot a$. (5)

c) From the experimental data we see that for small r_i the forces $M \cdot a$ and T are in opposite direction and that they are in the same direction for large r_i .

For small values of r the torque produced by the string-tensions is not large enough to provide the angular acceleration required to prevent slipping. The interaction force between cylinder and plane acts into the direction opposite to the motion of the center of mass and thereby delivers an additional torque.

For large values of r the torque produced by string-tension is too large and the interaction force has such a direction that an opposed torque is produced.

From the rotary-impulse theorem we find

$$T \cdot r + F \cdot R = I \cdot \ddot{\theta} = I \cdot \frac{a}{R},$$

where I is the moment of inertia of the cylinder.

With (3) and (5) you may eliminate T and a from this equation. If the moment of inertia of the cylinder is taken as $I = \frac{1}{2} \cdot M \cdot R^2$ (neglecting the step-up cones) we find after some arithmetical transformations

$$F = mg \cdot \frac{1 - 2 \cdot \frac{r}{R}}{3 + 2 \cdot \frac{m}{M} \cdot \left(1 + \frac{r}{R} \right)^2}.$$

For $r = 0 \rightarrow \quad F = \dfrac{m \cdot g}{3 + 2 \cdot \frac{m}{M}} > 0.$

For $r = R \Rightarrow \quad F = \dfrac{-m \cdot g}{3 + 8 \cdot \frac{m}{M}} < 0.$

Because $\dfrac{m}{M} \ll 1$ it is approximately $F = \dfrac{1}{3} m \cdot g - \dfrac{2}{3} \cdot \dfrac{r}{R}$.

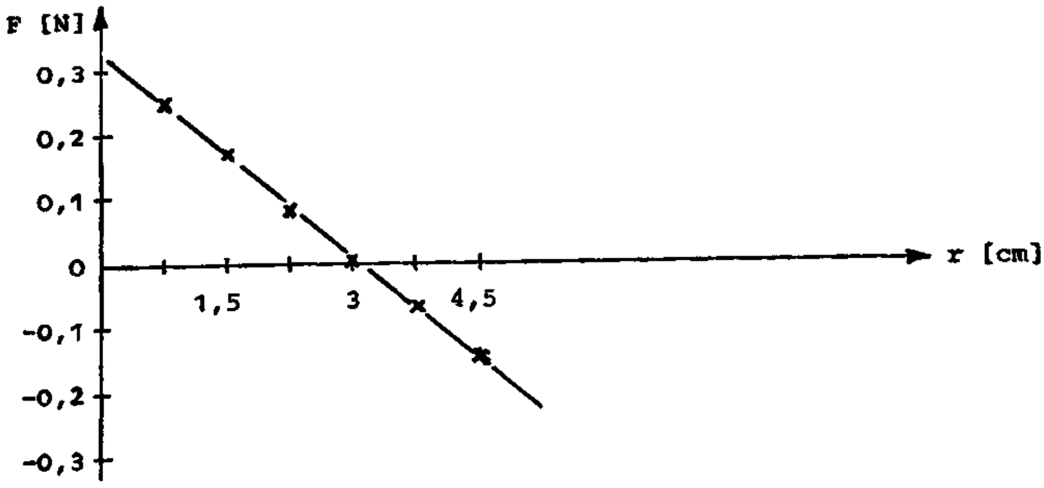
That means: the dependence of F from r is approximately linear. F will be zero if $\frac{r}{R} = \frac{m \cdot g}{2}$.

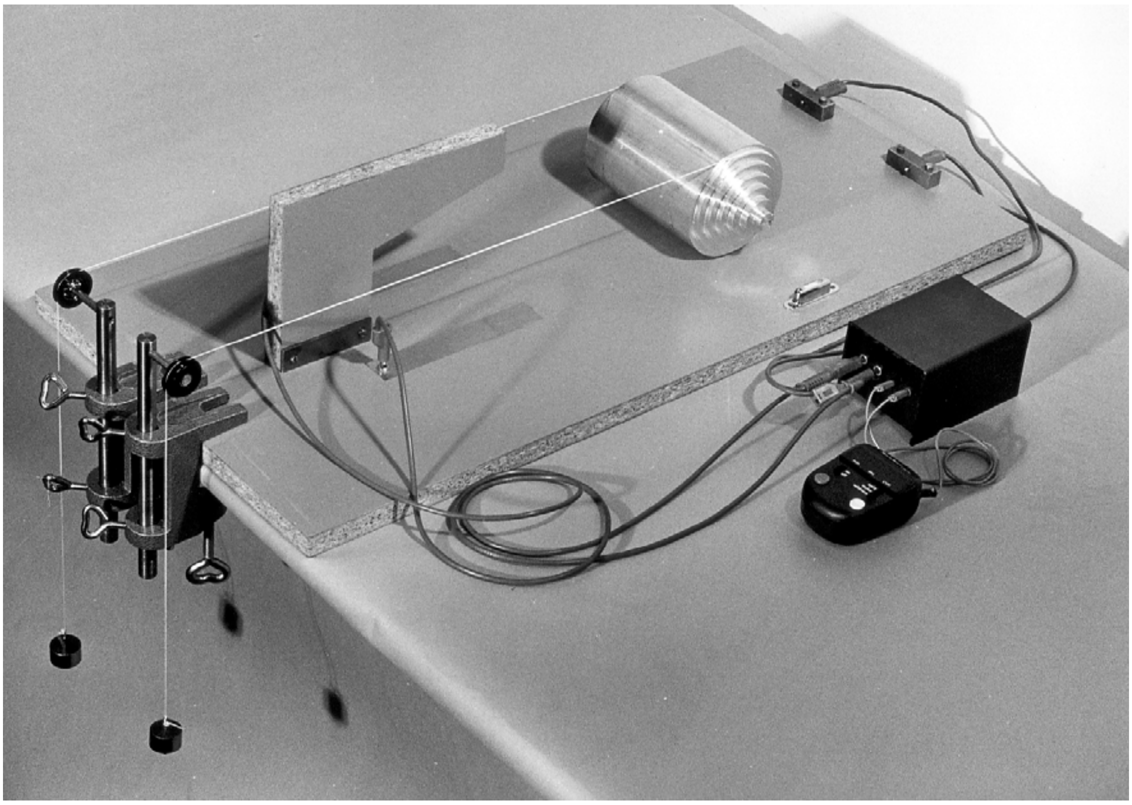
Experimental results:

$$s = L - (2 \cdot R \cdot D + D^2)^{\frac{1}{2}} - (2 \cdot R \cdot d - d^2)^{\frac{1}{2}}$$

$$s = L - 4.5 \text{ cm} = 39.2 \text{ cm} - 4.5 \text{ cm} = 34.7 \text{ cm}$$

r [cm]	t [s]			$\overline{t}$ [s]	a [m/s^2]	F [N]
0.75	1.81	1.82	1.82	1.816	0.211	0.266
1.50	1.71	1.72	1.73	1.720	0.235	0.181
2.25	1.63	1.63	1.64	1.633	0.261	0.090
3.00	1.56	1.56	1.57	1.563	0.284	0.004
3.75	1.51	1.51	1.52	1.513	0.304	- 0.066
4.50	1.46	1.46	1.46	1.456	0.328	- 0.154





Grading schemes

Theoretical problems

Problem 1: Fluorescent lamp	pts.
Part a	2
Part b	1
Part c	1
Part d	1
Part e	1
Part f	1
Part g	2
Part h	1
	10

Problem 2: Oscillating coat hanger	pts.
equation (1)	1,5
equation (2)	1,5
equation (4)	3
equation (5)	2
numerical value for T	1
	10

Problem 3: Hot-air-balloon	pts.
Part a	3
Part b	2
Part c	3
Part d	2
	10

Experimental problems

Problem 4: Lens experiment	pts.
correct description of experimental prodedure	1
selection of magnification one	0.5
parallaxe for verifying his magnification	1
$f_L = g = b$ with derivation	1
several measurements with suitable averaging or other determination of error interval	1
taking into account the lens thickness and computing f_L, including the error	0.5
idea of water lens	0.5
theory of lens combination	1
measurements of f'	0.5
calculation of n and correct result	1
	8

Problem 5: Motion of a rolling cylinder	pts.
Adjustment mentioned of strings a) horizontally and b) in direction of motion	0.5
Indication that angle offset of strings enters the formula for the acting force only quadratically, i.e. by its cosine	0.5
Explanation that with non-horizontal position, the force m·g is to be replaced by $m \cdot g \pm M \cdot g \cdot \sin\alpha$	1.0
Determination of the running length according for formula $s = L - \left(2 \cdot R \cdot D + D^2\right)^{1/2} - \left(2 \cdot R \cdot d + d^2\right)^{1/2}$ including correct numerical result	1.0
Reliable data for rolling time	1.0
accompanied by reasonable error estimate	0.5
Numerical evaluation of the F_i	0.5
Correct plot of $F_i\,(v_i)$	0.5
Qualitative interpretation of the result by intuitive consideration of the limiting cases $r = 0$ and $r = R$	1.0
Indication of a quantitative, theoretical interpretation using the concept of moment of inertia	1.0
Knowledge and application of the formula $a = 2\,s / t^2$	0.5
Force equation for small mass and tension of the string $m \cdot (g - a_m) = T$	1.0
Connection of tension, acceleration of cylinder and reaction force $T - F = M \cdot a$	1.0
Connection between rotary and translatory motion	
$x_m = (R + r) \cdot \theta$	0.5
$a_m = (1 + r / R) \cdot a$	0.5
Final formula for the reaction force $F = m \cdot g - \left(M + m \cdot (1 + r / R)\right) \cdot a$	1.0
If final formulae are given correctly, the knowledge for preceding equations must be assumed and is graded accordingly.	
	12

Mechanics – Problem I (8 points)

A particle moves along the positive axis Ox (one-dimensional situation) under a force having a projection $F_x = F_0$ on Ox, as represented, as function of x, in the figure 1.1. In the origin of the Ox axis is placed a perfectly reflecting wall.
A friction force, with a constant modulus $F_f = 1{,}00\,N$, acts everywhere on the particle.
The particle starts from the point $x = x_0 = 1{,}00\,m$ having the kinetic energy $E_c = 10{,}0\,J$.

a. Find the length of the path of the particle until its' final stop
b. Plot the potential energy $U(x)$ of the particle in the force field F_x.
c. Qualitatively plot the dependence of the particle's speed as function of its' x coordinate.

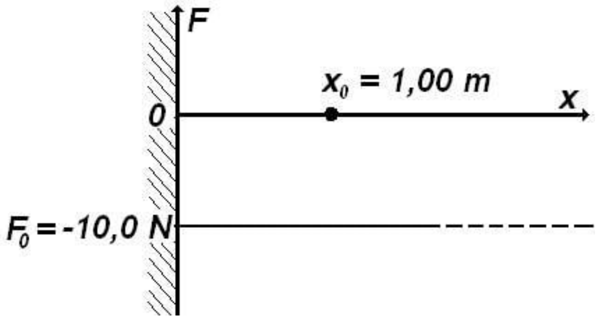


Figure 1.1

Problem I – Solution

a. *It is possible to make a model of the situation in the problem, considering the Ox axis vertically oriented having the wall in its' lower part. The conservative force F_x could be the weight of the particle. One may present the motion of the particle as the vertical motion of a small elastic ball elastically colliding with the ground and moving with constant friction through the medium. The friction force is smaller than the weight.*
The potential energy of the particle can be represented in analogy to the gravitational potential energy of the ball, $m \cdot g \cdot h$, considering $m \cdot g = |F_x|$; $h = x$. As is very well known, in the field of a conservative force, the variation of the potential energy depends only on the initial and final positions of the particle, being independent of the path between those positions.

For the situation in the problem, when the particle moves towards the wall, the force acting on it is directed towards the wall and has the modulus l

$$F_{\leftarrow} = |F_x| - F_f \quad (1.1)$$

$$F_{\leftarrow} = 9N \quad (1.2)$$

As a consequence, the motion of the particle towards the wall is a motion with a constant acceleration having the modulus

$$a_{\leftarrow} = \frac{F_{\leftarrow}}{m} = \frac{|F_x| - F_f}{m} \quad (1.3)$$

During the motion, the speed of the particle increases.

Hitting the wall, the particle starts moving in opposite direction with a speed equal in modulus with the one it had before the collision.
When the particle moves away from the wall, in the positive direction of the *Ox* axis, the acting force is again directed towards to the wall and has the magnitude

$$F_{\rightarrow} = |F_x| + F_f \qquad (1.4)$$

$$F_{\rightarrow} = 11N \qquad (1.5)$$

Correspondingly, the motion of the particle from the wall is slowed down and the magnitude of the acceleration is

$$a_{\rightarrow} = \frac{F_{\rightarrow}}{m} = \frac{|F_x| + F_f}{m} \qquad (1.6)$$

During this motion, the speed of the particle diminishes to zero.
Because during the motion a force acts on the particle, the body cannot have an equilibrium position in any point on axis – the origin making an exception as the potential energy vanishes there. The particle can definitively stop only in this point.
The work of a conservative force from the point having the coordinate $x_0 = 0$ to the point x , $L_{0\rightarrow x}$ is correlated with the variation of the potential energy of the particle $U(x) - U(0)$ as follows

$$\begin{cases} U(x) - U(0) = -L_{0\rightarrow x} \\ U(x) - U(0) = -\int_0^x \overrightarrow{F_x} \cdot \overrightarrow{dx} = \int_0^x |F_x| \cdot dx = |F_x| \cdot x \end{cases} \qquad (1.7)$$

Admitting that the potential energy of the particle vanishes for $x = 0$, the initial potential energy of the particle $U(x_0)$ in the field of conservative force

$$F_x(x) = F_0 \qquad (1.8)$$

can be written

$$U(x_0) = |F_0| \cdot x_0 \qquad (1.9)$$

The initial kinetic energy $E(x_0)$ of the particle is – as given

$$E(x_0) = E_c \qquad (1.10)$$

and, consequently the total energy of the particle $W(x_0)$ is

$$W(x_0) = U(x_0) + E_c \qquad (1.11)$$

The draw up of the particle occurs when the total energy of the particle is entirely exhausted by the work of the friction force. The distance covered by the particle before it stops, D , obeys

$$\begin{cases} W(x_0) = D \cdot F_f \\ U(x_0) + E_c = D \cdot F_f \\ |F_x| \cdot x_0 + E_c = D \cdot F_f \end{cases} \qquad (1.12)$$

so that ,

$$D = \frac{|F_x| \cdot x_0 + E_c}{F_f} \qquad (1.13)^*$$

and

$$D = 20m \qquad (1.14)^*$$

The relations (1.13) and (1.14) represent the answer to the question I.a.

b. The relation (1.7) written as

$$U(x) = |F_x| \cdot x \qquad (1.15)$$

gives the linear dependence of the potential energy to the position .
If the motion occurs without friction, the particle can reach a point A situated at the distance δ apart from the origin in which the kinetic energy vanishes. In the point A the energy of the particles is entirely potential.
The energy conservation law for the starting point and point A gives

$$\begin{cases} E_c + |F_x| \cdot x_0 = |F_x| \cdot \delta \\ \delta = x_0 + \dfrac{E_c}{|F_x|} \end{cases} \qquad (1.16)$$

The numerical value of the position of point A, furthest away from the origin, is
$\delta = 2m$
if the motion occurs without friction.
The representation of the dependence of the potential energy on the position in the domain $(0, \delta)$ is represented in the figure 1.2.

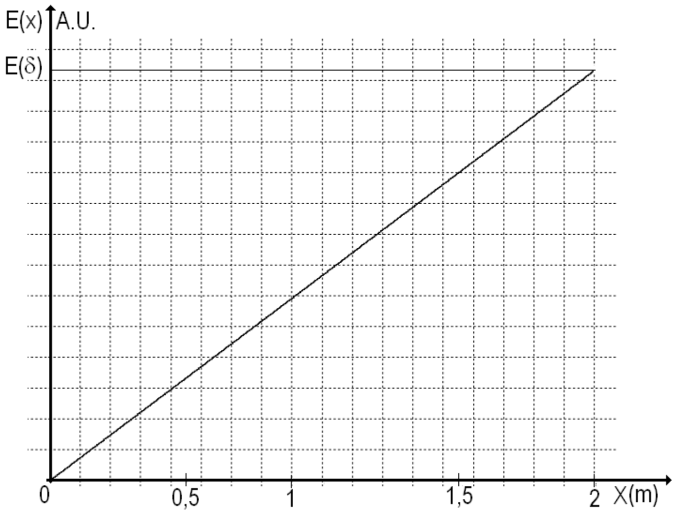


Figure 1.2

During the real motion of the particle (with friction) the extreme positions reached by the particle are smaller than δ (because of the leak of energy due to friction).
The graph in the figure 1.2 is the answer to the question I.b.

c. During the motion of the particle its energy decrease because of the dissipation work of the friction force. The speed of the particle has a local maximum near the wall. Denoting v_k the speed of the particle just before its' k[th] collision with the wall and v_{k+1} the speed just before its' next collision,

$v_k > v_{k+1}$

Among two successive collisions, the particle reaches its' x_k positions in which its' speed vanishes and the energy of the particle is purely potential. These positions are closer and closer to the wall because a part of the energy of the particle is dissipated through friction.

$$x_{k+1} < x_k \qquad (1.17)$$

<u>Case 1</u>

When the particle moves towards the wall, both its' speed and its' kinetic energy increases. The potential energy of the particle decreases. During the motion – independent of its' direction- energy is dissipated through the friction force.

The potential energy of the particle, $U(x)$, the kinetic energy $E(x)$ and the total energy of the particle during this part of the motion $W(x)$ obey the relation

$$W(x_0) - W(x) = F_f \cdot (x_0 - x) \qquad (1.18)$$

the position x lying in the domain

$$x \in (0, x_0) \qquad (1.19)$$

covered from x_0 towards origin. The relation (1.18) can be written as

$$\left[E_c + |F_x| \cdot x_0\right] - \left[\frac{m \cdot v^2}{2} + |F_x| \cdot x\right] = F_f \cdot (x_0 - x) \qquad (1.20)$$

so that

$$\begin{cases} v^2 = \dfrac{2}{m}\left[E_c + |F_x| \cdot x_0 - |F_x| \cdot x - F_f \cdot (x_0 - x)\right] \\ v^2 = \dfrac{2}{m}\left[E_c + x_0(|F_x| - F_f) - x(|F_x| - F_f)\right] \end{cases} \qquad (1.21)$$

and by consequence

$$v = -\sqrt{\frac{2}{m}\left[E_c + x_0(|F_x| - F_f) - x(|F_x| - F_f)\right]} \qquad (1.22)$$

The minus sign in front of the magnitude of the speed indicates that the motion of the particle occurs into the negative direction of the coordinate axis.

Using the problem data

$$\begin{cases} v^2 = \dfrac{2}{m}(19 - 9 \cdot x) \\ v = -\sqrt{\dfrac{2}{m}(19 - 9 \cdot x)} \end{cases} \qquad (1.23)$$

The speed of the particle at the first collision with the wall $v_{1\leftarrow}$ can be written as

$$v_{1\leftarrow} = -\sqrt{\frac{2}{m}\left[E_c + x_0\left(\left|F_x\right| - F_f\right)\right]} \tag{1.24}$$

and has the value

$$v_{1\leftarrow} = -\sqrt{\frac{2}{m}19} \tag{1.25}$$

The total energy near the wall, purely kinetic $E_{1\leftarrow}$, has the expression

$$E_{1\leftarrow} = E_c + x_0\left(\left|F_x\right| - F_f\right) \tag{1.26}$$

The numerical value of this energy is

$$E_{1\leftarrow} = 19J \tag{1.27}$$

The graph in the figure (1.3) gives the dependence on position of the square of the speed for the first part of the particle's motion.

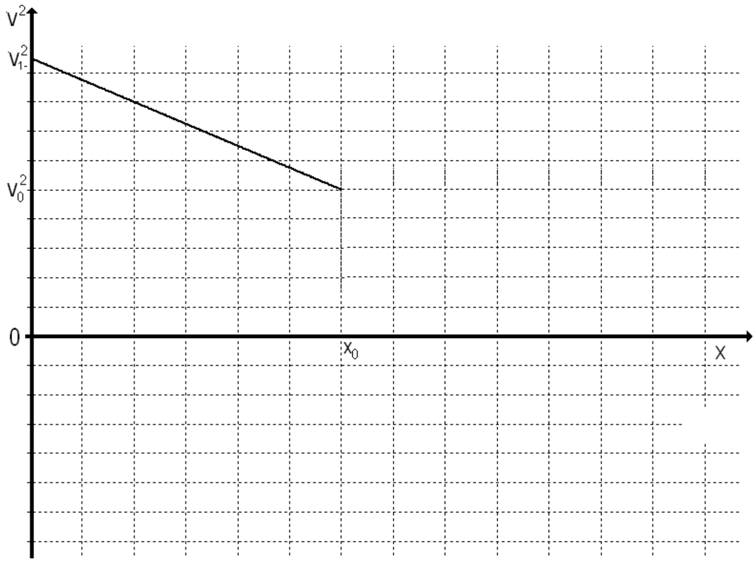


Figure 1.3

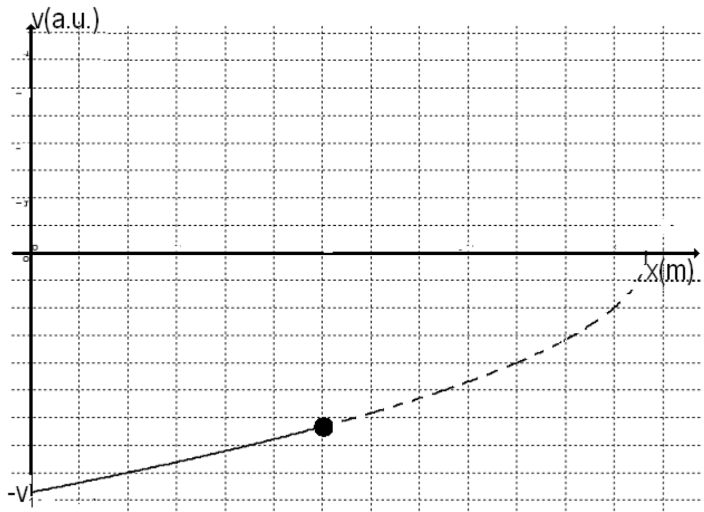


Figure 1.4

The graph in the figure (1.4) presents the speed's dependence on the position in this first part of the particle's motion (towards the wall).
After the collision with the wall, the speed of the particle, $v_{1\rightarrow}$, has the same magnitude as the speed just before the collision but it is directed in the opposite way. In the graphical representation of the speed as a function of position, the collision with the wall is represented as a jump of the speed from a point lying on negative side of the speed axis to a point lying on positive side of the speed axis. The absolute value of the speed just before and immediately after the collision is the same as represented in the figure 1.5.

$$v_{1\rightarrow} = \sqrt{\frac{2}{m}\left[E_c + x_0\left(|F_x| - F_f\right)\right]} \quad (1.28)$$

After the first collision, the motion of the particle is slowed down with a constant deceleration $a_{\rightarrow}$ and an initial speed $v_{1\rightarrow}$.
This motion continues to the position x_1 where the speed vanishes.
From Galileo law it can be inferred that

$$\begin{cases} 0 = v_{1\rightarrow}^2 - 2\cdot a_{\rightarrow}\cdot x_1 \\ x_1 = \dfrac{v_{1\rightarrow}^2}{2\cdot a_{\rightarrow}} = \dfrac{\frac{2}{m}\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{2\cdot\frac{|F_x| + F_f}{m}} = \dfrac{\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{|F_x| + F_f} \end{cases} \quad (1.29)$$

The numerical value of the position x_1 is

$$x_1 = \frac{19}{11}m \quad (1.30)$$

For the positions

$$x \in \left(0, x_1\right) \quad (1.31)$$

covered from the origin towards x_1 the total energy $W(x)$ has the expression

$$W(x) = \frac{m\cdot v^2}{2} + |F_x|\cdot x \quad (1.32)$$

From the wall, the energy of the particle diminishes because of the friction – that is

$$\begin{cases} E_{1\leftarrow} - W(x) = F_f\cdot x \\ E_c + x_0\left(|F_x| - F_f\right) - \dfrac{m\cdot v^2}{2} - |F_x|\cdot x = F_f\cdot x \end{cases} \quad (1.33)$$

The square of the magnitude of the speed is

$$\begin{cases} v^2 = \dfrac{2}{m}\left[E_c + x_0\left(|F_x| - F_f\right) - \left(|F_x| + F_f\right)\cdot x\right] \\ v^2 = \dfrac{2}{m}\left(|F_x| + F_f\right)\cdot\left(x_1 - x\right) \end{cases} \quad (1.34)$$

and the speed is

$$v = \sqrt{\frac{2}{m}\left[E_c + x_0\left(|F_x| - F_f\right) - \left(|F_x| + F_f\right)\cdot x\right]} \quad (1.35)$$

Using the furnished data results

$$v^2 = \frac{2}{m}\left[19 - 11\cdot x\right] \qquad (1.36)$$

and respectively

$$v = \sqrt{\frac{2}{m}\left[19 - 11\cdot x\right]} \qquad (1.37)$$

For the positions lying in the domain $x \in (0, x_1)$ - (which correspond to a second part of the motion of particle) the figure 1.5 gives the dependence of the speed on the position.

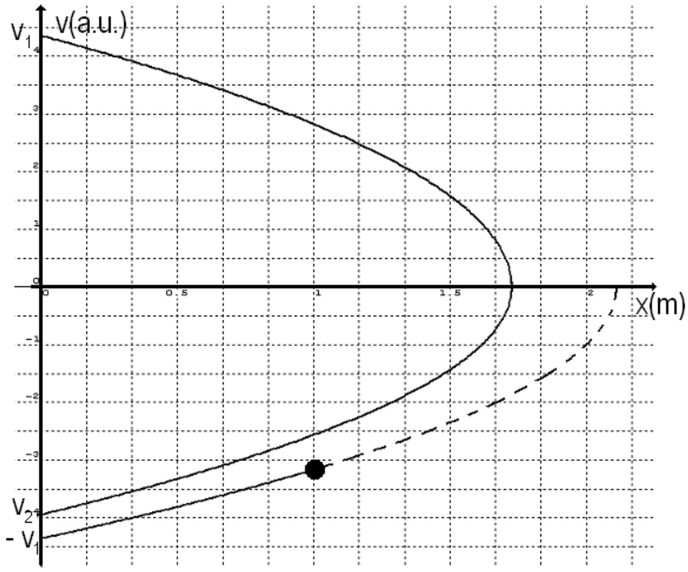


Figure 1.5

As can be observed in the figure, after reaching the furthest away position, x_1, the particle moves towards the origin, without an initial speed, in an accelerated motion having an acceleration with the magnitude of $a_{\leftarrow} = \left(|F_x| - F_f\right)/m$. After the collision with the wall, the particle has a velocity equal in magnitude but opposite in direction with the one it had just before the collision.

When the particle reaches a point in the domain $(0, x_1)$ moving from x_1 towards the origin its' total energy $W(x)$ has the expression (1.32).

Starting from x_1, because of the dissipation determined by the friction force, the energy changes to the value corresponding to the position with coordinate x.

$$\begin{cases} |F_x|\cdot x_1 - W(x) = F_f\cdot(x_1 - x) \\ |F_x|\cdot x_1 - \dfrac{m\cdot v^2}{2} - |F_x|\cdot x = F_f\cdot(x_1 - x) \end{cases} \qquad (1.38)$$

The square of the speed has the expression

$$\begin{cases} v^2 = \dfrac{2}{m}\left[\left(|F_x| - F_f\right)\cdot(x_1 - x)\right] \\ v^2 = \dfrac{2}{m}\left[\dfrac{\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{|F_x| + F_f} - x\right]\cdot\left(|F_x| - F_f\right) \end{cases} \qquad (1.39)$$

and the speed is

$$v = \sqrt{\frac{2}{m}\left[\frac{\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{|F_x| + F_f} - x\right]\cdot\left(|F_x| - F_f\right)} \qquad (1.40)$$

Using the given data, for a position in the domain $(0, x_1)$

$$v^2 = \frac{2}{m}\left[\frac{19}{11} - x\right]\cdot 9 \qquad (1.41)$$

respectively

$$v = -\sqrt{\frac{2}{m}\left[\frac{19}{11} - x\right]\cdot 9} \qquad (1.42)$$

The speed of the particle when it reaches for the second time the wall has - using (1.39) - the expression

$$v_{2\leftarrow} = -\sqrt{\frac{2}{m}\left\{\frac{\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{|F_x| + F_f}\cdot\left(|F_x| - F_f\right)\right\}} \qquad (1.43)$$

The resulting numerical value is

$$v_{2\leftarrow} = -\sqrt{\frac{2}{m}\frac{171}{11}} \qquad (1.44)$$

Concluding, after the first collision and first recoil, the particle moves away from the wall, reaches again a position where the speed vanishes and then comes back to the wall. The speed of the particle hitting again the wall is smaller than before – as in the figure 1.5.

As it was denoted before v_k is the speed of the particle just before its' k th run and x_k is the coordinate of the furthest away point reached during the k th run.

The energy of the particle starting from the wall is

$$E_k = \frac{v_k^2 \cdot m}{2} = W_k(0) \qquad (1.45)$$

In the point x_k, the furthest away from the origin after k th collision, the energy verifies the relation

$$U_k = x_k \cdot |F_x| = W_k(x_k) \qquad (1.46)$$

The variation of the energy between starting point and point x_k is

$$\frac{v_k^2 \cdot m}{2} - x_k \cdot |F_x| = F_f \cdot x_k \qquad (1.47)$$

so that

$$x_k = \frac{v_k^2 \cdot m}{2\cdot\left(|F_x| + F_f\right)} \qquad (1.48)$$

After the particle reaches point x_k the direction of the speed changes and, when the particle reaches again the wall

$$\frac{v_{k+1}^2 \cdot m}{2} = E_{k+1} = W_{k+1}(0) \qquad (1.49)$$

The energy conservation law for the x_k point and the state when the particle reaches again the wall gives

$$x_k \cdot |F_x| - \frac{v_{k+1}^2 \cdot m}{2} = F_f \cdot x_k \qquad (1.50)$$

so that

$$v_{k+1}^2 = \frac{2}{m} x_k \left(|F_x| - F_f\right) \qquad (1.51)$$

Considering (1.48), the relation (1.51) becomes

$$v_{k+1}^2 = v_k^2 \cdot \frac{|F_x| - F}{|F_x| + F} \qquad (1.52)$$

Between two consequent collisions the speed diminishes in a geometrical progression having the ratio q .This ratio has the expression

$$q = \sqrt{\frac{|F_x| - F}{|F_x| + F}} \qquad (1.53)$$

and the value

$$q = \sqrt{\frac{9}{11}} \qquad (1.54)$$

For the $k+1$ collision the relation (1.48) becomes

$$x_{k+1} = \frac{v_{k+1}^2 \cdot m}{2 \cdot \left(|F_x| + F_f\right)} \qquad (1.55)$$

Taking into account (1.52), the ratio of the successive extreme positions can be written as

$$\begin{cases} \dfrac{x_{k+1}}{x_k} = \dfrac{|F_x| - F_f}{|F_x| + F_f} = q^2 \\ x_{k+1} = q^2 \cdot x_k \end{cases} \qquad (1.56)$$

From the k run towards origin, (analogous to (1.39)), the dependence of the square of the speed on position can be written as $v_{(k,\leftarrow)}^2$

$$\begin{cases} v_{(k,\leftarrow)}^2 = \dfrac{2}{m}\left[\left(|F_x| - F_f\right) \cdot \left(x_k - x\right)\right] \\ v_{(k,\leftarrow)}^2 = \dfrac{2}{m}\left[\left(|F_x| - F_f\right) \cdot \left(x_1 \cdot q^{2k} - x\right)\right] \end{cases} \qquad (1.57)$$

or, using the data

$$v_{(k,\leftarrow)}^2 = \frac{2}{m}\left[9\cdot\left(\frac{19}{11}\cdot\left(\frac{9}{11}\right)^k - x\right)\right] \qquad (1.58)$$

For the k th run from the origin (analogous with (1.34)), the dependence on the position of the square of the magnitude of the speed $v_{(k,\rightarrow)}^2$ can be written as

$$\begin{cases} v_{(k,\rightarrow)}^2 = \dfrac{2}{m}\left[\left(|F_x| + F_f\right)\cdot\left(x_k - x\right)\right] \\ v_{(k,\rightarrow)}^2 = \dfrac{2}{m}\left[\left(|F_x| + F_f\right)\cdot\left(x_1\cdot q^{2k} - x\right)\right] \end{cases} \qquad (1.59)$$

Using given data

$$v_{(k,\rightarrow)}^2 = \frac{2}{m}\left[11\cdot\left(\frac{19}{11}\cdot\left(\frac{9}{11}\right)^k - x\right)\right] \qquad (1.60)$$

The evolution of the square of the speed as function of position is represented in the figure 1.6.

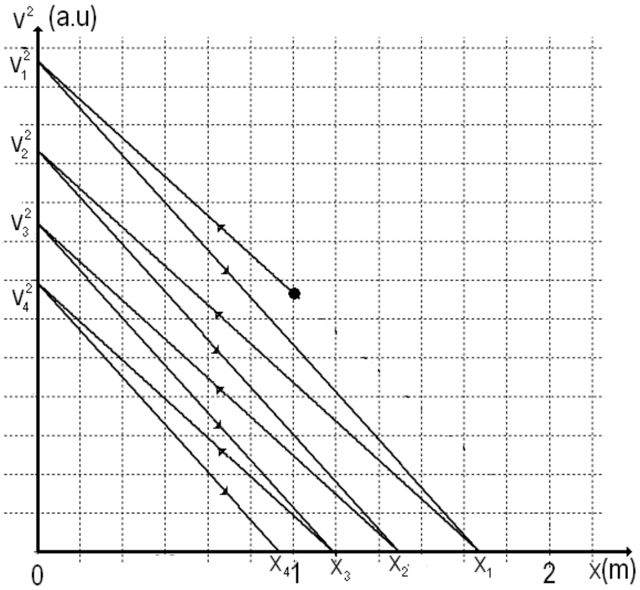


Figure 1.6

And the evolution of the speed as function of position is represented in the figure 1.7.

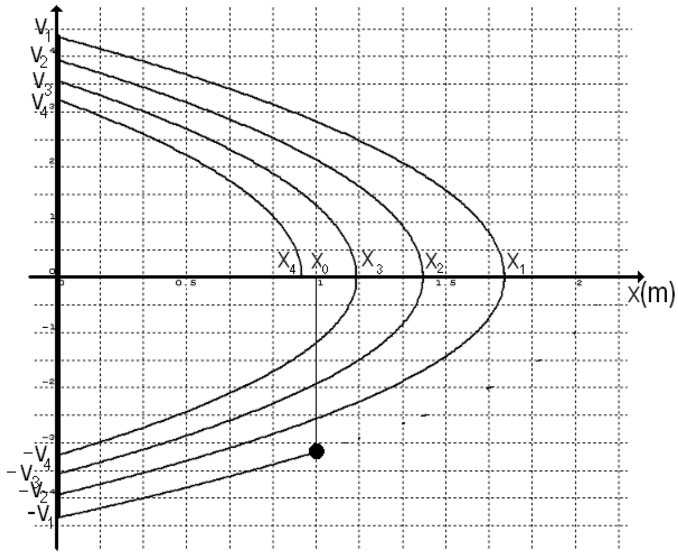


Figure 1.7

The sum of the progression given in (1.56) gives half of the distance covered by the particle after the first collision.

$$\sum_{k=1}^{\infty} x_k = x_1 \frac{1}{1-q^2} \quad (1.61)$$

Considering (1.53) and (1.29)

$$\sum_{k=1}^{\infty} x_k = \frac{E_c + x_0 \cdot \left(|F_x| - F_f\right)}{2 \cdot F_f} \quad (1.62)$$

Numerically,

$$\sum_{k=1}^{\infty} x_k = \frac{19}{2} m \quad (1.63)$$

The total covered distance is

$$\begin{cases} D = 2 \cdot \sum_{k=1}^{\infty} x_k + x_0 \\ D = 20m \end{cases} \quad (1.64)$$

which is the same with (1.14).

Case 2

If the particle starts from the x_0 position moving in the positive direction of the coordinate axis *Ox* its' speed diminishes and its' kinetic energy also diminishes while its' potential energy increases to a maximum in the x_1' position where the speed vanishes. During this motion the energy is dissipated due to the friction.

The total energy $W(x)$, for the positions x between x_0 and x_1' verify the relation

$$W(x_0) - W(x) = F_f \cdot (x - x_0) \quad (1.65)$$

the position x lying in the domain

$$x \in (x_0, x_1') \tag{1.66}$$

when the particle moves from x_0 in the positive direction of the axis. The relation (1.65) becomes

$$\left[E_c + |F_x| \cdot x_0\right] - \left[\frac{m \cdot v^2}{2} + |F_x| \cdot x\right] = F_f \cdot (x - x_0) \tag{1.67}$$

so that

$$\begin{cases} v^2 = \dfrac{2}{m}\left[E_c + |F_x| \cdot x_0 - |F_x| \cdot x - F_f \cdot (x - x_0)\right] \\ v^2 = \dfrac{2}{m}\left[E_c + x_0\left(|F_x| + F_f\right) - x\left(|F_x| + F_f\right)\right] \end{cases} \tag{1.68}$$

and

$$v = \sqrt{\frac{2}{m}\left[E_c + x_0\left(|F_x| + F_f\right) - x\left(|F_x| + F_f\right)\right]} \tag{1.69}$$

Using provided data

$$\begin{cases} v^2 = \dfrac{2}{m}(21 - 11 \cdot x) \\ v = \sqrt{\dfrac{2}{m}(21 - 11 \cdot x)} \end{cases} \tag{1.70}$$

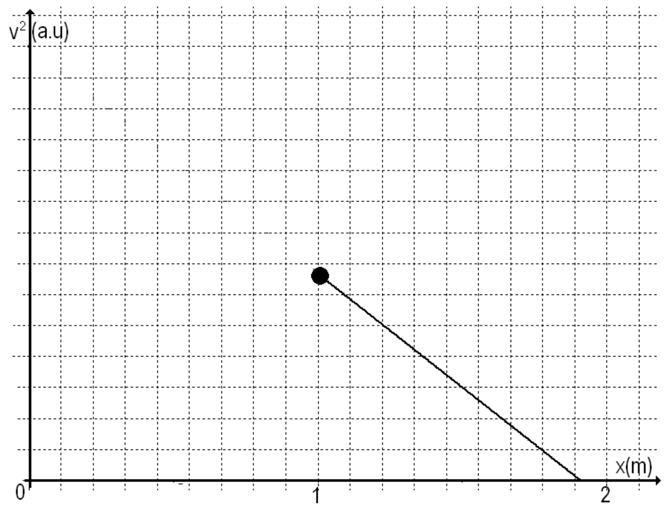


Figure 1.8

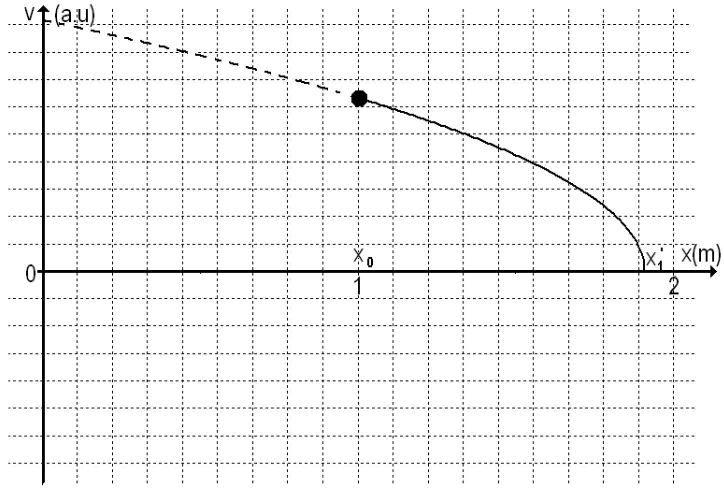


Figure 1.9

The graph in the figure (1.8) presents the dependence of the square speed on the position for the motion in the domain $x \in (x_0, x_1')$. The particle moves in the positive direction of the coordinate axis *Ox.* This motion occurs until the position x_1' - when the speed vanishes - is reached. From the relation (1.68), in which we take the modulus of the speed zero, results

$$x_1' = x_0 + \frac{E_c}{|F_x| + F_f} \qquad (1.71)$$

the numerical value for x_1' is

$$x_1' = \frac{21}{11} m \qquad (1.72)$$

After furthest away position x_1' is reached, the particle moves again towards the origin, without initial speed, in a speeded up motion having an acceleration of magnitude $a_{\leftarrow} = (|F_x| - F_f)/m$. After the collision with the wall, the particle has a velocity $v_{1\rightarrow}'$ equal in magnitude but opposite direction with the one it had before the collision $v_{1\leftarrow}'$.

When the particle is at a point lying in the domain $(0, x_1')$ running from x_1' to the origin, its' total energy $W(x)$ has the expression

$$W(x) = \frac{m \cdot v^2}{2} + |F_x| \cdot x \qquad (1.73)$$

Because of friction, the value of the energy decreases from the one it had at x_1' to the corresponding to the x position

$$\begin{cases} |F_x| \cdot x_1' - W(x) = F_f \cdot (x_1' - x) \\ |F_x| \cdot x_1' - \frac{m \cdot v^2}{2} - |F_x| \cdot x = F_f \cdot (x_1' - x) \end{cases} \qquad (1.74)$$

The square of the speed has the expression

$$v^2 = \frac{2}{m}\left[(|F_x| - F_f) \cdot (x_1' - x)\right] \qquad (1.75)$$

and the speed is

$$v = -\sqrt{\frac{2}{m}\left[\left(\left|F_x\right| - F_f\right)\cdot\left(x_1' - x\right)\right]} \qquad (1.76)$$

For the given data, in the domain, $\left(0, x_1'\right)$

$$v^2 = \frac{2}{m}\left[\frac{21}{11} - x\right]\cdot 9 \qquad (1.77)$$

respectively

$$v = -\sqrt{\frac{2}{m}\left[\frac{21}{11} - x\right]\cdot 9} \qquad (1.78)$$

The speed of the particle hitting a second time the wall is – according to (1.78)-

$$v_{1\leftarrow}' = -\sqrt{\frac{2}{m}\left[\left(\left|F_x\right| - F_f\right)\cdot x_1'\right]} \qquad (1.79)$$

and has the value

$$v_{1\leftarrow}' = -\sqrt{\frac{2}{m}\frac{189}{11}} \qquad (1.80)$$

Concluding, after the first collision and first recoil, the particle moves away from the wall, reaches again a position where the speed vanishes and then comes back to the wall. The speed of the particle hitting again the wall is smaller than before – as in the figure 1.11.

Denoting v_k' the speed at the beginning of the k th run and x_k' the coordinate of the furthest away point during the k th run, the energy of the particle leaving the wall is

$$E_k' = \frac{v'^2_k \cdot m}{2} = W_k'(0) \qquad (1.81)$$

In the position x_k' after the k departure from the wall, the energy is

$$U_k' = x_k' \cdot \left|F_x\right| = W_k'\left(x_k'\right) \qquad (1.82)$$

The variation of the total energy has the expression

$$\frac{v'^2_k \cdot m}{2} - x_k' \cdot \left|F_x\right| = F_f \cdot x_k' \qquad (1.83)$$

so that

$$x_k' = \frac{v'^2_k \cdot m}{2\cdot\left(\left|F_x\right| + F_f\right)} \qquad (1.84)$$

After the particle reaches the position x_k' the direction of the speed changes and, when the particle hits the wall,

$$\frac{v'^2_{k+1} \cdot m}{2} = E_{k+1}' = W_{k+1}'(0) \qquad (1.85)$$

The energy conservation law for the x_k' position and the point in which the particle hits the wall gives

$$x_k'\cdot\left|F_x\right|-\frac{v'^2_{k+1}\cdot m}{2}=F_f\cdot x_k' \qquad (1.86)$$

so that

$$v'^2_{k+1}=\frac{2}{m}x_k'\left(\left|F_x\right|-F_f\right) \qquad (1.87)$$

Considering (1.84), the relation (1.87) becomes

$$v'^2_{k+1}=v'^2_k\cdot\frac{\left|F_x\right|-F}{\left|F_x\right|+F} \qquad (1.88)$$

Between two successive collisions the speed diminishes in a geometrical progression with the ratio q

$$q=\sqrt{\frac{\left|F_x\right|-F}{\left|F_x\right|+F}} \qquad (1.89)$$

Using the data provided

$$q=\sqrt{\frac{9}{11}} \qquad (1.90)$$

From $(k+1)^{th}$, collision the relation (1.84) is written as

$$x_{k+1}'=\frac{v'^2_{k+1}\cdot m}{2\cdot\left(\left|F_x\right|+F_f\right)} \qquad (1.91)$$

Considering (1.84) and (1.91), the ratio of the extreme positions in two successive runs is

$$\begin{cases}\dfrac{x_{k+1}'}{x_k'}=\dfrac{\left|F_x\right|-F_f}{\left|F_x\right|+F_f}=q^2\\ x_{k+1}'=q^2\cdot x_k'\end{cases} \qquad (1.92)$$

For the k th run towards the origin, analogous to (1.57), one may write the dependence of the square speed $v'^2_{(k,\leftarrow)}$ as function of the position as

$$\begin{cases}v'^2_{(k,\leftarrow)}=\dfrac{2}{m}\left[\left(\left|F_x\right|-F_f\right)\cdot\left(x_k'-x\right)\right]\\ v'^2_{(k,\leftarrow)}=\dfrac{2}{m}\left[\left(\left|F_x\right|-F_f\right)\cdot\left(x_1'\cdot q^{2k}-x\right)\right]\end{cases} \qquad (1.93)$$

Or, using the data

$$v'^2_{(k,\leftarrow)}=\frac{2}{m}\left[9\cdot\left(\frac{21}{11}\cdot\left(\frac{9}{11}\right)^k-x\right)\right] \qquad (1.94)$$

From the k th run from the origin, analogous to (1.59), the dependence on the position of the square speed $v^2_{(k,\rightarrow)}$ can be written as

$$\begin{cases} v'^2_{(k,\rightarrow)} = \dfrac{2}{m}\left[\left(\left|F_x\right| + F_f\right)\cdot\left(x_k' - x\right)\right] \\ v'^2_{(k,\rightarrow)} = \dfrac{2}{m}\left[\left(\left|F_x\right| + F_f\right)\cdot\left(x_1'\cdot q^{2k} - x\right)\right] \end{cases} \qquad (1.95)$$

Using given data

$$v'^2_{(k,\rightarrow)} = \frac{2}{m}\left[11\cdot\left(\frac{21}{11}\cdot\left(\frac{9}{11}\right)^k - x\right)\right] \qquad (1.96)$$

The evolution of the square of the speed as function on position is presented in the figure 1.10.

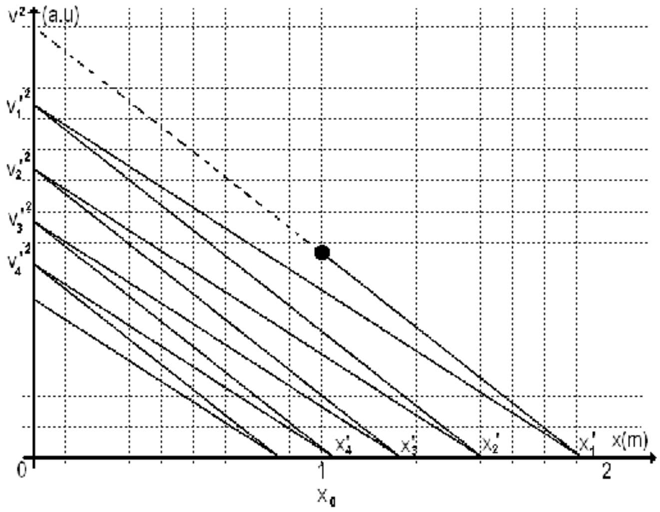


Figure 1.10

And the evolution of the speed as function of the position is presented in the figure 1.11.

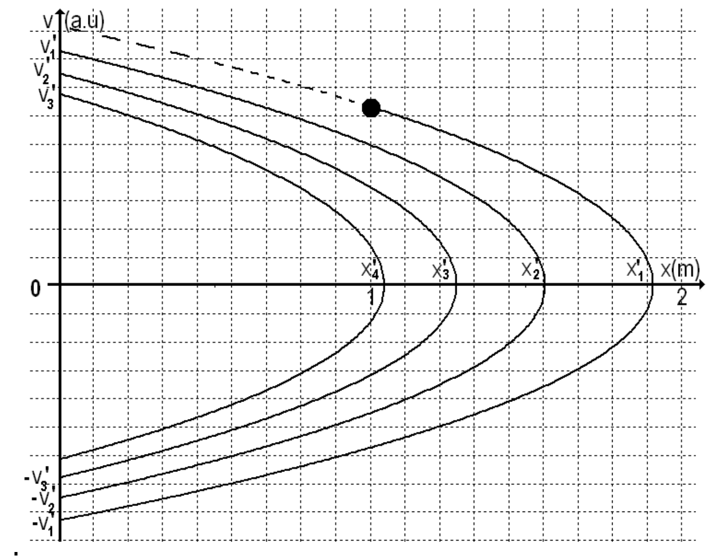


Figure 1.11

The sum of the geometrical progression (1.92) gives (after the doubling and then subtracting of the x_0) the total distance covered by the particle.

$$\sum_{k=1}^{\infty} x_k' = x_1' \frac{1}{1-q^2} \qquad (1.97)$$

Considering (1.97), (1.71) and (1.72) it results

$$\sum_{k=1}^{\infty} x_k' = \frac{21}{2} m \qquad (1.98)$$

The total distance covered by the particle is

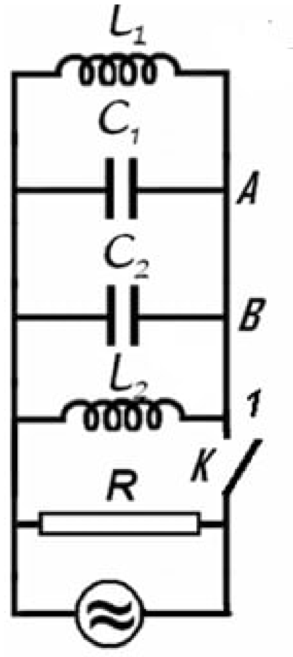
$$\begin{cases} D = 2 \cdot \sum_{k=1}^{\infty} x_k' - x_0 \\ D = 20m \end{cases} \qquad (1.99)$$

which allows us to find again the result (1.14).

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research- Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

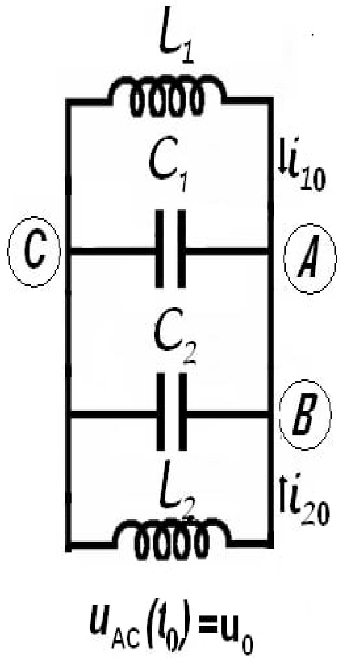
Electricity – Problem II (8 points)

Different kind of oscillation



Let's consider the electric circuit in the figure, for which $L_1 = 10\ mH$, $L_2 = 20\ mH$, $C_1 = 10\ nF$, $C_2 = 5\ nF$ and $R = 100 k\Omega$. The switch K being closed the circuit is coupled with a source of alternating current. The current furnished by the source has constant intensity while the frequency of the current may be varied.

a. Find the ratio of frequency f_m for which the active power in circuit has the maximum value P_m and the frequency difference $\Delta f = f_+ - f_-$ of the frequencies f_+ and f_- for which the active power in the circuit is half of the maximum power P_m.



The switch K is now open. In the moment t_0 immediately after the switch is open the intensities of the currents in the coils L_1 and $i_{01} = 0{,}1\ A$ and $i_{02} = 0{,}2\ A$ L_1 (the currents flow as in the figure); at the same moment, the potential difference on the capacitor with capacity C_1 is $u_0 = 40V$:

b. Calculate the frequency of electromagnetic oscillation in $L_1C_1C_2L_2$ circuit;

c. Determine the intensity of the electric current in the AB conductor;

d. Calculate the amplitude of the oscillation of the intensity of electric current in the coil L_1.

Neglect the mutual induction of the coils, and the electric resistance of the conductors. Neglect the fast transition phenomena occurring when the switch is closed or opened.

Problem II - Solution

a. As is very well known in the study of AC circuits using the formalism of complex numbers, a complex inductive reactance $\overline{X_L} = L \cdot \omega \cdot j$, ($j = \sqrt{-1}$) is attached to the inductance L - part of a circuit supplied with an alternative current having the pulsation ω .

Similar, a complex capacitive reactance $\overline{X_C} = -\dfrac{j}{C \cdot \omega}$ is attached to the capacity C .

A parallel circuit will be characterized by his complex admittance $\overline{Y}$.

The admittance of the AC circuit represented in the figure is

$$\begin{cases} \overline{Y} = \dfrac{1}{R} + \dfrac{1}{L_1 \cdot \omega \cdot j} + \dfrac{1}{L_2 \cdot \omega \cdot j} - \dfrac{C_1 \cdot \omega}{j} - \dfrac{C_2 \cdot \omega}{j} \\ \overline{Y} = \dfrac{1}{R} + j \cdot \left[(C_1 + C_2) - \left(\dfrac{1}{L_1} + \dfrac{1}{L_2} \right) \right] \end{cases} \quad (2.1)$$

The circuit behave as if has a parallel equivalent capacity C

$$C = C_1 + C_2 \quad (2.2)$$

and a parallel equivalent inductance L

$$\begin{cases} \dfrac{1}{L} = \dfrac{1}{L_1} + \dfrac{1}{L_2} \\ L = \dfrac{L_1 L_2}{L_1 + L_2} \end{cases} \quad (2.3)$$

The complex admittance of the circuit may be written as

$$\overline{Y} = \frac{1}{R} + j \cdot \left(C \cdot \omega - \frac{1}{L \cdot \omega} \right) \quad (2.4)$$

and the complex impedance of the circuit will be

$$\begin{cases} \overline{Z} = \dfrac{1}{\overline{Y}} \\ \overline{Z} = \dfrac{\dfrac{1}{R} + j \cdot \left(\dfrac{1}{L \cdot \omega} - C \cdot \omega \right)}{\sqrt{\left(\dfrac{1}{R} \right)^2 + \left(C \cdot \omega - \dfrac{1}{L \cdot \omega} \right)^2}} \end{cases} \quad (2.5)$$

The impedance Z of the circuit, the inverse of the admittance of the circuit Y is the modulus of the complex impedance $\overline{Z}$

$$Z = \left| \overline{Z} \right| = \frac{1}{\sqrt{\left(\dfrac{1}{R} \right)^2 + \left(C \cdot \omega - \dfrac{1}{L \cdot \omega} \right)^2}} = \frac{1}{Y} \quad (2.6)$$

The constant current source supplying the circuit furnish a current having a momentary value $i(t)$

$$i(t) = I \cdot \sqrt{2} \cdot \sin(\omega \cdot t), \quad (2.7)$$

where I is the effective intensity (constant), of the current and ω is the current pulsation (that can vary) . The potential difference at the jacks of the circuit has the momentary value $u(t)$

$$u(t) = U \cdot \sqrt{2} \cdot \sin(\omega \cdot t + \varphi) \quad (2.8)$$

where U is the effective value of the tension and φ is the phase difference between tension and current.

The effective values of the current and tension obey the relation

$$U = I \cdot Z \quad (2.9)$$

The active power in the circuit is

$$P = \frac{U^2}{R} = \frac{Z^2 \cdot I^2}{R} \quad (2.10)$$

Because as in the enounce,

$$\begin{cases} I = \text{constant} \\ \text{R} = \text{constant} \end{cases} \quad (2.11)$$

the maximal active power is realized for the maximum value of the impedance that is the minimal value of the admittance .
The admittance

$$Y = \sqrt{\left(\frac{1}{R}\right)^2 + \left(C \cdot \omega - \frac{1}{L \cdot \omega}\right)^2} \quad (2.12)$$

has– as function of the pulsation ω - an „the smallest value”

$$Y_{\min} = \frac{1}{R} \quad (2.13)$$

for the pulsation

$$\omega_m = \frac{1}{\sqrt{L \cdot C}} \quad (2.14)$$

In this case

$$\left(C \cdot \omega - \frac{1}{L \cdot \omega}\right) = 0. \quad (2.15)$$

So, the minimal active power in the circuit has the value

$$P_m = R \cdot I^2 \quad (2.16)$$

and occurs in the situation of alternative current furnished by the source at the frequency f_m

$$f_m = \frac{1}{2\pi}\omega_m = \frac{1}{2\pi \cdot \sqrt{C \cdot L}} \quad (2.17)$$

To ensure that the active power is half of the maximum power it is necessary that

$$\begin{cases} P = \frac{1}{2} P_m \\ \frac{Z^2 \cdot I^2}{R} = \frac{1}{2} R \cdot I^2 \\ \frac{2}{R^2} = \frac{1}{Z^2} = Y^2 \end{cases} \quad (2.18)$$

That is

$$\begin{cases} \dfrac{2}{R^2} = \dfrac{1}{R^2} + \left(C \cdot \omega - \dfrac{1}{L \cdot \omega} \right)^2 \\ \pm \dfrac{1}{R} = C \cdot \omega - \dfrac{1}{L \cdot \omega} \end{cases} \qquad (2.19)$$

The pulsation of the current ensuring an active power at half of the maximum power must satisfy one of the equations

$$\omega^2 \pm \frac{1}{R \cdot C} \omega - \frac{1}{L \cdot C} = 0 \qquad (2.20)$$

The two second degree equation may furnish the four solutions

$$\omega = \pm \frac{1}{2R \cdot C} \pm \frac{1}{2} \sqrt{\left(\frac{1}{R \cdot C} \right)^2 + \frac{4}{L \cdot C}} \qquad (2.21)$$

Because the pulsation is every time positive, and because

$$\sqrt{\left(\frac{1}{R \cdot C} \right)^2 + \frac{4}{L \cdot C}} > \frac{1}{R \cdot C} \qquad (2.22)$$

the only two valid solutions are

$$\omega_{\pm} = \frac{1}{2} \sqrt{\left(\frac{1}{R \cdot C} \right)^2 + \frac{4}{L \cdot C}} \pm \frac{1}{2R \cdot C} \qquad (2.23)$$

It exist two frequencies $f_{\pm} = \frac{1}{2\pi} \omega_{\pm}$ allowing to obtain in the circuit an active power representing half of the maximum power.

$$\begin{cases} f_{+} = \dfrac{1}{2\pi} \left(\dfrac{1}{2} \sqrt{\left(\dfrac{1}{R \cdot C} \right)^2 + \dfrac{4}{L \cdot C}} + \dfrac{1}{2R \cdot C} \right) \\ f_{-} = \dfrac{1}{2\pi} \left(\dfrac{1}{2} \sqrt{\left(\dfrac{1}{R \cdot C} \right)^2 + \dfrac{4}{L \cdot C}} - \dfrac{1}{2R \cdot C} \right) \end{cases} \qquad (2.24)$$

The difference of these frequencies is

$$\Delta f = f_{+} - f_{-} = \frac{1}{2\pi} \frac{1}{R \cdot C} \qquad (2.25)$$

the bandwidth of the circuit – the frequency interval around the resonance frequency having at the ends a signal representing $1/\sqrt{2}$ from the resonance signal. At the ends of the bandwidth the active power reduces at the half of his value at the resonance.

The asked ratio is

$$\begin{cases} \dfrac{f_m}{\Delta f} = \dfrac{R \cdot C}{\sqrt{L \cdot C}} = R\sqrt{\dfrac{C}{L}} \\ \dfrac{f_m}{\Delta f} = R\sqrt{\dfrac{(C_1 + C_2)\cdot(L_1 + L_2)}{L_1 \cdot L_2}} \end{cases} \qquad (2.26)^*$$

Because

$$\begin{cases} C = 15\ nF \\ L = \dfrac{20}{3}\ mH \end{cases}$$

it results that

$\omega_m = 10^5\ rad \cdot s^{-1}$

and

$$\frac{f_m}{\Delta f} = R\sqrt{\frac{C}{L}} = 100 \times 10^3 \cdot \sqrt{\frac{3 \cdot 15 \times 10^{-9}}{20 \times 10^{-3}}} = 150 \qquad (2.27)$$

The (2.26) relation is the answer at the question a.

b. The fact that immediately after the source is detached it is a current in the coils, allow as to admit that currents dependents on time will continue to flow through the coils.

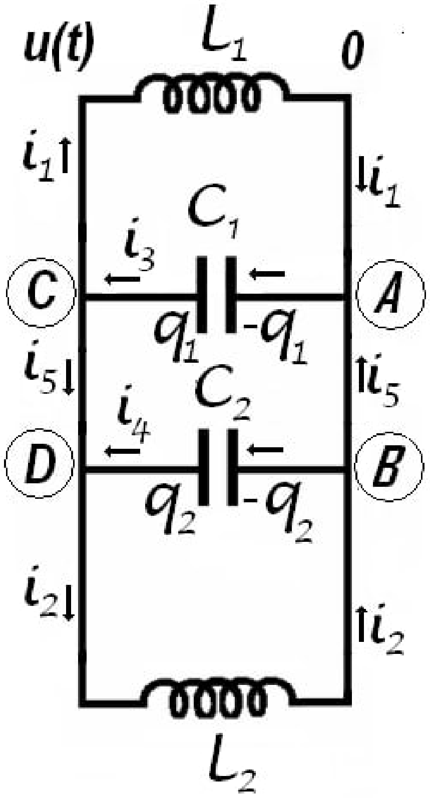


Figure 2.1

The capacitors will be charged with charges variable in time. The variation of the charges of the capacitors will results in currents flowing through the conductors linking the capacitors in the circuit.
The momentary tension on the jacks of the coils and capacitors – identical for all elements in circuit – is also dependent on time. Let's admit that the electrical potential of the points *C* and *D* is $u(t)$ and the

potential of the points A and B is zero. If through the inductance L_1 passes the variable current having the momentary value $i_1(t)$, the relation between the current and potentials is

$$u(t) - L_1 \frac{di_1}{dt} = 0 \qquad (2.28)$$

The current passing through the second inductance $i_2(t)$ has the expression,

$$u(t) - L_2 \frac{di_2}{dt} = 0 \qquad (2.29)$$

If on the positive plate of the capacitor having the capacity C_1 is stocked the charge $q_1(t)$, then at the jacks of the capacitor the electrical tension is $u(t)$ and

$$q_1 = C_1 \cdot u \qquad (2.30)$$

Deriving this relation it results

$$\frac{dq_1}{dt} = C_1 \cdot \frac{du}{dt} \qquad (2.31)$$

But

$$\frac{dq_1}{dt} = -i_3 \qquad (2.32)$$

because the electrical current appears because of the diminishing of the electrical charge on capacitor plate. Consequently

$$i_3 = -C_1 \cdot \frac{du}{dt} \qquad (2.33)$$

Analogous, for the other capacitor,

$$i_4 = -C_4 \cdot \frac{du}{dt} \qquad (2.34)$$

Considering all obtained results

$$\begin{cases} \dfrac{di_1}{dt} = \dfrac{u}{L_1} \\ \dfrac{di_2}{dt} = \dfrac{u}{L_2} \end{cases} \qquad (2.35)$$

respectively

$$\begin{cases} \dfrac{di_3}{dt} = -C_1 \dfrac{d^2u}{dt^2} \\ \dfrac{di_4}{dt} = C_2 \dfrac{d^2u}{dt^2} \end{cases} \qquad (2.36)$$

Denoting $i_5(t)$ the momentary intensity of the current flowing from point B to the point A, then the same momentary intensity has the current through the points C and D. For the point A the Kirchhoff rule of the currents gives

$$i_1 + i_5 = i_3 \qquad (2.37)$$

For B point the same rule produces

$$i_4 + i_5 = i_2 \qquad (2.38)$$

Considering (2.37) and (2.38) results

$$i_1 - i_3 = i_4 - i_2 \qquad (2.39)$$

and deriving

$$\frac{di_1}{dt} - \frac{di_3}{dt} = \frac{di_4}{dt} - \frac{di_2}{dt} \qquad (2.40)$$

that is

$$\begin{cases} -\dfrac{u}{L_1} - \dfrac{u}{L_2} = C_1 \dfrac{d^2u}{dt^2} + C_2 \dfrac{d^2u}{dt^2} \\ -u \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_2} \right) = \dfrac{d^2u}{dt^2} \cdot (C_1 + C_2) \end{cases} \qquad (2.41)$$

Using the symbols defined above

$$\begin{cases} -\dfrac{u}{L} = \dfrac{d^2u}{dt^2} \cdot C \\ \ddot{u} + \dfrac{1}{LC} u = 0 \end{cases} \qquad (2.42)$$

Because the tension obeys the relation above, it must have a harmonic dependence on time

$$u(t) = A \cdot \sin(\omega \cdot t + \delta) \qquad (2.43)$$

The pulsation of the tension is

$$\omega = \frac{1}{\sqrt{L \cdot C}} \qquad (2.44)$$

Taking into account the relations (2.43) and (2.36) it results that

$$\begin{cases} i_3 = -C_1 \dfrac{d}{dt}(A \cdot \sin(\omega \cdot t + \delta)) = -C_1 \cdot A \cdot \omega \cdot \cos(\omega \cdot t + \delta) \\ i_4 = -C_2 \dfrac{d}{dt}(A \cdot \sin(\omega \cdot t + \delta)) = -C_2 \cdot A \cdot \omega \cdot \cos(\omega \cdot t + \delta) \end{cases} \qquad (2.45)$$

and

$$\begin{cases} \dfrac{di_1}{dt} = \dfrac{u}{L_1} = \dfrac{1}{L_1} \cdot A \cdot \sin(\omega \cdot t + \delta) \\ \dfrac{di_2}{dt} = \dfrac{u}{L_2} = \dfrac{1}{L_2} \cdot A \cdot \sin(\omega \cdot t + \delta) \end{cases} \qquad (2.46)$$

It results that

$$\begin{cases} i_1 = \dfrac{1}{L_1 \cdot \omega} \cdot A \cdot \cos(\omega \cdot t + \delta) + M \\ i_2 = \dfrac{1}{L_2 \cdot \omega} \cdot A \cdot \cos(\omega \cdot t + \delta) + N \end{cases} \quad (2.47)$$

In the expression above, A , M , N and δ are constants that must be determined using initially conditions. It is remarkable that the currents through capacitors are sinusoidal but the currents through the coils are the sum of sinusoidal and constant currents.
In the first moment

$$\begin{cases} u(0) = u_0 = 40V \\ i_1(0) = i_{01} = 0{,}1A \\ i_2(0) = i_{02} = 0{,}2A \end{cases} \quad (2.48)$$

Because the values of the inductances and capacities are

$$\begin{cases} L_1 = 0{,}01H \\ L_2 = 0{,}02H \\ C_1 = 10nF \\ C_2 = 5nF \end{cases} \quad (2.49)$$

the equivalent inductance and capacity is

$$\begin{cases} \dfrac{1}{L} = \dfrac{1}{L_1} + \dfrac{1}{L_2} \\ L = \dfrac{L_1 \cdot L_2}{L_1 + L_2} \\ L = \dfrac{2 \times 10^{-4}}{3 \times 10^{-2}} H = \dfrac{1}{150} H \end{cases} \quad (2.50)$$

respectively

$$\begin{cases} C = C_1 + C_2 \\ C = 15nF \end{cases} . \quad (2.51)$$

From (2.44) results

$$\omega = \frac{1}{\sqrt{\dfrac{1}{150} \cdot 15 \times 10^{-9}}} = 10^5 rad \cdot s^{-1} \quad (2.52)^*$$

The value of the pulsation allows calculating the value of the requested frequency **b**. This frequency has the value f

$$f = \frac{\omega}{2\pi} = \frac{10^5}{2\pi} Hz \quad (2.53)^*$$

c. If the momentary tension on circuit is like in (2.43), one may write

$$\begin{cases} u(0) = A \cdot \sin(\delta) = u_0 \\ \sin(\delta) = \dfrac{u_0}{A} \end{cases} \tag{2.54}$$

From the currents (2.47) is possible to write

$$\begin{cases} i_{01} = \dfrac{1}{L_1 \cdot \omega} \cdot A \cdot \cos(\delta) + M \\ i_{02} = \dfrac{1}{L_2 \cdot \omega} \cdot A \cdot \cos(\delta) + N \end{cases} \tag{2.55}$$

On the other side is possible to express (2.39) as

$$\begin{cases} i_1 - i_3 = i_4 - i_2 \\ \dfrac{1}{L_1 \cdot \omega} \cdot A \cdot \cos(\omega \cdot t + \delta) + M + C_1 \cdot A \cdot \omega \cdot \cos(\omega \cdot t + \delta) = \\ -C_2 \cdot A \cdot \omega \cdot \cos(\omega \cdot t + \delta) - \dfrac{1}{L_2 \cdot \omega} \cdot A \cdot \cos(\omega \cdot t + \delta) - N \end{cases} \tag{2.56}$$

An identity as

$$A \cdot \cos\alpha + B \equiv C \cdot \cos\alpha + D \tag{2.57}$$

is valuable for any value of the argument α only if

$$\begin{cases} A = C \\ B = D \end{cases} \tag{2.58}$$

Considering (2.58), from (2.56) it results

$$\begin{cases} M + N = 0 \\ A \cdot \omega \cdot (C_1 + C_2) = -\dfrac{A}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_2} \right) \end{cases} \tag{2.59}$$

For the last equation it results that the circuit oscillate with the pulsation in the relation (2.44)
Adding relations (2.55) and considering (2.54) and (2.59) results that

$$\begin{cases} i_{01} + i_{02} = A \cdot \cos(\delta) \cdot \dfrac{1}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_1} \right) \\ A = \dfrac{i_{01} + i_{02}}{\cos(\delta) \cdot \dfrac{1}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_1} \right)} \\ \cos\delta = \dfrac{i_{01} + i_{02}}{A \cdot \dfrac{1}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_1} \right)} \\ \cos\delta = \dfrac{(i_{01} + i_{02}) \cdot L \cdot \omega}{A} \end{cases} \tag{2.60}$$

The numerical value of the amplitude of the electrical tension results by summing the last relations from (2.54) and (2.60)

$$\begin{cases} \sin(\delta) = \dfrac{u_0}{A} \\ \cos\delta = \dfrac{(i_{01} + i_{02}) \cdot L \cdot \omega}{A} \\ (\cos(\delta))^2 + (\sin(\delta))^2 = 1 \\ \left(\dfrac{u_0}{A}\right)^2 + \left(\dfrac{(i_{01} + i_{02}) \cdot L \cdot \omega}{A}\right)^2 = 1 \\ A = \sqrt{(u_0)^2 + ((i_{01} + i_{02}) \cdot L \cdot \omega)^2} \end{cases} \tag{2.61}$$

The numerical value of the electrical tension on the jacks of the circuit is

$$\begin{cases} A = \sqrt{(40)^2 + \left((0,3) \cdot \dfrac{1}{150} \cdot 10^5\right)^2} \\ A = \sqrt{(40)^2 + (200)^2} = 40\sqrt{26}\,V \end{cases} \tag{2.62}$$

And consequently from (2.54) results

$$\begin{cases} \sin(\delta) = \dfrac{u_0}{A} \\ \sin(\delta) = \dfrac{40}{40\sqrt{26}} = \dfrac{1}{\sqrt{26}} \end{cases} \tag{2.63}$$

and

$$\cos(\delta) = \frac{5}{\sqrt{26}} \tag{2.64}$$

Also

$$\begin{cases} tg(\delta) = \dfrac{1}{5} \\ \delta = arctg(1/5) \end{cases} \tag{2.65}$$

From (2.55)

$$\begin{cases} M = i_{01} - \dfrac{1}{L_1 \cdot \omega} \cdot A \cdot \cos(\delta) \\ N = i_{02} - \dfrac{1}{L_2 \cdot \omega} \cdot A \cdot \cos(\delta) \end{cases} \tag{2.66}$$

the corresponding numerical values are

$$\begin{cases} M = \left(0{,}1 - \dfrac{1}{0{,}01 \cdot 10^5} \cdot 40\sqrt{26} \cdot \dfrac{5}{\sqrt{26}} \right) A = -0{,}1 A \\ N = \left(0{,}2 - \dfrac{1}{0{,}02 \cdot 10^5} \cdot 40\sqrt{26} \cdot \dfrac{5}{\sqrt{26}} \right) A = 0{,}1 A \end{cases} \quad (2.67)^*$$

The relations (2.47) becomes

$$\begin{cases} i_1 = \left(\dfrac{4\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right) - 0{,}1 \right) A = \tilde{i}_1 - I_0 \\ i_2 = \left(\dfrac{2\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right) + 0{,}1 \right) A = \tilde{i}_2 + I_0 \end{cases} \quad (2.68)$$

The currents through the coils are the superposition of sinusoidal currents having different amplitudes and a direct current passing only through the coils. This direct current has the constant value

$$I_0 = 0{,}1 A \quad (2.69)^*$$

as in the figure 2.2.

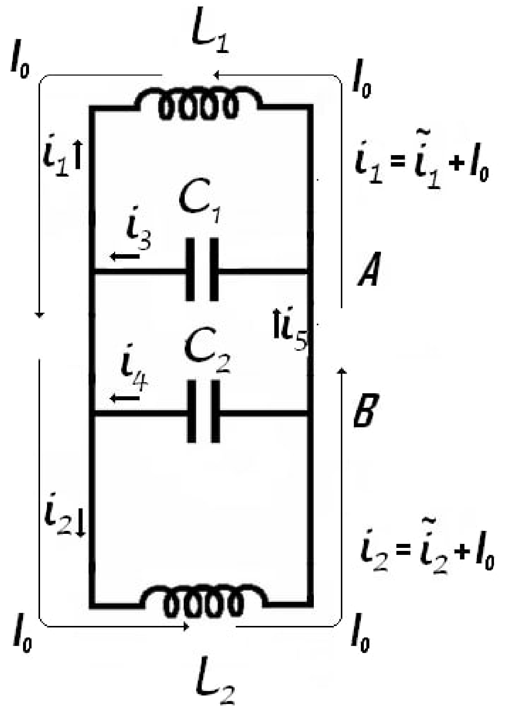


Figure 2.2

The alternative currents through the coils has the expressions

$$\begin{cases} \tilde{i}_1 = \left(\dfrac{4\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right) \right) A \\ \tilde{i}_2 = \left(\dfrac{2\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right) \right) A \end{cases} \quad (2.70)$$

The currents through the capacitors has the forms

$$\begin{cases} i_3 = \left(-10\times10^{-4}\cdot 40\sqrt{26}\cdot\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \\ i_3 = \left(-\frac{4\sqrt{26}}{100}\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \\ i_4 = \left(-5\times10^{-4}\cdot 40\sqrt{26}\cdot\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \\ i_4 = \left(-\frac{2\sqrt{26}}{100}\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \end{cases} \quad (2.71)$$

The current i_5 has the expression

$$\begin{cases} i_5 = i_3 - i_1 \\ i_5 = \left(-\frac{8\sqrt{26}}{100}\cos\left(10^5\cdot t + arctg(1/5)\right) + 0{,}1\right)A \end{cases} \quad (2.72)$$

The value of the intensity of i_5 current is the answer from the question **c**.
The initial value of this current is

$$i_5 = \left(-\frac{8\sqrt{26}}{100}\frac{5}{\sqrt{26}} + 0{,}1\right)A = -0{,}3\,A \quad (2.73)^*$$

d. The amplitude of the current through the inductance L_1 is

$$\max\left(\tilde{i}_1\right) = \max\left(\frac{4\sqrt{26}}{100}\cdot\cos\left(10^5\cdot t + arctg(1/5)\right)A\right) = \frac{4\sqrt{26}}{100}A \approx 0{,}2\,A \quad (2.74)^*$$

representing the answer at the question **d**.

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research- Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

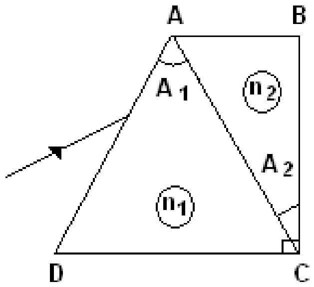
Optics – Problem III (7points)

Prisms

Two dispersive prisms having apex angles $\hat{A}_1 = 60°$ and $\hat{A}_2 = 30°$ are glued as in the figure ($\hat{C} = 90°$). The dependences of refraction indexes of the prisms on the wavelength are given by the relations

$$n_1(\lambda) = a_1 + \frac{b_1}{\lambda^2};$$

$$n_2(\lambda) = a_2 + \frac{b_2}{\lambda^2}$$



were

$a_1 = 1{,}1$, $b_1 = 1 \cdot 10^5\ nm^2$, $a_2 = 1{,}3$, $b_2 = 5 \cdot 10^4\ nm^2$.

a. Determine the wavelength λ_0 of the incident radiation that pass through the prisms without refraction on AC face at any incident angle; determine the corresponding refraction indexes of the prisms.

b. Draw the ray path in the system of prisms for three different radiations λ_{red}, λ_0, λ_{violet} incident on the system at the same angle.

c. Determine the minimum deviation angle in the system for a ray having the wavelength λ_0.

d. Calculate the wavelength of the ray that penetrates and exits the system along directions parallel to DC.

Problem III - Solution

a. The ray with the wavelength λ_0 pass trough the prisms system without refraction on AC face at any angle of incidence if :

$n_1(\lambda_0) = n_2(\lambda_0)$

Because the dependence of refraction indexes of prisms on wavelength has the form :

$$n_1(\lambda) = a_1 + \frac{b_1}{\lambda^2} \qquad (3.1)$$

$$n_2(\lambda) = a_2 + \frac{b_2}{\lambda^2} \qquad (3.2)$$

The relation (3.1) becomes:

$$a_1 + \frac{b_1}{\lambda_0^2} = a_2 + \frac{b_2}{\lambda_0^2} \qquad (3.3)$$

The wavelength λ_0 has correspondingly the form:

$$\lambda_0 = \sqrt{\frac{b_1 - b_2}{a_2 - a_1}} \qquad (3.4)$$

Substituting the furnished numerical values

$$\lambda_0 = 500\,nm \qquad (3.5)$$

The corresponding common value of indexes of refraction of prisms for the radiation with the wavelength λ_0 is:

$$n_1(\lambda_0) = n_2(\lambda_0) = 1,5 \qquad (3.6)$$

The relations (3.6) and (3.7) represent the answers of question **a**.

b. For the rays with different wavelength (λ_{red}, λ_0, λ_{violet}) having the same incidence angle on first prism, the paths are illustrated in the figure 1.1.

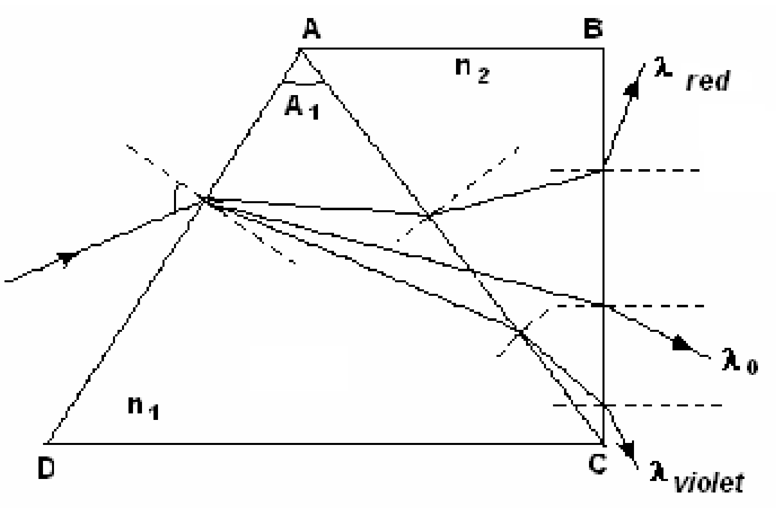


Figure 3.1

The draw illustrated in the figure 1.1 represents the answer of question **b**.

c. In the figure 1.2 is presented the path of ray with wavelength λ_0 at minimum deviation (the angle between the direction of incidence of ray and the direction of emerging ray is minimal).

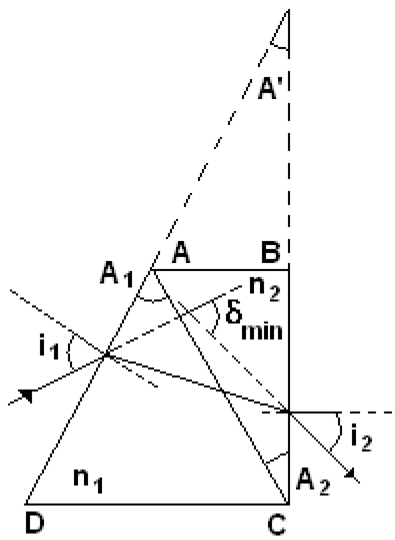


Figure 3.2

In this situation

$$n_1(\lambda_0) = n_2(\lambda_0) = \frac{\sin\frac{\delta_{\min} + A'}{2}}{\sin\frac{A'}{2}} \qquad (3.7)$$

where

$m(\hat{A}') = 30°$,

as in the figure 1.1

Substituting in (3.8) the values of refraction indexes the result is

$$\sin\frac{\delta_{\min} + A'}{2} = \frac{3}{2}\cdot\sin\frac{A'}{2} \qquad (3.8)$$

or

$$\delta_{\min} = 2\arcsin\left(\frac{3}{2}\cdot\sin\frac{A'}{2}\right) - \frac{A'}{2} \qquad (3.9)$$

Numerically

$$\delta_{\min} \cong 30{,}7° \qquad (3.10)$$

The relation (3.11) represents the answer of question **c**.

d. Using the figure 1.3 the refraction law on the AD face is

$$\sin i_1 = n_1 \cdot \sin r_1 \qquad (3.11)$$

The refraction law on the AC face is

$$n_1 \cdot \sin r_1' = n_2 \cdot \sin r_2 \qquad (3.12)$$

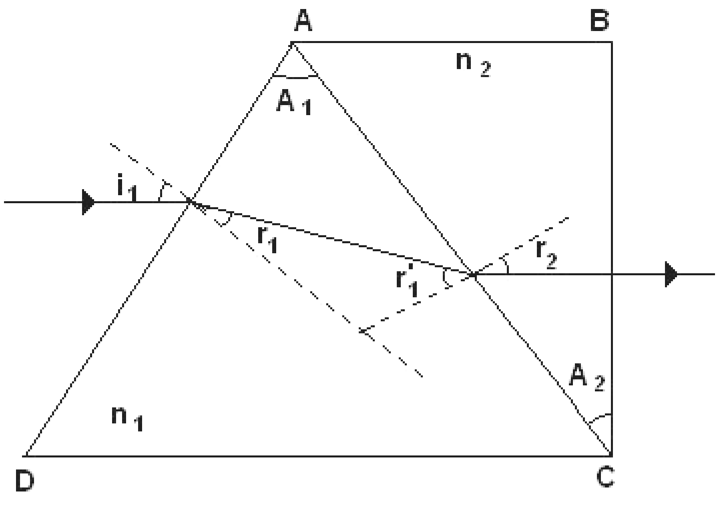


Figure 3.3

As it can be seen in the figure 1.3

$$r_2 = A_2 \qquad (3.13)$$

and

$$i_1 = 30° \qquad (3.14)$$

Also,

$$r_1 + r_1' = A_1 \qquad (3.15)$$

Substituting (3.16) and (3.14) in (3.13) it results

$$n_1 \cdot \sin(A_1 - r_1) = n_2 \cdot \sin A_2 \quad (3.16)$$

or

$$n_1 \cdot (\sin A_1 \cdot \cos r_1 - \sin r_1 \cdot \cos A_1) = n_2 \cdot \sin A_2 \quad (3.17)$$

Because of (3.12) and (3.15) it results that

$$\sin r_1 = \frac{1}{2n_1} \quad (3.18)$$

and

$$\cos r_1 = \frac{1}{2n_1}\sqrt{4n_1^2 - 1} \quad (3.19)$$

Putting together the last three relations it results

$$\sqrt{4n_1^2 - 1} = \frac{2n_2 \cdot \sin A_2 + \cos A_1}{\sin A_1} \quad (3.20)$$

Because

$\hat{A}_1 = 60°$

and

$\hat{A}_2 = 30°$

relation (3.21) can be written as

$$\sqrt{4n_1^2 - 1} = \frac{2n_2 + 1}{\sqrt{3}} \quad (3.21)$$

or

$$3 \cdot n_1^2 = 1 + n_2 + n_2^2 \quad (3.22)$$

Considering the relations (3.1), (3.2) and (3.23) and operating all calculus it results:

$$\lambda^4 \cdot (3a_1^2 - a_2^2 - a_2 - 1) + (6a_1 b_1 - b_2 - 2a_2 b_2) \cdot \lambda^2 + 3b_1^2 - b_2^2 = 0 \quad (3.23)$$

Solving the equation (3.24) one determine the wavelength λ of the ray that enter the prisms system having the direction parallel with DC and emerges the prism system having the direction again parallel with DC. That is

$$\lambda = 1194\, nm \quad (3.24)$$

or

$$\lambda \cong 1{,}2\, \mu m \quad (3.25)$$

The relation (3.26) represents the answer of question d.

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research- Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

Atomics - Problem IV (7 points)

Compton scattering

A photon of wavelength λ_i is scattered by a moving, free electron. As a result the electron stops and the resulting photon of wavelength λ_0 scattered at an angle $\theta = 60°$ with respect to the direction of the incident photon, is again scattered by a second free electron at rest. In this second scattering process a photon with wavelength of $\lambda_f = 1{,}25 \times 10^{-10}\, m$ emerges at an angle $\theta = 60°$ with respect to the direction of the photon of wavelength λ_0. Find the de Broglie wavelength for the first electron before the interaction. The following constants are known:
$h = 6{,}6 \times 10^{-34}\, J \cdot s$ - Planck's constant
$m = 9{,}1 \times 10^{-31}\, kg$ - mass oh the electron
$c = 3{,}0 \times 10^{8}\, m/s$ - speed of light in vacuum

Problem III - Solution

The purpose of the problem is to calculate the values of the speed, momentum and wavelength of the first electron.

To characterize the photons the following notation are used:

Table 4.1

	initial photon	photon – after the first scattering	final photon
momentum	$\vec{p}_i$	$\vec{p}_0$	$\vec{p}_f$
energy	E_i	E_0	E_f
wavelength	λ_i	λ_i	λ_f

To characterize the electrons one uses

Table 4.2

	first electron before collision	first electron after collision	second electron before collision	Second electron after collision
momentum	$\vec{p}_{1e}$	0	0	$\vec{p}_{2e}$
energy	E_{1e}	E_{0e}	E_{0e}	E_{2e}
speed	$\vec{v}_{1e}$	0	0	$\vec{v}_{2e}$

The image in figure 4.1 presents the situation before the first scattering of photon.

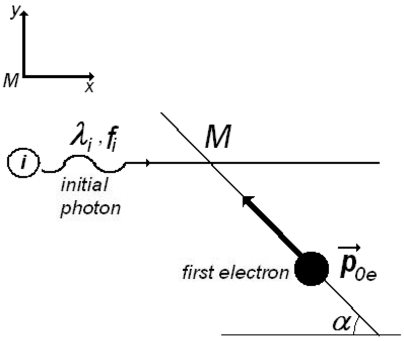


Figure 4.1

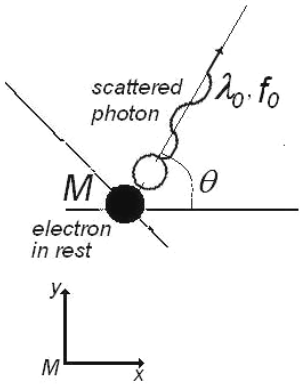


Figure 4.2

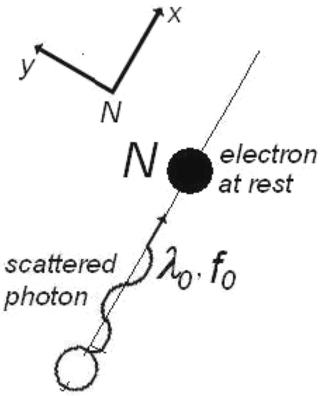


Figure 4.3

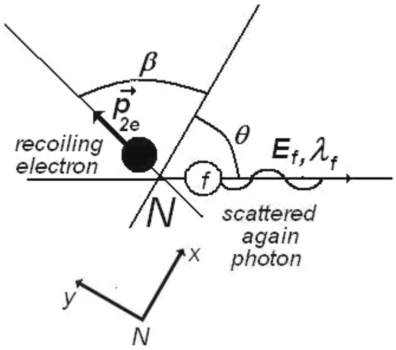


Figure 4.4

To characterize the initial photon we will use his momentum $\vec{p}_i$ and his energy E_i

$$\begin{cases} \vec{P}_i = \dfrac{h}{\lambda_i} = \dfrac{h \cdot f_i}{c} \\ E_i = h \cdot f_i \end{cases} \tag{4.1}$$

$$f_i = \frac{c}{\lambda_i} \tag{4.2}$$

is the frequency of initial photon.
For initial, free electron in motion the momentum $\vec{p}_{oe}$ and the energy E_{oe} are

$$\begin{cases} \vec{P}_{oe} = m \cdot \vec{v}_{1e} = \dfrac{m_0 \cdot \vec{v}_{1e}}{\sqrt{1-\beta^2}} \\ E_{oe} = m \cdot c^2 = \dfrac{m_0 \cdot c^2}{\sqrt{1-\beta^2}} \end{cases} \tag{4.3}$$

where m_0 is the rest mass of electron and m is the mass of moving electron. As usual, $\beta = \dfrac{v_{1e}}{c}$.
De Broglie wavelength of the first electron is

$$\lambda_{oe} = \frac{h}{p_{0e}} = \frac{h \cdot}{m_0 \cdot v_{1e}} \sqrt{1-\beta^2}$$

The situation after the scattering of photon is described in the figure 4.2.
To characterize the scattered photon we will use his momentum $\vec{p}_0$ and his energy E_0

$$\begin{cases} \vec{P}_o = \dfrac{h}{\lambda_o} = \dfrac{h \cdot f_o}{c} \\ E_o = h \cdot f_o \end{cases} \qquad (4.4).$$

where

$$f_o = \frac{c}{\lambda_0} \qquad (4.5)$$

is the frequency of scattered photon.

The magnitude of momentum of the electron (that remains in rest) after the scattering is zero; his energy is E_{1e}. The mass of electron after collision is m_0 - the rest mass of electron at rest. So,

$E_{1e} = m_0 \cdot c^2$

To determine the moment of the first moving electron, one can write the principles of conservation of moments and energy. That is

$$\vec{P}_i + \vec{p}_{oe} = \vec{p}_0 \qquad (4.6)$$

and

$$E_i + E_{0e} = E_0 + E_{1e} \qquad (4.7)$$

The conservation of moment on Ox direction is written as

$$\frac{h \cdot f_i}{c} + m \cdot v_{1e} \cdot \cos\alpha = \frac{h \cdot f_0}{c}\cos\theta \qquad (4.8)$$

and the conservation of moment on Oy is

$$m \cdot v_{1e} \cdot \sin\alpha = \frac{h \cdot f_0}{c}\sin\theta \qquad (4.9)$$

To eliminate α, the last two equation must be written again as

$$\begin{cases} (m \cdot v_{1e} \cdot \cos\alpha)^2 = \dfrac{h^2 \cdot}{c^2}(f_0 \cdot \cos\theta - f_i)^2 \\ (m \cdot v_{1e} \cdot \sin\alpha)^2 = \left(\dfrac{h \cdot f_0}{c}\sin\theta\right)^2 \end{cases} \qquad (4.10)$$

and then added.
The result is

$$m^2 \cdot v_{1e}^2 = \frac{h^2 \cdot}{c^2}\left(f_0^2 + f_1^2 - 2f_0 \cdot f_i \cdot \cos\theta\right) \qquad (4.11)$$

or

$$\frac{m_0^2 \cdot c^2}{1-\left(\frac{v_{1e}}{c}\right)^2} \cdot v_{1e}^2 = h^2 \cdot \left(f_0^2 + f_1^2 - 2f_0 \cdot f_i \cdot \cos\theta\right) \qquad (4.12)$$

The conservation of energy (4.7) can be written again as

$$m \cdot c^2 + h \cdot f_1 = m_0 \cdot c^2 + h \cdot f_0 \qquad (4.13)$$

or

$$\frac{m_0 \cdot c^2}{\sqrt{1-\left(\frac{v_{1e}}{c}\right)^2}} = m_0 \cdot c^2 + h \cdot \left(f_0 - f_1\right) \qquad (4.14)$$

Squaring the last relation results

$$\frac{m_0^2 \cdot c^4}{1-\left(\frac{v_{1e}}{c}\right)^2} = m_0^2 \cdot c^4 + h^2 \cdot \left(f_0 - f_1\right)^2 + m_0 \cdot h \cdot c^2 \cdot \left(f_0 - f_1\right) \qquad (4.15)$$

Subtracting (4.12) from (4.15) the result is

$$2m_0 \cdot c^2 \cdot h \cdot \left(f_0 - f_1\right) + 2h^2 \cdot f_1 \cdot f_0 \cdot \cos\theta - 2h^2 \cdot f_1 \cdot f = 0 \qquad (4.16)$$

or

$$\frac{h}{m_0 \cdot c}\left(1 - \cos\theta\right) = \frac{c}{f_1} - \frac{c}{f_0} \qquad (4.17)$$

Using

$$\Lambda = \frac{h}{m_0 \cdot c} \qquad (4.18)$$

the relation (4.17) becomes

$$\Lambda \cdot \left(1 - \cos\theta\right) = \lambda_i - \lambda_0 \qquad (4.19)$$

The wavelength of scattered photon is

$$\lambda_0 = \lambda_i - \Lambda \cdot \left(1 - \cos\theta\right) \qquad (4.20)$$

shorter than the wavelength of initial photon and consequently the energy of scattered photon is greater that the energy of initial photon.

$$\begin{cases} \lambda_i < \lambda_0 \\ E_i > E_0 \end{cases} \qquad (4.21)$$

Let's analyze now the second collision process that occurs in point N. To study that, let's consider a new referential having Ox direction on the direction of the photon scattered after the first collision.

The figure 4.3 presents the situation before the second collision and the figure 4.4 presents the situation after this scattering process. The conservation principle for moment in the scattering process gives

$$\begin{cases} \frac{h}{\lambda_0} = \frac{h}{\lambda_f}\cos\theta + m \cdot v_{2e} \cdot \cos\beta \\ \frac{h}{\lambda_f}\sin\theta - m \cdot v_{2e} \cdot \sin\beta = 0 \end{cases} \qquad (4.22)$$

To eliminate the unknown angle β must square and then add the equations (4.22)

That is

$$\begin{cases} \left(\frac{h}{\lambda_0} - \frac{h}{\lambda_f}\cos\theta\right)^2 = \left(m \cdot v_{2e} \cdot \cos\beta\right)^2 \\ \left(\frac{h}{\lambda_f}\sin\theta\right)^2 = \left(m \cdot v_{2e} \cdot \sin\beta\right)^2 \end{cases} \qquad (4.23)$$

or

$$\left(\frac{h}{\lambda_f}\right)^2 + \left(\frac{h}{\lambda_0}\right)^2 - \frac{2 \cdot h^2}{\lambda_0 \cdot \lambda_f}\cos\theta = \left(m \cdot v_{2e}\right)^2 \qquad (4.24)$$

The conservation principle of energy in the second scattering process gives

$$\frac{h \cdot c}{\lambda_0} + m_0 \cdot c^2 = \frac{h \cdot c}{\lambda_f} + m \cdot c^2 \qquad (4.25)$$

(4.24) and (4.25) gives

$$\frac{h^2 \cdot c^2}{\lambda_f^2} + \frac{h^2 \cdot c^2}{\lambda_0^2} - \frac{2 \cdot h^2 \cdot c^2}{\lambda_0 \cdot \lambda_f}\cos\theta = m^2 \cdot c^2 \cdot v_{2e}^2 \qquad (4.26)$$

and

$$h^2 \cdot c^2 \cdot \left(\frac{1}{\lambda_f} - \frac{1}{\lambda_0}\right)^2 + m_0^2 \cdot c^4 + 2h \cdot c^3 \cdot m_0 \cdot \left(\frac{1}{\lambda_f} - \frac{1}{\lambda_0}\right) = m^2 \cdot c^4 \qquad (4.27)$$

Subtracting (4.26) from (1.27), one obtain

$$\begin{cases} \frac{h}{m_0 \cdot c} \cdot \left(1 - \cos\theta\right) = \lambda_f - \lambda_0 \\ \lambda_f - \lambda_0 = \Lambda \cdot \left(1 - \cos\theta\right) \end{cases} \qquad (4.28)$$

That is

$$\begin{cases} \lambda_f > \lambda_0 \\ E_f < E_0 \end{cases} \qquad (4.29)$$

Because the value of λ_f is know and Λ can be calculate as

$$\begin{cases} \lambda_f = 1{,}25 \times 10^{-10}\, m \\ \Lambda = \dfrac{6{,}6 \times 10^{-34}}{9{,}1 \times 10^{-31} \cdot 3 \times 10^{8}}\, m = 2{,}41 \times 10^{-12}\, m = 0{,}02 \times 10^{-10}\, m \end{cases} \qquad (4.30)$$

the value of wavelength of photon before the second scattering is

$$\lambda_0 = 1{,}23 \times 10^{-10}\, m \qquad (4.31)$$

Comparing (4.28) written as:

$$\lambda_f = \lambda_0 + \Lambda \cdot (1 - \cos\theta) \qquad (4.32)$$

and (4.20) written as

$$\lambda_i = \lambda_0 + \Lambda \cdot (1 - \cos\theta) \qquad (4.33)$$

clearly results

$$\lambda_i = \lambda_f \qquad (4.34)$$

The energy of the double scattered photon is the same as the energy of initial photon. The direction of "final photon" is the same as the direction of "initial" photon. Concluding, the final photon is identical with the initial photon. The result is expected because of the symmetry of the processes.

Extending the symmetry analyze on electrons, the first moving electron that collides the initial photon and after that remains at rest, must have the same momentum and energy as the second electron after the collision – because this second electron is at rest before the collision.

That is

$$\begin{cases} \vec{p}_{1e} = \vec{p}_{2e} \\ E_{1e} = E_{2e} \end{cases} \qquad (4.35)$$

Taking into account (4.24), the moment of final electron is

$$p_{2e} = h\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{(\lambda_f - \Lambda(1-\cos\theta))^2} - \frac{2\cdot\cos\theta}{\lambda_f \cdot (\lambda_f - \Lambda(1-\cos\theta))}} \qquad (4.36)$$

The de Broglie wavelength of second electron after scattering (and of first electron before scattering) is

$$\lambda_{1e} = \lambda_{2e} = 1 \Big/ \left(\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{(\lambda_f - \Lambda(1-\cos\theta))^2} - \frac{2\cdot\cos\theta}{\lambda_f \cdot (\lambda_f - \Lambda(1-\cos\theta))}}\right) \qquad (4.37)$$

Numerical value of this wavelength is

$$\lambda_{1e} = \lambda_{2e} = 1{,}24 \times 10^{-10}\, m \qquad (4.38)$$

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research- Bucharest, Romania

Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

IPhO's LOGO – Problem V

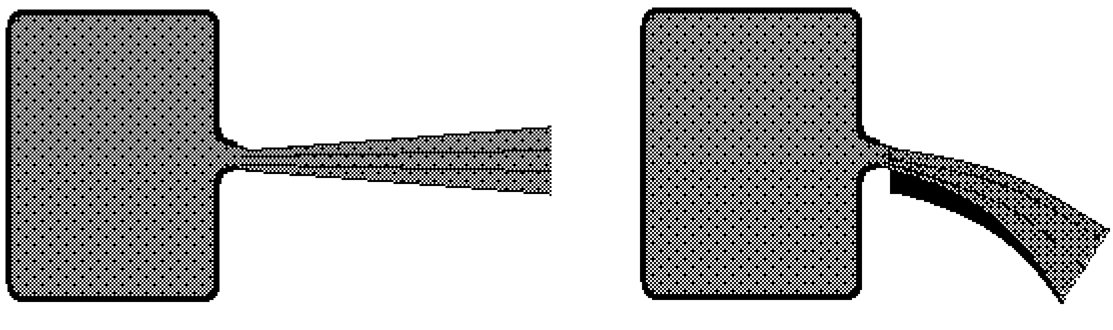
The Logo of the International Physics Olympiad is represented in the figure below.
The figure presents the phenomenon of the curving of the trajectory of a jet of fluid around the shape of a cylindrical surface. The trajectory of fluid is not like the expected dashed line but as the circular solid line.
Qualitatively explain this phenomenon (first observed by Romanian engineer Henry Coanda in 1936).

This problem will be not considered in the general score of the Olympiad. The best solution will be awarded a special prize.

Figure 5.1

Problem V -Solution

Suppose a fluid is in a recipient at a constant pressure. If a thin jet of fluid (gas or liquid) having a small circular or rectangular cross section leaves the recipient through a nozzle entering the medium, the particles belonging to the medium will be carried out by the jet. Other particles belonging to the medium will be attracted to the jet.
If the jet flows over a large surface, the particles belonging to the medium over the jet and the particles leaving between the jet and the surface will be carried out by the jet. The density of particles over the jet remains constant because of newly arriving particles, but the particles between the surface and the jet cannot be replaced. A pressure difference appears between the upper and lower side of the jet, pushing the jet to the surface. If the surface is curved, the jet will follow its shape.
The left image in the figure below presents the normal flow of a fluid jet leaving through a nozzle of a recipient with a high, constant pressure. The final pressure of the fluid is of medium pressure.
The right image in the figure below presents the flow of a fluid over the large surface. The jet is "stuck"



against the surface.

The process of deflection of the jet increases the speed of the jet without any variation of the pressure and temperature of the jet.
During the tests of the first jet plane in Paris, December 1936, the Romanian engineer Henry Coanda was the first to observe this phenomenon, occurring when the flames of the engine passed through a flap.

The logo of the Olympiad illustrates the Coanda flow of a fluid.

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research- Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

Mechanics – Problem I (8 points)

Jumping particle

A particle moves along the positive axis Ox (one-dimensional situation) under a force that's projection on Ox is $F_x = F_0$ as represented in the figure below (as function of x). At the origin of Ox axis is placed a perfectly reflecting wall.

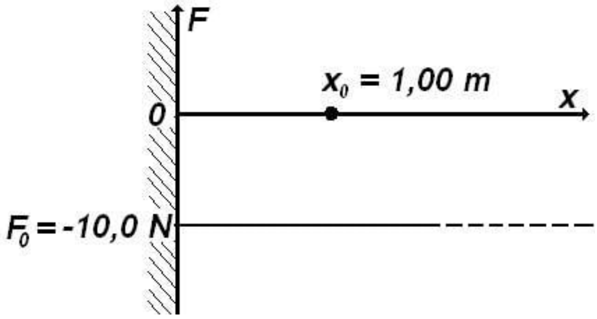
A friction force of constant modulus $F_f = 1{,}00\,N$ acts anywhere the particle is situated.

The particle starts from the point $x = x_0 = 1{,}00\,m$ having the kinetic energy $E_c = 10{,}0\,J$.

a. Find the length of the path of the particle before it comes to a final stop

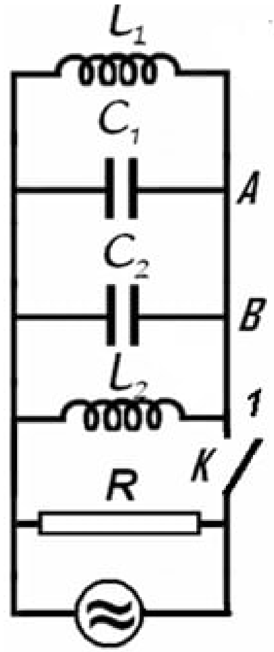
b. Sketch the potential energy $U(x)$ of the particle in the force field F_x.

c. Draw qualitatively the dependence of the particle speed as function of his coordinate x.



Electricity – Problem II (8 points)

Different kind of oscillation



Let's consider the electric circuit in the figure, for which $L_1 = 10\ mH$, $L_2 = 20\ mH$, $C_1 = 10\ nF$, $C_2 = 5\ nF$ and $R = 100\,k\Omega$. The switch K being closed the circuit is coupled with a source of alternating current. The current furnished by the source has constant intensity while the frequency of the current may be varied.

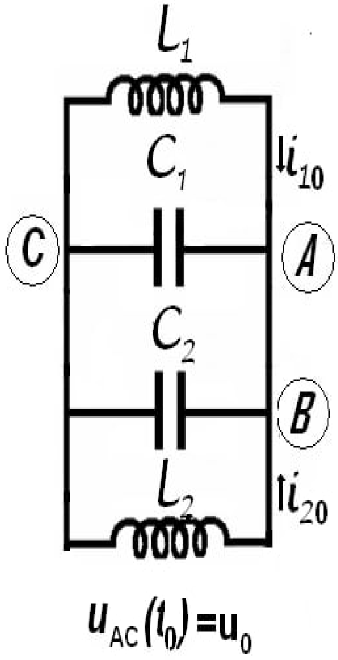
a. Find the ratio of frequency f_m for which the active power in circuit has the maximum value P_m and the frequency difference $\Delta f = f_+ - f_-$ of the frequencies f_+ and f_- for which the active power in the circuit is half of the maximum power P_m.

The switch K is now open. In the moment t_0 immediately after the switch is open the intensities of the currents in the coils L_1 and $i_{01} = 0{,}1\ A$ and

$i_{02} = 0{,}2\ A$ L_1 (the currents flow as in the figure); at the same moment, the potential difference on the capacitor with capacity C_1 is $u_0 = 40V$:

b. Calculate the frequency of electromagnetic oscillation in $L_1C_1C_2L_2$ circuit;
c. Determine the intensity of the electric current in the *AB* conductor;
d. Calculate the amplitude of the oscillation of the intensity of electric current in the coil L_1.

Neglect the mutual induction of the coils, and the electric resistance of the conductors. Neglect the fast transition phenomena occurring when the switch is closed or opened.



Optics – Problem III (7points)

Prisms

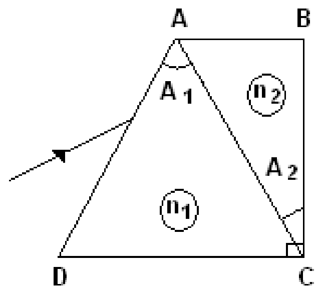
Two dispersive prisms having apex angles $\hat{A}_1 = 60°$ and $\hat{A}_2 = 30°$ are glued as in the figure below ($\hat{C} = 90°$). The dependences of refraction indexes of the prisms on the wavelength are given by the relations

$$n_1(\lambda) = a_1 + \frac{b_1}{\lambda^2};$$

$$n_2(\lambda) = a_2 + \frac{b_2}{\lambda^2}$$

were

$a_1 = 1{,}1$, $b_1 = 1 \cdot 10^5\ nm^2$, $a_2 = 1{,}3$, $b_2 = 5 \cdot 10^4\ nm^2$.



a. Determine the wavelength λ_0 of the incident radiation that pass through the prisms without refraction on AC face at any incident angle; determine the corresponding refraction indexes of the prisms.
b. Draw the ray path in the system of prisms for three different radiations λ_{red}, λ_0, λ_{violet} incident on the system at the same angle.
c. Determine the minimum deviation angle in the system for a ray having the wavelength λ_0.
d. Calculate the wavelength of the ray that penetrates and exits the system along directions parallel to DC.

Atomics - Problem IV (7 points)

Compton scattering

A photon of wavelength λ_i is scattered by a moving, free electron. As a result the electron stops and the resulting photon of wavelength λ_0 scattered at an angle $\theta = 60°$ with respect to the direction of the incident photon, is again scattered by a second free electron at rest. In this second scattering process a photon with wavelength of $\lambda_f = 1{,}25 \times 10^{-10} m$ emerges at an angle $\theta = 60°$ with respect to the direction of the photon of wavelength λ_0. Find the de Broglie wavelength for the first electron before the interaction. The following constants are known:

$h = 6{,}6 \times 10^{-34} J \cdot s$ - Planck's constant

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$c = 3{,}0 \times 10^{8} m/s$ - speed of light in vacuum

The purpose of the problem is to calculate the values of the speed, momentum and wavelength of the first electron.

To characterize the photons the following notation are used:

Table 4.1

	initial photon	photon – after the first scattering	final photon
momentum	$\vec{p}_i$	$\vec{p}_0$	$\vec{p}_f$
energy	E_i	E_0	E_f
wavelength	λ_i	λ_i	λ_f

To characterize the electrons one uses

Table 4.2

	first electron before collision	first electron after collision	second electron before collision	Second electron after collision
momentum	$\vec{p}_{1e}$	0	0	$\vec{p}_{2e}$
energy	E_{1e}	E_{0e}	E_{0e}	E_{2e}
speed	$\vec{v}_{1e}$	0	0	$\vec{v}_{2e}$

IPhO's LOGO – Problem V

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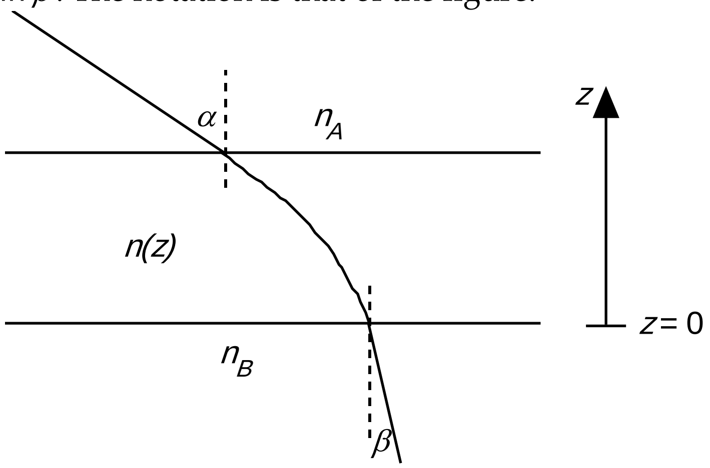
Problems of the XV International Physics Olympiad (Sigtuna, 1984)

Lars Gislén
Department of Theoretical Physics, University of Lund, Sweden

Theoretical problems

Problem 1

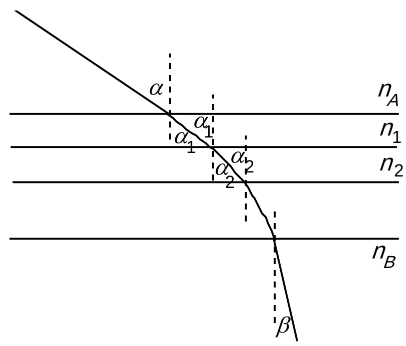
a) Consider a plane-parallel transparent plate, where the refractive index, n, varies with distance, z, from the lower surface (see figure). Show that $n_A \sin\alpha = n_B \sin\beta$. The notation is that of the figure.



b) Assume that you are standing in a large flat desert. At some distance you see what appears to be a water surface. When you approach the "water" is seems to move away such that the distance to the "water" is always constant. Explain the phenomenon.

c) Compute the temperature of the air close to the ground in b) assuming that your eyes are located 1.60 m above the ground and that the distance to the "water" is 250 m. The refractive index of the air at 15 ˚C and at normal air pressure (101.3 kPa) is 1.000276. The temperature of the air more than 1 m above the ground is assumed to be constant and equal to 30 ˚C. The atmospheric pressure is assumed to be normal. The refractive index, n, is such that $n - 1$ is proportional to the density of the air. Discuss the accuracy of your result.

Solution:



a) From the figure we get

$n_A \sin\alpha = n_1 \sin\alpha_1 = n_2 \sin\alpha_2 = \ldots = n_B \sin\beta$

b) The phenomenon is due to total reflexion in a warm layer of air when $\beta = 90˚$. This gives

$$n_A \sin\alpha = n_B$$

c) As the density, ρ, of the air is inversely proportional to the absolute temperature, T, for fixed pressure we have

$$n(T) = 1 + k\cdot\rho = 1 + k/T$$

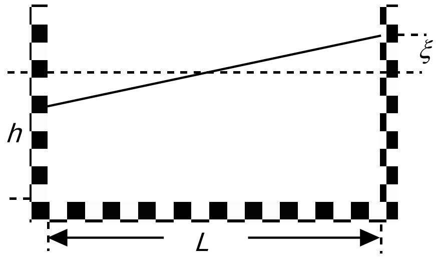
The value given at 15 ˚C determines the value of k = 0.0795.
In order to have total reflexion we have $n_{30}\sin\alpha = n_T$ or

$$\left(1+\frac{k}{303}\right)\cdot\frac{L}{\sqrt{h^2+L^2}} = \left(1+\frac{k}{T}\right)$$ with h = 1.6 m and L = 250 m

As $h \ll L$ we can use a power expansion in h/L:

$$T = \frac{303}{\left(\frac{303}{k}+1\right)\frac{1}{\sqrt{1+h^2/L^2}}-\frac{303}{k}} \approx 303\left(1+\frac{303h^2}{2kL^2}\right) = 328\,\mathrm{K} = 56\,^\circ\mathrm{C}$$

Problem 2



In certain lakes there is a strange phenomenon called "seiching" which is an oscillation of the water. Lakes in which you can see this phenomenon are normally long compared with the depth and also narrow. It is natural to see waves in a lake but not something like the seiching, where the entire water volume oscillates, like the coffee in a cup that you carry to a waiting guest.

In order to create a model of the seiching we look at water in a rectangular container. The length of the container is L and the depth of the water is h. Assume that the surface of the water to begin with makes a small angle with the horizontal. The seiching will then start, and we assume that the water surface continues to be plane but oscillates around an axis in the horizontal plane and located in the middle of the container.

Create a model of the movement of the water and derive a formula for the oscillation period T. The starting conditions are given in figure above. Assume that $\xi << h$. The table below shows experimental oscillation periods for different water depths in two containers of different lengths. Check in some reasonable way how well the formula that you have derived agrees with the experimental data. Give your opinion on the quality of your model.

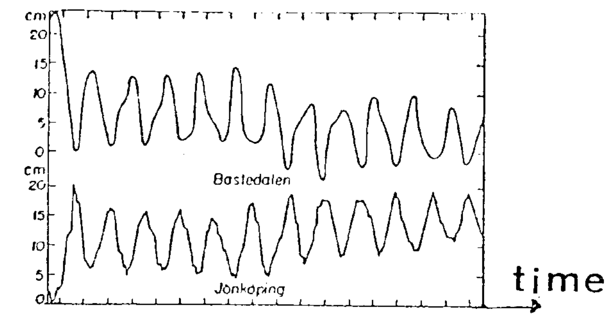
Table 1. L = 479 mm

h/ mm	30	50	69	88	107	124	142
T/ s	1.78	1.40	1.18	1.08	1.00	0.91	0.82

Table 2. L = 143 mm

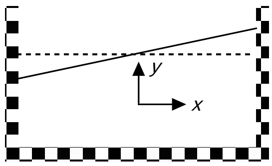
h/ mm	31	38	58	67	124
T/ s	0.52	0.52	0.43	0.35	0.28

The graph below shows results from measurements in lake Vättern in Sweden. This lake has a length of 123 km and a mean depth of 50 m. What is the time scale in the graph?



The water surface level in Bastudalen (northern end of lake Vättern) and Jönköping (southern end).

Solution:



In the coordinate system of the figure, we have for the centre of mass coordinates of the two triangular parts of the water

$$(x_1, y_1) = (L/3, h/2 + \xi/3) \quad (x_2, y_2) = (-L/3, h/2 - \xi/3).$$

For the entire water mass the centre of mass coordinates will then be

$$(x_{CoM}, y_{CoM}) = \left(\frac{\xi L}{6h}, \frac{\xi^2}{6h}\right)$$

Due to that the y component is quadratic in ξ will be much much smaller than the x component.

The velocities of the water mass are

$$(v_x, v_y) = \left(\frac{\dot{\xi} L}{6h}, \frac{\xi\dot{\xi}}{3h}\right),$$

and again the vertical component is much smaller the the horizontal one.

We now in our model neglect the vertical components. The total energy (kinetic + potential) will then be

$$W = W_K + W_P = \tfrac{1}{2} M \frac{\dot{\xi}^2 L^2}{36h^2} + Mg\frac{\xi^2}{6h^2}$$

For a harmonic oscillator we have

$$W = W_K + W_P = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} m\omega^2 x^2$$

Identifying gives

$$\omega = \sqrt{\frac{12gh}{L}} \text{ or } T_{model} = \frac{\pi L}{\sqrt{3h}}.$$

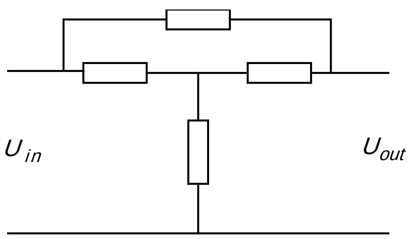
Comparing with the experimental data we find $T_{experiment} \approx 1.1 \cdot T_{model}$, our model gives a slight underestimation of the oscillation period.

Applying our corrected model on the Vättern data we have that the oscillation period of the seiching is about 3 hours.

Many other models are possible and give equivalent results.

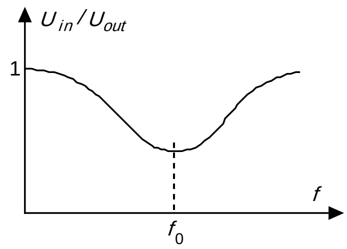
Problem 3

An electronic frequency filter consists of four components coupled as in the upper figure. The impedance of the source can be neglected and the impedance of the load can be taken as infinite. The filter should be such that the voltage ratio U_{out}/U_{in} has a frequency dependence shown in the lower where U_{in} is the input voltage and U_{out} is the output voltage. At frequency f_0 the phase lag between the two voltages is zero.



In order to build the filter you can choose from the following components:

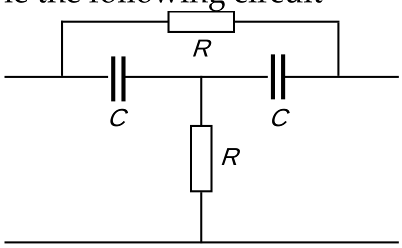
2 resistors, 10 kΩ
2 capacitors, 10 nF
2 solenoids, 160 mH (iron-free and with neglible resistance)



Construct, by combining four of these components, a filter that fulfils the stated conditions. Determine the frequency f_0 and the ratio U_{out}/U_{in} at this frequency for as many component combinations as possible.

Solution:

The conditions at very high and very low frequencies can be satisfied with for example the following circuit



Using either the graphic vector method or the analytic jω method we can show that the minimum occurs för a frequency $f_0 = \dfrac{1}{2\pi RC}$ when the ratio between the output and input voltages is 2/3. Switching the resistors and the capacitors gives a new circuit with the same frequency f_0. Another two possibilities is to exchange the capacitors for solenoids where we get $f_0 = \dfrac{R}{2\pi L}$. There are further eight solutions with unsymmetric patterns of the electronic components.

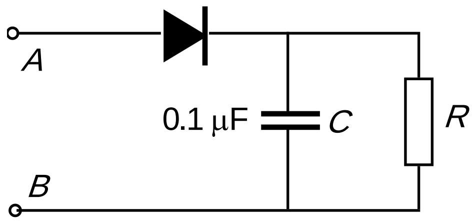
Experimental problems

Problem 1

You have at your disposal the following material:

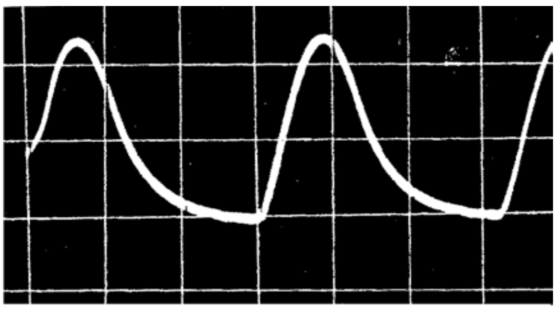
(1) A sine wave voltage generator set to a frequency of 0,20 kHz.
(2) A dual ray oscilloscope.
(3) Millimeter graph paper.
(4) A diod.
(5) A capacitor of 0.10 μF (square and black).
(6) An unknown resistor *R* (red).
(7) A coupling plate.
(8) Coupling wires.

Build the circuit shown in the figure.



Connect the terminals *A* and *B* to the sine wave generator set to a frequency of 0.20 kHz. Determine experimentally the mean power developed in the resistor *R* when the amplitude of the generator voltage is 2.0 V (that is the peak-to-peak voltage is 4.0 V).

Solution:



The picture to the right shows the oscilloscope voltage over the resistor. The period of the sine wave is 5 ms and this gives the relation 1 horizontal division = 1.5 ms. The actual vertical scale was 0.85 V / division. The first rising part of the curve is a section of a sine wave, the second falling part is an exponential decay determined by the time constant of the resistor and capacitor. Reading from the display the "half-life" $t_{1/2} = RC \cdot \ln 2$ turns out to be 0.5 ms. This gives *R* = 7.2 kΩ. The mean power developed in the resistor is

$\langle P \rangle = \frac{1}{T} \int_0^T \frac{U^2(t)}{R} dt$. Numerical integration (counting squares) gives

$\int_0^T U^2(t) dt = 4{,}5 \cdot 10^{-3}\ \mathrm{V^2 s}$ from which $\langle P \rangle \approx 0.1$ mW.

Problem 2

Material:
(1) A glow discharge lamp connected to 220 V, alternating current.
(2) A laser producing light of unknown wavelength.
(3) A grating.
(4) A transparent "micro-ruler", 1 mm long with 100 subdivisions, the ruler is situated exactly in the centre of the circle.
(5) A 1 m long ruler
(6) Writing material.

The spectrum of the glow discharge lamp has a number of spectral lines in the region yellow-orange-red. On of the yellow lines in the short wavelength part of this spectrum is very strong. Determine the wavelength of this spectral line. Estimate the accuracy of your measurement.

Note: If you happen to know the wavelength of the laser light beforehand you are not allowed to use that value in your computation.

Warning. Do not look into the laser beam. Do not touch the surface of the grating or the surface of the transparent micro-ruler.

Solution:

Using the micro-ruler with we can determine the wavelength of the laser light. Knowing this wavelength we can calibrate the grating and then use it to determine the unknown wavelength from the glow discharge lamp. We cannot use the micro-ruler to determine this wavelength because the intensity of the light from the lamp is too weak.

Problems and solutions of the 16th IPhO*
Portorož, Slovenia, (Former Yugoslavia), 1985

Contents

1 Problems

1.1 Theoretical competition

Problem 1

A young radio amateur maintains a radio link with two girls living in two towns. He positions an aerial array such that when the girl living in town A receives a maximum signal, the girl living in town B receives no signal and vice versa. The array is built from two vertical rod aerials transmitting with equal intensities uniformly in all directions in the horizontal plane.

*Edited by B. Golli, Faculty of Education, University of Ljubljana, and J. Stefan Institute, Ljubljana, Slovenia, e-mail:bojan.golli@ijs.si

a) Find the parameters of the array, i. e. the distance between the rods, its orientation and the phase shift between the electrical signals supplied to the rods, such that the distance between the rods is minimum.

b) Find the numerical solution if the boy has a radio station transmitting at 27 MHz and builds up the aerial array at Portorož. Using the map he has found that the angles between the north and the direction of A (Koper) and of B (small town of Buje in Istria) are 72° and 157°, respectively.

Problem 2

In a long bar having the shape of a rectangular parallelepiped with sides a, b, and c ($a \gg b \gg c$), made from the semiconductor InSb flows a current I parallel to the edge a. The bar is in an external magnetic field B which is parallel to the edge c. The magnetic field produced by the current I can be neglected. The current carriers are electrons. The average velocity of electrons in a semiconductor in the presence of an electric field only is $v = \mu E$, where μ is called mobility. If the magnetic field is also present, the electric field is no longer parallel to the current. This phenomenon is known as the Hall effect.

a) Determine what the magnitude and the direction of the electric field in the bar is, to yield the current described above.

b) Calculate the difference of the electric potential between the opposite points on the surfaces of the bar in the direction of the edge b.

c) Find the analytic expression for the DC component of the electric potential difference in case b) if the current and the magnetic field are alternating (AC); $I = I_0 \sin \omega t$ and $B = B_0 \sin(\omega t + \delta)$.

d) Design and explain an electric circuit which would make possible, by exploiting the result c), to measure the power consumption of an electric apparatus connected with the AC network.

   Data: The electron mobility in InSb is $7.8\ \mathrm{m^2T/Vs}$, the electron concentration in InSb is $2.5 \cdot 10^{22}\ \mathrm{m^{-3}}$, $I = 1.0$ A, $B = 0.10$ T, $b = 1.0$ cm, $c = 1.0$ mm, $e_0 = -1.6 \cdot 10^{-19}$ As.

Problem 3

In a space research project two schemes of launching a space probe out of the Solar system are discussed. The first scheme (i) is to launch the probe with a velocity large enough to escape from the Solar system directly. According to the second one (ii), the probe is to approach one of the outer planets, and with its help change its direction of motion and reach the velocity necessary to escape from the Solar system. Assume that the probe moves under the gravitational field of only the Sun or the planet, depending on whichever field is stronger at that point.

a) Determine the minimum velocity and its direction relative to the Earth's motion that should be given to the probe on launching according to scheme (i).

b) Suppose that the probe has been launched in the direction determined in a) but with another velocity. Determine the velocity of the probe when it crosses the orbit of Mars, i. e., its parallel and perpendicular components with respect to this orbit. Mars is not near the point of crossing, when crossing occurs.

c) Let the probe enter the gravitational field of Mars. Find the minimum launching velocity from the Earth necessary for the probe to escape from the Solar system.

   Hint: From the result a) you know the optimal magnitude and the direction of the velocity of the probe that is necessary to escape from the Solar system after leaving the gravitational field of Mars. (You do not have to worry about the precise position of Mars during the encounter.) Find the relation between this velocity and the velocity components before the probe enters the gravitational field of Mars; i. e., the components you determined in b). What about the conservation of energy of the probe?

d) Estimate the maximum possible fractional saving of energy in scheme (ii) with respect to scheme (i). Notes: Assume that all the planets revolve round the Sun in circles, in the same direction and in the same plane. Neglect the air resistance, the rotation of the Earth around its axis as well as the energy used in escaping from the Earth's gravitational field.

Data: Velocity of the Earth round the Sun is 30 km/s, and the ratio of the distances of the Earth and Mars from the Sun is 2/3.

1.2 Experimental competition

Exercise A

Follow the acceleration and the deceleration of a brass disk, driven by an AC electric motor. From the measured times of half turns, plot the angle, angular velocity and angular acceleration of the disk as functions of time. Determine the torque and power of the motor as functions of angular velocity.

Instrumentation

1. AC motor with switch and brass disk
2. Induction sensor
3. Multichannel stop-watch (computer)

Instruction

The induction sensor senses the iron pegs, mounted on the disk, when they are closer than 0.5 mm and sends a signal to the stop-watch. The stop-watch is programmed on a computer so that it registers the time at which the sensor senses the approaching peg and stores it in memory. You run the stop-watch by giving it simple numerical commands, i. e. pressing one of the following numbers:

5 – MEASURE.
The measurement does not start immediately. The stop-watch waits until you specify the number of measurements, that is, the number of successive detections of the pegs:

3 – 30 measurements

6 – 60 measurements
Either of these commands starts the measurement. When a measurement is completed, the computer displays the results in graphic form. The vertical axis represents the length of the interval between detection of the pegs and the horizontal axis is the number of the interval.

7 – display results in numeric form.

The first column is the number of times a peg has passed the detector, the second is the time elapsed from the beginning of the measurement and the third column is the length of the time interval between the detection of the two pegs.

In the case of 60 measurements:

8 – displays the first page of the table

2 – displays the second page of the table

4 – displays the results graphically.

A measurement can be interrupted before the prescribed number of measurements by pressing any key and giving the disk another half turn.

The motor runs on 25 V AC. You start it with a switch on the mounting base. It may sometimes be necessary to give the disk a light push or to tap the base plate to start the disk.

The total moment of inertia of all the rotating parts is: $(14.0 \pm 0.5) \cdot 10^{-6}$ kgm^2.

Exercise B

Locate the position of the centers and determine the orientations of a number of identical permanent magnets hidden in the black painted block. A diagram of one such magnet is given in Figure 1. The coordinates x, y and z should be measured from the red corner point, as indicated in Figure 2.

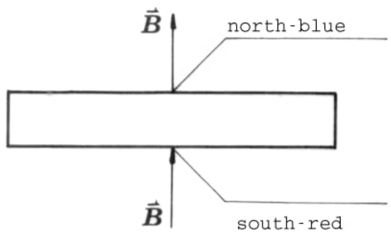
Determine the z component of the magnetic induction vector $\vec{B}$ in the (x, y) plane at $z = 0$ by calibrating the measuring system beforehand.

Find the greatest magnetic induction B obtainable from the magnet supplied.

Instrumentation

1. Permanent magnet given is identical to the hidden magnets in the block.

2. Induction coil; 1400 turns, $R = 230\ \Omega$



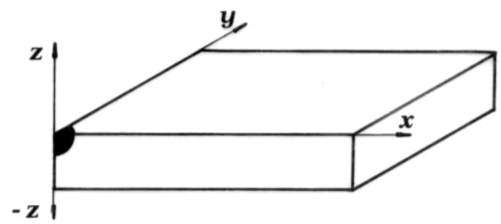


Fig. 1 Fig. 2

3. Field generating coils, 8800 turns, $R = 990\ \Omega$, 2 pieces
4. Black painted block with hidden magnets
5. Voltmeter (ranges 1 V, 3 V and 10 V recommended)
6. Electronic circuit (recommended supply voltage 24 V)
7. Ammeter
8. Variable resistor 3.3 kΩ
9. Variable stabilized power supply 0 – 25 V, with current limiter
10. Four connecting wires
11. Supporting plate with fixing holes
12. Rubber bands, multipurpose (e. g. for coil fixing)
13. Tooth picks
14. Ruler
15. Thread

Instructions

For the magnet-search only nondestructive methods are acceptable. The final report should include results, formulae, graphs and diagrams. The diagrams should be used instead of comments on the methods used wherever possible.

The proper use of the induced voltage measuring system is shown in Figure 3.

This device is capable of responding to the magnetic field. The peak voltage is proportional to the change of the magnetic flux through the coil.

The variable stabilized power supply is switched ON (1) or OFF (0) by the lower left pushbutton. By the (U) knob the output voltage is increased through the clockwise rotation. The recommended voltage is 24 V. Therefore switch the corresponding toggle switch to the 12 V – 25 V position. With this instrument either the output voltage U or the output current I is measured, with respect to the position of the corresponding toggle switch (V,A). However, to get the output voltage the upper right switch should be in the 'Vklop' position. By the knob (I) the output current is limited bellow the preset value. When rotated clockwise the power supply can provide 1.5 A at most.

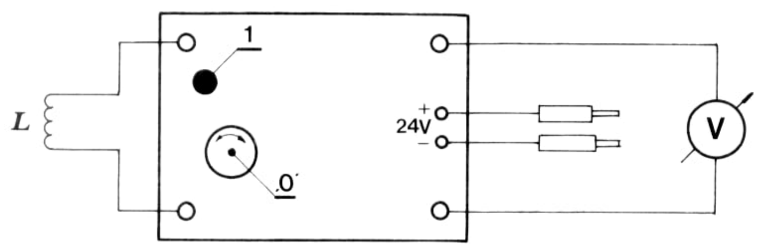


Fig. 3 '0' zero adjust dial, '1' push reset button

Note: permeability of empty space $\mu_0 = 1.2 \cdot 10^{-6}$ Vs/Am.

2 Solutions

2.1 Theoretical competition

Problem 1

a) Let the electrical signals supplied to rods 1 and 2 be $E_1 = E_0 \cos \omega t$ and $E_2 = E_0 \cos(\omega t + \delta)$, respectively. The condition for a maximum signal in direction ϑ_A (Fig. 4) is:

$$\frac{2\pi a}{\lambda} \sin \vartheta_A - \delta = 2\pi N$$

and the condition for a minimum signal in direction ϑ_B:

$$\frac{2\pi a}{\lambda} \sin \vartheta_B - \delta = 2\pi N' + \pi \qquad \text{(2p.)}$$

where N and N' are arbitrary integers. In addition, $\vartheta_A - \vartheta_B = \varphi$, where

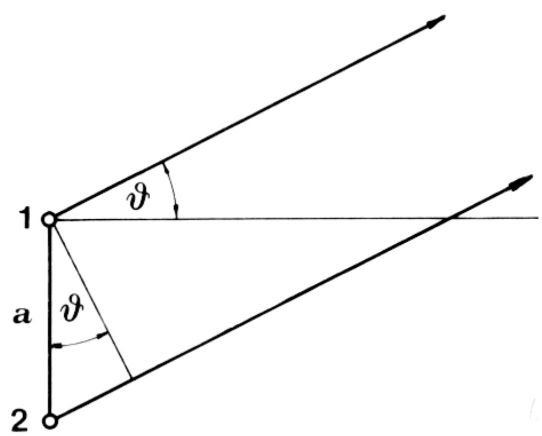


Fig. 4

φ is given. The problem can now be formulated as follows: Find the parameters a, ϑ_A, ϑ_B, δ, N, and N' satisfying the above equations such, that a is minimum.

We first eliminate δ by subtracting the second equation from the first one:

$$a \sin \vartheta_A - a \sin \vartheta_B = \lambda(N - N' - \tfrac{1}{2}) \, .$$

Using the sine addition theorem and the relation $\vartheta_B = \vartheta_A - \varphi$:

$$2a\cos(\vartheta_A - \tfrac{1}{2}\varphi)\sin\tfrac{1}{2}\varphi = \lambda(N - N' - \tfrac{1}{2})$$

or

$$a = \frac{\lambda(N - N' - \frac{1}{2})}{2\cos(\vartheta_A - \frac{1}{2}\varphi)\sin\frac{1}{2}\varphi}\,.$$

The minimum of a is obtained for the greatest possible value of the denominator, i. e.:

$$\cos(\vartheta_A - \tfrac{1}{2}\varphi) = 1\,, \qquad \vartheta_A = \tfrac{1}{2}\varphi\,,$$

and the minimum value of the numerator, i. e.:

$$N - N' = 1\,.$$

The solution is therefore:

$$a = \frac{\lambda}{4\sin\frac{1}{2}\varphi}\,, \quad \vartheta_A = \tfrac{1}{2}\varphi\,, \quad \vartheta_B = -\tfrac{1}{2}\varphi \quad \text{and} \quad \delta = \tfrac{1}{2}\pi - 2\pi N\,. \qquad \text{(6p.)}$$

(N = 0 can be assumed throughout without loosing any physically relevant solution.)

b) The wavelength $\lambda = c/\nu = 11.1$ m, and the angle between directions A and B, $\varphi = 157^\circ - 72^\circ = 85^\circ$. The minimum distance between the rods is $a = 4.1$ m, while the direction of the symmetry line of the rods is $72^\circ + 42.5^\circ = 114.5^\circ$ measured from the north. (2 p.)

Problem 2

a) First the electron velocity is calculated from the current I:

$$I = jS = ne_0 v bc, \qquad v = \frac{I}{ne_0 bc} = 25 \text{ m/s}.$$

The components of the electric field are obtained from the electron velocity. The component in the direction of the current is

$$E_{\parallel} = \frac{v}{\mu} = 3.2 \text{ V/m}. \qquad (0.5\text{p.})$$

The component of the electric field in the direction b is equal to the Lorentz force on the electron divided by its charge:

$$E_{\perp} = vB = 2.5 \text{ V/m}. \qquad (1\text{p.})$$

The magnitude of the electric field is

$$E = \sqrt{E_{\parallel}^2 + E_{\perp}^2} = 4.06 \text{ V/m}. \qquad (0.5\text{p.})$$

while its direction is shown in Fig. 5 (Note that the electron velocity is in the opposite direction with respect to the current.) (1.5 p.)

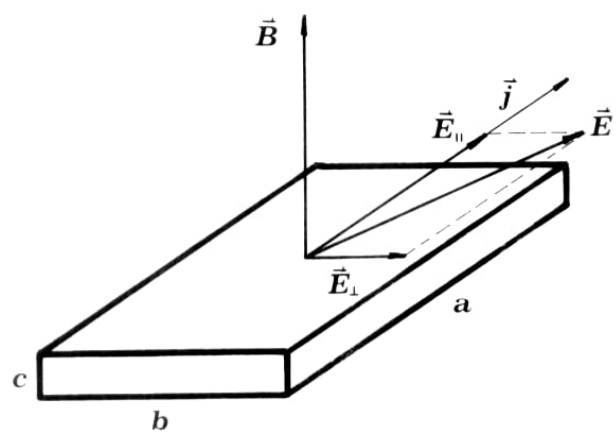


Fig. 5

b) The potential difference is

$$U_H = E_{\perp} b = 25 \text{ mV} \qquad (1\text{p.})$$

c) The potential difference U_H is now time dependent:

$$U_H = \frac{IBb}{ne_0bc} = \frac{I_0B_0}{ne_0c}\,\sin\omega t\,\sin(\omega t + \delta)\,.$$

The DC component of U_H is

$$\overline{U}_H = \frac{I_0B_0}{2ne_0c}\,\cos\delta\,. \qquad (3p.)$$

d) A possible experimental setup is-shown in Fig. 6

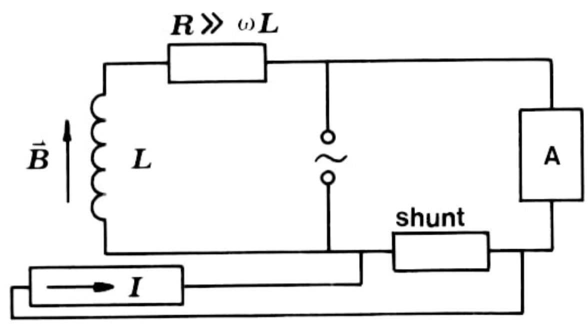


Fig. 6

(2 p.)

Problem 3

a) The necessary condition for the space-probe to escape from the Solar system is that the sum of its kinetic and potential energy in the Sun's gravitational field is larger than or equal to zero:

$$\tfrac{1}{2}mv_a^2 - \frac{GmM}{R_E} \geq 0\,,$$

where m is the mass of the probe, v_a its velocity relative to the Sun, M the mass of the Sun, R_E the distance of the Earth from the Sun and G the gravitational constant. Using the expression for the velocity of the Earth, $v_E = \sqrt{GM/R_E}$, we can eliminate G and M from the above condition:

$$v_a^2 \geq \frac{2GM}{R_E} = 2v_E^2\,. \qquad \text{(1p.)}$$

Let v_a' be the velocity of launching relative to the Earth and ϑ the angle between $\vec{v}_E$ and $\vec{v}_a'$ (Fig. 7). Then from $\vec{v}_a = \vec{v}_a' + \vec{v}_E$ and $v_a^2 = 2v_E^2$ it

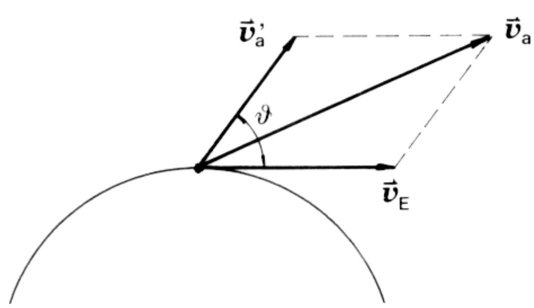


Fig. 7

follows:

$$v_a'^{\,2} + 2v_a' v_E \cos\vartheta - v_E^2 = 0$$

and

$$v_a' = v_E\left[-\cos\vartheta + \sqrt{1+\cos^2\vartheta}\right].$$

The minimum velocity is obtained for $\vartheta = 0$:

$$v_a' = v_E(\sqrt{2}-1) = 12.3\ \text{km/s}\,. \qquad \text{(1p.)}$$

b) Let v_b' and v_b be the velocities of launching the probe in the Earth's and Sun's system of reference respectively. For the solution (a), $v_b = v_b' + v_E$. From the conservation of angular momentum of the probe:

$$mv_bR_E = mv_{\parallel}R_M \qquad \text{(1p.)}$$

and the conservation of energy:

$$\tfrac{1}{2}mv_b^2 - \frac{GmM}{R_E} = \tfrac{1}{2}m(v_\parallel^2 + v_\perp^2) - \frac{GmM}{R_M} \qquad \text{(1p.)}$$

we get for the, parallel component of the velocity (Fig. 8):

$$v_\parallel = (v_b' + v_E)k\,,$$

and for the perpendicular component:

$$v_\perp = \sqrt{(v_b' + v_E)^2(1-k^2) - 2v_E^2(1-k)}\,. \qquad \text{(1p.)}$$

where $k = R_E/R_M$.

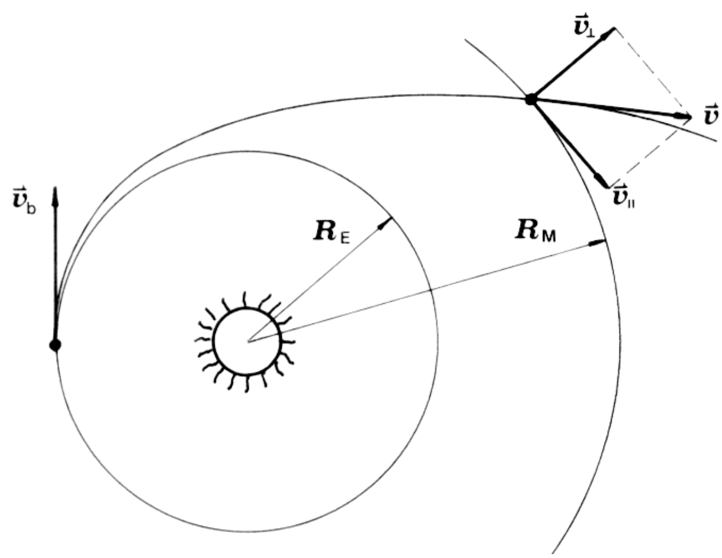


Fig. 8

c) The minimum velocity of the probe in the Mars' system of reference to escape from the Solar system, is $v_s'' = v_M(\sqrt{2}-1)$, in the direction parallel to the Mars orbit (v_M is the Mars velocity around the Sun). The role of Mars is therefore to change the velocity of the probe so that it leaves its gravitational field with this velocity.

(1 p.)

In the Mars' system, the energy of the probe is conserved. That is, however, not true in the Sun's system in which this encounter can be considered as an elastic collision between Mars and the probe. The velocity of the probe before it enters the gravitational field of Mars is therefore, in the Mars' system, equal to the velocity with which the probe leaves its gravitational field. The components of the former velocity are $v''_\perp = v_\perp$ and $v''_\parallel = v_\parallel - v_M$, hence:

$$v'' = \sqrt{{v''_\parallel}^2 + {v''_\perp}^2} = \sqrt{v_\perp^2 + (v_\parallel - v_M)^2} = v''_s\,. \qquad \text{(1p.)}$$

Using the expressions for $v_\perp$ and $v_\parallel$ from (b), we can now find the relation between the launching velocity from the Earth, v'_b, and the velocity v''_s, $v''_s = v_M(\sqrt{2} - 1)$:

$$(v'_b+v_E)^2(1-k^2)-2v_E^2(1-k)+(v'_b+v_E)^2k^2-2v_M(v'_b+v_E)k = v_M^2(2-2\sqrt{2})\,.$$

The velocity of Mars round the Sun is $v_M = \sqrt{GM/R_M} = \sqrt{k}\,v_E$, and the equation for v'_b takes the form:

$$(v'_b + v_E)^2 - 2\sqrt{k}^3 v_E(v'_b + v_E) + (2\sqrt{2}\,k - 2)v_E^2 = 0\,. \qquad \text{(1p.)}$$

The physically relevant solution is:

$$v'_b = v_E\left[\sqrt{k}^3 - 1 + \sqrt{k^3 + 2 - 2\sqrt{2}\,k}\right] = 5.5\ \text{km/s}\,. \qquad \text{(1p.)}$$

d) The fractional saving of energy is:

$$\frac{W_a - W_b}{W_a} = \frac{{v'_a}^2 - {v'_b}^2}{{v'_a}^2} = 0.80\,,$$

where W_a and W_b are the energies of launching in scheme (i) and in scheme (ii), respectively. (1 p.)

2.2 Experimental competition

Exercise A

The plot of the angle as a function of time for a typical measurement of the acceleration of the disk is shown in Fig. 9.

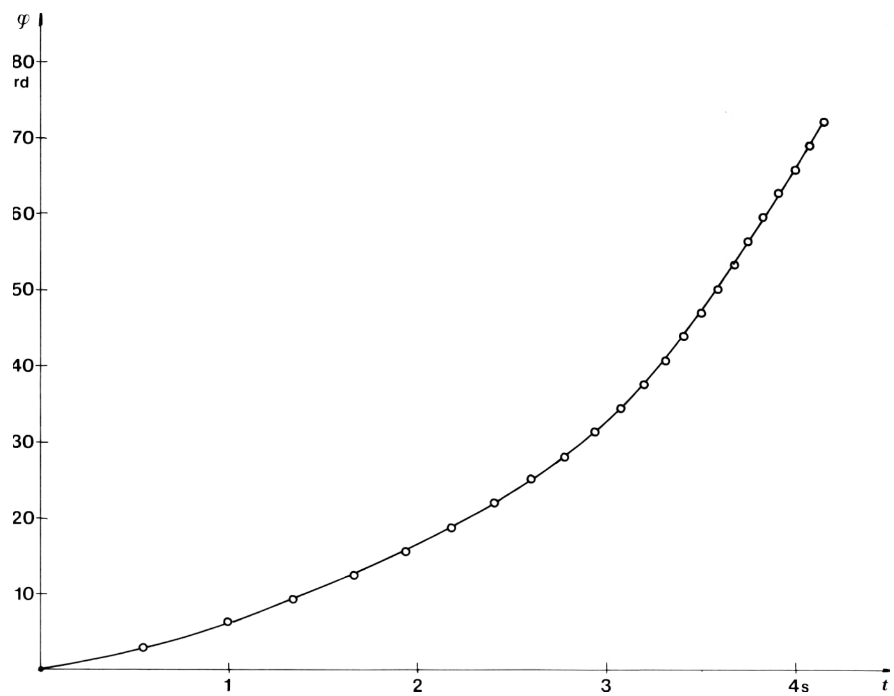


Fig. 9 Angle vs. time

The angular velocity is calculated using the formula:

$$\omega_i(t_i') = \frac{\pi}{(t_{i+i} - t_i)}$$

and corresponds to the time in the middle of the interval (t_i, t_{i+1}): $t_i' = \frac{1}{2}(t_{i+1} + t_i)$. The calculated values are displayed in Table 1 and plotted in Fig. 10.

Observing the time intervals of half turns when the constant angular velocity is reached, one can conclude that the iron pegs are not positioned perfectly symmetrically. This systematic error can be neglected in the calculation of angular velocity, but not in the calculation of angular acceleration. To avoid this error we use the time intervals of full turns:

$$\alpha_i(t_i'') = \frac{\Delta\omega_i}{\Delta t_i},$$

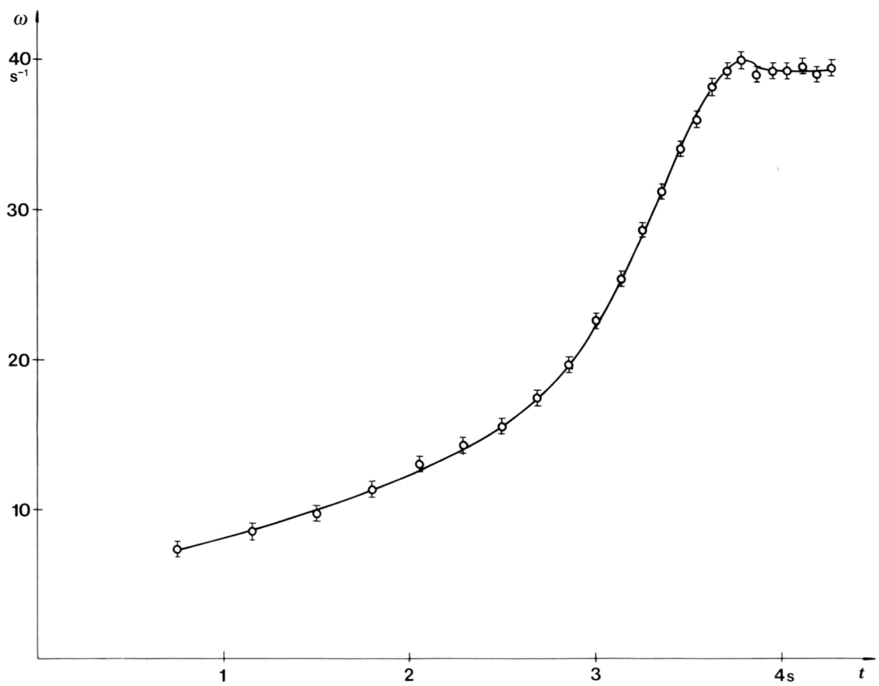


Fig. 10 Angular velocity vs. time

where $\Delta t_i = t_{2i+2} - t_{2i}$,

$$\Delta\omega_i = \frac{2\pi}{(t_{2i+3} - t_{2i+1})} - \frac{2\pi}{(t_{2i+1} - t_{2i-1})}$$

and $t_i'' = t_{2i+1}'$.

The angular acceleration as a function of time is plotted in Fig. 11.

The torque, M, and the power, P, necessary to drive the disk (net torque and net power), are calculated using the relation:

$$M(t) = I\alpha(t)$$

and

$$P(t) = M(t)\omega(t)$$

where the moment of inertia, $I = (14.0 \pm 0.5) \cdot 10^{-6}$ kgm^2, is given. The corresponding angular velocity is determined from the plot in Fig. 10 by interpolation. This plot is used also to find the torque and the power as functions of angular velocity (Fig. 12 and 13).

i	t	δt	φ	t'	ω	α
	ms	ms	rd	ms	s^{-1}	s^{-2}
1	0.0		0.0			
				272.0	5.78	
2	543.9	543.9	3.14			
				758.7	7.31	
3	973.5	429.6	6.28			3.38
				1156.3	8.60	
4	1339.0	365.5	9.42			
				1499.9	9.76	
5	1660.8	327.8	12.57			5.04
				1798.6	11.40	
6	1936.3	275.5	15.71			
				2057.1	13.01	
7	2177.8	241.5	18.85			5.96
				2287.2	14.36	
8	2396.6	218.8	21.99			
				2498.1	15.48	
9	2599.6	203.0	25.73			9.40
				2689.6	17.46	
10	2779.5	179.9	28.27			
				2859.4	19.66	
11	2939.3	159.8	31.42			18.22
				3008.6	22.65	
12	3078.0	138.7	34.56			
				3139.9	25.38	
13	3201.8	123.8	37.70			25.46
				3256.6	28.66	
14	3311.4	109.6	40.84			
				3361.8	31.20	
15	3472.1	100.7	43.98			26.89
				3458.2	34.11	
16	3504.2	92.1	47.12			
				3547.8	36.07	
17	3591.3	87.1	50.27			21.72
				3632.4	38.27	
18	3673.4	82.1	53.41			
				3713.5	39.22	
19	3753.5	80.1	56.55			4.76
				3792.8	39.97	
20	3832.7	78.6	59.69			
				3872.4	39.03	
21	3912.6	80.5	62.83			−1.69
				3952.7	39.22	
22	3992.7	80.1	65.97			
				4032.8	39.22	
23	4072.8	80.1	69.12			0.77
				4112.4	39.67	
24	4152.0	79.2	72.26			
				4192.3	39.03	
25	4232.5	80.5	75.40			−0.15
				4272.4	39.42	
26	4312.3	79.7	78.54			

Table 1

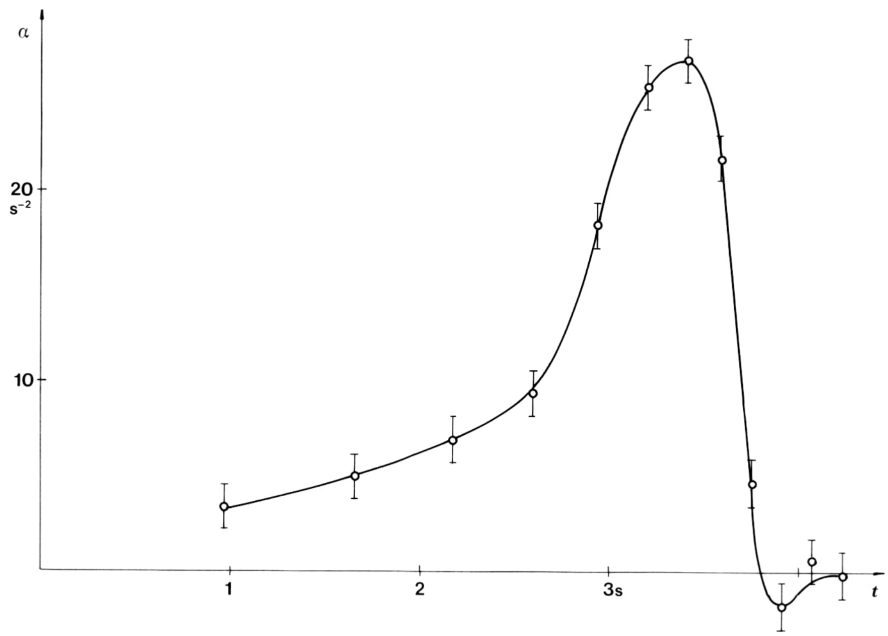


Fig. 11 Angular acceleration vs. time

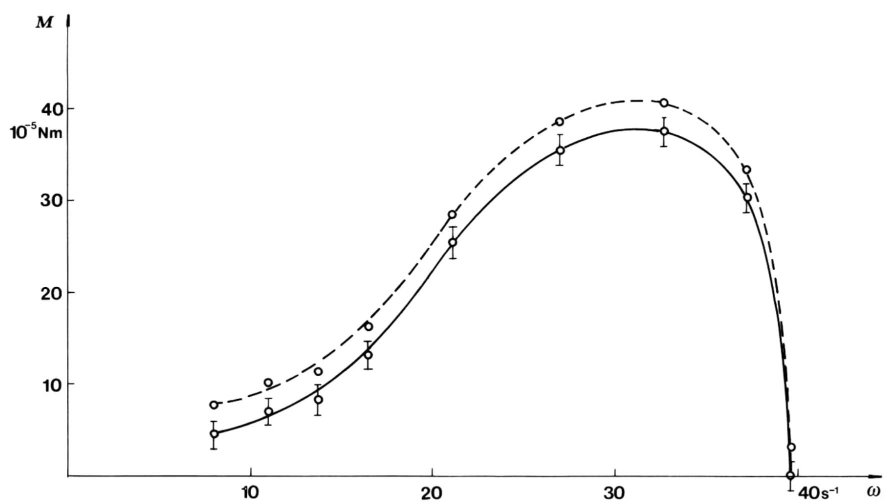


Fig. 12 Net torque (full line) and total torque (dashed line) vs. angular velocity

To find the total torque and the power of the motor, the torque and the power losses due to the friction forces have to be determined and added to the corresponding values of net torque and power. By measuring the angular velocity during the deceleration of the disk after the motor has been switched off (Fig. 14), we can determine the torque of friction which is approximately constant and is equal to $M' = (3.1 \pm 0.3) \cdot 10^{-5}$ Nm.

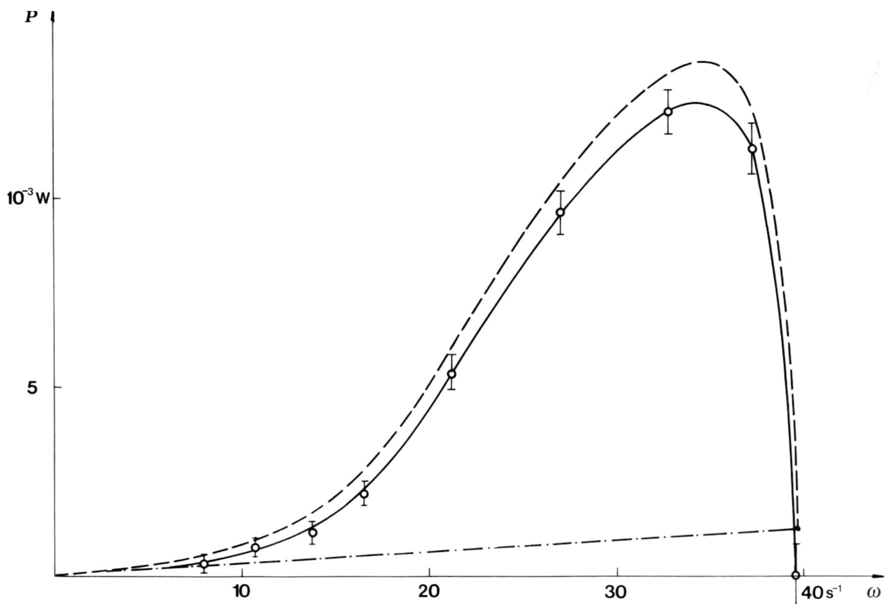


Fig. 13 Net power (full line), power loses (dashed and dotted line) and total power (dashed line) vs. angular velocity

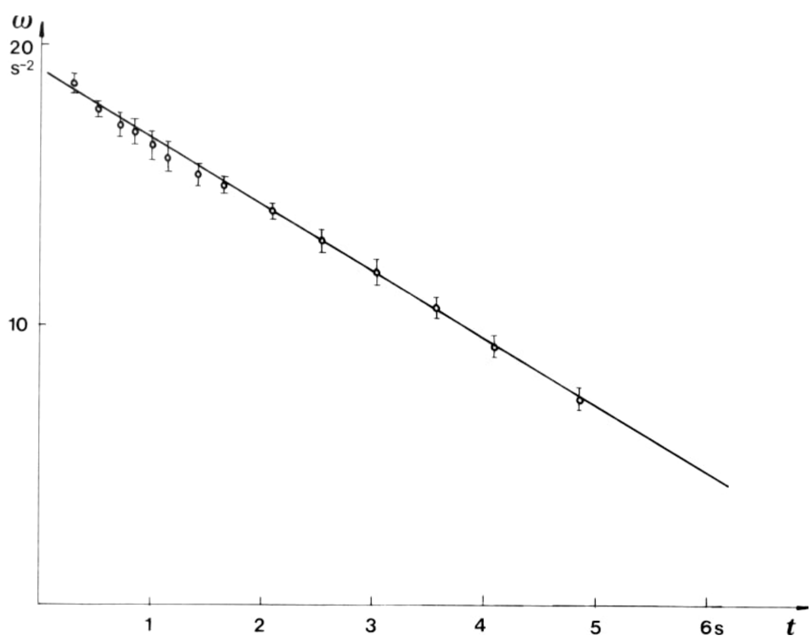


Fig. 14 Angular velocity vs. time during deceleration

The total torque and the total power are shown in Fig. 12 and 13.

Marking scheme

a) Determination of errors 1 p.

b) Plot of angle vs. time 1 p.

c) Plot of angular velocity and acceleration 3 p.

d) Correct times for angular velocity 1 p.

e) Plot of net torque vs. angular velocity 2 p. (Plot of torque vs. time only, 1 p.)

f) Plot of net power vs. angular velocity 1 p.

g) Determination of friction 1 p.

Exercise B

Two permanent magnets having the shape of rectangular parallelepipeds with sides 50 mm, 20 mm and 8 mm are hidden in a block of polystyrene foam with dimension 50 cm, 31 cm and 4.0 cm. Their sides are parallel to the sides of the block. One of the hidden magnets (A) is positioned so that its $\vec{B}$ (Fig. 15) points in z direction and the other (B) with its $\vec{B}$ in x or y direction (Fig. 15).

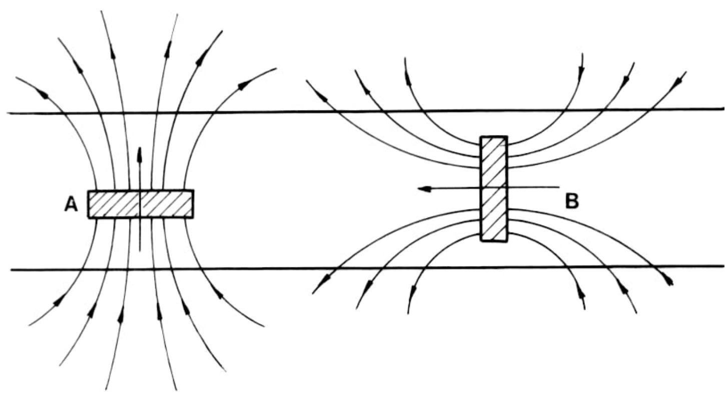


Fig. 15 A typical implementation of the magnets in the block

The positions and the orientations of the magnets should be determined on the basis of observations of forces acting on the extra magnet. The best way to do this is to hang the extra magnet on the thread and move it above the surface to be explored. Three areas of strong forces are revealed when the extra magnet is in the horizontal position i. e. its $\vec{B}$ is parallel to z axis, suggesting that three magnets are hidden. Two of these areas producing an attractive force in position P (Fig. 16) and a repulsive force in position R are closely together.

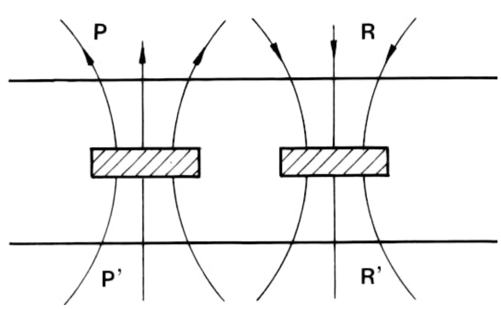


Fig: 16 Two 'ghost' magnets appearing in the place of magnet B

However, by inspecting the situation on the other side of the block, again an attractive force in area P' is found, and a repulsive one in area R'. This is in the contradiction with the supposed magnets layout in Fig. 16 but corresponds to the force distribution of magnet B in Fig. 15.

To determine the z position of the hidden magnets one has to measure the z component of $\vec{B}$ on the surface of the block and compare it to the measurement of B_z of the extra magnet as a function of distance from its center (Fig. 18). To achieve this the induction coil of the measuring system is removed from the point in which the magnetic field is measured to a distance in which the magnetic field is practically zero, and the peak voltage is measured.

In order to make the absolute calibration of the measuring system, the response of the system to the known magnetic field should be measured. The best defined magnetic field is produced in the gap between two field generating coils. The experimental layout is displayed in Fig. 17.

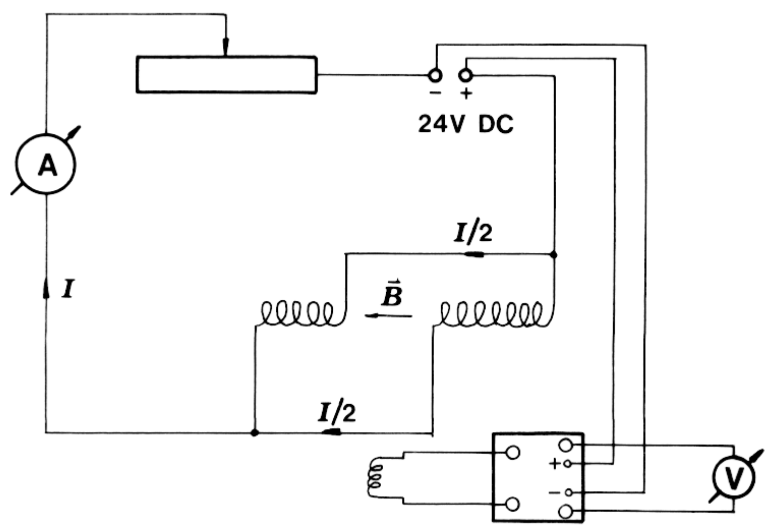


Fig. 17 Calibration of the measuring system

The magnetic induction in the gap between the field generating coils is calculated using the formula:

$$B = \frac{\mu_O N I}{(2l + d)} .$$

Here N is the number of the turns of one of the coils, l its length , d the width of the gap, and I the current through the ammeter. The peak voltage, U, is measured when the induction coil is removed from the gap.

Plotting the magnetic induction B as a function of peak voltage, we can determine the sensitivity of our measuring system:

$$\frac{B}{U} = 0.020 \text{ T/V}.$$

(More precise calculation of the magnetic field in the gap, which is beyond the scope of the exercise, shows that the true value is only 60 % of the value calculated above.)

The greatest value of B is 0.21 T.

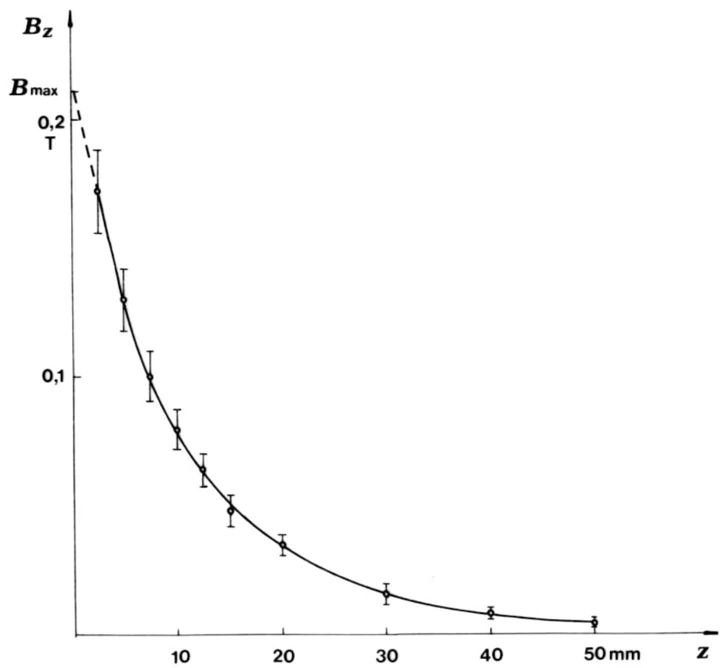


Fig. 18 Magnetic induction vs. distance

Marking scheme:

a) determination of x, y position of magnets (±1 cm) 1 p.

b) determination of the orientations 1 p.

c) depth of magnets (±4 mm) 2 p.

d) calibration (±50 %) 3 p.

e) mapping of the magnetic field 2 p.

f) determination of B_{max} (±50 %) 1 p.

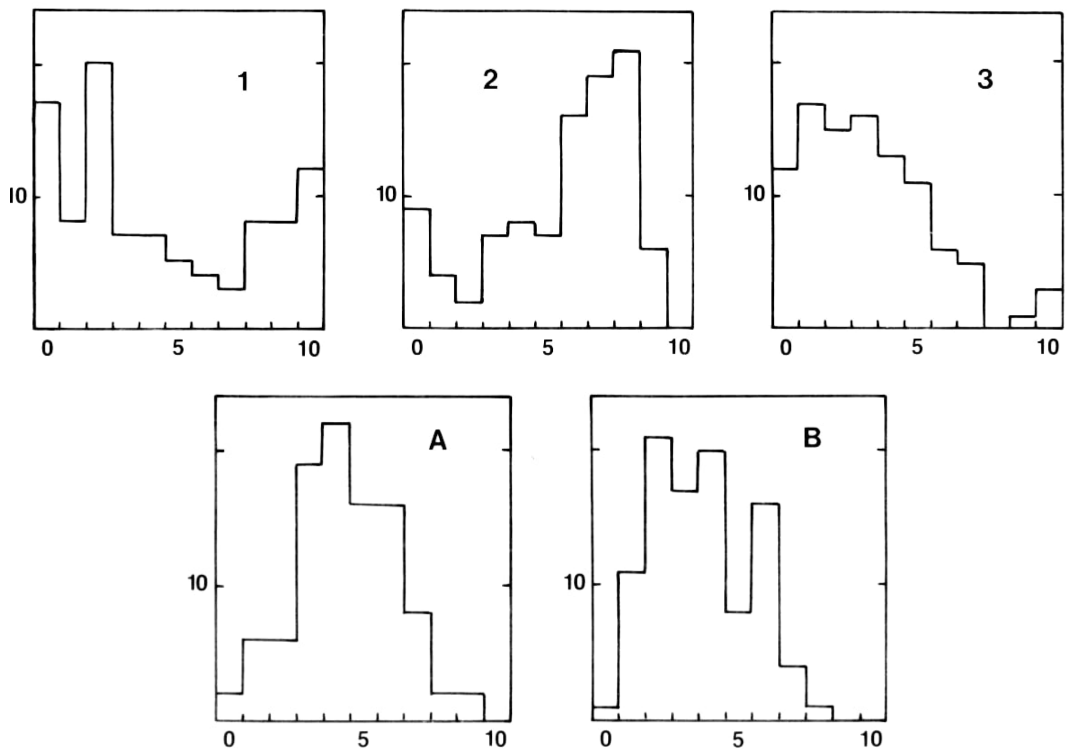


Fig. 19 Distribution of marks for the theoretical (1,2,3) and the experimental exercises. The highest mark for each exercise is 10 points.

1986 INTERNATIONAL PHYSICS OLYMPIAD

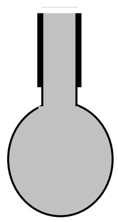
EXPERIMENT 1.

2½ hrs

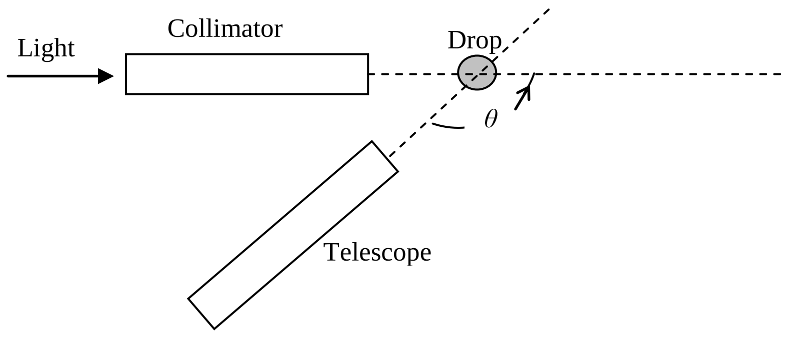
APPARATUS

1. Spectrometer with collimator and telescope.
2. 3 syringes; one for water, one for liquid A and one for liquid *B.*
3. A beaker of water plus two sample tubes containing liquids A and *B.*
4 3 retort stands with clamps.
5. 12V shielded source of white light.
6. Black card, plasticine, and black tape.
7. 2 plastic squares with holes to act as stops to be placed over the ends of the telescope, with the use of 2 elastic bands.
8. Sheets of graph paper.
9. Three dishes to collect water plus liquids A and B lost from syringes.

Please complete synopsis sheet in addition to answering this experimental problem.



Pendant drop



Plan of Apparatus

INSTRUCTIONS AND INFORMATION

1. Adjust collimator to produce parallel light. This may be performed by the following sequence of operations:

(a) Focus the telescope on a distant object, using adjusting knob on telescope, so that the cross hairs and object are both in focus.
(b) Position the telescope so that it is opposite the collimator with slit illuminated so that the slit can be viewed through the telescope.
(c) Adjust the position of the collimator lens, using the adjusting knob on the collimator, so that the image of the slit is in focus on the cross hairs of the telescope's eyepiece.
(d) Lock the spectrometer table, choosing an appropriate 'zero' on the vernier scale, so that subsequent angular measurements of the telescope's position can conveniently be made.

2. Remove the eyepiece from telescope and place black plastic stops symmetrically over both ends of the telescope, using the elastic bands, so that the angle of view is reduced.

3. Open up collimator slit.

4. Use the syringes to suspend, vertically, a pendant drop symmetrically above the centre of the spectrometer table so that it is fully illuminated by the light from the collimator and can be viewed by telescope.

5. The central horizontal region of the suspended drop will produce rainbows as a result of two reflections and k (k = 1,2,...) internal reflections of the light. The first order rainbow corresponds to one internal reflection. The second order rainbow corresponds to two Internal reflections. The k'th order rainbow corresponds to k internal reflections. Each rainbow contains all the colours of the spectrum. These can be observed directly by eye and their angular positions can bed accurately measured using the telescope. Each rainbow is due to white light rays incident on the drop at a well determined angle of incidence, that is different for each rainbow.

6. The first order rainbow can be recognized as it has the greatest intensity and appears on the right hand* side of the drop. The second order rainbow appears with the greatest intensity on the left hand* side of the drop. These two rainbows are within an angular separation of 20° of each other for water droplets. The weak intensity fifth order rainbow can be observed on the right hand side of the drop located somewhere between the other two, 'blue', extreme ends of the first and second order rainbows.

7. Light reflected directly from the external surface of the drop and that refracted twice but not internally reflected, will produce bright white glare spots that will hinder observations.

8. The refractive indices, *n*, of the liquids are:

Water	$n_w = 1.333$
Liquid A	$n_A = 1.467$
Liquid B	$n_B = 1.534$

In addition to the experimental report please complete the summary sheet.

Footnote: This statement is correct if the collimator is to the left of the telescope, as indicated in the diagram. If the collimator is on the righthand side of the telescope the first order rainbow will appear on the lefthand side of the drop and the second order rainbow on the righthand side of the drop.

Measurements

1) Observe, by eye, the first and second order water rainbows. Measure the angle θ through which the telescope has to be rotated, from the initial direction for observing the parallel light from the collimator, to observe, using a pendant water droplet, the red light at the extreme end of the visible spectrum from:

(a) the first order rainbow on the right of the drop ($k = 1$);
(b) the second order rainbow on the left of the drop ($k = 2$);
(c) the weak fifth order rainbow ($k = 5$), between the first and second order rainbows.

One of these angles may not be capable of measurement by the rotation of the telescope due to the mechanical constraints limiting the range of θ. If this is found to be the case, use a straight edge in place of the telescope to measure θ.
(Place the appropriate dish on the spectrometer table to catch any falling droplets.)

Deduce the angle of deviation, ϕ , that is the angle the incident light is rotated by the two reflections and k reflections at the drop's internal surface, for (a), (b) and (c). Plot a graph of ϕ against k.

2.Determine ϕ for the second order rainbows produced by liquids A and B using the red visible light at the extreme end of the visible spectrum. (Place respective dishes on table below to catch any falling liquid as the quantities of liquid are limited).
Using graph paper plot $\cos\frac{\phi}{6}$ against $\frac{1}{n}$, n being the refractive index, for all three liquids and insert the additional point for $n = 1$. Obtain the best straight line through these points; measure its gradient and the value of ϕ for which $n = 2$.

EXPERIMENT 2

Apparatus
RML Nimbus computer

Ten sheets of graph paper.

Please complete synopsis sheet in addition to answering this experimental problem.

THIS IS A TWO AND A HALF HOUR EXAMINATION

INFORMATION

The microcomputer has been programmed to solve the Newtonian equations of motion for a two-dimensional system of 25 interacting particles, in the *xy* plane. It is able to generate the positions and velocities of all particles at discrete, equally spaced time intervals. By depressing appropriate keys (which will be described), access to dynamic information about the system can be obtained.
The system of particles is confined to a box which is initially (at time $t = 0$) arranged in a two-dimensional square lattice. A picture of the system is displayed on the screen together with the numerical data requested. All particles are identical; the colours are to enable the particles to be distinguished. As the system evolves in time the positions and velocities of the particles will change. If a particle is seen to leave the box the program automatically generates a new particle that enters the box at the opposite face with the same velocity, thus conserving the number of particles in the box.

Any two particles i and j, separated by a distance r_{ij} interact with a well-defined potential U_{ij},

It is convenient to use dimensionless quantities throughout the computation. The quantities given below are used throughout the calculations.

Variable	**Symbol**
Distance	r^*
Velocity	v^*
Time	t^*
Energy	E^*
Mass of particle	$M^* = 48$
Potential	U_{ij}^*
Temperature	T^*
Kinetic Energy	$E_k^* = \frac{1}{2} m^* v^{*2}$

INSTRUCTIONS

The computer program allows you to access three distinct sets of numerical information and display them on the screen. Access is controlled by the grey function keys on the left-hand side of the keyboard, labelled F1, F2, F3, F4, and F10. These keys should be pressed and released - do not hold down a key, nor press it repeatedly. The program may take up to 1 second to respond.

FIRST INFORMATION SET. PROBLEMS 1 – 5

$$< v_x, n > = \frac{1}{25}\sum_{i=1}^{25}\left(v_{ix}^*\right)^n$$

$$< v_y, n > = \frac{1}{25}\sum_{i=1}^{25}\left(v_{iy}^*\right)^n$$

and

$$< U > = \frac{1}{25}\sum_{j=1}^{25}\ \sum_{i=1}^{25} U_{ij}^* \qquad (i \neq j)$$

where

v_{ix}^* is the dimensionless x – component of the velocity for the i'th particle,

v_{iy}^* is the dimensionless y – component of the velocity for the i'th particle,

and n is and integer with $n \geq 1$.

[Note: the summation over U_{iy}^* excludes the cases in which $i = j$]

After depressing F1 it is necessary to input the integer $n\,(n \geq 1)$ by depressing one of the white keys in the top row of the keyboard, before the information appears on the screen.

The information is displayed in dimensionless time intervals Δt at dimensionless times

S Δt^{**} (S = 0, 1, 2, ...)

Δt^{**} is set by the computer program to the value Δt^{**} = 0•100000.

The value of S is displayed at the bottom right hand of the screen. Initially it has the value S = 0. The word "waiting" on the screen indicates that the calculation has halted and information concerning the value of S is displayed.

Depressing the long bar (the "space" bar) at the bottom of the keyboard will allow the calculation of the evolution of the system to proceed in time steps $\Delta^{**}t$. The current value of S is always displayed on the screen. Whilst the calculation is proceeding the word "running" is displayed on the screen.

Depressing F1 again will stop the calculation at the time integer indicated by S on the screen, and display the current values of

$<v_x,n>$, $<v_y,n>$ and $<U>$

after depressing the integer n. The evolution of the system continues on pressing the long bar.
The system can, if required, be reset to its original state at S = 0 by pressing F10 TWICE.

SECOND INFORMATION SET: PROBLEM 6

Depressing F2 initiates the computer program for the compilation of the histogram in problem 6. This program generates a histogram table of the accumulated number ΔN, of particle velocity components as a function of dimensionless velocity. The dimensionless velocity components, v_x and v_y are referred to collectively by v_c. The dimensionless velocity range is divided into equal intervals $\Delta v_c = 0.05$. The centres of the dimensionless velocity "bins" have magnitudes

$$v_c^{\ *} = B\Delta v_c^{\ *} \qquad (B = 0, \pm 1\ \pm 2, ...)$$

When the long bar on the keyboard is pressed the 2 x 25 dimensionless velocity components are calculated at the current time step, and the program adds one, for each velocity component, into the appropriate velocity 'bin'. This process is continued, for each time step, until F3 is depressed. Once F3 is depressed the (accumulated) histogram is displayed. The accumulation of counts can then be continued by pressing the long bar. (Alternatively if you wish to return to the initial situation, with zero in all bins, press F2).

The accumulation of histogram data should continue for about 200 time steps after initiation.

In the thermodynamic equilibrium the histogram can be approximated by the relation

$$\Delta N = Ae^{\left[\frac{-24(v_c^{\ *})^2}{\alpha}\right]}$$

where α is a constant associated with the temperature of the system, and A depends on the total number of accumulated velocity components.

THIRD INFORMATION SET: PROBLEM 7

Depressing F4 followed by the long bar at any time during the evolution of the system will initiate the program for Problem 7. The program will take some 30 seconds, in real time, before displaying a table containing the two
Quantities

$$< RX,2 >= \frac{1}{25}\sum_{i=1}^{25}\left[x_i^*(S) - x_i^*(SR)\right]^2$$

and

$$< RY,2 >= \frac{1}{25}\sum_{i=1}^{25}\left[y_i^*(S) - y_i^*(SR)\right]^2$$

where x_i^* and y_i^* are the dimensionless position components for the i'th particle. S is the integer time unit and SR is the fixed initial integer time at which the programme is initiated by depressing F4. It is convenient to introduce integer

$SZ = S - SR$..

The programme displays a table of $<RX,2>$ and $<RY,2>$ for

SZ = 0, 2, 4..……24.

Prior to the display appearing on the screen a notice 'Running' will appear on
the screen indicating that a computation is proceeding. Depressing F4, followed by the long bar, again will initiate a new table with SR advanced to the point at which F4 was depressed.

COMPUTATIONAL PROBLEMS

1 Verify that the dimensionless total linear momentum of the system is conserved for the times given by

S = 0, 40, 80, 120, 160.

State the accuracy of the computer calculation.

2. Plot the variation in dimensionless kinetic energy of the system with time using the time sequence

S = 0, 2, 4, 6, 12, 18, 24, 30, 50, 70, 90, 130, 180.

3. Plot the variation in dimensionless potential energy of the system with time using the time sequence in 2.

4. Obtain the dimensionless total energy of the system at times indicated in 2. Does the system conserve energy? State the accuracy of the total energy calculation.

5. The system is initially (at $S = 0$) NOT in thermodynamic equilibrium. After a period of time the system reaches thermodynamic equilibrium in which the total dimensionless kinetic energy fluctuates about a mean value of E_k^*. Determine this value of E_k^* and indicate the time, *SD*, after which the system is in thermodynamic equilibrium.

6. Using the dimensionless accumulated velocity data, during thermodynamic equilibrium, draw up a histogram giving the number ΔN of velocity components against dimensionless velocity component, using the constant velocity component interval $\Delta V_c^* = 0.05$, specified in the table available from the SECOND INFORMATION SET. Data accumulated from approximately 200 time steps should be used and the starting time integer *S* should be recorded.

Verify that ΔN satisfies the relation

$$\Delta N = Ae^{-\left[\frac{24\left(v_C^*\right)^2}{\alpha}\right]}$$

where *C* and *A* are constants. Determine the value of α.

7. For the system of particles in thermodynamic equilibrium evaluate the average value of R^2, $< R^2 >$, where R is the straight line distance between the position of a particle at a fixed initial time number *SR* and time number *S*. The time number difference $SZ = (S - SR)$ takes the values

SZ = O, 2, 4, ….. 24.

Plot $<R^2>$ against *SZ* for any appropriate value of *SR*. Calculate the gradient of the function in the linear region and specify the time number range for which this gradient is valid.
In order to improve the accuracy of the plot repeat the previous calculations for three (additional) different values of SR and determine the AVERAGE $<R^2>$ for the four sets of results together with the 'linear' gradient and time number range.
Deduce, with appropriate reasoning, the thermodynamic equilibrium state of the system, either solid or liquid.

SUMMARY SHEET

EXPERIMENT 1

1. FOR WATER AND RED LIGHT AT EXTREME END OF SPECTRUM

k = 1	First Order Rainbow	θ_1 =129.0°	ϕ_1 = 137.0 ±5.0°
k = 2	Second Order Rainbow	θ_2= 129.0°	ϕ_2 = 231.0 ±3.0°
k = 5	Fifth Order Rainbow	θ_5= 126.0°	ϕ_5 = 486.0 ±4.0°

2. LIQUIDS A AND B USING SECOND ORDER RAINBOWS

For Liquid A	θ_2 = 105.0°	ϕ_2= 255.0 ±3.0°
For Liquid B	θ_2 = 89.5°	ϕ_2= 270.5 ±3.0°
For n = 1	θ_2 = 0.0..°	ϕ_2= 0.0..°
Gradient of graph		= 0.84 ±0.07
Extrapolated, n =2,	θ_2, value of ϕ	= 304 ±25°

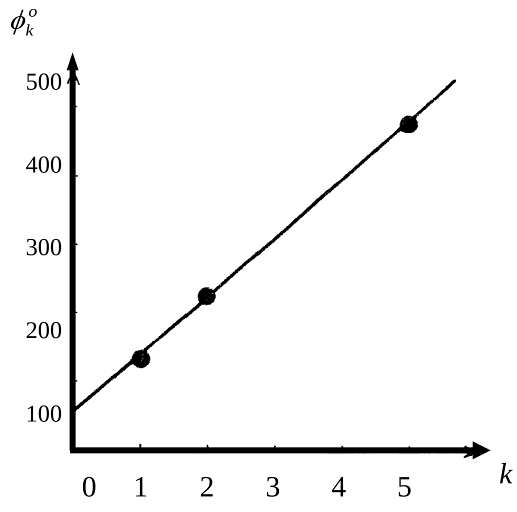


Figure E 1.1.

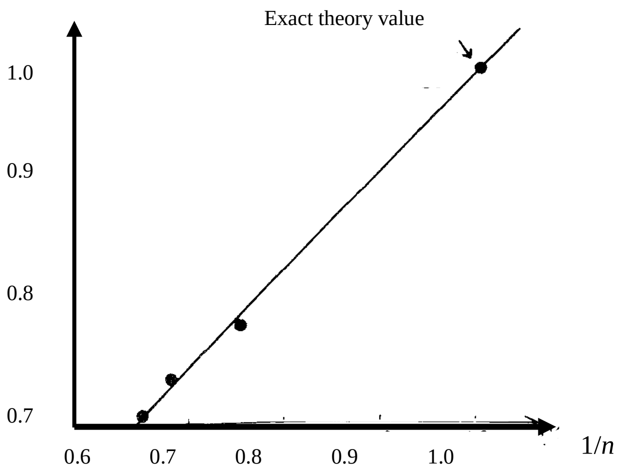


Figure E 1.2

SUMMARY SHEET

EXPERIMENT 2

Is the total momentum conserved? YES / ~~NO~~

Accuracy of computer calculation $100\frac{0.0000018}{0.1} \approx 0.002\%$ 0:1

(RMS velocity = 0.1)

Time	Total Energy
0	-1.61499
2	-1.62886
4	-1.62878
6	-1.62301
12	-1.62882
18	-1.62599
24	-1.62796
30	-1.62703
50	-1.62753
70	-1.62676
90	-1.62580
130	-1.62713
180	-1.62409

Does the system conserve energy? YES/~~NO~~ (~ ±1%)

Equilibrium value of E_k* (Average 24 to 180) = 0.534 ±0.05

Equilibrium time SD (see Fig. E 2.1) $\cong$ (10 to 20) $\times 0.1$

Value of S recorded > 20, e.g. 60

Value of α (for SD=60) (see Fig. E2.2) = 0.503

Accuracy of α = ±0.02

For what time number range is graph, obtained using first value of SR, linear? SZ = 18 to 24

Gradient of this graph in linear region	≅ 0.027 to 0.47
Accuracy of gradient	= 0.002
Gradient of AVERAGE <R^2> in linear region	= 0.035
Accuracy of this gradient	= ± 0.01

* delete as appropriate

Is the system a liquid/solid?	Liquid/~~Solid~~*

Mean Momentum of the system at requested steps (*S*)

S	<*VX,1*>	<*VY,1*>	<*PX*>	<*PY*>
0	0.0000000	0.0000000	0.000000	0.000000
40	0.0000010	0.0000016	0.000048	0.000077
80	0.0000018	0.0000001	0.000086	0.000005
120	0.0000014	0.0000007	0.000067	0.000034
160	0.0000016	0.0000010	0.000077	0.000048

Energy of the system at requested steps (*S*)

S	<*VX,2*>	<*VY,2*>	<*KE*> = *T**	<*U*>	<*E*>= Total Energy
0	0.0173874	0.0142851	0.760140	-4.7502660	-1.61499
2	0.0162506	0.0131025	0.704474	-4.6666675	-1.62886
4	0.0124966	0.0089562	0.514867	-4.2873015	-1.62878
6	0.0077405	0.0039113	0.279643	-3.8053113	-1.62301
12	0.0118740	0.0120959	0.575278	-4.4081878	-1.62882
18	0.0099579	0.0075854	0.421039	-4.0940627	-1.62599
24	0.0108577	0.0116978	0.541332	-4.3385782	-1.62796
30	0.0126065	01000340	0.543372	-4.3407997	-1.62703
50	0.0127138	0.0103334	0.553133	-4.3613165	-1.62753
70	0.0088657	0.0158292	0.592678	-4.4388669	-1.62676
90	0.0107740	0.0076446	0.442087	-4.1357699	-1.62580
130	0.0073008	0.0177446	0.601090	-4.4564333	-1.62713
180	0.0097161	0.0096426	0.464609	-4.1773882	-1.62409

All values are in reduced units. <*KE*> is the mean kinetic energy per atom. <*U**> is twice the potential energy. <*VX*,2> and <*VY*,2> are the mean values of the squares of the *X* and *Y* velocity components, as described in the question. Similarly <*VX*,1> and <*VY*,1> are the mean values of the velocity components. <*PX*> and <*PY*> are the mean momentum per particle.

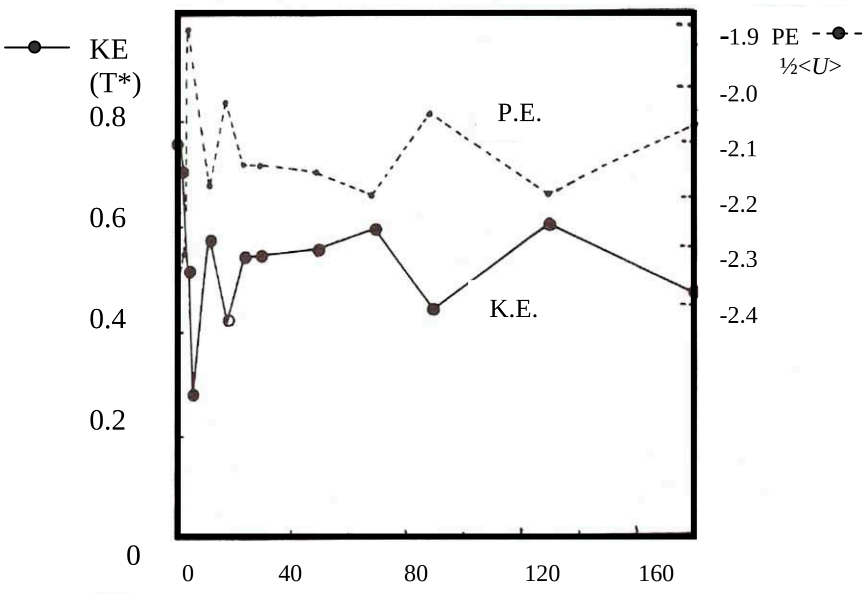


Figure E 2.1

Variation of K.E and P.E.

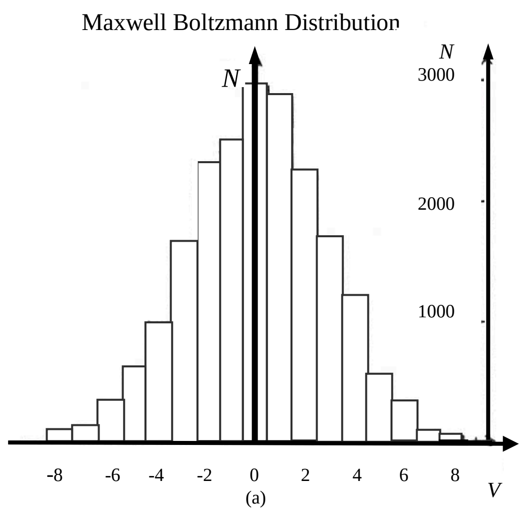
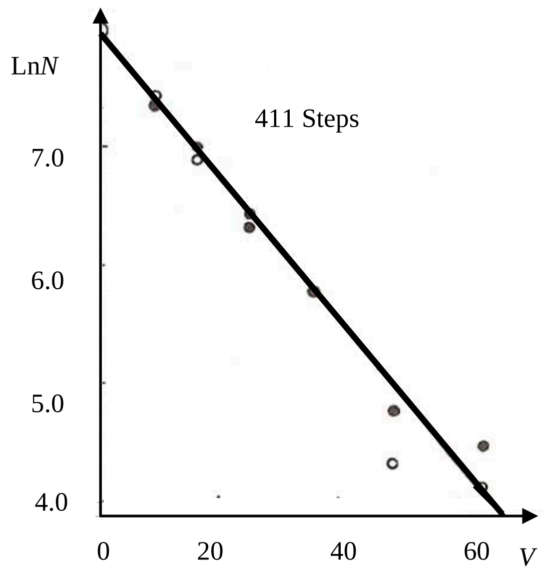
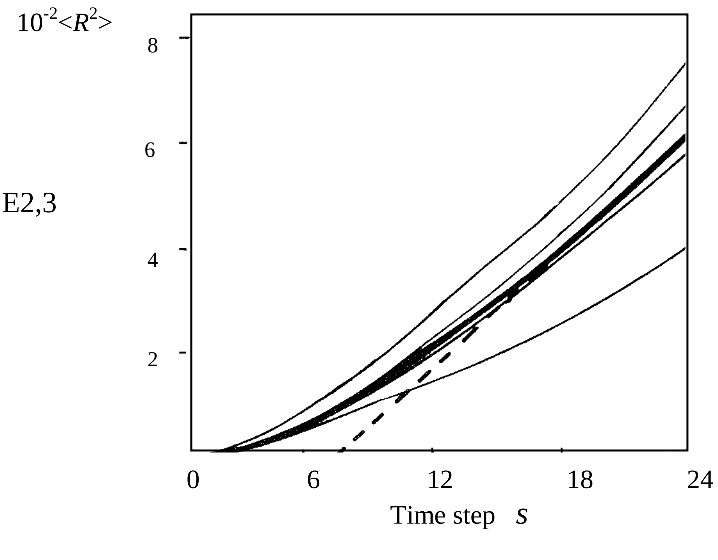


Figure E2.2



(b)

Figure E2,3



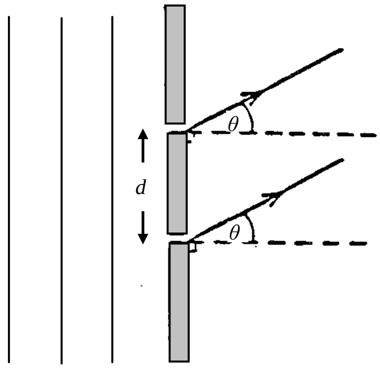
<R^z> curves as a function of time

TYPICAL RESULTS : NOTE THE LARGE VARIATIONS IN THE VALUES OF $\langle R^2 \rangle$

Time Number SZ - S-SR	SR = 261 $\langle R^2 \rangle$	SR = 301 $\langle R^2 \rangle$	SR = 334 $\langle R^2 \rangle$	SR = 370 $\langle R^2 \rangle$	AVERAGE $\langle R^2 \rangle$
0	0	0	0	0	0
2	0.00088	0.00067	0.00091	0.00079	0.00081
4	0.00287	0.00276	0.00382	0.00298	0.00311
6	0.00523	0.00628	0.00858	0.00623	0.00658
8	0.00797	0.01101	0.01449	0.01039	0.01097
10	0.01143	0.01656	0.02095	0.01523	0.01604
12	0.01528	0.02235	0.02768	0.02022	0.02138
14	0.01874	0.02845	0.03453	0.02564	0.02684
16	0.02184	0.03539	0.04157	0.03160	0.03260
18	0.02526	0.04293	0.04902	0.03833	0.03889
20	0.02979	0.05080	0.05718	0.04532	0.04577
22	0.03538	0.05918	0.06605	0.0510	0.05303
24	0.04063	0.06784	0.07533	0.05569	0.05987

Q1

Figure 1.1



A plane monochromatic light wave, wavelength λ and frequency f, is incident normally on two identical narrow slits, separated by a distance d, as indicated in Figure 1.1. The light wave emerging at each slit is given, at a distance x in a direction θ at time t, by

$$y = a\cos[2\pi(ft - x/\lambda)]$$

where the amplitude a is the same for both waves. (Assume x is much larger than d).

(i) Show that the two waves observed at an angle θ to a normal to the slits, have a resultant amplitude A which can be obtained by adding two vectors, each having magnitude a, and each with an associated direction determined by the phase of the light wave.

Verify geometrically, from the vector diagram, that

$$A = 2a\cos\theta$$

where

$$\beta = \frac{\pi}{\lambda} d\sin\theta$$

(ii) The double slit is replaced by a diffraction grating with N equally spaced slits, adjacent slits being separated by a distance d. Use the vector method of adding amplitudes to show that the vector amplitudes, each of magnitude a, form a part of a regular polygon with vertices on a circle of radius R given by

$$R = \frac{a}{2\sin\beta},$$

Deduce that the resultant amplitude is

$$\frac{a\sin N\beta}{\sin\beta}$$

and obtain the resultant phase difference relative to that of the light from the slit at the edge of the grating.

(iii) Sketch, in the same graph, sin $N\beta$ and (1/sinβ) as a function of β. On a separate graph show how the intensity of the resultant wave varies as a function of β.

(iv) Determine the intensities of the principal intensity maxima.

(v) Show that the number of principal maxima cannot exceed

$$\left(\frac{2d}{\lambda}+1\right)$$

(vi) Show that two wavelengths λ and $\lambda+\delta\lambda$, where $\delta\lambda << \lambda$, produce principal maxima with an angular separation given by

$$\Delta\theta = \frac{n\Delta\lambda}{d\cos\theta} \quad \text{where } n = 0, \pm 1, \pm 2....\text{etc}$$

Calculate this angular separation for the sodium D lines for which

$$\lambda = 589.0\text{nm}, \quad \lambda+\Delta\lambda = 589.6\text{nm}, \quad n = 2, \text{ and } d = 1.2\times 10^{-6} \text{ m.}$$

$$\left[\text{reminder:} \quad \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{\text{A - B}}{2}\right)\right]$$

Q2

International Physics Olympiad 1956

2. Early this century a model of the earth was proposed in which it was assumed to be a sphere of radius R consisting of a homogeneous isotropic solid mantle down to radius R_c. The core region within radius R_c contained a liquid. Figure 2.1

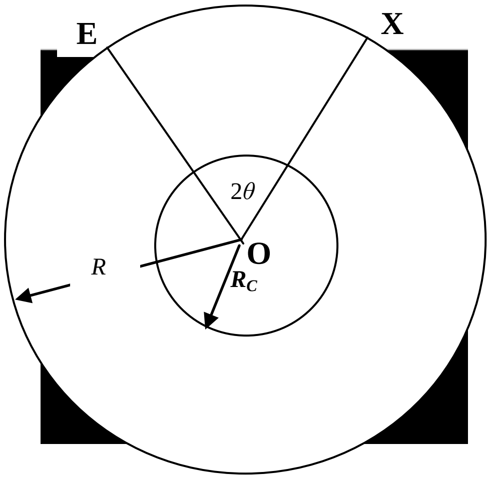


Figure 2.1

The velocities of longitudinal and transverse seismic waves P and S waves respectively, are constant, V_P, and V_S within the mantle. In the core, longitudinal waves have a constant velocity V_{CP}, $< V_P$, and transverse waves are not propagated.

An earthquake at E on the surface of the Earth produces seismic waves that travel through the Earth and are observed by a surface observer who can set up his seismometer at any point X on the Earth's surface. The angular separation between E and X, 2θ given by

$$2\theta = \text{Angle}\, EOX$$

where O is the centre of the Earth.

(i) Show that the seismic waves that travel through the mantle in a straight line will arrive at X at a time t (the travel time after the earthquake), is given by

$$t = \frac{2R\sin\theta}{v}, \qquad \text{for } \theta > \arccos\left[\frac{R_c}{R}\right],$$

where $v = v_P$ for the P waves and $v = v_S$ for the S waves.

(ii) For some of the positions of X such that the seismic P waves arrive at the observer after two refractions at the mantle-core interface. Draw the path of such a seismic P wave. Obtain a relation between θ and i, the angle of incidence of the seismic P wave at the mantle-core interface, for P waves.

(iii) Using the data

$$
\begin{aligned}
R &= 6370 \text{ km} \\
R_C &= 3470 \text{ km} \\
v_{CP} &= 10.85 \text{ km s}^{-1} \\
v_S &= 6.31 \text{ km s}^{-1} \\
v_{CP} &= 9.02 \text{ km s}^{-1}
\end{aligned}
$$

and the result obtained in (ii),draw a graph of θ against i. Comment on the physical consequences of the form of this graph for observers stationed at different points on the Earth's surface. Sketch the variation of the travel time taken by the P and S waves as a function of θ for $0 \leq \theta \leq 90$ degrees.

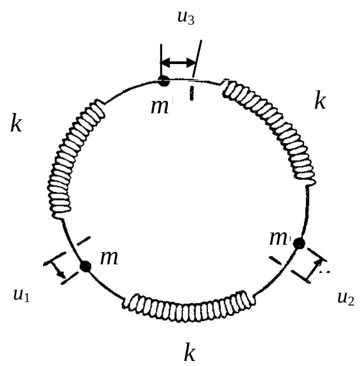
(iv) After an earthquake an observer measures the time delay between the arrival of the S wave, following the P wave, as 2 minutes 11 seconds. Deduce the angular separation of the earthquake from the observer using the data given in Section (iii).

(v) The observer in the previous measurement notices that some time after the arrival of the P and S waves there are two further recordings on the seismometer separated by a time interval of 6 minutes 37 seconds. Explain this result and verify that it is indeed associated with the angular separation determined in the previous section.

Q3

Three particles, each of mass *m*, are in equilibrium and joined by unstretched massless springs, each with Hooke's Law spring constant *k*. They are constrained to move in a circular path as indicated in Figure 3.1.

Figure 3.1



(i) If each mass is displaced from equilibrium by small displacements u_1, u_2 and u_3 respectively, write down the equation of motion for each mass.

(ii) Verify that the system has simple harmonic solutions of the form

$$u_n = a_n \cos \omega t \ ,$$

with accelerations, $(-\omega^2 u_n)$ where $a_n \ (n = 1,2,3)$ are constant amplitudes, and ω, the angular frequency, can have 3 possible values,

$$\omega_o \sqrt{3}, \omega_o \sqrt{3} \text{ and } 0. \text{ where } \omega_o^2 = \frac{k}{m}.$$

(iii) The system of alternate springs and masses is extended to *N* particles, each mass *m* is joined by springs to its neighbouring masses. Initially the springs are unstretched and in equilibrium. Write down the equation of motion of the *n*th mass (n = 1,2...N) in terms of its displacement and those of the adjacent masses when the particles are displaced from equilibrium.

$$u_n(t) = a_s \sin\left(\frac{2ns\pi}{N} + \phi\right)\cos \omega_s t,$$

are oscillatory solutions where s = 1, 2,...N , n = 1, 2, ...N and where ϕ is an arbitrary phase, providing the angular frequencies are given by

$$\omega_s = 2\omega_o \sin\left(\frac{s\pi}{N}\right),$$

where $a_s \ (s = 1,......N)$ are constant amplitudes independent of *n*.

State the range of possible frequencies for a chain containing an infinite number of masses.

(iv) Determine the ratio

$$u_n / u_{n+1}$$

for large N, in the two cases:

(a) low frequency solutions

(b) $\omega = \omega_{\max}$, where $\omega_{\max}$ is the maximum frequency solution.

Sketch typical graphs indicating the displacements of the particles against particle number along the chain at time t for cases (a) and (b).

(v) If one of the masses is replaced by a mass $m' << m$ estimate any major change one would expect to occur to the angular frequency distribution.

Describe qualitatively the form of the frequency spectrum one would predict for a diatomic chain with alternate masses m and m' on the basis of the previous result.

Reminder

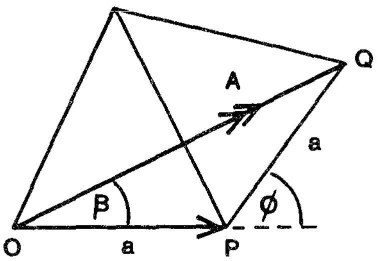
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

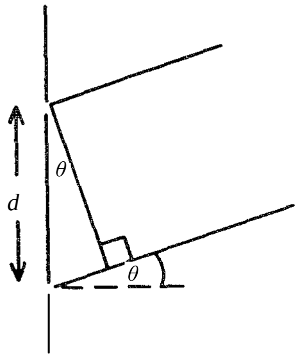
$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$2\sin^2 A = 1 - \cos 2A$$

Answers Question 1

(i) Vector Diagram





If the phase of the light from the first slit is zero, the phase from second slit is

$$\phi = \frac{2\pi}{\lambda} d \sin\theta$$

Adding the two waves with phase difference ϕ where $\xi = 2\pi\left(ft - \frac{x}{\lambda}\right)$,

$$a\cos(\xi+\phi) + a\cos(\xi) = 2a\cos(\phi/2)(\xi+\phi/2)$$
$$a\cos(\xi+\phi) + a\cos(\xi) = 2a\cos\beta\{\cos(\xi+\beta)\}$$

This is a wave of amplitude $A = 2a\cos\beta$ and phase β. From vector diagram, in isosceles triangle OPQ,

$$\beta = \frac{1}{2}\phi = \frac{\pi}{\lambda} d \sin\theta \qquad (NB \quad \phi = 2\beta)$$

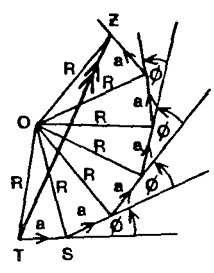
and

$$A = 2a\cos\beta.$$

Thus the sum of the two waves can be obtained by the addition of two vectors of amplitude a and angular directions 0 and ϕ.

(ii) Each slit in diffraction grating produces a wave of amplitude a with phase 2β relative to previous slit wave. The vector diagram consists of a 'regular' polygon with sides of constant length a and with constant angles between adjacent sides.
Let O be the centre of circumscribing circle passing through the vertices of the polygon. Then radial lines such as OS have length R and bisect the internal angles of the polygon. Figure 1.2.

Figure 1.2



$$O\hat{S}T = O\hat{T}S = \frac{1}{2}(180-\phi)$$

$$\text{and} \quad T\hat{O}S = \phi$$

In the triangle *TOS*, for example

$$a = 2R\sin(\phi/2) = 2R\sin\beta \text{ as } (\phi = 2\beta)$$

$$\therefore R = \frac{a}{2\sin\beta} \quad (1)$$

As the polygon has *N* faces then:

$$T\hat{O}Z = N(T\hat{O}Z) = N\phi = 2N\beta$$

Therefore in isosceles triangle TOZ, the amplitude of the resultant wave, TZ, is given by

$$2R\sin N\beta \, .$$

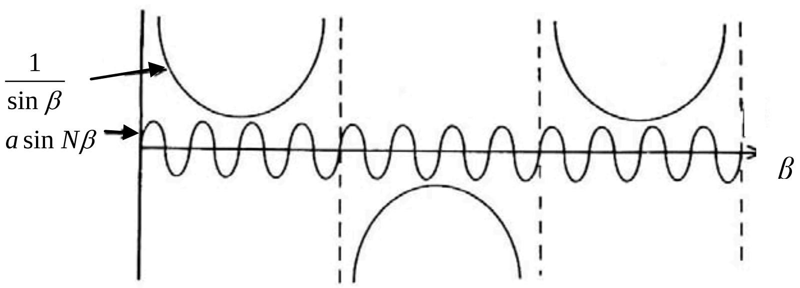
Hence form (1) this amplitude is

$$\frac{a\sin N\beta}{\sin\beta}$$

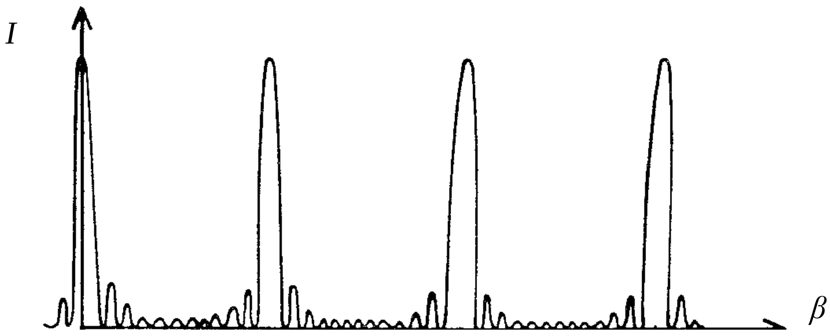
Resultant phase is

$$= Z\hat{T}S$$
$$= O\hat{T}S - O\hat{T}Z$$
$$\left(90-\frac{\phi}{2}\right)-\frac{1}{2}(180-N\phi)$$
$$-\frac{1}{2}(N-1)\phi$$
$$= (N-1)\beta$$

(iii)



Intensity $I = \dfrac{a^2\sin^2 N\beta}{\sin^2\beta}$



(iv) For the principle maxima $\beta = \pi p$ where $p = 0 \pm 1 \pm 2$........

$$I_{\max} = a^2 \left(\frac{N\beta'}{\beta'} \right) = N^2 a^2 \qquad \beta' = 0 \text{ and } \beta = \pi p + \beta'$$

(v) Adjacent max. estimate I_1 :

$$\sin^2 N\beta = 1, \quad \beta = 2\pi p \mp \frac{3\pi}{2N} \text{ i.e } \beta = \pm \frac{3\pi}{2N}$$

$\left[\beta = \pi p \pm \frac{\pi}{2N} \right]$ does not give a maximum as can be observed from the graph.

$$I_1 = a^2 \frac{1}{\frac{3\pi^2}{2n}} = \frac{a^2 N^2}{23} \text{ for } N >> 1$$

Adjacent zero intensity occurs for $\beta = \pi \rho \pm \frac{\pi}{N}$ i.e. $\delta = \pm \frac{\pi}{N}$

For phase differences much greater than δ, $\quad I = a^2 \left(\frac{\sin N\beta}{\sin \beta} \right) = a^2$.

(vi)

$\beta = n\pi$ for a principle maximum

i.e. $\frac{\pi}{\lambda} d \sin\theta = n\pi \qquad n = 0, \pm 1, \pm 2$..........

Differentiating w.r.t, λ

$d \cos\theta \Delta\theta = n \Delta\lambda$

$$\Delta\theta = \frac{n\Delta\lambda}{d \cos\theta}$$

Substituting $\lambda = 589.0\text{nm}$, $\lambda + \Delta\lambda = 589.6\,\text{nm}$. n = 2 and $d = 1.2 \times 10^{-6}$ m.

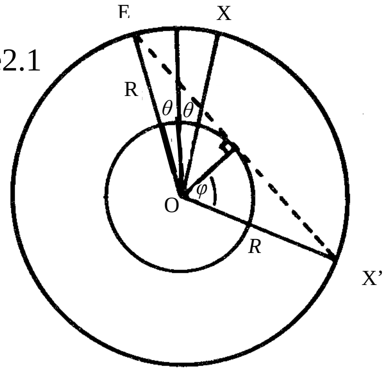
$$\Delta\theta = \frac{n\Delta\lambda}{d\sqrt{1 - \left(\frac{n\lambda}{d} \right)^2}} \quad \text{as } \sin\theta = \frac{n\lambda}{d} \text{ and } \cos\theta = \sqrt{1 - \left(\frac{n\lambda}{d} \right)^2}$$

$\Rightarrow \Delta\theta = 5.2 \times 10^{-3}\, rads$ or 0.30^0

Answers Q2

2.(i)

Figure2.1



$$\text{EX} = 2R\sin\theta \quad \therefore t = \frac{2R\sin\theta}{v}$$

where $v = v_P$ for P waves and $v = v_S$ for S waves.

This is valid providing X is at an angular separation less than or equal to X', the tangential ray to the liquid core. X' has an angular separation given by, from the diagram,

$$2\phi = 2\cos^{-1}\left(\frac{R_C}{R}\right),$$

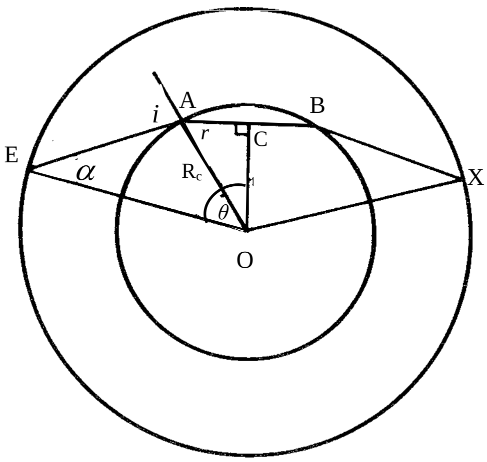
Thus

$$t = \frac{2R\sin\theta}{v} \quad , \qquad \text{for } \theta \le \cos^{-1}\left(\frac{R_C}{R}\right),$$

where $v = v_P$ for P waves and $v = v_S$ for shear waves.

(ii) $\frac{R_C}{R} = 0.5447$ and $\frac{v_{CP}}{v_P} = 0.831.3$

Figure 2.2



From Figure 2.2

$$\theta = A\hat{O}C + E\hat{O}A \Rightarrow \theta = (90 - r) + (1 - \alpha) \qquad (1)$$

(ii) Continued

Snell's Law gives:

$$\frac{\sin i}{\sin r} = \frac{v_P}{v_{CP}}. \qquad (2)$$

From the triangle EAO, sine rule gives

$$\frac{R_C}{\sin x} = \frac{R}{\sin i}. \qquad (3)$$

Substituting (2) and (3) into (1)

$$\theta = \left[90 - \sin^{-1}\left(\frac{v_{CP}}{v_P}\sin i\right) + i - \sin^{-1}\left(\frac{R_C}{R}\sin i\right)\right] \qquad (4)$$

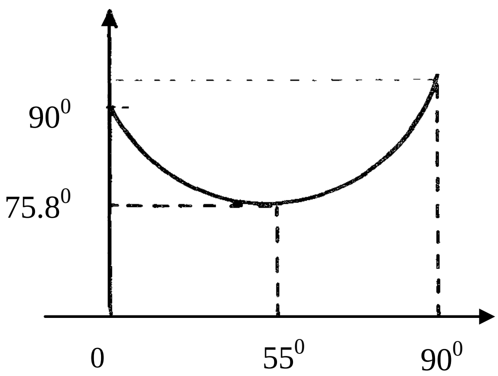
(iii)

For Information Only

For minimum $\theta, \frac{d\theta}{di} = 0$. $\Rightarrow 1 - \frac{\left(\frac{v_{CP}}{v_P}\right)\cos i}{\sqrt{1-\left(\frac{v_{CP}}{v_P}\sin i\right)^2}} - \frac{\left(\frac{R_C}{R}\right)\cos i}{\sqrt{1-\left(\frac{R_C}{R}\sin i\right)^2}} = 0$

Substituting $i = 55.0^o$ gives LHS=0, this verifying the minimum occurs at this value of *i*. Substituting $i = 55.0^o$ into (4) gives $\theta = 75.8^0$.

Plot of θ against i.



Substituting into 4:

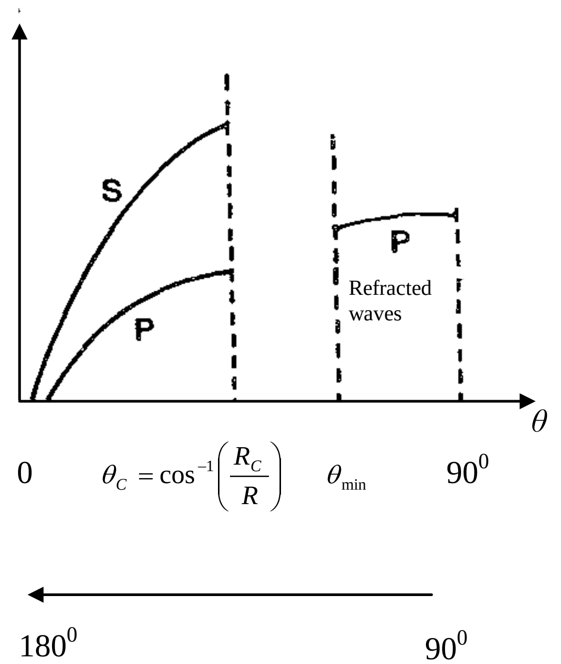
$i = 0$ gives $\theta = 90$

$i = 90°$ gives $\theta = 90.8°$

Substituting numerical values for $i = 0 \rightarrow 90°$ one finds a minimum value at $i = 55^{\circ}$; the minimum values of 0, $\theta_{MIN} = 75•8°$.

Physical Consequence

As θ has a minimum value of 75•8° observers at position for which 2 θ <151•6° will not observe the earthquake as seismic waves are not deviated by angles of less than 151•6°. However for 2 $\theta \leq$ 114° the direct, non-refracted, seismic waves will reach the observer.



(iv) Using the result

$$t = \frac{2r\sin\theta}{v}$$

the time delay Δt is given by

$$\Delta t = 2R\sin\theta\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

Substituting the given data

$$131 = 2(6370)\left[\frac{1}{6.31} - \frac{1}{10.85}\right]\sin\theta$$

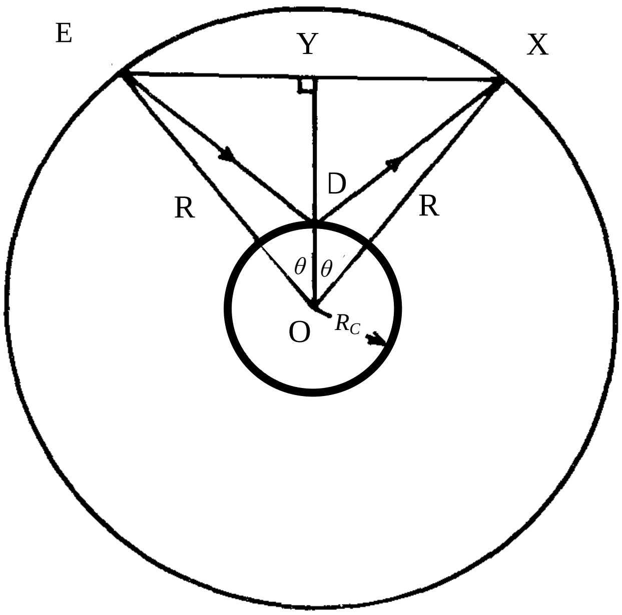
Therefore the angular separation of E and X is

$$2\theta = 17.84^o$$

This result is less than $2\cos^{-1}\left(\frac{R_C}{R}\right) = 2\cos^{-1}\left(\frac{3470}{6370}\right) = 114^o$

And consequently the seismic wave is not refracted through the core.

(v)



The observations are most likely due to reflections from the mantle-core interface. Using the symbols given in the diagram, the time delay is given by

$$\Delta t' = (\text{ED} + \text{DX})\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

$$\Delta t' = 2(\text{ED})\left[\frac{1}{v_S} - \frac{1}{v_P}\right] \text{ as ED} = \text{EX by symmetry}$$

In the triangle EYD,

$$(\text{ED})^2 = (R\sin\theta)^2 + (R\cos\theta - R_C)^2$$

$$(\text{ED})^2 = R^2 + R_C^2 - 2RR_C\cos\theta \qquad \sin^2\theta + \cos^2\theta = 1$$

Therefore

$$\Delta t' = 2\sqrt{R^2 + R_C^2 - 2RR_C\cos\theta}\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

Using (ii)

$$\Delta t' = \frac{\Delta t}{R\sin\theta}\sqrt{R^2 + {R_C}^2 - 2RR_C\cos\theta}$$

$$\Rightarrow 396.7s \text{ or } 6\text{m } 37\text{s}$$

Thus the subsequent time interval, produced by the reflection of seismic waves at the mantle core interface, is consistent with angular separation of 17.84^0.

Answer Q3

Equations of motion:

$$m\frac{d^2u_1}{dt^2} = k(u_2 - u_1) + k(u_3 - u_1)$$
$$m\frac{d^2u_2}{dt^2} = k(u_3 - u_2) + k(u_1 - u_2)$$
$$m\frac{d^2u_3}{dt^2} = k(u_1 - u_3) + k(u_2 - u_3)$$

Substituting $u_n(t) = u_n(0)\cos\omega t$ and $\omega_o^2 = \frac{k}{m}$:

$$(2\omega_o^2 - \omega^2)u_1(0) - \omega_o^2 u_2(0) - \omega_o^2 u_3(0) = 0 \quad \text{(a)}$$
$$-\omega_o^2 u_1(0) + (2\omega_o^2 - \omega^2)u_2(0) - \omega_o^2 u_3(0) = 0 \quad \text{(b)}$$
$$-\omega_o^2 u_1(0) - \omega_o^2 u_2(0) + (2\omega_o^2 - \omega^2)u_3(0) = 0 \quad \text{(c)}$$

Solving for $u_1(0)$ and $u_2(0)$ in terms of $u_3(0)$ using (a) and (b) and substituting into (c) gives the equation equivalent to

$$(3\omega_o^2 - \omega^2)^2\omega^2 = 0$$
$$\omega^2 = 3\omega_o^2,\ 3\omega_o^2 \text{ and } 0$$
$$\omega = \sqrt{3}\omega_o,\ \sqrt{3}\omega_o \text{ and } 0$$

(ii) Equation of motion of the n'th particle:

$$m\frac{d^2u_n}{dt^2} = k(u_{1+n} - u_n) + k(u_{n-1} - u_n)$$
$$\frac{d^2u_n}{dt^2} = k(u_{1+n} - u_n) + \omega_o^2(u_{n-1} - u_n)$$

$n = 1,2\ldots\ldots N$

Substituting $u_n(t) = u_n(0)\sin\left(2ns\frac{\pi}{N}\right)\cos\omega_s t$

$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = \omega_o^2\left[\sin\left(2(n+1)s\frac{\pi}{N}\right) - 2\sin\left(2ns\frac{\pi}{N}\right) + \sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o^2\left[\frac{1}{2}\sin\left(2(n+1)s\frac{\pi}{N}\right) + \sin\left(2ns\frac{\pi}{N}\right) - \frac{1}{2}\sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o^2\left[\sin\left(2ns\frac{\pi}{N}\right)\cos\left(2s\frac{\pi}{N}\right) - \sin\left(2ns\frac{\pi}{N}\right)\right]$$
$$\therefore \omega_s^2 = 2\omega_o^2\left[1 - \cos\left(2s\frac{\pi}{N}\right)\right]: \quad (s = 1,2,\ldots\ldots N)$$

As $2\sin^2\theta = 1 - \cos 2\theta$

This gives

$$\omega_s = 2\omega_o\sin\left(\frac{s\pi}{N}\right) \quad (s = 1,2,\ldots N)$$

ω_s can have values from 0 to $2\omega_o = 2\sqrt{\frac{k}{m}}$ when $N \to \infty$; corresponding to range $s = 1$ to $\frac{N}{2}$.

(iv) For s'th mode

$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\frac{\pi}{N}\right)}{\sin\left(2(n+1)s\frac{\pi}{N}\right)}$$

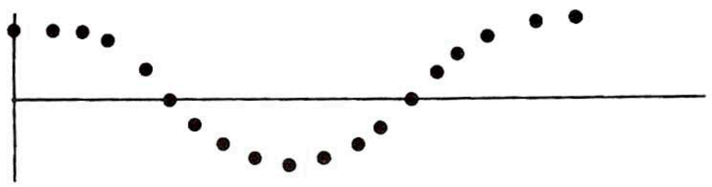
$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\frac{\pi}{N}\right)}{\sin\left(2ns\frac{\pi}{N}\right)\cos\left(2s\frac{\pi}{N}\right)+\cos\left(2ns\frac{\pi}{N}\right)\sin\left(2s\frac{\pi}{N}\right)}$$

(a) For small ω, $\left(\frac{s}{N}\right) \approx 0$, thus $\cos\left(2ns\frac{\pi}{N}\right) \cong 1$ and $= \sin\left(2ns\frac{\pi}{N}\right) \approx 0$, and so $\frac{u_n}{u_{n+1}} \cong 1$.

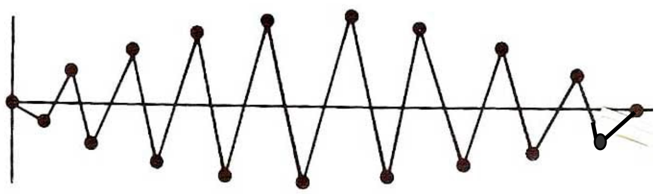
(b) The highest mode, $\omega_{\max} = 2\omega_o$, corresponds to $s = N/2$

$$\therefore \frac{u_n}{u_{n+1}} = -1 \quad \text{as} \quad \frac{\sin(2n\pi)}{\sin(2(n+1)\pi)} = -1$$

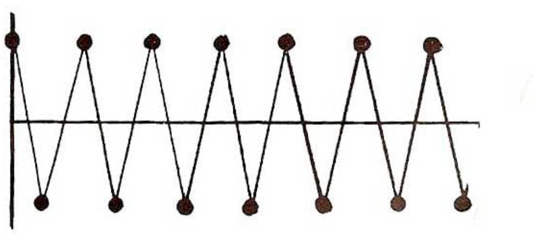
Case (a)



Case (b)
N odd

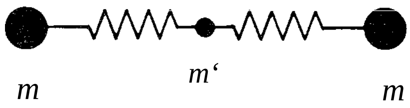


N even



(vi) If $m' << m$, one can consider the frequency associated with m' as due to vibration of m' between two adjacent, much heavier, masses which can be considered stationary relative to m'.

The normal mode frequency of m', in this approximation, is given by

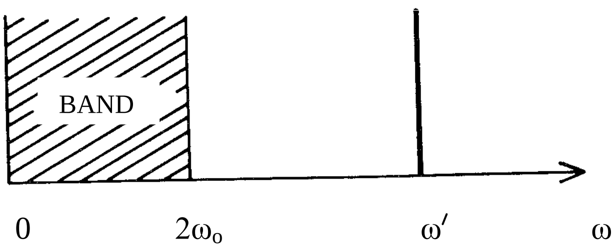


$$m'\ddot{x} = -2kx$$

$$\omega'^2 = \frac{2k}{m}$$

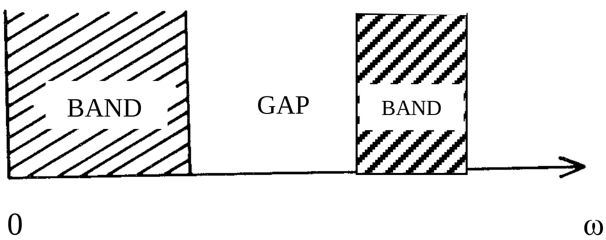
$$\omega' = \sqrt{\frac{2k}{m'}}$$

For small m', ω' will be much greater than ω_{max},



DIATOMIC SYSTEM

More light masses, m', will increase the number of frequencies in region of ω' giving a band-gap-band spectrum.



Problems of the 18th International Physics Olympiad
(Jena, 1987)

Gunnar Friege & Gunter Lind[1]

Leibniz-Institute for Science Education (IPN) at the University of Kiel, Germany

Abstract

The 18th International Physics Olympiad took place in 1987 in the German Democratic Republic (GDR). This article contains the competition problems, their solutions and also a (rough) grading scheme.

Introduction

The 18th international Physics olympics in 1987 was the second International Physics Olympiad hosted by the German Democratic Republic (GDR) . The organisation was lead by the ministry for education and the problems were formulated by a group of professors of different universities. However, the main part of the work was done by the physics department of the university of Jena. The company Carl-Zeiss and a special scientific school in Jena were involved also.

In the competition three theoretical and one experimental problem had to be solved. The theoretical part was quite difficult. Only the first of the three problems ("ascending moist air") had a medium level of difficulty. The points given in the markings were equal distributed. Therefore, there were lots of good but also lots of unsatisfying solutions. The other two theoretical problems were rather difficult. About half of the pupils even did not find an adequate start in solving these problems. The third problem ("infinite LC-grid") revealed quite a few complete solutions. The high level of difficulty can probably be explained with the fact that many pupils nearly had no experience with the subject. Concerning the second problem ("electrons in a magnetic field") only a few pupils worked on the last part 3 (see below).

The experimental problem ("refracting indices") was much more easier than the theoretical problems. There were lots of different possibilities of solution and most of the pupils

[1] Contact: Leibniz-Institute for Science Education (IPN) at the University of Kiel
Olshausenstrasse 62, 24098 Kiel, Germany
ipho@ipn.uni-kiel.de

managed to come up with partial or complete solutions. Over the half of all teams got more points in the experimental part than in the theoretical part of the competition.

The problems and their solutions are based on the original German and English versions of the competition problems. Only minor changes have been made. Despite the fact that nowadays almost all printed figures are generated with the aid of special computer programmes, the original hand-made figures are published here.

Theoretical Problems

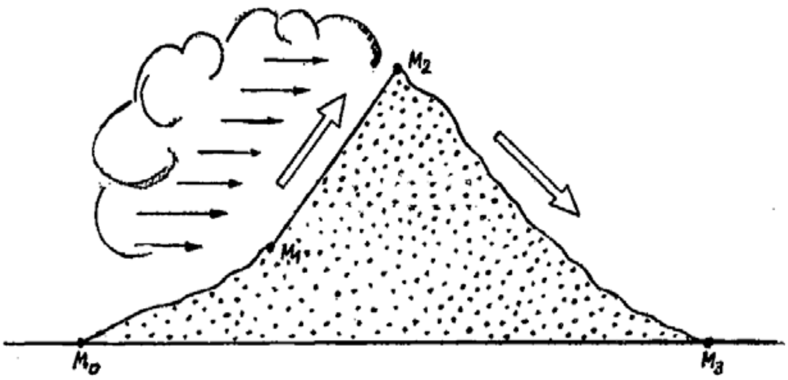
Problem 1: Ascending moist air

Moist air is streaming adiabatically across a mountain range as indicated in the figure.

Equal atmospheric pressures of 100 kPa are measured at meteorological stations M_0 and M_3 and a pressure of 70 kPa at station M_2. The temperature of the air at M_0 is 20° C.

As the air is ascending, cloud formation sets in at 84.5 kPa.

Consider a quantity of moist air ascending the mountain with a mass of 2000 kg over each square meter. This moist air reaches the mountain ridge (station M_2) after 1500 seconds. During that rise an amount of 2.45 g of water per kilogram of air is precipitated as rain.



1. Determine temperature T_1 at M_1 where the cloud ceiling forms.
2. What is the height h_1 (at M_1) above station M_0 of the cloud ceiling assuming a linear decrease of atmospheric density?
3. What temperature T_2 is measured at the ridge of the mountain range?
4. Determine the height of the water column (precipitation level) precipitated by the air stream in 3 hours, assuming a homogeneous rainfall between points M_1 and M_2.

5. What temperature T_3 is measured in the back of the mountain range at station M_3? Discuss the state of the atmosphere at station M_3 in comparison with that at station M_0.

Hints and Data

The atmosphere is to be dealt with as an ideal gas. Influences of the water vapour on the specific heat capacity and the atmospheric density are to be neglected; the same applies to the temperature dependence of the specific latent heat of vaporisation. The temperatures are to be determined to an accuracy of 1 K, the height of the cloud ceiling to an accuracy of 10 m and the precipitation level to an accuracy of 1 mm.

Specific heat capacity of the atmosphere in the pertaining temperature range:

$c_p = 1005\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}}$

Atmospheric density for p_0 and T_0 at station M_0: $\rho_0 = 1.189\ \mathrm{kg \cdot m^{-3}}$

Specific latent heat of vaporisation of the water within the volume of the cloud:

$L_v = 2500\ \mathrm{kJ \cdot kg^{-1}}$

$\frac{c_p}{c_v} = \chi = 1.4$ and $g = 9.81\ \mathrm{m \cdot s^{-2}}$

Solution of problem 1:

1. Temperature T_1 where the cloud ceiling forms

$$T_1 = T_0 \cdot \left(\frac{p_1}{p_0}\right)^{1-\frac{1}{\chi}} = 279\,\mathrm{K} \qquad (1)$$

2. Height h_1 of the cloud ceiling:

$$p_0 - p_1 = \frac{\rho_0 + \rho_1}{2} \cdot g \cdot h_1, \text{ with } \rho_1 = \rho_0 \cdot \frac{p_1}{p_0} \cdot \frac{T_0}{T_1}.$$

$$h_1 = 1410\,\mathrm{m} \qquad (2)$$

3. Temperature T_2 at the ridge of the mountain.

   The temperature difference when the air is ascending from the cloud ceiling to the mountain ridge is caused by two processes:

   – adiabatic cooling to temperature T_x,

– heating by ΔT by condensation.

$$T_2 = T_x + \Delta T \tag{3}$$

$$T_x = T_1 \cdot \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\chi}} = 265\,\text{K} \tag{4}$$

For each kg of air the heat produced by condensation is $L_v \cdot 2.45$ g = 6.125 kJ.

$$\Delta T = \frac{6.125}{c_p} \cdot \frac{\text{kJ}}{\text{kg}} = 6.1\,\text{K} \tag{5}$$

$$T_2 = 271\text{ K} \tag{6}$$

4. Height of precipitated water column

$$h = 35\text{ mm} \tag{7}$$

5. Temperature T_3 behind the mountain

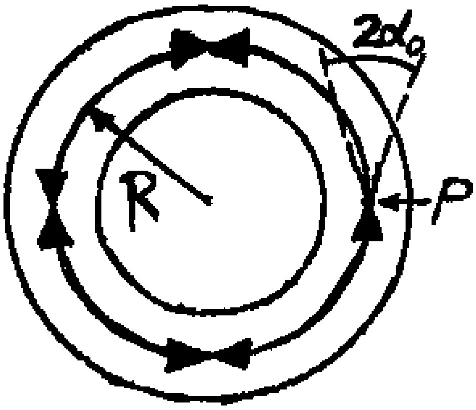
$$T_3 = T_2 \cdot \left(\frac{p_3}{p_2}\right)^{1-\frac{1}{\chi}} = 300\,\text{K} \tag{8}$$

The air has become warmer and dryer. The temperature gain is caused by condensation of vapour.

Problem 2: Electrons in a magnetic field

A beam of electrons emitted by a point source P enters the magnetic field $\vec{B}$ of a toroidal coil (toroid) in the direction of the lines of force. The angle of the aperture of the beam $2 \cdot \alpha_0$ is assumed to be small ($2 \cdot \alpha_0 \ll 1$). The injection of the electrons occurs on the mean radius R of the toroid with acceleration voltage V_0.

Neglect any interaction between the electrons. The magnitude of $\vec{B}$, B, is assumed to be constant.



1. To guide the electron in the toroidal field a homogeneous magnetic deflection field $\vec{B}_1$ is required. Calculate $\vec{B}_1$ for an electron moving on a circular orbit of radius R in the torus.

2. Determine the value of $\vec{B}$ which gives four focussing points separated by $\pi/2$ as indicated in the diagram.

   Note: When considering the electron paths you may disregard the curvature of the magnetic field.

3. The electron beam cannot stay in the toroid without a deflection field $\vec{B}_1$, but will leave it with a systematic motion (drift) perpendicular to the plane of the toroid.

   a) Show that the radial deviation of the electrons from the injection radius is finite.

   b) Determine the direction of the drift velocity.

   Note: The angle of aperture of the electron beam can be neglected. Use the laws of conservation of energy and of angular momentum.

Data:

$\frac{e}{m} = 1.76 \cdot 10^{11}\,\mathrm{C \cdot kg^{-1}}$; $V_0 = 3$ kV; R = 50 mm

Solution of problem 2:

1. Determination of B:

   The vector of the velocity of any electron is divided into components parallel with and perpendicular to the magnetic field $\vec{B}$:

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp} \tag{1}$$

The Lorentz force $\vec{F} = -e \cdot (\vec{v} \times \vec{B})$ influences only the perpendicular component, it acts as a radial force:

$$m \cdot \frac{v_{\perp}^2}{r} = e \cdot v_{\perp} \cdot B \tag{2}$$

Hence the radius of the circular path that has been travelled is

$$r = \frac{m}{e} \cdot \frac{v_{\perp}}{B} \tag{3}$$

and the period of rotation which is independent of $v_\perp$ is

$$T = \frac{2 \cdot \pi \cdot r}{v_\perp} = \frac{2 \cdot \pi \cdot m}{B \cdot e} \qquad (4)$$

The parallel component of the velocity does not vary. Because of $\alpha_0 << 1$ it is approximately equal for all electrons:

$$v_{\parallel 0} = v_0 \cdot \cos \alpha_0 \approx v_0 \qquad (5)$$

Hence the distance b between the focusing points, using eq. (5), is

$$b = v_{\parallel 0} \cdot T \approx v_0 \cdot T \qquad (6)$$

From the law of conservation of energy follows the relation between the acceleration voltage V_0 and the velocity v_0:

$$\frac{m}{2} \cdot v_0^{\,2} = e \cdot V_0 \qquad (7)$$

Using eq. (7) and eq. (4) one obtains from eq. (6)

$$b = \frac{2 \cdot \pi}{B} \cdot \sqrt{2 \cdot \frac{m}{e} \cdot V_0} \qquad (8)$$

and because of $b = \frac{2 \cdot \pi \cdot R}{4}$ one obtains

$$B = \frac{4}{R} \cdot \sqrt{2 \cdot \frac{m}{e} \cdot V_0} = 1.48 \cdot 10^{-2} \, \frac{Vs}{m^2} \qquad (9)$$

2. Determination of B_1:

Analogous to eq. (2)

$$m \cdot \frac{v_0^2}{R} = e \cdot v_0 \cdot B_1 \qquad (10)$$

must hold.

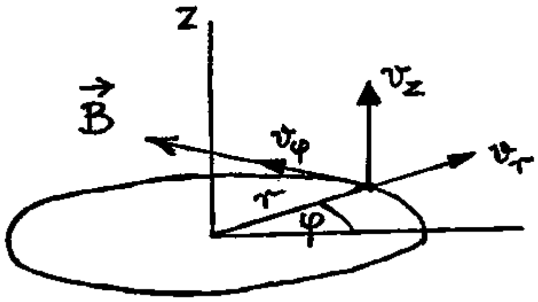
From eq. (7) follows

$$B_1 = \frac{1}{R} \cdot \sqrt{2 \cdot \frac{m}{e} \cdot V_0} = 0.37 \cdot 10^{-2} \, \frac{Vs}{m^2} \qquad (11)$$

3. Finiteness of r_1 and direction of the drift velocity

In the magnetic field the lines of force are circles with their centres on the symmetry axis (z-axis) of the toroid.

In accordance with the symmetry of the problem, polar coordinates r and φ are introduced into the plane perpendicular to the z-axis (see figure below) and the occurring vector quantities (velocity, magnetic field $\vec{B}$, Lorentz force) are divided into the corresponding components.



Since the angle of aperture of the beam can be neglected examine a single electron injected tangentially into the toroid with velocity v_0 on radius R.

In a static magnetic field the kinetic energy is conserved, thus

$$E = \frac{m}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) = \frac{m}{2} v_0^2 \tag{12}$$

The radial points of inversion of the electron are defined by the condition

$$v_r = 0$$

Using eq. (12) one obtains

$$v_0^2 = v_\varphi^2 + v_z^2 \tag{13}$$

Such an inversion point is obviously given by

$$r = R \cdot \left(v_\varphi = v_0,\ v_r = 0,\ v_z = 0\right).$$

To find further inversion points and thus the maximum radial deviation of the electron the components of velocity v_φ and v_z in eq. (13) have to be expressed by the radius.

v_φ will be determined by the law of conservation of angular momentum. The Lorentz force obviously has no component in the φ - direction (parallel to the magnetic field). Therefore it cannot produce a torque around the z-axis. From this follows that the z-component of the angular momentum is a constant, i.e. $L_z = m \cdot v_\varphi \cdot r = m \cdot v_0 \cdot R$ and

therefore $v_\varphi = v_0 \cdot \frac{R}{r}$ (14)

v_z will be determined from the equation of motion in the z-direction. The z-component of the Lorentz force is $F_z = - e \cdot B \cdot v_r$. Thus the acceleration in the z-direction is

$$a_z = -\frac{e}{m} \cdot B \cdot v_r \, . \qquad (15).$$

That means, since B is assumed to be constant, a change of v_z is related to a change of r as follows:

$$\Delta v_z = -\frac{e}{m} \cdot B \cdot \Delta r$$

Because of $\Delta r = r - R$ and $\Delta v_z = v_z$ one finds

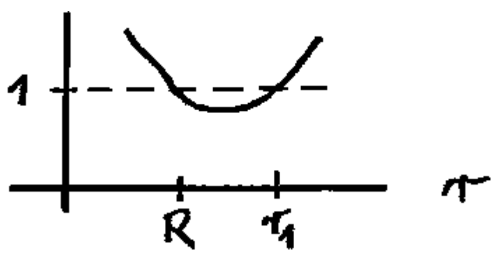
$$v_z = -\frac{e}{m} \cdot B \cdot (r - R) \qquad (16)$$

Using eq. (14) and eq. (15) one obtains for eq. (13)

$$1 = \left(\frac{R}{r}\right)^2 + A^2 \cdot \left(\frac{r}{R} - 1\right)^2 \qquad (17)$$

where $A = \frac{e}{m} \cdot B \cdot \frac{R}{v_0}$

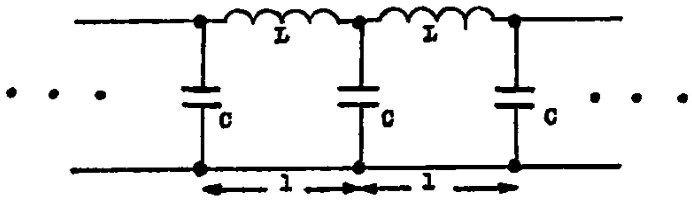
Discussion of the curve of the right side of eq. (17) gives the qualitative result shown in the following diagram:



Hence r_1 is finite. Since $R \le r \le r_1$ eq. (16) yields $v_z < 0$. Hence the drift is in the direction of the negative z-axis.

Problem 3: Infinite LC-grid

When sine waves propagate in an infinite LC-grid (see the figure below) the phase of the ac-voltage across two successive capacitors differs by Φ.



a) Determine how Φ depends on ω, L and C (ω is the angular frequency of the sine wave).

b) Determine the velocity of propagation of the waves if the length of each unit is ℓ .

c) State under what conditions the propagation velocity of the waves is almost independent of ω. Determine the velocity in this case.

d) Suggest a simple mechanical model which is an analogue to the above circuit and derive equations which establish the validity of your model.

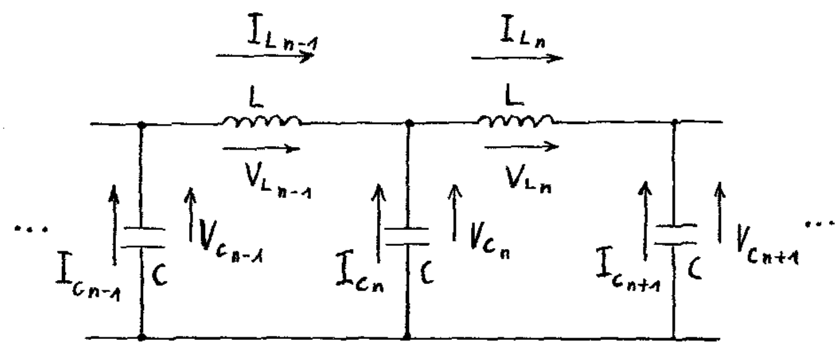
Formulae:

$$\cos\alpha - \cos\beta = -2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

Solution of problem 3:

a)



Current law: $\quad I_{L_{n-1}} + I_{C_n} - I_{L_n} = 0 \quad (1)$

Voltage law: $\quad V_{C_{n-1}} + V_{L_{n-1}} - V_{C_n} = 0 \quad (2)$

Capacitive voltage drop: $V_{C_{n-1}} = \frac{1}{\omega \cdot C} \cdot \tilde{I}_{C_{n-1}}$ (3)

Note: In eq. (3) $\tilde{I}_{C_{n-1}}$ is used instead of $I_{C_{n-1}}$ because the current leads the voltage by 90°.

Inductive voltage drop: $V_{L_{n-1}} = \omega \cdot L \cdot \tilde{I}_{L_{n-1}}$ (4)

Note: In eq. (4) $\tilde{I}_{L_{n-1}}$ is used instead of $I_{L_{n-1}}$ because the current lags behind the voltage by 90°.

The voltage V_{C_n} is given by: $V_{C_n} = V_0 \cdot \sin\left(\omega \cdot t + n \cdot \varphi\right)$ (5)

Formula (5) follows from the problem.

From eq. (3) and eq. (5): $I_{C_n} = \omega \cdot C \cdot V_0 \cdot \cos\left(\omega \cdot t + n \cdot \varphi\right)$ (6)

From eq. (4) and eq. (2) and with eq. (5)

$$I_{L_{n-1}} = \frac{V_0}{\omega \cdot L} \cdot \left[2 \cdot \sin\left(\omega \cdot t + \left(n - \frac{1}{2} \right) \cdot \varphi \right) \cdot \sin\frac{\varphi}{2} \right] \quad (7)$$

$$I_{L_n} = \frac{V_0}{\omega \cdot L} \cdot \left[2 \cdot \sin\left(\omega \cdot t + \left(n + \frac{1}{2} \right) \cdot \varphi \right) \cdot \sin\frac{\varphi}{2} \right] \quad (8)$$

Eqs. (6), (7) and (8) must satisfy the current law. This gives the dependence of φ on ω, L and C.

$$0 = V_0 \cdot \omega \cdot C \cdot \cos\left(\omega \cdot t + n \cdot \varphi\right) + 2 \cdot \frac{V_0}{\omega \cdot L} \cdot \sin\frac{\varphi}{2} \cdot \left[2 \cdot \cos\left(\omega \cdot t + n \cdot \varphi\right) \cdot \sin\left(-\frac{\varphi}{2} \right) \right]$$

This condition must be true for any instant of time. Therefore it is possible to divide by $V_0 \cdot \cos(\omega \cdot t + n \cdot \varphi)$.

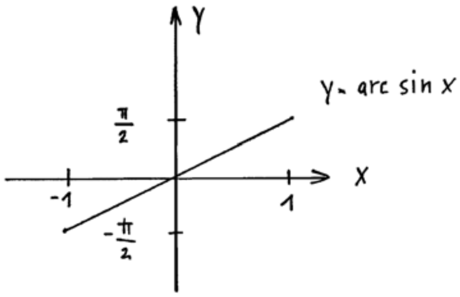
Hence $\omega^2 \cdot L \cdot C = 4 \cdot \sin^2\left(\frac{\varphi}{2}\right)$. The result is

$$\varphi = 2 \cdot \arcsin\left(\frac{\omega \cdot \sqrt{L \cdot C}}{2} \right) \quad \text{with} \quad 0 \le \omega \le \frac{2}{\sqrt{L \cdot C}} \quad (9).$$

b) The distance ℓ is covered in the time Δt thus the propagation velocity is

$$v = \frac{\ell}{\Delta t} = \frac{\omega \cdot \ell}{\varphi} \quad \text{or} \quad v = \frac{\omega \cdot \ell}{2 \cdot \arcsin\left(\frac{\omega \cdot \sqrt{L \cdot C}}{2} \right)} \quad (10)$$

c)



Slightly dependent means arc sin $\left(\frac{\omega \cdot \sqrt{L \cdot C}}{2}\right) \sim \omega$, since v is constant in that case.

This is true only for small values of ω. That means $\frac{\omega \cdot \sqrt{L \cdot C}}{2} \ll 1$ and therefore

$$v_0 = \frac{\ell}{\sqrt{L \cdot C}} \tag{11}$$

d) The energy is conserved since only inductances and capacitances are involved. Using the terms of a) one obtains the capacitive energy

$$W_C = \sum_n \frac{1}{2} \cdot C \cdot {V_{C_n}}^2 \tag{12}$$

and the inductive energy

$$W_L = \sum_n \frac{1}{2} \cdot L \cdot {I_{L_n}}^2 \tag{13}$$

From this follows the standard form of the law of conservation of energy

$$W_C = \sum_n \frac{1}{2} \left(C \cdot {V_{C_n}}^2 + L \cdot {I_{L_n}}^2 \right) \tag{14}$$

The relation to mechanics is not recognizable in this way since two different physical quantities (V_{C_n} and I_{L_n}) are involved and there is nothing that corresponds to the relation between the locus x and the velocity v = $\dot{x}$.

To produce an analogy to mechanics the energy has to be described in terms of the charge Q, the current I = $\dot{Q}$ and the constants L and C. For this purpose the voltage V_{C_n} has to be expressed in terms of the charges Q_{L_n} passing through the coil.

One obtains:

$$W = \sum_n \left[\underbrace{\frac{L}{2} \cdot \dot{Q}_{L_n}^{\,2}}_{A} + \underbrace{\frac{1}{2 \cdot C} \left(Q_{L_n} - Q_{L_{n-1}} \right)^2}_{B} \right] \tag{15}$$

Mechanical analogue:

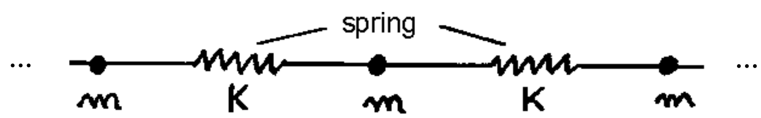
A (kinetic part): $\dot{Q}_{L_n} \longrightarrow v_n$; $L \longrightarrow m$

B (potential part): $Q_{L_n} \longrightarrow x_n$

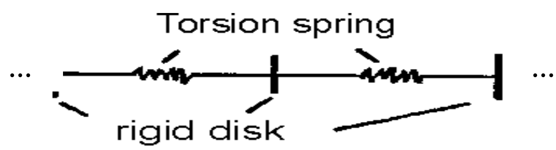
x_n: displacement and v_n: velocity.

However, Q_{L_n} could equally be another quantity (e.g. an angle). L could be e.g. a moment of inertia.

From the structure of the problems follows: Interaction only with the nearest neighbour (the force rises linearly with the distance). A possible model could be:



Another model is:



Experimental Problems

Problem 4: Refractive indices

Find the refractive indices of a prism, n_p, and a liquid, n_l. Ignore dispersion.

a) Determine the refractive index n_p of a single prism by <u>two</u> different experimental methods.

Illustrate your solution with accurate diagrams and deduce the relations necessary to calculate the refractive index. (One prism only should be used).

b) Use two identical prisms to determine the refractive index n_L of a liquid with $n_L < n_p$. Illustrate your solution with accurate diagrams and deduce the relations necessary to calculate the refractive index.

Apparatus:

Two identical prisms with angles of 30°, 60° and 90°; a set square, a glass dish, a round table, a liquid, sheets of graph paper, other sheets of paper and a pencil.

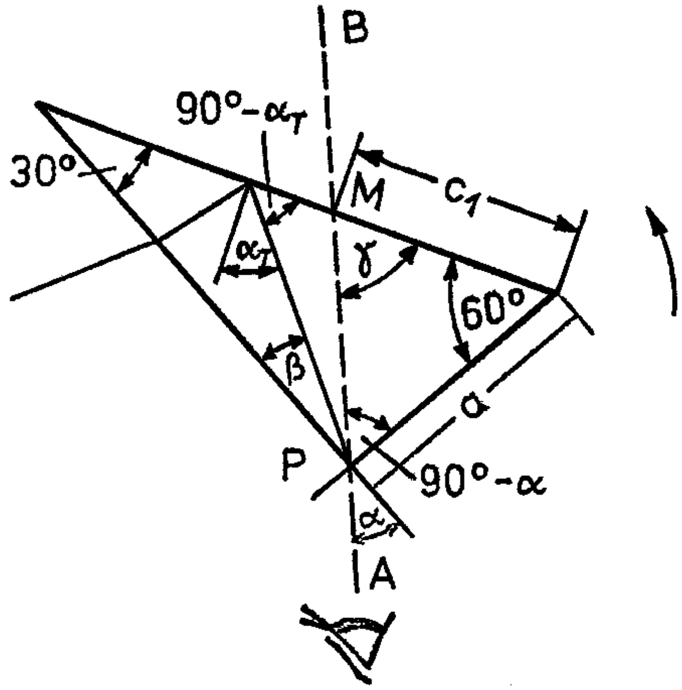
Formulae: $\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$

Additional remarks: You may mark the opaque sides of the prisms with a pencil. The use of the lamp is optional.

Solution of problem 4:

a) Calculation of the refractive index of the prism

First method:



Draw a straight line A – B on a sheet of paper and let this be your line of sight. Place the prism with its rectangular edge facing you onto the line (at point P on the line). Now turn the prism in the direction of the arrow until the dark edge of total reflection which can be seen in the short face of the prism coincides with the 90° edge of the prism. Mark a point M and measure the length c_1. Measure also the length of the short face of the prism.

The following equations apply:

$$\sin\alpha_T = \frac{1}{n_p} \tag{1}$$

$$\frac{\sin\alpha}{\sin\beta} = n_p \tag{2}$$

$$\beta = 60° - \alpha_T \tag{3}$$

$$\gamma = 30° + \alpha \tag{4}$$

$$\frac{\sin\gamma}{\sin(90° - \alpha)} = \frac{a}{c_1} \tag{5}$$

From eq. (5) follows with eq. (4) and the given formulae:

$$\frac{a}{c_1}\cdot\cos\alpha = \sin(30° + \alpha) = \frac{1}{2}\cdot\cos\alpha + \frac{1}{2}\cdot\sqrt{3}\cdot\sin\alpha$$

$$\sin\alpha = \frac{2a - c_1}{2\cdot\sqrt{a^2 - a\cdot c_1 + c_1^{\,2}}} \tag{6}$$

From eqs. (2), (3) and (1) follows:

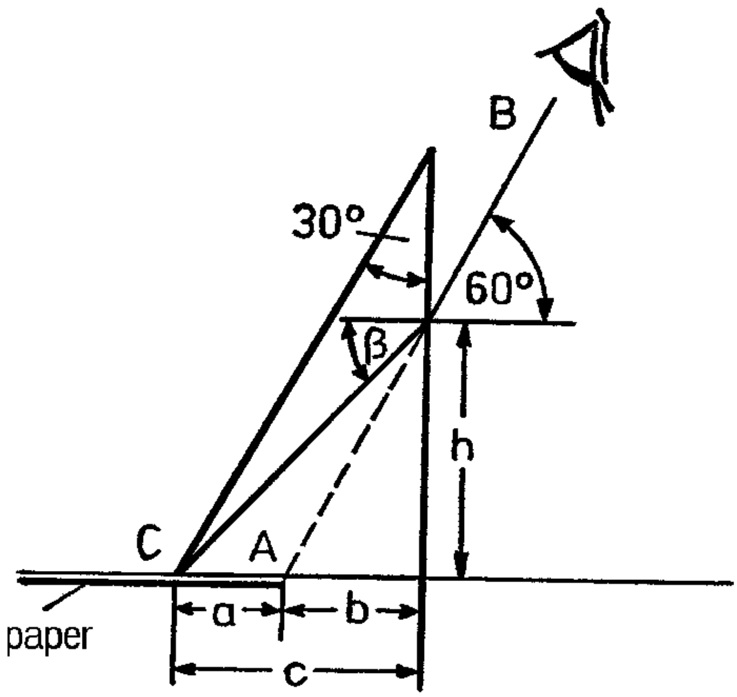
$$\sin\alpha = n_p\cdot\sin(60° - \alpha_T) = \frac{n_p}{2}\cdot\left(\sqrt{3}\cdot\cos\alpha_T - \sin\alpha_T\right)$$

$$n_p = +\left\{\frac{1}{3}\cdot(2\cdot\sin\alpha + 1)^2 + 1\right\}^{1/2} \tag{7}$$

When measuring c_1 and a one notices that within the error limits of ± 1 mm a equals c_1.

Hence: $\sin\alpha = \frac{1}{2}$ and $n_p = 1.53$. (8)

Second method:



Place edge C of the prism on edge A of a sheet of paper and look along the prism hypotenuse at edge A so that your direction of sight B-A and the table surface form an angle of 60°. Then shift the prism over the edge of the paper into the position shown, such that prism edge C can be seen inside the prism collinear with edge A of the paper outside the prism. The direction of sight must not be changed while the prism is being displaced.

The following equations apply:

$$\left.\begin{array}{l}\tan\beta = \frac{h}{c} \\ \tan 60° = \sqrt{3} = \frac{h}{b}\end{array}\right\} => h = b \cdot \sqrt{3} = \frac{c \cdot \sin\beta}{\sqrt{1-\sin^2\beta}} \tag{9}$$

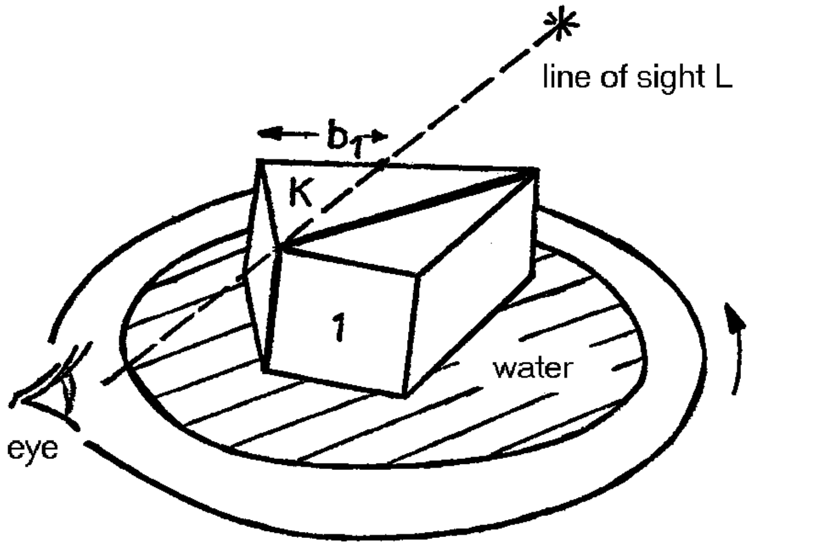
$$\sin\beta = \sin 60° \cdot \frac{1}{n_p} = \frac{\sqrt{3}}{2 \cdot n_p} \tag{10}$$

$$n_p = \frac{1}{2} \cdot \sqrt{\left(\frac{c}{b}\right)^2 + 3} \tag{11}$$

With the measured values c = 29 mm and b = 11.5 mm, it follows

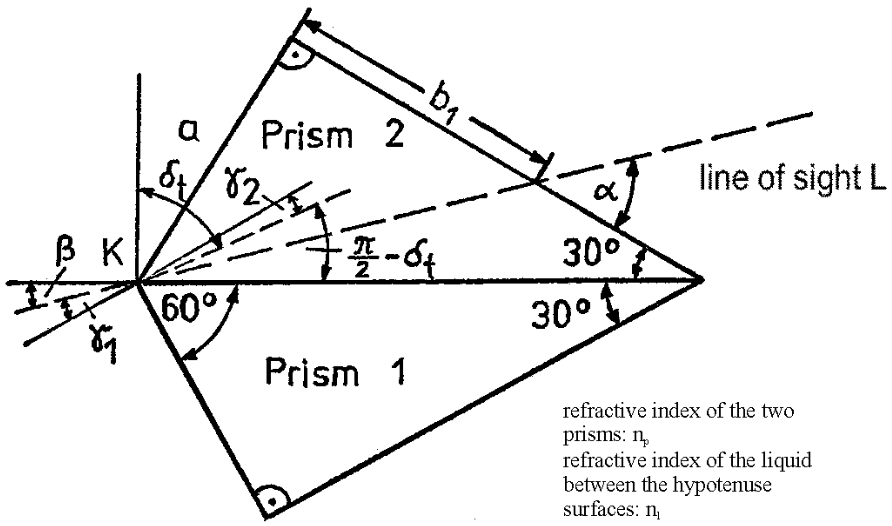
$$n_p = 1.53. \tag{12}$$

b) Determination of the refractive index of the liquid by means of two prisms



Place the two prisms into a glass dish filled with water as shown in the figure above. Some water will rise between the hypotenuse surfaces. By pressing and moving the prisms slightly against each other the water can be made to cover the whole surface. Look over the 60° edges of the prisms along a line of sight L (e.g. in the direction of a fixed point on an illuminated wall). Turn the glass dish together with the two prisms in such a way that the dark shadow of total reflection which can be seen in the short face of prism 1 coincides with the 60° edge of that prism (position shown in the figure below).

While turning the arrangement take care to keep the 60° edge (point K) on the line of sight L. In that position measure the length b_1 with a ruler (marking, reading). The figure below illustrates the position described.



refractive index of the two prisms: n_p
refractive index of the liquid between the hypotenuse surfaces: n_l

If the refractive index of the prism is known (see part a) the refractive index of the liquid may be calculated as follows:

$$\sin\alpha = \frac{a}{\sqrt{a^2 + b_1^{\,2}}} \qquad (13)$$

$$\beta = \alpha - 30°;\ \ \gamma_1 = 30° - \beta = 60° - \alpha \qquad (14, 15)$$

$$\frac{\sin\gamma_1}{\sin\gamma_2} = n_p \quad \text{refraction at the short face of prism 1.} \qquad (16)$$

The angle of total reflection δ_t at the hypotenuse surface of prism 1 in the position described is:

$$\frac{\pi}{2} - \delta_t = 30° - \gamma_2 \qquad (17)$$

$$\delta_t = 60° + \arcsin\left(\frac{\sin\gamma_1}{n_p}\right) \qquad (18)$$

From this we can easily obtain n_1:

$$n_1 = n_p \cdot \sin\delta_t = n_p \cdot \sin\left\{60° + \arcsin\frac{\sin\gamma_1}{n_p}\right\} \qquad (19)$$

Numerical example for water as liquid:

b_1 = 1.9 cm; α = 55.84°; γ_1 = 4.16°; δ_t = 62.77°; a = 2.8 cm; with n_p = 1.5 follows

n_1= 1.33. (20)

Grading Scheme

Theoretical problems

Problem 1: Ascending moist art	
part 1	2
part 2	2
part 3	2
part 4	2
part 5	2
	10

Problem 2: Electron in a magnetic field	
part 1	3
part 2	1
part 3	6
	10

Problem 3: Infinite LC-grid	
part a	4
part b	1
part c	1
part d	4
	10

Problem 4: Refractive indices	
part a, first method	5
part a, second method	5
part b	10
	20

19th International Physics Olympiad - 1988

Bad Ischl / Austria

THEORY 1
Spectroscopy of Particle Velocities

Basic Data

The absorption and emission of a photon is a reversible process. A good example is to be found in the excitation of an atom from the ground state to a higher energy state and the atoms´ subsequent return to the ground state. In such a case we may detect the absorption of a photon from the phenomenon of spontaneous emission or fluorescence. Some of the more modern instrumentation make use of this principle to identify atoms, and also to measure or calculate the value of the velocity in the velocity spectrum of the electron beam.

In an idealised experiment (see fig. 19.1) a single-charged ion travels in the opposite direction to light from a laser source with velocity v. The wavelength of light from the laser source is adjustable. An ion with velocity Zero can be excited to a higher energy state by the application of laser light having a wavelength of $\lambda = 600$ nm. If we excite a moving ion, our knowledge on Dopplers´ effect tells us that we need to apply laser light of a wavelength other than the value given above.

There is given a velocity spectrum embracing velocity magnitude from $v_1 = 0\,\frac{m}{s}$ to $v_2 = 6{,}000\,\frac{m}{s}$. (see fig. 19.1)

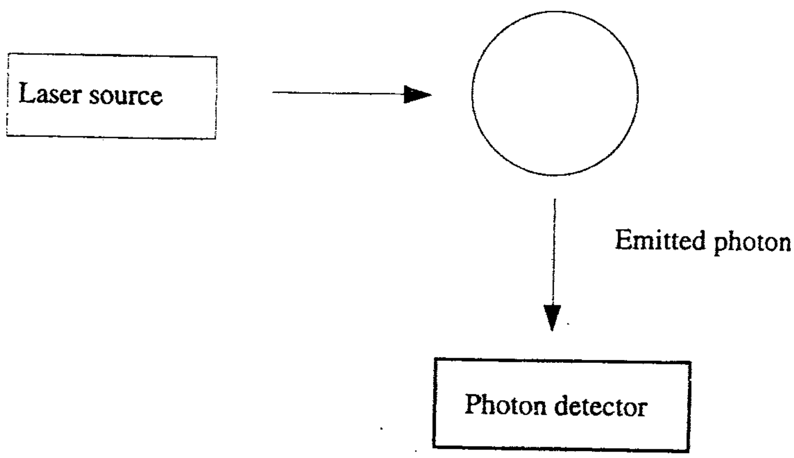


Fig. 19.1

Questions

1.1

1.1.1

What range of wavelength of the laser beam must be used to excite ions of all velocities in the velocity spectrum given above ?

1.1.2

A rigorous analysis of the problem calls for application of the principle from the theory of special relativity

$$\nu' = \nu \cdot \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Determine the error when the classical formula for Dopplers´ effect is used to solve the problem.

1.2

Assuming the ions are accelerated by a potential U before excited by the laser beam, determine the relationship between the width of the velocity spectrum of the ion beam and the accelerating potential. Does the accelerating voltage increase or decrease the velocity spectrum width ?

1.3

Each ion has the value $\frac{e}{m} = 4 \cdot 10^6 \frac{A \cdot s}{kg}$, two energy levels corresponding to wavelength $\lambda^{1)} = 600 \text{ nm}$ and wavelength $\lambda^{2)} = \lambda^{1)} + 10^{-3} \text{ nm}$. Show that lights of the two wavelengths used to excite ions overlap when no accelerating potential is applied. Can accelerating voltage be used to separate the two spectra of laser light used to excite ions so that they no longer overlap ? If the answer is positive, calculate the minimum value of the voltage required.

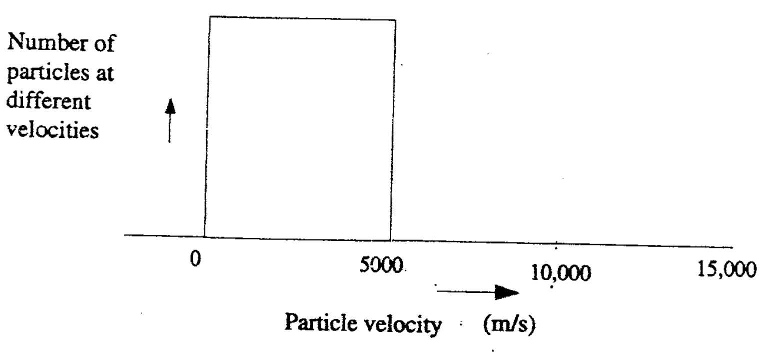


Fig. 19.2

Solution

1.1

1.1.1

Let v be the velocity of the ion towards the laser source relative to the laser source,
ν' the frequency of the laser light as observed by the observer moving with the ion (e.g. in the frame in which the velocity of the ion is 0) and
ν the frequency of the laser light as observed by the observer at rest with respect to the laser source.

Classical formula for Doppler´s effect is given as

$$\nu' = \nu \cdot \left(1 + \frac{v}{c}\right) \qquad (1)$$

Let ν^* be the frequency absorbed by an ion (characteristic of individual ions) and
ν_L be the frequency of the laser light used to excite an ion at rest,
hence:

$\nu^* = \nu_L$

For a moving ion, the frequency used to excite ions must be lower than ν^*.

Let ν_H be the frequency used to excite the moving ion.

When no accelerating voltage is applied

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
ν_H	0	ν^*	λ_1
ν_L	$v = 6 \,.\, 10^3$ m/s	ν^*	λ_2

$\nu_L < \nu_H$

$\nu_L = \nu^*$

Calculation of frequency ν_H absorbed by moving ions.

$$\nu^* = \nu_L \cdot \left(1 + \frac{v}{c}\right) \qquad \text{where } \nu^* = \nu_H = 5 \,.\, 10^{14} \text{ Hz and } v = 6 \,.\, 10^3 \text{ m/s} \qquad (2)$$

The difference in the values of the frequency absorbed by the stationary ion and the ion moving with the velocity v $\Delta\nu = \nu_H - \nu_L$
The difference in the values of the wavelengths absorbed by the stationary ion and the ion moving with the velocity v $\Delta\lambda = \lambda_L - \lambda_H$
(higher frequency implies shorter wavelength)

$$\lambda_L - \lambda_H = \frac{c}{\nu_L} - \frac{c}{\nu_H}$$

from (2)

$$\lambda_L - \lambda_H = \frac{c}{\nu^*}\cdot\left(1+\frac{v}{c}\right) - \frac{c}{\nu^*} = \frac{v}{\nu^*}$$

In this case

$$\boxed{\lambda_L - \lambda_H = \frac{6\cdot 10^3}{5\cdot 10^{14}}\ \mathrm{m} = 12\cdot 10^{-3}\ \mathrm{nm}}$$

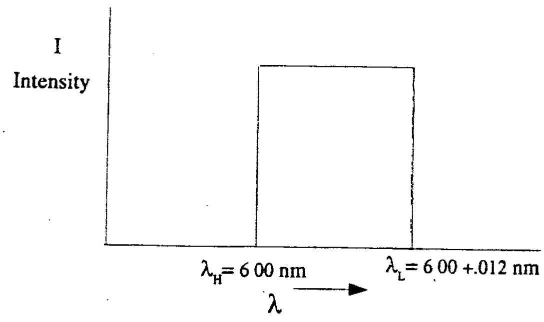


Fig. 19.3 Spectrum of laser light used to excite ions

1.1.2

The formula for calculation of ν´ as observed by the observer moving towards light source based on the principle of the theory of special relativity,

$$\nu' = \nu\cdot\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

where v is the magnitude of the velocity of the observer towards the light source,
ν´ is the frequency absorbed by the ion moving with the velocity v towards the light source (also observed by the observer moving with velocity v towards the laser source) and
ν is the frequency of laser light as observed by an observer at rest.

(To put in a metaphoric way, the moving ion "sees" the laser light of frequency ν´ even though the scientist who operates the laser source insists that he is sending a laser beam of frequency ν).

$$\nu' = \nu\cdot\sqrt{\left(1+\frac{v}{c}\right)\cdot\left(1+\frac{v}{c}+\frac{v^2}{c^2}+..............\right)} = \nu\cdot\sqrt{\left(1+\frac{v}{c}\right)^2+\left(1+\frac{v}{c}\right)\cdot\frac{v^2}{c^2}+.............}$$

$$\nu' = \nu\cdot\left(1+\frac{v}{c}\right)\cdot\left[1+\frac{v^2}{c^2}\cdot\frac{1}{1+\frac{v}{c}}+..........\right]^{\frac{1}{2}} = \nu\cdot\left(1+\frac{v}{c}\right)\cdot\left[1+\frac{v^2}{2\cdot c^2}\cdot\frac{1}{\left(1+\frac{v}{c}\right)}+..........\right]$$

The second term in the brackets represents the error if the classical formula for Doppler´s effect is employed.

$$\frac{v}{c} = 2\cdot 10^{-5}$$

$$\boxed{\frac{v^2}{2\cdot c^2}\cdot\frac{1}{\left(1+\frac{v}{c}\right)} = \frac{1}{2}\cdot\frac{4\cdot 10^{-10}}{1+2\cdot 10^{-5}} \approx 2\cdot 10^{-10}}$$

The error in the application of classical formula for Doppler´s effect however is of the order of the factor 2.10^{-10}. This means that classical formula for Doppler´s effect can be used to analyze the problem without loosing accuracy.

1.2 When acceleration voltage is used

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
ν_H'	v_H'	$\nu^* = 5 \cdot 10^{14}$ Hz	λ_H'
ν_L'	v_L'	$\nu^* = 5 \cdot 10^{14}$ Hz	λ_L'

Lowest limit of the kinetic energy of ions $\frac{1}{2}\cdot m\cdot(v_L')^2 = e\cdot U$ and $v_L' = \sqrt{\frac{2\cdot e\cdot U}{m}}$

Highest limit of the kinetic energy of ions $\frac{1}{2}\cdot m\cdot(v_H')^2 = \frac{1}{2}\cdot m\cdot v^2 + e\cdot U$

and $v_H' = \sqrt{v^2 + \frac{2\cdot e\cdot U}{m}}$

Spectrum width of velocity spectrum $\boxed{v_H' - v_L' = \sqrt{v^2 + \frac{2\cdot e\cdot U}{m}} - \sqrt{\frac{2\cdot e\cdot U}{m}}}$ .... (3)

(Note that the final velocity of accelerated ions is not the sum of v and $\sqrt{\frac{2\cdot e\cdot U}{m}}$ as velocity changes with time).

In equation (3) if $\sqrt{\frac{2\cdot e\cdot U}{m}}$ is negligibly small, the change in the width of the spectrum is negligible, by the same token of argument if $\sqrt{\frac{2\cdot e\cdot U}{m}}$ is large or approaches ∞, the width of the spectrum of the light used in exciting the ions becomes increasingly narrow and approaches 0.

1.3

Given two energy levels of the ion, corresponding to wavelength $\lambda^{(1)} = 600$ nm and $\lambda^{(2)} = 600+10^{-2}$ nm

For the sake of simplicity, the following sign notations will be adopted:

The superscript in the bracket indicates energy level (1) or (2) as the case may be. The sign ´ above denotes the case when accelerating voltage is applied, and also the subscripts H and L apply to absorbed frequencies (and also wavelengths) correspond to the high velocity and low velocity ends of the velocity spectrum of the ion beam respectively.

The subscript following λ (or ν) can be either 1 or 2, with number 1 corresponding to lowest velocity of the ion and number 2 the highest velocity of the ion. When no accelerating voltage is applied, the subscript 1 implies that minimum velocity of the ion is 0, and the highest velocity of the ion is 6000 m/s. If accelerating voltage U is applied, number 1 indicates that the wavelength of laser light pertains to the ion of lowest velocity and number 2 indicates the ion of the highest velocity.

Finally the sign * indicates the value of the wavelength (λ^*) or frequency (ν^*) absorbed by the ion (characteristic absorbed frequency).

When no accelerating voltage is applied:

For the first energy level:

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
$\nu_H^{(1)}$ $\nu_L^{(1)}$	0 v=6.10^3 m/s	$\nu^{(1)*} = 5 \,.\, 10^{14}$ Hz $\nu^{(1)*} = 5 \,.\, 10^{14}$ Hz	$\lambda_1^{(1)}$ $\lambda_2^{(1)}$

$\nu_H^{(1)*} = \nu_L^{(1)*} = \nu^{(1)*} = 5 \,.\, 10^{14}$ Hz

Differences in frequencies of laser light used to excite ions $= \nu_H^{(1)} - \nu_L^{(1)}$
Differences of wavelengths of laser light used to excite ions $= \lambda_L^{(1)} - \lambda_H^{(1)}$

$$\frac{v}{\nu_L^{(1)*}} = \frac{6000}{5 \cdot 10^{14}} = 0{,}012 \text{ nm}$$

For the second energy level:

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
$\nu_H^{(2)}$ $\nu_L^{(2)}$	0 v = 6000 m/s	$\nu^{(2)*} = 5 \,.\, 10^{14}$ Hz $\nu^{(2)*} = 5 \,.\, 10^{14}$ Hz	$\lambda_H^{(2)}$ $\lambda_L^{(2)}$

$\nu_H^{(2)*} = \nu_L^{(2)*} = \nu^{(2)*} = 5 \,.\, 10^{14}$ Hz

Differences in frequencies of laser light used to excite ions $\nu_H^{(2)} - \nu_L^{(2)}$
Differences in wavelengths of laser light used to excite ions $\lambda_L^{(2)} - \lambda_H^{(2)}$

This gives $\frac{6000}{5 \cdot 10^{14}} = 0{,}012 \text{ nm}$

Hence the spectra of laser light (absorption spectrum) used to excite an ion at two energy levels overlap as shown in fig. 19.4.

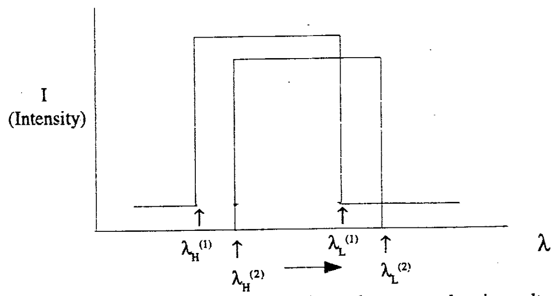


Fig. 19.4 Spectrum of laser light used to excite ions when no accelerating voltage is applied(Absorption Spectrum)

When accelerating voltage is applied:

Let $\lambda_H^{(1)\prime}$ and $\lambda_L^{(1)\prime}$ be the range of the wavelengths used to excite ions in the first energy level, when accelerating voltage is applied. (Note the prime sign to denote the situation in which the accelerating voltage is used), and let $\lambda_H^{(2)\prime}$ and $\lambda_L^{(2)\prime}$ represent the range of the wavelengths used to excite ions in the second energy level also when an accelerating voltage is applied.

Condition for the two spectra not to overlap:

$$\lambda_H^{(2)\prime} \geq \lambda_L^{(1)\prime} \qquad \text{(see fig. 19.4)} \qquad (4)$$

(Keep in mind that lower energy means longer wavelengths and vice versa).

From condition (3):
$$\lambda_L - \lambda_H = \frac{v}{\nu^*} \qquad (5)$$

The meanings of this equation is if the velocity of the ion is v, the wavelength which the ion "sees" is λ_L, when λ_H is the wavelength which the ion of zero-velocity "sees".

Equation (5) may be rewritten in the context of the applications of accelerating voltage in order for the two spectra of laser light will not overlap as follows:

$$\lambda_L^{(N)\prime} - \lambda_H^{(N)\prime} = \frac{v'}{\nu^*} \qquad \text{where N is the order of the energy level} \qquad (6)$$

The subscript L relates λ to lowest velocity of the ion which "sees" frequency ν^*. The lowest velocity in this case is $\sqrt{\frac{2 \cdot e \cdot U}{m}}$ and the subscript H relates λ to the highest velocity of the ion, in this case $\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}$.

Equation (6) will be used to calculate

- width of velocity spectrum of the ion accelerated by voltage U
- potential U which results in condition given by (4)

Let us take up the second energy level (lower energy level of the two ones) of the ion first:

$$\lambda_L^{(2)\prime} - \lambda_H^{(2)} = \frac{v'}{\nu^*} \qquad (7)$$

substitute

$$v' = \sqrt{\frac{2 \cdot e \cdot U}{m}}$$

$\lambda_H^{(1)} = 600 + 10^{-3}$ nm

$\nu^* = 5 \,.\, 10^{14}$ Hz

v = 0 m/s

$$\lambda_H^{(2)\prime} = (600 + 0{,}001) \cdot 10^{-9} + \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \text{ m} \qquad (8)$$

Considering the first energy level of the ion

$$\lambda_L^{(1)'} - \lambda_H^{(1)} = \frac{v'}{\nu^*} \quad \text{.......} \quad (9)$$

In this case

$$v' = \sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}$$

$\nu^* = 5 \,.\, 10^{14}$ Hz

v = 6000 m/s

$\lambda_H^{(1)} = 600 \,.\, 10^{-9}$ m

$$\lambda_L^{(1)'} = 600 \cdot 10^{-9} + \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \text{ m} \quad \text{.......} \quad (10)$$

Substitute $\lambda_H^{(2)'}$ from (8) and $\lambda_L^{(1)'}$ from (10) in (4) one gets

$$(600 + 0{,}001) \cdot 10^{-9} + \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \geq 600 \cdot 10^{-9} + \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}}$$

$$500 \geq \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} - \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}}$$

$$500 \geq \sqrt{36 \cdot 10^6 + 2 \cdot 4 \cdot 10^6 \cdot U} - \sqrt{2 \cdot 4 \cdot 10^6 \cdot U}$$

assume that U is of the order of 100 and over,

then $$\sqrt{8 \cdot 10^6 \cdot U} \cdot \left(1 + \frac{9}{4 \cdot U}\right) - \sqrt{8 \cdot 10^6 \cdot U} \leq 500$$

$$\frac{1}{\sqrt{2 \cdot U}} \cdot 9 \cdot 10^3 \leq 500$$

$$\sqrt{2 \cdot U} \geq 324$$

$$\boxed{U \geq 162 \text{ V}}$$

The minimum value of accelerating voltage to avoid overlapping of absorption spectra is approximately 162 V

THEORY 2
Maxwell´s Wheel

Introduction

A cylindrical wheel of uniform density, having the mass M = 0,40 kg, the radius R = 0,060 m and the thickness d = 0,010 m is suspended by means of two light strings of the same length from the ceiling. Each string is wound around the axle of the wheel. Like the strings, the mass of the axle is negligible. When the wheel is turned manually, the strings are wound up until the centre of mass is raised 1,0 m above the floor. If the wheel is allowed to move downward vertically under the pulling force of the gravity, the strings are unwound to the full length of the strings and the wheel reaches the lowest point. The strings then begin to wound in the opposite sense resulting in the wheel being raised upwards.

Analyze and answer the following questions, assuming that the strings are in vertical position and the points where the strings touch the axle are directly below their respective suspending points (see fig. 19.5).

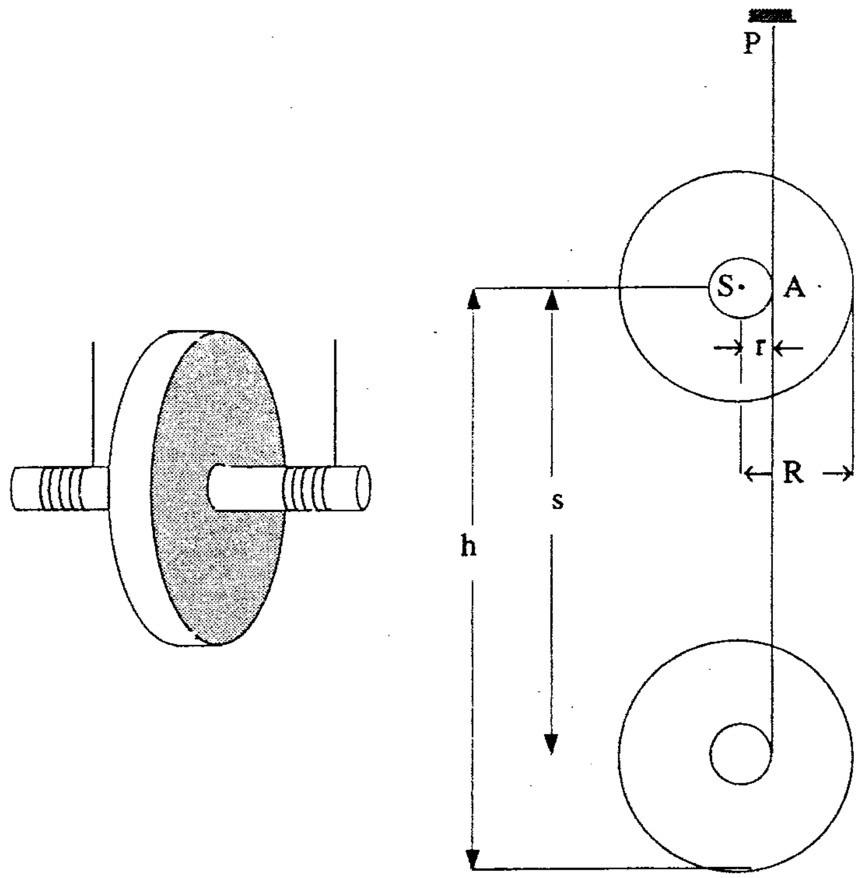


Fig. 19.5

Questions

2.1

Determine the angular speed of the wheel when the centre of mass of the wheel covers the vertical distance s.

2.2

Determine the kinetic energy of the linear motion of the centre of mass E_r after the wheel travels a distance s = 0,50 m, and calculate the ratio between E_r and the energy in any other form in this problem up to this point.

Radius of the axle = 0,0030 m

2.3

Determine the tension in the string while the wheel is moving downward.

2.4

Calculate the angular speed ω' as a function of the angle Φ when the strings begin to unwind themselves in opposite sense as depicted in fig. 19.6.
Sketch a graph of variables which describe the motion (in cartesian system which suits the problem) and also the speed of the centre of mass as a function of Φ.

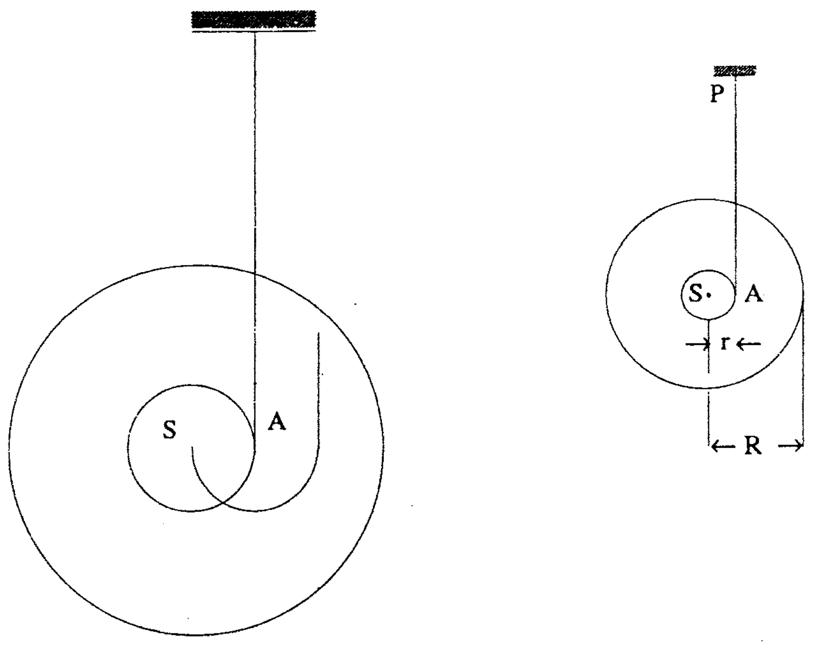


Fig. 19.6

2.5

If the string can withstand a maximum tension T_m = 10 N, find the maximum length of the string which may be unwound without breaking by the wheel.

Solution

2.1

conservation of energy: $$M \cdot g \cdot s = \frac{1}{2} \cdot I_A \cdot \omega^2 \qquad (1)$$

where ω is the angular speed of the wheel and I_A is the moment of inertia about the axis through A.

Note: If we would take the moment of inertia about S instead of A we would have

$$M \cdot g \cdot s = \frac{1}{2} \cdot I_S \cdot \omega^2 + \frac{1}{2} \cdot m \cdot v^2$$

where v is the speed of the centre of mass along the vertical.
This equation is the same as the above one in meanings since

$I_A = I_S + M \cdot r^2$ and $I_S = M \cdot R^2$

From (1) we get $$\omega = \sqrt{\frac{2 \cdot M \cdot g \cdot s}{I_A}}$$

substitute $$I_A = \frac{1}{2} \cdot M \cdot r^2 + M \cdot R^2$$

$$\omega = \sqrt{\frac{2 \cdot g \cdot s}{r^2 + \frac{R^2}{2}}}$$

Putting in numbers we get $$\omega = \sqrt{\frac{2 \cdot 9{,}81 \cdot 0{,}50}{9 \cdot 10^{-6} + \frac{1}{2} \cdot 36 \cdot 10^{-4}}} \approx 72{,}4 \ \frac{\text{rad}}{\text{s}}$$

2.2

Kinetic energy of linear motion of the centre of mass of the wheel is

$$E_T = \frac{1}{2} \cdot M \cdot v^2 = \frac{1}{2} \cdot M \cdot \omega^2 \cdot r^2 = \frac{1}{2} \cdot 0{,}40 \cdot 72{,}4^2 \cdot 9 \cdot 10^{-6} = 9{,}76 \cdot 10^{-3} \text{ J}$$

Potential energy of the wheel

$$E_P = M \cdot g \cdot s = 0{,}40 \cdot 9{,}81 \cdot 0{,}50 = 1{,}962 \text{ J}$$

Rotational kinetic energy of the wheel

$$E_R = \frac{1}{2} \cdot I_S \cdot \omega^2 = \frac{1}{2} \cdot 0{,}40 \cdot 1{,}81 \cdot 10^{-3} \cdot 72{,}4^2 = 1{,}899 \text{ J}$$

$$\frac{E_T}{E_R} = \frac{9{,}76 \cdot 10^{-3}}{1{,}899} = 5{,}13 \cdot 10^{-3}$$

2.3

Let $\frac{T}{2}$ be the tension in each string.

Torque τ which causes the rotation is given by $\tau = M \cdot g \cdot r = I_A \cdot \alpha$

where α is the angular acceleration $\alpha = \frac{M \cdot g \cdot r}{I_A}$

The equation of the motion of the wheel is M.g – T = M.a

Substituting $a = \alpha \cdot r$ and $I_A = \frac{1}{2} \cdot M \cdot r^2 + M \cdot R^2$ we get

$$T = M \cdot g + \frac{M \cdot g \cdot r^2}{\frac{1}{2} \cdot M \cdot R^2 + M \cdot r^2} = M \cdot g \cdot \left(1 + \frac{2 \cdot r^2}{R^2 + 2 \cdot r^2}\right)$$

Thus for the tension $\frac{T}{2}$ in each string we get

$$\frac{T}{2} = \frac{M \cdot g}{2} \cdot \left(1 + \frac{2 \cdot r^2}{R^2 + 2 \cdot r^2}\right) = \frac{0{,}40 \cdot 9{,}81}{2} \cdot \left(1 + \frac{2 \cdot 9 \cdot 10^{-6}}{3{,}6 \cdot 10^{-3} + 2 \cdot 9 \cdot 10^{-6}}\right) = 1{,}96\ \text{N}$$

$$\boxed{\frac{T}{2} = 1{,}96\ \text{N}}$$

2.4

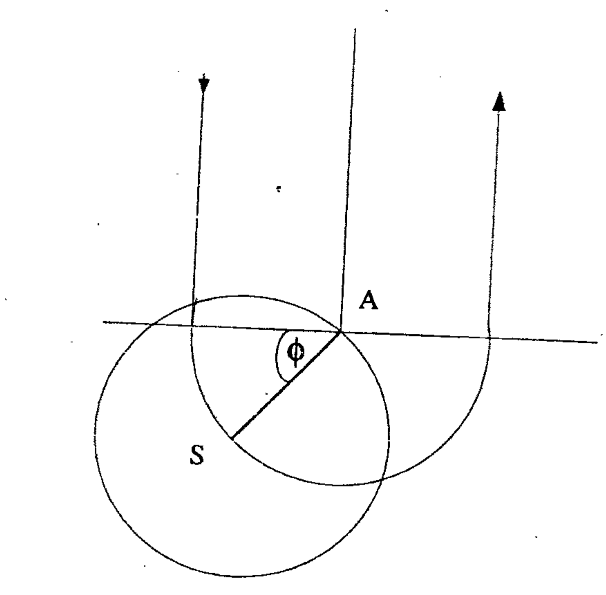


Fig 19.7

After the whole length of the strings is completely unwound, the wheel continues to rotate about A (which is at rest for some interval to be discussed). Let $\dot{\Phi}$ be the angular speed of the centre of mass about the axis through A. The equation of the rotational motion of the wheel about A may be written as $|\tau| = I_A \cdot \ddot{\Phi}$,

where τ is the torque about A, I_A is the moment of inertia about the axis A and $\ddot{\Phi}$ is the angular acceleration about the axis through A.

Hence $M \cdot g \cdot r \cdot \cos\Phi = I_A \cdot \ddot{\phi}$

and $\ddot{\phi} = \frac{M \cdot g \cdot r \cdot \cos\Phi}{I_A}$

Multiplied with $\dot{\Phi}$ gives:

$$\dot{\Phi} \cdot \ddot{\Phi} = \frac{M \cdot g \cdot r \cdot \cos\Phi \cdot \dot{\Phi}}{I_A} \quad \text{or} \quad \frac{1}{2} \cdot \frac{d(\dot{\Phi})^2}{dt} = \frac{M \cdot g \cdot r \cdot \cos\Phi}{I_A} \cdot \frac{d\Phi}{dt}$$

this gives

$$\left(\dot{\Phi}\right)^2 = \frac{2 \cdot M \cdot g \cdot r \cdot \sin\Phi}{I_A} + C$$ [C = arbitrary constant]

If $\Phi = 0$ [s = H] than is $\dot{\Phi} = \omega$

That gives $\omega = \frac{2 \cdot M \cdot g \cdot H}{I_A}$ and therefore $C = \frac{2 \cdot M \cdot g \cdot H}{I_A}$

Putting these results into the equation above one gets

$$\dot{\Phi} = \omega = \sqrt{\frac{2 \cdot M \cdot g \cdot H \cdot \sin\Phi}{I_A} \cdot \left(1 + \frac{r}{H}\right)}$$

For $\frac{r}{H}$ <<< 1 we get:

$$\omega = \omega'_{MAX} = \sqrt{\frac{2 \cdot M \cdot g \cdot H}{I_A}}$$

and

$$v = r \cdot \omega'_{MAX} = r \cdot \sqrt{\frac{2 \cdot M \cdot g \cdot H}{I_A}}$$

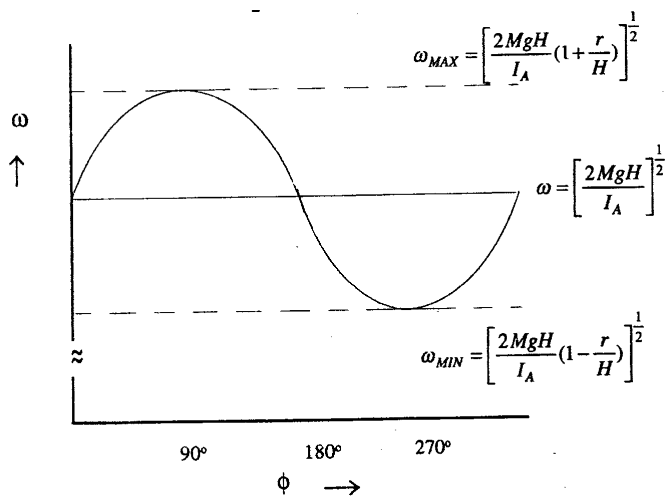


Fig.19.8

Component of the displacement
along x-axis is x = r.sin Φ-r
along y-axis is y = r.cos Φ-r

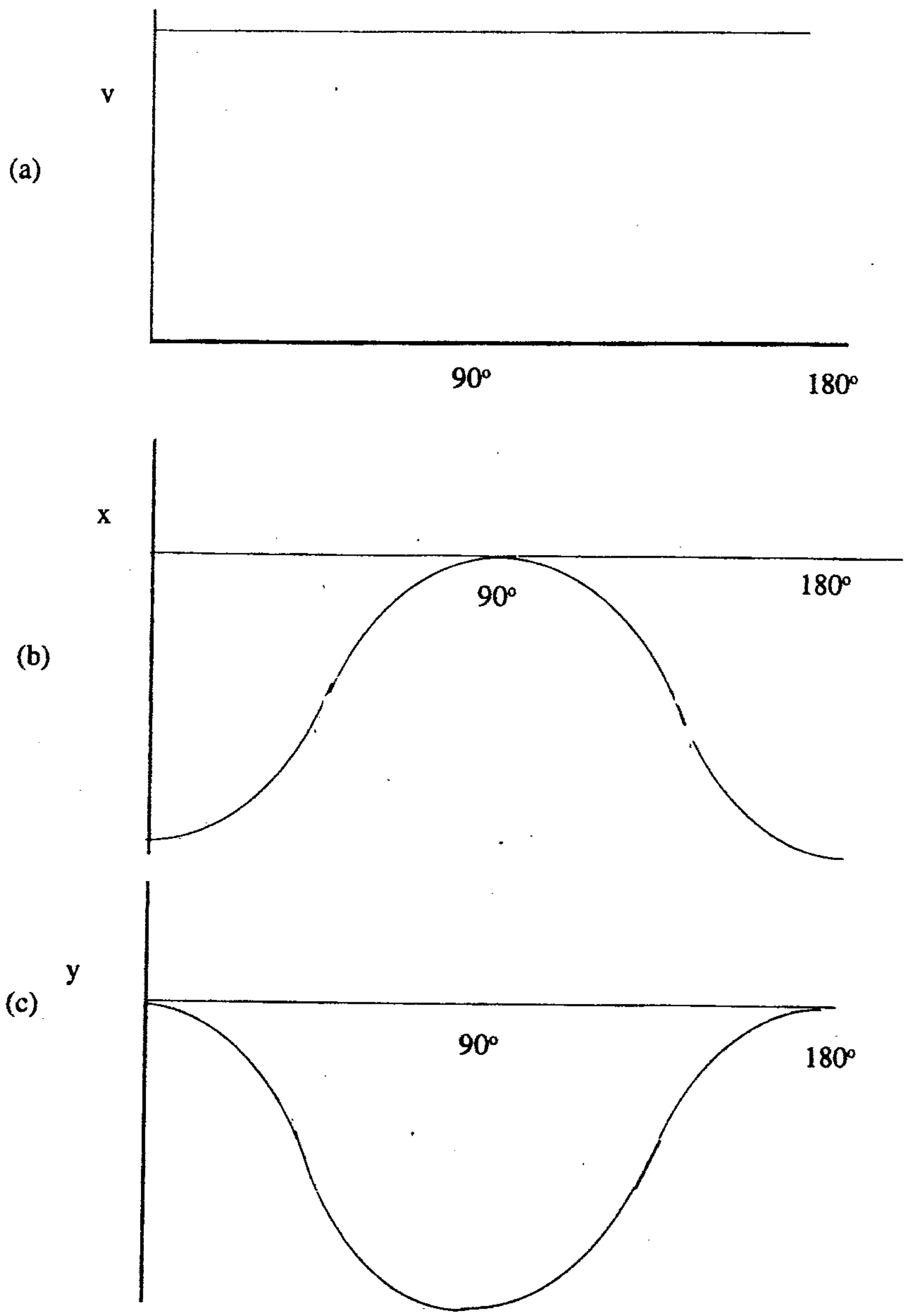


Fig.19.9

2.5

Maximum tension in each string occurs $\dot{\Phi} = \omega'_{MAX}$

The equation of the motion is $T_{MAX} - M \cdot g = M \cdot (\omega'_{MAX})^2 \cdot r$

Putting in T = 20 N and $\omega'_{MAX} = \sqrt{\frac{2 \cdot M \cdot g \cdot s}{I_A}}$ (where s is the maximum length of the strings supporting the wheel without breaking) and $I_A = M \cdot \left(\frac{R^2}{2} + r^2 \right)$ the numbers one gets:

$$20 = 0{,}40 \cdot 9{,}81 \cdot \left(1 + \frac{4 \cdot 3 \cdot 10^{-3} \cdot s}{36 \cdot 10^{-4} + 2 \cdot 9 \cdot 10^{-6}} \right)$$

This gives: s = 1,24 m

The maximum length of the strings which support maximum tension without breaking is

1,24 m .

THEORY 3
Recombination of Positive and Negative Ions in Ionized Gas

Introduction

A gas consists of positive ions of some element (at high temperature) and electrons. The positive ion belongs to an atom of unknown mass number Z. It is known that this ion has only one electron in the shell (orbit).

Let this ion be represented by the symbol $A^{(Z-1)+}$

Constants:

electric field constant	$\varepsilon_O = 8{,}85 \cdot 10^{-12} \ \frac{A \cdot s}{V \cdot m}$
elementary charge	$e = \pm 1{,}602 \cdot 10^{-19} \ A \cdot s$
	$q^2 = \frac{e^2}{4 \cdot \pi \cdot \varepsilon_O} = 2{,}037 \cdot 10^{-28} \ J \cdot m$
Planck´s constant	$\hbar = 1{,}054 \cdot 10^{-34} \ J \cdot s$
(rest) mass of an electron	$m_e = 9{,}108 \cdot 10^{-31} \ kg$
Bohr´s atomic radius	$r_B = \frac{\hbar}{m \cdot q^2} = 5{,}92 \cdot 10^{-11} \ m$
Rydberg´s energy	$E_R = \frac{q^2}{2 \cdot r_B} = 2{,}180 \cdot 10^{-18} \ J$
(rest) mass of a proton	$m_P \cdot c^2 = 1{,}503 \cdot 10^{-10} \ J$

Questions:

3.1

Assume that the ion which has just one electron left the shell.
$A^{(Z-1)+}$ is in the ground state.

In the lowest energy state, the square of the average distance of the electron from the nucleus or r^2 with components along x-, y- and z-axis being $(\Delta x)^2$, $(\Delta y)^2$ and $(\Delta z)^2$ respectively and $r_O^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ and also the square of the average momentum by $p_O^2 = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2$, whereas $\Delta p_x \ge \frac{\hbar}{2 \cdot \Delta x}$, $\Delta p_y \ge \frac{\hbar}{2 \cdot \Delta y}$ and $\Delta p_z \ge \frac{\hbar}{2 \cdot \Delta z}$.

Write inequality involving $(p_O)^2 \cdot (r_O)^2$ in a complete form.

3.2

The ion represented by $A^{(Z-1)+}$ may capture an additional electron and consequently emits a photon.
Write down an equation which is to be used for calculation the frequency of an emitted photon.

3.3

Calculate the energy of the ion $A^{(Z-1)+}$ using the value of the lowest energy. The calculation should be approximated based on the following principles:

3.3.A

The potential energy of the ion should be expressed in terms of the average value of $\frac{1}{r}$.

(ie. $\frac{1}{r_o}$; r_0 is given in the problem).

3.3.B

In calculating the kinetic energy of the ion, use the average value of the square of the momentum given in 3.1 after being simplified by $(p_o)^2 \cdot (r_o)^2 \approx (\hbar)^2$

3.4

Calculate the energy of the ion $A^{(Z-2)+}$ taken to be in the ground state, using the same principle as the calculation of the energy of $A^{(Z-1)+}$. Given the average distance of each of the two electrons in the outermost shell (same as r_0 given in 3.3) denoted by r_1 and r_2, assume the average distance between the two electrons is given by r_1+r_2 and the average value of the square of the momentum of each electron obeys the principle of uncertainty ie.

$p_1^2 \cdot r_1^2 \approx \hbar^2$ and $p_2^2 \cdot r_2^2 \approx \hbar^2$

hint: Make use of the information that in the ground state $r_1 = r_2$

3.5

Consider in particular the ion $A^{(Z-2)+}$ is at rest in the ground state when capturing an additional electron and the captured electron is also at rest prior to the capturing. Determine the numerical value of Z, if the frequency of the emitted photon accompanying electron capturing is $2{,}057 \cdot 10^{17}$ rad/s. Identify the element which gives rise to the ion.

Solution

3.1

$r_0^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

$p_0^2 = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2$

since

$\Delta p_x \geq \frac{\hbar}{2 \cdot \Delta x} \qquad \Delta p_y \geq \frac{\hbar}{2 \cdot \Delta y} \qquad \Delta p_z \geq \frac{\hbar}{2 \cdot \Delta z}$

gives

$$p_0^2 \geq \frac{\hbar^2}{4} \cdot \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right]$$

and

$$(\Delta x)^2 = (\Delta y)^2 = (\Delta z)^2 = \frac{r_0^2}{3}$$

thus $\boxed{p_0^2 \cdot r_0^2 \geq \frac{9}{4} \cdot \hbar^2}$

3.2

$|\vec{v}_e|$ ...... speed of the external electron before the capture

$|\vec{V}_i|$ .... speed of $A^{(Z-1)+}$ before capturing

$|\vec{V}_f|$ ...... speed of $A^{(Z-1)+}$ after capturing

$E_n = h.\nu$ ...... energy of the emitted photon

conservation of energy:

$$\frac{1}{2} \cdot m_e \cdot v_e^2 + \frac{1}{2} \cdot (M + m_e) \cdot V_i^2 + E\left[A^{(Z-1)+}\right] = \frac{1}{2} \cdot (M + 2 \cdot m_e) \cdot V_f^2 + E\left[A^{(Z-2)+}\right]$$

where $E[A^{(Z-1)+})$ and $E[A^{(Z-2)+}]$ denotes the energy of the electron in the outermost shell of ions $A^{(Z-1)+}$ and $A^{(Z-2)+}$ respectively.

conservation of momentum:

$$m_e \cdot \vec{v}_e + (M + m) \cdot \vec{V}_i = (M + 2 \cdot m_e) \cdot \vec{V}_f + \frac{h \cdot \nu}{c} \cdot \vec{1}$$

where $\vec{1}$ is the unit vector pointing in the direction of the motion of the emitted photon.

3.3

Determination of the energy of $A^{(Z-1)+}$:

potential energy $= -\frac{Z \cdot e^2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_0} = -\frac{Z \cdot q^2}{r_0}$

kinetic energy $= \frac{p^2}{2 \cdot m}$

If the motion of the electrons is confined within the x-y-plane, principles of uncertainty in 3.1 can be written as

$r_0^2 = (\Delta x)^2 + (\Delta y)^2$

$p_0^2 = (\Delta p_x)^2 + (\Delta p_y)^2$

$p_0^2 = \frac{\hbar^2}{4} \cdot \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right] = \frac{\hbar^2}{4} \cdot \left[\frac{2}{r_0^2} + \frac{2}{r_0^2} \right] = \frac{\hbar^2}{4} \cdot \frac{4}{r_0^2}$

thus

$p_0^2 \cdot r_0^2 = \hbar^2$

$$E\left[A^{(Z-1)+}\right] = \frac{p_0^2}{2 \cdot m_e} - \frac{Z \cdot q^2}{r_0} = \frac{\hbar^2}{2 \cdot m_e \cdot r_e} - \frac{Z \cdot q^2}{r_0}$$

Energy minimum exists, when $\frac{dE}{dr_0} = 0$.

Hence

$\frac{dE}{dr_0} = -\frac{\hbar^2}{m_e \cdot r_e^3} + \frac{Z \cdot q^2}{r_0^2} = 0$

this gives $\frac{1}{r_0} = \frac{Z \cdot q^2 \cdot m_e}{\hbar^2}$

hence

$$E\left[A^{(Z-1)+}\right] = \frac{\hbar^2}{2 \cdot m_e} \cdot \left(\frac{Z \cdot q^2 \cdot m_e}{\hbar} \right)^2 - Z \cdot q^2 \cdot \frac{Z \cdot q^2 \cdot m_e}{\hbar^2} = -\frac{m_e}{2} \cdot \left(\frac{Z \cdot q^2}{\hbar} \right)^2 = -\frac{q^2 \cdot Z^2}{2 \cdot r_B} = -E_R \cdot Z^2$$

$$\boxed{E\left[A^{(Z-1)+}\right] = -E_R \cdot Z^2}$$

3.4

In the case of $A^{(Z-1)+}$ ion captures a second electron

potential energy of both electrons = $-2\cdot\frac{Z\cdot q^2}{r_0}$

kinetic energy of the two electrons = $2\cdot\frac{p^2}{2\cdot m}=\frac{\hbar^2}{m_e\cdot r_0^2}$

potential energy due to interaction between the two electrons = $\frac{q^2}{\left|\vec{r}_1-\vec{r}_2\right|}=\frac{q^2}{2\cdot r_0}$

$$E\left[A^{(Z-2)+}\right]=\frac{\hbar^2}{m_e\cdot r_0^2}-\frac{2\cdot Z\cdot q^2}{r_0^2}+\frac{q^2}{2\cdot r_0}$$

total energy is lowest when $\frac{dE}{dr_0}=0$

hence

$$0=-\frac{2\cdot\hbar^2}{m_e\cdot r_0^3}+\frac{2\cdot Z\cdot q^2}{r_0^3}-\frac{q^2}{2\cdot r_0^2}$$

hence

$$\frac{1}{r_0}=\frac{q^2\cdot m_e}{2\cdot\hbar^2}\cdot\left(2\cdot Z-\frac{1}{2}\right)=\frac{1}{r_B}\cdot\left(Z-\frac{1}{4}\right)$$

$$E\left[A^{(Z-2)+}\right]=\frac{\hbar^2}{m_e}\cdot\left(\frac{q^2\cdot m_e}{2\cdot\hbar^2}\right)^2-\frac{q^2\cdot\left(2\cdot Z-\frac{1}{2}\right)}{\hbar}\cdot\frac{q^2\cdot m_e\cdot\left(2\cdot Z-\frac{1}{2}\right)}{2\cdot\hbar}$$

$$E\left[A^{(Z-2)+}\right]=-\frac{m_e}{4}\cdot\left[\frac{q^2\cdot\left(2\cdot Z-\frac{1}{2}\right)}{\hbar}\right]^2=-\frac{m_e\cdot\left[q^2\cdot\left(Z-\frac{1}{4}\right)\right]^2}{\hbar^2}=-\frac{q^2\cdot\left(Z-\frac{1}{4}\right)^2}{\hbar^2}$$

this gives

$$\boxed{E\left[A^{(Z-2)+}\right]=-2\cdot E_R\cdot\left(Z-\frac{1}{4}\right)^2}$$

3.5

The ion $A^{(Z-1)+}$ is at rest when it captures the second electron also at rest before capturing. From the information provided in the problem, the frequency of the photon emitted is given by

$$\nu = \frac{\omega}{2\cdot\pi} = \frac{2{,}057\cdot 10^{17}}{2\cdot\pi}\ \text{Hz}$$

The energy equation can be simplified to $E\left[A^{(Z-1)+}\right] - E\left[A^{(Z-2)+}\right] = \hbar\cdot\omega = h\cdot\nu$

that is

$$-E_R\cdot Z^2 - \left[-2\cdot E_R\cdot\left(Z-\frac{1}{4}\right)^2\right] = \hbar\cdot\omega$$

putting in known numbers follows

$$2{,}180\cdot 10^{-18}\cdot\left[-Z^2 + 2\cdot\left(Z-\frac{1}{4}\right)^2\right] = 1{,}05\cdot 10^{-34}\cdot 2{,}607\cdot 10^{17}$$

this gives

$$Z^2 - Z - 12{,}7 = 0$$

with the physical sensuous result $Z = \dfrac{1+\sqrt{1+51}}{2} = 4{,}1$

This implies Z = 4, and that means Beryllium

EXPERIMENTS

EXPERIMENT 1: Polarized Light

General Information

Equipment:

- one electric tungsten bulb made of frosedt-surface glass complete with mounting stand, 1 set
- 3 wooden clamps, each of which contains a slit for light experiment
- 2 glass plates; one of which is rectangular and the other one is square-shaped
- 1 polaroid sheet (circular-shaped)
- 1 red film or filter
- 1 roll self adhesive tape
- 6 pieces of self-adhesive labelling tape
- 1 cellophane sheet
- 1 sheet of black paper
- 1 drawing triangle with a handle
- 1 unerasable luminocolour pen 312, extra fine and black colour
- 1 lead pencil type F
- 1 lead pencil type H
- 1 pencil sharpener
- 1 eraser
- 1 pair of scissors

Important Instructions to be Followed

1. There are 4 pieces of labelling tape coded for each contestant. Stick the tape one each on the instrument marked with the sign #. Having done this, the contestant may proceed to perform the experiment to answer the questions.
2. Cutting, etching, scraping or folding the polaroid is strictly forbidden.
3. If marking is to be made on the polaroid, use the lumino-colour pen provided and put the cap back in place after finishing.
4. When marking is to be made on white paper sheet, use the white tape.
5. Use lead pencils to draw or sketch a graph.
6. Black paper may be cut into pieces for use in the experiment, but the best way of using the black paper is to roll it into a cylinder as to form a shield around the electric bulb. An aperture of proper size may be cut into the side of the cylinder to form an outlet for light used in the experiment.
7. Red piece of paper is to be folded to form a double layer.

The following four questions will be answered by performing the experiment:

Questions

1.1

1.1.a

Locate the axis of the light transmission of the polaroid film. This may be done by observing light reflected from the surface of the rectangular glass plate provided. (Light transmitting axis is the direction of vibration of the electric field vector of light wave transmitted through the polaroid). Draw a straight line along the light transmission axis as exactly as possible on the polaroid film. (#)

1.1.b

Set up the apparatus on the graph paper for the experiment to determine the refractive index of the glass plate for white light.

When unpolarized light is reflected at the glass plate, reflected light is partially polarized. Polarization of the reflected light is a maximum if the tangens of incident angle is equal to the refractive index of the glass plate, or: $\tan\alpha = n$.
Draw lines or dots that are related to the determination of the refractive index on the graph paper. (#)

1.2

Assemble a polariscope to observe birefringence in birefringent glass plate when light is normally incident on the plastic sheet and the glass plates.

A birefringent object is the object which splits light into two components, with the electric field vectors of the two components perpendicular to each other. The two directions of the electric field vectors are known as birefringent axes characteristic of birefringent material. These two components of light travel with different velocity.

Draw a simple sketch depicting design and functions of the polariscope assembled.

Insert a sheet of clear cellophane in the path of light in the polariscope. Draw lines to indicate birefringent axes (#). Comment briefly but concisely on what is observed, and describe how berefringent axes are located.

1.3

1.3.a

Stick 10 layers of self-adhesive tape provided on the glass plate as shown below. Make sure that each layer recedes in equal steps.

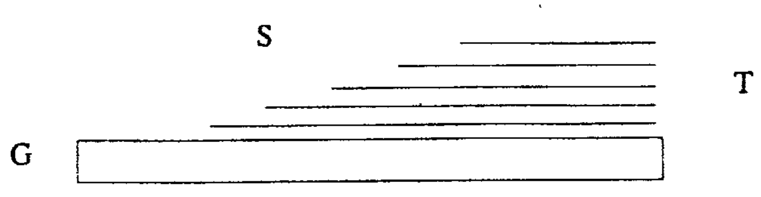


Fig. 19.10

G square glass plate as a substrate for the cellophane layers
T 10 layers of cellophane sheet
S steps about 3 mm up to 4 mm wide

Insert the assembled square plate into the path of light in the polariscope. Describe conditions for observing colours. How can these colours be changed ? Comment on the observations from this experiment.

1.3.b

Prepare monochromatic red light by placing doubly-folded red plastic sheet in the path of white light. Mark on the assembled square plate to show the steps which allow the determination of the difference of the optical paths of the two components of light from berefringent phenomenon, described under 1.2 (#).

Estimate the difference of the optical paths from two consecutive steps.

1.4

1.4.a

With the polariscope assembled, examine the central part of the drawing triangle provided. Describe relevant optical properties of the drawing triangle pertaining to birefringence.

1.4.b

Comment on the results observed. Draw conclusions about the physical properties of the material of which the triangle is made.

Additional Cautions

Be sure that the following items affixed with the coded labels provided accompany the report.

1. (#) Polarized film with the position of the transmission axis clearly marked.
2. (#) Graph paper with lines and dots denoting experimental setup for determining refractive index.
3. (#) Sheet of cellophane paper with marking indicating the positions of birefringent axis.
4. (#) Square glass plate affixed with self-adhesive tape with markings to indicate the positions of birefringent axis.

Solution

In this experiment the results from one experimental stage are used to solve problems in the following experimental stages. Without actually performing all parts of the experiment, solution cannot be meaningfully discussed.

It suffices that some transparent crystals are anisotropic, meaning their optical properties vary with the direction. Crystals which have this property are said to be doubly refracting or exhibit birefringece.

This phenomenon can be understood on the basis of wave theory. When a wavefront enters a birefringent material, two sets of Huygens wavelets propagate from every point of the entering wavefront causing the incident light to split into two components of two different velocities. In some crystals there is a particular direction (or rather a set of parallel directions) in which the velocities of the two components are the same. This direction is known as optic axes. the former is said to be uniaxial, and the latter biaxial.

If a plane polarized light (which may be white light or monochromatic light) is allowed to enter a uniaxial birefringed material, with its plane of polarization making some angle, say 45° with the optic axis, the incident light is splitted into two components (ordinary and extraordinary) travelling with two different velocities. Because of different velocities their phases different.

Upon emerging from the crystal, the two components recombine to from a resultant wave. The phase difference between the two components causes the resultant wave to be either linearly or circularly or elliptical polarized depending on the phase difference between the two components. The type of polarization can be determined by means of an analyser which is a second polaroid sheet provided for this experiment.

EXPERIMENT 2: Electron Tube

Introduction

Free electrons in a metal may be thought of as being "electron gas" confined in potential or energy walls. Under normal conditions or even when a voltage is applied near the surface of the metal, these electrons cannot leave the potential walls (see fig. 19.11)
If however the metal or the electron gas is heated, the electrons have enough thermal energy (kinetic energy) to overcome the energy barrier W (W is known as "work function"). If a voltage is applied across the metal and the anode, these thermally activated electrons may reach the anode.
The number of electrons arriving at the anode per unit time depends on the nature of the cathode and the temperature, i.e. all electrons freed from the potential wall will reach the anode no longer increase with applied voltage (see fig. 19.11)
The saturated current corresponding to the number of thermally activated electrons freed from the metal surface per unit time obeys what is generally known as Richardson´s equation i.e.

$$I_B = C \cdot T^2 \cdot e^{-\frac{W}{k \cdot T}}$$

where
C is a constant
T temperature of the cathode in Kelvin
k Boltzmann´s constant = 1,38 . 10^{-23} J/K

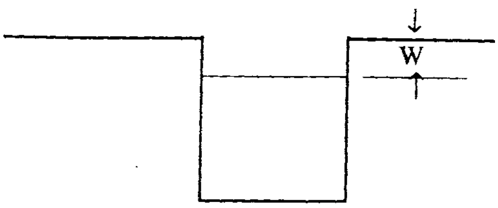


Fig 19.11

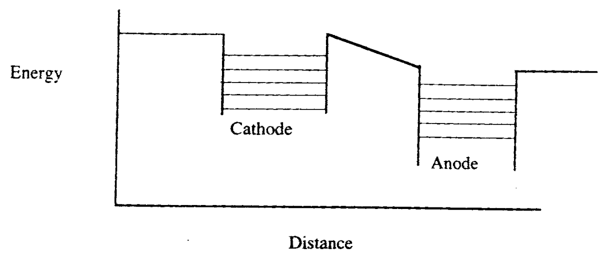


Fig.19.12

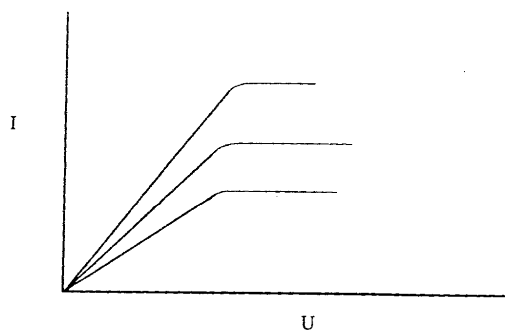


Fig 19.13 Graph of current as a function of voltage across anode-cathode

Determine the value of the work function W of tungsten metal in the form of heating filament of the vacuum tube provided.

The following items of equipment are placed at the disposal of the contestants:

- Electron tube AZ 41 which is a high-vacuum, full-wave rectifying diode. The cathode is made from a coated tungsten filament the work function of which is to be ascertained. According to the manual prepared by its manufacturer, no more than 4 V should be used when applying heating current to the cathode. Since the tube has two anodes, it is most desirable to have them connected for all measurements. The diagram in fig. 19.14 is a guide to identifying the anodes and the cathode.

- multimeter 1 unit, internal resistance for voltage measurement: 10 MΩ
- battery 1,5 V (together with a spare)
- battery 9 V; four units can be connected in series as shown in fig. 19.15
- connectors
- resistors; each of which has specifications as follows:
  1000 Ω ± 2% (brown, black, black, brown, brown, red)
  100 Ω ± 2% (brown, black, black, black, brown, red)
  47,5 Ω ± 1% (yellow, violet, green, gold, brown)
- resistors; 4 units, each of which has the resistance of about 1 Ω and coded
- connecting wires
- screw driver
- graph paper (1 sheet)
- graph of specific resistance of tungsten as a function of temperature; 1 sheet

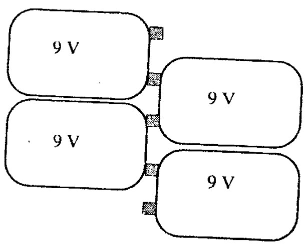


Fig 19.15

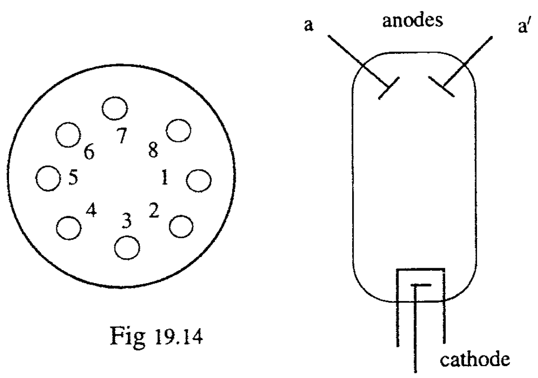


Fig 19.14

Solve the following problems:

2.1
Determine the resistance of 4 numerically-coded resistors. Under no circumstances must the multimeter be used as an ohmmeter.

2.2
Determine the saturated current for 4 different values of cathode temperatures, using 1,5 V battery to heat the cathode filament. A constant value of voltage between 35 V – 40 V between the anode and the cathode is sufficient to produce a saturated current. Obtain this value of voltage by connecting the four 9 V batteries in series. Describe how the different values of temperature are determined.

2.3
Determine the value of W. Explain the procedures used.

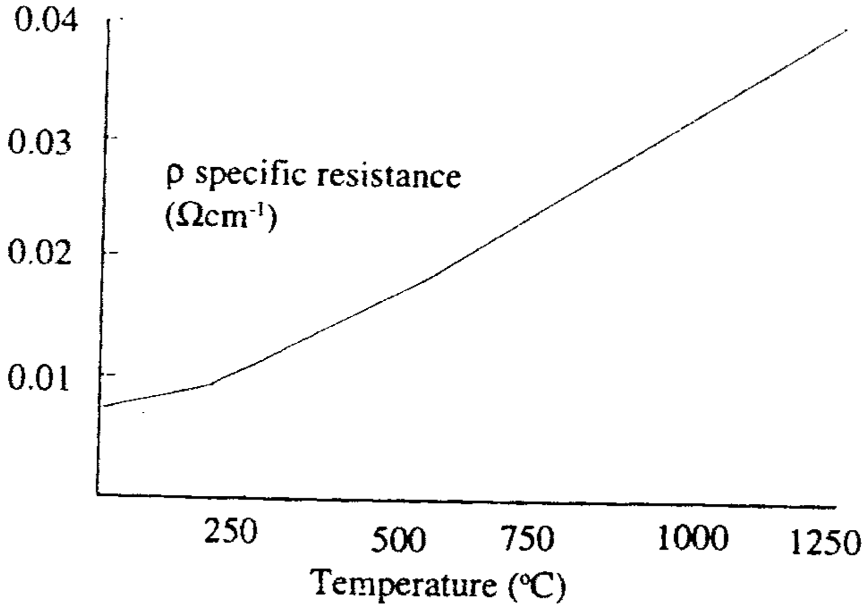


Fig 19.16

Solution

2.1

Connect the circuit as shown in fig. 19.17

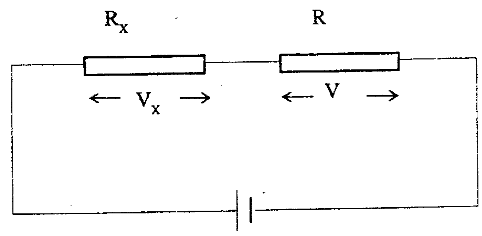


Fig. 19.17

R_X .... resistance to be determined
R ...... known value of resistance

Measure potential difference across R_X and R. Chose the value of R which gives comparable value of potential difference across R_X.

In this particular case R = 47,5 Ω

$$\frac{R_X}{R} = \frac{V_X}{V}$$

where V_X and V are values of potential differences across R_X and R respectively.
R_X can be calculated from the above equation.
(The error in R_X depends on the errors of V_X and V_R).

2.2

Connect the circuit as shown in fig. 19.18

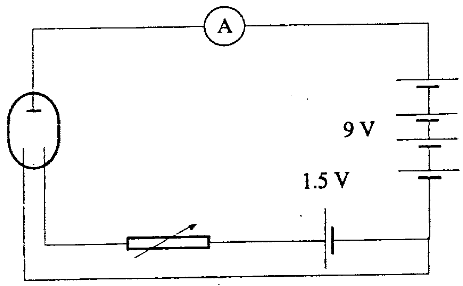


Fig 19.18

- Begin the experiment by measuring the resistance R_0 of the tungsten cathode when there is no heating current
- Add resistor R = 1000 Ω into the cathode circuit, determine resistance R_1 of the tungsten cathode, calculate the resistance of the current-carrying cathode.
- Repeat the experiment, using the resistor R = 100 Ω in the cathode circuit, determine resistance R_2 of tungsten cathode with heating current in the circuit.
- Repeat the experiment, using the resistor R = 47,5 Ω in the cathode circuit, determine resistance R_3 of tungsten cathode with heating current in the circuit.
- Plot a graph of $\frac{R_1}{R_0}, \frac{R_2}{R_0}$ and $\frac{R_3}{R_0}$ as a function of temperature, put the value of $\frac{R_0}{R_0} = 1$ to coincide with room temperature i.e. 18°C approximately and draw the remaining part of the graph parallel to the graph of specific resistance as a function of temperature provided in the problem. From the graph, read values of the temperature of the cathode T_1, T_2 and T_3 in Kelvin.

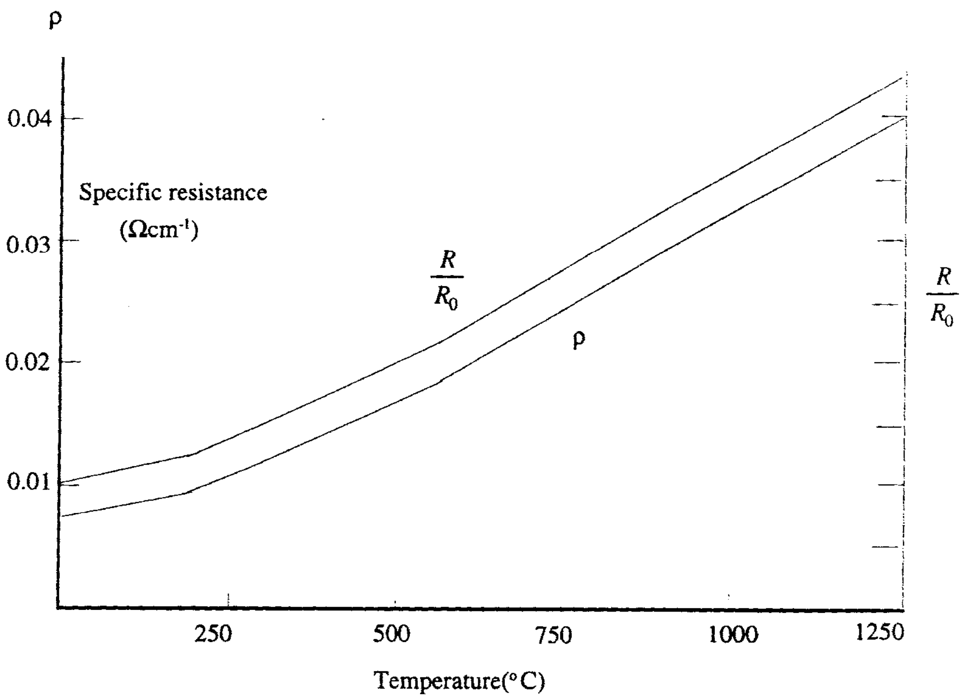


Fig 19.19

From the equation $I = C \cdot T^2 \cdot e^{-\frac{W}{k \cdot T}}$

we get $\ln \frac{I}{T^2} = -\frac{W}{k \cdot T} + \ln C$

Plot a graph of $\ln \frac{I}{T^2}$ against $\frac{1}{T}$.

The curve is linear. Determine the slope m from this graph. $-m = -\frac{W}{k}$

Work function W can be calculated using known values of m and k (given in the problem).

Error in W depends on the error of T which in turn depends on the error of measured R.

Problems of the 20th International Physics Olympiad [1] (Warsaw, 1989)

Waldemar Gorzkowski

Institute of Physics, Polish Academy of Sciences, Warsaw, Poland [2]

Abstract

The article contains problems given at the 20th International Physics Olympiad (1989) and their solutions. The 20th IPhO was the third IPhO organized in Warsaw, Poland.

Logo

The emblem of the XX International Physics Olympiad contains a picture that is a historical record of the first hypernuclear event observed and interpreted in Warsaw by M. Danysz and J. Pniewski[3]. The collision of a high-energy particle with a heavy nucleus was registered in nuclear emulsion. Tracks of the secondary particles emitted in the event, seen in the picture (upper star), consist of tracks due to fast pions ("thin tracks") and to much slower fragments of the target nucleus ("black tracks"). The "black track" connecting the upper star (greater) with the lower star (smaller) in the figure is due to a hypernuclear fragment, in this case due to a part of the primary nucleus containing an unstable hyperon Λ instead of a nucleon. Hyperfragments (hypernuclei) are a new kind of matter in which the nuclei contain not only protons and neutrons but also some other heavy particles.

In the event observed above the hyperon Λ, bound with nucleon, decays like a free particle through a week (slow) process only. This fact strongly suggested the existence of a new quantum number that could explain suppression of the decay, even in presence of nucleons. Indeed, this was one of the observations that, 30 months later, led to the concept of strangeness.

Introduction

Theoretical problems (including solutions and marking schemes) were prepared especially for the 20th IPhO by Waldemar Gorzkowski. The experimental problem (including the solution and marking scheme) was prepared especially for this Olympiad by Andrzej Kotlicki. The problems were refereed independently (and many times) by at least two persons

[1] This article has been sent for publication in *Physics Competitions* in October 2003

[2] e-mail: gorzk@ifpan.edu.pl

[3] M. Danysz and J. Pniewski, *Bull. Acad. Polon. Sci.*, **3(1)** 42 (1952) and *Phil. Mag.*, **44**, 348 (1953). Later the same physicists, Danysz and Pniewski, discovered the first case of a nucleus with two hyperons (double hyperfragment).

after any change was made in the text to avoid unexpected difficulties at the competition. This work was done by:

First Problem:
Andrzej Szadkowski, Andrzej Szymacha, Włodzimierz Ungier

Second Problem:
Andrzej Szadkowski, Andrzej Szymacha, Włodzimierz Ungier, Stanisław Woronowicz

Third Problem:
Andrzej Rajca, Andrzej Szymacha, Włodzimierz Ungier

Experimental Problem:
Krzysztof Korona, Anna Lipniacka, Jerzy Łusakowski, Bruno Sikora

Several English versions of the texts of the problems were given to the English-speaking students. As far as I know it happened for the first time (at present it is typical). The original English version was accepted (as a version for the students) by the leaders of the Australian delegation only. The other English-speaking delegations translated the English originals into English used in their countries. The net result was that there were at least four English versions. Of course, physics contained in them was exactly the same, while wording and spelling were somewhat different (the difference, however, were not too great).

This article is based on the materials quoted at the end of the article and on personal notes of the author.

THEORETICAL PROBLEMS

Problem 1

Consider two liquids A and B insoluble in each other. The pressures p_i (i = A or B) of their saturated vapors obey, to a good approximation, the formula:

$$\ln(p_i / p_o) = \frac{\alpha_i}{T} + \beta_i ; \quad i = \text{A or B},$$

where p_o denotes the normal atmospheric pressure, T – the absolute temperature of the vapor, and α_i and β_i (i = A or B) – certain constants depending on the liquid. (The symbol ln denotes the natural logarithm, i.e. logarithm with base e = 2.7182818…)

The values of the ratio p_i/p_0 for the liquids A and B at the temperature 40°C and 90°C are given in Tab. 1.1.

Table 1.1

t [°C]	p_i/p_0	
	i = A	i = B
40	0.284	0.07278
90	1.476	0.6918

The errors of these values are negligible.

A. Determine the boiling temperatures of the liquids A and B under the pressure p_0.

B. The liquids A and B were poured into a vessel in which the layers shown in Fig. 1.1 were formed. The surface of the liquid B has been covered with a thin layer of a non-volatile liquid C, which is insoluble in the liquids A and B and vice versa, thereby preventing any free evaporation from the upper surface of the liquid B, The ratio of molecular masses of the liquids A and B (in the gaseous phase) is:

$$\gamma = \mu_A / \mu_B = 8.$$

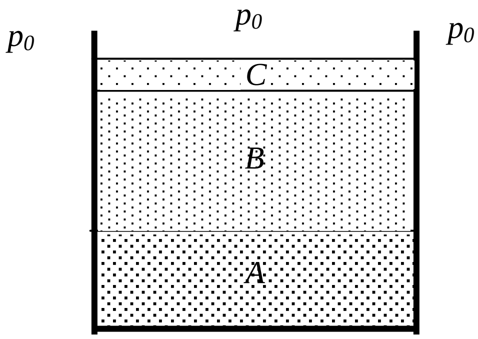


Fig. 1.1

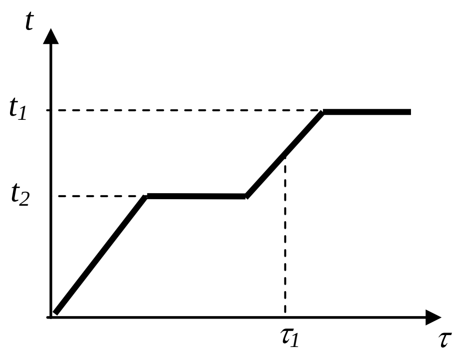


Fig. 1.2

The masses of the liquids A and B were initially the same, each equal to m = 100g. The heights of the layers of the liquids in the vessel and the densities of the liquids are small enough to make the assumption that the pressure in any point in the vessel is practically equal to the normal atmospheric pressure p_0.

The system of liquids in the vessel is slowly, but continuously and uniformly, heated. It was established that the temperature t of the liquids changed with time τ as shown schematically in the Fig. 1.2.

Determine the temperatures t_1 and t_2 corresponding to the horizontal parts of the diagram and the masses of the liquids A and B at the time τ_1. The temperatures should be rounded to the nearest degree (in °C) and the masses of the liquids should be determined to one-tenth of gram.

REMARK: Assume that the vapors of the liquids, to a good approximation,

(1) obey the Dalton law stating that the pressure of a mixture of gases is equal to the sum of the partial pressures of the gases forming the mixture and

(2) can be treated as perfect gases up to the pressures corresponding to the saturated vapors.

Solution

PART A

The liquid boils when the pressure of its saturated vapor is equal to the external pressure. Thus, in order to find the boiling temperature of the liquid i (i - A or B), one should determine such a temperature T_{bi} (or t_{bi}) for which $p_i/p_0 = 1$.

Then $\ln(p_i / p_0) = 0$, and we have:

$$T_{bi} = -\frac{\alpha_i}{\beta_i}.$$

The coefficients α_i and β_i are not given explicitly. However, they can be calculated from the formula given in the text of the problem. For this purpose one should make use of the numerical data given in the Tab. 1.1.

For the liquid A, we have:

$$\ln 0.284 = \frac{\alpha_A}{(40 + 273.15)\mathrm{K}} + \beta_A,$$

$$\ln 1.476 = \frac{\alpha_A}{(90 + 273.15)\mathrm{K}} + \beta_A.$$

After subtraction of these equations, we get:

$$\ln 0.284 - \ln 1.476 = \alpha_A \left(\frac{1}{40 + 273.15} - \frac{1}{90 + 273.15} \right) \mathrm{K}^{-1}.$$

$$\alpha_A = \frac{\ln \frac{0.284}{1.476}}{\frac{1}{40 + 273.15} - \frac{1}{90 + 273.15}} \mathrm{K} \approx -3748.49\mathrm{K}.$$

Hence,

$$\beta_A = \ln 0.284 - \frac{\alpha_A}{(40 + 273.15)\mathrm{K}} \approx 10.711.$$

Thus, the boiling temperature of the liquid A is equal to

$$T_{bA} = 3748.49\mathrm{K}/10.711 \approx 349.95\ \mathrm{K}.$$

In the Celsius scale the boiling temperature of the liquid A is

$$t_{bA} = (349.95 - 273.15)°\mathrm{C} = 76.80°\mathrm{C} \approx 77°\mathrm{C}.$$

For the liquid B, in the same way, we obtain:

$$\alpha_B \approx -5121.64\ \mathrm{K},$$
$$\beta_B \approx 13.735,$$
$$T_{bB} \approx 372\text{-}89\ \mathrm{K},$$
$$t_{bB} \approx 99.74°\mathrm{C} \approx 100°\mathrm{C}.$$

PART B

As the liquids are in thermal contact with each other, their temperatures increase in time in the same way.

At the beginning of the heating, what corresponds to the left sloped part of the diagram, no evaporation can occur. The free evaporation from the upper surface of the liquid B cannot occur - it is impossible due to the layer of the non-volatile liquid C. The evaporation from the inside of the system is considered below.

Let us consider a bubble formed in the liquid A or in the liquid B or on the surface that separates these liquids. Such a bubble can be formed due to fluctuations or for many other reasons, which will not be analyzed here.

The bubble can get out of the system only when the pressure inside it equals to the external pressure p_0 (or when it is a little bit higher than p_0). Otherwise, the bubble will collapse.

The pressure inside the bubble formed in the volume of the liquid A or in the volume of the liquid B equals to the pressure of the saturated vapor of the liquid A or B, respectively. However, the pressure inside the bubble formed on the surface separating the liquids A and B is equal to the sum of the pressures of the saturated vapors of both these liquids, as then the bubble is in a contact with the liquids A and B at the same time. In the case considered the pressure inside the bubble is greater than the pressures of the saturated vapors of each of the liquids A and B (at the same temperature).

Therefore, when the system is heated, the pressure p_0 is reached first in the bubbles that were formed on the surface separating the liquids. Thus, the temperature t_1 corresponds to a kind of common boiling of both liquids that occurs in the region of their direct contact. The temperature t_1 is for sure lower than the boiling temperatures of the liquids A and B as then the pressures of the saturated vapors of the liquids A and B are less then p_0 (their sum equals to p_0 and each of them is greater than zero).

In order to determine the value of t_1 with required accuracy, we can calculate the values of the sum of the saturated vapors of the liquids A and B for several values of the temperature t and look when one gets the value p_0.

From the formula given in the text of the problem, we have:

$$\frac{p_A}{p_0} = e^{\frac{\alpha_A}{T}+\beta_A}, \tag{1}$$

$$\frac{p_B}{p_0} = e^{\frac{\alpha_B}{T}+\beta_B}. \tag{2}$$

$p_A + p_B$ equals to p_0 if

$$\frac{p_A}{p_0} + \frac{p_B}{p_0} = 1.$$

Thus, we have to calculate the values of the following function:

$$y(x) = e^{\frac{\alpha_A}{t+t_0}+\beta_A} + e^{\frac{\alpha_B}{t+t_0}+\beta_B},$$

(where $t_0 = 273.15$ °C) and to determine the temperature $t = t_1$, at which $y(t)$ equals to 1. When calculating the values of the function $y(t)$ we can divide the intervals of the temperatures t by 2 (approximately) and look whether the results are greater or less than 1.

We have:

Table 1.2

t	$y(t)$
40°C	< 1 (see Tab. 1.1)
77°C	> 1 (as t_1 is less than t_{bA})
59°C	0.749 < 1
70°C	1.113 > 1
66°C	0.966 < 1
67°C	1.001 > 1
66.5°C	0.983 < 1

Therefore, $t_1 \approx$ 67° C (with required accuracy).

Now we calculate the pressures of the saturated vapors of the liquids A and B at the temperature $t_1 \approx$ 67°C, i.e. the pressures of the saturated vapors of the liquids A and B in each bubble formed on the surface separating the liquids. From the equations (1) and (2), we get:

$$p_A \approx 0.734\, p_0 ,$$
$$p_B \approx 0.267\, p_0 ,$$
$$(p_A + p_B = 1.001 p_0 \approx p_0) .$$

These pressures depend only on the temperature and, therefore, they remain constant during the motion of the bubbles through the liquid B. The volume of the bubbles during this motion also cannot be changed without violation of the relation $p_A + p_B = p_0$. It follows from the above remarks that the mass ratio of the saturated vapors of the liquids A and B in each bubble is the same. This conclusion remains valid as long as both liquids are in the system. After total evaporation of one of the liquids the temperature of the system will increase again (second sloped part of the diagram). Then, however, the mass of the system remains constant until the temperature reaches the value t_2 at which the boiling of the liquid (remained in the vessel) starts. Therefore, the temperature t_2 (the higher horizontal part of the diagram) corresponds to the boiling of the liquid remained in the vessel.

The mass ratio m_A / m_B of the saturated vapors of the liquids A and B in each bubble leaving the system at the temperature t_1 is equal to the ratio of the densities of these vapors ρ_A / ρ_B. According to the assumption 2, stating that the vapors can be treated as ideal gases, the last ratio equals to the ratio of the products of the pressures of the saturated vapors by the molecular masses:

$$\frac{m_A}{m_B} = \frac{\rho_A}{\rho_B} = \frac{p_A \mu_A}{p_B \mu_B} = \frac{p_A}{p_B} \mu .$$

Thus,

$$\frac{m_A}{m_B} \approx 22.0 .$$

We see that the liquid A evaporates 22 times faster than the liquid B. The evaporation of 100 g of the liquid A during the “surface boiling” at the temperature t_1 is associated with the evaporation of 100 g / 22 $\approx$4.5 g of the liquid B. Thus, at the time τ_1 the vessel contains 95.5 g of the liquid B (and no liquid A). The temperature t_2 is equal to the boiling temperature of the liquid B: $t_2 = 100$°C.

Marking Scheme

1. physical condition for boiling — 1 point
2. boiling temperature of the liquid A (numerical value) — 1 point
3. boiling temperature of the liquid B (numerical value) — 1 point
4. analysis of the phenomena at the temperature t_1 — 3 points
5. numerical value of t_1 — 1 point
6. numerical value of the mass ratio of the saturated vapors in the bubble — 1 point
7. masses of the liquids at the time τ_1 — 1 point
8. determination of the temperature t_2 — 1 point

REMARK: As the sum of the logarithms is not equal to the logarithm of the sum, the formula given in the text of the problem should not be applied to the mixture of the saturated vapors in the bubbles formed on the surface separating the liquids. However, the numerical data have been chosen in such a way that even such incorrect solution of the problem gives the correct value of the temperature t_1 (within required accuracy). The purpose of that was to allow the pupils to solve the part B of the problem even if they determined the temperature t_1 in a wrong way. Of course, one cannot receive any points for an incorrect determination of the temperature t_1 even if its numerical value is correct.

Typical mistakes in the pupils' solutions

Nobody has received the maximum possible number of points for this problem, although several solutions came close. Only two participants tried to analyze proportion of pressures of the vapors during the upward movement of the bubble trough the liquid B. Part of the students confused Celsius degrees with Kelvins. Many participants did not take into account the boiling on the surface separating the liquids A and B, although this effect was the essence of the problem. Part of the students, who did notice this effect, assumed a priori that the liquid with lower boiling temperature "must" be the first to evaporate. In general, this need not be true: if γ were, for example, 1/8 instead 8, then liquid A rather than B would remain in the vessel. As regards the boiling temperatures, practically nobody had any essential difficulties.

Problem 2

Three non-collinear points P_1, P_2 and P_3, with known masses m_1, m_2 and m_3, interact with one another through their mutual gravitational forces only; they are isolated in free space and do not interact with any other bodies. Let σ denote the axis going through the center-of-mass of the three masses, and perpendicular to the triangle $P_1P_2P_3$. What conditions should the angular velocities ω of the system (around the axis σ) and the distances:

$$P_1P_2 = a_{12}, \quad P_2P_3 = a_{23}, \quad P_1P_3 = a_{13},$$

fulfill to allow the shape and size of the triangle $P_1P_2P_3$ unchanged during the motion of the system, i.e. under what conditions does the system rotate around the axis σ as a rigid body?

Solution

As the system is isolated, its total energy, i.e. the sum of the kinetic and potential energies, is conserved. The total potential energy of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 in the inertial system (i.e. when there are no inertial forces) is equal to the sum of the gravitational potential energies of all the pairs of points (P_1,P_2), (P_2,P_3) and (P_1,P_3). It depends only on the distances a_{12}, a_{23} and a_{23} which are constant in time. Thus, the total potential energy of the system is constant. As a consequence the kinetic energy of the system is constant too. The moment of inertia of the system with respect to the axis σ depends only on the distances from the points P_1, P_2 and P_3 to the axis σ which, for fixed a_{12}, a_{23} and a_{23} do not depend on time. This means that the moment of inertia I is constant. Therefore, the angular velocity of the system must also be constant:

$$\omega = \text{const.} \tag{1}$$

This is the first condition we had to find. The other conditions will be determined by using three methods described below. However, prior to performing calculations, it is desirable to specify a convenient coordinates system in which the calculations are expected to be simple.

Let the positions of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 be given by the vectors $\mathbf{r}_1$, $\mathbf{r_2}$ and $\mathbf{r_3}$. For simplicity we assume that the origin of the coordinate system is localized at the center of mass of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 and that all the vectors $\mathbf{r}_1$, $\mathbf{r_2}$ and $\mathbf{r_3}$ are in the same coordinate plane, e.g. in the plane (*x*,*y*). Then the axis σ is the axis z.

In this coordinate system, according to the definition of the center of mass, we have:

$$m_1\mathbf{r_1} + m_2\mathbf{r_2} + m_3\mathbf{r_2} = 0 \tag{2}$$

Now we will find the second condition by using several methods.

FIRST METHOD

Consider the point P_1 with the mass m_1. The points P_2 and P_3 act on it with the forces:

$$\mathbf{F}_{21} = G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1), \tag{3}$$

$$\mathbf{F}_{31} = G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1). \tag{4}$$

where *G* denotes the gravitational constant.

In the inertial frame the sum of these forces is the centripetal force

$$\mathbf{F}_{r1} = -m_1\omega^2\mathbf{r}_1,$$

which causes the movement of the point P_1 along a circle with the angular velocity ω. (The moment of this force with respect to the axis σ is equal to zero.) Thus, we have:

$$\mathbf{F}_{21} + \mathbf{F}_{31} = \mathbf{F}_{r1}. \tag{5}$$

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the sum of the forces (3), (4) and the centrifugal force

$$\mathbf{F}'_{r1} = m_1 \omega^2 \mathbf{r}_1$$

should be equal to zero:

$$\mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}'_{r1} = 0. \qquad (6)$$

(The moment of this sum with respect to any axis equals to zero.)

The conditions (5) and (6) are equivalent. They give the same vector equality:

$$G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1) + m_1 \omega^2 \mathbf{r}_1 = 0, \qquad (7')$$

$$G\frac{m_1}{a_{12}^3} m_2 \mathbf{r}_2 + G\frac{m_1}{a_{13}^3} m_3 \mathbf{r}_3 + m_1 \mathbf{r}_1 \left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} \right) = 0 \qquad (7'')$$

From the formula (2), we get:

$$m_2 \mathbf{r}_2 = -m_1 \mathbf{r}_1 - m_3 \mathbf{r}_3 \qquad (8)$$

Using this relation, we write the formula (7) in the following form:

$$G\frac{m_1}{a_{12}^3}(-m_1 \mathbf{r}_1 - m_3 \mathbf{r}_3) + G\frac{m_1}{a_{13}^3} m_3 \mathbf{r}_3 + m_1 \mathbf{r}_1 \left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} \right) = 0,$$

i.e.

$$\mathbf{r}_1 m_1 \left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3} \right) + \mathbf{r}_3 \left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3} \right) G m_1 m_3 = 0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-col1inear. Therefore, the coefficients in the last formula must be equal to zero:

$$\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3} \right) G m_1 m_3 = 0,$$

$$m_1 \left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3} \right) = 0.$$

The first equality leads to:

$$\frac{1}{a_{13}^3} = \frac{1}{a_{12}^3}$$

and hence,

$$a_{13} = a_{12}.$$

Let $a_{13} = a_{12} = a$. Then the second equality gives:

$$\omega^2 a^3 = GM \qquad (9)$$

where

$$M = m_1 + m_2 + m_3 \qquad (10)$$

denotes the total mass of the system.

In the same way, for the points P_2 and P_3, one gets the relations:

a) the point P_2:

$$a_{23} = a_{12}; \qquad \omega^2 a^3 = GM$$

b) the point P_3:

$$a_{13} = a_{23}; \qquad \omega^2 a^3 = GM$$

Summarizing, the system can rotate as a rigid body if all the distances between the masses are equal:

$$a_{12} = a_{23} = a_{13} = a, \tag{11}$$

the angular velocity ω is constant and the relation (9) holds.

SECOND METHOD

At the beginning we find the moment of inertia I of the system with respect to the axis σ. Using the relation (2), we can write:

$$0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)^2 = m_1^2\mathbf{r}_1^2 + m_2^2\mathbf{r}_2^2 + m_3^2\mathbf{r}_3^2 + 2m_1m_2\mathbf{r}_1\mathbf{r}_2 + 2m_1m_3\mathbf{r}_1\mathbf{r}_3 + 2m_3m_2\mathbf{r}_3\mathbf{r}_2.$$

Of course,

$$\mathbf{r}_i^2 = r_i^2 \qquad\qquad i = 1, 2, 3$$

The quantities $2\mathbf{r}_i\mathbf{r}_j$ (i, j = 1, 2, 3) can be determined from the following obvious relation:

$$a_{ij}^2 = \left|\mathbf{r}_i - \mathbf{r}_j\right|^2 = (\mathbf{r}_i - \mathbf{r}_j)^2 = \mathbf{r}_i^2 + \mathbf{r}_j^2 - 2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - 2\mathbf{r}_i\mathbf{r}_j.$$

We get:

$$2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - a_{ij}^2.$$

With help of this relation, after simple transformations, we obtain:

$$0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)^2 = (m_1 + m_2 + m_3)(m_1r_1^2 + m_2r_2^2 + m_3r_3^2) - \sum_{i<j} m_i m_j a_{ij}^2.$$

The moment of inertia I of the system with respect to the axis σ, *a*ccording to the definition of this quantity, is equal to

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2.$$

The last two formulae lead to the following expression:

$$I = \frac{1}{M}\sum_{i<j} m_i m_j a_{ij}^2$$

where M (the total mass of the system) is defined by the formula (10).

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the total potential energy V_{tot} is the sum of the gravitational potential energies

$$V_{ij} = -G\frac{m_i m_j}{a_{ij}}; \qquad\qquad i, j = 1, 2, 3;\ i < j$$

of all the masses and the potential energies

$$V_i = -\tfrac{1}{2}\omega^2 m_i r_i^2\,; \qquad\qquad i = 1, 2, 3$$

of the masses m_i (i = 1, 2, 3) in the field of the centrifugal force:

$$V_{tot} = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2\sum_{i=1}^{3} m_i r_i^2 = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2 I = G\sum_{i<j}\frac{m_i m_j}{a_{ij}} - \tfrac{1}{2}\omega^2\frac{1}{M}\sum_{i<j} m_i m_j a_{ij}^2 =$$
$$= -\sum_{i<j} m_i m_j\left(\frac{\omega^2}{2M}a_{ij}^2 + \frac{G}{a_{ij}}\right)$$

i.e.

$$V_{tot} = -\sum_{i<j} m_i m_j\left(\frac{\omega^2}{2M}a_{ij}^2 + \frac{G}{a_{ij}}\right).$$

A mechanical system is in equilibrium if its total potential energy has an extremum. In our case the total potential energy V_{tot} is a sum of three terms. Each of them is proportional to:

$$f(a) = \frac{\omega^2}{2M}a^2 + \frac{G}{a}.$$

The extrema of this function can be found by taking its derivative with respect to a and requiring this derivative to be zero. We get:

$$\frac{\omega^2}{M}a - \frac{G}{a^2} = 0.$$

It leads to:

$$\omega^2 a^3 = GM \quad \text{or} \quad \omega^2 a^3 = G(m_1 + m_2 + m_3).$$

We see that all the terms in V_{tot} have extrema at the same values of $a_{ij} = a$. (In addition, the values of a and ω should obey the relation written above.) It is easy to show that it is a maximum. Thus, the quantity V_{tot} has a maximum at $a_{ij} = a$.

This means that our three masses can remain in fixed distances only if these distances are equal to each other:

$$a_{12} = a_{23} = a_{13} = a$$

and if the relation

$$\omega^2 a^3 = GM\ ,$$

where M the total mass of the system, holds. We have obtained the conditions (9) and (11) again.

THIRD METHOD

Let us consider again the point P_1 with the mass m_1 and the forces $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$ given by the formulae (3) and (4). It follows from the text of the problem that the total moment (with respect to any fixed point or with respect to the mass center) of the forces acting on the point P_1 must be equal to zero. Thus, we have:

$$\mathbf{F}_{21}\times\mathbf{r}_1 + \mathbf{F}_{31}\times\mathbf{r}_1 = 0$$

where the symbol $\times$ denotes the vector product. Therefore

$$G\frac{m_1m_2}{a_{12}^3}(\mathbf{r}_2-\mathbf{r}_1)\times\mathbf{r}_1+G\frac{m_1m_3}{a_{13}^3}(\mathbf{r}_3-\mathbf{r}_1)\times\mathbf{r}_1=0.$$

But

$$\mathbf{r}_1\times\mathbf{r}_1=0.$$

Thus:

$$\frac{m_2}{a_{12}^3}\mathbf{r}_2\times\mathbf{r}_1+\frac{m_3}{a_{13}^3}\mathbf{r}_3\times\mathbf{r}_1=0.$$

Using the formula (8), the last relation can be written as follows:

$$\frac{1}{a_{12}^3}(-m_1\mathbf{r}_1-m_3\mathbf{r}_3)\times\mathbf{r}_1+\frac{m_3}{a_{13}^3}\mathbf{r}_3\times\mathbf{r}_1=0,$$

$$-\frac{m_3}{a_{12}^3}\mathbf{r}_3\times\mathbf{r}_1+\frac{m_3}{a_{13}^3}\mathbf{r}_3\times\mathbf{r}_1=0,$$

$$\left(\frac{1}{a_{13}^3}-\frac{1}{a_{12}^3}\right)\mathbf{r}_3\times\mathbf{r}_1=0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-col1inear (and different from 0). Therefore

$$\mathbf{r}_3\times\mathbf{r}_1\neq 0$$

and

$$\frac{1}{a_{13}^3}-\frac{1}{a_{12}^3}=0,$$

hence,

$$a_{12}=a_{13}.$$

Similarly, one gets:

$$a_{12}=a_{23}\ \ (=a).$$

We have re-derived the condition (11).

Taking into account that all the distances a_{ij} have the same value a, from the equation (7) concerning the point P_1, using the relation (2) we obtain:

$$G\frac{m_1m_2}{a^3}(\mathbf{r}_2-\mathbf{r}_1)+G\frac{m_1m_3}{a^3}(\mathbf{r}_3-\mathbf{r}_1)+m_1\omega^2\mathbf{r}_1=0,$$

$$-\left(G\frac{m_1}{a^3}+G\frac{m_2}{a^3}G\frac{m_3}{a^3}\right)m_1\mathbf{r}_1+m_1\omega^2\mathbf{r}_1=0,$$

$$\frac{GM}{a^3}=\omega^2.$$

This is the condition (9). The same condition is got in result of similar calculations for the points P_2 and P_3.

The method described here does not differ essentially from the first method. In fact they are slight modifications of each other. However, it is interesting to notice how application of a proper mathematical language, e.g. the vector product, simplifies the calculations.

The relation (9) can be called a "generalized Kepler's law" as, in fact, it is very similar to the Kepler's law but with respect to the many-body system. As far as I know this generalized Kepler's law was presented for the first time right at the 20th IPhO.

Marking scheme

1. the proof that $\omega = \text{const}$	1 point
2. the conditions at the equilibrium (conditions for the forces and their moments or extremum of the total potential energy)	3 points
3. the proof of the relation $a_{ij} = a$	4 points
4. the proof of the relation $\omega^2 a^3 = GM$	2 points

Remarks and typical mistakes in the pupils' solutions

No type of error was observed as predominant in the pupils' solutions. Practically all the mistakes can be put down to the students' scant experience in calculations and general lack of skill. Several students misunderstood the text of the problem and attempted to prove that the three masses should be equal. Of course, this was impossible. Moreover, it was pointless, since the masses were given. Almost all the participants tried to solve the problem by analyzing equilibrium of forces and/or their moments. Only one student tried to solve the problem by looking for a minimum of the total potential energy (unfortunately, his solution was not fully correct). Several participants solved the problem using a convenient reference system: one mass in the origin and one mass on the *x*-axis. One of them received a special prize.

Problem 3

The problem concerns investigation of transforming the electron microscope with magnetic guiding of the electron beam (which is accelerated with the potential difference U = 511 kV) into a proton microscope (in which the proton beam is accelerated with the potential difference $-U$). For this purpose, solve the following two problems:

A. An electron after leaving a device, which accelerated it with the potential difference U, falls into a region with an inhomogeneous field $\boldsymbol{B}$ generated with a system of stationary coils $L_1, L_2, \dots, L_n$. The known currents in the coils are $i_1, i_2, \dots, i_n$, respectively.

What should the currents $i_1', i_2', \dots, i_n'$ in the coils $L_1, L_2, \dots, L_n$ be, in order to guide the proton (initially accelerated with the potential difference $-U$) along the same trajectory (and in the same direction) as that of the electron?

HINT: The problem can be solved by finding a condition under which the equation describing the trajectory is the same in both cases. It may be helpful to use the relation:

$$\boldsymbol{p}\frac{d}{dt}\boldsymbol{p} = \frac{1}{2}\frac{d}{dt}\boldsymbol{p}^2 = \frac{1}{2}\frac{d}{dt}p^2.$$

B. How many times would the resolving power of the above microscope increase or decrease if the electron beam were replaced with the proton beam? Assume that the resolving power of the microscope (i.e. the smallest distance between two point objects whose circular images can be just separated) depends only on the wave properties of the particles.

Assume that the velocities of the electrons and protons before their acceleration are zero, and that there is no interaction between own magnetic moment of either electrons or protons and the magnetic field. Assume also that the electromagnetic radiation emitted by the moving particles can be neglected.

NOTE: Very often physicists use 1 electron-volt (1 eV), and its derivatives such as 1 keV or 1 MeV, as a unit of energy. 1 electron-volt is the energy gained by the electron that passed the potential difference equal to 1 V.

Perform the calculations assuming the following data:

Rest energy of electron:	$E_e = m_e c^2 = 511$ keV
Rest energy of proton:	$E_p = m_p c^2 = 938$ MeV

Solution

PART A

At the beginning one should notice that the kinetic energy of the electron accelerated with the potential difference $U = 511$ kV equals to its rest energy E_0. Therefore, at least in the case of the electron, the laws of the classical physics cannot be applied. It is necessary to use relativistic laws.

The relativistic equation of motion of a particle with the charge e in the magnetic field $\mathbf{B}$ has the following form:

$$\frac{d}{dt}\mathbf{p} = \mathbf{F}_L$$

where $\mathbf{p} = m_0\gamma\mathbf{v}$ denotes the momentum of the particle (vector) and

$$\mathbf{F}_L = e\mathbf{v}\times\mathbf{B}$$

is the Lorentz force (its value is evB and its direction is determined with the right hand rule). m_0 denotes the (rest) mass of the particle and v denotes the velocity of the particle. The quantity γ is given by the formula:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity $\mathbf{v}$ of the particle and to its momentum $\mathbf{p} = m_0\gamma\mathbf{v}$. Hence,

$$\mathbf{F}_L \cdot \mathbf{v} = \mathbf{F}_L \cdot \mathbf{p} = 0\,.$$

Multiplying the equation of motion by $\mathbf{p}$ and making use of the hint given in the text of the problem, we get:

$$\frac{1}{2}\frac{d}{dt}p^2 = 0\,.$$

It means that the value of the particle momentum (and the value of the velocity) is constant during the motion:

$$p = m_0 v\gamma = \text{ const;} \qquad v = \text{const.}$$

The same result can be obtained without any formulae in the following way:

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity $\mathbf{v}$ (and to the momentum p as $\mathbf{p} = m_0\gamma\mathbf{v}$) and, as a consequence, to the trajectory of the particle. Therefore, there is no force that could change the component of the momentum tangent to the trajectory. Thus, this component, whose value is equal to the length of $\mathbf{p}$, should be constant: $\mathbf{p} = \text{const}$. (The same refers to the component of the velocity tangent to the trajectory as $\mathbf{p} = m_0\gamma\mathbf{v}$).

Let s denotes the path passed by the particle along the trajectory. From the definition of the velocity, we have:

$$\frac{ds}{dt} = v.$$

Using this formula, we can rewrite the equation of motion as follows:

$$v\frac{d}{ds}\mathbf{p} = \frac{ds}{dt}\frac{d}{ds}\mathbf{p} = \frac{d}{dt}\mathbf{p} = \mathbf{F}_L\,,$$

$$\frac{d}{ds}\mathbf{p} = \frac{\mathbf{F}_L}{v}\,.$$

Dividing this equation by p and making use of the fact that p = const, we obtain:

$$v\frac{d}{ds}\frac{\mathbf{p}}{p} = \frac{\mathbf{F}_L}{vp}$$

and hence

$$\frac{d}{ds}\mathbf{t} = \frac{\mathbf{F}_L}{vp}$$

where $\mathbf{t} = \mathbf{p}/p = \mathbf{v}/v$ is the versor (unit vector) tangent to the trajectory. The above equation is exactly the same for both electrons and protons if and only if the vector quantity:

$$\frac{\mathbf{F}_L}{vp}$$

is the same in both cases.

Denoting corresponding quantities for protons with the same symbols as for the electrons, but with primes, one gets that the condition, under which both electrons and protons can move along the same trajectory, is equivalent to the equality:

$$\frac{\mathbf{F}_L}{vp} = \frac{\mathbf{F}'_L}{v'p'}\,.$$

However, the Lorentz force is proportional to the value of the velocity of the particle, and the directions of any two vectors of the following three: $\mathbf{t}$ (or $\mathbf{v}$), $\mathbf{F}_L$, $\mathbf{B}$ determine the direction of the third of them (right hand rule). Therefore, the above condition can be written in the following form:

$$\frac{e\mathbf{B}}{p} = \frac{e'\mathbf{B}'}{p'}\,.$$

Hence,

$$\mathbf{B}' = \frac{e}{e'}\frac{p'}{p}\mathbf{B} = \frac{p'}{p}\mathbf{B}.$$

This means that at any point the direction of the field **B** should be conserved, its orientation should be changed into the opposite one, and the value of the field should be multiplied by the same factor *p*'/*p*. The magnetic field **B** is a vector sum of the magnetic fields of the coils that are arbitrarily distributed in the space. Therefore, each of this fields should be scaled with the same factor -*p*'/*p*. However, the magnetic field of any coil is proportional to the current flowing in it. This means that the required scaling of the fields can only be achieved by the scaling of all the currents with the same factor -*p*'/*p*:

$$i'_n = -\frac{p'}{p} i_n.$$

Now we shall determine the ratio *p*'/*p*. The kinetic energies of the particles in both cases are the same; they are equal to $E_k = e|U| = 511$ keV. The general relativistic relation between the total energy *E* of the particle with the rest energy E_0 and its momentum *p* has the following form:

$$E^2 = E_0^2 + p^2c^2$$

where c denotes the velocity of light.

The total energy of considered particles is equal to the sum of their rest and kinetic energies:

$$E = E_0 + E_k.$$

Using these formulae and knowing that in our case $E_k = e|U| = E_e$, we determine the momenta of the electrons (*p*) and the protons (*p*'). We get:

a) electrons:

$$(E_e + E_e)^2 = E_e^2 + p^2c^2,$$

$$p = \frac{E_e}{c}\sqrt{3}.$$

b) protons

$$(E_p + E_e)^2 = E_p^2 + p'^2c^2,$$

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e} + 1\right)^2 - \left(\frac{E_p}{E_e}\right)^2}.$$

Hence,

$$\frac{p'}{p} = \frac{1}{\sqrt{3}}\sqrt{\left(\frac{E_p}{E} + 1\right)^2 - \left(\frac{E_p}{E_e}\right)^2} \approx 35.0$$

and

$$i'_n = -35.0\, i_n.$$

It is worthwhile to notice that our protons are 'almost classical', because their kinetic energy $E_k (= E_e)$ is small compared to the proton rest energy E_p. Thus, one can expect that the momentum of the proton can be determined, with a good accuracy, from the classical considerations. We have:

$$E_e = E_k = \frac{p'^2}{2m_p} = \frac{p'^2 c^2}{2m_p c^2} = \frac{p'^2 c^2}{2E_p},$$

$$p' = \frac{1}{c}\sqrt{2E_e E_p}.$$

On the other hand, the momentum of the proton determined from the relativistic formulae can be written in a simpler form since E_p/E_e » 1. We get:

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2} = \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}+1} \approx \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}} = \frac{1}{c}\sqrt{2E_e E_p}.$$

In accordance with our expectations, we have obtained the same result as above.

PART B

The resolving power of the microscope (in the meaning mentioned in the text of the problem) is proportional to the wavelength, in our case to the length of the de Broglie wave:

$$\lambda = \frac{h}{p}$$

where h denotes the Planck constant and p is the momentum of the particle. We see that λ is inversely proportional to the momentum of the particle. Therefore, after replacing the electron beam with the proton beam the resolving power will be changed by the factor $p/p' \approx 1/35$. It means that our proton microscope would allow observation of the objects about 35 times smaller than the electron microscope.

Marking scheme

1. the relativistic equation of motion	1 point
2. independence of p and v of the time	1 point
3. identity of $e\mathbf{B}/p$ in both cases	2 points
4. scaling of the fields and the currents with the same factor	1 point
5. determination of the momenta (relativistically)	1 point
6. the ratio of the momenta (numerically)	1 point
7. proportionality of the resolving power to λ	1 point
8. inverse proportionality of λ to p	1 point
9. scaling of the resolving power	1 point

Remarks and typical mistakes in the pupils' solutions

Some of the participants tried to solve the problem by using laws of classical mechanics only. Of course, this approach was entirely wrong. Some students tried to find the required condition by equating "accelerations" of particles in both cases. They understood the "acceleration" of the particle as a ratio of the force acting on the particle to the "relativistic" mass of the particle. This approach is incorrect. First, in relativistic physics the relationship between force and acceleration is more complicated. It deals with not one "relativistic" mass, but with two "relativistic" masses: transverse and longitudinal. Secondly, identity of trajectories need not require equality of accelerations.

The actual condition, i.e. the identity of $e\mathbf{B}/p$ in both cases, can be obtained from the following two requirements:

1° in any given point of the trajectory the curvature should be the same in both cases;
2° in the vicinity of any given point the plane containing a small arc of the trajectory should be oriented in space in both cases in the same way.

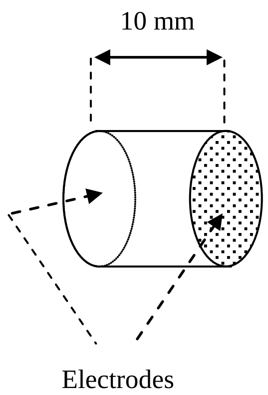
Most of the students followed the approach described just above. Unfortunately, many forgot about the second requirement (they neglected the vector character of the quantity $e\mathbf{B}/p$).

EXPERIMENTAL PROBLEM[1]

The following equipment is provided:

1. Two piezoelectric discs of thickness 10 mm with evaporated electrodes (Fig. 4.1) fixed in holders on the jaws of the calipers;

Fig. 4.1



2. The calibrated sine wave oscillator with a photograph of the control panel, explaining the functions of the switches and regulators;
3. A double channel oscilloscope with a photograph of the control panel, explaining the functions of the switches and regulators;
4. Two closed plastic bags containing liquids;
5. A beaker with glycerin (for wetting the discs surfaces to allow better mechanical coupling);
6. Cables and a three way connector;
7. A stand for support the bags with the liquids;
8. Support and calipers.

A piezoelectric material changes its linear dimensions under the influence of an electric field and vice-versa, the distortion of a piezoelectric material induces an electrical field. Therefore, it is possible to excite the mechanical vibrations in a piezoelectric material by applying an alternating electric field, and also to induce an alternating electric field by mechanical vibrations.

[1] The Organizing Committee planned to give another experimental problem: a problem on high T_c superconductivity. Unfortunately, the samples of superconductors, prepared that time by a factory, were of very poor quality. Moreover, they were provided after a long delay. Because of that the organizers decided to use this problem, which was also prepared, but considered as a second choice.

A. Knowing that the velocity of longitudinal ultrasonic waves in the material of the disc is about $4 \cdot 10^3$ m/s, estimate roughly the resonant frequency of the mechanical vibrations parallel to the disc axis. Assume that the disc holders do no restrict the vibrations. (Note that other types of resonant vibrations with lower or higher frequencies may occur in the discs.)

Using your estimation, determine experimentally the frequency for which the piezoelectric discs work best as a transmitter-receiver set for ultrasound in the liquid. Wetting surfaces of the discs before putting them against the bags improves penetration of the liquid in the bag by ultrasound.

B. Determine the velocity of ultrasound for both liquids without opening the bags and estimate the error.

C. Determine the ratio of the ultrasound velocities for both liquids and its error.

Complete carefully the synopsis sheet. Your report should, apart from the synopsis sheet, contain the descriptions of:

- method of resonant frequency estimation;
- methods of measurements;
- methods of estimating errors of the measured quantities and of final results.

Remember to define all the used quantities and to explain the symbols.

<table>
<tr><th colspan="2">Synopsis Sheet[1]</th></tr>
<tr><td rowspan="2">A</td><td>Formula for estimating the resonant frequency: Results (with units):</td></tr>
<tr><td>Measured best transmitter frequency (with units): Error:</td></tr>
<tr><td rowspan="4">B</td><td>Definition of measured quantity: Symbol: Results: Error:</td></tr>
<tr><td>Final formula for ultrasound velocity in liquid:</td></tr>
<tr><td>Velocity of ultrasound (with units): Error:</td></tr>
<tr><td>Liquid A
Liquid B</td></tr>
<tr><td></td><td>Ratio of velocities: Error:</td></tr>
</table>

Solution (draft)[1]

[1] In the real Synopsis Sheet the students had more space for filling.

A. As the holders do not affect vibrations of the disc we may expect antinodes on the flat surfaces of the discs (Fig. 4.2; geometric proportions not conserved). One of the frequencies is expected for

$$l = \tfrac{1}{2}\lambda = \frac{v}{2f},$$

where v denotes the velocity of longitudinal ultrasonic wave (its value is given in the text of the problem), f - the frequency and l - the thickness of the disc. Thus:

$$f = \frac{v}{2l}.$$

Numerically $f = 2 \cdot 10^5$ Hz = 200 kHz.

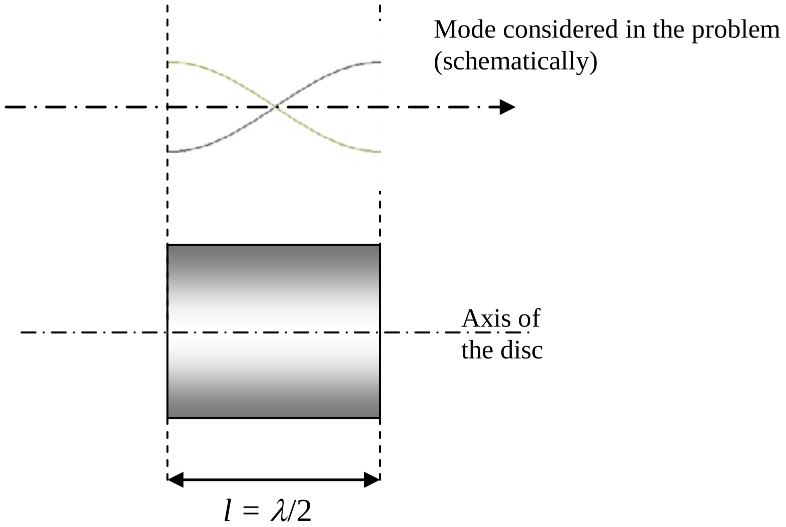


Fig. 4.2

One should stress out that different modes of vibrations can be excited in the disc with height comparable to its diameter. We confine our considerations to the modes related to longitudinal waves moving along the axis of the disc as the sound waves in liquids are longitudinal. We neglect coupling between different modes and require antinodes exactly at the flat parts of the disc. We assume also that the piezoelectric effect does not affect velocity of ultrasound. For these reasons the frequency just determined should be treated as only a rough approximation. However, it indicates that one should look for the resonance in vicinity of 200 kHz.

The experimental set-up is shown in Fig. 4.3. The oscillator (generator) is connected to one of the discs that works as a transmitter and to one channel of the oscilloscope. The second disc is connected to the second channel of the oscilloscope and works as a receiver. Both discs are placed against one of the bags with liquid (Fig. 4.4). The distance d can be varied.

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[1] This draft solution is based on the camera-ready text of the more detailed solution prepared by Dr. Andrzej Kotlicki and published in the proceedings [3]

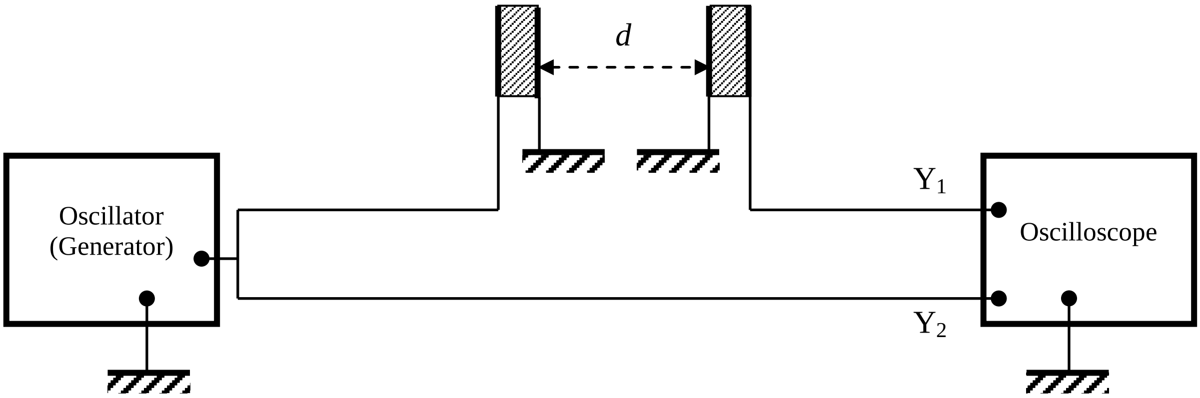


Fig. 4.3

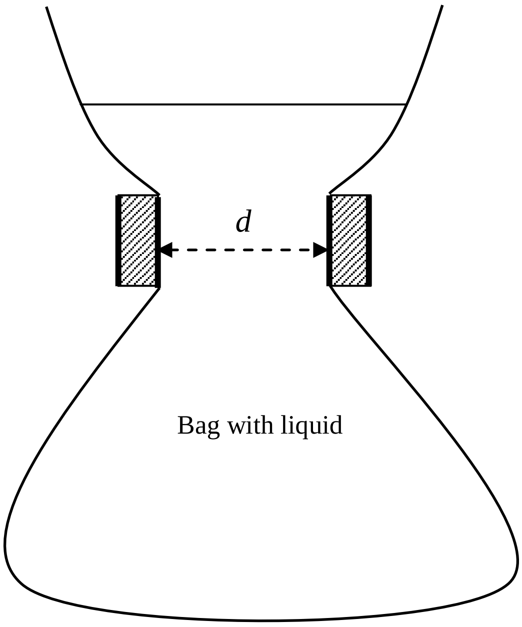


Fig. 4.4

One searches for the resonance by slowly changing the frequency of the oscillator in the range 100 – 1000 kHz and watching the signal on the oscilloscope. In this way the students could find a strong resonance at frequency $f \approx 220$ kHz. Other resonance peaks could be found at about 110 kHz and 670 kHz. They should have been neglected as they are substantially weaker. (They correspond to some other modes of vibrations.) Accuracy of these measurements was 10 kHz (due to the width of the resonance and the accuracy of the scale on the generator).

B. The ultrasonic waves pass through the liquid and generate an electric signal in the receiver. Using the same set-up (Fig. 4.3 and 4.4) we can measure dependence of the phase shift between the signals at Y_1 and Y_2 vs. distance between the piezoelectric discs d at the constant frequency found in point **A**. This phase shift is $\Delta\varphi = 2\pi df / v_l + \varphi_0$, where v_l denotes velocity of ultrasound in the liquid. φ_0 denotes a constant phase shift occurring when ultrasound passes trough the bag walls (possibly zero). The graph representing dependence $d(\Delta\varphi)$ should be a straight line. Its slope allows to determine v_l and its error. In general, the measurements of $\Delta\varphi$ are difficult for many reflections in the bag, which perturb the signal. One of the best ways is to measure d only for $\Delta\varphi = n\pi$ (n - integer) as such points can be found rather easy. Many technical details concerning measurements can be found in [3] (pp. 37 and 38).

The liquids given to the students were water and glycerin. In the standard solution the author of the problem received the following values:

$$v_{\text{water}} = (1.50 \pm 0.10)\cdot 10^3 \text{ m/s}; \quad v_{\text{glycerin}} = (1.96 \pm 0.10)\cdot 10^3 \text{ m/s}.$$

The ratio of these values was 1.31 ± 0.15.

The ultrasonic waves are partly reflected or scattered by the walls of the bag. This effect somewhat affects measurements of the phase shift. To minimize its role one can measure the phase shift (for a given distance) or distance (at the same phase shift) several times, each time changing the shape of the bag. As regards errors in determination of velocities it is worth to mention that the most important factor affecting them was the error in determination of the frequency. This error, however, practically does not affect the ratio of velocities.

Marking Scheme

Frequency estimation

1. Formula — 1 point
2. Result (with units) — 1 point
3. Method of experimental determining the resonance frequency — 1 point
4. Result (if within 5% of standard value) — 2 points
5. Error — 1 point

Measurements of velocities

1. Explanation of the method — 2 points
2. Proper number of measurements in each series — 3 points
3. Result for velocity in the first liquid (if within 5% of standard value) — 2 points
4. Error of the above — 1 point
5. Result for velocity in the second liquid (if within 5% of standard value) — 2 points
6. Error of the above — 1 point

Ratio of velocities

1. Result (if within 3% of standard value) — 2 points
2. Error of the above — 1 point

Typical mistakes

The results of this problem were very good (more than a half of competitors obtained more than 15 points). Nevertheless, many students encountered some difficulties in estimation of the frequency. Some of them assumed presence of nodes at the flat surfaces of the discs (this assumption is not adequate to the situation, but accidentally gives proper formula). In part B some students tried to find distances between nodes and antinodes for ultrasonic standing wave in the liquid. This approach gave false results as the pattern of standing waves in the bag is very complicated and changes when the shape of the bag is changed.

Acknowledgement

I would like to thank very warmly to Prof. Jan Mostowski and Dr. Andrzej Wysmołek for reading the text of this article and for valuable critical remarks. I express special thanks to Dr. Andrzej Kotlicki for critical reviewing the experimental part of the article and for a number of very important improvements.

Literature

[1] **Waldemar Gorzkowski** and **Andrzej Kotlicki**, *XX Międzynarodowa Olimpiada Fizyczna - cz. I*, Fizyka w Szkole *nr* **1/90**, pp. 34 - 39

[2] **Waldemar Gorzkowski**, *XX Międzynarodowa Olimpiada Fizyczna - cz. II*, Fizyka w Szkole *nr* **2/3-90**, pp. 23 - 32

[3] ***XX International Physics Olympiad*** - *Proceedings of the XX International Physics Olympiad, Warsaw (Poland), July 16 - 24, 1989*, ed. by W. Gorzkowski, World Scientific Publishing Company, Singapore 1990 [ISBN 981-02-0084-6]

THE 21st INTERNATIONAL PHYSICS OLYMPIAD - 1990
GRONINGEN, THE NETHERLANDS

Hans Jordens
Department of Science and Technology, University of Groningen, The Netherlands.

Question 1. X-ray Diffraction from a crystal.

We wish to study X-ray diffraction by a cubic crystal lattice. To do this we start with the diffraction of a plane, monochromatic wave that falls perpendicularly on a 2-dimensional grid that consists of N_1 x N_2 slits with separations d_1 and d_2. The diffraction pattern is observed on a screen at a distance L from the grid. The screen is parallel to the grid and L is much larger than d_1 and d_2.

a - Determine the positions and widths of the principal maximum on the screen. The width is defined as the distance between the minima on either side of the maxima.

We consider now a cubic crystal, with lattice spacing a and size $N_0.a$ x $N_0.a$ x $N_1.a$. N_1 is much smaller than N_0. The crystal is placed in a parallel X-ray beam along the z-axis at an angle Θ (see Fig. 1). The diffraction pattern is again observed on a screen at a great distance from the crystal.

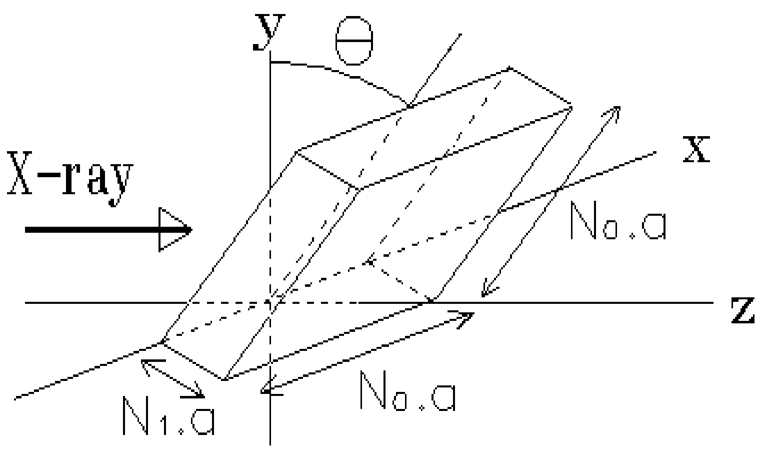


Figure 1 Diffraction of a parallel X-ray beam along the z-axis.
The angle between the crystal and the y-axis is Θ.

b - Calculate the position and width of the maxima as a function of the angle Θ (for small Θ).
- What in particular are the consequences of the fact that $N_1 << N_0$.

The diffraction pattern can also be derived by means of Bragg's theory, in which it is assumed that the X-rays are reflected from atomic planes in the lattice. The diffraction pattern then arises from interference of these reflected rays with each other.

c - Show that this so-called Bragg reflection yields the same conditions for the maxima as those that you found in b.

In some measurements the so-called powder method is employed. A beam of X-rays is scattered by a powder of very many, small crystals. (Of course the sizes of the crystals are much larger than the lattice spacing, a).
Scattering of X-rays of wavelength 0.15 nm by Potassium Chloride [KCl] (which has a cubic lattice, see Fig.2) results in the production of concentric dark circles on a photographic plate. The distance between the crystals and the plate is 0.10 m, and the radius of the smallest circle is 0.053 m (see Fig. 3). K^+ and Cl^- ions have almost the same size, and they may be treated as identical scattering centres.

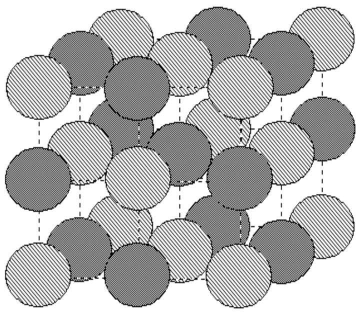


Figure 2. The cubic latice of Potassium Chloride in which the K^+ and Cl^-ions have almost the same size.

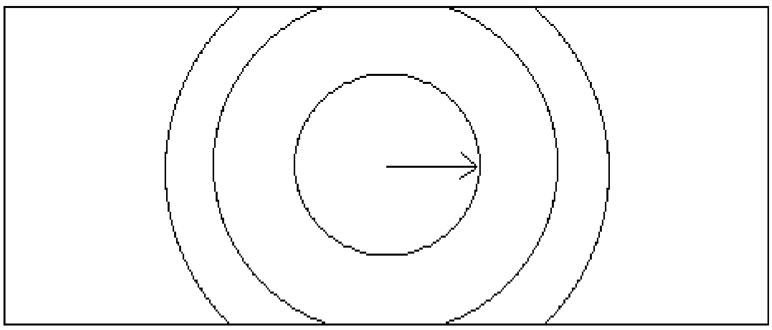


Figure 3. Scattering of X-rays by a powder of KCl crystals results in the production of concentric dark circles on a photographic plate.

d - Calculate the distance between two neighbouring K ions in the crystal.

Question 2. Electric experiments in the magnetosphere of the earth.

In May 1991 the spaceship Atlantis will be placed in orbit around the earth. We shall assume that this orbit will be circular and that it lies in the earth's equatorial plane.
At some predetermined moment the spaceship will release a satellite S, which is attached to a conducting rod of length L. We suppose that the rod is rigid, has negligible mass, and is covered by an electrical insulator. We also neglect all friction. Let α be the angle that the rod makes to the line between the Atlantis and the centre of the earth. (see Fig. 1).
S also lies in the equatorial plane. Assume that the mass of the satellite is much smaller than that of the Atlantis, and that L is much smaller than the radius of the orbit.

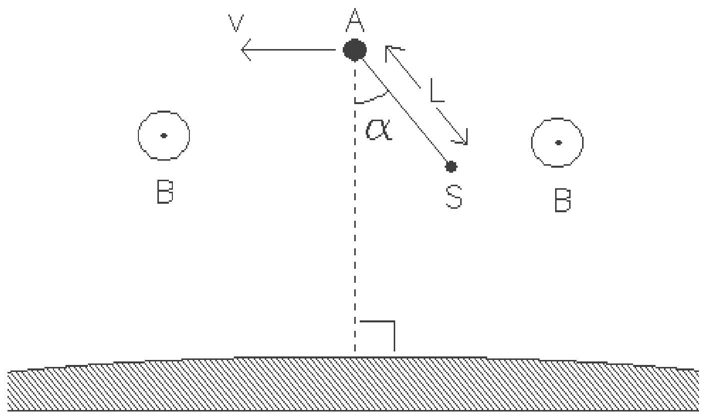


Figure 1 The spaceship Atlantis (A) with a satellite (S) in an orbit around the earth. The orbit lies in the earth's equatorial plane.
The magnetic field (B) is perpendicular to the diagram and is directed towards the reader.

a_1 - Deduce for which value(s) of α the configuration of the spaceship and satellite remain unchanged (with respect to the earth)? In other words, for which value(s) of α is α constant?

a_2 - Discuss the stability of the equilibrium for each case.

Suppose that, at a given moment,
the rod deviates from the stable configuration by a small angle. The system will begin to swing like a pendulum.

b - Express the period of the swinging in terms of the period of revolution of the system around the earth.

In Fig. 1 the magnetic field of the earth is perpendicular to the diagram and is directed towards the reader. Due to the orbital velocity of the rod, a potential difference arises between its ends. The environment (the magnetosphere) is a rarefied, ionised gas with a very good electrical conductivity. Contact with the ionised gas is made by means of electrodes in A (the Atlantis) and S (the satellite). As a consequence of the motion, a current, I, flows through the rod.

c_1 - In which direction does the current flow through the rod? (Take $\alpha = 0$)

Data:	- the period of the orbit	$T = 5{,}4 \cdot 10^3$ s
	- length of the rod	$L = 2{,}0 \cdot 10^4$ m
	- magnetic field strength of the eart at the height of the satellite	$B = 5{,}0 \cdot 10^{-5}$ Wb.m^{-2}
	- the mass of the shuttle Atlantis	$m = 1{,}0 \cdot 10^5$ kg

Next, a current source inside the shuttle is included in the circuit, which maintaines a net direct current of 0.1 A in the opposite direction.

c_2 - How long must this current be maintained to change the altitude of the orbit by 10 m.
Assume that α remains zero. Ignore all contributions from currents in the magnetosphere.
- Does the altitude decrease or increase?

Question 3. The rotating neutron star.

A 'millisecond pulsar' is a source of radiation in the universe that emits very short pulses with a period of one to several milliseconds. This radiation is in the radio range of wavelengths; and a suitable radio receiver can be used to detect the separate pulses and thereby to measure the period with great accuracy.

These radio pulses originate from the surface of a particular sort of star, the so-called neutron star. These stars are very compact: they have a mass of the same order of magnitude as that of the sun, but their radius is only a few tens of kilometers. They spin very quickly. Because of the fast rotation, a neutron star is slightly flattened (oblate). Assume the axial cross-section of the surface to be an ellipse with almost equal axes. Let r_p be the polar and r_e the equatorial radii; and let us define the flattening factor by:

$$\epsilon = \frac{(r_e - r_p)}{r_p}$$

Consider a neutron star with a mass of $2.0 \cdot 10^{30}$ kg,
an average radius of $1.0 \cdot 10^{4}$ m,
and a rotation period of $2.0 \cdot 10^{-2}$ s.

a - Calculate the flattening factor, given that the gravitational constant is $6.67 \cdot 10^{-11}$ $N.m^2.kg^{-2}$.

In the long run (over many years) the rotation of the star slows down, due to energy loss, and this leads to a decrease in the flattening. The star has however a solid crust that floats on a liquid interior. The solid crust resists a continuous adjustment to equilibrium shape. Instead, starquakes occur with sudden changes in the shape of the crust towards equilibrium. During and after such a star-quake the angular velocity is observed to change according to figure 1.

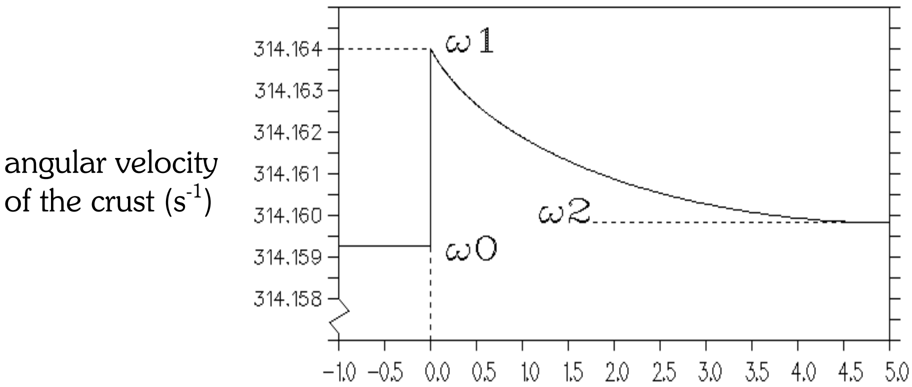


Figure 1

A sudden change in the shape of the crust of a neutron star results in a sudden change of the angular velocity.

b - Calculate the average radius of the liquid interior, using the data of Fig. 1. Make the approximation that the densities of the crust and the interior are the same. (Ignore the change in shape of the interior).

Question 4. Determination of the efficiency of a LED.

Introduction

In this experiment we shall use two modern semiconductors: the light-emitting diode (LED) and the photo-diode (PD). In a LED, part of the electrical energy is used to excite electrons to higher energy levels. When such an excited electron falls back to a lower energy level, a photon with energy E_{photon} is emitted, where

$$E_{photon} = \frac{h.c}{\lambda}$$

Here h is Planck's constant, c is the speed of light, and λ is the wavelength of the emitted light. The efficiency of the LED is defined to be the ratio between the radiated power, ϕ , and the electrical power used, P_{LED}:

$$\eta = \frac{\phi}{P_{LED}}$$

In a photo-diode, radiant energy is transformed into electrical energy. When light falls on the sensitive surface of a photo-diode, some (but not all) of the photons free some (but not all) of the electrons from the crystal structure. The ratio between the number of incoming photons per second, N_p, and the number of freed electrons per second, N_e, is called the quantum efficiency, q_p

$$q_p = \frac{N_e}{N_p}$$

The experiment

The purpose of this experiment is to determine the efficiency of a LED as a function of the current that flows through the LED. To do this, we will measure the intensity of the emitted light with a photo-diode. The LED and the PD have been mounted in two boxes, and they are connected to a circuit panel (Fig. 1). By measuring the potential difference across the LED, and across the resistors R_1 and R_3, one can determine both the potential differences across, and the currents flowing through the LED and the PD.

We use the multimeter to measure VOLTAGES only!! This is done by turning the knob to position 'V'. The meter selects the appropriate sensitivity range automatically. If the display is not on "AUTO" switch "off" and push on "V" again. Connection: "COM" and "V-Ω".

The box containing the photo-diode and the box containing the LED can be moved freely over the board. If both boxes are positioned opposite to each other, then the LED, the PD and the hole in the box containing the PD remain in a straight line.

Data:- The quantum efficiency of the photo-diode	$q_p = 0.88$
- The detection surface of the PD is	2.75×2.75 mm^2
- The wave-length of the light emitted from the LED is	635 nm.
- The internal resistance of the voltmeter is:	100 MΩ in the range up to 200 mV 10 MΩ in the other ranges.
The range is indicated by small numbers on the display.	
- Planck's constant	$h = 6.63 \cdot 10^{-34}$ J.s
- The elementary quantum of charge	$e = 1.6 \cdot 10^{-19}$ C
- The speed of light in vacuo	$c = 3.00 \cdot 10^{8}$ m.s^{-1}

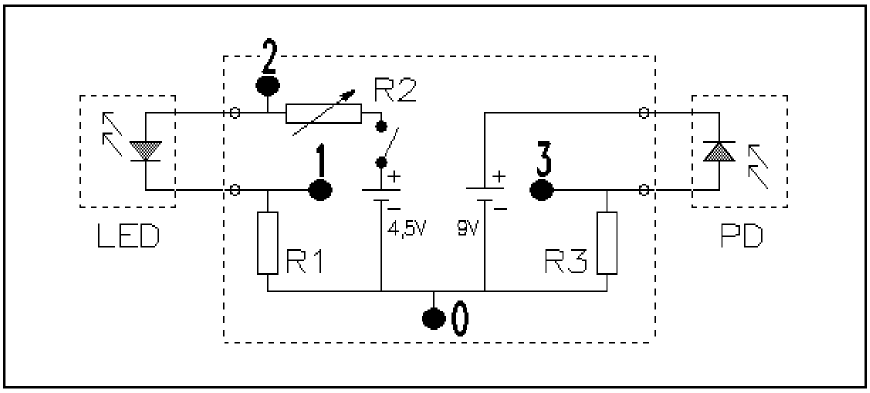


Figure 1.

R_1 = 100 Ω
R_2 = variable resistor
R_3 = 1 MΩ

The points labelled 0, 1, 2 and 3 are measuring points.

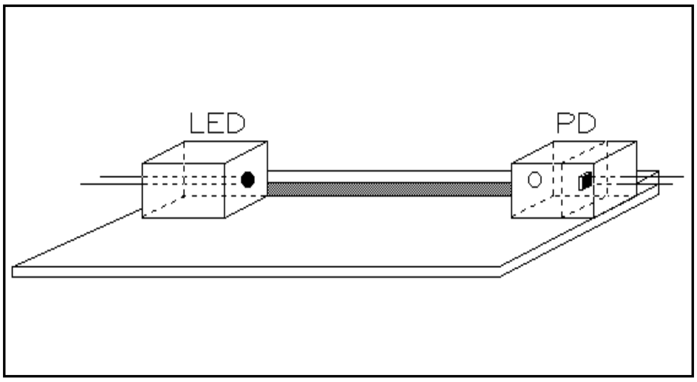


Figure 2 The experimental setup: a board and the two boxes containing the LED and the photo-diode.

Instructions

1. Before we can determine the efficiency of the LED, we must first calibrate the photo-diode. The problem is that we know nothing about the LED.

   Show experimentally that the relation between the current flowing through the photo-diode and the intensity of light falling on it, I [$J.s^{-1}.m^{-2}$], is linear.

2. Determine the current for which the LED has maximal efficiency.

3. Carry out an experiment to measure the maximal (absolute) efficiency of the LED.

No marks (points) will be allocated for an error analysis (in THIS experiment only). Please summarize data in tables and graphs with clear indications of quantities (and units).

Question 5. Determination of the ratio of the magnetic field strengths of two different magnets.

Introduction

When a conductor moves in a magnetic field, currents are induced: these are the so-called eddy currents. As a consequence of the interaction between the magnetic field and the induced currents, the moving conductor suffers a reactive force. Thus an aluminium disk that rotates in the neighbourhood of a stationary magnet experiences a braking force.

Material available

1. A stand.
2. A clamp.
3. An homogenius aluminium disk on an axle, in a holder, that can rotate.
4. Two magnets. The geometry of each is the same (up to 1%); each consists of a clip containing two small magnets of identical magnetization and area, the whole producing a homogenius field, B_1 or B_2.
5. Two weights. One weight has twice the mass (up to 1%) of the other.
6. A stop-watch.
7. A ruler.

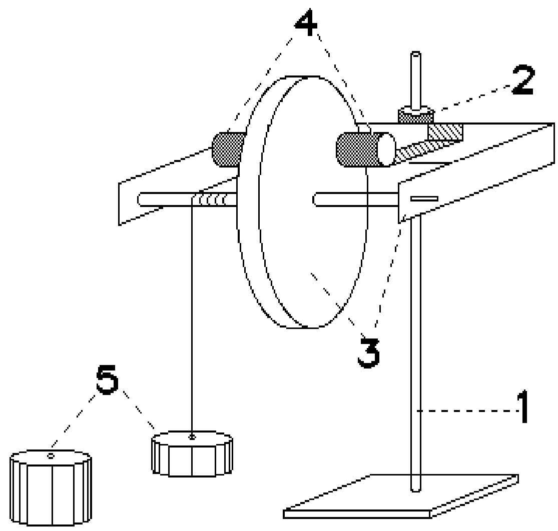


Figure 1.

The experiment

The aluminium disk is fixed to an axle, around which a cord is wrapped. A weight hangs from the cord; and when the weight is released, the disk accelerates until a constant angular velocity is reached. The terminal speed depends, among other things, on the magnitude of the magnetic field strength of the magnet.
Two magnets of different field strengths B_1 or B_2, are available. Either can be fitted on to the holder that carries the aluminium disk: they may be interchanged.

Instructions

1. Think of an experiment in which the ratio of the magnetic field strengths B_1 and B_2, of the two magnets can be measured as accurately as possible.

2. Give a - short - theoretical treatment, indicating how one can obtain the ratio from the measurements.

3. Carry out the experiment and determine the ratio.

4. GIVE AN ERROR ESTIMATION.

Use of the stopwatch

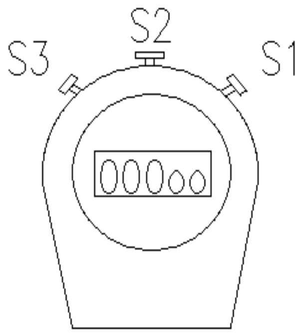


Figure 2.

The stop-watch has three buttons: S_1, S_2 and S_3 (see Fig. 2).

Button S_2 toggles between the date-time and the stop-watch modes. Switch to the stop-watch mode. One should see this:

OOOoo

On pressing S_1 once, the stop-watch begins timing. To stop it, press S_1 a second time.

The stop-watch can be reset to zero by pressing S_3 once.

Solution of question 1.

a - Consider first the x-direction. If waves coming from neighbouring slits (with separation d_1) traverse paths of lengths that differ by:

$$\Delta_1 = n_1.\lambda$$

where n_1 is an integer, then a principal maximum occurs. The position on the screen (in the x-direction) is:

$$x_{n_1} = \frac{n_1.\lambda.L}{d_1}$$

since $d_1 << d_2$.

The path difference between the middle slit and one of the slits at the edge is then:

$$\Delta_{(\frac{N_1}{2})} = \frac{N_1}{2}.n_1.\lambda$$

If on the other hand this path difference is:

$$\Delta_{(\frac{N_1}{2})} = \frac{N_1}{2}.n_1.\lambda + \frac{\lambda}{2}$$

then the first minimum, next to the principal maximum, occurs. The position of this minimum on the screen is given by:

$$x_{n_1} + \Delta x = \frac{\left(\frac{N_1}{2}.n_1.\lambda + \frac{\lambda}{2}\right).L}{\frac{N_1}{2}.d_1} = \frac{n_1.\lambda.L}{d_1} + \frac{\lambda.L}{N_1.d_1}$$

$$\rightarrow \quad \Delta x = \frac{\lambda.L}{N_1.d_1}$$

The width of the principal maximum is accordingly:

$$2.\Delta x = 2.\frac{\lambda.L}{N_1.d_1}$$

A similar treatment can be made for the y-direction, in which there are N_2 slits with separation d_2. The positions and widths of the principal maximal are:

$$(x_{n_1}, y_{n_2}) = (\frac{n_1.\lambda.L}{d_1}, \frac{n_2.\lambda.L}{d_2})$$

$$2.\Delta x = 2.\frac{\lambda.L}{N_1.d_1} \quad ; \qquad 2.\Delta y = 2.\frac{\lambda.L}{N_2.d_2}$$

An alternative method of solution is to calculate the intensity for the 2-dimensional grid as a function of the angle that the beam makes with the screen.

b - In the x-direction the beam 'sees' a grid with spacing a, so that in this direction we have:

$$x_{n_1} = \frac{n_1.\lambda.L}{a} \qquad \Delta x = 2.\frac{\lambda.L}{N_0.a}$$

In the y-direction, the beam 'sees' a grid with effective spacing a.cos(Θ). Analogously, we obtain:

$$y_{n_2} = \frac{n_2.\lambda.L}{a.\cos(\theta)} \qquad \Delta y = 2.\frac{\lambda.L}{N_0.a.\cos(\theta)}$$

In the z-direction, the beam 'sees' a grid with effective spacing a.sin(Θ). This gives rise to principal maxima with position and width:

$$y'_{n_3} = \frac{n_3.\lambda.L}{a.\sin(\theta)} \qquad \Delta y' = 2.\frac{\lambda.L}{N_1.a.\sin(\theta)}$$

This pattern is superimposed on the previous one. Since sin(Θ) is very small, only the zeroth-order pattern will be seen, and it is very broad, since $N_1.\sin(\Theta) << N_0$. The diffraction pattern from a plane wave falling on a thin plate of a cubic crystal, at a small angle of incidence to the normal, will be almost identical to that from a two-dimensional grid.

c - In Bragg reflection, the path difference for constructive interference between neighbouring planes:

$$\Delta = 2.a.\sin(\phi) \approx 2.a.\phi = n.\lambda \quad \rightarrow \quad \frac{x}{L} \approx 2.\phi \approx \frac{n.\lambda}{a} \quad \rightarrow \quad x \approx \frac{n.\lambda.L}{a}$$

Here ϕ is the angle of diffraction.
This is the same condition for a maximum as in section b.

d - For the distance, √2.a, between neighbouring K ions we have:

$$tg(2\phi) = \frac{x}{L} = \frac{0{,}053}{0{,}1} \approx 0{,}53 \rightarrow a = \frac{\lambda}{2.\sin(\phi)} \approx \frac{0{,}15.10^{-9}}{2\ .\ 0{,}24} \approx 0{,}31\ nm$$

$$K-K \approx \sqrt{2}.0{,}31 \approx 0{,}44\ nm$$

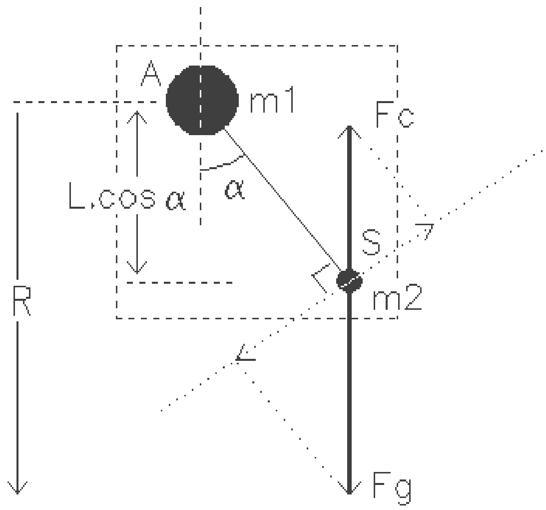
Marking Breakdown

a	position of principal maxima	:1
	width of principal maxima	:3
b	lattice constants	:1
	effect of thickness	:2
c	Bragg reflection	:2
d	Calculation of K-K spacing	:1

Solution of question 2.

a_1 - Since $m_2 << m_1$, the Atlantis travels around the earth with a constant speed. The motion of the satellite is composed of the circular motion of the Atlantis about the earth and (possibly) a circular motion of the satellite about the Atlantis.
For m_1 we have:

$$m_1.\Omega^2.R = \frac{G.m_1.m_a}{R^2} \rightarrow \Omega^2 = \frac{G.m_a}{R^3}$$



For m_2 we have:

$$m_2.L.\ddot{\alpha} = -(F_g - F_c).\sin(\alpha) = -\left(\frac{G.m_2.m_a}{(R - L.\cos(\alpha))^2} - m_2.\Omega^2.(R - L.\cos(\alpha))\right).\sin(\alpha)$$

Using the approximation:

$$\frac{1}{(R - L.\cos(\alpha))^2} \approx \frac{1}{R^2} + \frac{2.L.\cos(\alpha)}{R^3}$$

and equation (1), one finds:

$$L.\ddot{\alpha} = -\left(\frac{G.m_a}{R^2} + \frac{2.G.m_a}{R^3}.L.\cos(\alpha) - \frac{G.m_a}{R^3}.R + \frac{G.m_a}{R^3}.L.\cos(\alpha)\right).\sin(\alpha)$$

so:

$$\ddot{\alpha} + 3.\Omega^2.\sin(\alpha).\cos(\alpha) = 0 \qquad (2)$$

If α is constant: $\ddot{\alpha} = 0$ --> $\sin(\alpha) = 0$ --> $\alpha = 0$; $\alpha = \pi$
--> $\cos(\alpha) = 0$ --> $\alpha = \pi/2$; $\alpha = 3\pi/2$

a_2 - The situation is stable if the moment $M = m_2.L.\ddot{\alpha}.L = m_2.L^2.\ddot{\alpha}$ changes sign in a manner opposed to that in which the sign of $\alpha - \alpha_0$ changes:

sign($\alpha - \alpha_0$)	- +	- +	- +	- +	- +
α	0	$\pi/2$	π	$3\pi/2$	2π
sign(M)	+ -	- +	+ -	- +	+ -
α	0	$\pi/2$	π	$3\pi/2$	2π

The equilibrium about the angles 0 en π is thus stable, whereas that around $\pi/2$ and $3\pi/2$ is unstable.

b - For small values of α equation (2) becomes:

$$\ddot{\alpha} + 3.\Omega^2.\alpha = 0$$

This is the equation of a simple harmonic motion.
The square of the angular frequency is:

$$\omega^2 = 3.\Omega^2$$

so:

$$\omega = \Omega.\sqrt{3} \quad \rightarrow \quad T_1 = \frac{2\pi}{\omega} = \frac{1}{3}\sqrt{3}.\left(\frac{2\pi}{\Omega}\right) \approx 0{,}58.T_0$$

c_1 - According to Lenz's law, there will be a current from the satellite (S) towards the shuttle (A).

c_2 - For the total energy of the system we have:

$$U = U_{kin} + U_{pot} = \frac{1}{2}.m.\Omega^2.R^2 - \frac{G.m.m_a}{R} = -\frac{1}{2}.\frac{G.m.m_a}{R}$$

A small change in the radius of the orbit corresponds to a change in the energy of:

$$\Delta U = \frac{1}{2}.\frac{G.m.m_a}{R^2}.\Delta R = \frac{1}{2}.m.\Omega^2.R.\Delta R$$

In the situation under c_1 energy is absorbed from the system as a consequence of which the radius of the orbit will decrease.
Is a current source inside the shuttle included in the circuit, which maintains a net current in the opposite direction, energy is absorbed by the system as a consequence of which the radius of the orbit will increase.
From the assumptions in c_2 we have:

$$\Delta U = F_l.v.t = B.I.L.\Omega.R.t = \frac{1}{2}.m.\Omega^2.R.\Delta R \quad \rightarrow \quad t = \frac{1}{2}.\frac{m.\Omega.\Delta R}{B.I.L}$$

Numerical application gives for the time: $t \approx 5{,}8 \cdot 10^3$ s; which is about the period of the system.

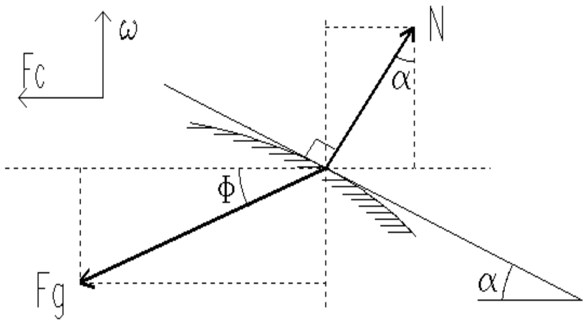
Marking breakdown:

a_1		: 1
a_2		: 1
b	- Atlantis in uniform circular motion	: 0,5
	- calculation of the period Ω	: 0,5
	- equation of motion of the satellite	: 2,5
	- equation of motion for small angles	: 0,5
	- period of oscillations	: 1
c_1 -		: 1
c_2	- calculation of the time the current has to be maintained	: 1,5
	- increase or decrease of the radius of the orbit	: 0,5

Solution of question 3.

a - *1st method*

For equilibrium we have $F_c = F_g + N$ where N is normal to the surface.



Resolving into horizontal and vertical components, we find:

$$\begin{array}{l} F_g.\cos(\phi) = F_c + N.\sin(\alpha) \\ F_g.\sin(\phi) = N.\cos(\alpha) \end{array} \rightarrow F_g.\cos(\phi) = F_c + F_g.\sin(\phi).tg(\alpha)$$

From:

$$F_g = \frac{G.M}{r^2} \ , \quad F_c = \omega^2.r \ , \quad x = r.\cos(\phi) \ , \quad y = r.\sin(\phi) \ \ en \quad tg(\alpha) = \frac{dy}{dx}$$

we find:

$$y.dy + \left(1 - \frac{\omega^2.r^3}{G.M}\right).x.dx = 0$$

where:

$$\frac{\omega^2.r^3}{G.M} \approx 7.10^{-4}$$

This means that, although r depends on x and y, the change in the factor in front of xdx is so slight that we can take it to be constant. The solution of Eq. (1) is then an ellipse:

$$\frac{x^2}{r_e^2} + \frac{y^2}{r_p^2} = 1 \rightarrow \frac{r_p}{r_e} = \sqrt{1 - \frac{\omega^2.r^3}{G.M}} \approx 1 - \frac{\omega^2.r^3}{2.G.M}$$

and from this it follows that:

$$\epsilon = \frac{r_e - r_p}{r_e} = \frac{\omega^2.r^3}{2.G.M} \approx 3{,}7.10^{-4}$$

2nd method

For a point mass of 1 kg on the surface,

$$U_{pot} = -\frac{G.M}{r} \qquad U_{kin} = \frac{1}{2}.\omega^2.r^2.\cos^2(\phi)$$

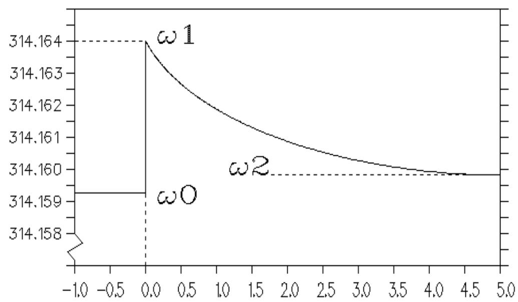
The form of the surface is such that U_{pot} - U_{kin} = constant. For the equator ($\Phi = 0$, $r = r_e$) and for the pole ($\Phi = \pi/2$, $r = r_p$) we have:

$$\frac{G.M}{r_p} = \frac{G.M}{r_e} + \frac{1}{2}.\omega^2.r_e^2 \rightarrow \frac{r_e}{r_p} = 1 + \frac{\omega^2.r_e^3}{2.G.M}$$

Thus:

$$\epsilon = \frac{r_e - r_p}{r_e} = \frac{1 + \frac{\omega^2.r_e^3}{2.G.M} - 1}{1 + \frac{\omega^2.r_e^3}{2.G.M}} \approx \frac{\omega^2.r_e^3}{2.G.M} \approx 3{,}7.10^{-4}$$

b - As a consequence of the star-quake, the moment of inertia of the crust I_m decreases by ΔI_m.



From the conservation of angular momentum, we have:

$$I_m.\omega_0 = (I_m - \Delta I_m).\omega_1 \rightarrow \Delta I_m = I_m.\frac{\omega_1 - \omega_0}{\omega_1}$$

After the internal friction has equalized the angular velocities of the crust and the core, we have:

$$(I_m + I_c).\omega_0 = (I_m + I_c - \Delta I_m).\omega_2 \rightarrow \Delta I_m = (I_m + I_c).\frac{\omega_2 - \omega_0}{\omega_2}$$

$$\frac{I_m}{I_m + I_c} = \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2} \rightarrow 1 - \frac{I_c}{I_m + I_c} = \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2}$$

$$I \;(:)\; R^2$$

$$\rightarrow \frac{I_c}{I_m + I_c} = \frac{r_c^2}{r^2} \rightarrow \frac{r_c}{r} = \sqrt{1 - \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2}} \approx 0.95$$

Marking breakdown

a	1st method	- expressions for the forces	:1
		- equation for the surface	:2
		- equation of ellipse	:1
		- flattening factor	:1
	2nd method	- energy equation	:4
		- flattening factor	:1
b	- conservation of angular momentum for crust		:1.5
	- conservation of angular momentum for crust and core		:1.5
	- moment of inertia for a sphere		:1
	- ratio r_c/r		:1

Solution of question 4.

1. *The linearity of the photo-diode.*

The linearity of the photo-diode can be checked by using the inverse square law between distance and intensity. Suppose that the measured distance between the LED and the (box containing the) PD is x. The intensity of the light falling on the PD satisfies:

$$I(x) = \frac{I_0}{x^2}$$

If the intensity is indeed proportional to the current flowing through the PD, it will also be proportional to the voltage, V(x), measured across the resistor R3. From (1) it then follows that:

$$\frac{1}{\sqrt{V(x)}} \propto x$$

To obtain the correct value of V(x), one should subtract from the measured voltage V_1 the voltage V_2 that one measures when the LED is turned off (but the LED box is still in place in front of the PD).

x (cm)	V_1 (V)	V_2 (V)	i_1 (μA)	i_2 (μA)	$1/[i_1(x) - i_2(x)]^{1/2}$ (μA$^{-1/2}$)
1.0	5.66	.003	6.23	.003	0.40
2.0	4.07	.004	4.48	.005	0.47
3.0	3.03	.005	3.33	.005	0.55
4.0	2.32	.006	2.55	.006	0.63
5.0	1.83	.006	2.01	.006	0.71
6.0	1.48	.007	1.63	.007	0.79
7.0	1.23	.007	1.35	.007	0.86
8.0	1.006	.008	1.107	.008	0.95
9.0	0.859	.009	0.945	.009	1.03
10.0	0.744	.009	0.818	.009	1.11
11.0	0.648	.010	0.713	.010	1.19
12.0	0.570	.011	0.627	.011	1.27
13.0	0.507	.012	0.558	.012	1.35
14.0	0.456	.012	0.502	.012	1.43
15.0	0.414	.013	0.455	.013	1.50
16.0	0.373	.013	0.410	.014	1.59
17.0	0.341	.014	0.375	.014	1.66
18.0	0.312	.014	0.343	.014	1.74
19.0	0.291	.015	0.320	.015	1.81
20.0	0.272	.015	0.299	.015	1.88

Plotted on a graph, one finds a perfect straight line.

2. *The light intensity as a function of the electrical power of the LED*

The photo-current i_{PD} is determined from the voltage V over R3 = 1MΩ. The meter itself has an internal resistance of 100 MΩ in the 200 mV range and 10 MΩ in the other ranges. We have then: i_{PD} = 1.01 V resp. i_{PD} = 1.1 V where V is in volts and i_{PD} in μA. The current in ampères through the LED is the voltage over R1 in volts, divided by 100.

|---------- PD ----------| |---------------------- LED -------------------------|
[x = 5 cm]

V_1 (V)	V_2(V)	$i_1 - i_2$ (μA)	i_{LED} (10^{-2} A)	V_{LED} (V)	P_{LED} (10^{-2} W)	$(i_1 - i_2)/P_{LED}$
1.806	.0061	1.98	2.70	1.752	4.73	0.419
1.637	.0061	1.79	2.30	1.742	4.01	0.446
1.511	.0061	1.66	2.08	1.735	3.61	0.460
1.225	.0061	1.34	1.606	1.722	2.77	0.484
1.117	.0061	1.22	1.433	1.718	2.46	0.496
0.903	.0061	0.99	1.123	1.705	1.91	0.518
0.711	.0061	0.78	0.889	1.708	1.52	0.513
0.448	.0061	0.49	0.555	1.673	0.93	0.527
0.315	.0061	0.34	0.410	1.659	0.68	0.5
0.192	.0061	0.21	0.258	1.637	0.42	0.2

The efficiency is proportional to $(i_1 - i_2)/P_{LED}$. In the graph of $(i_1 - i_2)/P_{LED}$ against i_{LED} the maximal efficiency corresponds to $i_{LED} = 0{,}6 \cdot 10^{-2}$ A. (See figure 2.)

3. *Determination of the maximal efficiency.*

The LED emits a conical beam with cylindrical symmetry. Suppose we measure the light intensity with a PD of sensitive area d^2 at a distance r_i from the axis of symmetry. Let the intensity of the light there be $\Phi(r_i)$, then we have:

$$i(r_i) = N_e.e = N_f.q_f.e = \frac{\Phi(r_i)}{h.\nu}.q_f.e$$

$$\Phi = \sum_i \Phi(r_i).\frac{2.\pi.r_i.d}{d^2} = \frac{2.\pi}{d}.\sum_i \Phi(r_i).r_i = \frac{2.\pi}{d}.\frac{h.\nu}{q_f.e}.\sum_i i(r_i).r_i$$

r_i (mm)	V_1 (V)	V_2 (V)	$(i_1 - i_2).r_i$ (x 10^{-9} Am)	r_i (mm)	V_1 (V)	V_2 (V)	$(i_1 - i_2).r_i$ (x 10^{-9} Am)
0	1.833	0.006	0	39	0.097	0.006	
3	1.906	0.006	6.27	42	0.089	0.006	4.16
6	1.846	0.006	12.54	45	0.082	0.006	3.86
9	1.750	0.006	17.28	48	0.071	0.006	3.79
12	1.347	0.006	17.76	51	0.066	0.006	3.48
15	0.997	0.006	16.20	54	0.050	0.006	3.39
18	0.643	0.006	12.60	57	0.045	0.006	2.52
21	0.313	0.006	7.14	60	0.037	0.006	2.45
24	0.343	0.006	8.88	63	0.032	0.006	2.08
27	0.637	0.006	18.90	66	0.023	0.006	1.83
30	0.681	0.006	22.20	69	0.017	0.006	1.27
33	0.266	0.006	9.57	72	0.014	0.006	0.88
36	0.119	0.006	4.48	75	0.011	0.006	0.68
							0.49

The efficiency = $\Phi/P_{LED} \approx 0.001$

Marking breakdown

1 linearity of the PD
- inverse square law :1.5
- number of measuring points [1,3>; [3,5>; [5,..> :0.5/1.0/1.5
- dark current :0.5
- correct graph :1

2 determination of current at maximal effiency
- principle :0.5
- number of measuring points [1,3>; [3,5>; [5,..> :0.5/1.0/1.5
- graph efficiency-current :0.5
- determination of current at maximal efficiency :0.5

3 determination of the maximal efficiency
- determination of the emitted light intensity :1.5
  - via estimation of the cone cross-section :0.5
  - via measurement of the intensity distribution :1.5
- determination of the maximum efficiency :1

Solution of question 5.

1. Theory

Let
- the moment of inertia of the disk be : I
- the mass of the weight : m
- the moment of the frictional force : M_f
- magnetic field strength : B
- the radius of the axle : r
- the moment of the magnetic force : M_B

For the motion of the rotating disk we have:

$$I.\alpha = (m.g - m.a).r - M_f - M_B$$

We suppose that M_f is constant but not negligible. Because the disk moves in the magnetic field, eddy currents are set up in the disk. The magnitude of these currents is proportional to B and to the angular velocity. The Lorentz force as a result of the eddy currents and the magnetic field is thus proportional to the square of B and to the angular velocity, i.e.

$$M_B = c.B^2.\omega$$

Substituting this into Eq. (1), we find:

$$I.\alpha = (m.g - m.a).r - M_f - c.B^2.\omega$$

$$v_e = (\frac{g.r^2}{c.B^2}).\left(m - \frac{M_f}{g.r}\right)$$

After some time, the disk will reach its final constant angular velocity; the angular acceleration is now zero and for the final velocity v_e we find:

The final constant velocity is thus a linear function of m.

2. The experiment

The final constant speed is determined by measuring the time taken to fall the last 21 cm [this is the width of a sheet of paper].

In the first place it is necessary to check that the final speed has been reached. This is done by allowing the weight to fall over different heights. It is clear that, with the weaker magnet, the necessary height before the constant speed is attained will be larger. Measurements for the weak magnet system:

---------------- time taken to fall ----------------

height (m)	smaller weight	larger weight
0.30	5.04 ± 0.02 (s)	2.00 ± 0.01 (s)
0.40	4.67 ± 0.04 (s)	1.71 ± 0.02 (s)
0.50	4.59 ± 0.05 (s)	1.55 ± 0.02 (s)
0.60	4.44 ± 0.06 (s)	1.48 ± 0.01 (s)
0.70	4.49 ± 0.05 (s)	1.44 ± 0.04 (s)
0.80	4.43 ± 0.03 (s)	1.38 ± 0.03 (s)
0.90	4.43 ± 0.04 (s)	1.35 ± 0.02 (s)
1.10	- - -	1.34 ± 0.05 (s)
1.30	- - -	1.33 ± 0.04 (s)

3. *Final constant speed measurements for both magnet systems and for several choices of weight.*

Measurements for the weak magnet:

weight	T (s)	T (s)	T (s)	T (s)	<T> (s)	<v> (m/s)
small	4.42	4.23	4.24	4.33	4.31 ± 0.09	4.9 ± 0.1
large	1.89	1.91	1.98	1.92	1.93 ± 0.04	10.9 ± 0.2
both	1.29	1.32	1.23	1.30	1.29 ± 0.04	16.3 ± 0.5

Measurements for the strong magnet:

weight	T (s)	T (s)	T (s)	T (s)	<T> (s)	<v> (m/s)
small	8.93	9.01	9.17	8.91	9.0 ± 0.1	2.33 ± 0.03
large	4.03	3.92	4.03	3.95	3.98 ± 0.06	5.28 ± 0.08
both	2.53	2.52	2.53	2.48	2.52 ± 0.03	8.3 ± 0.1

4. *Discussion of the results:*

- A graph between v_e and the weight should be made.
- From Eq. (2) we observe that:
    - both straight lines should intersect on the horizontal axis.
    - from the square-root of the ratio of the slopes we have immediately the ratio of the magnetic field strengths.
    - For the above measurements we find:

$$\frac{B_1}{B_2} = \sqrt{\frac{7.22}{15}} \approx 0.69 \quad \rightarrow \quad \frac{\Delta\left(\frac{B_1}{B_2}\right)}{\left(\frac{B_1}{B_2}\right)} = \frac{1}{2}.\sqrt{\left(\frac{\Delta r_1}{r_1}\right)^2 + \left(\frac{\Delta r_2}{r_2}\right)^2} \approx 0.05$$

$$\frac{B_1}{B_2} = 0.69 \pm 0.03$$

Marking Breakdown

1	$M_B = c.B^2.\omega$		: 1
	Eq. (2)		: 1
2	Investigation of the range in which the speed is constant		: 2
3	Number of timing measurements [1,2,3,...]		: 0,1,2
		Error estimation	: 0.5
4	graph	- quality	: 0.5
		- the lines intersect each other on the mass-axis	: 1
		- calculation of B_1/B_2	: 1
		- Error calculation	: 1