

International Competition in Mathematics for  
Universtiy Students  
in  
Plovdiv, Bulgaria  
1994

## PROBLEMS AND SOLUTIONS

First day — July 29, 1994

**Problem 1.** (13 points)

a) Let  $A$  be a  $n \times n$ ,  $n \geq 2$ , symmetric, invertible matrix with real positive elements. Show that  $z_n \leq n^2 - 2n$ , where  $z_n$  is the number of zero elements in  $A^{-1}$ .

b) How many zero elements are there in the inverse of the  $n \times n$  matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 1 & 2 & \dots & \dots \end{pmatrix} ?$$

**Solution.** Denote by  $a_{ij}$  and  $b_{ij}$  the elements of  $A$  and  $A^{-1}$ , respectively. Then for  $k \neq m$  we have  $\sum_{i=0}^n a_{ki}b_{im} = 0$  and from the positivity of  $a_{ij}$  we conclude that at least one of  $\{b_{im} : i = 1, 2, \dots, n\}$  is positive and at least one is negative. Hence we have at least two non-zero elements in every column of  $A^{-1}$ . This proves part a). For part b) all  $b_{ij}$  are zero except  $b_{1,1} = 2$ ,  $b_{n,n} = (-1)^n$ ,  $b_{i,i+1} = b_{i+1,i} = (-1)^i$  for  $i = 1, 2, \dots, n-1$ .

**Problem 2.** (13 points)

Let  $f \in C^1(a, b)$ ,  $\lim_{x \rightarrow a^+} f(x) = +\infty$ ,  $\lim_{x \rightarrow b^-} f(x) = -\infty$  and  $f'(x) + f^2(x) \geq -1$  for  $x \in (a, b)$ . Prove that  $b - a \geq \pi$  and give an example where  $b - a = \pi$ .

**Solution.** From the inequality we get

$$\frac{d}{dx}(\operatorname{arctg} f(x) + x) = \frac{f'(x)}{1 + f^2(x)} + 1 \geq 0$$

for  $x \in (a, b)$ . Thus  $\operatorname{arctg} f(x) + x$  is non-decreasing in the interval and using the limits we get  $\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b$ . Hence  $b - a \geq \pi$ . One has equality for  $f(x) = \cotg x$ ,  $a = 0$ ,  $b = \pi$ .

**Problem 3.** (13 points)

Given a set  $S$  of  $2n - 1$ ,  $n \in \mathbb{N}$ , different irrational numbers. Prove that there are  $n$  different elements  $x_1, x_2, \dots, x_n \in S$  such that for all non-negative rational numbers  $a_1, a_2, \dots, a_n$  with  $a_1 + a_2 + \dots + a_n > 0$  we have that  $a_1x_1 + a_2x_2 + \dots + a_nx_n$  is an irrational number.

**Solution.** Let  $\mathbb{I}$  be the set of irrational numbers,  $\mathbb{Q}$  – the set of rational numbers,  $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$ . We work by induction. For  $n = 1$  the statement is trivial. Let it be true for  $n - 1$ . We start to prove it for  $n$ . From the induction argument there are  $n - 1$  different elements  $x_1, x_2, \dots, x_{n-1} \in S$  such that

$$(1) \quad \begin{aligned} & a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} \in \mathbb{I} \\ & \text{for all } a_1, a_2, \dots, a_n \in \mathbb{Q}^+ \text{ with } a_1 + a_2 + \dots + a_{n-1} > 0. \end{aligned}$$

Denote the other elements of  $S$  by  $x_n, x_{n+1}, \dots, x_{2n-1}$ . Assume the statement is not true for  $n$ . Then for  $k = 0, 1, \dots, n - 1$  there are  $r_k \in \mathbb{Q}$  such that

$$(2) \quad \sum_{i=1}^{n-1} b_{ik}x_i + c_kx_{n+k} = r_k \quad \text{for some } b_{ik}, c_k \in \mathbb{Q}^+, \sum_{i=1}^{n-1} b_{ik} + c_k > 0.$$

Also

$$(3) \quad \sum_{k=0}^{n-1} d_kx_{n+k} = R \quad \text{for some } d_k \in \mathbb{Q}^+, \sum_{k=0}^{n-1} d_k > 0, \quad R \in \mathbb{Q}.$$

If in (2)  $c_k = 0$  then (2) contradicts (1). Thus  $c_k \neq 0$  and without loss of generality one may take  $c_k = 1$ . In (2) also  $\sum_{i=1}^{n-1} b_{ik} > 0$  in view of  $x_{n+k} \in \mathbb{I}$ .

Replacing (2) in (3) we get

$$\sum_{k=0}^{n-1} d_k \left( - \sum_{i=1}^{n-1} b_{ik}x_i + r_k \right) = R \quad \text{or} \quad \sum_{i=1}^{n-1} \left( \sum_{k=0}^{n-1} d_k b_{ik} \right) x_i \in \mathbb{Q},$$

which contradicts (1) because of the conditions on  $b$ 's and  $d$ 's.

**Problem 4.** (18 points)

Let  $\alpha \in \mathbb{R} \setminus \{0\}$  and suppose that  $F$  and  $G$  are linear maps (operators) from  $\mathbb{R}^n$  into  $\mathbb{R}^n$  satisfying  $F \circ G - G \circ F = \alpha F$ .

- a) Show that for all  $k \in \mathbb{N}$  one has  $F^k \circ G - G \circ F^k = \alpha k F^k$ .
- b) Show that there exists  $k \geq 1$  such that  $F^k = 0$ .

**Solution.** For a) using the assumptions we have

$$\begin{aligned}
 F^k \circ G - G \circ F^k &= \sum_{i=1}^k \left( F^{k-i+1} \circ G \circ F^{i-1} - F^{k-i} \circ G \circ F^i \right) = \\
 &= \sum_{i=1}^k F^{k-i} \circ (F \circ G - G \circ F) \circ F^{i-1} = \\
 &= \sum_{i=1}^k F^{k-i} \circ \alpha F \circ F^{i-1} = \alpha k F^k.
 \end{aligned}$$

b) Consider the linear operator  $L(F) = F \circ G - G \circ F$  acting over all  $n \times n$  matrices  $F$ . It may have at most  $n^2$  different eigenvalues. Assuming that  $F^k \neq 0$  for every  $k$  we get that  $L$  has infinitely many different eigenvalues  $\alpha k$  in view of a) – a contradiction.

**Problem 5.** (18 points)

a) Let  $f \in C[0, b]$ ,  $g \in C(\mathbb{R})$  and let  $g$  be periodic with period  $b$ . Prove that  $\int_0^b f(x)g(nx)dx$  has a limit as  $n \rightarrow \infty$  and

$$\lim_{n \rightarrow \infty} \int_0^b f(x)g(nx)dx = \frac{1}{b} \int_0^b f(x)dx \cdot \int_0^b g(x)dx.$$

b) Find

$$\lim_{n \rightarrow \infty} \int_0^\pi \frac{\sin x}{1 + 3\cos^2 nx} dx.$$

**Solution.** Set  $\|g\|_1 = \int_0^b |g(x)|dx$  and

$$\omega(f, t) = \sup \{ |f(x) - f(y)| : x, y \in [0, b], |x - y| \leq t \}.$$

In view of the uniform continuity of  $f$  we have  $\omega(f, t) \rightarrow 0$  as  $t \rightarrow 0$ . Using the periodicity of  $g$  we get

$$\begin{aligned}
 \int_0^b f(x)g(nx)dx &= \sum_{k=1}^n \int_{b(k-1)/n}^{bk/n} f(x)g(nx)dx \\
 &= \sum_{k=1}^n f(bk/n) \int_{b(k-1)/n}^{bk/n} g(nx)dx + \sum_{k=1}^n \int_{b(k-1)/n}^{bk/n} \{f(x) - f(bk/n)\}g(nx)dx \\
 &= \frac{1}{n} \sum_{k=1}^n f(bk/n) \int_0^b g(x)dx + O(\omega(f, b/n)\|g\|_1)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b} \sum_{k=1}^n \int_{b(k-1)/n}^{bk/n} f(x) dx \int_0^b g(x) dx \\
&\quad + \frac{1}{b} \sum_{k=1}^n \left( \frac{b}{n} f(bk/n) - \int_{b(k-1)/n}^{bk/n} f(x) dx \right) \int_0^b g(x) dx + O(\omega(f, b/n) \|g\|_1) \\
&= \frac{1}{b} \int_0^b f(x) dx \int_0^b g(x) dx + O(\omega(f, b/n) \|g\|_1).
\end{aligned}$$

This proves a). For b) we set  $b = \pi$ ,  $f(x) = \sin x$ ,  $g(x) = (1 + 3\cos^2 x)^{-1}$ . From a) and

$$\int_0^\pi \sin x dx = 2, \quad \int_0^\pi (1 + 3\cos^2 x)^{-1} dx = \frac{\pi}{2}$$

we get

$$\lim_{n \rightarrow \infty} \int_0^\pi \frac{\sin x}{1 + 3\cos^2 nx} dx = 1.$$

**Problem 6.** (25 points)

Let  $f \in C^2[0, N]$  and  $|f'(x)| < 1$ ,  $f''(x) > 0$  for every  $x \in [0, N]$ . Let  $0 \leq m_0 < m_1 < \dots < m_k \leq N$  be integers such that  $n_i = f(m_i)$  are also integers for  $i = 0, 1, \dots, k$ . Denote  $b_i = n_i - n_{i-1}$  and  $a_i = m_i - m_{i-1}$  for  $i = 1, 2, \dots, k$ .

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \dots < \frac{b_k}{a_k} < 1.$$

b) Prove that for every choice of  $A > 1$  there are no more than  $N/A$  indices  $j$  such that  $a_j > A$ .

c) Prove that  $k \leq 3N^{2/3}$  (i.e. there are no more than  $3N^{2/3}$  integer points on the curve  $y = f(x)$ ,  $x \in [0, N]$ ).

**Solution.** a) For  $i = 1, 2, \dots, k$  we have

$$b_i = f(m_i) - f(m_{i-1}) = (m_i - m_{i-1})f'(x_i)$$

for some  $x_i \in (m_{i-1}, m_i)$ . Hence  $\frac{b_i}{a_i} = f'(x_i)$  and so  $-1 < \frac{b_i}{a_i} < 1$ . From the convexity of  $f$  we have that  $f'$  is increasing and  $\frac{b_i}{a_i} = f'(x_i) < f'(x_{i+1}) = \frac{b_{i+1}}{a_{i+1}}$  because of  $x_i < m_i < x_{i+1}$ .

b) Set  $S_A = \{j \in \{0, 1, \dots, k\} : a_j > A\}$ . Then

$$N \geq m_k - m_0 = \sum_{i=1}^k a_i \geq \sum_{j \in S_A} a_j > A|S_A|$$

and hence  $|S_A| < N/A$ .

c) All different fractions in  $(-1, 1)$  with denominators less or equal  $A$  are no more  $2A^2$ . Using b) we get  $k < N/A + 2A^2$ . Put  $A = N^{1/3}$  in the above estimate and get  $k < 3N^{2/3}$ .

*Second day — July 30, 1994*

**Problem 1.** (14 points)

Let  $f \in C^1[a, b]$ ,  $f(a) = 0$  and suppose that  $\lambda \in \mathbb{R}$ ,  $\lambda > 0$ , is such that

$$|f'(x)| \leq \lambda|f(x)|$$

for all  $x \in [a, b]$ . Is it true that  $f(x) = 0$  for all  $x \in [a, b]$ ?

**Solution.** Assume that there is  $y \in (a, b]$  such that  $f(y) \neq 0$ . Without loss of generality we have  $f(y) > 0$ . In view of the continuity of  $f$  there exists  $c \in [a, y)$  such that  $f(c) = 0$  and  $f(x) > 0$  for  $x \in (c, y]$ . For  $x \in (c, y]$  we have  $|f'(x)| \leq \lambda f(x)$ . This implies that the function  $g(x) = \ln f(x) - \lambda x$  is not increasing in  $(c, y]$  because of  $g'(x) = \frac{f'(x)}{f(x)} - \lambda \leq 0$ . Thus  $\ln f(x) - \lambda x \geq \ln f(y) - \lambda y$  and  $f(x) \geq e^{\lambda x - \lambda y} f(y)$  for  $x \in (c, y]$ . Thus

$$0 = f(c) = f(c + 0) \geq e^{\lambda c - \lambda y} f(y) > 0$$

— a contradiction. Hence one has  $f(x) = 0$  for all  $x \in [a, b]$ .

**Problem 2.** (14 points)

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be given by  $f(x, y) = (x^2 - y^2)e^{-x^2 - y^2}$ .

a) Prove that  $f$  attains its minimum and its maximum.

b) Determine all points  $(x, y)$  such that  $\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0$  and determine for which of them  $f$  has global or local minimum or maximum.

**Solution.** We have  $f(1, 0) = e^{-1}$ ,  $f(0, 1) = -e^{-1}$  and  $te^{-t} \leq 2e^{-2}$  for  $t \geq 2$ . Therefore  $|f(x, y)| \leq (x^2 + y^2)e^{-x^2 - y^2} \leq 2e^{-2} < e^{-1}$  for  $(x, y) \notin M = \{(u, v) : u^2 + v^2 \leq 2\}$  and  $f$  cannot attain its minimum and its

maximum outside  $M$ . Part a) follows from the compactness of  $M$  and the continuity of  $f$ . Let  $(x, y)$  be a point from part b). From  $\frac{\partial f}{\partial x}(x, y) = 2x(1 - x^2 + y^2)e^{-x^2 - y^2}$  we get

$$(1) \quad x(1 - x^2 + y^2) = 0.$$

Similarly

$$(2) \quad y(1 + x^2 - y^2) = 0.$$

All solutions  $(x, y)$  of the system (1), (2) are  $(0, 0)$ ,  $(0, 1)$ ,  $(0, -1)$ ,  $(1, 0)$  and  $(-1, 0)$ . One has  $f(1, 0) = f(-1, 0) = e^{-1}$  and  $f$  has global maximum at the points  $(1, 0)$  and  $(-1, 0)$ . One has  $f(0, 1) = f(0, -1) = -e^{-1}$  and  $f$  has global minimum at the points  $(0, 1)$  and  $(0, -1)$ . The point  $(0, 0)$  is not an extrema point because of  $f(x, 0) = x^2e^{-x^2} > 0$  if  $x \neq 0$  and  $f(y, 0) = -y^2e^{-y^2} < 0$  if  $y \neq 0$ .

**Problem 3.** (14 points)

Let  $f$  be a real-valued function with  $n + 1$  derivatives at each point of  $\mathbb{R}$ . Show that for each pair of real numbers  $a, b$ ,  $a < b$ , such that

$$\ln \left( \frac{f(b) + f'(b) + \cdots + f^{(n)}(b)}{f(a) + f'(a) + \cdots + f^{(n)}(a)} \right) = b - a$$

there is a number  $c$  in the open interval  $(a, b)$  for which

$$f^{(n+1)}(c) = f(c).$$

Note that  $\ln$  denotes the natural logarithm.

**Solution.** Set  $g(x) = (f(x) + f'(x) + \cdots + f^{(n)}(x))e^{-x}$ . From the assumption one get  $g(a) = g(b)$ . Then there exists  $c \in (a, b)$  such that  $g'(c) = 0$ . Replacing in the last equality  $g'(x) = (f^{(n+1)}(x) - f(x))e^{-x}$  we finish the proof.

**Problem 4.** (18 points)

Let  $A$  be a  $n \times n$  diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \cdots (x - c_k)^{d_k},$$

where  $c_1, c_2, \dots, c_k$  are distinct (which means that  $c_1$  appears  $d_1$  times on the diagonal,  $c_2$  appears  $d_2$  times on the diagonal, etc. and  $d_1 + d_2 + \cdots + d_k = n$ ).

Let  $V$  be the space of all  $n \times n$  matrices  $B$  such that  $AB = BA$ . Prove that the dimension of  $V$  is

$$d_1^2 + d_2^2 + \cdots + d_k^2.$$

**Solution.** Set  $A = (a_{ij})_{i,j=1}^n$ ,  $B = (b_{ij})_{i,j=1}^n$ ,  $AB = (x_{ij})_{i,j=1}^n$  and  $BA = (y_{ij})_{i,j=1}^n$ . Then  $x_{ij} = a_{ii}b_{ij}$  and  $y_{ij} = a_{jj}b_{ij}$ . Thus  $AB = BA$  is equivalent to  $(a_{ii} - a_{jj})b_{ij} = 0$  for  $i, j = 1, 2, \dots, n$ . Therefore  $b_{ij} = 0$  if  $a_{ii} \neq a_{jj}$  and  $b_{ij}$  may be arbitrary if  $a_{ii} = a_{jj}$ . The number of indices  $(i, j)$  for which  $a_{ii} = a_{jj} = c_m$  for some  $m = 1, 2, \dots, k$  is  $d_m^2$ . This gives the desired result.

**Problem 5.** (18 points)

Let  $x_1, x_2, \dots, x_k$  be vectors of  $m$ -dimensional Euclidian space, such that  $x_1 + x_2 + \cdots + x_k = 0$ . Show that there exists a permutation  $\pi$  of the integers  $\{1, 2, \dots, k\}$  such that

$$\left\| \sum_{i=1}^n x_{\pi(i)} \right\| \leq \left( \sum_{i=1}^k \|x_i\|^2 \right)^{1/2} \quad \text{for each } n = 1, 2, \dots, k.$$

Note that  $\|\cdot\|$  denotes the Euclidian norm.

**Solution.** We define  $\pi$  inductively. Set  $\pi(1) = 1$ . Assume  $\pi$  is defined for  $i = 1, 2, \dots, n$  and also

$$(1) \quad \left\| \sum_{i=1}^n x_{\pi(i)} \right\|^2 \leq \sum_{i=1}^n \|x_{\pi(i)}\|^2.$$

Note (1) is true for  $n = 1$ . We choose  $\pi(n+1)$  in a way that (1) is fulfilled with  $n+1$  instead of  $n$ . Set  $y = \sum_{i=1}^n x_{\pi(i)}$  and  $A = \{1, 2, \dots, k\} \setminus \{\pi(i) : i = 1, 2, \dots, n\}$ . Assume that  $(y, x_r) > 0$  for all  $r \in A$ . Then  $\left( y, \sum_{r \in A} x_r \right) > 0$  and in view of  $y + \sum_{r \in A} x_r = 0$  one gets  $-(y, y) > 0$ , which is impossible. Therefore there is  $r \in A$  such that

$$(2) \quad (y, x_r) \leq 0.$$

Put  $\pi(n+1) = r$ . Then using (2) and (1) we have

$$\left\| \sum_{i=1}^{n+1} x_{\pi(i)} \right\|^2 = \|y + x_r\|^2 = \|y\|^2 + 2(y, x_r) + \|x_r\|^2 \leq \|y\|^2 + \|x_r\|^2 \leq$$

$$\leq \sum_{i=1}^n \|x_{\pi(i)}\|^2 + \|x_r\|^2 = \sum_{i=1}^{n+1} \|x_{\pi(i)}\|^2,$$

which verifies (1) for  $n + 1$ . Thus we define  $\pi$  for every  $n = 1, 2, \dots, k$ . Finally from (1) we get

$$\left\| \sum_{i=1}^n x_{\pi(i)} \right\|^2 \leq \sum_{i=1}^n \|x_{\pi(i)}\|^2 \leq \sum_{i=1}^k \|x_i\|^2.$$

**Problem 6.** (22 points)

Find  $\lim_{N \rightarrow \infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)}$ . Note that  $\ln$  denotes the natural logarithm.

**Solution.** Obviously

$$(1) \quad A_N = \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} \geq \frac{\ln^2 N}{N} \cdot \frac{N-3}{\ln^2 N} = 1 - \frac{3}{N}.$$

Take  $M$ ,  $2 \leq M < N/2$ . Then using that  $\frac{1}{\ln k \cdot \ln(N-k)}$  is decreasing in  $[2, N/2]$  and the symmetry with respect to  $N/2$  one get

$$\begin{aligned} A_N &= \frac{\ln^2 N}{N} \left\{ \sum_{k=2}^M + \sum_{k=M+1}^{N-M-1} + \sum_{k=N-M}^{N-2} \right\} \frac{1}{\ln k \cdot \ln(N-k)} \leq \\ &\leq \frac{\ln^2 N}{N} \left\{ 2 \frac{M-1}{\ln 2 \cdot \ln(N-2)} + \frac{N-2M-1}{\ln M \cdot \ln(N-M)} \right\} \leq \\ &\leq \frac{2}{\ln 2} \cdot \frac{M \ln N}{N} + \left(1 - \frac{2M}{N}\right) \frac{\ln N}{\ln M} + O\left(\frac{1}{\ln N}\right). \end{aligned}$$

Choose  $M = \left\lfloor \frac{N}{\ln^2 N} \right\rfloor + 1$  to get

$$(2) \quad A_N \leq \left(1 - \frac{2}{N \ln^2 N}\right) \frac{\ln N}{\ln N - 2 \ln \ln N} + O\left(\frac{1}{\ln N}\right) \leq 1 + O\left(\frac{\ln \ln N}{\ln N}\right).$$

Estimates (1) and (2) give

$$\lim_{N \rightarrow \infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} = 1.$$

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## PROBLEMS AND SOLUTIONS

*First day*

**Problem 1.** (10 points)

Let  $X$  be a nonsingular matrix with columns  $X_1, X_2, \dots, X_n$ . Let  $Y$  be a matrix with columns  $X_2, X_3, \dots, X_n, 0$ . Show that the matrices  $A = YX^{-1}$  and  $B = X^{-1}Y$  have rank  $n - 1$  and have only 0's for eigenvalues.

**Solution.** Let  $J = (a_{ij})$  be the  $n \times n$  matrix where  $a_{ij} = 1$  if  $i = j + 1$  and  $a_{ij} = 0$  otherwise. The rank of  $J$  is  $n - 1$  and its only eigenvalues are 0's. Moreover  $Y = XJ$  and  $A = YX^{-1} = XJX^{-1}$ ,  $B = X^{-1}Y = J$ . It follows that both  $A$  and  $B$  have rank  $n - 1$  with only 0's for eigenvalues.

**Problem 2.** (15 points)

Let  $f$  be a continuous function on  $[0, 1]$  such that for every  $x \in [0, 1]$  we have  $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ . Show that  $\int_0^1 f^2(t)dt \geq \frac{1}{3}$ .

**Solution.** From the inequality

$$0 \leq \int_0^1 (f(x) - x)^2 dx = \int_0^1 f^2(x)dx - 2 \int_0^1 xf(x)dx + \int_0^1 x^2 dx$$

we get

$$\int_0^1 f^2(x)dx \geq 2 \int_0^1 xf(x)dx - \int_0^1 x^2 dx = 2 \int_0^1 xf(x)dx - \frac{1}{3}.$$

From the hypotheses we have  $\int_0^1 \int_x^1 f(t)dt dx \geq \int_0^1 \frac{1-x^2}{2} dx$  or  $\int_0^1 tf(t)dt \geq \frac{1}{3}$ . This completes the proof.

**Problem 3.** (15 points)

Let  $f$  be twice continuously differentiable on  $(0, +\infty)$  such that  $\lim_{x \rightarrow 0^+} f'(x) = -\infty$  and  $\lim_{x \rightarrow 0^+} f''(x) = +\infty$ . Show that

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{f'(x)} = 0.$$

**Solution.** Since  $f'$  tends to  $-\infty$  and  $f''$  tends to  $+\infty$  as  $x$  tends to  $0+$ , there exists an interval  $(0, r)$  such that  $f'(x) < 0$  and  $f''(x) > 0$  for all  $x \in (0, r)$ . Hence  $f$  is decreasing and  $f'$  is increasing on  $(0, r)$ . By the mean value theorem for every  $0 < x < x_0 < r$  we obtain

$$f(x) - f(x_0) = f'(\xi)(x - x_0) > 0,$$

for some  $\xi \in (x, x_0)$ . Taking into account that  $f'$  is increasing,  $f'(x) < f'(\xi) < 0$ , we get

$$x - x_0 < \frac{f'(\xi)}{f'(x)}(x - x_0) = \frac{f(x) - f(x_0)}{f'(x)} < 0.$$

Taking limits as  $x$  tends to  $0+$  we obtain

$$-x_0 \leq \liminf_{x \rightarrow 0+} \frac{f(x)}{f'(x)} \leq \limsup_{x \rightarrow 0+} \frac{f(x)}{f'(x)} \leq 0.$$

Since this happens for all  $x_0 \in (0, r)$  we deduce that  $\lim_{x \rightarrow 0+} \frac{f(x)}{f'(x)}$  exists and

$$\lim_{x \rightarrow 0+} \frac{f(x)}{f'(x)} = 0.$$

**Problem 4.** (15 points)

Let  $F : (1, \infty) \rightarrow \mathbb{R}$  be the function defined by

$$F(x) := \int_x^{x^2} \frac{dt}{\ln t}.$$

Show that  $F$  is one-to-one (i.e. injective) and find the range (i.e. set of values) of  $F$ .

**Solution.** From the definition we have

$$F'(x) = \frac{x-1}{\ln x}, \quad x > 1.$$

Therefore  $F'(x) > 0$  for  $x \in (1, \infty)$ . Thus  $F$  is strictly increasing and hence one-to-one. Since

$$F(x) \geq (x^2 - x) \min \left\{ \frac{1}{\ln t} : x \leq t \leq x^2 \right\} = \frac{x^2 - x}{\ln x^2} \rightarrow \infty$$

as  $x \rightarrow \infty$ , it follows that the range of  $F$  is  $(F(1+), \infty)$ . In order to determine  $F(1+)$  we substitute  $t = e^v$  in the definition of  $F$  and we get

$$F(x) = \int_{\ln x}^{2 \ln x} \frac{e^v}{v} dv.$$

Hence

$$F(x) < e^{2 \ln x} \int_{\ln x}^{2 \ln x} \frac{1}{v} dv = x^2 \ln 2$$

and similarly  $F(x) > x \ln 2$ . Thus  $F(1+) = \ln 2$ .

**Problem 5.** (20 points)

Let  $A$  and  $B$  be real  $n \times n$  matrices. Assume that there exist  $n + 1$  different real numbers  $t_1, t_2, \dots, t_{n+1}$  such that the matrices

$$C_i = A + t_i B, \quad i = 1, 2, \dots, n + 1,$$

are nilpotent (i.e.  $C_i^n = 0$ ).

Show that both  $A$  and  $B$  are nilpotent.

**Solution.** We have that

$$(A + tB)^n = A^n + tP_1 + t^2P_2 + \dots + t^{n-1}P_{n-1} + t^nB^n$$

for some matrices  $P_1, P_2, \dots, P_{n-1}$  not depending on  $t$ .

Assume that  $a, p_1, p_2, \dots, p_{n-1}, b$  are the  $(i, j)$ -th entries of the corresponding matrices  $A^n, P_1, P_2, \dots, P_{n-1}, B^n$ . Then the polynomial

$$bt^n + p_{n-1}t^{n-1} + \dots + p_2t^2 + p_1t + a$$

has at least  $n + 1$  roots  $t_1, t_2, \dots, t_{n+1}$ . Hence all its coefficients vanish. Therefore  $A^n = 0$ ,  $B^n = 0$ ,  $P_i = 0$ ; and  $A$  and  $B$  are nilpotent.

**Problem 6.** (25 points)

Let  $p > 1$ . Show that there exists a constant  $K_p > 0$  such that for every  $x, y \in \mathbb{R}$  satisfying  $|x|^p + |y|^p = 2$ , we have

$$(x - y)^2 \leq K_p (4 - (x + y)^2).$$

**Solution.** Let  $0 < \delta < 1$ . First we show that there exists  $K_{p,\delta} > 0$  such that

$$f(x, y) = \frac{(x - y)^2}{4 - (x + y)^2} \leq K_{p,\delta}$$

for every  $(x, y) \in D_\delta = \{(x, y) : |x - y| \geq \delta, |x|^p + |y|^p = 2\}$ .

Since  $D_\delta$  is compact it is enough to show that  $f$  is continuous on  $D_\delta$ . For this we show that the denominator of  $f$  is different from zero. Assume the contrary. Then  $|x + y| = 2$ , and  $\left|\frac{x + y}{2}\right|^p = 1$ . Since  $p > 1$ , the function  $g(t) = |t|^p$  is strictly convex, in other words  $\left|\frac{x + y}{2}\right|^p < \frac{|x|^p + |y|^p}{2}$  whenever  $x \neq y$ . So for some  $(x, y) \in D_\delta$  we have  $\left|\frac{x + y}{2}\right|^p < \frac{|x|^p + |y|^p}{2} = 1 = \left|\frac{x + y}{2}\right|^p$ . We get a contradiction.

If  $x$  and  $y$  have different signs then  $(x, y) \in D_\delta$  for all  $0 < \delta < 1$  because then  $|x - y| \geq \max\{|x|, |y|\} \geq 1 > \delta$ . So we may further assume without loss of generality that  $x > 0$ ,  $y > 0$  and  $x^p + y^p = 2$ . Set  $x = 1 + t$ . Then

$$\begin{aligned} y &= (2 - x^p)^{1/p} = (2 - (1 + t)^p)^{1/p} = \left(2 - (1 + pt + \frac{p(p-1)}{2}t^2 + o(t^2))\right)^{1/p} \\ &= \left(1 - pt - \frac{p(p-1)}{2}t^2 + o(t^2)\right)^{1/p} \\ &= 1 + \frac{1}{p} \left(-pt - \frac{p(p-1)}{2}t^2 + o(t^2)\right) + \frac{1}{2p} \left(\frac{1}{p} - 1\right) (-pt + o(t))^2 + o(t^2) \\ &= 1 - t - \frac{p-1}{2}t^2 + o(t^2) - \frac{p-1}{2}t^2 + o(t^2) \\ &= 1 - t - (p-1)t^2 + o(t^2). \end{aligned}$$

We have

$$(x - y)^2 = (2t + o(t))^2 = 4t^2 + o(t^2)$$

and

$$4 - (x + y)^2 = 4 - (2 - (p-1)t^2 + o(t^2))^2 = 4 - 4 + 4(p-1)t^2 + o(t^2) = 4(p-1)t^2 + o(t^2).$$

So there exists  $\delta_p > 0$  such that if  $|t| < \delta_p$  we have  $(x - y)^2 < 5t^2$ ,  $4 - (x + y)^2 > 3(p - 1)t^2$ . Then

$$(*) \quad (x - y)^2 < 5t^2 = \frac{5}{3(p-1)} \cdot 3(p-1)t^2 < \frac{5}{3(p-1)}(4 - (x + y)^2)$$

if  $|x - 1| < \delta_p$ . From the symmetry we have that (\*) also holds when  $|y - 1| < \delta_p$ .

To finish the proof it is enough to show that  $|x - y| \geq 2\delta_p$  whenever  $|x - 1| \geq \delta_p$ ,  $|y - 1| \geq \delta_p$  and  $x^p + y^p = 2$ . Indeed, since  $x^p + y^p = 2$  we have that  $\max\{x, y\} \geq 1$ . So let  $x - 1 \geq \delta_p$ . Since  $\left(\frac{x+y}{2}\right)^p \leq \frac{x^p + y^p}{2} = 1$  we get  $x + y \leq 2$ . Then  $x - y \geq 2(x - 1) \geq 2\delta_p$ .

*Second day*

**Problem 1.** (10 points)

Let  $A$  be  $3 \times 3$  real matrix such that the vectors  $Au$  and  $u$  are orthogonal for each column vector  $u \in \mathbb{R}^3$ . Prove that:

- $A^\top = -A$ , where  $A^\top$  denotes the transpose of the matrix  $A$ ;
- there exists a vector  $v \in \mathbb{R}^3$  such that  $Au = v \times u$  for every  $u \in \mathbb{R}^3$ , where  $v \times u$  denotes the vector product in  $\mathbb{R}^3$ .

**Solution.** a) Set  $A = (a_{ij})$ ,  $u = (u_1, u_2, u_3)^\top$ . If we use the orthogonality condition

$$(1) \quad (Au, u) = 0$$

with  $u_i = \delta_{ik}$  we get  $a_{kk} = 0$ . If we use (1) with  $u_i = \delta_{ik} + \delta_{im}$  we get

$$a_{kk} + a_{km} + a_{mk} + a_{mm} = 0$$

and hence  $a_{km} = -a_{mk}$ .

- Set  $v_1 = -a_{23}$ ,  $v_2 = a_{13}$ ,  $v_3 = -a_{12}$ . Then

$$Au = (v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1)^\top = v \times u.$$

**Problem 2.** (15 points)

Let  $\{b_n\}_{n=0}^\infty$  be a sequence of positive real numbers such that  $b_0 = 1$ ,  $b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}$ . Calculate

$$\sum_{n=1}^{\infty} b_n 2^n.$$

**Solution.** Put  $a_n = 1 + \sqrt{b_n}$  for  $n \geq 0$ . Then  $a_n > 1$ ,  $a_0 = 2$  and

$$a_n = 1 + \sqrt{1 + a_{n-1} - 2\sqrt{a_{n-1}}} = \sqrt{a_{n-1}},$$

so  $a_n = 2^{2^{-n}}$ . Then

$$\begin{aligned} \sum_{n=1}^N b_n 2^n &= \sum_{n=1}^N (a_n - 1)^2 2^n = \sum_{n=1}^N [a_n^2 2^n - a_n 2^{n+1} + 2^n] \\ &= \sum_{n=1}^N [(a_{n-1} - 1)2^n - (a_n - 1)2^{n+1}] \\ &= (a_0 - 1)2^1 - (a_N - 1)2^{N+1} = 2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}}. \end{aligned}$$

Put  $x = 2^{-N}$ . Then  $x \rightarrow 0$  as  $N \rightarrow \infty$  and so

$$\sum_{n=1}^{\infty} b_n 2^n = \lim_{N \rightarrow \infty} \left( 2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}} \right) = \lim_{x \rightarrow 0} \left( 2 - 2\frac{2^x - 1}{x} \right) = 2 - 2 \ln 2.$$

**Problem 3.** (15 points)

Let all roots of an  $n$ -th degree polynomial  $P(z)$  with complex coefficients lie on the unit circle in the complex plane. Prove that all roots of the polynomial

$$2zP'(z) - nP(z)$$

lie on the same circle.

**Solution.** It is enough to consider only polynomials with leading coefficient 1. Let  $P(z) = (z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n)$  with  $|\alpha_j| = 1$ , where the complex numbers  $\alpha_1, \alpha_2, \dots, \alpha_n$  may coincide.

We have

$$\begin{aligned} \tilde{P}(z) \equiv 2zP'(z) - nP(z) &= (z + \alpha_1)(z - \alpha_2) \dots (z - \alpha_n) + \\ &\quad + (z - \alpha_1)(z + \alpha_2) \dots (z - \alpha_n) + \dots + (z - \alpha_1)(z - \alpha_2) \dots (z + \alpha_n). \end{aligned}$$

Hence,  $\frac{\tilde{P}(z)}{P(z)} = \sum_{k=1}^n \frac{z + \alpha_k}{z - \alpha_k}$ . Since  $\operatorname{Re} \frac{z + \alpha}{z - \alpha} = \frac{|z|^2 - |\alpha|^2}{|z - \alpha|^2}$  for all complex  $z$ ,

$\alpha, z \neq \alpha$ , we deduce that in our case  $\operatorname{Re} \frac{\tilde{P}(z)}{P(z)} = \sum_{k=1}^n \frac{|z|^2 - 1}{|z - \alpha_k|^2}$ . From  $|z| \neq 1$

it follows that  $\operatorname{Re} \frac{\tilde{P}(z)}{P(z)} \neq 0$ . Hence  $\tilde{P}(z) = 0$  implies  $|z| = 1$ .

**Problem 4.** (15 points)

a) Prove that for every  $\varepsilon > 0$  there is a positive integer  $n$  and real numbers  $\lambda_1, \dots, \lambda_n$  such that

$$\max_{x \in [-1, 1]} \left| x - \sum_{k=1}^n \lambda_k x^{2k+1} \right| < \varepsilon.$$

b) Prove that for every odd continuous function  $f$  on  $[-1, 1]$  and for every  $\varepsilon > 0$  there is a positive integer  $n$  and real numbers  $\mu_1, \dots, \mu_n$  such that

$$\max_{x \in [-1, 1]} \left| f(x) - \sum_{k=1}^n \mu_k x^{2k+1} \right| < \varepsilon.$$

Recall that  $f$  is odd means that  $f(x) = -f(-x)$  for all  $x \in [-1, 1]$ .

**Solution.** a) Let  $n$  be such that  $(1 - \varepsilon^2)^n \leq \varepsilon$ . Then  $|x(1 - x^2)^n| < \varepsilon$  for every  $x \in [-1, 1]$ . Thus one can set  $\lambda_k = (-1)^{k+1} \binom{n}{k}$  because then

$$x - \sum_{k=1}^n \lambda_k x^{2k+1} = \sum_{k=0}^n (-1)^k \binom{n}{k} x^{2k+1} = x(1 - x^2)^n.$$

b) From the Weierstrass theorem there is a polynomial, say  $p \in \Pi_m$ , such that

$$\max_{x \in [-1, 1]} |f(x) - p(x)| < \frac{\varepsilon}{2}.$$

Set  $q(x) = \frac{1}{2}\{p(x) - p(-x)\}$ . Then

$$f(x) - q(x) = \frac{1}{2}\{f(x) - p(x)\} - \frac{1}{2}\{f(-x) - p(-x)\}$$

and

$$(1) \max_{|x| \leq 1} |f(x) - q(x)| \leq \frac{1}{2} \max_{|x| \leq 1} |f(x) - p(x)| + \frac{1}{2} \max_{|x| \leq 1} |f(-x) - p(-x)| < \frac{\varepsilon}{2}.$$

But  $q$  is an odd polynomial in  $\Pi_m$  and it can be written as

$$q(x) = \sum_{k=0}^m b_k x^{2k+1} = b_0 x + \sum_{k=1}^m b_k x^{2k+1}.$$

If  $b_0 = 0$  then (1) proves b). If  $b_0 \neq 0$  then one applies a) with  $\frac{\varepsilon}{2|b_0|}$  instead of  $\varepsilon$  to get

$$(2) \quad \max_{|x| \leq 1} \left| b_0 x - \sum_{k=1}^n b_0 \lambda_k x^{2k+1} \right| < \frac{\varepsilon}{2}$$

for appropriate  $n$  and  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Now b) follows from (1) and (2) with  $\max\{n, m\}$  instead of  $n$ .

**Problem 5.** (10+15 points)

a) Prove that every function of the form

$$f(x) = \frac{a_0}{2} + \cos x + \sum_{n=2}^N a_n \cos(nx)$$

with  $|a_0| < 1$ , has positive as well as negative values in the period  $[0, 2\pi)$ .

b) Prove that the function

$$F(x) = \sum_{n=1}^{100} \cos(n^{\frac{3}{2}}x)$$

has at least 40 zeros in the interval  $(0, 1000)$ .

**Solution.** a) Let us consider the integral

$$\int_0^{2\pi} f(x)(1 \pm \cos x) dx = \pi(a_0 \pm 1).$$

The assumption that  $f(x) \geq 0$  implies  $a_0 \geq 1$ . Similarly, if  $f(x) \leq 0$  then  $a_0 \leq -1$ . In both cases we have a contradiction with the hypothesis of the problem.

b) We shall prove that for each integer  $N$  and for each real number  $h \geq 24$  and each real number  $y$  the function

$$F_N(x) = \sum_{n=1}^N \cos(xn^{\frac{3}{2}})$$

changes sign in the interval  $(y, y+h)$ . The assertion will follow immediately from here.

Consider the integrals

$$I_1 = \int_y^{y+h} F_N(x) dx, \quad I_2 = \int_y^{y+h} F_N(x) \cos x dx.$$

If  $F_N(x)$  does not change sign in  $(y, y+h)$  then we have

$$|I_2| \leq \int_y^{y+h} |F_N(x)| dx = \left| \int_y^{y+h} F_N(x) dx \right| = |I_1|.$$

Hence, it is enough to prove that

$$|I_2| > |I_1|.$$

Obviously, for each  $\alpha \neq 0$  we have

$$\left| \int_y^{y+h} \cos(\alpha x) dx \right| \leq \frac{2}{|\alpha|}.$$

Hence

$$(1) \quad |I_1| = \left| \sum_{n=1}^N \int_y^{y+h} \cos(xn^{\frac{3}{2}}) dx \right| \leq 2 \sum_{n=1}^N \frac{1}{n^{\frac{3}{2}}} < 2 \left( 1 + \int_1^{\infty} \frac{dt}{t^{\frac{3}{2}}} \right) = 6.$$

On the other hand we have

$$\begin{aligned} I_2 &= \sum_{n=1}^N \int_y^{y+h} \cos x \cos(xn^{\frac{3}{2}}) dx \\ &= \frac{1}{2} \int_y^{y+h} (1 + \cos(2x)) dx + \\ &\quad + \frac{1}{2} \sum_{n=2}^N \int_y^{y+h} \left( \cos(x(n^{\frac{3}{2}} - 1)) + \cos(x(n^{\frac{3}{2}} + 1)) \right) dx \\ &= \frac{1}{2}h + \Delta, \end{aligned}$$

where

$$|\Delta| \leq \frac{1}{2} \left( 1 + 2 \sum_{n=2}^N \left( \frac{1}{n^{\frac{3}{2}} - 1} + \frac{1}{n^{\frac{3}{2}} + 1} \right) \right) \leq \frac{1}{2} + 2 \sum_{n=2}^N \frac{1}{n^{\frac{3}{2}} - 1}.$$

We use that  $n^{\frac{3}{2}} - 1 \geq \frac{2}{3}n^{\frac{3}{2}}$  for  $n \geq 3$  and we get

$$|\Delta| \leq \frac{1}{2} + \frac{2}{2^{\frac{3}{2}} - 1} + 3 \sum_{n=3}^N \frac{1}{n^{\frac{3}{2}}} < \frac{1}{2} + \frac{2}{2\sqrt{2} - 1} + 3 \int_2^{\infty} \frac{dt}{t^{\frac{3}{2}}} < 6.$$

Hence

$$(2) \quad |I_2| > \frac{1}{2}h - 6.$$

We use that  $h \geq 24$  and inequalities (1), (2) and we obtain  $|I_2| > |I_1|$ . The proof is completed.

**Problem 6.** (20 points)

Suppose that  $\{f_n\}_{n=1}^{\infty}$  is a sequence of continuous functions on the interval  $[0, 1]$  such that

$$\int_0^1 f_m(x)f_n(x)dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

and

$$\sup\{|f_n(x)| : x \in [0, 1] \text{ and } n = 1, 2, \dots\} < +\infty.$$

Show that there exists no subsequence  $\{f_{n_k}\}$  of  $\{f_n\}$  such that  $\lim_{k \rightarrow \infty} f_{n_k}(x)$  exists for all  $x \in [0, 1]$ .

**Solution.** It is clear that one can add some functions, say  $\{g_m\}$ , which satisfy the hypothesis of the problem and the closure of the finite linear combinations of  $\{f_n\} \cup \{g_m\}$  is  $L_2[0, 1]$ . Therefore without loss of generality we assume that  $\{f_n\}$  generates  $L_2[0, 1]$ .

Let us suppose that there is a subsequence  $\{n_k\}$  and a function  $f$  such that

$$f_{n_k}(x) \xrightarrow[k \rightarrow \infty]{} f(x) \text{ for every } x \in [0, 1].$$

Fix  $m \in \mathbb{N}$ . From Lebesgue's theorem we have

$$0 = \int_0^1 f_m(x)f_{n_k}(x)dx \xrightarrow[k \rightarrow \infty]{} \int_0^1 f_m(x)f(x)dx.$$

Hence  $\int_0^1 f_m(x)f(x)dx = 0$  for every  $m \in \mathbb{N}$ , which implies  $f(x) = 0$  almost everywhere. Using once more Lebesgue's theorem we get

$$1 = \int_0^1 f_{n_k}^2(x)dx \xrightarrow[k \rightarrow \infty]{} \int_0^1 f^2(x)dx = 0.$$

The contradiction proves the statement.

International Competition in Mathematics for  
Universtiy Students  
in  
Plovdiv, Bulgaria  
1996

## PROBLEMS AND SOLUTIONS

*First day — August 2, 1996*

**Problem 1.** (10 points)

Let for  $j = 0, \dots, n$ ,  $a_j = a_0 + jd$ , where  $a_0, d$  are fixed real numbers.

Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_0 & a_1 & \dots & a_{n-1} \\ a_2 & a_1 & a_0 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}.$$

Calculate  $\det(A)$ , where  $\det(A)$  denotes the determinant of  $A$ .

**Solution.** Adding the first column of  $A$  to the last column we get that

$$\det(A) = (a_0 + a_n) \det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ a_1 & a_0 & a_1 & \dots & 1 \\ a_2 & a_1 & a_0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & a_{n-2} & \dots & 1 \end{pmatrix}.$$

Subtracting the  $n$ -th row of the above matrix from the  $(n+1)$ -st one,  $(n-1)$ -st from  $n$ -th,  $\dots$ , first from second we obtain that

$$\det(A) = (a_0 + a_n) \det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ d & -d & -d & \dots & 0 \\ d & d & -d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & 0 \end{pmatrix}.$$

Hence,

$$\det(A) = (-1)^n (a_0 + a_n) \det \begin{pmatrix} d & -d & -d & \dots & -d \\ d & d & -d & \dots & -d \\ d & d & d & \dots & -d \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & d \end{pmatrix}.$$

Adding the last row of the above matrix to the other rows we have

$$\det(A) = (-1)^n (a_0 + a_n) \det \begin{pmatrix} 2d & 0 & 0 & \dots & 0 \\ 2d & 2d & 0 & \dots & 0 \\ 2d & 2d & 2d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & d \end{pmatrix} = (-1)^n (a_0 + a_n) 2^{n-1} d^n.$$

**Problem 2.** (10 points)

Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx,$$

where  $n$  is a natural number.

**Solution.** We have

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx \\ &= \int_0^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx + \int_{-\pi}^0 \frac{\sin nx}{(1+2^x)\sin x} dx. \end{aligned}$$

In the second integral we make the change of variable  $x = -x$  and obtain

$$\begin{aligned} I_n &= \int_0^{\pi} \frac{\sin nx}{(1+2^x)\sin x} dx + \int_0^{\pi} \frac{\sin nx}{(1+2^{-x})\sin x} dx \\ &= \int_0^{\pi} \frac{(1+2^x)\sin nx}{(1+2^x)\sin x} dx \\ &= \int_0^{\pi} \frac{\sin nx}{\sin x} dx. \end{aligned}$$

For  $n \geq 2$  we have

$$\begin{aligned} I_n - I_{n-2} &= \int_0^{\pi} \frac{\sin nx - \sin (n-2)x}{\sin x} dx \\ &= 2 \int_0^{\pi} \cos (n-1)x dx = 0. \end{aligned}$$

The answer

$$I_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \pi & \text{if } n \text{ is odd} \end{cases}$$

follows from the above formula and  $I_0 = 0$ ,  $I_1 = \pi$ .

**Problem 3.** (15 points)

The linear operator  $A$  on the vector space  $V$  is called an involution if  $A^2 = E$  where  $E$  is the identity operator on  $V$ . Let  $\dim V = n < \infty$ .

(i) Prove that for every involution  $A$  on  $V$  there exists a basis of  $V$  consisting of eigenvectors of  $A$ .

(ii) Find the maximal number of distinct pairwise commuting involutions on  $V$ .

**Solution.**

(i) Let  $B = \frac{1}{2}(A + E)$ . Then

$$B^2 = \frac{1}{4}(A^2 + 2AE + E) = \frac{1}{4}(2AE + 2E) = \frac{1}{2}(A + E) = B.$$

Hence  $B$  is a projection. Thus there exists a basis of eigenvectors for  $B$ , and the matrix of  $B$  in this basis is of the form  $\text{diag}(1, \dots, 1, 0, \dots, 0)$ .

Since  $A = 2B - E$  the eigenvalues of  $A$  are  $\pm 1$  only.

(ii) Let  $\{A_i : i \in I\}$  be a set of commuting diagonalizable operators on  $V$ , and let  $A_1$  be one of these operators. Choose an eigenvalue  $\lambda$  of  $A_1$  and denote  $V_\lambda = \{v \in V : A_1 v = \lambda v\}$ . Then  $V_\lambda$  is a subspace of  $V$ , and since  $A_1 A_i = A_i A_1$  for each  $i \in I$  we obtain that  $V_\lambda$  is invariant under each  $A_i$ . If  $V_\lambda = V$  then  $A_1$  is either  $E$  or  $-E$ , and we can start with another operator  $A_i$ . If  $V_\lambda \neq V$  we proceed by induction on  $\dim V$  in order to find a common eigenvector for all  $A_i$ . Therefore  $\{A_i : i \in I\}$  are simultaneously diagonalizable.

If they are involutions then  $|I| \leq 2^n$  since the diagonal entries may equal 1 or -1 only.

**Problem 4.** (15 points)

Let  $a_1 = 1$ ,  $a_n = \frac{1}{n} \sum_{k=1}^{n-1} a_k a_{n-k}$  for  $n \geq 2$ . Show that

(i)  $\limsup_{n \rightarrow \infty} |a_n|^{1/n} < 2^{-1/2}$ ;

(ii)  $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \geq 2/3$ .

**Solution.**

(i) We show by induction that

$$(*) \quad a_n \leq q^n \quad \text{for } n \geq 3,$$

where  $q = 0.7$  and use that  $0.7 < 2^{-1/2}$ . One has  $a_1 = 1$ ,  $a_2 = \frac{1}{2}$ ,  $a_3 = \frac{1}{3}$ ,  $a_4 = \frac{11}{48}$ . Therefore (\*) is true for  $n = 3$  and  $n = 4$ . Assume (\*) is true for  $n \leq N - 1$  for some  $N \geq 5$ . Then

$$a_N = \frac{2}{N}a_{N-1} + \frac{1}{N}a_{N-2} + \frac{1}{N} \sum_{k=3}^{N-3} a_k a_{N-k} \leq \frac{2}{N}q^{N-1} + \frac{1}{N}q^{N-2} + \frac{N-5}{N}q^N \leq q^N$$

because  $\frac{2}{q} + \frac{1}{q^2} \leq 5$ .

(ii) We show by induction that

$$a_n \geq q^n \quad \text{for } n \geq 2,$$

where  $q = \frac{2}{3}$ . One has  $a_2 = \frac{1}{2} > \left(\frac{2}{3}\right)^2 = q^2$ . Going by induction we have for  $N \geq 3$

$$a_N = \frac{2}{N}a_{N-1} + \frac{1}{N} \sum_{k=2}^{N-2} a_k a_{N-k} \geq \frac{2}{N}q^{N-1} + \frac{N-3}{N}q^N = q^N$$

because  $\frac{2}{q} = 3$ .

**Problem 5.** (25 points)

(i) Let  $a, b$  be real numbers such that  $b \leq 0$  and  $1 + ax + bx^2 \geq 0$  for every  $x$  in  $[0, 1]$ . Prove that

$$\lim_{n \rightarrow +\infty} n \int_0^1 (1 + ax + bx^2)^n dx = \begin{cases} -\frac{1}{a} & \text{if } a < 0, \\ +\infty & \text{if } a \geq 0. \end{cases}$$

(ii) Let  $f : [0, 1] \rightarrow [0, \infty)$  be a function with a continuous second derivative and let  $f''(x) \leq 0$  for every  $x$  in  $[0, 1]$ . Suppose that  $L = \lim_{n \rightarrow \infty} n \int_0^1 (f(x))^n dx$  exists and  $0 < L < +\infty$ . Prove that  $f'$  has a constant sign and  $\min_{x \in [0, 1]} |f'(x)| = L^{-1}$ .

**Solution.** (i) With a linear change of the variable (i) is equivalent to:

(i') Let  $a, b, A$  be real numbers such that  $b \leq 0$ ,  $A > 0$  and  $1 + ax + bx^2 > 0$  for every  $x$  in  $[0, A]$ . Denote  $I_n = n \int_0^A (1 + ax + bx^2)^n dx$ . Prove that

$$\lim_{n \rightarrow +\infty} I_n = -\frac{1}{a} \quad \text{when } a < 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} I_n = +\infty \quad \text{when } a \geq 0.$$

Let  $a < 0$ . Set  $f(x) = e^{ax} - (1 + ax + bx^2)$ . Using that  $f(0) = f'(0) = 0$  and  $f''(x) = a^2 e^{ax} - 2b$  we get for  $x > 0$  that

$$0 < e^{ax} - (1 + ax + bx^2) < cx^2$$

where  $c = \frac{a^2}{2} - b$ . Using the mean value theorem we get

$$0 < e^{anx} - (1 + ax + bx^2)^n < cx^2 n e^{a(n-1)x}.$$

Therefore

$$0 < n \int_0^A e^{anx} dx - n \int_0^A (1 + ax + bx^2)^n dx < cn^2 \int_0^A x^2 e^{a(n-1)x} dx.$$

Using that

$$n \int_0^A e^{anx} dx = \frac{e^{anA} - 1}{a} \xrightarrow{n \rightarrow \infty} -\frac{1}{a}$$

and

$$\int_0^A x^2 e^{a(n-1)x} dx < \frac{1}{|a|^3 (n-1)^3} \int_0^\infty t^2 e^{-t} dt$$

we get (i') in the case  $a < 0$ .

Let  $a \geq 0$ . Then for  $n > \max\{A^{-2}, -b\} - 1$  we have

$$\begin{aligned} n \int_0^A (1 + ax + bx^2)^n dx &> n \int_0^{\frac{1}{\sqrt{n+1}}} (1 + bx^2)^n dx \\ &> n \cdot \frac{1}{\sqrt{n+1}} \cdot \left(1 + \frac{b}{n+1}\right)^n \\ &> \frac{n}{\sqrt{n+1}} e^b \xrightarrow{n \rightarrow \infty} \infty. \end{aligned}$$

(i) is proved.

(ii) Denote  $I_n = n \int_0^1 (f(x))^n dx$  and  $M = \max_{x \in [0,1]} f(x)$ .

For  $M < 1$  we have  $I_n \leq nM^n \xrightarrow{n \rightarrow \infty} 0$ , a contradiction.

If  $M > 1$  since  $f$  is continuous there exists an interval  $I \subset [0, 1]$  with  $|I| > 0$  such that  $f(x) > 1$  for every  $x \in I$ . Then  $I_n \geq n|I| \xrightarrow{n \rightarrow \infty} +\infty$ , a contradiction. Hence  $M = 1$ . Now we prove that  $f'$  has a constant sign. Assume the opposite. Then  $f'(x_0) = 0$  for some  $x \in (0, 1)$ . Then

$f(x_0) = M = 1$  because  $f'' \leq 0$ . For  $x_0 + h$  in  $[0, 1]$ ,  $f(x_0 + h) = 1 + \frac{h^2}{2}f''(\xi)$ ,  $\xi \in (x_0, x_0 + h)$ . Let  $m = \min_{x \in [0,1]} f''(x)$ . So,  $f(x_0 + h) \geq 1 + \frac{h^2}{2}m$ .

Let  $\delta > 0$  be such that  $1 + \frac{\delta^2}{2}m > 0$  and  $x_0 + \delta < 1$ . Then

$$I_n \geq n \int_{x_0}^{x_0+\delta} (f(x))^n dx \geq n \int_0^\delta \left(1 + \frac{m}{2}h^2\right)^n dh \xrightarrow{n \rightarrow \infty} \infty$$

in view of (i') – a contradiction. Hence  $f$  is monotone and  $M = f(0)$  or  $M = f(1)$ .

Let  $M = f(0) = 1$ . For  $h$  in  $[0, 1]$

$$1 + hf'(0) \geq f(h) \geq 1 + hf'(0) + \frac{m}{2}h^2,$$

where  $f'(0) \neq 0$ , because otherwise we get a contradiction as above. Since  $f(0) = M$  the function  $f$  is decreasing and hence  $f'(0) < 0$ . Let  $0 < A < 1$  be such that  $1 + Af'(0) + \frac{m}{2}A^2 > 0$ . Then

$$n \int_0^A (1 + hf'(0))^n dh \geq n \int_0^A (f(x))^n dx \geq n \int_0^A \left(1 + hf'(0) + \frac{m}{2}h^2\right)^n dh.$$

From (i') the first and the third integral tend to  $-\frac{1}{f'(0)}$  as  $n \rightarrow \infty$ , hence so does the second.

Also  $n \int_A^1 (f(x))^n dx \leq n(f(A))^n \xrightarrow{n \rightarrow \infty} 0$  ( $f(A) < 1$ ). We get  $L = -\frac{1}{f'(0)}$  in this case.

If  $M = f(1)$  we get in a similar way  $L = \frac{1}{f'(1)}$ .

**Problem 6.** (25 points)

Upper content of a subset  $E$  of the plane  $\mathbb{R}^2$  is defined as

$$\mathcal{C}(E) = \inf \left\{ \sum_{i=1}^n \text{diam}(E_i) \right\}$$

where inf is taken over all finite families of sets  $E_1, \dots, E_n$ ,  $n \in \mathbb{N}$ , in  $\mathbb{R}^2$  such that  $E \subset \bigcup_{i=1}^n E_i$ .

Lower content of  $E$  is defined as

$$\mathcal{K}(E) = \sup \{ \text{lenght}(L) \quad : \quad L \text{ is a closed line segment} \\ \text{onto which } E \text{ can be contracted} \}.$$

Show that

- (a)  $\mathcal{C}(L) = \text{lenght}(L)$  if  $L$  is a closed line segment;
- (b)  $\mathcal{C}(E) \geq \mathcal{K}(E)$ ;
- (c) the equality in (b) needs not hold even if  $E$  is compact.

**Hint.** If  $E = T \cup T'$  where  $T$  is the triangle with vertices  $(-2, 2)$ ,  $(2, 2)$  and  $(0, 4)$ , and  $T'$  is its reflexion about the  $x$ -axis, then  $\mathcal{C}(E) = 8 > \mathcal{K}(E)$ .

**Remarks:** All *distances* used in this problem are Euclidian. *Diameter* of a set  $E$  is  $\text{diam}(E) = \sup\{\text{dist}(x, y) : x, y \in E\}$ . *Contraction* of a set  $E$  to a set  $F$  is a mapping  $f : E \mapsto F$  such that  $\text{dist}(f(x), f(y)) \leq \text{dist}(x, y)$  for all  $x, y \in E$ . A set  $E$  can be contracted *onto* a set  $F$  if there is a contraction  $f$  of  $E$  to  $F$  which is onto, i.e., such that  $f(E) = F$ . *Triangle* is defined as the union of the three segments joining its vertices, i.e., it does not contain the interior.

**Solution.**

(a) The choice  $E_1 = L$  gives  $\mathcal{C}(L) \leq \text{lenght}(L)$ . If  $E \subset \cup_{i=1}^n E_i$  then  $\sum_{i=1}^n \text{diam}(E_i) \geq \text{lenght}(L)$ : By induction,  $n=1$  obvious, and assuming that  $E_{n+1}$  contains the end point  $a$  of  $L$ , define the segment  $L_\varepsilon = \{x \in L : \text{dist}(x, a) \geq \text{diam}(E_{n+1}) + \varepsilon\}$  and use induction assumption to get  $\sum_{i=1}^{n+1} \text{diam}(E_i) \geq \text{lenght}(L_\varepsilon) + \text{diam}(E_{n+1}) \geq \text{lenght}(L) - \varepsilon$ ; but  $\varepsilon > 0$  is arbitrary.

(b) If  $f$  is a contraction of  $E$  onto  $L$  and  $E \subset \cup_{i=1}^n E_i$ , then  $L \subset \cup_{i=1}^n f(E_i)$  and  $\text{lenght}(L) \leq \sum_{i=1}^n \text{diam}(f(E_i)) \leq \sum_{i=1}^n \text{diam}(E_i)$ .

(c1) Let  $E = T \cup T'$  where  $T$  is the triangle with vertices  $(-2, 2)$ ,  $(2, 2)$  and  $(0, 4)$ , and  $T'$  is its reflexion about the  $x$ -axis. Suppose  $E \subset \bigcup_{i=1}^n E_i$ . If no set among  $E_i$  meets both  $T$  and  $T'$ , then  $E_i$  may be partitioned into covers of segments  $[(-2, 2), (2, 2)]$  and  $[(-2, -2), (2, -2)]$ , both of length 4, so  $\sum_{i=1}^n \text{diam}(E_i) \geq 8$ . If at least one set among  $E_i$ , say  $E_k$ , meets both  $T$  and  $T'$ , choose  $a \in E_k \cap T$  and  $b \in E_k \cap T'$  and note that the sets  $E'_i = E_i$  for  $i \neq k$ ,  $E'_k = E_k \cup [a, b]$  cover  $T \cup T' \cup [a, b]$ , which is a set of upper content

at least 8, since its orthogonal projection onto  $y$ -axis is a segment of length

8. Since  $\text{diam}(E_j) = \text{diam}(E'_j)$ , we get  $\sum_{i=1}^n \text{diam}(E_i) \geq 8$ .

(c2) Let  $f$  be a contraction of  $E$  onto  $L = [a', b']$ . Choose  $a = (a_1, a_2)$ ,  $b = (b_1, b_2) \in E$  such that  $f(a) = a'$  and  $f(b) = b'$ . Since  $\text{lenght}(L) = \text{dist}(a', b') \leq \text{dist}(a, b)$  and since the triangles have diameter only 4, we may assume that  $a \in T$  and  $b \in T'$ . Observe that if  $a_2 \leq 3$  then  $a$  lies on one of the segments joining some of the points  $(-2, 2)$ ,  $(2, 2)$ ,  $(-1, 3)$ ,  $(1, 3)$ ; since all these points have distances from vertices, and so from points, of  $T_2$  at most  $\sqrt{50}$ , we get that  $\text{lenght}(L) \leq \text{dist}(a, b) \leq \sqrt{50}$ . Similarly if  $b_2 \geq -3$ . Finally, if  $a_2 > 3$  and  $b_2 < -3$ , we note that every vertex, and so every point of  $T$  is in the distance at most  $\sqrt{10}$  for  $a$  and every vertex, and so every point, of  $T'$  is in the distance at most  $\sqrt{10}$  of  $b$ . Since  $f$  is a contraction, the image of  $T$  lies in a segment containing  $a'$  of length at most  $\sqrt{10}$  and the image of  $T'$  lies in a segment containing  $b'$  of length at most  $\sqrt{10}$ . Since the union of these two images is  $L$ , we get  $\text{lenght}(L) \leq 2\sqrt{10} \leq \sqrt{50}$ . Thus  $\mathcal{K}(E) \leq \sqrt{50} < 8$ .

*Second day — August 3, 1996*

**Problem 1.** (10 points)

Prove that if  $f : [0, 1] \rightarrow [0, 1]$  is a continuous function, then the sequence of iterates  $x_{n+1} = f(x_n)$  converges if and only if

$$\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = 0.$$

**Solution.** The “only if” part is obvious. Now suppose that  $\lim_{n \rightarrow \infty} (x_{n+1} - x_n) = 0$  and the sequence  $\{x_n\}$  does not converge. Then there are two cluster points  $K < L$ . There must be points from the interval  $(K, L)$  in the sequence. There is an  $x \in (K, L)$  such that  $f(x) \neq x$ . Put  $\varepsilon = \frac{|f(x) - x|}{2} > 0$ . Then from the continuity of the function  $f$  we get that for some  $\delta > 0$  for all  $y \in (x - \delta, x + \delta)$  it is  $|f(y) - y| > \varepsilon$ . On the other hand for  $n$  large enough it is  $|x_{n+1} - x_n| < 2\delta$  and  $|f(x_n) - x_n| = |x_{n+1} - x_n| < \varepsilon$ . So the sequence cannot come into the interval  $(x - \delta, x + \delta)$ , but also cannot jump over this interval. Then all cluster points have to be at most  $x - \delta$  (a contradiction with  $L$  being a cluster point), or at least  $x + \delta$  (a contradiction with  $K$  being a cluster point).

**Problem 2.** (10 points)

Let  $\theta$  be a positive real number and let  $\cosh t = \frac{e^t + e^{-t}}{2}$  denote the hyperbolic cosine. Show that if  $k \in \mathbb{N}$  and both  $\cosh k\theta$  and  $\cosh (k+1)\theta$  are rational, then so is  $\cosh \theta$ .

**Solution.** First we show that

(1) If  $\cosh t$  is rational and  $m \in \mathbb{N}$ , then  $\cosh mt$  is rational.

Since  $\cosh 0.t = \cosh 0 = 1 \in \mathbb{Q}$  and  $\cosh 1.t = \cosh t \in \mathbb{Q}$ , (1) follows inductively from

$$\cosh (m+1)t = 2\cosh t.\cosh mt - \cosh (m-1)t.$$

The statement of the problem is obvious for  $k = 1$ , so we consider  $k \geq 2$ . For any  $m$  we have

$$\begin{aligned} (2) \quad \cosh \theta &= \cosh ((m+1)\theta - m\theta) = \\ &= \cosh (m+1)\theta.\cosh m\theta - \sinh (m+1)\theta.\sinh m\theta \\ &= \cosh (m+1)\theta.\cosh m\theta - \sqrt{\cosh^2(m+1)\theta - 1}.\sqrt{\cosh^2 m\theta - 1} \end{aligned}$$

Set  $\cosh k\theta = a$ ,  $\cosh (k+1)\theta = b$ ,  $a, b \in \mathbb{Q}$ . Then (2) with  $m = k$  gives

$$\cosh \theta = ab - \sqrt{a^2 - 1}\sqrt{b^2 - 1}$$

and then

$$(3) \quad \begin{aligned} (a^2 - 1)(b^2 - 1) &= (ab - \cosh \theta)^2 \\ &= a^2b^2 - 2abc\cosh \theta + \cosh^2 \theta. \end{aligned}$$

Set  $\cosh (k^2 - 1)\theta = A$ ,  $\cosh k^2\theta = B$ . From (1) with  $m = k - 1$  and  $t = (k+1)\theta$  we have  $A \in \mathbb{Q}$ . From (1) with  $m = k$  and  $t = k\theta$  we have  $B \in \mathbb{Q}$ . Moreover  $k^2 - 1 > k$  implies  $A > a$  and  $B > b$ . Thus  $AB > ab$ . From (2) with  $m = k^2 - 1$  we have

$$(4) \quad \begin{aligned} (A^2 - 1)(B^2 - 1) &= (AB - \cosh \theta)^2 \\ &= A^2B^2 - 2AB\cosh \theta + \cosh^2 \theta. \end{aligned}$$

So after we cancel the  $\cosh^2 \theta$  from (3) and (4) we have a non-trivial linear equation in  $\cosh \theta$  with rational coefficients.

**Problem 3.** (15 points)

Let  $G$  be the subgroup of  $GL_2(\mathbb{R})$ , generated by  $A$  and  $B$ , where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let  $H$  consist of those matrices  $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$  in  $G$  for which  $a_{11}=a_{22}=1$ .

- (a) Show that  $H$  is an abelian subgroup of  $G$ .  
 (b) Show that  $H$  is not finitely generated.

**Remarks.**  $GL_2(\mathbb{R})$  denotes, as usual, the group (under matrix multiplication) of all  $2 \times 2$  invertible matrices with real entries (elements). *Abelian* means commutative. A group is *finitely generated* if there are a finite number of elements of the group such that every other element of the group can be obtained from these elements using the group operation.

**Solution.**

- (a) All of the matrices in  $G$  are of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

So all of the matrices in  $H$  are of the form

$$M(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

so they commute. Since  $M(x)^{-1} = M(-x)$ ,  $H$  is a subgroup of  $G$ .

(b) A generator of  $H$  can only be of the form  $M(x)$ , where  $x$  is a binary rational, i.e.,  $x = \frac{p}{2^n}$  with integer  $p$  and non-negative integer  $n$ . In  $H$  it holds

$$\begin{aligned} M(x)M(y) &= M(x+y) \\ M(x)M(y)^{-1} &= M(x-y). \end{aligned}$$

The matrices of the form  $M\left(\frac{1}{2^n}\right)$  are in  $H$  for all  $n \in \mathbb{N}$ . With only finite number of generators all of them cannot be achieved.

**Problem 4.** (20 points)

Let  $B$  be a bounded closed convex symmetric (with respect to the origin) set in  $\mathbb{R}^2$  with boundary the curve  $\Gamma$ . Let  $B$  have the property that the ellipse of maximal area contained in  $B$  is the disc  $D$  of radius 1 centered at the origin with boundary the circle  $C$ . Prove that  $A \cap \Gamma \neq \emptyset$  for any arc  $A$  of  $C$  of length  $l(A) \geq \frac{\pi}{2}$ .

**Solution.** Assume the contrary – there is an arc  $A \subset C$  with length  $l(A) = \frac{\pi}{2}$  such that  $A \subset B \setminus \Gamma$ . Without loss of generality we may assume that the ends of  $A$  are  $M = (1/\sqrt{2}, 1/\sqrt{2})$ ,  $N = (1/\sqrt{2}, -1/\sqrt{2})$ .  $A$  is compact and  $\Gamma$  is closed. From  $A \cap \Gamma = \emptyset$  we get  $\delta > 0$  such that  $\text{dist}(x, y) > \delta$  for every  $x \in A$ ,  $y \in \Gamma$ .

Given  $\varepsilon > 0$  with  $E_\varepsilon$  we denote the ellipse with boundary:  $\frac{x^2}{(1+\varepsilon)^2} + \frac{y^2}{b^2} = 1$ , such that  $M, N \in E_\varepsilon$ . Since  $M \in E_\varepsilon$  we get

$$b^2 = \frac{(1+\varepsilon)^2}{2(1+\varepsilon)^2 - 1}.$$

Then we have

$$\text{area } E_\varepsilon = \pi \frac{(1+\varepsilon)^2}{\sqrt{2(1+\varepsilon)^2 - 1}} > \pi = \text{area } D.$$

In view of the hypotheses,  $E_\varepsilon \setminus B \neq \emptyset$  for every  $\varepsilon > 0$ . Let  $S = \{(x, y) \in \mathbb{R}^2 : |x| > |y|\}$ . From  $E_\varepsilon \setminus S \subset D \subset B$  it follows that  $E_\varepsilon \setminus B \subset S$ . Taking  $\varepsilon < \delta$  we get that

$$\emptyset \neq E_\varepsilon \setminus B \subset E_\varepsilon \cap S \subset D_{1+\varepsilon} \cap S \subset B$$

– a contradiction (we use the notation  $D_t = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq t^2\}$ ).

*Remark.* The ellipse with maximal area is well known as John's ellipse. Any coincidence with the President of the Jury is accidental.

**Problem 5.** (20 points)

(i) Prove that

$$\lim_{x \rightarrow +\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2 + x)^2} = \frac{1}{2}.$$

(ii) Prove that there is a positive constant  $c$  such that for every  $x \in [1, \infty)$  we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2 + x)^2} - \frac{1}{2} \right| \leq \frac{c}{x}.$$

**Solution.**

(i) Set  $f(t) = \frac{t}{(1+t^2)^2}$ ,  $h = \frac{1}{\sqrt{x}}$ . Then

$$\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = h \sum_{n=1}^{\infty} f(nh) \xrightarrow{h \rightarrow 0} \int_0^{\infty} f(t) dt = \frac{1}{2}.$$

The convergence holds since  $h \sum_{n=1}^{\infty} f(nh)$  is a Riemann sum of the integral  $\int_0^{\infty} f(t) dt$ . There are no problems with the infinite domain because  $f$  is integrable and  $f \downarrow 0$  for  $x \rightarrow \infty$  (thus  $h \sum_{n=N}^{\infty} f(nh) \geq \int_{nN}^{\infty} f(t) dt \geq h \sum_{n=N+1}^{\infty} f(nh)$ ).

(ii) We have

$$(1) \quad \left| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \right| = \left| \sum_{n=1}^{\infty} \left( hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t) dt \right) - \int_0^{\frac{h}{2}} f(t) dt \right| \\ \leq \sum_{n=1}^{\infty} \left| hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t) dt \right| + \int_0^{\frac{h}{2}} f(t) dt$$

Using twice integration by parts one has

$$(2) \quad 2bg(a) - \int_{a-b}^{a+b} g(t) dt = -\frac{1}{2} \int_0^b (b-t)^2 (g''(a+t) + g''(a-t)) dt$$

for every  $g \in C^2[a-b, a+b]$ . Using  $f(0) = 0$ ,  $f \in C^2[0, h/2]$  one gets

$$(3) \quad \int_0^{h/2} f(t) dt = O(h^2).$$

From (1), (2) and (3) we get

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \right| \leq \sum_{n=1}^{\infty} h^2 \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} |f''(t)| dt + O(h^2) = \\ = h^2 \int_{\frac{h}{2}}^{\infty} |f''(t)| dt + O(h^2) = O(h^2) = O(x^{-1}).$$

**Problem 6.** (Carleman's inequality) (25 points)

(i) Prove that for every sequence  $\{a_n\}_{n=1}^{\infty}$ , such that  $a_n > 0$ ,  $n = 1, 2, \dots$  and  $\sum_{n=1}^{\infty} a_n < \infty$ , we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

where  $e$  is the natural log base.

(ii) Prove that for every  $\varepsilon > 0$  there exists a sequence  $\{a_n\}_{n=1}^{\infty}$ , such that  $a_n > 0$ ,  $n = 1, 2, \dots$ ,  $\sum_{n=1}^{\infty} a_n < \infty$  and

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} > (e - \varepsilon) \sum_{n=1}^{\infty} a_n.$$

**Solution.**

(i) Put for  $n \in \mathbb{N}$

$$(1) \quad c_n = (n+1)^n / n^{n-1}.$$

Observe that  $c_1 c_2 \cdots c_n = (n+1)^n$ . Hence, for  $n \in \mathbb{N}$ ,

$$\begin{aligned} (a_1 a_2 \cdots a_n)^{1/n} &= (a_1 c_1 a_2 c_2 \cdots a_n c_n)^{1/n} / (n+1) \\ &\leq (a_1 c_1 + \cdots + a_n c_n) / n(n+1). \end{aligned}$$

Consequently,

$$(2) \quad \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq \sum_{n=1}^{\infty} a_n c_n \left( \sum_{m=n}^{\infty} (m(m+1))^{-1} \right).$$

Since

$$\sum_{m=n}^{\infty} (m(m+1))^{-1} = \sum_{m=n}^{\infty} \left( \frac{1}{m} - \frac{1}{m+1} \right) = 1/n$$

we have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n c_n \left( \sum_{m=n}^{\infty} (m(m+1))^{-1} \right) &= \sum_{n=1}^{\infty} a_n c_n / n \\ &= \sum_{n=1}^{\infty} a_n ((n+1)/n)^n < e \sum_{n=1}^{\infty} a_n \end{aligned}$$

(by (1)). Combining the last inequality with (2) we get the result.

(ii) Set  $a_n = n^{n-1}(n+1)^{-n}$  for  $n = 1, 2, \dots, N$  and  $a_n = 2^{-n}$  for  $n > N$ , where  $N$  will be chosen later. Then

$$(3) \quad (a_1 \cdots a_n)^{1/n} = \frac{1}{n+1}$$

for  $n \leq N$ . Let  $K = K(\varepsilon)$  be such that

$$(4) \quad \left(\frac{n+1}{n}\right)^n > e - \frac{\varepsilon}{2} \text{ for } n > K.$$

Choose  $N$  from the condition

$$(5) \quad \sum_{n=1}^K a_n + \sum_{n=1}^{\infty} 2^{-n} \leq \frac{\varepsilon}{(2e - \varepsilon)(e - \varepsilon)} \sum_{n=K+1}^N \frac{1}{n},$$

which is always possible because the harmonic series diverges. Using (3), (4) and (5) we have

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^K a_n + \sum_{n=N+1}^{\infty} 2^{-n} + \sum_{n=K+1}^N \frac{1}{n} \left(\frac{n}{n+1}\right)^n < \\ &< \frac{\varepsilon}{(2e - \varepsilon)(e - \varepsilon)} \sum_{n=K+1}^N \frac{1}{n} + \left(e - \frac{\varepsilon}{2}\right)^{-1} \sum_{n=K+1}^N \frac{1}{n} = \\ &= \frac{1}{e - \varepsilon} \sum_{n=K+1}^N \frac{1}{n} \leq \frac{1}{e - \varepsilon} \sum_{n=1}^{\infty} (a_1 \cdots a_n)^{1/n}. \end{aligned}$$

**FOURTH INTERNATIONAL COMPETITION  
FOR UNIVERSITY STUDENTS IN MATHEMATICS  
July 30 – August 4, 1997, Plovdiv, BULGARIA**

**First day — August 1, 1997**

*Problems and Solutions*

**Problem 1.**

Let  $\{\varepsilon_n\}_{n=1}^{\infty}$  be a sequence of positive real numbers, such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . Find

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} + \varepsilon_n \right),$$

where  $\ln$  denotes the natural logarithm.

**Solution.**

It is well known that

$$-1 = \int_0^1 \ln x dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} \right)$$

(Riemman's sums). Then

$$\frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} + \varepsilon_n \right) \geq \frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} \right) \xrightarrow{n \rightarrow \infty} -1.$$

Given  $\varepsilon > 0$  there exist  $n_0$  such that  $0 < \varepsilon_n \leq \varepsilon$  for all  $n \geq n_0$ . Then

$$\frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} + \varepsilon_n \right) \leq \frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} + \varepsilon \right).$$

Since

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} + \varepsilon \right) &= \int_0^1 \ln(x + \varepsilon) dx \\ &= \int_{\varepsilon}^{1+\varepsilon} \ln x dx \end{aligned}$$

we obtain the result when  $\varepsilon$  goes to 0 and so

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left( \frac{k}{n} + \varepsilon_n \right) = -1.$$

**Problem 2.**

Suppose  $\sum_{n=1}^{\infty} a_n$  converges. Do the following sums have to converge as well?

a)  $a_1 + a_2 + a_4 + a_3 + a_8 + a_7 + a_6 + a_5 + a_{16} + a_{15} + \dots + a_9 + a_{32} + \dots$

b)  $a_1 + a_2 + a_3 + a_4 + a_5 + a_7 + a_6 + a_8 + a_9 + a_{11} + a_{13} + a_{15} + a_{10} + a_{12} + a_{14} + a_{16} + a_{17} + a_{19} + \dots$

Justify your answers.

**Solution.**

a) Yes. Let  $S = \sum_{n=1}^{\infty} a_n$ ,  $S_n = \sum_{k=1}^n a_k$ . Fix  $\varepsilon > 0$  and a number  $n_0$  such that  $|S_n - S| < \varepsilon$  for  $n > n_0$ . The partial sums of the permuted series have the form  $L_{2^{n-1}+k} = S_{2^{n-1}} + S_{2^n} - S_{2^{n-k}}$ ,  $0 \leq k < 2^{n-1}$  and for  $2^{n-1} > n_0$  we have  $|L_{2^{n-1}+k} - S| < 3\varepsilon$ , i.e. the permuted series converges.

b) No. Take  $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$ . Then  $L_{3 \cdot 2^{n-2}} = S_{2^{n-1}} + \sum_{k=2^{n-2}}^{2^{n-1}-1} \frac{1}{\sqrt{2k+1}}$  and  $L_{3 \cdot 2^{n-2}} - S_{2^{n-1}} \geq 2^{n-2} \frac{1}{\sqrt{2^n}} \xrightarrow{n \rightarrow \infty} \infty$ , so  $L_{3 \cdot 2^{n-2}} \xrightarrow{n \rightarrow \infty} \infty$ .

**Problem 3.**

Let  $A$  and  $B$  be real  $n \times n$  matrices such that  $A^2 + B^2 = AB$ . Prove that if  $BA - AB$  is an invertible matrix then  $n$  is divisible by 3.

**Solution.**

Set  $S = A + \omega B$ , where  $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ . We have

$$\begin{aligned} S\bar{S} &= (A + \omega B)(A + \bar{\omega}B) = A^2 + \omega BA + \bar{\omega}AB + B^2 \\ &= AB + \omega BA + \bar{\omega}AB = \omega(BA - AB), \end{aligned}$$

because  $\bar{\omega} + 1 = -\omega$ . Since  $\det(S\bar{S}) = \det S \cdot \det \bar{S}$  is a real number and  $\det \omega(BA - AB) = \omega^n \det(BA - AB)$  and  $\det(BA - AB) \neq 0$ , then  $\omega^n$  is a real number. This is possible only when  $n$  is divisible by 3.

**Problem 4.**

Let  $\alpha$  be a real number,  $1 < \alpha < 2$ .

a) Show that  $\alpha$  has a unique representation as an infinite product

$$\alpha = \left(1 + \frac{1}{n_1}\right) \left(1 + \frac{1}{n_2}\right) \dots$$

where each  $n_i$  is a positive integer satisfying

$$n_i^2 \leq n_{i+1}.$$

b) Show that  $\alpha$  is rational if and only if its infinite product has the following property:

For some  $m$  and all  $k \geq m$ ,

$$n_{k+1} = n_k^2.$$

**Solution.**

a) We construct inductively the sequence  $\{n_i\}$  and the ratios

$$\theta_k = \frac{\alpha}{\prod_{i=1}^k \left(1 + \frac{1}{n_i}\right)}$$

so that

$$\theta_k > 1 \text{ for all } k.$$

Choose  $n_k$  to be the least  $n$  for which

$$1 + \frac{1}{n} < \theta_{k-1}$$

( $\theta_0 = \alpha$ ) so that for each  $k$ ,

$$(1) \quad 1 + \frac{1}{n_k} < \theta_{k-1} \leq 1 + \frac{1}{n_k - 1}.$$

Since

$$\theta_{k-1} \leq 1 + \frac{1}{n_k - 1}$$

we have

$$1 + \frac{1}{n_{k+1}} < \theta_k = \frac{\theta_{k-1}}{1 + \frac{1}{n_k}} \leq \frac{1 + \frac{1}{n_k - 1}}{1 + \frac{1}{n_k}} = 1 + \frac{1}{n_k^2 - 1}.$$

Hence, for each  $k$ ,  $n_{k+1} \geq n_k^2$ .

Since  $n_1 \geq 2$ ,  $n_k \rightarrow \infty$  so that  $\theta_k \rightarrow 1$ . Hence

$$\alpha = \prod_1^{\infty} \left(1 + \frac{1}{n_k}\right).$$

The uniqueness of the infinite product will follow from the fact that on every step  $n_k$  has to be determined by (1).

Indeed, if for some  $k$  we have

$$1 + \frac{1}{n_k} \geq \theta_{k-1}$$

then  $\theta_k \leq 1$ ,  $\theta_{k+1} < 1$  and hence  $\{\theta_k\}$  does not converge to 1.

Now observe that for  $M > 1$ ,

$$(2) \quad \left(1 + \frac{1}{M}\right) \left(1 + \frac{1}{M^2}\right) \left(1 + \frac{1}{M^4}\right) \cdots = 1 + \frac{1}{M} + \frac{1}{M^2} + \frac{1}{M^3} + \cdots = 1 + \frac{1}{M-1}.$$

Assume that for some  $k$  we have

$$1 + \frac{1}{n_k - 1} < \theta_{k-1}.$$

Then we get

$$\begin{aligned} \frac{\alpha}{\left(1 + \frac{1}{n_1}\right)\left(1 + \frac{1}{n_2}\right) \cdots} &= \frac{\theta_{k-1}}{\left(1 + \frac{1}{n_k}\right)\left(1 + \frac{1}{n_{k+1}}\right) \cdots} \\ &\geq \frac{\theta_{k-1}}{\left(1 + \frac{1}{n_k}\right)\left(1 + \frac{1}{n_k^2}\right) \cdots} = \frac{\theta_{k-1}}{1 + \frac{1}{n_k - 1}} > 1 \end{aligned}$$

– a contradiction.

b) From (2)  $\alpha$  is rational if its product ends in the stated way.

Conversely, suppose  $\alpha$  is the rational number  $\frac{p}{q}$ . Our aim is to show that for some  $m$ ,

$$\theta_{m-1} = \frac{n_m}{n_m - 1}.$$

Suppose this is not the case, so that for every  $m$ ,

$$(3) \quad \theta_{m-1} < \frac{n_m}{n_m - 1}.$$

For each  $k$  we write

$$\theta_k = \frac{p_k}{q_k}$$

as a fraction (not necessarily in lowest terms) where

$$p_0 = p, \quad q_0 = q$$

and in general

$$p_k = p_{k-1}n_k, \quad q_k = q_{k-1}(n_k + 1).$$

The numbers  $p_k - q_k$  are positive integers: to obtain a contradiction it suffices to show that this sequence is strictly decreasing. Now,

$$\begin{aligned} p_k - q_k - (p_{k-1} - q_{k-1}) &= n_k p_{k-1} - (n_k + 1)q_{k-1} - p_{k-1} + q_{k-1} \\ &= (n_k - 1)p_{k-1} - n_k q_{k-1} \end{aligned}$$

and this is negative because

$$\frac{p_{k-1}}{q_{k-1}} = \theta_{k-1} < \frac{n_k}{n_k - 1}$$

by inequality (3).

**Problem 5.** For a natural  $n$  consider the hyperplane

$$R_0^n = \left\{ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0 \right\}$$

and the lattice  $Z_0^n = \{y \in R_0^n : \text{all } y_i \text{ are integers}\}$ . Define the (quasi-)norm in  $\mathbb{R}^n$  by  $\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$  if  $0 < p < \infty$ , and  $\|x\|_\infty = \max_i |x_i|$ .

a) Let  $x \in R_0^n$  be such that

$$\max_i x_i - \min_i x_i \leq 1.$$

For every  $p \in [1, \infty]$  and for every  $y \in Z_0^n$  prove that

$$\|x\|_p \leq \|x + y\|_p.$$

b) For every  $p \in (0, 1)$ , show that there is an  $n$  and an  $x \in R_0^n$  with  $\max_i x_i - \min_i x_i \leq 1$  and an  $y \in Z_0^n$  such that

$$\|x\|_p > \|x + y\|_p.$$

**Solution.**

a) For  $x = 0$  the statement is trivial. Let  $x \neq 0$ . Then  $\max_i x_i > 0$  and  $\min_i x_i < 0$ . Hence  $\|x\|_\infty < 1$ . From the hypothesis on  $x$  it follows that:

- i) If  $x_j \leq 0$  then  $\max_i x_i \leq x_j + 1$ .
- ii) If  $x_j \geq 0$  then  $\min_i x_i \geq x_j - 1$ .

Consider  $y \in Z_0^n$ ,  $y \neq 0$ . We split the indices  $\{1, 2, \dots, n\}$  into five sets:

$$I(0) = \{i : y_i = 0\},$$

$$I(+, +) = \{i : y_i > 0, x_i \geq 0\}, \quad I(+, -) = \{i : y_i > 0, x_i < 0\},$$

$$I(-, +) = \{i : y_i < 0, x_i > 0\}, \quad I(-, -) = \{i : y_i < 0, x_i \leq 0\}.$$

As least one of the last four index sets is not empty. If  $I(+, +) \neq \emptyset$  or  $I(-, -) \neq \emptyset$  then  $\|x + y\|_\infty \geq 1 > \|x\|_\infty$ . If  $I(+, +) = I(-, -) = \emptyset$  then  $\sum y_i = 0$  implies  $I(+, -) \neq \emptyset$  and  $I(-, +) \neq \emptyset$ . Therefore i) and ii) give  $\|x + y\|_\infty \geq \|x\|_\infty$  which completes the case  $p = \infty$ .

Now let  $1 \leq p < \infty$ . Then using i) for every  $j \in I(+, -)$  we get  $|x_j + y_j| = y_j - 1 + x_j + 1 \geq |y_j| - 1 + \max_i x_i$ . Hence

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \quad \text{for every } k \in I(-, +) \text{ and } j \in I(+, -).$$

Similarly

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \quad \text{for every } k \in I(+, -) \text{ and } j \in I(-, +);$$

$$|x_j + y_j|^p \geq |y_j| + |x_j|^p \quad \text{for every } j \in I(+, +) \cup I(-, -).$$

Assume that  $\sum_{j \in I(+, -)} 1 \geq \sum_{j \in I(-, +)} 1$ . Then

$$\begin{aligned} & \|x + y\|_p^p - \|x\|_p^p \\ = & \sum_{j \in I(+, +) \cup I(-, -)} (|x_j + y_j|^p - |x_j|^p) + \left( \sum_{j \in I(+, -)} |x_j + y_j|^p - \sum_{k \in I(-, +)} |x_k|^p \right) \\ & + \left( \sum_{j \in I(-, +)} |x_j + y_j|^p - \sum_{k \in I(+, -)} |x_k|^p \right) \\ \geq & \sum_{j \in I(+, +) \cup I(-, -)} |y_j| + \sum_{j \in I(+, -)} (|y_j| - 1) \end{aligned}$$

$$\begin{aligned}
& + \left( \sum_{j \in I(-,+)} (|y_j| - 1) - \sum_{j \in I(+,-)} 1 + \sum_{j \in I(-,+)} 1 \right) \\
= & \sum_{i=1}^n |y_i| - 2 \sum_{j \in I(+,-)} 1 = 2 \sum_{j \in I(+,-)} (y_j - 1) + 2 \sum_{j \in I(+,+)} y_j \geq 0.
\end{aligned}$$

The case  $\sum_{j \in I(+,-)} 1 \leq \sum_{j \in I(-,+)} 1$  is similar. This proves the statement.

b) Fix  $p \in (0, 1)$  and a rational  $t \in (\frac{1}{2}, 1)$ . Choose a pair of positive integers  $m$  and  $l$  such that  $mt = l(1 - t)$  and set  $n = m + l$ . Let

$$\begin{aligned}
x_i &= t, \quad i = 1, 2, \dots, m; \quad x_i = t - 1, \quad i = m + 1, m + 2, \dots, n; \\
y_i &= -1, \quad i = 1, 2, \dots, m; \quad y_{m+1} = m; \quad y_i = 0, \quad i = m + 2, \dots, n.
\end{aligned}$$

Then  $x \in R_0^n$ ,  $\max_i x_i - \min_i x_i = 1$ ,  $y \in Z_0^n$  and

$$\|x\|_p^p - \|x + y\|_p^p = m(t^p - (1 - t)^p) + (1 - t)^p - (m - 1 + t)^p,$$

which is positive for  $m$  big enough.

**Problem 6.** Suppose that  $F$  is a family of finite subsets of  $\mathbb{N}$  and for any two sets  $A, B \in F$  we have  $A \cap B \neq \emptyset$ .

a) Is it true that there is a finite subset  $Y$  of  $\mathbb{N}$  such that for any  $A, B \in F$  we have  $A \cap B \cap Y \neq \emptyset$ ?

b) Is the statement a) true if we suppose in addition that all of the members of  $F$  have the same size?

Justify your answers.

**Solution.**

a) No. Consider  $F = \{A_1, B_1, \dots, A_n, B_n, \dots\}$ , where  $A_n = \{1, 3, 5, \dots, 2n - 1, 2n\}$ ,  $B_n = \{2, 4, 6, \dots, 2n, 2n + 1\}$ .

b) Yes. We will prove inductively a stronger statement:

Suppose  $F, G$  are

two families of finite subsets of  $\mathbb{N}$  such that:

1) For every  $A \in F$  and  $B \in G$  we have  $A \cap B \neq \emptyset$ ;

2) All the elements of  $F$  have the same size  $r$ , and elements of  $G$  – size  $s$ . (we shall write  $\#(F) = r$ ,  $\#(G) = s$ ).

Then there is a finite set  $Y$  such that  $A \cup B \cup Y \neq \emptyset$  for every  $A \in F$  and  $B \in G$ .

The problem b) follows if we take  $F = G$ .

**Proof of the statement:** The statement is obvious for  $r = s = 1$ . Fix the numbers  $r, s$  and suppose the statement is proved for all pairs  $F', G'$  with  $\#(F') < r, \#(G') < s$ . Fix  $A_0 \in F, B_0 \in G$ . For any subset  $C \subset A_0 \cup B_0$ , denote

$$F(C) = \{A \in F : A \cap (A_0 \cup B_0) = C\}.$$

Then  $F = \bigcup_{\emptyset \neq C \subset A_0 \cup B_0} F(C)$ . It is enough to prove that for any pair of non-empty sets  $C, D \subset A_0 \cup B_0$  the families  $F(C)$  and  $G(D)$  satisfy the statement.

Indeed, if we denote by  $Y_{C,D}$  the corresponding finite set, then the finite set  $\bigcup_{C,D \subset A_0 \cup B_0} Y_{C,D}$  will satisfy the statement for  $F$  and  $G$ . The proof for  $F(C)$  and  $G(D)$ .

If  $C \cap D \neq \emptyset$ , it is trivial.

If  $C \cap D = \emptyset$ , then any two sets  $A \in F(C), B \in G(D)$  must meet outside  $A_0 \cup B_0$ . Then if we denote  $\tilde{F}(C) = \{A \setminus C : A \in F(C)\}, \tilde{G}(D) = \{B \setminus D : B \in G(D)\}$ , then  $\tilde{F}(C)$  and  $\tilde{G}(D)$  satisfy the conditions 1) and 2) above, with  $\#(\tilde{F}(C)) = \#(F) - \#C < r, \#(\tilde{G}(D)) = \#(G) - \#D < s$ , and the inductive assumption works.

**FOURTH INTERNATIONAL COMPETITION  
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July 30 – August 4, 1997, Plovdiv, BULGARIA**

**Second day — August 2, 1997**

*Problems and Solutions*

**Problem 1.**

Let  $f$  be a  $C^3(\mathbb{R})$  non-negative function,  $f(0)=f'(0)=0$ ,  $0 < f''(0)$ .

Let

$$g(x) = \left( \frac{\sqrt{f(x)}}{f'(x)} \right)'$$

for  $x \neq 0$  and  $g(0) = 0$ . Show that  $g$  is bounded in some neighbourhood of 0. Does the theorem hold for  $f \in C^2(\mathbb{R})$ ?

**Solution.**

Let  $c = \frac{1}{2}f''(0)$ . We have

$$g = \frac{(f')^2 - 2ff''}{2(f')^2\sqrt{f}},$$

where

$$f(x) = cx^2 + O(x^3), \quad f'(x) = 2cx + O(x^2), \quad f''(x) = 2c + O(x).$$

Therefore  $(f'(x))^2 = 4c^2x^2 + O(x^3)$ ,

$$2f(x)f''(x) = 4c^2x^2 + O(x^3)$$

and

$$2(f'(x))^2\sqrt{f(x)} = 2(4c^2x^2 + O(x^3))|x|\sqrt{c + O(x)}.$$

$g$  is bounded because

$$\frac{2(f'(x))^2\sqrt{f(x)}}{|x|^3} \xrightarrow{x \rightarrow 0} 8c^{5/2} \neq 0$$

and  $f'(x)^2 - 2f(x)f''(x) = O(x^3)$ .

The theorem does not hold for some  $C^2$ -functions.

Let  $f(x) = (x + |x|^{3/2})^2 = x^2 + 2x^2\sqrt{|x|} + |x|^3$ , so  $f$  is  $C^2$ . For  $x > 0$ ,

$$g(x) = \frac{1}{2} \left( \frac{1}{1 + \frac{3}{2}\sqrt{x}} \right)' = -\frac{1}{2} \cdot \frac{1}{(1 + \frac{3}{2}\sqrt{x})^2} \cdot \frac{3}{4} \cdot \frac{1}{\sqrt{x}} \xrightarrow{x \rightarrow 0} -\infty.$$

**Problem 2.**

Let  $M$  be an invertible matrix of dimension  $2n \times 2n$ , represented in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

Show that  $\det M \cdot \det H = \det A$ .

**Solution.**

Let  $I$  denote the identity  $n \times n$  matrix. Then

$$\det M \cdot \det H = \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \det \begin{bmatrix} I & F \\ 0 & H \end{bmatrix} = \det \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} = \det A.$$

**Problem 3.**

Show that  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin(\log n)}{n^\alpha}$  converges if and only if  $\alpha > 0$ .

**Solution.**

Set  $f(t) = \frac{\sin(\log t)}{t^\alpha}$ . We have

$$f'(t) = \frac{-\alpha}{t^{\alpha+1}} \sin(\log t) + \frac{\cos(\log t)}{t^{\alpha+1}}.$$

So  $|f'(t)| \leq \frac{1+\alpha}{t^{\alpha+1}}$  for  $\alpha > 0$ . Then from Mean value theorem for some  $\theta \in (0, 1)$  we get  $|f(n+1) - f(n)| = |f'(n+\theta)| \leq \frac{1+\alpha}{n^{\alpha+1}}$ . Since  $\sum \frac{1+\alpha}{n^{\alpha+1}} < +\infty$  for  $\alpha > 0$  and  $f(n) \xrightarrow{n \rightarrow \infty} 0$  we get that  $\sum_{n=1}^{\infty} (-1)^{n-1} f(n) = \sum_{n=1}^{\infty} (f(2n-1) - f(2n))$  converges.

Now we have to prove that  $\frac{\sin(\log n)}{n^\alpha}$  does not converge to 0 for  $\alpha \leq 0$ . It suffices to consider  $\alpha = 0$ .

We show that  $a_n = \sin(\log n)$  does not tend to zero. Assume the contrary. There exist  $k_n \in \mathbb{N}$  and  $\lambda_n \in \left[-\frac{1}{2}, \frac{1}{2}\right]$  for  $n > e^2$  such that  $\frac{\log n}{\pi} = k_n + \lambda_n$ . Then  $|a_n| = \sin \pi |\lambda_n|$ . Since  $a_n \rightarrow 0$  we get  $\lambda_n \rightarrow 0$ .

We have  $k_{n+1} - k_n =$

$$= \frac{\log(n+1) - \log n}{\pi} - (\lambda_{n+1} - \lambda_n) = \frac{1}{\pi} \log\left(1 + \frac{1}{n}\right) - (\lambda_{n+1} - \lambda_n).$$

Then  $|k_{n+1} - k_n| < 1$  for all  $n$  big enough. Hence there exists  $n_0$  so that  $k_n = k_{n_0}$  for  $n > n_0$ . So  $\frac{\log n}{\pi} = k_{n_0} + \lambda_n$  for  $n > n_0$ . Since  $\lambda_n \rightarrow 0$  we get contradiction with  $\log n \rightarrow \infty$ .

**Problem 4.**

a) Let the mapping  $f : M_n \rightarrow \mathbb{R}$  from the space  $M_n = \mathbb{R}^{n^2}$  of  $n \times n$  matrices with real entries to reals be linear, i.e.:

$$(1) \quad f(A+B) = f(A) + f(B), \quad f(cA) = cf(A)$$

for any  $A, B \in M_n, c \in \mathbb{R}$ . Prove that there exists a unique matrix  $C \in M_n$  such that  $f(A) = \text{tr}(AC)$  for any  $A \in M_n$ . (If  $A = \{a_{ij}\}_{i,j=1}^n$  then  $\text{tr}(A) = \sum_{i=1}^n a_{ii}$ ).

b) Suppose in addition to (1) that

$$(2) \quad f(A.B) = f(B.A)$$

for any  $A, B \in M_n$ . Prove that there exists  $\lambda \in \mathbb{R}$  such that  $f(A) = \lambda \cdot \text{tr}(A)$ .

**Solution.**

a) If we denote by  $E_{ij}$  the standard basis of  $M_n$  consisting of elementary matrix (with entry 1 at the place  $(i, j)$  and zero elsewhere), then the entries  $c_{ij}$  of  $C$  can be defined by  $c_{ij} = f(E_{ji})$ . b) Denote by  $L$  the  $n^2 - 1$ -dimensional linear subspace of  $M_n$  consisting of all matrices with zero trace. The elements  $E_{ij}$  with  $i \neq j$  and the elements  $E_{ii} - E_{nn}, i = 1, \dots, n-1$  form a linear basis for  $L$ . Since

$$\begin{aligned} E_{ij} &= E_{ij}.E_{jj} - E_{jj}.E_{ij}, \quad i \neq j \\ E_{ii} - E_{nn} &= E_{in}.E_{ni} - E_{ni}.E_{in}, \quad i = 1, \dots, n-1, \end{aligned}$$

then the property (2) shows that  $f$  is vanishing identically on  $L$ . Now, for any  $A \in M_n$  we have  $A - \frac{1}{n}\text{tr}(A).E \in L$ , where  $E$  is the identity matrix, and therefore  $f(A) = \frac{1}{n}f(E).\text{tr}(A)$ .

**Problem 5.**

Let  $X$  be an arbitrary set, let  $f$  be an one-to-one function mapping  $X$  onto itself. Prove that there exist mappings  $g_1, g_2 : X \rightarrow X$  such that  $f = g_1 \circ g_2$  and  $g_1 \circ g_1 = id = g_2 \circ g_2$ , where  $id$  denotes the identity mapping on  $X$ .

**Solution.**

Let  $f^n = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}$ ,  $f^0 = id$ ,  $f^{-n} = (f^{-1})^n$  for every natural number  $n$ . Let  $T(x) = \{f^n(x) : n \in \mathbb{Z}\}$  for every  $x \in X$ . The sets  $T(x)$  for different  $x$ 's either coincide or do not intersect. Each of them is mapped by  $f$  onto itself. It is enough to prove the theorem for every such set. Let  $A = T(x)$ . If  $A$  is finite, then we can think that  $A$  is the set of all vertices of a regular  $n$  polygon and that  $f$  is rotation by  $\frac{2\pi}{n}$ . Such rotation can be obtained as a composition of 2 symmetries mapping the  $n$  polygon onto itself (if  $n$  is even then there are axes of symmetry making  $\frac{\pi}{n}$  angle; if  $n = 2k + 1$  then there are axes making  $k\frac{2\pi}{n}$  angle). If  $A$  is infinite then we can think that  $A = \mathbb{Z}$  and  $f(m) = m + 1$  for every  $m \in \mathbb{Z}$ . In this case we define  $g_1$  as a symmetry relative to  $\frac{1}{2}$ ,  $g_2$  as a symmetry relative to 0.

**Problem 6.**

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Say that  $f$  "crosses the axis" at  $x$  if  $f(x) = 0$  but in any neighbourhood of  $x$  there are  $y, z$  with  $f(y) < 0$  and  $f(z) > 0$ .

a) Give an example of a continuous function that "crosses the axis" infinitely often.

b) Can a continuous function "cross the axis" uncountably often? Justify your answer.

**Solution.**

a)  $f(x) = x \sin \frac{1}{x}$ .

b) Yes. The Cantor set is given by

$$C = \{x \in [0, 1) : x = \sum_{j=1}^{\infty} b_j 3^{-j}, b_j \in \{0, 2\}\}.$$

There is an one-to-one mapping  $f : [0, 1) \rightarrow C$ . Indeed, for  $x = \sum_{j=1}^{\infty} a_j 2^{-j}$ ,

$a_j \in \{0, 1\}$  we set  $f(x) = \sum_{j=1}^{\infty} (2a_j) 3^{-j}$ . Hence  $C$  is uncountable.

For  $k = 1, 2, \dots$  and  $i = 0, 1, 2, \dots, 2^{k-1} - 1$  we set

$$a_{k,i} = 3^{-k} \left( 6 \sum_{j=0}^{k-2} a_j 3^j + 1 \right), \quad b_{k,i} = 3^{-k} \left( 6 \sum_{j=0}^{k-2} a_j 3^j + 2 \right),$$

where  $i = \sum_{j=0}^{k-2} a_j 2^j$ ,  $a_j \in \{0, 1\}$ . Then

$$[0, 1] \setminus C = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{2^{k-1}-1} (a_{k,i}, b_{k,i}),$$

i.e. the Cantor set consists of all points which have a trinary representation with 0 and 2 as digits and the points of its compliment have some 1's in their trinary representation. Thus,  $\bigcup_{i=0}^{2^{k-1}-1} (a_{k,i}, b_{k,i})$  are all points (except  $a_{k,i}$ ) which have 1 on  $k$ -th place and 0 or 2 on the  $j$ -th ( $j < k$ ) places.

Noticing that the points with at least one digit equals to 1 are everywhere dense in  $[0,1]$  we set

$$f(x) = \sum_{k=1}^{\infty} (-1)^k g_k(x).$$

where  $g_k$  is a piece-wise linear continuous functions with values at the knots

$$g_k \left( \frac{a_{k,i} + b_{k,i}}{2} \right) = 2^{-k}, \quad g_k(0) = g_k(1) = g_k(a_{k,i}) = g_k(b_{k,i}) = 0,$$

$i = 0, 1, \dots, 2^{k-1} - 1$ .

Then  $f$  is continuous and  $f$  "crosses the axis" at every point of the Cantor set.

**5<sup>th</sup> INTERNATIONAL MATHEMATICS COMPETITION FOR UNIVERSITY STUDENTS**

**July 29 - August 3, 1998, Blagoevgrad, Bulgaria**

**First day**

*PROBLEMS AND SOLUTIONS*

**Problem 1.** (20 points) Let  $V$  be a 10-dimensional real vector space and  $U_1$  and  $U_2$  two linear subspaces such that  $U_1 \subseteq U_2$ ,  $\dim_{\mathbf{R}} U_1 = 3$  and  $\dim_{\mathbf{R}} U_2 = 6$ . Let  $\mathcal{E}$  be the set of all linear maps  $T : V \rightarrow V$  which have  $U_1$  and  $U_2$  as invariant subspaces (i.e.,  $T(U_1) \subseteq U_1$  and  $T(U_2) \subseteq U_2$ ). Calculate the dimension of  $\mathcal{E}$  as a real vector space.

**Solution** First choose a basis  $\{v_1, v_2, v_3\}$  of  $U_1$ . It is possible to extend this basis with vectors  $v_4, v_5$  and  $v_6$  to get a basis of  $U_2$ . In the same way we can extend a basis of  $U_2$  with vectors  $v_7, \dots, v_{10}$  to get as basis of  $V$ .

Let  $T \in \mathcal{E}$  be an endomorphism which has  $U_1$  and  $U_2$  as invariant subspaces. Then its matrix, relative to the basis  $\{v_1, \dots, v_{10}\}$  is of the form

$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * & * & * \\ & & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix}.$$

So  $\dim_{\mathbf{R}} \mathcal{E} = 9 + 18 + 40 = 67$ .

**Problem 2.** Prove that the following proposition holds for  $n = 3$  (5 points) and  $n = 5$  (7 points), and does not hold for  $n = 4$  (8 points).

“For any permutation  $\pi_1$  of  $\{1, 2, \dots, n\}$  different from the identity there is a permutation  $\pi_2$  such that any permutation  $\pi$  can be obtained from  $\pi_1$  and  $\pi_2$  using only compositions (for example,  $\pi = \pi_1 \circ \pi_1 \circ \pi_2 \circ \pi_1$ ).”

**Solution**

Let  $S_n$  be the group of permutations of  $\{1, 2, \dots, n\}$ .

1) When  $n = 3$  the proposition is obvious: if  $x = (12)$  we choose  $y = (123)$ ; if  $x = (123)$  we choose  $y = (12)$ .

2)  $n = 4$ . Let  $x = (12)(34)$ . Assume that there exists  $y \in S_n$ , such that  $S_4 = \langle x, y \rangle$ . Denote by  $K$  the invariant subgroup

$$K = \{id, (12)(34), (13)(24), (14)(23)\}.$$

By the fact that  $x$  and  $y$  generate the whole group  $S_4$ , it follows that the factor group  $S_4/K$  contains only powers of  $\bar{y} = yK$ , i.e.,  $S_4/K$  is cyclic. It is easy to see that this factor-group is not commutative (something more this group is not isomorphic to  $S_3$ ).

3)  $n = 5$

a) If  $x = (12)$ , then for  $y$  we can take  $y = (12345)$ .

b) If  $x = (123)$ , we set  $y = (124)(35)$ . Then  $y^3xy^3 = (125)$  and  $y^4 = (124)$ . Therefore  $(123), (124), (125) \in \langle x, y \rangle$ - the subgroup generated by  $x$  and  $y$ . From the fact that  $(123), (124), (125)$  generate the alternating subgroup  $A_5$ , it follows that  $A_5 \subset \langle x, y \rangle$ . Moreover  $y$  is an odd permutation, hence  $\langle x, y \rangle = S_5$ .

c) If  $x = (123)(45)$ , then as in b) we see that for  $y$  we can take the element  $(124)$ .

d) If  $x = (1234)$ , we set  $y = (12345)$ . Then  $(yx)^3 = (24) \in \langle x, y \rangle$ ,  $x^2(24) = (13) \in \langle x, y \rangle$  and  $y^2 = (13524) \in \langle x, y \rangle$ . By the fact  $(13) \in \langle x, y \rangle$  and  $(13524) \in \langle x, y \rangle$ , it follows that  $\langle x, y \rangle = S_5$ .

e) If  $x = (12)(34)$ , then for  $y$  we can take  $y = (1354)$ . Then  $y^2x = (125)$ ,  $y^3x = (124)(53)$  and by c)  $S_5 = \langle x, y \rangle$ .

f) If  $x = (12345)$ , then it is clear that for  $y$  we can take the element  $y = (12)$ .

**Problem 3.** Let  $f(x) = 2x(1 - x)$ ,  $x \in \mathbb{R}$ . Define

$$f_n = \overbrace{f \circ \dots \circ f}^n.$$

a) (10 points) Find  $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx$ .

b) (10 points) Compute  $\int_0^1 f_n(x) dx$  for  $n = 1, 2, \dots$

**Solution.** a) Fix  $x = x_0 \in (0, 1)$ . If we denote  $x_n = f_n(x_0)$ ,  $n = 1, 2, \dots$  it is easy to see that  $x_1 \in (0, 1/2]$ ,  $x_1 \leq f(x_1) \leq 1/2$  and  $x_n \leq f(x_n) \leq 1/2$  (by induction). Then  $(x_n)_n$  is a bounded non-decreasing sequence and, since  $x_{n+1} = 2x_n(1 - x_n)$ , the limit  $l = \lim_{n \rightarrow \infty} x_n$  satisfies  $l = 2l(1 - l)$ , which implies  $l = 1/2$ . Now the monotone convergence theorem implies that

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 1/2.$$

b) We prove by induction that

$$(1) \quad f_n(x) = \frac{1}{2} - 2^{2^n - 1} \left( x - \frac{1}{2} \right)^{2^n}$$

holds for  $n = 1, 2, \dots$ . For  $n = 1$  this is true, since  $f(x) = 2x(1 - x) = \frac{1}{2} - 2(x - \frac{1}{2})^2$ . If (1) holds for some  $n = k$ , then we have

$$\begin{aligned} f_{k+1}(x) &= f_k(f(x)) = \frac{1}{2} - 2^{2^k - 1} \left( \left( \frac{1}{2} - 2(x - \frac{1}{2})^2 \right) - \frac{1}{2} \right)^{2^k} \\ &= \frac{1}{2} - 2^{2^k - 1} \left( -2(x - \frac{1}{2})^2 \right)^{2^k} \\ &= \frac{1}{2} - 2^{2^{k+1} - 1} \left( x - \frac{1}{2} \right)^{2^{k+1}} \end{aligned}$$

which is (2) for  $n = k + 1$ .

Using (1) we can compute the integral,

$$\int_0^1 f_n(x) dx = \left[ \frac{1}{2}x - \frac{2^{2^n - 1}}{2^n + 1} \left( x - \frac{1}{2} \right)^{2^n + 1} \right]_{x=0}^1 = \frac{1}{2} - \frac{1}{2(2^n + 1)}.$$

**Problem 4.** (20 points) The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is twice differentiable and satisfies  $f(0) = 2$ ,  $f'(0) = -2$  and  $f(1) = 1$ . Prove that there exists a real number  $\xi \in (0, 1)$  for which

$$f(\xi) \cdot f'(\xi) + f''(\xi) = 0.$$

**Solution.** Define the function

$$g(x) = \frac{1}{2}f^2(x) + f'(x).$$

Because  $g(0) = 0$  and

$$f(x) \cdot f'(x) + f''(x) = g'(x),$$

it is enough to prove that there exists a real number  $0 < \eta \leq 1$  for which  $g(\eta) = 0$ .

a) If  $f$  is never zero, let

$$h(x) = \frac{x}{2} - \frac{1}{f(x)}.$$

Because  $h(0) = h(1) = -\frac{1}{2}$ , there exists a real number  $0 < \eta < 1$  for which  $h'(\eta) = 0$ . But  $g = f^2 \cdot h'$ , and we are done.

b) If  $f$  has at least one zero, let  $z_1$  be the first one and  $z_2$  be the last one. (The set of the zeros is closed.) By the conditions,  $0 < z_1 \leq z_2 < 1$ .

The function  $f$  is positive on the intervals  $[0, z_1)$  and  $(z_2, 1]$ ; this implies that  $f'(z_1) \leq 0$  and  $f'(z_2) \geq 0$ .

Then  $g(z_1) = f'(z_1) \leq 0$  and  $g(z_2) = f'(z_2) \geq 0$ , and there exists a real number  $\eta \in [z_1, z_2]$  for which  $g(\eta) = 0$ .

**Remark.** For the function  $f(x) = \frac{2}{x+1}$  the conditions hold and  $f \cdot f' + f''$  is constantly 0.

**Problem 5.** Let  $P$  be an algebraic polynomial of degree  $n$  having only real zeros and real coefficients.

a) (15 points) Prove that for every real  $x$  the following inequality holds:

$$(2) \quad (n-1)(P'(x))^2 \geq nP(x)P''(x).$$

b) (5 points) Examine the cases of equality.

**Solution.** Observe that both sides of (2) are identically equal to zero if  $n = 1$ . Suppose that  $n > 1$ . Let  $x_1, \dots, x_n$  be the zeros of  $P$ . Clearly (2) is true when  $x = x_i$ ,  $i \in \{1, \dots, n\}$ , and equality is possible only if  $P'(x_i) = 0$ , i.e., if  $x_i$  is a multiple zero of  $P$ . Now suppose that  $x$  is not a zero of  $P$ . Using the identities

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^n \frac{1}{x-x_i}, \quad \frac{P''(x)}{P(x)} = \sum_{1 \leq i < j \leq n} \frac{2}{(x-x_i)(x-x_j)},$$

we find

$$(n-1) \left( \frac{P'(x)}{P(x)} \right)^2 - n \frac{P''(x)}{P(x)} = \sum_{i=1}^n \frac{n-1}{(x-x_i)^2} - \sum_{1 \leq i < j \leq n} \frac{2}{(x-x_i)(x-x_j)}.$$

But this last expression is simply

$$\sum_{1 \leq i < j \leq n} \left( \frac{1}{x-x_i} - \frac{1}{x-x_j} \right)^2,$$

and therefore is positive. The inequality is proved. In order that (2) holds with equality sign for every real  $x$  it is necessary that  $x_1 = x_2 = \dots = x_n$ . A direct verification shows that indeed, if  $P(x) = c(x-x_1)^n$ , then (2) becomes an identity.

**Problem 6.** Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function with the property that for any  $x$  and  $y$  in the interval,

$$xf(y) + yf(x) \leq 1.$$

a) (15 points) Show that

$$\int_0^1 f(x) dx \leq \frac{\pi}{4}.$$

b) (5 points) Find a function, satisfying the condition, for which there is equality.

**Solution** Observe that the integral is equal to

$$\int_0^{\frac{\pi}{2}} f(\sin \theta) \cos \theta d\theta$$

and to

$$\int_0^{\frac{\pi}{2}} f(\cos \theta) \sin \theta d\theta$$

So, twice the integral is at most

$$\int_0^{\frac{\pi}{2}} 1 d\theta = \frac{\pi}{2}.$$

Now let  $f(x) = \sqrt{1-x^2}$ . If  $x = \sin \theta$  and  $y = \sin \phi$  then

$$xf(y) + yf(x) = \sin \theta \cos \phi + \sin \phi \cos \theta = \sin(\theta + \phi) \leq 1.$$

5<sup>th</sup> INTERNATIONAL MATHEMATICS COMPETITION FOR UNIVERSITY STUDENTS

July 29 - August 3, 1998, Blagoevgrad, Bulgaria

Second day

PROBLEMS AND SOLUTION

**Problem 1.** (20 points) Let  $V$  be a real vector space, and let  $f, f_1, f_2, \dots, f_k$  be linear maps from  $V$  to  $\mathbb{R}$ . Suppose that  $f(x) = 0$  whenever  $f_1(x) = f_2(x) = \dots = f_k(x) = 0$ . Prove that  $f$  is a linear combination of  $f_1, f_2, \dots, f_k$ .

**Solution.** We use induction on  $k$ . By passing to a subset, we may assume that  $f_1, \dots, f_k$  are linearly independent.

Since  $f_k$  is independent of  $f_1, \dots, f_{k-1}$ , by induction there exists a vector  $a_k \in V$  such that  $f_1(a_k) = \dots = f_{k-1}(a_k) = 0$  and  $f_k(a_k) \neq 0$ . After normalising, we may assume that  $f_k(a_k) = 1$ . The vectors  $a_1, \dots, a_{k-1}$  are defined similarly to get

$$f_i(a_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For an arbitrary  $x \in V$  and  $1 \leq i \leq k$ ,  $f_i(x - f_1(x)a_1 - f_2(x)a_2 - \dots - f_k(x)a_k) = f_i(x) - \sum_{j=1}^k f_j(x)f_i(a_j) = f_i(x) - f_i(x)f_i(a_i) = 0$ , thus  $f(x - f_1(x)a_1 - \dots - f_k(x)a_k) = 0$ . By the linearity of  $f$  this implies  $f(x) = f_1(x)f(a_1) + \dots + f_k(x)f(a_k)$ , which gives  $f(x)$  as a linear combination of  $f_1(x), \dots, f_k(x)$ .

**Problem 2.** (20 points) Let

$$\mathcal{P} = \left\{ f : f(x) = \sum_{k=0}^3 a_k x^k, a_k \in \mathbb{R}, |f(\pm 1)| \leq 1, |f(\pm \frac{1}{2})| \leq 1 \right\}.$$

Evaluate

$$\sup_{f \in \mathcal{P}} \max_{-1 \leq x \leq 1} |f''(x)|$$

and find all polynomials  $f \in \mathcal{P}$  for which the above "sup" is attained.

**Solution.** Denote  $x_0 = 1, x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}, x_3 = 1$ ,

$$w(x) = \prod_{i=0}^3 (x - x_i),$$

$$w_k(x) = \frac{w(x)}{x - x_k}, \quad k = 0, \dots, 3,$$

$$l_k(x) = \frac{w_k(x)}{w_k(x_k)}.$$

Then for every  $f \in \mathcal{P}$

$$f''(x) = \sum_{k=0}^3 l_k''(x)f(x_k),$$

$$|f''(x)| \leq \sum_{k=0}^3 |l_k''(x)|.$$

Since  $f''$  is a linear function  $\max_{-1 \leq x \leq 1} |f''(x)|$  is attained either at  $x = -1$  or at  $x = 1$ . Without loss of generality let the maximum point is  $x = 1$ . Then

$$\sup_{f \in \mathcal{P}} \max_{-1 \leq x \leq 1} |f''(x)| = \sum_{k=0}^3 |l_k''(1)|.$$

In order to have equality for the extremal polynomial  $f_*$  there must hold

$$f_*(x_k) = \text{sign} l_k''(1), \quad k = 0, 1, 2, 3.$$

It is easy to see that  $\{l_k''(1)\}_{k=0}^3$  alternate in sign, so  $f_*(x_k) = (-1)^{k-1}$ ,  $k = 0, \dots, 3$ . Hence  $f_*(x) = T_3(x) = 4x^3 - 3x$ , the Chebyshev polynomial of the first kind, and  $f_*''(1) = 24$ . The other extremal polynomial, corresponding to  $x = -1$ , is  $-T_3$ .

**Problem 3.** (20 points) Let  $0 < c < 1$  and

$$f(x) = \begin{cases} \frac{x}{c} & \text{for } x \in [0, c], \\ \frac{1-x}{1-c} & \text{for } x \in [c, 1]. \end{cases}$$

We say that  $p$  is an  $n$ -periodic point if

$$\underbrace{f(f(\dots f(p)))}_n = p$$

and  $n$  is the smallest number with this property. Prove that for every  $n \geq 1$  the set of  $n$ -periodic points is non-empty and finite.

**Solution.** Let  $f_n(x) = \underbrace{f(f(\dots f(x)))}_n$ . It is easy to see that  $f_n(x)$  is a piecewise monotone function and its graph contains  $2^n$  linear segments; one endpoint is always on  $\{(x, y) : 0 \leq x \leq 1, y = 0\}$ , the other is on  $\{(x, y) : 0 \leq x \leq 1, y = 1\}$ . Thus the graph of the identity function intersects each segment once, so the number of points for which  $f_n(x) = x$  is  $2^n$ .

Since for each  $n$ -periodic points we have  $f_n(x) = x$ , the number of  $n$ -periodic points is finite.

A point  $x$  is  $n$ -periodic if  $f_n(x) = x$  but  $f_k(x) \neq x$  for  $k = 1, \dots, n-1$ . But as we saw before  $f_k(x) = x$  holds only at  $2^k$  points, so there are at most  $2^1 + 2^2 + \dots + 2^{n-1} = 2^n - 2$  points  $x$  for which  $f_k(x) = x$  for at least one  $k \in \{1, 2, \dots, n-1\}$ . Therefore at least two of the  $2^n$  points for which  $f_n(x) = x$  are  $n$ -periodic points.

**Problem 4.** (20 points) Let  $A_n = \{1, 2, \dots, n\}$ , where  $n \geq 3$ . Let  $\mathcal{F}$  be the family of all non-constant functions  $f: A_n \rightarrow A_n$  satisfying the following conditions:

- (1)  $f(k) \leq f(k+1)$  for  $k = 1, 2, \dots, n-1$ ,
- (2)  $f(k) = f(f(k+1))$  for  $k = 1, 2, \dots, n-1$ .

Find the number of functions in  $\mathcal{F}$ .

**Solution.** It is clear that  $id: A_n \rightarrow A_n$ , given by  $id(x) = x$ , does not verify condition (2). Since  $id$  is the only increasing injection on  $A_n$ ,  $\mathcal{F}$  does not contain injections. Let us take any  $f \in \mathcal{F}$  and suppose that  $\#(f^{-1}(k)) \geq 2$ . Since  $f$  is increasing, there exists  $i \in A_n$  such that  $f(i) = f(i+1) = k$ . In view of (2),  $f(k) = f(f(i+1)) = f(i) = k$ . If  $\{i < k : f(i) < k\} = \emptyset$ , then taking  $j = \max\{i < k : f(i) < k\}$  we get  $f(j) < f(j+1) = k = f(f(j+1))$ , a contradiction. Hence  $f(i) = k$  for  $i \leq k$ . If  $\#(f^{-1}(\{l\})) \geq 2$  for some  $l \geq k$ , then the similar consideration shows that  $f(i) = l = k$  for  $i \leq k$ . Hence  $\#(f^{-1}\{i\}) = 0$  or 1 for every  $i > k$ . Therefore  $f(i) \leq i$  for  $i > k$ . If  $f(l) = l$ , then taking  $j = \max\{i < l : f(i) < l\}$  we get  $f(j) < f(j+1) = l = f(f(j+1))$ , a contradiction. Thus,  $f(i) \leq i-1$  for  $i > k$ . Let  $m = \max\{i : f(i) = k\}$ . Since  $f$  is non-constant  $m \leq n-1$ . Since  $k = f(m) = f(f(m+1))$ ,  $f(m+1) \in [k+1, m]$ . If  $f(l) > l-1$  for some  $l > m+1$ , then  $l-1$  and  $f(l)$  belong to  $f^{-1}(f(l))$  and

this contradicts the facts above. Hence  $f(i) = i - 1$  for  $i > m + 1$ . Thus we show that every function  $f$  in  $\mathcal{F}$  is defined by natural numbers  $k, l, m$ , where  $1 \leq k < l = f(m + 1) \leq m \leq n - 1$ .

$$f(i) = \begin{cases} k & \text{if } i \leq m \\ l & \text{if } i = m \\ i - 1 & \text{if } i > m + 1. \end{cases}$$

Then

$$\#(\mathcal{F}) = \binom{n}{3}.$$

**Problem 5.** (20 points) Suppose that  $\mathcal{S}$  is a family of spheres (i.e., surfaces of balls of positive radius) in  $\mathbb{R}^n$ ,  $n \geq 2$ , such that the intersection of any two contains at most one point. Prove that the set  $M$  of those points that belong to at least two different spheres from  $\mathcal{S}$  is countable.

**Solution.** For every  $x \in M$  choose spheres  $S, T \in \mathcal{S}$  such that  $S \neq T$  and  $x \in S \cap T$ ; denote by  $U, V, W$  the three components of  $\mathbb{R}^n \setminus (S \cup T)$ , where the notation is such that  $\partial U = S$ ,  $\partial V = T$  and  $x$  is the only point of  $\overline{U} \cap \overline{V}$ , and choose points with rational coordinates  $u \in U$ ,  $v \in V$ , and  $w \in W$ . We claim that  $x$  is uniquely determined by the triple  $\langle u, v, w \rangle$ ; since the set of such triples is countable, this will finish the proof.

To prove the claim, suppose, that from some  $x' \in M$  we arrived to the same  $\langle u, v, w \rangle$  using spheres  $S', T' \in \mathcal{S}$  and components  $U', V', W'$  of  $\mathbb{R}^n \setminus (S' \cup T')$ . Since  $S \cap S'$  contains at most one point and since  $U \cap U' \neq \emptyset$ , we have that  $U \subset U'$  or  $U' \subset U$ ; similarly for  $V$ 's and  $W$ 's. Exchanging the role of  $x$  and  $x'$  and/or of  $U$ 's and  $V$ 's if necessary, there are only two cases to consider: (a)  $U \supset U'$  and  $V \supset V'$  and (b)  $U \subset U'$ ,  $V \supset V'$  and  $W \subset W'$ . In case (a) we recall that  $\overline{U} \cap \overline{V}$  contains only  $x$  and that  $x' \in \overline{U'} \cap \overline{V'}$ , so  $x = x'$ . In case (b) we get from  $W \subset W'$  that  $U' \subset \overline{U \cup V}$ ; so since  $U'$  is open and connected, and  $\overline{U} \cap \overline{V}$  is just one point, we infer that  $U' = U$  and we are back in the already proved case (a).

**Problem 6.** (20 points) Let  $f: (0, 1) \rightarrow [0, \infty)$  be a function that is zero except at the distinct points  $a_1, a_2, \dots$ . Let  $b_n = f(a_n)$ .

(a) Prove that if  $\sum_{n=1}^{\infty} b_n < \infty$ , then  $f$  is differentiable at at least one point  $x \in (0, 1)$ .

(b) Prove that for any sequence of non-negative real numbers  $(b_n)_{n=1}^{\infty}$ , with  $\sum_{n=1}^{\infty} b_n = \infty$ , there exists a sequence  $(a_n)_{n=1}^{\infty}$  such that the function  $f$  defined as above is nowhere differentiable.

**Solution**

a) We first construct a sequence  $c_n$  of positive numbers such that  $c_n \rightarrow \infty$  and  $\sum_{n=1}^{\infty} c_n b_n < \frac{1}{2}$ . Let  $B = \sum_{n=1}^{\infty} b_n$ , and for each  $k = 0, 1, \dots$  denote by  $N_k$  the first positive integer for which

$$\sum_{n=N_k}^{\infty} b_n \leq \frac{B}{4^k}.$$

Now set  $c_n = \frac{2^k}{5B}$  for each  $n$ ,  $N_k \leq n < N_{k+1}$ . Then we have  $c_n \rightarrow \infty$  and

$$\sum_{n=1}^{\infty} c_n b_n = \sum_{k=0}^{\infty} \sum_{N_k \leq n < N_{k+1}} c_n b_n \leq \sum_{k=0}^{\infty} \frac{2^k}{5B} \sum_{n=N_k}^{\infty} b_n \leq \sum_{k=0}^{\infty} \frac{2^k}{5B} \cdot \frac{B}{4^k} = \frac{2}{5}.$$

Consider the intervals  $I_n = (a_n - c_n b_n, a_n + c_n b_n)$ . The sum of their lengths is  $2 \sum c_n b_n < 1$ , thus there exists a point  $x_0 \in (0, 1)$  which is not contained in any  $I_n$ . We show that  $f$  is differentiable at  $x_0$ ,

and  $f'(x_0) = 0$ . Since  $x_0$  is outside of the intervals  $I_n$ ,  $x_0 \neq a_n$  for any  $n$  and  $f(x_0) = 0$ . For arbitrary  $x \in (0, 1) \setminus \{x_0\}$ , if  $x = a_n$  for some  $n$ , then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| = \frac{f(a_n) - 0}{|a_n - x_0|} \leq \frac{b_n}{c_n b_n} = \frac{1}{c_n},$$

otherwise  $\frac{f(x) - f(x_0)}{x - x_0} = 0$ . Since  $c_n \rightarrow \infty$ , this implies that for arbitrary  $\varepsilon > 0$  there are only finitely many  $x \in (0, 1) \setminus \{x_0\}$  for which

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| < \varepsilon$$

does not hold, and we are done.

**Remark.** The variation of  $f$  is finite, which implies that  $f$  is differentiable almost everywhere .

b) We remove the zero elements from sequence  $b_n$ . Since  $f(x) = 0$  except for a countable subset of  $(0, 1)$ , if  $f$  is differentiable at some point  $x_0$ , then  $f(x_0)$  and  $f'(x_0)$  must be 0.

It is easy to construct a sequence  $\beta_n$  satisfying  $0 < \beta_n \leq b_n$ ,  $b_n \rightarrow 0$  and  $\sum_{n=1}^{\infty} \beta_n = \infty$ .

Choose the numbers  $a_1, a_2, \dots$  such that the intervals  $I_n = (a_n - \beta_n, a_n + \beta_n)$  ( $n = 1, 2, \dots$ ) cover each point of  $(0, 1)$  infinitely many times (it is possible since the sum of lengths is  $2 \sum b_n = \infty$ ). Then for arbitrary  $x_0 \in (0, 1)$ ,  $f(x_0) = 0$  and  $\varepsilon > 0$  there is an  $n$  for which  $\beta_n < \varepsilon$  and  $x_0 \in I_n$  which implies

$$\frac{|f(a_n) - f(x_0)|}{|a_n - x_0|} > \frac{b_n}{\beta_n} \geq 1.$$

**6th INTERNATIONAL COMPETITION FOR UNIVERSITY  
STUDENTS IN MATHEMATICS**

Keszthely, 1999.

Problems and solutions on the first day

1. a) Show that for any  $m \in \mathbf{N}$  there exists a real  $m \times m$  matrix  $A$  such that  $A^3 = A + I$ , where  $I$  is the  $m \times m$  identity matrix. (6 points)  
 b) Show that  $\det A > 0$  for every real  $m \times m$  matrix satisfying  $A^3 = A + I$ . (14 points)

**Solution.** a) The diagonal matrix

$$A = \lambda I = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

is a solution for equation  $A^3 = A + I$  if and only if  $\lambda^3 = \lambda + 1$ , because  $A^3 - A - I = (\lambda^3 - \lambda - 1)I$ . This equation, being cubic, has real solution.

b) It is easy to check that the polynomial  $p(x) = x^3 - x - 1$  has a positive real root  $\lambda_1$  (because  $p(0) < 0$ ) and two conjugated complex roots  $\lambda_2$  and  $\lambda_3$  (one can check the discriminant of the polynomial, which is  $(\frac{-1}{3})^3 + (\frac{-1}{2})^2 = \frac{23}{108} > 0$ , or the local minimum and maximum of the polynomial).

If a matrix  $A$  satisfies equation  $A^3 = A + I$ , then its eigenvalues can be only  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$ . The multiplicity of  $\lambda_2$  and  $\lambda_3$  must be the same, because  $A$  is a real matrix and its characteristic polynomial has only real coefficients. Denoting the multiplicity of  $\lambda_1$  by  $\alpha$  and the common multiplicity of  $\lambda_2$  and  $\lambda_3$  by  $\beta$ ,

$$\det A = \lambda_1^\alpha \lambda_2^\beta \lambda_3^\beta = \lambda_1^\alpha \cdot (\lambda_2 \lambda_3)^\beta.$$

Because  $\lambda_1$  and  $\lambda_2 \lambda_3 = |\lambda_2|^2$  are positive, the product on the right side has only positive factors.

2. Does there exist a bijective map  $\pi : \mathbf{N} \rightarrow \mathbf{N}$  such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

(20 points)

**Solution 1.** No. For, let  $\pi$  be a permutation of  $\mathbf{N}$  and let  $N \in \mathbf{N}$ . We shall argue that

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} > \frac{1}{9}.$$

In fact, of the  $2N$  numbers  $\pi(N+1), \dots, \pi(3N)$  only  $N$  can be  $\leq N$  so that at least  $N$  of them are  $> N$ . Hence

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} \geq \frac{1}{(3N)^2} \sum_{n=N+1}^{3N} \pi(n) > \frac{1}{9N^2} \cdot N \cdot N = \frac{1}{9}.$$

**Solution 2.** Let  $\pi$  be a permutation of  $\mathbf{N}$ . For any  $n \in \mathbf{N}$ , the numbers  $\pi(1), \dots, \pi(n)$  are distinct positive integers, thus  $\pi(1) + \dots + \pi(n) \geq 1 + \dots + n = \frac{n(n+1)}{2}$ . By this inequality,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} &= \sum_{n=1}^{\infty} (\pi(1) + \dots + \pi(n)) \left( \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \geq \\ &\geq \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \cdot \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} \geq \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty. \end{aligned}$$

3. Suppose that a function  $f : \mathbf{R} \rightarrow \mathbf{R}$  satisfies the inequality

$$\left| \sum_{k=1}^n 3^k (f(x+ky) - f(x-ky)) \right| \leq 1 \quad (1)$$

for every positive integer  $n$  and for all  $x, y \in \mathbf{R}$ . Prove that  $f$  is a constant function. (20 points)

**Solution.** Writing (1) with  $n-1$  instead of  $n$ ,

$$\left| \sum_{k=1}^{n-1} 3^k (f(x+ky) - f(x-ky)) \right| \leq 1. \quad (2)$$

From the difference of (1) and (2),

$$|3^n (f(x+ny) - f(x-ny))| \leq 2;$$

which means

$$|f(x+ny) - f(x-ny)| \leq \frac{2}{3^n}. \quad (3)$$

For arbitrary  $u, v \in \mathbf{R}$  and  $n \in \mathbf{N}$  one can choose  $x$  and  $y$  such that  $x-ny = u$  and  $x+ny = v$ , namely  $x = \frac{u+v}{2}$  and  $y = \frac{v-u}{2n}$ . Thus, (3) yields

$$|f(u) - f(v)| \leq \frac{2}{3^n}$$

for arbitrary positive integer  $n$ . Because  $\frac{2}{3^n}$  can be arbitrary small, this implies  $f(u) = f(v)$ .

4. Find all strictly monotonic functions  $f : (0, +\infty) \rightarrow (0, +\infty)$  such that  $f\left(\frac{x^2}{f(x)}\right) \equiv x$ . (20 points)

**Solution.** Let  $g(x) = \frac{f(x)}{x}$ . We have  $g\left(\frac{x}{g(x)}\right) = g(x)$ . By induction it follows that  $g\left(\frac{x}{g^n(x)}\right) = g(x)$ , i.e.

$$(1) \quad f\left(\frac{x}{g^n(x)}\right) = \frac{x}{g^{n-1}(x)}, \quad n \in \mathbf{N}.$$

On the other hand, let substitute  $x$  by  $f(x)$  in  $f\left(\frac{x^2}{f(x)}\right) = x$ . From the injectivity of  $f$  we get  $\frac{f^2(x)}{f(f(x))} = x$ , i.e.  $g(xg(x)) = g(x)$ . Again by induction we deduce that  $g(xg^n(x)) = g(x)$  which can be written in the form

$$(2) \quad f(xg^n(x)) = xg^{n-1}(x), \quad n \in \mathbf{N}.$$

Set  $f^{(m)} = \underbrace{f \circ f \circ \dots \circ f}_{m \text{ times}}$ . It follows from (1) and (2) that

$$(3) \quad f^{(m)}(xg^n(x)) = xg^{n-m}(x), \quad m, n \in \mathbf{N}.$$

Now, we shall prove that  $g$  is a constant. Assume  $g(x_1) < g(x_2)$ . Then we may find  $n \in \mathbf{N}$  such that  $x_1 g^n(x_1) \leq x_2 g^n(x_2)$ . On the other hand, if  $m$  is even then  $f^{(m)}$  is strictly increasing and from (3) it follows that  $x_1^m g^{n-m}(x_1) \leq x_2^m g^{n-m}(x_2)$ . But when  $n$  is fixed the opposite inequality holds  $\forall m \gg 1$ . This contradiction shows that  $g$  is a constant, i.e.  $f(x) = Cx$ ,  $C > 0$ .

Conversely, it is easy to check that the functions of this type verify the conditions of the problem.

5. Suppose that  $2n$  points of an  $n \times n$  grid are marked. Show that for some  $k > 1$  one can select  $2k$  distinct marked points, say  $a_1, \dots, a_{2k}$ , such that  $a_1$  and  $a_2$  are in the same row,  $a_2$  and  $a_3$  are in the same column,  $\dots$ ,  $a_{2k-1}$  and  $a_{2k}$  are in the same row, and  $a_{2k}$  and  $a_1$  are in the same column. (20 points)

**Solution 1.** We prove the more general statement that if at least  $n + k$  points are marked in an  $n \times k$  grid, then the required sequence of marked points can be selected.

If a row or a column contains at most one marked point, delete it. This decreases  $n + k$  by 1 and the number of the marked points by at most 1, so the condition remains true. Repeat this step until each row and column contains at least two marked points. Note that the condition implies that there are at least two marked points, so the whole set of marked points cannot be deleted.

We define a sequence  $b_1, b_2, \dots$  of marked points. Let  $b_1$  be an arbitrary marked point. For any positive integer  $n$ , let  $b_{2n}$  be an other marked point in the row of  $b_{2n-1}$  and  $b_{2n+1}$  be an other marked point in the column of  $b_{2n}$ .

Let  $m$  be the first index for which  $b_m$  is the same as one of the earlier points, say  $b_m = b_l$ ,  $l < m$ .

If  $m - l$  is even, the line segments  $b_l b_{l+1}, b_{l+1} b_{l+2}, \dots, b_{m-1} b_l = b_{m-1} b_m$  are alternating horizontal and vertical. So one can choose  $2k = m - l$ , and  $(a_1, \dots, a_{2k}) = (b_l, \dots, b_{m-1})$  or  $(a_1, \dots, a_{2k}) = (b_{l+1}, \dots, b_m)$  if  $l$  is odd or even, respectively.

If  $m - l$  is odd, then the points  $b_l = b_m, b_{l+1}$  and  $b_{m-1}$  are in the same row/column. In this case chose  $2k = m - l - 1$ . Again, the line segments  $b_{l+1} b_{l+2}, b_{l+2} b_{l+3}, \dots, b_{m-1} b_{l+1}$  are alternating horizontal and vertical and one can choose  $(a_1, \dots, a_{2k}) = (b_{l+1}, \dots, b_{m-1})$  or  $(a_1, \dots, a_{2k}) = (b_{l+2}, \dots, b_{m-1}, b_{l+1})$  if  $l$  is even or odd, respectively.

**Solution 2.** Define the graph  $G$  in the following way: Let the vertices of  $G$  be the rows and the columns of the grid. Connect a row  $r$  and a column  $c$  with an edge if the intersection point of  $r$  and  $c$  is marked.

The graph  $G$  has  $2n$  vertices and  $2n$  edges. As is well known, if a graph of  $N$  vertices contains no circle, it can have at most  $N - 1$  edges. Thus  $G$  does contain a circle. A circle is an alternating sequence of rows and columns, and the intersection of each neighbouring row and column is a marked point. The required sequence consists of these intersection points.

- 6.** a) For each  $1 < p < \infty$  find a constant  $c_p < \infty$  for which the following statement holds: If  $f : [-1, 1] \rightarrow \mathbf{R}$  is a continuously differentiable function satisfying  $f(1) > f(-1)$  and  $|f'(y)| \leq 1$  for all  $y \in [-1, 1]$ , then there is an  $x \in [-1, 1]$  such that  $f'(x) > 0$  and  $|f(y) - f(x)| \leq c_p (f'(x))^{1/p} |y - x|$  for all  $y \in [-1, 1]$ . (10 points)  
 b) Does such a constant also exist for  $p = 1$ ? (10 points)

**Solution.** (a) Let  $g(x) = \max(0, f'(x))$ . Then  $0 < \int_{-1}^1 f'(x) dx = \int_{-1}^1 g(x) dx + \int_{-1}^1 (f'(x) - g(x)) dx$ , so we get  $\int_{-1}^1 |f'(x)| dx = \int_{-1}^1 g(x) dx + \int_{-1}^1 (g(x) - f'(x)) dx < 2 \int_{-1}^1 g(x) dx$ . Fix  $p$  and  $c$  (to be determined at the end). Given any  $t > 0$ , choose for every  $x$  such that  $g(x) > t$  an interval  $I_x = [x, y]$  such that  $|f(y) - f(x)| > cg(x)^{1/p} |y - x| > ct^{1/p} |I_x|$  and choose disjoint  $I_{x_i}$  that cover at least one third of the measure of the set  $\{g > t\}$ . For  $I = \bigcup_i I_i$  we thus have  $ct^{1/p} |I| \leq \int_I f'(x) dx \leq \int_{-1}^1 |f'(x)| dx < 2 \int_{-1}^1 g(x) dx$ ; so  $|\{g > t\}| \leq 3|I| < (6/c)t^{-1/p} \int_{-1}^1 g(x) dx$ . Integrating the inequality, we get  $\int_{-1}^1 g(x) dx = \int_0^1 |\{g > t\}| dt < (6/c)p/(p-1) \int_{-1}^1 g(x) dx$ ; this is a contradiction e.g. for  $c_p = (6p)/(p-1)$ .

(b) No. Given  $c > 1$ , denote  $\alpha = 1/c$  and choose  $0 < \varepsilon < 1$  such that  $((1 + \varepsilon)/(2\varepsilon))^{-\alpha} < 1/4$ . Let  $g : [-1, 1] \rightarrow [-1, 1]$  be continuous, even,  $g(x) = -1$  for  $|x| \leq \varepsilon$  and  $0 \leq g(x) < \alpha((|x| + \varepsilon)/(2\varepsilon))^{-\alpha-1}$  for  $\varepsilon < |x| \leq 1$  is chosen such that  $\int_{\varepsilon}^1 g(t) dt > -\varepsilon/2 + \int_{\varepsilon}^1 \alpha((|x| + \varepsilon)/(2\varepsilon))^{-\alpha-1} dt = -\varepsilon/2 + 2\varepsilon(1 - ((1 + \varepsilon)/(2\varepsilon))^{-\alpha}) > \varepsilon$ . Let  $f = \int g(t) dt$ . Then  $f(1) - f(-1) \geq -2\varepsilon + 2 \int_{\varepsilon}^1 g(t) dt > 0$ . If  $\varepsilon < x < 1$  and  $y = -\varepsilon$ , then  $|f(x) - f(y)| \geq 2\varepsilon - \int_{\varepsilon}^x g(t) dt \geq 2\varepsilon - \int_{\varepsilon}^x \alpha((t + \varepsilon)/(2\varepsilon))^{-\alpha-1} dt = 2\varepsilon((x + \varepsilon)/(2\varepsilon))^{-\alpha} > g(x)|x - y|/\alpha = f'(x)|x - y|/\alpha$ ; symmetrically for  $-1 < x < -\varepsilon$  and  $y = \varepsilon$ .

**6th INTERNATIONAL COMPETITION FOR UNIVERSITY  
STUDENTS IN MATHEMATICS**

Keszthely, 1999.

Problems and solutions on the second day

**1.** Suppose that in a not necessarily commutative ring  $R$  the square of any element is 0. Prove that  $abc + abc = 0$  for any three elements  $a, b, c$ . (20 points)

**Solution.** From  $0 = (a + b)^2 = a^2 + b^2 + ab + ba = ab + ba$ , we have  $ab = -(ba)$  for arbitrary  $a, b$ , which implies

$$abc = a(bc) = -((bc)a) = -(b(ca)) = (ca)b = c(ab) = -((ab)c) = -abc.$$

**2.** We throw a dice (which selects one of the numbers  $1, 2, \dots, 6$  with equal probability)  $n$  times. What is the probability that the sum of the values is divisible by 5? (20 points)

**Solution 1.** For all nonnegative integers  $n$  and modulo 5 residue class  $r$ , denote by  $p_n^{(r)}$  the probability that after  $n$  throwing the sum of values is congruent to  $r$  modulo 5. It is obvious that  $p_0^{(0)} = 1$  and  $p_0^{(1)} = p_0^{(2)} = p_0^{(3)} = p_0^{(4)} = 0$ .

Moreover, for any  $n > 0$  we have

$$p_n^{(r)} = \sum_{i=1}^6 \frac{1}{6} p_{n-1}^{(r-i)}. \quad (1)$$

From this recursion we can compute the probabilities for small values of  $n$  and can conjecture that  $p_n^{(r)} = \frac{1}{5} + \frac{4}{5 \cdot 6^n}$  if  $n \equiv r \pmod{5}$  and  $p_n^{(r)} = \frac{1}{5} - \frac{1}{5 \cdot 6^n}$  otherwise. From (1), this conjecture can be proved by induction.

**Solution 2.** Let  $S$  be the set of all sequences consisting of digits  $1, \dots, 6$  of length  $n$ . We create collections of these sequences.

Let a collection contain sequences of the form

$$\underbrace{66 \dots 6}_k XY_1 \dots Y_{n-k-1},$$

where  $X \in \{1, 2, 3, 4, 5\}$  and  $k$  and the digits  $Y_1, \dots, Y_{n-k-1}$  are fixed. Then each collection consists of 5 sequences, and the sums of the digits of sequences give a whole residue system mod 5.

Except for the sequence  $66 \dots 6$ , each sequence is the element of one collection. This means that the number of the sequences, which have a sum of digits divisible by 5, is  $\frac{1}{5}(6^n - 1) + 1$  if  $n$  is divisible by 5, otherwise  $\frac{1}{5}(6^n - 1)$ .

Thus, the probability is  $\frac{1}{5} + \frac{4}{5 \cdot 6^n}$  if  $n$  is divisible by 5, otherwise it is  $\frac{1}{5} - \frac{1}{5 \cdot 6^n}$ .

**Solution 3.** For arbitrary positive integer  $k$  denote by  $p_k$  the probability that the sum of values is  $k$ . Define the generating function

$$f(x) = \sum_{k=1}^{\infty} p_k x^k = \left( \frac{x + x^2 + x^3 + x^4 + x^5 + x^6}{6} \right)^n.$$

(The last equality can be easily proved by induction.)

Our goal is to compute the sum  $\sum_{k=1}^{\infty} p_{5k}$ . Let  $\varepsilon = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$  be the first 5th root of unity. Then

$$\sum_{k=1}^{\infty} p_{5k} = \frac{f(1) + f(\varepsilon) + f(\varepsilon^2) + f(\varepsilon^3) + f(\varepsilon^4)}{5}.$$

Obviously  $f(1) = 1$ , and  $f(\varepsilon^j) = \frac{\varepsilon^{jn}}{6^n}$  for  $j = 1, 2, 3, 4$ . This implies that  $f(\varepsilon) + f(\varepsilon^2) + f(\varepsilon^3) + f(\varepsilon^4)$  is  $\frac{4}{6^n}$  if  $n$  is divisible by 5, otherwise it is  $\frac{-1}{6^n}$ . Thus,  $\sum_{k=1}^{\infty} p_{5k}$  is  $\frac{1}{5} + \frac{4}{5 \cdot 6^n}$  if  $n$  is divisible by 5, otherwise it is  $\frac{1}{5} - \frac{1}{5 \cdot 6^n}$ .

3. Assume that  $x_1, \dots, x_n \geq -1$  and  $\sum_{i=1}^n x_i^3 = 0$ . Prove that  $\sum_{i=1}^n x_i \leq \frac{n}{3}$ . (20 points)

**Solution.** The inequality

$$0 \leq x^3 - \frac{3}{4}x + \frac{1}{4} = (x+1) \left(x - \frac{1}{2}\right)^2$$

holds for  $x \geq -1$ .

Substituting  $x_1, \dots, x_n$ , we obtain

$$0 \leq \sum_{i=1}^n \left(x_i^3 - \frac{3}{4}x_i + \frac{1}{4}\right) = \sum_{i=1}^n x_i^3 - \frac{3}{4} \sum_{i=1}^n x_i + \frac{n}{4} = 0 - \frac{3}{4} \sum_{i=1}^n x_i + \frac{n}{4},$$

so  $\sum_{i=1}^n x_i \leq \frac{n}{3}$ .

*Remark.* Equality holds only in the case when  $n = 9k$ ,  $k$  of the  $x_1, \dots, x_n$  are  $-1$ , and  $8k$  of them are  $\frac{1}{2}$ .

4. Prove that there exists no function  $f : (0, +\infty) \rightarrow (0, +\infty)$  such that  $f^2(x) \geq f(x+y)(f(x)+y)$  for any  $x, y > 0$ . (20 points)

**Solution.** Assume that such a function exists. The initial inequality can be written in the form  $f(x) - f(x+y) \geq f(x) - \frac{f^2(x)}{f(x)+y} = \frac{f(x)y}{f(x)+y}$ . Obviously,  $f$  is a decreasing function. Fix  $x > 0$  and choose  $n \in \mathbf{N}$  such that  $nf(x+1) \geq 1$ . For  $k = 0, 1, \dots, n-1$  we have

$$f\left(x + \frac{k}{n}\right) - f\left(x + \frac{k+1}{n}\right) \geq \frac{f\left(x + \frac{k}{n}\right)}{nf\left(x + \frac{k}{n}\right) + 1} \geq \frac{1}{2n}.$$

The addition of these inequalities gives  $f(x+1) \leq f(x) - \frac{1}{2}$ . From this it follows that  $f(x+2m) \leq f(x) - m$  for all  $m \in \mathbf{N}$ . Taking  $m \geq f(x)$ , we get a contradiction with the condition  $f(x) > 0$ .

5. Let  $S$  be the set of all words consisting of the letters  $x, y, z$ , and consider an equivalence relation  $\sim$  on  $S$  satisfying the following conditions: for arbitrary words  $u, v, w \in S$

- (i)  $uu \sim u$ ;
- (ii) if  $v \sim w$ , then  $uv \sim uw$  and  $vu \sim wu$ .

Show that every word in  $S$  is equivalent to a word of length at most 8. (20 points)

**Solution.** First we prove the following lemma: If a word  $u \in S$  contains at least one of each letter, and  $v \in S$  is an arbitrary word, then there exists a word  $w \in S$  such that  $uvw \sim u$ .

If  $v$  contains a single letter, say  $x$ , write  $u$  in the form  $u = u_1xu_2$ , and choose  $w = u_2$ . Then  $uvw = (u_1xu_2)u_2 = u_1((xu_2)(u_2)) \sim u_1(xu_2) = u$ .

In the general case, let the letters of  $v$  be  $a_1, \dots, a_k$ . Then one can choose some words  $w_1, \dots, w_k$  such that  $(ua_1)w_1 \sim u$ ,  $(ua_1a_2)w_2 \sim ua_1$ ,  $\dots$ ,  $(ua_1 \dots a_k)w_k \sim ua_1 \dots a_{k-1}$ . Then  $u \sim ua_1w_1 \sim ua_1a_2w_2w_1 \sim \dots \sim ua_1 \dots a_kw_k \dots w_1 = uv(w_k \dots w_1)$ , so  $w = w_k \dots w_1$  is a good choice.

Consider now an arbitrary word  $a$ , which contains more than 8 digits. We shall prove that there is a shorter word which is equivalent to  $a$ . If  $a$  can be written in the form  $uvvw$ , its length can be reduced by  $uvvw \sim uvw$ . So we can assume that  $a$  does not have this form.

Write  $a$  in the form  $a = bcd$ , where  $b$  and  $d$  are the first and last four letter of  $a$ , respectively. We prove that  $a \sim bd$ .

It is easy to check that  $b$  and  $d$  contains all the three letters  $x, y$  and  $z$ , otherwise their length could be reduced. By the lemma there is a word  $e$  such that  $b(cd)e \sim b$ , and there is a word  $f$  such that  $def \sim d$ . Then we can write

$$a = bcd \sim bc(def) \sim bc(dedef) = (bcde)(def) \sim bd.$$

*Remark.* Of course, it is enough to give for every word of length 9 an shortest shorter word. Assuming that the first letter is  $x$  and the second is  $y$ , it is easy (but a little long) to check that there are 18 words of length 9 which cannot be written in the form  $uvvw$ .

For five of these words there is a 2-step solution, for example

$$xyzzyzx \underline{zy} \sim xy \underline{xzyz} xzyzy \sim xyx \underline{zyzy} \sim xyxzy.$$

In the remaining 13 cases we need more steps. The general algorithm given by the Solution works for these cases as well, but needs also very long words. For example, to reduce the length of the word  $a = xyzyxzxyz$ , we have set  $b = xzyz$ ,  $c = x$ ,  $d = zxyz$ ,  $e = xyxzzyxyzy$ ,  $f = zyxyxzyxzxzxxyzyzyz$ . The longest word in the algorithm was

$$bcdedef = xyzyxzxyzxyxzzyxyzyzxyxzzyxyzyzyxyxzzyxzxzxxyzyzyz,$$

which is of length 46. This is not the shortest way: reducing the length of word  $a$  can be done for example by the following steps:

$$xyzyxz \underline{yz} \sim xyzyxz \underline{xzyz} z \sim xyzyxzxy \underline{zyx} yzyz \sim \underline{xyzyxz} \underline{xyzyxz} \underline{yx} \underline{yz} \underline{yz} \sim xy \underline{zyx} \underline{zyx} \underline{yz} \sim xyzyxzyz.$$

(The last example is due to Nayden Kambouchev from Sofia University.)

**6.** Let  $A$  be a subset of  $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$  containing at most  $\frac{1}{100} \ln n$  elements. Define the  $r$ th Fourier coefficient of  $A$  for  $r \in \mathbf{Z}_n$  by

$$f(r) = \sum_{s \in A} \exp\left(\frac{2\pi i}{n} sr\right).$$

Prove that there exists an  $r \neq 0$ , such that  $|f(r)| \geq \frac{|A|}{2}$ . (20 points)

**Solution.** Let  $A = \{a_1, \dots, a_k\}$ . Consider the  $k$ -tuples

$$\left(\exp \frac{2\pi i a_1 t}{n}, \dots, \exp \frac{2\pi i a_k t}{n}\right) \in \mathbf{C}^k, \quad t = 0, 1, \dots, n-1.$$

Each component is in the unit circle  $|z| = 1$ . Split the circle into 6 equal arcs. This induces a decomposition of the  $k$ -tuples into  $6^k$  classes. By the condition  $k \leq \frac{1}{100} \ln n$  we have  $n > 6^k$ , so there are two  $k$ -tuples in the same class say for  $t_1 < t_2$ . Set  $r = t_2 - t_1$ . Then

$$\operatorname{Re} \exp \frac{2\pi i a_j r}{n} = \cos\left(\frac{2\pi a_j t_2}{n} - \frac{2\pi a_j t_1}{n}\right) \geq \cos \frac{\pi}{3} = \frac{1}{2}$$

for all  $j$ , so

$$|f(r)| \geq \operatorname{Re} f(r) \geq \frac{k}{2}.$$

## Solutions for the first day problems at the IMC 2000

### Problem 1.

Is it true that if  $f : [0, 1] \rightarrow [0, 1]$  is

a) monotone increasing

b) monotone decreasing

then there exists an  $x \in [0, 1]$  for which  $f(x) = x$ ?

#### Solution.

a) Yes.

Proof: Let  $A = \{x \in [0, 1] : f(x) > x\}$ . If  $f(0) = 0$  we are done, if not then  $A$  is non-empty (0 is in  $A$ ) bounded, so it has supremum, say  $a$ . Let  $b = f(a)$ .

I. case:  $a < b$ . Then, using that  $f$  is monotone and  $a$  was the sup, we get  $b = f(a) \leq f((a+b)/2) \leq (a+b)/2$ , which contradicts  $a < b$ .

II. case:  $a > b$ . Then we get  $b = f(a) \geq f((a+b)/2) > (a+b)/2$  contradiction. Therefore we must have  $a = b$ .

b) No. Let, for example,

$$f(x) = 1 - x/2 \quad \text{if } x \leq 1/2$$

and

$$f(x) = 1/2 - x/2 \quad \text{if } x > 1/2$$

This is clearly a good counter-example.

### Problem 2.

Let  $p(x) = x^5 + x$  and  $q(x) = x^5 + x^2$ . Find all pairs  $(w, z)$  of complex numbers with  $w \neq z$  for which  $p(w) = p(z)$  and  $q(w) = q(z)$ .

**Short solution.** Let

$$P(x, y) = \frac{p(x) - p(y)}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + 1$$

and

$$Q(x, y) = \frac{q(x) - q(y)}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + x + y.$$

We need those pairs  $(w, z)$  which satisfy  $P(w, z) = Q(w, z) = 0$ .

From  $P - Q = 0$  we have  $w + z = 1$ . Let  $c = wz$ . After a short calculation we obtain  $c^2 - 3c + 2 = 0$ , which has the solutions  $c = 1$  and  $c = 2$ . From the system  $w + z = 1$ ,  $wz = c$  we obtain the following pairs:

$$\left( \frac{1 \pm \sqrt{3}i}{2}, \frac{1 \mp \sqrt{3}i}{2} \right) \quad \text{and} \quad \left( \frac{1 \pm \sqrt{7}i}{2}, \frac{1 \mp \sqrt{7}i}{2} \right).$$

**Problem 3.**

$A$  and  $B$  are square complex matrices of the same size and

$$\text{rank}(AB - BA) = 1.$$

Show that  $(AB - BA)^2 = 0$ .

Let  $C = AB - BA$ . Since  $\text{rank } C = 1$ , at most one eigenvalue of  $C$  is different from 0. Also  $\text{tr } C = 0$ , so all the eigenvalues are zero. In the Jordan canonical form there can only be one  $2 \times 2$  cage and thus  $C^2 = 0$ .

**Problem 4.**

a) Show that if  $(x_i)$  is a decreasing sequence of positive numbers then

$$\left( \sum_{i=1}^n x_i^2 \right)^{1/2} \leq \sum_{i=1}^n \frac{x_i}{\sqrt{i}}.$$

b) Show that there is a constant  $C$  so that if  $(x_i)$  is a decreasing sequence of positive numbers then

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left( \sum_{i=m}^{\infty} x_i^2 \right)^{1/2} \leq C \sum_{i=1}^{\infty} x_i.$$

**Solution.**

a)

$$\left( \sum_{i=1}^n \frac{x_i}{\sqrt{i}} \right)^2 = \sum_{i,j} \frac{x_i x_j}{\sqrt{i} \sqrt{j}} \geq \sum_{i=1}^n \frac{x_i}{\sqrt{i}} \sum_{j=1}^i \frac{x_i}{\sqrt{j}} \geq \sum_{i=1}^n \frac{x_i}{\sqrt{i}} i \frac{x_i}{\sqrt{i}} = \sum_{i=1}^n x_i^2$$

b)

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left( \sum_{i=m}^{\infty} x_i^2 \right)^{1/2} \leq \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{i=m}^{\infty} \frac{x_i}{\sqrt{i-m+1}}$$

by a)

$$= \sum_{i=1}^{\infty} x_i \sum_{m=1}^i \frac{1}{\sqrt{m} \sqrt{i-m+1}}$$

You can get a sharp bound on

$$\sup_i \sum_{m=1}^i \frac{1}{\sqrt{m} \sqrt{i-m+1}}$$

by checking that it is at most

$$\int_0^{i+1} \frac{1}{\sqrt{x} \sqrt{i+1-x}} dx = \pi$$

Alternatively you can observe that

$$\begin{aligned} \sum_{m=1}^i \frac{1}{\sqrt{m}\sqrt{i+1-m}} &= 2 \sum_{m=1}^{i/2} \frac{1}{\sqrt{m}\sqrt{i+1-m}} \leq \\ &\leq 2 \frac{1}{\sqrt{i/2}} \sum_{m=1}^{i/2} \frac{1}{\sqrt{m}} \leq 2 \frac{1}{\sqrt{i/2}} \cdot 2\sqrt{i/2} = 4 \end{aligned}$$

**Problem 5.**

Let  $R$  be a ring of characteristic zero (not necessarily commutative). Let  $e, f$  and  $g$  be idempotent elements of  $R$  satisfying  $e + f + g = 0$ . Show that  $e = f = g = 0$ .

( $R$  is of characteristic zero means that, if  $a \in R$  and  $n$  is a positive integer, then  $na \neq 0$  unless  $a = 0$ . An idempotent  $x$  is an element satisfying  $x = x^2$ .)

**Solution.** Suppose that  $e + f + g = 0$  for given idempotents  $e, f, g \in R$ . Then

$$g = g^2 = -(e + f)^2 = e + (ef + fe) + f = (ef + fe) - g,$$

i.e.  $ef + fe = 2g$ , whence the additive commutator

$$[e, f] = ef - fe = [e, ef + fe] = 2[e, g] = 2[e, -e - f] = -2[e, f],$$

i.e.  $ef = fe$  (since  $R$  has zero characteristic). Thus  $ef + fe = 2g$  becomes  $ef = g$ , so that  $e + f + ef = 0$ . On multiplying by  $e$ , this yields  $e + 2ef = 0$ , and similarly  $f + 2ef = 0$ , so that  $f = -2ef = e$ , hence  $e = f = g$  by symmetry. Hence, finally,  $3e = e + f + g = 0$ , i.e.  $e = f = g = 0$ .

For part (i) just omit some of this.

**Problem 6.**

Let  $f : \mathbb{R} \rightarrow (0, \infty)$  be an increasing differentiable function for which  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $f'$  is bounded.

Let  $F(x) = \int_0^x f$ . Define the sequence  $(a_n)$  inductively by

$$a_0 = 1, \quad a_{n+1} = a_n + \frac{1}{f(a_n)},$$

and the sequence  $(b_n)$  simply by  $b_n = F^{-1}(n)$ . Prove that  $\lim_{n \rightarrow \infty} (a_n - b_n) = 0$ .

**Solution.** From the conditions it is obvious that  $F$  is increasing and  $\lim_{n \rightarrow \infty} b_n = \infty$ .

By Lagrange's theorem and the recursion in (1), for all  $k \geq 0$  integers there exists a real number  $\xi \in (a_k, a_{k+1})$  such that

$$F(a_{k+1}) - F(a_k) = f(\xi)(a_{k+1} - a_k) = \frac{f(\xi)}{f(a_k)}. \tag{2}$$

By the monotonicity,  $f(a_k) \leq f(\xi) \leq f(a_{k+1})$ , thus

$$1 \leq F(a_{k+1}) - F(a_k) \leq \frac{f(a_{k+1})}{f(a_k)} = 1 + \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \quad (3)$$

Summing (3) for  $k = 0, \dots, n-1$  and substituting  $F(b_n) = n$ , we have

$$F(b_n) < n + F(a_0) \leq F(a_n) \leq F(b_n) + F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \quad (4)$$

From the first two inequalities we already have  $a_n > b_n$  and  $\lim_{n \rightarrow \infty} a_n = \infty$ .

Let  $\varepsilon$  be an arbitrary positive number. Choose an integer  $K_\varepsilon$  such that  $f(a_{K_\varepsilon}) > \frac{2}{\varepsilon}$ . If  $n$  is sufficiently large, then

$$\begin{aligned} & F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} = \\ & = \left( F(a_0) + \sum_{k=0}^{K_\varepsilon-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \right) + \sum_{k=K_\varepsilon}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} < \\ & < O_\varepsilon(1) + \frac{1}{f(a_{K_\varepsilon})} \sum_{k=K_\varepsilon}^{n-1} (f(a_{k+1}) - f(a_k)) < \\ & < O_\varepsilon(1) + \frac{\varepsilon}{2} (f(a_n) - f(a_{K_\varepsilon})) < \varepsilon f(a_n). \end{aligned} \quad (5)$$

Inequalities (4) and (5) together say that for any positive  $\varepsilon$ , if  $n$  is sufficiently large,

$$F(a_n) - F(b_n) < \varepsilon f(a_n).$$

Again, by Lagrange's theorem, there is a real number  $\zeta \in (b_n, a_n)$  such that

$$F(a_n) - F(b_n) = f(\zeta)(a_n - b_n) > f(b_n)(a_n - b_n), \quad (6)$$

thus

$$f(b_n)(a_n - b_n) < \varepsilon f(a_n). \quad (7)$$

Let  $B$  be an upper bound for  $f'$ . Apply  $f(a_n) < f(b_n) + B(a_n - b_n)$  in (7):

$$\begin{aligned} f(b_n)(a_n - b_n) & < \varepsilon(f(b_n) + B(a_n - b_n)), \\ (f(b_n) - \varepsilon B)(a_n - b_n) & < \varepsilon f(b_n). \end{aligned} \quad (8)$$

Due to  $\lim_{n \rightarrow \infty} f(b_n) = \infty$ , the first factor is positive, and we have

$$a_n - b_n < \varepsilon \frac{f(b_n)}{f(b_n) - \varepsilon B} < 2\varepsilon \quad (9)$$

for sufficiently large  $n$ .

Thus, for arbitrary positive  $\varepsilon$  we proved that  $0 < a_n - b_n < 2\varepsilon$  if  $n$  is sufficiently large.

## Solutions for the second day problems at the IMC 2000

### Problem 1.

a) Show that the unit square can be partitioned into  $n$  smaller squares if  $n$  is large enough.

b) Let  $d \geq 2$ . Show that there is a constant  $N(d)$  such that, whenever  $n \geq N(d)$ , a  $d$ -dimensional unit cube can be partitioned into  $n$  smaller cubes.

**Solution.** We start with the following lemma: If  $a$  and  $b$  be coprime positive integers then every sufficiently large positive integer  $m$  can be expressed in the form  $ax + by$  with  $x, y$  non-negative integers.

*Proof of the lemma.* The numbers  $0, a, 2a, \dots, (b-1)a$  give a complete residue system modulo  $b$ . Consequently, for any  $m$  there exists a  $0 \leq x \leq b-1$  so that  $ax \equiv m \pmod{b}$ . If  $m \geq (b-1)a$ , then  $y = (m - ax)/b$ , for which  $x + by = m$ , is a non-negative integer, too.

Now observe that any dissection of a cube into  $n$  smaller cubes may be refined to give a dissection into  $n + (a^d - 1)$  cubes, for any  $a \geq 1$ . This refinement is achieved by picking an arbitrary cube in the dissection, and cutting it into  $a^d$  smaller cubes. To prove the required result, then, it suffices to exhibit two relatively prime integers of form  $a^d - 1$ . In the 2-dimensional case,  $a_1 = 2$  and  $a_2 = 3$  give the coprime numbers  $2^2 - 1 = 3$  and  $3^2 - 1 = 8$ . In the general case, two such integers are  $2^d - 1$  and  $(2^d - 1)^d - 1$ , as is easy to check.

**Problem 2.** Let  $f$  be continuous and nowhere monotone on  $[0, 1]$ . Show that the set of points on which  $f$  attains local minima is dense in  $[0, 1]$ .

(A function is nowhere monotone if there exists no interval where the function is monotone. A set is dense if each non-empty open interval contains at least one element of the set.)

**Solution.** Let  $(x - \alpha, x + \alpha) \subset [0, 1]$  be an arbitrary non-empty open interval. The function  $f$  is not monotone in the intervals  $[x - \alpha, x]$  and  $[x, x + \alpha]$ , thus there exist some real numbers  $x - \alpha \leq p < q \leq x$ ,  $x \leq r < s \leq x + \alpha$  so that  $f(p) > f(q)$  and  $f(r) < f(s)$ .

By Weierstrass' theorem,  $f$  has a global minimum in the interval  $[p, s]$ . The values  $f(p)$  and  $f(s)$  are not the minimum, because they are greater than  $f(q)$  and  $f(s)$ , respectively. Thus the minimum is in the interior of the interval, it is a local minimum. So each non-empty interval  $(x - \alpha, x + \alpha) \subset [0, 1]$  contains at least one local minimum.

**Problem 3.** Let  $p(z)$  be a polynomial of degree  $n$  with complex coefficients. Prove that there exist at least  $n + 1$  complex numbers  $z$  for which  $p(z)$  is 0 or 1.

**Solution.** The statement is not true if  $p$  is a constant polynomial. We prove it only in the case if  $n$  is positive.

For an arbitrary polynomial  $q(z)$  and complex number  $c$ , denote by  $\mu(q, c)$  the largest exponent  $\alpha$  for which  $q(z)$  is divisible by  $(z - c)^\alpha$ . (With other words, if  $c$  is a root of  $q$ , then  $\mu(q, c)$  is the root's multiplicity. Otherwise 0.)

Denote by  $S_0$  and  $S_1$  the sets of complex numbers  $z$  for which  $p(z)$  is 0 or 1, respectively. These sets contain all roots of the polynomials  $p(z)$  and  $p(z) - 1$ , thus

$$\sum_{c \in S_0} \mu(p, c) = \sum_{c \in S_1} \mu(p - 1, c) = n. \quad (1)$$

The polynomial  $p'$  has at most  $n - 1$  roots ( $n > 0$  is used here). This implies that

$$\sum_{c \in S_0 \cup S_1} \mu(p', c) \leq n - 1. \quad (2)$$

If  $p(c) = 0$  or  $p(c) - 1 = 0$ , then

$$\mu(p, c) - \mu(p'c) = 1 \quad \text{or} \quad \mu(p - 1, c) - \mu(p'c) = 1, \quad (3)$$

respectively. Putting (1), (2) and (3) together we obtain

$$\begin{aligned} |S_0| + |S_1| &= \sum_{c \in S_0} (\mu(p, c) - \mu(p'c)) + \sum_{c \in S_1} (\mu(p - 1, c) - \mu(p'c)) = \\ &= \sum_{c \in S_0} \mu(p, c) + \sum_{c \in S_1} \mu(p - 1, c) - \sum_{c \in S_0 \cup S_1} \mu(p'c) \geq n + n - (n - 1) = n + 1. \end{aligned}$$

**Problem 4.** Suppose the graph of a polynomial of degree 6 is tangent to a straight line at 3 points  $A_1, A_2, A_3$ , where  $A_2$  lies between  $A_1$  and  $A_3$ .

a) Prove that if the lengths of the segments  $A_1A_2$  and  $A_2A_3$  are equal, then the areas of the figures bounded by these segments and the graph of the polynomial are equal as well.

b) Let  $k = \frac{A_2A_3}{A_1A_2}$ , and let  $K$  be the ratio of the areas of the appropriate figures. Prove that

$$\frac{2}{7}k^5 < K < \frac{7}{2}k^5.$$

**Solution.** a) Without loss of generality, we can assume that the point  $A_2$  is the origin of system of coordinates. Then the polynomial can be presented in the form

$$y = (a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4)x^2 + a_5x,$$

where the equation  $y = a_5x$  determines the straight line  $A_1A_3$ . The abscissas of the points  $A_1$  and  $A_3$  are  $-a$  and  $a$ ,  $a > 0$ , respectively. Since  $-a$  and  $a$  are points of tangency, the numbers  $-a$  and  $a$  must be double roots of the polynomial  $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ . It follows that the polynomial is of the form

$$y = a_0(x^2 - a^2)^2 + a_5x.$$

The equality follows from the equality of the integrals

$$\int_{-a}^0 a_0(x^2 - a^2)x^2 dx = \int_0^a a_0(x^2 - a^2)x^2 dx$$

due to the fact that the function  $y = a_0(x^2 - a^2)$  is even.

b) Without loss of generality, we can assume that  $a_0 = 1$ . Then the function is of the form

$$y = (x + a)^2(x - b)^2x^2 + a_5x,$$

where  $a$  and  $b$  are positive numbers and  $b = ka$ ,  $0 < k < \infty$ . The areas of the figures at the segments  $A_1A_2$  and  $A_2A_3$  are equal respectively to

$$\int_{-a}^0 (x + a)^2(x - b)^2x^2 dx = \frac{a^7}{210}(7k^2 + 7k + 2)$$

and

$$\int_0^b (x + a)^2(x - b)^2x^2 dx = \frac{a^7}{210}(2k^2 + 7k + 7)$$

Then

$$K = k^5 \frac{2k^2 + 7k + 7}{7k^2 + 7k + 2}.$$

The derivative of the function  $f(k) = \frac{2k^2 + 7k + 7}{7k^2 + 7k + 2}$  is negative for  $0 < k < \infty$ . Therefore  $f(k)$  decreases from  $\frac{7}{2}$  to  $\frac{2}{7}$  when  $k$  increases from 0 to  $\infty$ . Inequalities  $\frac{2}{7} < \frac{2k^2 + 7k + 7}{7k^2 + 7k + 2} < \frac{7}{2}$  imply the desired inequalities.

**Problem 5.** Let  $\mathbb{R}^+$  be the set of positive real numbers. Find all functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that for all  $x, y \in \mathbb{R}^+$

$$f(x)f(yf(x)) = f(x + y).$$

**First solution.** First, if we assume that  $f(x) > 1$  for some  $x \in \mathbb{R}^+$ , setting  $y = \frac{x}{f(x) - 1}$  gives the contradiction  $f(x) = 1$ . Hence  $f(x) \leq 1$  for each  $x \in \mathbb{R}^+$ , which implies that  $f$  is a decreasing function.

If  $f(x) = 1$  for some  $x \in \mathbb{R}^+$ , then  $f(x + y) = f(y)$  for each  $y \in \mathbb{R}^+$ , and by the monotonicity of  $f$  it follows that  $f \equiv 1$ .

Let now  $f(x) < 1$  for each  $x \in \mathbb{R}^+$ . Then  $f$  is strictly decreasing function, in particular injective. By the equalities

$$f(x)f(yf(x)) = f(x + y) =$$

$$= f(yf(x) + x + y(1 - f(x))) = f(yf(x))f\left((x + y(1 - f(x)))f(yf(x))\right)$$

we obtain that  $x = (x + y(1 - f(x)))f(yf(x))$ . Setting  $x = 1$ ,  $z = xf(1)$  and  $a = \frac{1 - f(1)}{f(1)}$ , we get  $f(z) = \frac{1}{1 + az}$ .

Combining the two cases, we conclude that  $f(x) = \frac{1}{1 + ax}$  for each  $x \in \mathbb{R}^+$ , where  $a \geq 0$ . Conversely, a direct verification shows that the functions of this form satisfy the initial equality.

**Second solution.** As in the first solution we get that  $f$  is a decreasing function, in particular differentiable almost everywhere. Write the initial equality in the form

$$\frac{f(x + y) - f(x)}{y} = f^2(x) \frac{f(yf(x)) - 1}{yf(x)}.$$

It follows that if  $f$  is differentiable at the point  $x \in \mathbb{R}^+$ , then there exists the limit  $\lim_{z \rightarrow 0^+} \frac{f(z) - 1}{z} =: -a$ . Therefore  $f'(x) = -af^2(x)$  for each  $x \in \mathbb{R}^+$ , i.e.  $\left(\frac{1}{f(x)}\right)' = a$ , which means that  $f(x) = \frac{1}{ax + b}$ . Substituting in the initial relation, we find that  $b = 1$  and  $a \geq 0$ .

**Problem 6.** For an  $m \times m$  real matrix  $A$ ,  $e^A$  is defined as  $\sum_{n=0}^{\infty} \frac{1}{n!} A^n$ . (The sum is convergent for all matrices.) Prove or disprove, that for all real polynomials  $p$  and  $m \times m$  real matrices  $A$  and  $B$ ,  $p(e^{AB})$  is nilpotent if and only if  $p(e^{BA})$  is nilpotent. (A matrix  $A$  is nilpotent if  $A^k = 0$  for some positive integer  $k$ .)

**Solution.** First we prove that for any polynomial  $q$  and  $m \times m$  matrices  $A$  and  $B$ , the characteristic polynomials of  $q(e^{AB})$  and  $q(e^{BA})$  are the same. It is easy to check that for any matrix  $X$ ,  $q(e^X) = \sum_{n=0}^{\infty} c_n X^n$  with some real numbers  $c_n$  which depend on  $q$ . Let

$$C = \sum_{n=1}^{\infty} c_n \cdot (BA)^{n-1} B = \sum_{n=1}^{\infty} c_n \cdot B(AB)^{n-1}.$$

Then  $q(e^{AB}) = c_0 I + AC$  and  $q(e^{BA}) = c_0 I + CA$ . It is well-known that the characteristic polynomials of  $AC$  and  $CA$  are the same; denote this polynomial by  $f(x)$ . Then the characteristic polynomials of matrices  $q(e^{AB})$  and  $q(e^{BA})$  are both  $f(x - c_0)$ .

Now assume that the matrix  $p(e^{AB})$  is nilpotent, i.e.  $(p(e^{AB}))^k = 0$  for some positive integer  $k$ . Chose  $q = p^k$ . The characteristic polynomial of the matrix  $q(e^{AB}) = 0$  is  $x^m$ , so the same holds for the matrix  $q(e^{BA})$ . By the theorem of Cayley and Hamilton, this implies that  $(q(e^{BA}))^m = (p(e^{BA}))^{km} = 0$ . Thus the matrix  $q(e^{BA})$  is nilpotent, too.

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*First day*

**Problem 1.**

Let  $n$  be a positive integer. Consider an  $n \times n$  matrix with entries  $1, 2, \dots, n^2$  written in order starting top left and moving along each row in turn left-to-right. We choose  $n$  entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

**Solution.** Since there are exactly  $n$  rows and  $n$  columns, the choice is of the form

$$\{(j, \sigma(j)) : j = 1, \dots, n\}$$

where  $\sigma \in S_n$  is a permutation. Thus the corresponding sum is equal to

$$\begin{aligned} \sum_{j=1}^n n(j-1) + \sigma(j) &= \sum_{j=1}^n nj - \sum_{j=1}^n n + \sum_{j=1}^n \sigma(j) \\ &= n \sum_{j=1}^n j - \sum_{j=1}^n n + \sum_{j=1}^n j = (n+1) \frac{n(n+1)}{2} - n^2 = \frac{n(n^2+1)}{2}, \end{aligned}$$

which shows that the sum is independent of  $\sigma$ .

**Problem 2.**

Let  $r, s, t$  be positive integers which are pairwise relatively prime. If  $a$  and  $b$  are elements of a commutative multiplicative group with unity element  $e$ , and  $a^r = b^s = (ab)^t = e$ , prove that  $a = b = e$ .

Does the same conclusion hold if  $a$  and  $b$  are elements of an arbitrary non-commutative group?

**Solution.** 1. There exist integers  $u$  and  $v$  such that  $us + vt = 1$ . Since  $ab = ba$ , we obtain

$$ab = (ab)^{us+vt} = (ab)^{us} \left( (ab)^t \right)^v = (ab)^{us} e = (ab)^{us} = a^{us} (b^s)^u = a^{us} e = a^{us}.$$

Therefore,  $b^r = eb^r = a^r b^r = (ab)^r = a^{usr} = (a^r)^{us} = e$ . Since  $xr + ys = 1$  for suitable integers  $x$  and  $y$ ,

$$b = b^{xr+ys} = (b^r)^x (b^s)^y = e.$$

It follows similarly that  $a = e$  as well.

2. This is not true. Let  $a = (123)$  and  $b = (34567)$  be cycles of the permutation group  $S_7$  of order 7. Then  $ab = (1234567)$  and  $a^3 = b^5 = (ab)^7 = e$ .

**Problem 3.** Find  $\lim_{t \nearrow 1} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n}$ , where  $t \nearrow 1$  means that  $t$  approaches 1 from below.

**Solution.**

$$\begin{aligned} \lim_{t \rightarrow 1-0} (1-t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n} &= \lim_{t \rightarrow 1-0} \frac{1-t}{-\ln t} \cdot (-\ln t) \sum_{n=1}^{\infty} \frac{t^n}{1+t^n} = \\ &= \lim_{t \rightarrow 1-0} (-\ln t) \sum_{n=1}^{\infty} \frac{1}{1+e^{-n \ln t}} = \lim_{h \rightarrow +0} h \sum_{n=1}^{\infty} \frac{1}{1+e^{nh}} = \int_0^{\infty} \frac{dx}{1+e^x} = \ln 2. \end{aligned}$$

**Problem 4.**

Let  $k$  be a positive integer. Let  $p(x)$  be a polynomial of degree  $n$  each of whose coefficients is  $-1$ ,  $1$  or  $0$ , and which is divisible by  $(x-1)^k$ . Let  $q$  be a prime such that  $\frac{q}{\ln q} < \frac{k}{\ln(n+1)}$ . Prove that the complex  $q$ th roots of unity are roots of the polynomial  $p(x)$ .

**Solution.** Let  $p(x) = (x-1)^k \cdot r(x)$  and  $\varepsilon_j = e^{2\pi i \cdot j/q}$  ( $j = 1, 2, \dots, q-1$ ). As is well-known, the polynomial  $x^{q-1} + x^{q-2} + \dots + x + 1 = (x-\varepsilon_1) \dots (x-\varepsilon_{q-1})$  is irreducible, thus all  $\varepsilon_1, \dots, \varepsilon_{q-1}$  are roots of  $r(x)$ , or none of them.

Suppose that none of  $\varepsilon_1, \dots, \varepsilon_{q-1}$  is a root of  $r(x)$ . Then  $\prod_{j=1}^{q-1} r(\varepsilon_j)$  is a rational integer, which is not 0 and

$$\begin{aligned} (n+1)^{q-1} &\geq \prod_{j=1}^{q-1} |p(\varepsilon_j)| = \left| \prod_{j=1}^{q-1} (1-\varepsilon_j)^k \right| \cdot \left| \prod_{j=1}^{q-1} r(\varepsilon_j) \right| \geq \\ &\geq \left| \prod_{j=1}^{q-1} (1-\varepsilon_j) \right|^k = (1^{q-1} + 1^{q-2} + \dots + 1^1 + 1)^k = q^k. \end{aligned}$$

This contradicts the condition  $\frac{q}{\ln q} < \frac{k}{\ln(n+1)}$ .

**Problem 5.**

Let  $A$  be an  $n \times n$  complex matrix such that  $A \neq \lambda I$  for all  $\lambda \in \mathbf{C}$ . Prove that  $A$  is similar to a matrix having at most one non-zero entry on the main diagonal.

**Solution.** The statement will be proved by induction on  $n$ . For  $n = 1$ , there is nothing to do. In the case  $n = 2$ , write  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $b \neq 0$ , and  $c \neq 0$  or  $b = c = 0$  then  $A$  is similar to

$$\begin{bmatrix} 1 & 0 \\ a/b & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a/b & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ c - ad/b & a + d \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -a/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & a/c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b - ad/c \\ c & a + d \end{bmatrix},$$

respectively. If  $b = c = 0$  and  $a \neq d$ , then  $A$  is similar to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & d-a \\ 0 & d \end{bmatrix},$$

and we can perform the step seen in the case  $b \neq 0$  again.

Assume now that  $n > 3$  and the problem has been solved for all  $n' < n$ . Let  $A = \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix}_n$ , where  $A'$  is  $(n-1) \times (n-1)$  matrix. Clearly we may assume that  $A' \neq \lambda'I$ , so the induction provides a  $P$  with, say,  $P^{-1}A'P = \begin{bmatrix} 0 & * \\ * & \alpha \end{bmatrix}_{n-1}$ .

But then the matrix

$$B = \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P^{-1}A'P & * \\ * & \beta \end{bmatrix}$$

is similar to  $A$  and its diagonal is  $(0, 0, \dots, 0, \alpha, \beta)$ . On the other hand, we may also view  $B$  as  $\begin{bmatrix} 0 & * \\ * & C \end{bmatrix}_n$ , where  $C$  is an  $(n-1) \times (n-1)$  matrix with diagonal  $(0, \dots, 0, \alpha, \beta)$ . If the inductive hypothesis is applicable to  $C$ , we would have  $Q^{-1}CQ = D$ , with  $D = \begin{bmatrix} 0 & * \\ * & \gamma \end{bmatrix}_{n-1}$  so that finally the matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \cdot B \cdot \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} 0 & * \\ * & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & D \end{bmatrix}$$

is similar to  $A$  and its diagonal is  $(0, 0, \dots, 0, \gamma)$ , as required.

The inductive argument can fail only when  $n-1 = 2$  and the resulting matrix applying  $P$  has the form

$$P^{-1}AP = \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix}$$

where  $d \neq 0$ . The numbers  $a, b, c, e$  cannot be 0 at the same time. If, say,  $b \neq 0$ ,  $A$  is similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -b & a & b \\ c & d & 0 \\ e-b-d & a & b+d \end{bmatrix}.$$

Performing half of the induction step again, the diagonal of the resulting matrix will be  $(0, d-b, d+b)$  (the trace is the same) and the induction step can be finished. The cases  $a \neq 0, c \neq 0$  and  $e \neq 0$  are similar.

**Problem 6.**

Suppose that the differentiable functions  $a, b, f, g : \mathbb{R} \rightarrow \mathbb{R}$  satisfy

$$f(x) \geq 0, f'(x) \geq 0, g(x) > 0, g'(x) > 0 \text{ for all } x \in \mathbb{R},$$

$$\lim_{x \rightarrow \infty} a(x) = A > 0, \quad \lim_{x \rightarrow \infty} b(x) = B > 0, \quad \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = \infty,$$

and

$$\frac{f'(x)}{g'(x)} + a(x) \frac{f(x)}{g(x)} = b(x).$$

Prove that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}.$$

**Solution.** Let  $0 < \varepsilon < A$  be an arbitrary real number. If  $x$  is sufficiently large then  $f(x) > 0$ ,  $g(x) > 0$ ,  $|a(x) - A| < \varepsilon$ ,  $|b(x) - B| < \varepsilon$  and

$$\begin{aligned}
 (1) \quad B - \varepsilon < b(x) &= \frac{f'(x)}{g'(x)} + a(x) \frac{f(x)}{g(x)} < \frac{f'(x)}{g'(x)} + (A + \varepsilon) \frac{f(x)}{g(x)} < \\
 &< \frac{(A + \varepsilon)(A + 1)}{A} \cdot \frac{f'(x)(g(x))^A + A \cdot f(x) \cdot (g(x))^{A-1} \cdot g'(x)}{(A + 1) \cdot (g(x))^A \cdot g'(x)} = \\
 &= \frac{(A + \varepsilon)(A + 1)}{A} \cdot \frac{\left(f(x) \cdot (g(x))^A\right)'}{\left((g(x))^{A+1}\right)'},
 \end{aligned}$$

thus

$$(2) \quad \frac{\left(f(x) \cdot (g(x))^A\right)'}{\left((g(x))^{A+1}\right)'} > \frac{A(B - \varepsilon)}{(A + \varepsilon)(A + 1)}.$$

It can be similarly obtained that, for sufficiently large  $x$ ,

$$(3) \quad \frac{\left(f(x) \cdot (g(x))^A\right)'}{\left((g(x))^{A+1}\right)'} < \frac{A(B + \varepsilon)}{(A - \varepsilon)(A + 1)}.$$

From  $\varepsilon \rightarrow 0$ , we have

$$\lim_{x \rightarrow \infty} \frac{\left(f(x) \cdot (g(x))^A\right)'}{\left((g(x))^{A+1}\right)'} = \frac{B}{A + 1}.$$

By l'Hospital's rule this implies

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f(x) \cdot (g(x))^A}{(g(x))^{A+1}} = \frac{B}{A + 1}.$$

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*Second day*

**Problem 1.**

Let  $r, s \geq 1$  be integers and  $a_0, a_1, \dots, a_{r-1}, b_0, b_1, \dots, b_{s-1}$  be real non-negative numbers such that

$$(a_0 + a_1x + a_2x^2 + \dots + a_{r-1}x^{r-1} + x^r)(b_0 + b_1x + b_2x^2 + \dots + b_{s-1}x^{s-1} + x^s) = 1 + x + x^2 + \dots + x^{r+s-1} + x^{r+s}.$$

Prove that each  $a_i$  and each  $b_j$  equals either 0 or 1.

**Solution.** Multiply the left hand side polynomials. We obtain the following equalities:

$$a_0b_0 = 1, \quad a_0b_1 + a_1b_0 = 1, \quad \dots$$

Among them one can find equations

$$a_0 + a_1b_{s-1} + a_2b_{s-2} + \dots = 1$$

and

$$b_0 + b_1a_{r-1} + b_2a_{r-2} + \dots = 1.$$

From these equations it follows that  $a_0, b_0 \leq 1$ . Taking into account that  $a_0b_0 = 1$  we can see that  $a_0 = b_0 = 1$ .

Now looking at the following equations we notice that all  $a$ 's must be less than or equal to 1. The same statement holds for the  $b$ 's. It follows from  $a_0b_1 + a_1b_0 = 1$  that one of the numbers  $a_1, b_1$  equals 0 while the other one must be 1. Follow by induction.

**Problem 2.**

Let  $a_0 = \sqrt{2}$ ,  $b_0 = 2$ ,  $a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}$ ,  $b_{n+1} = \frac{2b_n}{2 + \sqrt{4 + b_n^2}}$ .

- a) Prove that the sequences  $(a_n)$ ,  $(b_n)$  are decreasing and converge to 0.
- b) Prove that the sequence  $(2^n a_n)$  is increasing, the sequence  $(2^n b_n)$  is decreasing and that these two sequences converge to the same limit.
- c) Prove that there is a positive constant  $C$  such that for all  $n$  the following inequality holds:  $0 < b_n - a_n < \frac{C}{8^n}$ .

**Solution.** Obviously  $a_2 = \sqrt{2 - \sqrt{2}} < \sqrt{2}$ . Since the function  $f(x) = \sqrt{2 - \sqrt{4 - x^2}}$  is increasing on the interval  $[0, 2]$  the inequality  $a_1 > a_2$  implies that  $a_2 > a_3$ . Simple induction ends the proof of monotonicity of  $(a_n)$ . In the same way we prove that  $(b_n)$  decreases (just notice that  $g(x) = \frac{2x}{2 + \sqrt{4 + x^2}} = 2 / \left( 2/x + \sqrt{1 + 4/x^2} \right)$ ). It is a matter of simple manipulation to prove that  $2f(x) > x$  for all  $x \in (0, 2)$ , this implies that the sequence  $(2^n a_n)$  is strictly

increasing. The inequality  $2g(x) < x$  for  $x \in (0, 2)$  implies that the sequence  $(2^n b_n)$  strictly decreases. By an easy induction one can show that  $a_n^2 = \frac{4b_n^2}{4+b_n^2}$  for positive integers  $n$ . Since the limit of the *decreasing* sequence  $(2^n b_n)$  of *positive* numbers is finite we have

$$\lim 4^n a_n^2 = \lim \frac{4 \cdot 4^n b_n^2}{4 + b_n^2} = \lim 4^n b_n^2.$$

We know already that the limits  $\lim 2^n a_n$  and  $\lim 2^n b_n$  are equal. The first of the two is positive because the sequence  $(2^n a_n)$  is strictly increasing. The existence of a number  $C$  follows easily from the equalities

$$2^n b_n - 2^n a_n = \left(4^n b_n^2 - \frac{4^{n+1} b_n^2}{4 + b_n^2}\right) / (2^n b_n + 2^n a_n) = \frac{(2^n b_n)^4}{4 + b_n^2} \cdot \frac{1}{4^n} \cdot \frac{1}{2^n(b_n + a_n)}$$

and from the existence of positive limits  $\lim 2^n b_n$  and  $\lim 2^n a_n$ .

**Remark.** The last problem may be solved in a much simpler way by someone who is able to make use of sine and cosine. It is enough to notice that  $a_n = 2 \sin \frac{\pi}{2^{n+1}}$  and  $b_n = 2 \tan \frac{\pi}{2^{n+1}}$ .

**Problem 3.**

Find the maximum number of points on a sphere of radius 1 in  $\mathbb{R}^n$  such that the distance between any two of these points is strictly greater than  $\sqrt{2}$ .

**Solution.** The unit sphere in  $\mathbb{R}^n$  is defined by

$$S_{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^n x_k^2 = 1 \right\}.$$

The distance between the points  $X = (x_1, \dots, x_n)$  and  $Y = (y_1, \dots, y_n)$  is:

$$d^2(X, Y) = \sum_{k=1}^n (x_k - y_k)^2.$$

We have

$$\begin{aligned} d(X, Y) > \sqrt{2} &\Leftrightarrow d^2(X, Y) > 2 \\ &\Leftrightarrow \sum_{k=1}^n x_k^2 + \sum_{k=1}^n y_k^2 + 2 \sum_{k=1}^n x_k y_k > 2 \\ &\Leftrightarrow \sum_{k=1}^n x_k y_k < 0 \end{aligned}$$

Taking account of the symmetry of the sphere, we can suppose that

$$A_1 = (-1, 0, \dots, 0).$$

For  $X = A_1$ ,  $\sum_{k=1}^n x_k y_k < 0$  implies  $y_1 > 0$ ,  $\forall Y \in M_n$ .

Let  $X = (x_1, \overline{X})$ ,  $Y = (y_1, \overline{Y}) \in M_n \setminus \{A_1\}$ ,  $\overline{X}, \overline{Y} \in \mathbb{R}^{n-1}$ .

We have

$$\sum_{k=1}^n x_k y_k < 0 \Rightarrow x_1 y_1 + \sum_{k=1}^{n-1} \bar{x}_k \bar{y}_k < 0 \Leftrightarrow \sum_{k=1}^{n-1} x'_k y'_k < 0,$$

where

$$x'_k = \frac{\bar{x}_k}{\sqrt{\sum \bar{x}_k^2}}, \quad y'_k = \frac{\bar{y}_k}{\sqrt{\sum \bar{y}_k^2}}.$$

therefore

$$(x'_1, \dots, x'_{n-1}), (y'_1, \dots, y'_{n-1}) \in S_{n-2}$$

and verifies  $\sum_{k=1}^n x_k y_k < 0$ .

If  $a_n$  is the search number of points in  $\mathbb{R}^n$  we obtain  $a_n \leq 1 + a_{n-1}$  and  $a_1 = 2$  implies that  $a_n \leq n + 1$ .

We show that  $a_n = n + 1$ , giving an example of a set  $M_n$  with  $(n + 1)$  elements satisfying the conditions of the problem.

$$\begin{aligned} A_1 &= (-1, 0, 0, 0, \dots, 0, 0) \\ A_2 &= \left(\frac{1}{n}, -c_1, 0, 0, \dots, 0, 0\right) \\ A_3 &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, -c_2, 0, \dots, 0, 0\right) \\ A_4 &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \frac{1}{n-1} \cdot c_2, -c_3, \dots, 0, 0\right) \\ A_{n-1} &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \frac{1}{n-2} \cdot c_2, \frac{1}{n-3} \cdot c_3, \dots, -c_{n-2}, 0\right) \\ A_n &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \frac{1}{n-2} \cdot c_2, \frac{1}{n-3} \cdot c_3, \dots, \frac{1}{2} \cdot c_{n-2}, -c_{n-1}\right) \\ A_{n+1} &= \left(\frac{1}{n}, \frac{1}{n-1} \cdot c_1, \frac{1}{n-2} \cdot c_2, \frac{1}{n-3} \cdot c_3, \dots, \frac{1}{2} \cdot c_{n-2}, c_{n-1}\right) \end{aligned}$$

where

$$c_k = \sqrt{\left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{n-k+1}\right)}, \quad k = \overline{1, n-1}.$$

We have  $\sum_{k=1}^n x_k y_k = -\frac{1}{n} < 0$  and  $\sum_{k=1}^n x_k^2 = 1, \quad \forall X, Y \in \{A_1, \dots, A_{n+1}\}$ .

These points are on the unit sphere in  $\mathbb{R}^n$  and the distance between any two points is equal to

$$d = \sqrt{2} \sqrt{1 + \frac{1}{n}} > \sqrt{2}.$$

**Remark.** For  $n = 2$  the points form an equilateral triangle in the unit circle; for  $n = 3$  the four points form a regular tetrahedron and in  $\mathbb{R}^n$  the points form an  $n$  dimensional regular simplex.

#### Problem 4.

Let  $A = (a_{k,\ell})_{k,\ell=1,\dots,n}$  be an  $n \times n$  complex matrix such that for each  $m \in \{1, \dots, n\}$  and  $1 \leq j_1 < \dots < j_m \leq n$  the determinant of the matrix  $(a_{j_k, j_\ell})_{k,\ell=1,\dots,m}$  is zero. Prove that  $A^n = 0$  and that there exists a permutation  $\sigma \in S_n$  such that the matrix

$$(a_{\sigma(k), \sigma(\ell)})_{k,\ell=1,\dots,n}$$

has all of its nonzero elements above the diagonal.

**Solution.** We will only prove (2), since it implies (1). Consider a directed graph  $G$  with  $n$  vertices  $V_1, \dots, V_n$  and a directed edge from  $V_k$  to  $V_\ell$  when  $a_{k,\ell} \neq 0$ . We shall prove that it is acyclic.

Assume that there exists a cycle and take one of minimum length  $m$ . Let  $j_1 < \dots < j_m$  be the vertices the cycle goes through and let  $\sigma_0 \in S_n$  be a permutation such that  $a_{j_k, j_{\sigma_0(k)}} \neq 0$  for  $k = 1, \dots, m$ . Observe that for any other  $\sigma \in S_n$  we have  $a_{j_k, j_{\sigma(k)}} = 0$  for some  $k \in \{1, \dots, m\}$ , otherwise we would obtain a different cycle through the same set of vertices and, consequently, a shorter cycle. Finally

$$\begin{aligned} 0 &= \det(a_{j_k, j_\ell})_{k, \ell=1, \dots, m} \\ &= (-1)^{\text{sign } \sigma_0} \prod_{k=1}^m a_{j_k, j_{\sigma_0(k)}} + \sum_{\sigma \neq \sigma_0} (-1)^{\text{sign } \sigma} \prod_{k=1}^m a_{j_k, j_{\sigma(k)}} \neq 0, \end{aligned}$$

which is a contradiction.

Since  $G$  is acyclic there exists a topological ordering i.e. a permutation  $\sigma \in S_n$  such that  $k < \ell$  whenever there is an edge from  $V_{\sigma(k)}$  to  $V_{\sigma(\ell)}$ . It is easy to see that this permutation solves the problem.

**Problem 5.** Let  $\mathbb{R}$  be the set of real numbers. Prove that there is no function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(0) > 0$ , and such that

$$f(x+y) \geq f(x) + yf(f(x)) \quad \text{for all } x, y \in \mathbb{R}.$$

**Solution.** Suppose that there exists a function satisfying the inequality. If  $f(f(x)) \leq 0$  for all  $x$ , then  $f$  is a decreasing function in view of the inequalities  $f(x+y) \geq f(x) + yf(f(x)) \geq f(x)$  for any  $y \leq 0$ . Since  $f(0) > 0 \geq f(f(x))$ , it implies  $f(x) > 0$  for all  $x$ , which is a contradiction. Hence there is a  $z$  such that  $f(f(z)) > 0$ . Then the inequality  $f(z+x) \geq f(z) + xf(f(z))$  shows that  $\lim_{x \rightarrow \infty} f(x) = +\infty$  and therefore  $\lim_{x \rightarrow \infty} f(f(x)) = +\infty$ . In particular, there exist  $x, y > 0$  such that  $f(x) \geq 0$ ,  $f(f(x)) > 1$ ,  $y \geq \frac{x+1}{f(f(x))-1}$  and  $f(f(x+y+1)) \geq 0$ . Then  $f(x+y) \geq f(x) + yf(f(x)) \geq x+y+1$  and hence

$$\begin{aligned} f(f(x+y)) &\geq f(x+y+1) + (f(x+y) - (x+y+1))f(f(x+y+1)) \geq \\ &\geq f(x+y+1) \geq f(x+y) + f(f(x+y)) \geq \\ &\geq f(x) + yf(f(x)) + f(f(x+y)) > f(f(x+y)). \end{aligned}$$

This contradiction completes the solution of the problem.

**Problem 6.**

For each positive integer  $n$ , let  $f_n(\vartheta) = \sin \vartheta \cdot \sin(2\vartheta) \cdot \sin(4\vartheta) \cdots \sin(2^n \vartheta)$ .  
For all real  $\vartheta$  and all  $n$ , prove that

$$|f_n(\vartheta)| \leq \frac{2}{\sqrt{3}} |f_n(\pi/3)|.$$

**Solution.** We prove that  $g(\vartheta) = |\sin \vartheta| |\sin(2\vartheta)|^{1/2}$  attains its maximum value  $(\sqrt{3}/2)^{3/2}$  at points  $2^k \pi/3$  (where  $k$  is a positive integer). This can be seen by using derivatives or a classical bound like

$$\begin{aligned} |g(\vartheta)| &= |\sin \vartheta| |\sin(2\vartheta)|^{1/2} = \frac{\sqrt{2}}{\sqrt[4]{3}} \left( \sqrt[4]{|\sin \vartheta| \cdot |\sin \vartheta| \cdot |\sin \vartheta| \cdot |\sqrt{3} \cos \vartheta|} \right)^2 \\ &\leq \frac{\sqrt{2}}{\sqrt[4]{3}} \cdot \frac{3 \sin^2 \vartheta + 3 \cos^2 \vartheta}{4} = \left( \frac{\sqrt{3}}{2} \right)^{3/2}. \end{aligned}$$

Hence

$$\begin{aligned} \left| \frac{f_n(\vartheta)}{f_n(\pi/3)} \right| &= \left| \frac{g(\vartheta) \cdot g(2\vartheta)^{1/2} \cdot g(4\vartheta)^{3/4} \cdots g(2^{n-1}\vartheta)^E}{g(\pi/3) \cdot g(2\pi/3)^{1/2} \cdot g(4\pi/3)^{3/4} \cdots g(2^{n-1}\pi/3)^E} \right| \cdot \left| \frac{\sin(2^n \vartheta)}{\sin(2^n \pi/3)} \right|^{1-E/2} \\ &\leq \left| \frac{\sin(2^n \vartheta)}{\sin(2^n \pi/3)} \right|^{1-E/2} \leq \left( \frac{1}{\sqrt{3}/2} \right)^{1-E/2} \leq \frac{2}{\sqrt{3}}. \end{aligned}$$

where  $E = \frac{2}{3}(1 - (-1/2)^n)$ . This is exactly the bound we had to prove.

Solutions for problems in the  
9<sup>th</sup> International Mathematics Competition  
for University Students

Warsaw, July 19 - July 25, 2002

First Day

**Problem 1.** A standard parabola is the graph of a quadratic polynomial  $y = x^2 + ax + b$  with leading coefficient 1. Three standard parabolas with vertices  $V_1, V_2, V_3$  intersect pairwise at points  $A_1, A_2, A_3$ . Let  $A \mapsto s(A)$  be the reflection of the plane with respect to the  $x$  axis.

Prove that standard parabolas with vertices  $s(A_1), s(A_2), s(A_3)$  intersect pairwise at the points  $s(V_1), s(V_2), s(V_3)$ .

*Solution.* First we show that the standard parabola with vertex  $V$  contains point  $A$  if and only if the standard parabola with vertex  $s(A)$  contains point  $s(V)$ .

Let  $A = (a, b)$  and  $V = (v, w)$ . The equation of the standard parabola with vertex  $V = (v, w)$  is  $y = (x - v)^2 + w$ , so it contains point  $A$  if and only if  $b = (a - v)^2 + w$ . Similarly, the equation of the parabola with vertex  $s(A) = (a, -b)$  is  $y = (x - a)^2 - b$ ; it contains point  $s(V) = (v, -w)$  if and only if  $-w = (v - a)^2 - b$ . The two conditions are equivalent.

Now assume that the standard parabolas with vertices  $V_1$  and  $V_2, V_1$  and  $V_3, V_2$  and  $V_3$  intersect each other at points  $A_3, A_2, A_1$ , respectively. Then, by the statement above, the standard parabolas with vertices  $s(A_1)$  and  $s(A_2), s(A_1)$  and  $s(A_3), s(A_2)$  and  $s(A_3)$  intersect each other at points  $V_3, V_2, V_1$ , respectively, because they contain these points.

**Problem 2.** Does there exist a continuously differentiable function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for every  $x \in \mathbb{R}$  we have  $f(x) > 0$  and  $f'(x) = f(f(x))$ ?

*Solution.* Assume that there exists such a function. Since  $f'(x) = f(f(x)) > 0$ , the function is strictly monotone increasing.

By the monotonicity,  $f(x) > 0$  implies  $f(f(x)) > f(0)$  for all  $x$ . Thus,  $f(0)$  is a lower bound for  $f'(x)$ , and for all  $x < 0$  we have  $f(x) < f(0) + x \cdot f(0) = (1 + x)f(0)$ . Hence, if  $x \leq -1$  then  $f(x) \leq 0$ , contradicting the property  $f(x) > 0$ .

So such function does not exist.

**Problem 3.** Let  $n$  be a positive integer and let

$$a_k = \frac{1}{\binom{n}{k}}, \quad b_k = 2^{k-n}, \quad \text{for } k = 1, 2, \dots, n.$$

Show that

$$\frac{a_1 - b_1}{1} + \frac{a_2 - b_2}{2} + \dots + \frac{a_n - b_n}{n} = 0. \quad (1)$$

*Solution.* Since  $k \binom{n}{k} = n \binom{n-1}{k-1}$  for all  $k \geq 1$ , (1) is equivalent to

$$\frac{2^n}{n} \left[ \frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \dots + \frac{1}{\binom{n-1}{n-1}} \right] = \frac{2^1}{1} + \frac{2^2}{2} + \dots + \frac{2^n}{n}. \quad (2)$$

We prove (2) by induction. For  $n = 1$ , both sides are equal to 2.

Assume that (2) holds for some  $n$ . Let

$$x_n = \frac{2^n}{n} \left[ \frac{1}{\binom{n-1}{0}} + \frac{1}{\binom{n-1}{1}} + \dots + \frac{1}{\binom{n-1}{n-1}} \right];$$

then

$$\begin{aligned} x_{n+1} &= \frac{2^{n+1}}{n+1} \sum_{k=0}^n \frac{1}{\binom{n}{k}} = \frac{2^n}{n+1} \left( 1 + \sum_{k=0}^{n-1} \left( \frac{1}{\binom{n}{k}} + \frac{1}{\binom{n}{k+1}} \right) + 1 \right) = \\ &= \frac{2^n}{n+1} \sum_{k=0}^{n-1} \frac{\frac{n-k}{n} + \frac{k+1}{n}}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = \frac{2^n}{n} \sum_{k=0}^{n-1} \frac{1}{\binom{n-1}{k}} + \frac{2^{n+1}}{n+1} = x_n + \frac{2^{n+1}}{n+1}. \end{aligned}$$

This implies (2) for  $n + 1$ .

**Problem 4.** Let  $f: [a, b] \rightarrow [a, b]$  be a continuous function and let  $p \in [a, b]$ . Define  $p_0 = p$  and  $p_{n+1} = f(p_n)$  for  $n = 0, 1, 2, \dots$ . Suppose that the set  $T_p = \{p_n: n = 0, 1, 2, \dots\}$  is closed, i.e., if  $x \notin T_p$  then there is a  $\delta > 0$  such that for all  $x' \in T_p$  we have  $|x' - x| \geq \delta$ . Show that  $T_p$  has finitely many elements.

*Solution.* If for some  $n > m$  the equality  $p_m = p_n$  holds then  $T_p$  is a finite set. Thus we can assume that all points  $p_0, p_1, \dots$  are distinct. There is a convergent subsequence  $p_{n_k}$  and its limit  $q$  is in  $T_p$ . Since  $f$  is continuous  $p_{n_k+1} = f(p_{n_k}) \rightarrow f(q)$ , so all, except for finitely many, points  $p_n$  are accumulation points of  $T_p$ . Hence we may assume that all of them are accumulation points of  $T_p$ . Let  $d = \sup\{|p_m - p_n|: m, n \geq 0\}$ . Let  $\delta_n$  be

positive numbers such that  $\sum_{n=0}^{\infty} \delta_n < \frac{d}{2}$ . Let  $I_n$  be an interval of length less than  $\delta_n$  centered at  $p_n$  such that there are infinitely many  $k$ 's such that  $p_k \notin \bigcup_{j=0}^n I_j$ , this can be done by induction. Let  $n_0 = 0$  and  $n_{m+1}$  be the

smallest integer  $k > n_m$  such that  $p_k \notin \bigcup_{j=0}^{n_m} I_j$ . Since  $T_p$  is closed the limit

of the subsequence  $(p_{n_m})$  must be in  $T_p$  but it is impossible because of the definition of  $I_n$ 's, of course if the sequence  $(p_{n_m})$  is not convergent we may replace it with its convergent subsequence. The proof is finished.

*Remark.* If  $T_p = \{p_1, p_2, \dots\}$  and each  $p_n$  is an accumulation point of  $T_p$ , then  $T_p$  is the countable union of nowhere dense sets (i.e. the single-element sets  $\{p_n\}$ ). If  $T$  is closed then this contradicts the Baire Category Theorem.

**Problem 5.** Prove or disprove the following statements:

(a) There exists a monotone function  $f: [0, 1] \rightarrow [0, 1]$  such that for each  $y \in [0, 1]$  the equation  $f(x) = y$  has uncountably many solutions  $x$ .

(b) There exists a continuously differentiable function  $f: [0, 1] \rightarrow [0, 1]$  such that for each  $y \in [0, 1]$  the equation  $f(x) = y$  has uncountably many solutions  $x$ .

*Solution.* *a.* It does not exist. For each  $y$  the set  $\{x: y = f(x)\}$  is either empty or consists of 1 point or is an interval. These sets are pairwise disjoint, so there are at most countably many of the third type.

*b.* Let  $f$  be such a map. Then for each value  $y$  of this map there is an  $x_0$  such that  $y = f(x)$  and  $f'(x) = 0$ , because an uncountable set  $\{x: y = f(x)\}$  contains an accumulation point  $x_0$  and clearly  $f'(x_0) = 0$ . For every  $\varepsilon > 0$  and every  $x_0$  such that  $f'(x_0) = 0$  there exists an open interval  $I_{x_0}$  such that if  $x \in I_{x_0}$  then  $|f'(x)| < \varepsilon$ . The union of all these intervals  $I_{x_0}$  may be written as a union of pairwise disjoint open intervals  $J_n$ . The image of each  $J_n$  is an interval (or a point) of length  $< \varepsilon \cdot \text{length}(J_n)$  due to Lagrange Mean Value Theorem. Thus the image of the interval  $[0, 1]$  may be covered with the intervals such that the sum of their lengths is  $\varepsilon \cdot 1 = \varepsilon$ . This is not possible for  $\varepsilon < 1$ .

*Remarks.* 1. The proof of part **b** is essentially the proof of the easy part of A. Sard's theorem about measure of the set of critical values of a smooth map.

2. If only continuity is required, there exists such a function, e.g. the first co-ordinate of the very well known Peano curve which is a continuous map from an interval onto a square.

**Problem 6.** For an  $n \times n$  matrix  $M$  with real entries let  $\|M\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mx\|_2}{\|x\|_2}$ ,

where  $\|\cdot\|_2$  denotes the Euclidean norm on  $\mathbb{R}^n$ . Assume that an  $n \times n$  matrix  $A$  with real entries satisfies  $\|A^k - A^{k-1}\| \leq \frac{1}{2002k}$  for all positive integers  $k$ . Prove that  $\|A^k\| \leq 2002$  for all positive integers  $k$ .

**Solution.**

*Lemma 1.* Let  $(a_n)_{n \geq 0}$  be a sequence of non-negative numbers such that  $a_{2k} - a_{2k+1} \leq a_k^2$ ,  $a_{2k+1} - a_{2k+2} \leq a_k a_{k+1}$  for any  $k \geq 0$  and  $\limsup n a_n < 1/4$ . Then  $\limsup \sqrt[n]{a_n} < 1$ .

*Proof.* Let  $c_l = \sup_{n \geq 2^l} (n+1)a_n$  for  $l \geq 0$ . We will show that  $c_{l+1} \leq 4c_l^2$ . Indeed, for any integer  $n \geq 2^{l+1}$  there exists an integer  $k \geq 2^l$  such that  $n = 2k$  or  $n = 2k+1$ . In the first case there is  $a_{2k} - a_{2k+1} \leq a_k^2 \leq \frac{c_l^2}{(k+1)^2} \leq \frac{4c_l^2}{2k+1} - \frac{4c_l^2}{2k+2}$ , whereas in the second case there is  $a_{2k+1} - a_{2k+2} \leq a_k a_{k+1} \leq \frac{c_l^2}{(k+1)(k+2)} \leq \frac{4c_l^2}{2k+2} - \frac{4c_l^2}{2k+3}$ .

Hence a sequence  $(a_n - \frac{4c_l^2}{n+1})_{n \geq 2^{l+1}}$  is non-decreasing and its terms are non-positive since it converges to zero. Therefore  $a_n \leq \frac{4c_l^2}{n+1}$  for  $n \geq 2^{l+1}$ , meaning that  $c_{l+1}^2 \leq 4c_l^2$ . This implies that a sequence  $((4c_l)^{2^{-l}})_{l \geq 0}$  is non-increasing and therefore bounded from above by some number  $q \in (0, 1)$  since all its terms except finitely many are less than 1. Hence  $c_l \leq q^{2^l}$  for  $l$  large enough. For any  $n$  between  $2^l$  and  $2^{l+1}$  there is  $a_n \leq \frac{c_l}{n+1} \leq q^{2^l} \leq (\sqrt{q})^n$  yielding  $\limsup \sqrt[n]{a_n} \leq \sqrt{q} < 1$ , yielding  $\limsup \sqrt[n]{a_n} \leq \sqrt{q} < 1$ , which ends the proof.

*Lemma 2.* Let  $T$  be a linear map from  $\mathbb{R}^n$  into itself. Assume that  $\limsup n \|T^{n+1} - T^n\| < 1/4$ . Then  $\limsup \|T^{n+1} - T^n\|^{1/n} < 1$ . In particular  $T^n$  converges in the operator norm and  $T$  is power bounded.

*Proof.* Put  $a_n = \|T^{n+1} - T^n\|$ . Observe that

$$T^{k+m+1} - T^{k+m} = (T^{k+m+2} - T^{k+m+1}) - (T^{k+1} - T^k)(T^{m+1} - T^m)$$

implying that  $a_{k+m} \leq a_{k+m+1} + a_k a_m$ . Therefore the sequence  $(a_m)_{m \geq 0}$  satisfies assumptions of Lemma 1 and the assertion of Proposition 1 follows.

*Remarks.* 1. The theorem proved above holds in the case of an operator  $T$  which maps a normed space  $X$  into itself,  $X$  does not have to be finite dimensional.

2. The constant  $1/4$  in Lemma 1 cannot be replaced by any greater number since a sequence  $a_n = \frac{1}{4n}$  satisfies the inequality  $a_{k+m} - a_{k+m+1} \leq a_k a_m$  for any positive integers  $k$  and  $m$  whereas it does not have exponential decay.

3. The constant  $1/4$  in Lemma 2 cannot be replaced by any number greater than  $1/e$ . Consider an operator  $(Tf)(x) = xf(x)$  on  $L^2([0, 1])$ . One can easily

check that  $\limsup \|T^{n+1} - T^n\| = 1/e$ , whereas  $T^n$  does not converge in the operator norm. The question whether in general  $\limsup n\|T^{n+1} - T^n\| < \infty$  implies that  $T$  is power bounded remains open.

*Remark* The problem was incorrectly stated during the competition: instead of the inequality  $\|A^k - A^{k-1}\| \leq \frac{1}{2002k}$ , the inequality  $\|A^k - A^{k-1}\| \leq \frac{1}{2002n}$  was assumed. If  $A = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$  then  $A^k = \begin{pmatrix} 1 & k\varepsilon \\ 0 & 1 \end{pmatrix}$ . Therefore

$A^k - A^{k-1} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$ , so for sufficiently small  $\varepsilon$  the condition is satisfied although the sequence  $(\|A^k\|)$  is clearly unbounded.

Solutions for problems in the  
9<sup>th</sup> International Mathematics Competition  
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Warsaw, July 19 - July 25, 2002

Second Day

**Problem 1.** Compute the determinant of the  $n \times n$  matrix  $A = [a_{ij}]$ ,

$$a_{ij} = \begin{cases} (-1)^{|i-j|}, & \text{if } i \neq j, \\ 2, & \text{if } i = j. \end{cases}$$

*Solution.* Adding the second row to the first one, then adding the third row to the second one, ..., adding the  $n$ th row to the  $(n-1)$ th, the determinant does not change and we have

$$\det(A) = \begin{vmatrix} 2 & -1 & +1 & \dots & \pm 1 & \mp 1 \\ -1 & 2 & -1 & \dots & \mp 1 & \pm 1 \\ +1 & -1 & 2 & \dots & \pm 1 & \mp 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mp 1 & \pm 1 & \mp 1 & \dots & 2 & -1 \\ \pm 1 & \mp 1 & \pm 1 & \dots & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ \pm 1 & \mp 1 & \pm 1 & \mp 1 & \dots & -1 & 2 \end{vmatrix}.$$

Now subtract the first column from the second, then subtract the resulting second column from the third, ..., and at last, subtract the  $(n-1)$ th column from the  $n$ th column. This way we have

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & n+1 \end{vmatrix} = n+1.$$

**Problem 2.** Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

*Solution.* For each pair of students, consider the set of those problems which was not solved by them. There exist  $\binom{200}{2} = 19900$  sets; we have to prove that at least one set is empty.

For each problem, there are at most 80 students who did not solve it. From these students at most  $\binom{80}{2} = 3160$  pairs can be selected, so the problem can belong to at most 3160 sets. The 6 problems together can belong to at most  $6 \cdot 3160 = 18960$  sets.

Hence, at least  $19900 - 18960 = 940$  sets must be empty.

**Problem 3.** For each  $n \geq 1$  let

$$a_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}, \quad b_n = \sum_{k=0}^{\infty} (-1)^k \frac{k^n}{k!}.$$

Show that  $a_n \cdot b_n$  is an integer.

*Solution.* We prove by induction on  $n$  that  $a_n/e$  and  $b_n e$  are integers, we prove this for  $n = 0$  as well. (For  $n = 0$ , the term  $0^0$  in the definition of the sequences must be replaced by 1.)

From the power series of  $e^x$ ,  $a_n = e^1 = e$  and  $b_n = e^{-1} = 1/e$ .

Suppose that for some  $n \geq 0$ ,  $a_0, a_1, \dots, a_n$  and  $b_0, b_1, \dots, b_n$  are all multipliers of  $e$  and  $1/e$ , respectively. Then, by the binomial theorem,

$$\begin{aligned} a_{n+1} &= \sum_{k=0}^n \frac{(k+1)^{n+1}}{(k+1)!} = \sum_{k=0}^{\infty} \frac{(k+1)^n}{k!} = \sum_{k=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \frac{k^m}{k!} = \\ &= \sum_{m=0}^n \binom{n}{m} \sum_{k=0}^{\infty} \frac{k^m}{k!} = \sum_{m=0}^n \binom{n}{m} a_m \end{aligned}$$

and similarly

$$\begin{aligned} b_{n+1} &= \sum_{k=0}^n (-1)^{k+1} \frac{(k+1)^{n+1}}{(k+1)!} = - \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)^n}{k!} = \\ &= - \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^n \binom{n}{m} \frac{k^m}{k!} = - \sum_{m=0}^n \binom{n}{m} \sum_{k=0}^{\infty} (-1)^k \frac{k^m}{k!} = - \sum_{m=0}^n \binom{n}{m} b_m. \end{aligned}$$

The numbers  $a_{n+1}$  and  $b_{n+1}$  are expressed as linear combinations of the previous elements with integer coefficients which finishes the proof.

**Problem 4.** In the tetrahedron  $OABC$ , let  $\angle BOC = \alpha$ ,  $\angle COA = \beta$  and  $\angle AOB = \gamma$ . Let  $\sigma$  be the angle between the faces  $OAB$  and  $OAC$ , and let  $\tau$  be the angle between the faces  $OBA$  and  $OBC$ . Prove that

$$\gamma > \beta \cdot \cos \sigma + \alpha \cdot \cos \tau.$$

*Solution.* We can assume  $OA = OB = OC = 1$ . Intersect the unit sphere with center  $O$  with the angle domains  $AOB$ ,  $BOC$  and  $COA$ ; the intersections are "slices" and their areas are  $\frac{1}{2}\gamma$ ,  $\frac{1}{2}\alpha$  and  $\frac{1}{2}\beta$ , respectively.

Now project the slices  $AOC$  and  $COB$  to the plane  $OAB$ . Denote by  $C'$  the projection of vertex  $C$ , and denote by  $A'$  and  $B'$  the reflections of vertices  $A$  and  $B$  with center  $O$ , respectively. By the projection,  $OC' < 1$ .

The projections of arcs  $AC$  and  $BC$  are segments of ellipses with long axes  $AA'$  and  $BB'$ , respectively. (The ellipses can be degenerate if  $\sigma$  or  $\tau$  is right angle.) The two ellipses intersect each other in 4 points; both half ellipses connecting  $A$  and  $A'$  intersect both half ellipses connecting  $B$  and  $B'$ . There exist no more intersection, because two different conics cannot have more than 4 common points.

The signed areas of the projections of slices  $AOC$  and  $COB$  are  $\frac{1}{2}\alpha \cdot \cos \tau$  and  $\frac{1}{2}\beta \cdot \cos \sigma$ , respectively. The statement says that the sum of these signed areas is less than the area of slice  $BOA$ .

There are three significantly different cases with respect to the signs of  $\cos \sigma$  and  $\cos \tau$  (see Figure). If both signs are positive (case (a)), then the projections of slices  $OAC$  and  $OBC$  are subsets of slice  $OBC$  without common interior point, and they do not cover the whole slice  $OBC$ ; this implies the statement. In cases (b) and (c) where at least one of the signs is negative, projections with positive sign are subsets of the slice  $OBC$ , so the statement is obvious again.

**Problem 5.** Let  $A$  be an  $n \times n$  matrix with complex entries and suppose that  $n > 1$ . Prove that

$$A\bar{A} = I_n \iff \exists S \in GL_n(\mathbb{C}) \text{ such that } A = S\bar{S}^{-1}.$$

(If  $A = [a_{ij}]$  then  $\bar{A} = [\bar{a}_{ij}]$ , where  $\bar{a}_{ij}$  is the complex conjugate of  $a_{ij}$ ;  $GL_n(\mathbb{C})$  denotes the set of all  $n \times n$  invertible matrices with complex entries, and  $I_n$  is the identity matrix.)

*Solution.* The direction  $\Leftarrow$  is trivial, since if  $A = S\bar{S}^{-1}$ , then  $A\bar{A} = S\bar{S}^{-1} \cdot \bar{S}S^{-1} = I_n$ .

For the direction  $\Rightarrow$ , we must prove that there exists an invertible matrix  $S$  such that  $A\bar{S} = S$ .

Let  $w$  be an arbitrary complex number which is not 0. Choosing  $S = wA + \bar{w}I_n$ , we have  $A\bar{S} = A(\bar{w}\bar{A} + \bar{w}I_n) = \bar{w}I_n + wA = S$ . If  $S$  is singular, then  $\frac{1}{w}S = A + (\bar{w}/w)I_n$  is singular as well, so  $\bar{w}/w$  is an eigenvalue of  $A$ . Since  $A$  has finitely many eigenvalues and  $\bar{w}/w$  can be any complex number on the unit circle, there exist such  $w$  that  $S$  is invertible.

**Problem 6.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function whose gradient  $\nabla f = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$  exists at every point of  $\mathbb{R}^n$  and satisfies the condition

$$\exists L > 0 \quad \forall x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\| \leq L\|x_1 - x_2\|.$$

Prove that

$$\forall x_1, x_2 \in \mathbb{R}^n \quad \|\nabla f(x_1) - \nabla f(x_2)\|^2 \leq L\langle \nabla f(x_1) - \nabla f(x_2), x_1 - x_2 \rangle. \quad (1)$$

In this formula  $\langle a, b \rangle$  denotes the scalar product of the vectors  $a$  and  $b$ .

*Solution.* Let  $g(x) = f(x) - f(x_1) - \langle \nabla f(x_1), x - x_1 \rangle$ . It is obvious that  $g$  has the same properties. Moreover,  $g(x_1) = \nabla g(x_1) = 0$  and, due to convexity,  $g$  has 0 as the absolute minimum at  $x_1$ . Next we prove that

$$g(x_2) \geq \frac{1}{2L} \|\nabla g(x_2)\|^2. \quad (2)$$

Let  $y_0 = x_2 - \frac{1}{L} \|\nabla g(x_2)\|$  and  $y(t) = y_0 + t(x_2 - y_0)$ . Then

$$\begin{aligned} g(x_2) &= g(y_0) + \int_0^1 \langle \nabla g(y(t)), x_2 - y_0 \rangle dt = \\ &= g(y_0) + \langle \nabla g(x_2), x_2 - y_0 \rangle - \int_0^1 \langle \nabla g(x_2) - \nabla g(y(t)), x_2 - y_0 \rangle dt \geq \\ &\geq 0 + \frac{1}{L} \|\nabla g(x_2)\|^2 - \int_0^1 \|\nabla g(x_2) - \nabla g(y(t))\| \cdot \|x_2 - y_0\| dt \geq \\ &\geq \frac{1}{L} \|\nabla g(x_2)\|^2 - \|x_2 - y_0\| \int_0^1 L \|x_2 - y(t)\| dt = \\ &= \frac{1}{L} \|\nabla g(x_2)\|^2 - L \|x_2 - y_0\|^2 \int_0^1 t dt = \frac{1}{2L} \|\nabla g(x_2)\|^2. \end{aligned}$$

Substituting the definition of  $g$  into (2), we obtain

$$f(x_2) - f(x_1) - \langle \nabla f(x_1), x_2 - x_1 \rangle \geq \frac{1}{2L} \|\nabla f(x_2) - \nabla f(x_1)\|^2,$$

$$\|\nabla f(x_2) - \nabla f(x_1)\|^2 \leq 2L\langle \nabla f(x_1), x_1 - x_2 \rangle + 2L(f(x_2) - f(x_1)). \quad (3)$$

Exchanging variables  $x_1$  and  $x_2$ , we have

$$\|\nabla f(x_2) - \nabla f(x_1)\|^2 \leq 2L\langle \nabla f(x_2), x_2 - x_1 \rangle + 2L(f(x_1) - f(x_2)). \quad (4)$$

The statement (1) is the average of (3) and (4).

10<sup>th</sup> International Mathematical Competition for University Students  
Cluj-Napoca, July 2003

Day 1

1. (a) Let  $a_1, a_2, \dots$  be a sequence of real numbers such that  $a_1 = 1$  and  $a_{n+1} > \frac{3}{2}a_n$  for all  $n$ . Prove that the sequence

$$\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$$

has a finite limit or tends to infinity. (10 points)

- (b) Prove that for all  $\alpha > 1$  there exists a sequence  $a_1, a_2, \dots$  with the same properties such that

$$\lim \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha.$$

(10 points)

*Solution.* (a) Let  $b_n = \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$ . Then  $a_{n+1} > \frac{3}{2}a_n$  is equivalent to  $b_{n+1} > b_n$ , thus the sequence

$(b_n)$  is strictly increasing. Each increasing sequence has a finite limit or tends to infinity.

- (b) For all  $\alpha > 1$  there exists a sequence  $1 = b_1 < b_2 < \dots$  which converges to  $\alpha$ . Choosing  $a_n = \left(\frac{3}{2}\right)^{n-1} b_n$ , we obtain the required sequence  $(a_n)$ .

2. Let  $a_1, a_2, \dots, a_{51}$  be non-zero elements of a field. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence  $b_1, \dots, b_{51}$ . If this new sequence is a permutation of the original one, what can be the characteristic of the field? (The characteristic of a field is  $p$ , if  $p$  is the smallest positive integer such that  $\underbrace{x + x + \dots + x}_p = 0$  for any element  $x$

of the field. If there exists no such  $p$ , the characteristic is 0.) (20 points)

*Solution.* Let  $S = a_1 + a_2 + \dots + a_{51}$ . Then  $b_1 + b_2 + \dots + b_{51} = 50S$ . Since  $b_1, b_2, \dots, b_{51}$  is a

permutation of  $a_1, a_2, \dots, a_{51}$ , we get  $50S = S$ , so  $49S = 0$ . Assume that the characteristic of the field is not equal to 7. Then  $49S = 0$  implies that  $S = 0$ . Therefore  $b_i = -a_i$  for  $i = 1, 2, \dots, 51$ . On the other hand,  $b_i = a_{\varphi(i)}$ , where  $\varphi \in S_{51}$ . Therefore, if the characteristic is not 2, the sequence  $a_1, a_2, \dots, a_{51}$  can be partitioned into pairs  $\{a_i, a_{\varphi(i)}\}$  of additive inverses. But this is impossible, since 51 is an odd number. It follows that the characteristic of the field is 7 or 2.

The characteristic can be either 2 or 7. For the case of 7,  $x_1 = \dots = x_{51} = 1$  is a possible choice. For the case of 2, any elements can be chosen such that  $S = 0$ , since then  $b_i = -a_i = a_i$ .

3. Let  $A$  be an  $n \times n$  real matrix such that  $3A^3 = A^2 + A + I$  ( $I$  is the identity matrix). Show that the sequence  $A^k$  converges to an idempotent matrix. (A matrix  $B$  is called idempotent if  $B^2 = B$ .) (20 points)

*Solution.* The minimal polynomial of  $A$  is a divisor of  $3x^3 - x^2 - x - 1$ . This polynomial has three different roots. This implies that  $A$  is diagonalizable:  $A = C^{-1}DC$  where  $D$  is a diagonal matrix. The eigenvalues of the matrices  $A$  and  $D$  are all roots of polynomial  $3x^3 - x^2 - x - 1$ . One of the three roots is 1, the remaining two roots have smaller absolute value than 1. Hence, the diagonal elements of  $D^k$ , which are the  $k$ th powers of the eigenvalues, tend to either 0 or 1 and the limit  $M = \lim D^k$  is idempotent. Then  $\lim A^k = C^{-1}MC$  is idempotent as well.

4. Determine the set of all pairs  $(a, b)$  of positive integers for which the set of positive integers can be decomposed into two sets  $A$  and  $B$  such that  $a \cdot A = b \cdot B$ . (20 points)

*Solution.* Clearly  $a$  and  $b$  must be different since  $A$  and  $B$  are disjoint.

Let  $\{a, b\}$  be a solution and consider the sets  $A, B$  such that  $a \cdot A = b \cdot B$ . Denoting  $d = (a, b)$  the greatest common divisor of  $a$  and  $b$ , we have  $a = d \cdot a_1$ ,  $b = d \cdot b_1$ ,  $(a_1, b_1) = 1$  and  $a_1 \cdot A = b_1 \cdot B$ . Thus  $\{a_1, b_1\}$  is a solution and it is enough to determine the solutions  $\{a, b\}$  with  $(a, b) = 1$ .

If  $1 \in A$  then  $a \in a \cdot A = b \cdot B$ , thus  $b$  must be a divisor of  $a$ . Similarly, if  $1 \in B$ , then  $a$  is a divisor of  $b$ . Therefore, in all solutions, one of numbers  $a, b$  is a divisor of the other one.

Now we prove that if  $n \geq 2$ , then  $(1, n)$  is a solution. For each positive integer  $k$ , let  $f(k)$  be the largest non-negative integer for which  $n^{f(k)} | k$ . Then let  $A = \{k : f(k) \text{ is odd}\}$  and  $B = \{k : f(k) \text{ is even}\}$ . This is a decomposition of all positive integers such that  $A = n \cdot B$ .

**5.** Let  $g : [0, 1] \rightarrow \mathbb{R}$  be a continuous function and let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be a sequence of functions defined by  $f_0(x) = g(x)$  and

$$f_{n+1}(x) = \frac{1}{x} \int_0^x f_n(t) dt \quad (x \in (0, 1], n = 0, 1, 2, \dots).$$

Determine  $\lim_{n \rightarrow \infty} f_n(x)$  for every  $x \in (0, 1]$ . (20 points)

**B.** We shall prove in two different ways that  $\lim_{n \rightarrow \infty} f_n(x) = g(0)$  for every  $x \in (0, 1]$ . (The second one is more lengthy but it tells us how to calculate  $f_n$  directly from  $g$ .)

**Proof I.** First we prove our claim for non-decreasing  $g$ . In this case, by induction, one can easily see that

1. each  $f_n$  is non-decreasing as well, and
2.  $g(x) = f_0(x) \geq f_1(x) \geq f_2(x) \geq \dots \geq g(0) \quad (x \in (0, 1])$ .

Then (2) implies that there exists

$$h(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (x \in (0, 1]).$$

Clearly  $h$  is non-decreasing and  $g(0) \leq h(x) \leq f_n(x)$  for any  $x \in (0, 1], n = 0, 1, 2, \dots$ . Therefore to show that  $h(x) = g(0)$  for any  $x \in (0, 1]$ , it is enough to prove that  $h(1)$  cannot be greater than  $g(0)$ .

Suppose that  $h(1) > g(0)$ . Then there exists a  $0 < \delta < 1$  such that  $h(1) > g(\delta)$ . Using the definition, (2) and (1) we get

$$f_{n+1}(1) = \int_0^1 f_n(t) dt \leq \int_0^\delta g(t) dt + \int_\delta^1 f_n(t) dt \leq \delta g(\delta) + (1 - \delta) f_n(1).$$

Hence

$$f_n(1) - f_{n+1}(1) \geq \delta(f_n(1) - g(\delta)) \geq \delta(h(1) - g(\delta)) > 0,$$

so  $f_n(1) \rightarrow -\infty$ , which is a contradiction.

Similarly, we can prove our claim for non-increasing continuous functions as well.

Now suppose that  $g$  is an arbitrary continuous function on  $[0, 1]$ . Let

$$M(x) = \sup_{t \in [0, x]} g(t), \quad m(x) = \inf_{t \in [0, x]} g(t) \quad (x \in [0, 1])$$

Then on  $[0, 1]$   $m$  is non-increasing,  $M$  is non-decreasing, both are continuous,  $m(x) \leq g(x) \leq M(x)$  and  $M(0) = m(0) = g(0)$ . Define the sequences of functions  $M_n(x)$  and  $m_n(x)$  in the same way as  $f_n$  is defined but starting with  $M_0 = M$  and  $m_0 = m$ .

Then one can easily see by induction that  $m_n(x) \leq f_n(x) \leq M_n(x)$ . By the first part of the proof,  $\lim_n m_n(x) = m(0) = g(0) = M(0) = \lim_n M_n(x)$  for any  $x \in (0, 1]$ . Therefore we must have  $\lim_n f_n(x) = g(0)$ .

**Proof II.** To make the notation clearer we shall denote the variable of  $f_j$  by  $x_j$ . By definition (and Fubini theorem) we get that

$$\begin{aligned} f_{n+1}(x_{n+1}) &= \frac{1}{x_{n+1}} \int_0^{x_{n+1}} \frac{1}{x_n} \int_0^{x_n} \frac{1}{x_{n-1}} \int_0^{x_{n-1}} \cdots \int_0^{x_2} \frac{1}{x_1} \int_0^{x_1} g(x_0) dx_0 dx_1 \dots dx_n \\ &= \frac{1}{x_{n+1}} \iiint_{0 \leq x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1}} g(x_0) \frac{dx_0 dx_1 \dots dx_n}{x_1 \dots x_n} \\ &= \frac{1}{x_{n+1}} \int_0^{x_{n+1}} g(x_0) \left( \iint_{x_0 \leq x_1 \leq \dots \leq x_n \leq x_{n+1}} \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \right) dx_0. \end{aligned}$$

Therefore with the notation

$$h_n(a, b) = \iint_{a \leq x_1 \leq \dots \leq x_n \leq b} \frac{dx_1 \dots dx_n}{x_1 \dots x_n}$$

and  $x = x_{n+1}, t = x_0$  we have

$$f_{n+1}(x) = \frac{1}{x} \int_0^x g(t) h_n(t, x) dt.$$

Using that  $h_n(a, b)$  is the same for any permutation of  $x_1, \dots, x_n$  and the fact that the integral is 0 on any hyperplanes ( $x_i = x_j$ ) we get that

$$\begin{aligned} n! h_n(a, b) &= \iint_{a \leq x_1, \dots, x_n \leq b} \frac{dx_1 \dots dx_n}{x_1 \dots x_n} = \int_a^b \cdots \int_a^b \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \\ &= \left( \int_a^b \frac{dx}{x} \right)^n = (\log(b/a))^n. \end{aligned}$$

Therefore

$$f_{n+1}(x) = \frac{1}{x} \int_0^x g(t) \frac{(\log(x/t))^n}{n!} dt.$$

Note that if  $g$  is constant then the definition gives  $f_n = g$ . This implies on one hand that we must have

$$\frac{1}{x} \int_0^x \frac{(\log(x/t))^n}{n!} dt = 1$$

and on the other hand that, by replacing  $g$  by  $g - g(0)$ , we can suppose that  $g(0) = 0$ .

Let  $x \in (0, 1]$  and  $\varepsilon > 0$  be fixed. By continuity there exists a  $0 < \delta < x$  and an  $M$  such that  $|g(t)| < \varepsilon$  on  $[0, \delta]$  and  $|g(t)| \leq M$  on  $[0, 1]$ . Since

$$\lim_{n \rightarrow \infty} \frac{(\log(x/\delta))^n}{n!} = 0$$

there exists an  $n_0$  such that  $\log(x/\delta)^n/n! < \varepsilon$  whenever  $n \geq n_0$ . Then, for any  $n \geq n_0$ , we have

$$\begin{aligned} |f_{n+1}(x)| &\leq \frac{1}{x} \int_0^x |g(t)| \frac{(\log(x/t))^n}{n!} dt \\ &\leq \frac{1}{x} \int_0^\delta \varepsilon \frac{(\log(x/t))^n}{n!} dt + \frac{1}{x} \int_\delta^x |g(t)| \frac{(\log(x/\delta))^n}{n!} dt \\ &\leq \frac{1}{x} \int_0^x \varepsilon \frac{(\log(x/t))^n}{n!} dt + \frac{1}{x} \int_\delta^x M \varepsilon dt \\ &\leq \varepsilon + M\varepsilon. \end{aligned}$$

Therefore  $\lim_n f(x) = 0 = g(0)$ .

6. Let  $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  be a polynomial with real coefficients. Prove that if all roots of  $f$  lie in the left half-plane  $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$  then

$$a_k a_{k+3} < a_{k+1} a_{k+2}$$

holds for every  $k = 0, 1, \dots, n-3$ . (20 points)

*Solution.* The polynomial  $f$  is a product of linear and quadratic factors,  $f(z) = \prod_i (k_i z + l_i) \cdot$

$\prod_j (p_j z^2 + q_j z + r_j)$ , with  $k_i, l_i, p_j, q_j, r_j \in \mathbb{R}$ . Since all roots are in the left half-plane, for each  $i$ ,  $k_i$  and  $l_i$  are of the same sign, and for each  $j$ ,  $p_j, q_j, r_j$  are of the same sign, too. Hence, multiplying  $f$  by  $-1$  if necessary, the roots of  $f$  don't change and  $f$  becomes the polynomial with all positive coefficients.

For the simplicity, we extend the sequence of coefficients by  $a_{n+1} = a_{n+2} = \dots = 0$  and  $a_{-1} = a_{-2} = \dots = 0$  and prove the same statement for  $-1 \leq k \leq n-2$  by induction.

For  $n \leq 2$  the statement is obvious:  $a_{k+1}$  and  $a_{k+2}$  are positive and at least one of  $a_{k-1}$  and  $a_{k+3}$  is 0; hence,  $a_{k+1} a_{k+2} > a_k a_{k+3} = 0$ .

Now assume that  $n \geq 3$  and the statement is true for all smaller values of  $n$ . Take a divisor of  $f(z)$  which has the form  $z^2 + pz + q$  where  $p$  and  $q$  are positive real numbers. (Such a divisor can be obtained from a conjugate pair of roots or two real roots.) Then we can write

$$f(z) = (z^2 + pz + q)(b_{n-2} z^{n-2} + \dots + b_1 z + b_0) = (z^2 + pz + q)g(x). \quad (1)$$

The roots polynomial  $g(z)$  are in the left half-plane, so we have  $b_{k+1} b_{k+2} < b_k b_{k+3}$  for all  $-1 \leq k \leq n-4$ . Defining  $b_{n-1} = b_n = \dots = 0$  and  $b_{-1} = b_{-2} = \dots = 0$  as well, we also have  $b_{k+1} b_{k+2} \leq b_k b_{k+3}$  for all integer  $k$ .

Now we prove  $a_{k+1} a_{k+2} > a_k a_{k+3}$ . If  $k = -1$  or  $k = n-2$  then this is obvious since  $a_{k+1} a_{k+2}$  is positive and  $a_k a_{k+3} = 0$ . Thus, assume  $0 \leq k \leq n-3$ . By an easy computation,

$$\begin{aligned} & a_{k+1} a_{k+2} - a_k a_{k+3} = \\ & = (qb_{k+1} + pb_k + b_{k-1})(qb_{k+2} + pb_{k+1} + b_k) - (qb_k + pb_{k-1} + b_{k-2})(qb_{k+3} + pb_{k+2} + b_{k+1}) = \\ & = (b_{k-1} b_k - b_{k-2} b_{k+1}) + p(b_k^2 - b_{k-2} b_{k+2}) + q(b_{k-1} b_{k+2} - b_{k-2} b_{k+3}) + \\ & + p^2(b_k b_{k+1} - b_{k-1} b_{k+2}) + q^2(b_{k+1} b_{k+2} - b_k b_{k+3}) + pq(b_{k+1}^2 - b_{k-1} b_{k+3}). \end{aligned}$$

We prove that all the six terms are non-negative and at least one is positive. Term  $p^2(b_k b_{k+1} - b_{k-1} b_{k+2})$  is positive since  $0 \leq k \leq n-3$ . Also terms  $b_{k-1} b_k - b_{k-2} b_{k+1}$  and  $q^2(b_{k+1} b_{k+2} - b_k b_{k+3})$  are non-negative by the induction hypothesis.

To check the sign of  $p(b_k^2 - b_{k-2} b_{k+2})$  consider

$$b_{k-1}(b_k^2 - b_{k-2} b_{k+2}) = b_{k-2}(b_k b_{k+1} - b_{k-1} b_{k+2}) + b_k(b_{k-1} b_k - b_{k-2} b_{k+1}) \geq 0.$$

If  $b_{k-1} > 0$  we can divide by it to obtain  $b_k^2 - b_{k-2} b_{k+2} \geq 0$ . Otherwise, if  $b_{k-1} = 0$ , either  $b_{k-2} = 0$  or  $b_{k+2} = 0$  and thus  $b_k^2 - b_{k-2} b_{k+2} = b_k^2 \geq 0$ . Therefore,  $p(b_k^2 - b_{k-2} b_{k+2}) \geq 0$  for all  $k$ . Similarly,  $pq(b_{k+1}^2 - b_{k-1} b_{k+3}) \geq 0$ .

The sign of  $q(b_{k-1} b_{k+2} - b_{k-2} b_{k+3})$  can be checked in a similar way. Consider

$$b_{k+1}(b_{k-1} b_{k+2} - b_{k-2} b_{k+3}) = b_{k-1}(b_{k+1} b_{k+2} - b_k b_{k+3}) + b_{k+3}(b_{k-1} b_k - b_{k-2} b_{k+1}) \geq 0.$$

If  $b_{k+1} > 0$ , we can divide by it. Otherwise either  $b_{k-2} = 0$  or  $b_{k+3} = 0$ . In all cases, we obtain  $b_{k-1} b_{k+2} - b_{k-2} b_{k+3} \geq 0$ .

Now the signs of all terms are checked and the proof is complete.

10<sup>th</sup> International Mathematical Competition for University Students  
Cluj-Napoca, July 2003

Day 2

1. Let  $A$  and  $B$  be  $n \times n$  real matrices such that  $AB + A + B = 0$ . Prove that  $AB = BA$ .

**Solution.** Since  $(A + I)(B + I) = AB + A + B + I = I$  ( $I$  is the identity matrix), matrices  $A + I$  and  $B + I$  are inverses of each other. Then  $(A + I)(B + I) = (B + I)(A + I)$  and  $AB + BA$ .

2. Evaluate the limit

$$\lim_{x \rightarrow 0^+} \int_x^{2x} \frac{\sin^m t}{t^n} dt \quad (m, n \in \mathbb{N}).$$

**Solution.** We use the fact that  $\frac{\sin t}{t}$  is decreasing in the interval  $(0, \pi)$  and  $\lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1$ .

For all  $x \in (0, \frac{\pi}{2})$  and  $t \in [x, 2x]$  we have  $\frac{\sin 2x}{2} x < \frac{\sin t}{t} < 1$ , thus

$$\begin{aligned} \left(\frac{\sin 2x}{2x}\right)^m \int_x^{2x} \frac{t^m}{t^n} dt &< \int_x^{2x} \frac{\sin^m t}{t^n} dt < \int_x^{2x} \frac{t^m}{t^n} dt, \\ \int_x^{2x} \frac{t^m}{t^n} dt &= x^{m-n+1} \int_1^2 u^{m-n} du. \end{aligned}$$

The factor  $\left(\frac{\sin 2x}{2x}\right)^m$  tends to 1. If  $m - n + 1 < 0$ , the limit of  $x^{m-n+1}$  is infinity; if  $m - n + 1 > 0$  then 0. If  $m - n + 1 = 0$  then  $x^{m-n+1} \int_1^2 u^{m-n} du = \ln 2$ . Hence,

$$\lim_{x \rightarrow 0^+} \int_x^{2x} \frac{\sin^m t}{t^n} dt = \begin{cases} 0, & m \geq n \\ \ln 2, & n - m = 1 \\ +\infty, & n - m > 1. \end{cases}$$

3. Let  $A$  be a closed subset of  $\mathbb{R}^n$  and let  $B$  be the set of all those points  $b \in \mathbb{R}^n$  for which there exists exactly one point  $a_0 \in A$  such that

$$|a_0 - b| = \inf_{a \in A} |a - b|.$$

Prove that  $B$  is dense in  $\mathbb{R}^n$ ; that is, the closure of  $B$  is  $\mathbb{R}^n$ .

**Solution.** Let  $b_0 \notin A$  (otherwise  $b_0 \in A \subset B$ ),  $\varrho = \inf_{a \in A} |a - b_0|$ . The intersection of the ball of radius  $\varrho + 1$  with centre  $b_0$  with set  $A$  is compact and there exists  $a_0 \in A$ :  $|a_0 - b_0| = \varrho$ .

Denote by  $\mathbf{B}_r(a) = \{x \in R^n : |x - a| \leq r\}$  and  $\partial\mathbf{B}_r(a) = \{x \in R^n : |x - a| = r\}$  the ball and the sphere of center  $a$  and radius  $r$ , respectively.

If  $a_0$  is not the unique nearest point then for any point  $a$  on the open line segment  $(a_0, b_0)$  we have  $\mathbf{B}_{|a-a_0|}(a) \subset \mathbf{B}_\rho(b_0)$  and  $\partial\mathbf{B}_{|a-a_0|}(a) \cap \partial\mathbf{B}_\rho(b_0) = \{a_0\}$ , therefore  $(a_0, b_0) \subset B$  and  $b_0$  is an accumulation point of set  $B$ .

4. Find all positive integers  $n$  for which there exists a family  $\mathcal{F}$  of three-element subsets of  $S = \{1, 2, \dots, n\}$  satisfying the following two conditions:

- (i) for any two different elements  $a, b \in S$ , there exists exactly one  $A \in \mathcal{F}$  containing both  $a, b$ ;
- (ii) if  $a, b, c, x, y, z$  are elements of  $S$  such that if  $\{a, b, x\}, \{a, c, y\}, \{b, c, z\} \in \mathcal{F}$ , then  $\{x, y, z\} \in \mathcal{F}$ .

**Solution.** The condition (i) of the problem allows us to define a (well-defined) operation  $*$  on the set  $S$  given by

$$a * b = c \text{ if and only if } \{a, b, c\} \in \mathcal{F}, \text{ where } a \neq b.$$

We note that this operation is still not defined completely (we need to define  $a * a$ ), but nevertheless let us investigate its features. At first, due to (i), for  $a \neq b$  the operation obviously satisfies the following three conditions:

- (a)  $a \neq a * b \neq b$ ;
- (b)  $a * b = b * a$ ;
- (c)  $a * (a * b) = b$ .

What does the condition (ii) give? It claims that

$$(e') \quad x * (a * c) = x * y = z = b * c = (x * a) * c$$

for any three different  $x, a, c$ , i.e. that the operation is associative if the arguments are different. Now we can complete the definition of  $*$ . In order to save associativity for non-different arguments, i.e. to make  $b = a * (a * b) = (a * a) * b$  hold, we will add to  $S$  an extra element, call it 0, and define

$$(d) \quad a * a = 0 \text{ and } a * 0 = 0 * a = a.$$

Now it is easy to check that, for any  $a, b, c \in S \cup \{0\}$ , (a),(b),(c) and (d), still hold, and

$$(e) \quad a * b * c := (a * b) * c = a * (b * c).$$

We have thus obtained that  $(S \cup \{0\}, *)$  has the structure of a finite Abelian group, whose elements are all of order two. Since the order of every such group is a power of 2, we conclude that  $|S \cup \{0\}| = n + 1 = 2^m$  and  $n = 2^m - 1$  for some integer  $m \geq 1$ .

Given  $n = 2^m - 1$ , according to what we have proven till now, we will construct a family of three-element subsets of  $S$  satisfying (i) and (ii). Let us define the operation  $*$  in the following manner:

if  $a = a_0 + 2a_1 + \dots + 2^{m-1}a_{m-1}$  and  $b = b_0 + 2b_1 + \dots + 2^{m-1}b_{m-1}$ , where  $a_i, b_i$  are either 0 or 1, we put  $a * b = |a_0 - b_0| + 2|a_1 - b_1| + \dots + 2^{m-1}|a_{m-1} - b_{m-1}|$ .

It is simple to check that this  $*$  satisfies (a),(b),(c) and (e'). Therefore, if we include in  $F$  all possible triples  $a, b, a * b$ , the condition (i) follows from (a),(b) and (c), whereas the condition (ii) follows from (e')

The answer is:  $n = 2^m - 1$ .

**5.** (a) Show that for each function  $f : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$  there exists a function  $g : \mathbb{Q} \rightarrow \mathbb{R}$  such that  $f(x, y) \leq g(x) + g(y)$  for all  $x, y \in \mathbb{Q}$ .

(b) Find a function  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  for which there is no function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x, y) \leq g(x) + g(y)$  for all  $x, y \in \mathbb{R}$ .

*Solution.* a) Let  $\varphi : \mathbb{Q} \rightarrow \mathbb{N}$  be a bijection. Define  $g(x) = \max\{|f(s, t)| : s, t \in \mathbb{Q}, \varphi(s) \leq \varphi(x), \varphi(t) \leq \varphi(x)\}$ . We have  $f(x, y) \leq \max\{g(x), g(y)\} \leq g(x) + g(y)$ .

b) We shall show that the function defined by  $f(x, y) = \frac{1}{|x-y|}$  for  $x \neq y$  and  $f(x, x) = 0$  satisfies the problem. If, by contradiction there exists a function  $g$  as above, it results, that  $g(y) \geq \frac{1}{|x-y|} - f(x)$  for  $x, y \in \mathbb{R}$ ,  $x \neq y$ ; one obtains that for each  $x \in \mathbb{R}$ ,  $\lim_{y \rightarrow x} g(y) = \infty$ . We show, that there exists no function  $g$  having an infinite limit at each point of a bounded and closed interval  $[a, b]$ .

For each  $k \in \mathbb{N}^+$  denote  $A_k = \{x \in [a, b] : |g(x)| \leq k\}$ .

We have obviously  $[a, b] = \cup_{k=1}^{\infty} A_k$ . The set  $[a, b]$  is uncountable, so at least one of the sets  $A_k$  is infinite (in fact uncountable). This set  $A_k$  being infinite, there exists a sequence in  $A_k$  having distinct terms. This sequence will contain a convergent subsequence  $(x_n)_{n \in \mathbb{N}}$  convergent to a point  $x \in [a, b]$ . But  $\lim_{y \rightarrow x} g(y) = \infty$  implies that  $g(x_n) \rightarrow \infty$ , a contradiction because  $|g(x_n)| \leq k$ ,  $\forall n \in \mathbb{N}$ .

*Second solution for part (b).* Let  $S$  be the set of all sequences of real numbers. The cardinality of  $S$  is  $|S| = |\mathbb{R}|^{\aleph_0} = 2^{\aleph_0} = 2^{\aleph_0} = |\mathbb{R}|$ . Thus, there exists a bijection  $h : \mathbb{R} \rightarrow S$ . Now define the function  $f$  in the following way. For any real  $x$  and positive integer  $n$ , let  $f(x, n)$  be the  $n$ th element of sequence  $h(x)$ . If  $y$  is not a positive integer then let  $f(x, y) = 0$ . We prove that this function has the required property.

Let  $g$  be an arbitrary  $\mathbb{R} \rightarrow \mathbb{R}$  function. We show that there exist real numbers  $x, y$  such that  $f(x, y) > g(x) + g(y)$ . Consider the sequence  $(n + g(n))_{n=1}^{\infty}$ . This sequence is an element of  $S$ , thus  $(n + g(n))_{n=1}^{\infty} = h(x)$  for a certain real  $x$ . Then for an arbitrary positive integer  $n$ ,  $f(x, n)$  is the  $n$ th element,  $f(x, n) = n + g(n)$ . Choosing  $n$  such that  $n > g(x)$ , we obtain  $f(x, n) = n + g(n) > g(x) + g(n)$ .

**6.** Let  $(a_n)_{n \in \mathbb{N}}$  be the sequence defined by

$$a_0 = 1, \quad a_{n+1} = \frac{1}{n+1} \sum_{k=0}^n \frac{a_k}{n-k+2}.$$

Find the limit

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{a_k}{2^k},$$

if it exists.

**Solution.** Consider the generating function  $f(x) = \sum_{n=0}^{\infty} a_n x^n$ . By induction  $0 < a_n \leq 1$ , thus this series is absolutely convergent for  $|x| < 1$ ,  $f(0) = 1$  and the function is positive in the interval  $[0, 1)$ . The goal is to compute  $f(\frac{1}{2})$ .

By the recurrence formula,

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{a_k}{n-k+2} x^n = \\ &= \sum_{k=0}^{\infty} a_k x^k \sum_{n=k}^{\infty} \frac{x^{n-k}}{n-k+2} = f(x) \sum_{m=0}^{\infty} \frac{x^m}{m+2}. \end{aligned}$$

Then

$$\begin{aligned} \ln f(x) &= \ln f(x) - \ln f(0) = \int_0^x \frac{f'}{f} = \sum_{m=0}^{\infty} \frac{x^{m+1}}{(m+1)(m+2)} = \\ &= \sum_{m=0}^{\infty} \left( \frac{x^{m+1}}{(m+1)} - \frac{x^{m+1}}{(m+2)} \right) = 1 + \left( 1 - \frac{1}{x} \right) \sum_{m=0}^{\infty} \frac{x^{m+1}}{(m+1)} = 1 + \left( 1 - \frac{1}{x} \right) \ln \frac{1}{1-x}, \\ \ln f\left(\frac{1}{2}\right) &= 1 - \ln 2, \end{aligned}$$

and thus  $f(\frac{1}{2}) = \frac{e}{2}$ .

Solutions for problems on Day 1

**Problem 1.** Let  $S$  be an infinite set of real numbers such that  $|s_1 + s_2 + \dots + s_k| < 1$  for every finite subset  $\{s_1, s_2, \dots, s_k\} \subset S$ . Show that  $S$  is countable. [20 points]

*Solution.* Let  $S_n = S \cap (\frac{1}{n}, \infty)$  for any integer  $n > 0$ . It follows from the inequality that  $|S_n| < n$ . Similarly, if we define  $S_{-n} = S \cap (-\infty, -\frac{1}{n})$ , then  $|S_{-n}| < n$ . Any nonzero  $x \in S$  is an element of some  $S_n$  or  $S_{-n}$ , because there exists an  $n$  such that  $x > \frac{1}{n}$ , or  $x < -\frac{1}{n}$ . Then  $S \subset \{0\} \cup \bigcup_{n \in \mathbb{N}} (S_n \cup S_{-n})$ ,  $S$  is a countable union of finite sets, and hence countable.

**Problem 2.** Let  $P(x) = x^2 - 1$ . How many distinct real solutions does the following equation have:

$$\underbrace{P(P(\dots(P(x))\dots))}_{2004} = 0 \quad [20 \text{ points}]$$

*Solution.* Put  $P_n(x) = \underbrace{P(P(\dots(P(x))\dots))}_n$ . As  $P_1(x) \geq -1$ , for each  $x \in R$ , it must be that  $P_{n+1}(x) = P_1(P_n(x)) \geq -1$ , for each  $n \in N$  and each  $x \in R$ . Therefore the equation  $P_n(x) = a$ , where  $a < -1$  has no real solutions. Let us prove that the equation  $P_n(x) = a$ , where  $a > 0$ , has exactly two distinct real solutions. To this end we use mathematical induction by  $n$ . If  $n = 1$  the assertion follows directly. Assuming that the assertion holds for a  $n \in N$  we prove that it must also hold for  $n + 1$ . Since  $P_{n+1}(x) = a$  is equivalent to  $P_1(P_n(x)) = a$ , we conclude that  $P_n(x) = \sqrt{a+1}$  or  $P_n(x) = -\sqrt{a+1}$ . The equation  $P_n(x) = \sqrt{a+1}$ , as  $\sqrt{a+1} > 1$ , has exactly two distinct real solutions by the inductive hypothesis, while the equation  $P_n(x) = -\sqrt{a+1}$  has no real solutions (because  $-\sqrt{a+1} < -1$ ). Hence the equation  $P_{n+1}(x) = a$ , has exactly two distinct real solutions.

Let us prove now that the equation  $P_n(x) = 0$  has exactly  $n + 1$  distinct real solutions. Again we use mathematical induction. If  $n = 1$  the solutions are  $x = \pm 1$ , and if  $n = 2$  the solutions are  $x = 0$  and  $x = \pm\sqrt{2}$ , so in both cases the number of solutions is equal to  $n + 1$ . Suppose that the assertion holds for some  $n \in N$ . Note that  $P_{n+2}(x) = P_2(P_n(x)) = P_n^2(x)(P_n^2(x) - 2)$ , so the set of all real solutions of the equation  $P_{n+2} = 0$  is exactly the union of the sets of all real solutions of the equations  $P_n(x) = 0$ ,  $P_n(x) = \sqrt{2}$  and  $P_n(x) = -\sqrt{2}$ . By the inductive hypothesis the equation  $P_n(x) = 0$  has exactly  $n + 1$  distinct real solutions, while the equations  $P_n(x) = \sqrt{2}$  and  $P_n(x) = -\sqrt{2}$  have two and no distinct real solutions, respectively. Hence, the sets above being pairwise disjoint, the equation  $P_{n+2}(x) = 0$  has exactly  $n + 3$  distinct real solutions. Thus we have proved that, for each  $n \in N$ , the equation  $P_n(x) = 0$  has exactly  $n + 1$  distinct real solutions, so the answer to the question posed in this problem is 2005.

**Problem 3.** Let  $S_n$  be the set of all sums  $\sum_{k=1}^n x_k$ , where  $n \geq 2$ ,  $0 \leq x_1, x_2, \dots, x_n \leq \frac{\pi}{2}$  and

$$\sum_{k=1}^n \sin x_k = 1.$$

- a) Show that  $S_n$  is an interval. [10 points]
- b) Let  $l_n$  be the length of  $S_n$ . Find  $\lim_{n \rightarrow \infty} l_n$ . [10 points]

*Solution.* (a) Equivalently, we consider the set

$$Y = \{y = (y_1, y_2, \dots, y_n) \mid 0 \leq y_1, y_2, \dots, y_n \leq 1, y_1 + y_2 + \dots + y_n = 1\} \subset R^n$$

and the image  $f(Y)$  of  $Y$  under

$$f(y) = \arcsin y_1 + \arcsin y_2 + \dots + \arcsin y_n.$$

Note that  $f(Y) = S_n$ . Since  $Y$  is a connected subspace of  $R^n$  and  $f$  is a continuous function, the image  $f(Y)$  is also connected, and we know that the only connected subspaces of  $R$  are intervals. Thus  $S_n$  is an interval.

(b) We prove that

$$n \arcsin \frac{1}{n} \leq x_1 + x_2 + \dots + x_n \leq \frac{\pi}{2}.$$

Since the graph of  $\sin x$  is concave down for  $x \in [0, \frac{\pi}{2}]$ , the chord joining the points  $(0, 0)$  and  $(\frac{\pi}{2}, 1)$  lies below the graph. Hence

$$\frac{2x}{\pi} \leq \sin x \text{ for all } x \in [0, \frac{\pi}{2}]$$

and we can deduce the right-hand side of the claim:

$$\frac{2}{\pi}(x_1 + x_2 + \dots + x_n) \leq \sin x_1 + \sin x_2 + \dots + \sin x_n = 1.$$

The value 1 can be reached choosing  $x_1 = \frac{\pi}{2}$  and  $x_2 = \dots = x_n = 0$ .

The left-hand side follows immediately from Jensen's inequality, since  $\sin x$  is concave down for  $x \in [0, \frac{\pi}{2}]$  and  $0 \leq \frac{x_1 + x_2 + \dots + x_n}{n} < \frac{\pi}{2}$

$$\frac{1}{n} = \frac{\sin x_1 + \sin x_2 + \dots + \sin x_n}{n} \leq \sin \frac{x_1 + x_2 + \dots + x_n}{n}.$$

Equality holds if  $x_1 = \dots = x_n = \arcsin \frac{1}{n}$ .

Now we have computed the minimum and maximum of interval  $S_n$ ; we can conclude that  $S_n = [n \arcsin \frac{1}{n}, \frac{\pi}{2}]$ . Thus  $l_n = \frac{\pi}{2} - n \arcsin \frac{1}{n}$  and

$$\lim_{n \rightarrow \infty} l_n = \frac{\pi}{2} - \lim_{n \rightarrow \infty} \frac{\arcsin(1/n)}{1/n} = \frac{\pi}{2} - 1.$$

**Problem 4.** Suppose  $n \geq 4$  and let  $M$  be a finite set of  $n$  points in  $\mathbb{R}^3$ , no four of which lie in a plane. Assume that the points can be coloured black or white so that any sphere which intersects  $M$  in at least four points has the property that exactly half of the points in the intersection of  $M$  and the sphere are white. Prove that all of the points in  $M$  lie on one sphere. [20 points]

*Solution.* Define  $f : M \rightarrow \{-1, 1\}$ ,  $f(X) = \begin{cases} -1, & \text{if } X \text{ is white} \\ 1, & \text{if } X \text{ is black} \end{cases}$ . The given condition becomes  $\sum_{X \in S} f(X) = 0$  for any sphere  $S$  which passes through at least 4 points of  $M$ . For any 3 given points  $A, B, C$  in  $M$ , denote by  $S(A, B, C)$  the set of all spheres which pass through  $A, B, C$  and at least one other point of  $M$  and by  $|S(A, B, C)|$  the number of these spheres. Also, denote by  $\sum$  the sum  $\sum_{X \in M} f(X)$ .

We have

$$0 = \sum_{S \in S(A, B, C)} \sum_{X \in S} f(X) = (|S(A, B, C)| - 1)(f(A) + f(B) + f(C)) + \sum \quad (1)$$

since the values of  $A, B, C$  appear  $|S(A, B, C)|$  times each and the other values appear only once.

If there are 3 points  $A, B, C$  such that  $|S(A, B, C)| = 1$ , the proof is finished.

If  $|S(A, B, C)| > 1$  for any distinct points  $A, B, C$  in  $M$ , we will prove at first that  $\sum = 0$ .

Assume that  $\sum > 0$ . From (1) it follows that  $f(A) + f(B) + f(C) < 0$  and summing by all  $\binom{n}{3}$  possible choices of  $(A, B, C)$  we obtain that  $\binom{n}{3} \sum < 0$ , which means  $\sum < 0$  (contradicts the starting assumption). The same reasoning is applied when assuming  $\sum < 0$ .

Now, from  $\sum = 0$  and (1), it follows that  $f(A) + f(B) + f(C) = 0$  for any distinct points  $A, B, C$  in  $M$ . Taking another point  $D \in M$ , the following equalities take place

$$\begin{aligned} f(A) + f(B) + f(C) &= 0 \\ f(A) + f(B) + f(D) &= 0 \\ f(A) + f(C) + f(D) &= 0 \\ f(B) + f(C) + f(D) &= 0 \end{aligned}$$

which easily leads to  $f(A) = f(B) = f(C) = f(D) = 0$ , which contradicts the definition of  $f$ .

**Problem 5.** Let  $X$  be a set of  $\binom{2k-4}{k-2} + 1$  real numbers,  $k \geq 2$ . Prove that there exists a monotone sequence  $\{x_i\}_{i=1}^k \subseteq X$  such that

$$|x_{i+1} - x_1| \geq 2|x_i - x_1|$$

for all  $i = 2, \dots, k-1$ . [20 points]

*Solution.* We prove a more general statement:

*Lemma.* Let  $k, l \geq 2$ , let  $X$  be a set of  $\binom{k+l-4}{k-2} + 1$  real numbers. Then either  $X$  contains an increasing sequence  $\{x_i\}_{i=1}^k \subseteq X$  of length  $k$  and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \quad \forall i = 2, \dots, k-1,$$

or  $X$  contains a decreasing sequence  $\{x_i\}_{i=1}^l \subseteq X$  of length  $l$  and

$$|x_{i+1} - x_1| \geq 2|x_i - x_1| \quad \forall i = 2, \dots, l-1.$$

*Proof of the lemma.* We use induction on  $k+l$ . In case  $k=2$  or  $l=2$  the lemma is obviously true.

Now let us make the induction step. Let  $m$  be the minimal element of  $X$ ,  $M$  be its maximal element. Let

$$X_m = \left\{x \in X : x \leq \frac{m+M}{2}\right\}, \quad X_M = \left\{x \in X : x > \frac{m+M}{2}\right\}.$$

Since  $\binom{k+l-4}{k-2} = \binom{k+(l-1)-4}{k-2} + \binom{(k-1)+l-4}{(k-1)-2}$ , we can see that either

$$|X_m| \geq \binom{(k-1)+l-4}{(k-1)-2} + 1, \quad \text{or} \quad |X_M| \geq \binom{k+(l-1)-4}{k-2} + 1.$$

In the first case we apply the inductive assumption to  $X_m$  and either obtain a decreasing sequence of length  $l$  with the required properties (in this case the inductive step is made), or obtain an increasing sequence  $\{x_i\}_{i=1}^{k-1} \subseteq X_m$  of length  $k-1$ . Then we note that the sequence  $\{x_1, x_2, \dots, x_{k-1}, M\} \subseteq X$  has length  $k$  and all the required properties.

In the case  $|X_M| \geq \binom{k+(l-1)-4}{k-2} + 1$  the inductive step is made in a similar way. Thus the lemma is proved.

The reader may check that the number  $\binom{k+l-4}{k-2} + 1$  cannot be smaller in the lemma.

**Problem 6.** For every complex number  $z \notin \{0, 1\}$  define

$$f(z) := \sum (\log z)^{-4},$$

where the sum is over all branches of the complex logarithm.

a) Show that there are two polynomials  $P$  and  $Q$  such that  $f(z) = P(z)/Q(z)$  for all  $z \in \mathbb{C} \setminus \{0, 1\}$ . [10 points]

b) Show that for all  $z \in \mathbb{C} \setminus \{0, 1\}$

$$f(z) = z \frac{z^2 + 4z + 1}{6(z-1)^4}. \quad [10 \text{ points}]$$

*Solution 1.* It is clear that the left hand side is well defined and independent of the order of summation, because we have a sum of the type  $\sum n^{-4}$ , and the branches of the logarithms do not matter because all branches are taken. It is easy to check that the convergence is locally uniform on  $\mathbb{C} \setminus \{0, 1\}$ ; therefore,  $f$  is a holomorphic function on the complex plane, except possibly for isolated singularities at 0 and 1. (We omit the detailed estimates here.)

The function  $\log$  has its only (simple) zero at  $z=1$ , so  $f$  has a quadruple pole at  $z=1$ .

Now we investigate the behavior near infinity. We have  $\operatorname{Re}(\log(z)) = \log|z|$ , hence (with  $c := \log|z|$ )

$$\begin{aligned} \left| \sum (\log z)^{-4} \right| &\leq \sum |\log z|^{-4} = \sum (\log|z| + 2\pi i n)^{-4} + O(1) \\ &= \int_{-\infty}^{\infty} (c + 2\pi i x)^{-4} dx + O(1) \\ &= c^{-4} \int_{-\infty}^{\infty} (1 + 2\pi i x/c)^{-4} dx + O(1) \\ &= c^{-3} \int_{-\infty}^{\infty} (1 + 2\pi i t)^{-4} dt + O(1) \\ &\leq \alpha (\log|z|)^{-3} \end{aligned}$$

for a universal constant  $\alpha$ . Therefore, the infinite sum tends to 0 as  $|z| \rightarrow \infty$ . In particular, the isolated singularity at  $\infty$  is not essential, but rather has (at least a single) zero at  $\infty$ .

The remaining singularity is at  $z = 0$ . It is readily verified that  $f(1/z) = f(z)$  (because  $\log(1/z) = -\log(z)$ ); this implies that  $f$  has a zero at  $z = 0$ .

We conclude that the infinite sum is holomorphic on  $\mathbb{C}$  with at most one pole and without an essential singularity at  $\infty$ , so it is a rational function, i.e. we can write  $f(z) = P(z)/Q(z)$  for some polynomials  $P$  and  $Q$  which we may as well assume coprime. This solves the first part.

Since  $f$  has a quadruple pole at  $z = 1$  and no other poles, we have  $Q(z) = (z - 1)^4$  up to a constant factor which we can as well set equal to 1, and this determines  $P$  uniquely. Since  $f(z) \rightarrow 0$  as  $z \rightarrow \infty$ , the degree of  $P$  is at most 3, and since  $P(0) = 0$ , it follows that  $P(z) = z(az^2 + bz + c)$  for yet undetermined complex constants  $a, b, c$ .

There are a number of ways to compute the coefficients  $a, b, c$ , which turn out to be  $a = c = 1/6$ ,  $b = 2/3$ . Since  $f(z) = f(1/z)$ , it follows easily that  $a = c$ . Moreover, the fact  $\lim_{z \rightarrow 1} (z - 1)^4 f(z) = 1$  implies  $a + b + c = 1$  (this fact follows from the observation that at  $z = 1$ , all summands cancel pairwise, except the principal branch which contributes a quadruple pole). Finally, we can calculate

$$f(-1) = \pi^{-4} \sum_{n \text{ odd}} n^{-4} = 2\pi^{-4} \sum_{n \geq 1 \text{ odd}} n^{-4} = 2\pi^{-4} \left( \sum_{n \geq 1} n^{-4} - \sum_{n \geq 1 \text{ even}} n^{-4} \right) = \frac{1}{48}.$$

This implies  $a - b + c = -1/3$ . These three equations easily yield  $a, b, c$ .

Moreover, the function  $f$  satisfies  $f(z) + f(-z) = 16f(z^2)$ : this follows because the branches of  $\log(z^2) = \log((-z)^2)$  are the numbers  $2\log(z)$  and  $2\log(-z)$ . This observation supplies the two equations  $b = 4a$  and  $a = c$ , which can be used instead of some of the considerations above.

Another way is to compute  $g(z) = \sum \frac{1}{(\log z)^2}$  first. In the same way,  $g(z) = \frac{dz}{(z-1)^2}$ . The unknown coefficient  $d$  can be computed from  $\lim_{z \rightarrow 1} (z - 1)^2 g(z) = 1$ ; it is  $d = 1$ . Then the exponent 2 in the denominator can be increased by taking derivatives (see Solution 2). Similarly, one can start with exponent 3 directly.

A more straightforward, though tedious way to find the constants is computing the first four terms of the Laurent series of  $f$  around  $z = 1$ . For that branch of the logarithm which vanishes at 1, for all  $|w| < \frac{1}{2}$  we have

$$\log(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} - \frac{w^4}{4} + O(|w|^5);$$

after some computation, one can obtain

$$\frac{1}{\log(1 + w)^4} = w^{-4} + 2w^{-2} + \frac{7}{6}w^{-2} + \frac{1}{6}w^{-1} + O(1).$$

The remaining branches of logarithm give a bounded function. So

$$f(1 + w) = w^{-4} + 2w^{-2} + \frac{7}{6}w^{-2} + \frac{1}{6}w^{-1}$$

(the remainder vanishes) and

$$f(z) = \frac{1 + 2(z - 1) + \frac{7}{6}(z - 1)^2 + \frac{1}{6}(z - 1)^3}{(z - 1)^4} = \frac{z(z^2 + 4z + 1)}{6(z - 1)^4}.$$

*Solution 2.* From the well-known series for the cotangent function,

$$\lim_{N \rightarrow \infty} \sum_{k=-N}^N \frac{1}{w + 2\pi i \cdot k} = \frac{i}{2} \cot \frac{iw}{2}$$

and

$$\lim_{N \rightarrow \infty} \sum_{k=-N}^N \frac{1}{\log z + 2\pi i \cdot k} = \frac{i}{2} \cot \frac{i \log z}{2} = \frac{i}{2} \cdot i \frac{e^{2i \cdot \frac{i \log z}{2}} + 1}{e^{2i \cdot \frac{i \log z}{2}} - 1} = \frac{1}{2} + \frac{1}{z - 1}.$$

Taking derivatives we obtain

$$\sum \frac{1}{(\log z)^2} = -z \cdot \left( \frac{1}{2} + \frac{1}{z - 1} \right)' = \frac{z}{(z - 1)^2},$$

$$\sum \frac{1}{(\log z)^3} = -\frac{z}{2} \cdot \left( \frac{z}{(z - 1)^2} \right)' = \frac{z(z + 1)}{2(z - 1)^3}$$

and

$$\sum \frac{1}{(\log z)^4} = -\frac{z}{3} \cdot \left( \frac{z(z + 1)}{2(z - 1)^3} \right)' = \frac{z(z^2 + 4z + 1)}{2(z - 1)^4}.$$

Solutions for problems on Day 2

1. Let  $A$  be a real  $4 \times 2$  matrix and  $B$  be a real  $2 \times 4$  matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Find  $BA$ . [20 points]

*Solution.* Let  $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$  and  $B = (B_1 \ B_2)$  where  $A_1, A_2, B_1, B_2$  are  $2 \times 2$  matrices. Then

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} (B_1 \ B_2) = \begin{pmatrix} A_1 B_1 & A_1 B_2 \\ A_2 B_1 & A_2 B_2 \end{pmatrix}$$

therefore,  $A_1 B_1 = A_2 B_2 = I_2$  and  $A_1 B_2 = A_2 B_1 = -I_2$ . Then  $B_1 = A_1^{-1}$ ,  $B_2 = -A_1^{-1}$  and  $A_2 = B_2^{-1} = -A_1$ . Finally,

$$BA = (B_1 \ B_2) \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = B_1 A_1 + B_2 A_2 = 2I_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

2. Let  $f, g: [a, b] \rightarrow [0, \infty)$  be continuous and non-decreasing functions such that for each  $x \in [a, b]$  we have

$$\int_a^x \sqrt{f(t)} dt \leq \int_a^x \sqrt{g(t)} dt$$

and  $\int_a^b \sqrt{f(t)} dt = \int_a^b \sqrt{g(t)} dt$ .

Prove that  $\int_a^b \sqrt{1+f(t)} dt \geq \int_a^b \sqrt{1+g(t)} dt$ . [20 points]

*Solution.* Let  $F(x) = \int_a^x \sqrt{f(t)} dt$  and  $G(x) = \int_a^x \sqrt{g(t)} dt$ . The functions  $F, G$  are convex,  $F(a) = 0 = G(a)$  and  $F(b) = G(b)$  by the hypothesis. We are supposed to show that

$$\int_a^b \sqrt{1+(F'(t))^2} dt \geq \int_a^b \sqrt{1+(G'(t))^2} dt$$

i.e. The length of the graph of  $F$  is  $\geq$  the length of the graph of  $G$ . This is clear since both functions are convex, their graphs have common ends and the graph of  $F$  is below the graph of  $G$  — the length of the graph of  $F$  is the least upper bound of the lengths of the graphs of piecewise linear functions whose values at the points of non-differentiability coincide with the values of  $F$ , if a convex polygon  $P_1$  is contained in a polygon  $P_2$  then the perimeter of  $P_1$  is  $\leq$  the perimeter of  $P_2$ .

3. Let  $D$  be the closed unit disk in the plane, and let  $p_1, p_2, \dots, p_n$  be fixed points in  $D$ . Show that there exists a point  $p$  in  $D$  such that the sum of the distances of  $p$  to each of  $p_1, p_2, \dots, p_n$  is greater than or equal to 1. [20 points]

*Solution.* considering as vectors, choose  $p$  to be the unit vector which points into the opposite direction as  $\sum_{i=1}^n p_i$ . Then, by the triangle inequality,

$$\sum_{i=1}^n |p - p_i| \geq \left| np - \sum_{i=1}^n p_i \right| = n + \left| \sum_{i=1}^n p_i \right| \geq n..$$

4. For  $n \geq 1$  let  $M$  be an  $n \times n$  complex matrix with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ , with multiplicities  $m_1, m_2, \dots, m_k$ , respectively. Consider the linear operator  $L_M$  defined by  $L_M(X) = MX + XM^T$ , for any complex  $n \times n$  matrix  $X$ . Find its eigenvalues and their multiplicities. ( $M^T$  denotes the transpose of  $M$ ; that is, if  $M = (m_{k,l})$ , then  $M^T = (m_{l,k})$ .) [20 points]

*Solution.* We first solve the problem for the special case when the eigenvalues of  $M$  are distinct and all sums  $\lambda_r + \lambda_s$  are different. Let  $\lambda_r$  and  $\lambda_s$  be two eigenvalues of  $M$  and  $\vec{v}_r, \vec{v}_s$  eigenvectors associated to them, i.e.  $M\vec{v}_j = \lambda_j\vec{v}_j$  for  $j = r, s$ . We have  $M\vec{v}_r(\vec{v}_s)^T + \vec{v}_r(\vec{v}_s)^T M^T = (M\vec{v}_r)(\vec{v}_s)^T + \vec{v}_r(M\vec{v}_s)^T = \lambda_r\vec{v}_r(\vec{v}_s)^T + \lambda_s\vec{v}_r(\vec{v}_s)^T$ , so  $\vec{v}_r(\vec{v}_s)^T$  is an eigenmatrix of  $L_M$  with the eigenvalue  $\lambda_r + \lambda_s$ .

Notice that if  $\lambda_r \neq \lambda_s$  then vectors  $\vec{u}, \vec{w}$  are linearly independent and matrices  $\vec{u}(\vec{w})^T$  and  $\vec{w}(\vec{u})^T$  are linearly independent, too. This implies that the eigenvalue  $\lambda_r + \lambda_s$  is double if  $r \neq s$ .

The map  $L_M$  maps  $n^2$ -dimensional linear space into itself, so it has at most  $n^2$  eigenvalues. We already found  $n^2$  eigenvalues, so there exists no more and the problem is solved for the special case.

In the general case, matrix  $M$  is a limit of matrices  $M_1, M_2, \dots$  such that each of them belongs to the special case above. By the continuity of the eigenvalues we obtain that the eigenvalues of  $L_M$  are

- $2\lambda_r$  with multiplicity  $m_r^2$  ( $r = 1, \dots, k$ );
- $\lambda_r + \lambda_s$  with multiplicity  $2m_r m_s$  ( $1 \leq r < s \leq k$ ).

(It can happen that the sums  $\lambda_r + \lambda_s$  are not pairwise different; for those multiple values the multiplicities should be summed up.)

5. Prove that

$$\int_0^1 \int_0^1 \frac{dx dy}{x^{-1} + |\ln y| - 1} \leq 1. \quad [20 \text{ points}]$$

*Solution 1.* First we use the inequality

$$x^{-1} - 1 \geq |\ln x|, \quad x \in (0, 1],$$

which follows from

$$\begin{aligned} (x^{-1} - 1)|_{x=1} &= |\ln x|_{x=1} = 0, \\ (x^{-1} - 1)' &= -\frac{1}{x^2} \leq -\frac{1}{x} = |\ln x|', \quad x \in (0, 1]. \end{aligned}$$

Therefore

$$\int_0^1 \int_0^1 \frac{dx dy}{x^{-1} + |\ln y| - 1} \leq \int_0^1 \int_0^1 \frac{dx dy}{|\ln x| + |\ln y|} = \int_0^1 \int_0^1 \frac{dx dy}{|\ln(x \cdot y)|}.$$

Substituting  $y = u/x$ , we obtain

$$\int_0^1 \int_0^1 \frac{dx dy}{|\ln(x \cdot y)|} = \int_0^1 \left( \int_u^1 \frac{dx}{x} \right) \frac{du}{|\ln u|} = \int_0^1 |\ln u| \cdot \frac{du}{|\ln u|} = 1.$$

*Solution 2.* Substituting  $s = x^{-1} - 1$  and  $u = s - \ln y$ ,

$$\int_0^1 \int_0^1 \frac{dx dy}{x^{-1} + |\ln y| - 1} = \int_0^\infty \int_s^\infty \frac{e^{s-u}}{(s+1)^2 u} du ds = \int_0^\infty \left( \int_0^u \frac{e^s}{(s+1)^2} ds \right) \frac{e^{-u}}{u} ds du.$$

Since the function  $\frac{e^s}{(s+1)^2}$  is convex,

$$\int_0^u \frac{e^s}{(s+1)^2} ds \leq \frac{u}{2} \left( \frac{e^u}{(u+1)^2} + 1 \right)$$

so

$$\int_0^1 \int_0^1 \frac{dx dy}{x^{-1} + |\ln y| - 1} \leq \int_0^\infty \frac{u}{2} \left( \frac{e^u}{(u+1)^2} + 1 \right) \frac{e^{-u}}{u} du = \frac{1}{2} \left( \int_0^\infty \frac{du}{(u+1)^2} + \int_0^\infty e^{-u} du \right) = 1.$$

6. For  $n \geq 0$  define matrices  $A_n$  and  $B_n$  as follows:  $A_0 = B_0 = (1)$  and for every  $n > 0$

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \text{ and } B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}.$$

Denote the sum of all elements of a matrix  $M$  by  $S(M)$ . Prove that  $S(A_n^{k-1}) = S(A_k^{n-1})$  for every  $n, k \geq 1$ . [20 points]

*Solution.* The quantity  $S(A_n^{k-1})$  has a special combinatorial meaning. Consider an  $n \times k$  table filled with 0's and 1's such that no  $2 \times 2$  contains only 1's. Denote the number of such fillings by  $F_{nk}$ . The filling of each row of the table corresponds to some integer ranging from 0 to  $2^n - 1$  written in base 2.  $F_{nk}$  equals to the number of  $k$ -tuples of integers such that every two consecutive integers correspond to the filling of  $n \times 2$  table without  $2 \times 2$  squares filled with 1's.

Consider binary expansions of integers  $i$  and  $j$   $\overline{i_n i_{n-1} \dots i_1}$  and  $\overline{j_n j_{n-1} \dots j_1}$ . There are two cases:

1. If  $i_n j_n = 0$  then  $i$  and  $j$  can be consecutive iff  $\overline{i_{n-1} \dots i_1}$  and  $\overline{j_{n-1} \dots j_1}$  can be consecutive.
2. If  $i_n = j_n = 1$  then  $i$  and  $j$  can be consecutive iff  $i_{n-1} j_{n-1} = 0$  and  $\overline{i_{n-2} \dots i_1}$  and  $\overline{j_{n-2} \dots j_1}$  can be consecutive.

Hence  $i$  and  $j$  can be consecutive iff  $(i+1, j+1)$ -th entry of  $A_n$  is 1. Denoting this entry by  $a_{i,j}$ , the sum  $S(A_n^{k-1}) = \sum_{i_1=0}^{2^n-1} \dots \sum_{i_k=0}^{2^n-1} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k}$  counts the possible fillings. Therefore  $F_{nk} = S(A_n^{k-1})$ .

The obvious statement  $F_{nk} = F_{kn}$  completes the proof.

12<sup>th</sup> International Mathematics Competition for University Students  
Blagoevgrad, July 22 - July 28, 2005

First Day

**Problem 1.** Let  $A$  be the  $n \times n$  matrix, whose  $(i, j)$ <sup>th</sup> entry is  $i + j$  for all  $i, j = 1, 2, \dots, n$ . What is the rank of  $A$ ?

*Solution 1.* For  $n = 1$  the rank is 1. Now assume  $n \geq 2$ . Since  $A = (i)_{i,j=1}^n + (j)_{i,j=1}^n$ , matrix  $A$  is the sum of two matrixes of rank 1. Therefore, the rank of  $A$  is at most 2. The determinant of the top-left  $2 \times 2$  minor is  $-1$ , so the rank is exactly 2.

Therefore, the rank of  $A$  is 1 for  $n = 1$  and 2 for  $n \geq 2$ .

*Solution 2.* Consider the case  $n \geq 2$ . For  $i = n, n - 1, \dots, 2$ , subtract the  $(i - 1)$ <sup>th</sup> row from the  $n$ <sup>th</sup> row. Then subtract the second row from all lower rows.

$$\text{rank} \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & & \ddots & \vdots \\ n+1 & n+2 & \dots & 2n \end{pmatrix} = \text{rank} \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = \text{rank} \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 2.$$

**Problem 2.** For an integer  $n \geq 3$  consider the sets

$$S_n = \{(x_1, x_2, \dots, x_n) : \forall i \ x_i \in \{0, 1, 2\}\}$$

$$A_n = \{(x_1, x_2, \dots, x_n) \in S_n : \forall i \leq n - 2 \ |\{x_i, x_{i+1}, x_{i+2}\}| \neq 1\}$$

and

$$B_n = \{(x_1, x_2, \dots, x_n) \in S_n : \forall i \leq n - 1 \ (x_i = x_{i+1} \Rightarrow x_i \neq 0)\}.$$

Prove that  $|A_{n+1}| = 3 \cdot |B_n|$ .

( $|A|$  denotes the number of elements of the set  $A$ .)

*Solution 1.* Extend the definitions also for  $n = 1, 2$ . Consider the following sets

$$A'_n = \{(x_1, x_2, \dots, x_n) \in A_n : x_{n-1} = x_n\}, \quad A''_n = A_n \setminus A'_n,$$

$$B'_n = \{(x_1, x_2, \dots, x_n) \in B_n : x_n = 0\}, \quad B''_n = B_n \setminus B'_n$$

and denote  $a_n = |A_n|$ ,  $a'_n = |A'_n|$ ,  $a''_n = |A''_n|$ ,  $b_n = |B_n|$ ,  $b'_n = |B'_n|$ ,  $b''_n = |B''_n|$ .

It is easy to observe the following relations between the  $a$ -sequences

$$\begin{cases} a_n &= a'_n + a''_n \\ a'_{n+1} &= a''_n \\ a''_{n+1} &= 2a'_n + 2a''_n \end{cases},$$

which lead to  $a_{n+1} = 2a_n + 2a_{n-1}$ .

For the  $b$ -sequences we have the same relations

$$\begin{cases} b_n &= b'_n + b''_n \\ b'_{n+1} &= b''_n \\ b''_{n+1} &= 2b'_n + 2b''_n \end{cases},$$

therefore  $b_{n+1} = 2b_n + 2b_{n-1}$ .

By computing the first values of  $(a_n)$  and  $(b_n)$  we obtain

$$\begin{cases} a_1 = 3, & a_2 = 9, & a_3 = 24 \\ & b_1 = 3, & b_2 = 8 \end{cases}$$

which leads to

$$\begin{cases} a_2 = 3b_1 \\ a_3 = 3b_2 \end{cases}$$

Now, reasoning by induction, it is easy to prove that  $a_{n+1} = 3b_n$  for every  $n \geq 1$ .

*Solution 2.* Regarding  $x_i$  to be elements of  $\mathbb{Z}_3$  and working “modulo 3”, we have that

$$(x_1, x_2, \dots, x_n) \in A_n \Rightarrow (x_1 + 1, x_2 + 1, \dots, x_n + 1) \in A_n, (x_1 + 2, x_2 + 2, \dots, x_n + 2) \in A_n$$

which means that  $1/3$  of the elements of  $A_n$  start with 0. We establish a bijection between the subset of all the vectors in  $A_{n+1}$  which start with 0 and the set  $B_n$  by

$$\begin{aligned} (0, x_1, x_2, \dots, x_n) \in A_{n+1} &\longmapsto (y_1, y_2, \dots, y_n) \in B_n \\ y_1 = x_1, y_2 = x_2 - x_1, y_3 = x_3 - x_2, \dots, y_n = x_n - x_{n-1} \end{aligned}$$

(if  $y_k = y_{k+1} = 0$  then  $x_k - x_{k-1} = x_{k+1} - x_k = 0$  (where  $x_0 = 0$ ), which gives  $x_{k-1} = x_k = x_{k+1}$ , which is not possible because of the definition of the sets  $A_p$ ; therefore, the definition of the above function is correct).

The inverse is defined by

$$\begin{aligned} (y_1, y_2, \dots, y_n) \in B_n &\longmapsto (0, x_1, x_2, \dots, x_n) \in A_{n+1} \\ x_1 = y_1, x_2 = y_1 + y_2, \dots, x_n = y_1 + y_2 + \dots + y_n \end{aligned}$$

**Problem 3.** Let  $f : \mathbb{R} \rightarrow [0, \infty)$  be a continuously differentiable function. Prove that

$$\left| \int_0^1 f^3(x) dx - f^2(0) \int_0^1 f(x) dx \right| \leq \max_{0 \leq x \leq 1} |f'(x)| \left( \int_0^1 f(x) dx \right)^2.$$

*Solution 1.* Let  $M = \max_{0 \leq x \leq 1} |f'(x)|$ . By the inequality  $-M \leq f'(x) \leq M$ ,  $x \in [0, 1]$  it follows:

$$-Mf(x) \leq f(x)f'(x) \leq Mf(x), \quad x \in [0, 1].$$

By integration

$$\begin{aligned} -M \int_0^x f(t) dt &\leq \frac{1}{2}f^2(x) - \frac{1}{2}f^2(0) \leq M \int_0^x f(t) dt, \quad x \in [0, 1] \\ -Mf(x) \int_0^x f(t) dt &\leq \frac{1}{2}f^3(x) - \frac{1}{2}f^2(0)f(x) \leq Mf(x) \int_0^x f(t) dt, \quad x \in [0, 1]. \end{aligned}$$

Integrating the last inequality on  $[0, 1]$  it follows that

$$\begin{aligned} -M \left( \int_0^1 f(x) dx \right)^2 &\leq \int_0^1 f^3(x) dx - f^2(0) \int_0^1 f(x) dx \leq M \left( \int_0^1 f(x) dx \right)^2 \Leftrightarrow \\ \left| \int_0^1 f^3(x) dx - f^2(0) \int_0^1 f(x) dx \right| &\leq M \left( \int_0^1 f(x) dx \right)^2. \end{aligned}$$

*Solution 2.* Let  $M = \max_{0 \leq x \leq 1} |f'(x)|$  and  $F(x) = -\int_x^1 f$ ; then  $F' = f$ ,  $F(0) = -\int_0^1 f$  and  $F(1) = 0$ .

Integrating by parts,

$$\begin{aligned} \int_0^1 f^3 &= \int_0^1 f^2 \cdot F' = [f^2 F]_0^1 - \int_0^1 (f^2)' F = \\ &= f^2(1)F(1) - f^2(0)F(0) - \int_0^1 2Fff' = f^2(0) \int_0^1 f - \int_0^1 2Fff'. \end{aligned}$$

Then

$$\left| \int_0^1 f^3(x) dx - f^2(0) \int_0^1 f(x) dx \right| = \left| \int_0^1 2Fff' \right| \leq \int_0^1 2Ff|f'| \leq M \int_0^1 2Ff = M \cdot [F^2]_0^1 = M \left( \int_0^1 f \right)^2.$$

**Problem 4.** Find all polynomials  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  ( $a_n \neq 0$ ) satisfying the following two conditions:

(i)  $(a_0, a_1, \dots, a_n)$  is a permutation of the numbers  $(0, 1, \dots, n)$

and

(ii) all roots of  $P(x)$  are rational numbers.

*Solution 1.* Note that  $P(x)$  does not have any positive root because  $P(x) > 0$  for every  $x > 0$ . Thus, we can represent them in the form  $-\alpha_i$ ,  $i = 1, 2, \dots, n$ , where  $\alpha_i \geq 0$ . If  $a_0 \neq 0$  then there is a  $k \in \mathbb{N}$ ,  $1 \leq k \leq n-1$ , with  $a_k = 0$ , so using Viète's formulae we get

$$\alpha_1 \alpha_2 \dots \alpha_{n-k-1} \alpha_{n-k} + \alpha_1 \alpha_2 \dots \alpha_{n-k-1} \alpha_{n-k+1} + \dots + \alpha_{k+1} \alpha_{k+2} \dots \alpha_{n-1} \alpha_n = \frac{a_k}{a_n} = 0,$$

which is impossible because the left side of the equality is positive. Therefore  $a_0 = 0$  and one of the roots of the polynomial, say  $\alpha_n$ , must be equal to zero. Consider the polynomial  $Q(x) = a_n x^{n-1} + a_{n-1} x^{n-2} + \dots + a_1$ . It has zeros  $-\alpha_i$ ,  $i = 1, 2, \dots, n-1$ . Again, Viète's formulae, for  $n \geq 3$ , yield:

$$\alpha_1 \alpha_2 \dots \alpha_{n-1} = \frac{a_1}{a_n} \quad (1)$$

$$\alpha_1 \alpha_2 \dots \alpha_{n-2} + \alpha_1 \alpha_2 \dots \alpha_{n-3} \alpha_{n-1} + \dots + \alpha_2 \alpha_3 \dots \alpha_{n-1} = \frac{a_2}{a_n} \quad (2)$$

$$\alpha_1 + \alpha_2 + \dots + \alpha_{n-1} = \frac{a_{n-1}}{a_n}. \quad (3)$$

Dividing (2) by (1) we get

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \dots + \frac{1}{\alpha_{n-1}} = \frac{a_2}{a_1}. \quad (4)$$

From (3) and (4), applying the AM-HM inequality we obtain

$$\frac{a_{n-1}}{(n-1)a_n} = \frac{\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}}{n-1} \geq \frac{n-1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \dots + \frac{1}{\alpha_{n-1}}} = \frac{(n-1)a_1}{a_2},$$

therefore  $\frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$ . Hence  $\frac{n^2}{2} \geq \frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$ , implying  $n \leq 3$ . So, the only polynomials possibly satisfying (i) and (ii) are those of degree at most three. These polynomials can easily be found and they are  $P(x) = x$ ,  $P(x) = x^2 + 2x$ ,  $P(x) = 2x^2 + x$ ,  $P(x) = x^3 + 3x^2 + 2x$  and  $P(x) = 2x^3 + 3x^2 + x$ .  $\square$

*Solution 2.* Consider the prime factorization of  $P$  in the ring  $\mathbb{Z}[x]$ . Since all roots of  $P$  are rational,  $P$  can be written as a product of  $n$  linear polynomials with rational coefficients. Therefore, all prime factor of  $P$  are linear and  $P$  can be written as

$$P(x) = \prod_{k=1}^n (b_k x + c_k)$$

where the coefficients  $b_k, c_k$  are integers. Since the leading coefficient of  $P$  is positive, we can assume  $b_k > 0$  for all  $k$ . The coefficients of  $P$  are nonnegative, so  $P$  cannot have a positive root. This implies  $c_k \geq 0$ . It is not possible that  $c_k = 0$  for two different values of  $k$ , because it would imply  $a_0 = a_1 = 0$ . So  $c_k > 0$  in at least  $n-1$  cases.

Now substitute  $x = 1$ .

$$P(1) = a_n + \dots + a_0 = 0 + 1 + \dots + n = \frac{n(n+1)}{2} = \prod_{k=1}^n (b_k + c_k) \geq 2^{n-1};$$

therefore it is necessary that  $2^{n-1} \leq \frac{n(n+1)}{2}$ , therefore  $n \leq 4$ . Moreover, the number  $\frac{n(n+1)}{2}$  can be written as a product of  $n-1$  integers greater than 1.

If  $n = 1$ , the only solution is  $P(x) = 1x + 0$ .

If  $n = 2$ , we have  $P(1) = 3 = 1 \cdot 3$ , so one factor must be  $x$ , the other one is  $x + 2$  or  $2x + 1$ . Both  $x(x + 2) = 1x^2 + 2x + 0$  and  $x(2x + 1) = 2x^2 + 1x + 0$  are solutions.

If  $n = 3$ , then  $P(1) = 6 = 1 \cdot 2 \cdot 3$ , so one factor must be  $x$ , another one is  $x+1$ , the third one is again  $x+2$  or  $2x+1$ . The two polynomials are  $x(x+1)(x+2) = 1x^3+3x^2+2x+0$  and  $x(x+1)(2x+1) = 2x^3+3x^2+1x+0$ , both have the proper set of coefficients.

In the case  $n = 4$ , there is no solution because  $\frac{n(n+1)}{2} = 10$  cannot be written as a product of 3 integers greater than 1.

Altogether we found 5 solutions:  $1x+0$ ,  $1x^2+2x+0$ ,  $2x^2+1x+0$ ,  $1x^3+3x^2+2x+0$  and  $2x^3+3x^2+1x+0$ .

**Problem 5.** Let  $f : (0, \infty) \rightarrow \mathbb{R}$  be a twice continuously differentiable function such that

$$|f''(x) + 2xf'(x) + (x^2 + 1)f(x)| \leq 1$$

for all  $x$ . Prove that  $\lim_{x \rightarrow \infty} f(x) = 0$ .

*Solution 1.* Let  $g(x) = f'(x) + xf(x)$ ; then  $f''(x) + 2xf'(x) + (x^2 + 1)f(x) = g'(x) + xg(x)$ .

We prove that if  $h$  is a continuously differentiable function such that  $h'(x) + xh(x)$  is bounded then  $\lim_{x \rightarrow \infty} h = 0$ . Applying this lemma for  $h = g$  then for  $h = f$ , the statement follows.

Let  $M$  be an upper bound for  $|h'(x) + xh(x)|$  and let  $p(x) = h(x)e^{x^2/2}$ . (The function  $e^{-x^2/2}$  is a solution of the differential equation  $u'(x) + xu(x) = 0$ .) Then

$$|p'(x)| = |h'(x) + xh(x)|e^{x^2/2} \leq Me^{x^2/2}$$

and

$$|h(x)| = \left| \frac{p(x)}{e^{x^2/2}} \right| = \left| \frac{p(0) + \int_0^x p'}{e^{x^2/2}} \right| \leq \frac{|p(0)| + M \int_0^x e^{x^2/2} dx}{e^{x^2/2}}.$$

Since  $\lim_{x \rightarrow \infty} e^{x^2/2} = \infty$  and  $\lim_{x \rightarrow \infty} \frac{\int_0^x e^{x^2/2} dx}{e^{x^2/2}} = 0$  (by L'Hospital's rule), this implies  $\lim_{x \rightarrow \infty} h(x) = 0$ .

*Solution 2.* Apply L'Hospital rule twice on the fraction  $\frac{f(x)e^{x^2/2}}{e^{x^2/2}}$ . (Note that L'Hospital rule is valid if the denominator converges to infinity, without any assumption on the numerator.)

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{f(x)e^{x^2/2}}{e^{x^2/2}} = \lim_{x \rightarrow \infty} \frac{(f'(x) + xf(x))e^{x^2/2}}{xe^{x^2/2}} = \lim_{x \rightarrow \infty} \frac{(f''(x) + 2xf'(x) + (x^2 + 1)f(x))e^{x^2/2}}{(x^2 + 1)e^{x^2/2}} = \\ &= \lim_{x \rightarrow \infty} \frac{f''(x) + 2xf'(x) + (x^2 + 1)f(x)}{x^2 + 1} = 0. \end{aligned}$$

**Problem 6.** Given a group  $G$ , denote by  $G(m)$  the subgroup generated by the  $m^{\text{th}}$  powers of elements of  $G$ . If  $G(m)$  and  $G(n)$  are commutative, prove that  $G(\gcd(m, n))$  is also commutative. ( $\gcd(m, n)$  denotes the greatest common divisor of  $m$  and  $n$ .)

*Solution.* Write  $d = \gcd(m, n)$ . It is easy to see that  $\langle G(m), G(n) \rangle = G(d)$ ; hence, it will suffice to check commutativity for any two elements in  $G(m) \cup G(n)$ , and so for any two generators  $a^m$  and  $b^n$ . Consider their commutator  $z = a^{-m}b^{-n}a^mb^n$ ; then the relations

$$z = (a^{-m}ba^m)^{-n}b^n = a^{-m}(b^{-n}ab^n)^m$$

show that  $z \in G(m) \cap G(n)$ . But then  $z$  is in the center of  $G(d)$ . Now, from the relation  $a^mb^n = b^na^mz$ , it easily follows by induction that

$$a^{ml}b^{nl} = b^{nl}a^{ml}z^{l^2}.$$

Setting  $l = m/d$  and  $l = n/d$  we obtain  $z^{(m/d)^2} = z^{(n/d)^2} = e$ , but this implies that  $z = e$  as well.

12<sup>th</sup> International Mathematics Competition for University Students

Blagoevgrad, July 22 - July 28, 2005

Second Day

**Problem 1.** Let  $f(x) = x^2 + bx + c$ , where  $b$  and  $c$  are real numbers, and let

$$M = \{x \in \mathbb{R} : |f(x)| < 1\}.$$

Clearly the set  $M$  is either empty or consists of disjoint open intervals. Denote the sum of their lengths by  $|M|$ . Prove that

$$|M| \leq 2\sqrt{2}.$$

*Solution.* Write  $f(x) = (x + \frac{b}{2})^2 + d$  where  $d = c - \frac{b^2}{4}$ . The absolute minimum of  $f$  is  $d$ .

If  $d \geq 1$  then  $f(x) \geq 1$  for all  $x$ ,  $M = \emptyset$  and  $|M| = 0$ .

If  $-1 < d < 1$  then  $f(x) > -1$  for all  $x$ ,

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \iff \left|x + \frac{b}{2}\right| < \sqrt{1-d}$$

so

$$M = \left(-\frac{b}{2} - \sqrt{1-d}, -\frac{b}{2} + \sqrt{1-d}\right)$$

and

$$|M| = 2\sqrt{1-d} < 2\sqrt{2}.$$

If  $d \leq -1$  then

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \iff \sqrt{|d|-1} < \left|x + \frac{b}{2}\right| < \sqrt{|d|+1}$$

so

$$M = (-\sqrt{|d|+1}, -\sqrt{|d|-1}) \cup (\sqrt{|d|-1}, \sqrt{|d|+1})$$

and

$$|M| = 2\left(\sqrt{|d|+1} - \sqrt{|d|-1}\right) = 2\frac{(|d|+1) - (|d|-1)}{\sqrt{|d|+1} + \sqrt{|d|-1}} \leq 2\frac{2}{\sqrt{1+1} + \sqrt{1-0}} = 2\sqrt{2}.$$

**Problem 2.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $(f(x))^n$  is a polynomial for every  $n = 2, 3, \dots$ . Does it follow that  $f$  is a polynomial?

*Solution 1.* Yes, it is even enough to assume that  $f^2$  and  $f^3$  are polynomials.

Let  $p = f^2$  and  $q = f^3$ . Write these polynomials in the form of

$$p = a \cdot p_1^{a_1} \cdot \dots \cdot p_k^{a_k}, \quad q = b \cdot q_1^{b_1} \cdot \dots \cdot q_l^{b_l},$$

where  $a, b \in \mathbb{R}$ ,  $a_1, \dots, a_k, b_1, \dots, b_l$  are positive integers and  $p_1, \dots, p_k, q_1, \dots, q_l$  are irreducible polynomials with leading coefficients 1. For  $p^3 = q^2$  and the factorisation of  $p^3 = q^2$  is unique we get that  $a^3 = b^2$ ,  $k = l$  and for some  $(i_1, \dots, i_k)$  permutation of  $(1, \dots, k)$  we have  $p_1 = q_{i_1}, \dots, p_k = q_{i_k}$  and  $3a_1 = 2b_{i_1}, \dots, 3a_k = 2b_{i_k}$ . Hence  $b_1, \dots, b_l$  are divisible by 3 let  $r = b^{1/3} \cdot q_1^{b_1/3} \cdot \dots \cdot q_l^{b_l/3}$  be a polynomial. Since  $r^3 = q = f^3$  we have  $f = r$ .

*Solution 2.* Let  $\frac{p}{q}$  be the simplest form of the rational function  $\frac{f^3}{f^2}$ . Then the simplest form of its square is  $\frac{p^2}{q^2}$ . On the other hand  $\frac{p^2}{q^2} = \left(\frac{f^3}{f^2}\right)^2 = f^2$  is a polynomial therefore  $q$  must be a constant and so  $f = \frac{f^3}{f^2} = \frac{p}{q}$  is a polynomial.

**Problem 3.** In the linear space of all real  $n \times n$  matrices, find the maximum possible dimension of a linear subspace  $V$  such that

$$\forall X, Y \in V \quad \text{trace}(XY) = 0.$$

(The trace of a matrix is the sum of the diagonal entries.)

*Solution.* If  $A$  is a nonzero symmetric matrix, then  $\text{trace}(A^2) = \text{trace}(A^t A)$  is the sum of the squared entries of  $A$  which is positive. So  $V$  cannot contain any symmetric matrix but 0.

Denote by  $S$  the linear space of all real  $n \times n$  symmetric matrices;  $\dim V = \frac{n(n+1)}{2}$ . Since  $V \cap S = \{0\}$ , we have  $\dim V + \dim S \leq n^2$  and thus  $\dim V \leq n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ .

The space of strictly upper triangular matrices has dimension  $\frac{n(n-1)}{2}$  and satisfies the condition of the problem.

Therefore the maximum dimension of  $V$  is  $\frac{n(n-1)}{2}$ .

**Problem 4.** Prove that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is three times differentiable, then there exists a real number  $\xi \in (-1, 1)$  such that

$$\frac{f'''(\xi)}{6} = \frac{f(1) - f(-1)}{2} - f'(0).$$

*Solution 1.* Let

$$g(x) = -\frac{f(-1)}{2}x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2}x^2(x+1) - f'(0)x(x-1)(x+1).$$

It is easy to check that  $g(\pm 1) = f(\pm 1)$ ,  $g(0) = f(0)$  and  $g'(0) = f'(0)$ .

Apply Rolle's theorem for the function  $h(x) = f(x) - g(x)$  and its derivatives. Since  $h(-1) = h(0) = h(1) = 0$ , there exist  $\eta \in (-1, 0)$  and  $\vartheta \in (0, 1)$  such that  $h'(\eta) = h'(\vartheta) = 0$ . We also have  $h'(0) = 0$ , so there exist  $\varrho \in (\eta, 0)$  and  $\sigma \in (0, \vartheta)$  such that  $h''(\varrho) = h''(\sigma) = 0$ . Finally, there exists a  $\xi \in (\varrho, \sigma) \subset (-1, 1)$  where  $h'''(\xi) = 0$ . Then

$$f'''(\xi) = g'''(\xi) = -\frac{f(-1)}{2} \cdot 6 - f(0) \cdot 0 + \frac{f(1)}{2} \cdot 6 - f'(0) \cdot 6 = \frac{f(1) - f(-1)}{2} - f'(0).$$

*Solution 2.* The expression  $\frac{f(1) - f(-1)}{2} - f'(0)$  is the divided difference  $f[-1, 0, 0, 1]$  and there exists a number  $\xi \in (-1, 1)$  such that  $f[-1, 0, 0, 1] = \frac{f'''(\xi)}{3!}$ .

**Problem 5.** Find all  $r > 0$  such that whenever  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a differentiable function such that  $|\text{grad } f(0, 0)| = 1$  and  $|\text{grad } f(u) - \text{grad } f(v)| \leq |u - v|$  for all  $u, v \in \mathbb{R}^2$ , then the maximum of  $f$  on the disk  $\{u \in \mathbb{R}^2 : |u| \leq r\}$  is attained at exactly one point. ( $\text{grad } f(u) = (\partial_1 f(u), \partial_2 f(u))$  is the gradient vector of  $f$  at the point  $u$ . For a vector  $u = (a, b)$ ,  $|u| = \sqrt{a^2 + b^2}$ .)

*Solution.* To get an upper bound for  $r$ , set  $f(x, y) = x - \frac{x^2}{2} + \frac{y^2}{2}$ . This function satisfies the conditions, since  $\text{grad } f(x, y) = (1 - x, y)$ ,  $\text{grad } f(0, 0) = (1, 0)$  and  $|\text{grad } f(x_1, y_1) - \text{grad } f(x_2, y_2)| = |(x_2 - x_1, y_1 - y_2)| = |(x_1, y_1) - (x_2, y_2)|$ .

In the disk  $D_r = \{(x, y) : x^2 + y^2 \leq r^2\}$

$$f(x, y) = \frac{x^2 + y^2}{2} - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \leq \frac{r^2}{2} + \frac{1}{4}.$$

If  $r > \frac{1}{2}$  then the absolute maximum is  $\frac{r^2}{2} + \frac{1}{4}$ , attained at the points  $\left(\frac{1}{2}, \pm\sqrt{r^2 - \frac{1}{4}}\right)$ . Therefore, it is necessary that  $r \leq \frac{1}{2}$  because if  $r > \frac{1}{2}$  then the maximum is attained twice.

Suppose now that  $r \leq 1/2$  and that  $f$  attains its maximum on  $D_r$  at  $u, v$ ,  $u \neq v$ . Since  $|\text{grad } f(z) - \text{grad } f(0)| \leq r$ ,  $|\text{grad } f(z)| \geq 1 - r > 0$  for all  $z \in D_r$ . Hence  $f$  may attain its maximum only at the boundary of  $D_r$ , so we must have  $|u| = |v| = r$  and  $\text{grad } f(u) = au$  and  $\text{grad } f(v) = bv$ , where  $a, b \geq 0$ . Since  $au = \text{grad } f(u)$  and  $bv = \text{grad } f(v)$  belong to the disk  $D$  with centre  $\text{grad } f(0)$  and radius  $r$ , they do not belong to the interior of  $D_r$ . Hence  $|\text{grad } f(u) - \text{grad } f(v)| = |au - bv| \geq |u - v|$  and this inequality is strict since  $D \cap D_r$  contains no more than one point. But this contradicts the assumption that  $|\text{grad } f(u) - \text{grad } f(v)| \leq |u - v|$ . So all  $r \leq \frac{1}{2}$  satisfies the condition.

**Problem 6.** Prove that if  $p$  and  $q$  are rational numbers and  $r = p + q\sqrt{7}$ , then there exists a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  with integer entries and with  $ad - bc = 1$  such that

$$\frac{ar + b}{cr + d} = r.$$

*Solution.* First consider the case when  $q = 0$  and  $r$  is rational. Choose a positive integer  $t$  such that  $r^2 t$  is an integer and set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + rt & -r^2 t \\ t & 1 - rt \end{pmatrix}.$$

Then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \quad \text{and} \quad \frac{ar + b}{cr + d} = \frac{(1 + rt)r - r^2 t}{tr + (1 - rt)} = r.$$

Now assume  $q \neq 0$ . Let the minimal polynomial of  $r$  in  $\mathbb{Z}[x]$  be  $ux^2 + vx + w$ . The other root of this polynomial is  $\bar{r} = p - q\sqrt{7}$ , so  $v = -u(r + \bar{r}) = -2up$  and  $w = ur\bar{r} = u(p^2 - 7q^2)$ . The discriminant is  $v^2 - 4uw = 7 \cdot (2uq)^2$ . The left-hand side is an integer, implying that also  $\Delta = 2uq$  is an integer.

The equation  $\frac{ar+b}{cr+d} = r$  is equivalent to  $cr^2 + (d-a)r - b = 0$ . This must be a multiple of the minimal polynomial, so we need

$$c = ut, \quad d - a = vt, \quad -b = wt$$

for some integer  $t \neq 0$ . Putting together these equalities with  $ad - bc = 1$  we obtain that

$$(a + d)^2 = (a - d)^2 + 4ad = 4 + (v^2 - 4uw)t^2 = 4 + 7\Delta^2 t^2.$$

Therefore  $4 + 7\Delta^2 t^2$  must be a perfect square. Introducing  $s = a + d$ , we need an integer solution  $(s, t)$  for the Diophantine equation

$$s^2 - 7\Delta^2 t^2 = 4 \tag{1}$$

such that  $t \neq 0$ .

The numbers  $s$  and  $t$  will be even. Then  $a + d = s$  and  $d - a = vt$  will be even as well and  $a$  and  $d$  will be really integers.

Let  $(8 \pm 3\sqrt{7})^n = k_n \pm l_n \sqrt{7}$  for each integer  $n$ . Then  $k_n^2 - 7l_n^2 = (k_n + l_n \sqrt{7})(k_n - l_n \sqrt{7}) = ((8 + 3\sqrt{7})^n (8 - 3\sqrt{7})^n) = 1$  and the sequence  $(l_n)$  also satisfies the linear recurrence  $l_{n+1} = 16l_n - l_{n-1}$ . Consider the residue of  $l_n$  modulo  $\Delta$ . There are  $\Delta^2$  possible residue pairs for  $(l_n, l_{n+1})$  so some are the same. Starting from such two positions, the recurrence shows that the sequence of residues is periodic in both directions. Then there are infinitely many indices such that  $l_n \equiv l_0 = 0 \pmod{\Delta}$ .

Taking such an index  $n$ , we can set  $s = 2k_n$  and  $t = 2l_n/\Delta$ .

*Remarks.* 1. It is well-known that if  $D > 0$  is not a perfect square then the Pell-like Diophantine equation

$$x^2 - Dy^2 = 1$$

has infinitely many solutions. Using this fact the solution can be generalized to all quadratic algebraic numbers.

2. It is also known that the continued fraction of a real number  $r$  is periodic from a certain point if and only if  $r$  is a root of a quadratic equation. This fact can lead to another solution.

13<sup>th</sup> International Mathematics Competition for University Students

Odessa, July 20-26, 2006

First Day

**Problem 1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a real function. Prove or disprove each of the following statements.

- (a) If  $f$  is continuous and  $\text{range}(f) = \mathbb{R}$  then  $f$  is monotonic.
- (b) If  $f$  is monotonic and  $\text{range}(f) = \mathbb{R}$  then  $f$  is continuous.
- (c) If  $f$  is monotonic and  $f$  is continuous then  $\text{range}(f) = \mathbb{R}$ .

(20 points)

**Solution.** (a) False. Consider function  $f(x) = x^3 - x$ . It is continuous,  $\text{range}(f) = \mathbb{R}$  but, for example,  $f(0) = 0$ ,  $f(\frac{1}{2}) = -\frac{3}{8}$  and  $f(1) = 0$ , therefore  $f(0) > f(\frac{1}{2})$ ,  $f(\frac{1}{2}) < f(1)$  and  $f$  is not monotonic.

(b) True. Assume first that  $f$  is non-decreasing. For an arbitrary number  $a$ , the limits  $\lim_{a^-} f$  and  $\lim_{a^+} f$  exist and  $\lim_{a^-} f \leq \lim_{a^+} f$ . If the two limits are equal, the function is continuous at  $a$ . Otherwise, if  $\lim_{a^-} f = b < \lim_{a^+} f = c$ , we have  $f(x) \leq b$  for all  $x < a$  and  $f(x) \geq c$  for all  $x > a$ ; therefore  $\text{range}(f) \subset (-\infty, b) \cup (c, \infty) \cup \{f(a)\}$  cannot be the complete  $\mathbb{R}$ .

For non-increasing  $f$  the same can be applied writing reverse relations or  $g(x) = -f(x)$ .

(c) False. The function  $g(x) = \arctan x$  is monotonic and continuous, but  $\text{range}(g) = (-\pi/2, \pi/2) \neq \mathbb{R}$ .

**Problem 2.** Find the number of positive integers  $x$  satisfying the following two conditions:

1.  $x < 10^{2006}$ ;
2.  $x^2 - x$  is divisible by  $10^{2006}$ .

(20 points)

**Solution 1.** Let  $S_k = \{0 < x < 10^k \mid x^2 - x \text{ is divisible by } 10^k\}$  and  $s(k) = |S_k|$ ,  $k \geq 1$ . Let  $x = a_{k+1}a_k \dots a_1$  be the decimal writing of an integer  $x \in S_{k+1}$ ,  $k \geq 1$ . Then obviously  $y = a_k \dots a_1 \in S_k$ . Now, let  $y = a_k \dots a_1 \in S_k$  be fixed. Considering  $a_{k+1}$  as a variable digit, we have  $x^2 - x = (a_{k+1}10^k + y)^2 - (a_{k+1}10^k + y) = (y^2 - y) + a_{k+1}10^k(2y - 1) + a_{k+1}^2 10^{2k}$ . Since  $y^2 - y = 10^k z$  for an integer  $z$ , it follows that  $x^2 - x$  is divisible by  $10^{k+1}$  if and only if  $z + a_{k+1}(2y - 1) \equiv 0 \pmod{10}$ . Since  $y \equiv 3 \pmod{10}$  is obviously impossible, the congruence has exactly one solution. Hence we obtain a one-to-one correspondence between the sets  $S_{k+1}$  and  $S_k$  for every  $k \geq 1$ . Therefore  $s(2006) = s(1) = 3$ , because  $S_1 = \{1, 5, 6\}$ .

**Solution 2.** Since  $x^2 - x = x(x - 1)$  and the numbers  $x$  and  $x - 1$  are relatively prime, one of them must be divisible by  $2^{2006}$  and one of them (may be the same) must be divisible by  $5^{2006}$ . Therefore,  $x$  must satisfy the following two conditions:

$$\begin{aligned} x &\equiv 0 \text{ or } 1 \pmod{2^{2006}}; \\ x &\equiv 0 \text{ or } 1 \pmod{5^{2006}}. \end{aligned}$$

Altogether we have 4 cases. The Chinese remainder theorem yields that in each case there is a unique solution among the numbers  $0, 1, \dots, 10^{2006} - 1$ . These four numbers are different because each two gives different residues modulo  $2^{2006}$  or  $5^{2006}$ . Moreover, one of the numbers is 0 which is not allowed.

Therefore there exist 3 solutions.

**Problem 3.** Let  $A$  be an  $n \times n$ -matrix with integer entries and  $b_1, \dots, b_k$  be integers satisfying  $\det A = b_1 \cdot \dots \cdot b_k$ . Prove that there exist  $n \times n$ -matrices  $B_1, \dots, B_k$  with integer entries such that  $A = B_1 \cdot \dots \cdot B_k$  and  $\det B_i = b_i$  for all  $i = 1, \dots, k$ .

(20 points)

**Solution.** By induction, it is enough to consider the case  $m = 2$ . Furthermore, we can multiply  $A$  with any integral matrix with determinant 1 from the right or from the left, without changing the problem. Hence we can assume  $A$  to be upper triangular.

*Lemma.* Let  $A$  be an integral upper triangular matrix, and let  $b, c$  be integers satisfying  $\det A = bc$ . Then there exist integral upper triangular matrices  $B, C$  such that  $\det B = b$ ,  $\det C = c$ ,  $A = BC$ .

*Proof.* The proof is done by induction on  $n$ , the case  $n = 1$  being obvious. Assume the statement is true for  $n - 1$ . Let  $A, b, c$  as in the statement of the lemma. Define  $B_{nn}$  to be the greatest common divisor of  $b$  and  $A_{nn}$ , and put  $C_{nn} = \frac{A_{nn}}{B_{nn}}$ . Since  $A_{nn}$  divides  $bc$ ,  $C_{nn}$  divides  $\frac{b}{B_{nn}}c$ , which divides  $c$ . Hence  $C_{nn}$  divides  $c$ . Therefore,  $b' = \frac{b}{B_{nn}}$  and  $c' = \frac{c}{C_{nn}}$  are integers. Define  $A'$  to be the upper-left  $(n - 1) \times (n - 1)$ -submatrix of  $A$ ; then  $\det A' = b'c'$ . By induction we can find the upper-left  $(n - 1) \times (n - 1)$ -part of  $B$  and  $C$  in such a way that  $\det B = b$ ,  $\det C = c$  and  $A = BC$  holds on the upper-left  $(n - 1) \times (n - 1)$ -submatrix of  $A$ . It remains to define  $B_{i,n}$  and  $C_{i,n}$  such that  $A = BC$  also holds for the  $(i, n)$ -th entry for all  $i < n$ .

First we check that  $B_{ii}$  and  $C_{nn}$  are relatively prime for all  $i < n$ . Since  $B_{ii}$  divides  $b'$ , it is certainly enough to prove that  $b'$  and  $C_{nn}$  are relatively prime, i.e.

$$\gcd\left(\frac{b}{\gcd(b, A_{nn})}, \frac{A_{nn}}{\gcd(b, A_{nn})}\right) = 1,$$

which is obvious. Now we define  $B_{j,n}$  and  $C_{j,n}$  inductively: Suppose we have defined  $B_{i,n}$  and  $C_{i,n}$  for all  $i = j + 1, j + 2, \dots, n - 1$ . Then  $B_{j,n}$  and  $C_{j,n}$  have to satisfy

$$A_{j,n} = B_{j,j}C_{j,n} + B_{j,j+1}C_{j+1,n} + \dots + B_{j,n}C_{n,n}$$

Since  $B_{j,j}$  and  $C_{n,n}$  are relatively prime, we can choose integers  $C_{j,n}$  and  $B_{j,n}$  such that this equation is satisfied. Doing this step by step for all  $j = n - 1, n - 2, \dots, 1$ , we finally get  $B$  and  $C$  such that  $A = BC$ .  $\square$

**Problem 4.** Let  $f$  be a rational function (i.e. the quotient of two real polynomials) and suppose that  $f(n)$  is an integer for infinitely many integers  $n$ . Prove that  $f$  is a polynomial. (20 points)

**Solution.** Let  $S$  be an infinite set of integers such that rational function  $f(x)$  is integral for all  $x \in S$ .

Suppose that  $f(x) = p(x)/q(x)$  where  $p$  is a polynomial of degree  $k$  and  $q$  is a polynomial of degree  $n$ . Then  $p, q$  are solutions to the simultaneous equations  $p(x) = q(x)f(x)$  for all  $x \in S$  that are not roots of  $q$ . These are linear simultaneous equations in the coefficients of  $p, q$  with rational coefficients. Since they have a solution, they have a rational solution.

Thus there are polynomials  $p', q'$  with rational coefficients such that  $p'(x) = q'(x)f(x)$  for all  $x \in S$  that are not roots of  $q$ . Multiplying this with the previous equation, we see that  $p'(x)q(x)f(x) = p(x)q'(x)f(x)$  for all  $x \in S$  that are not roots of  $q$ . If  $x$  is not a root of  $p$  or  $q$ , then  $f(x) \neq 0$ , and hence  $p'(x)q(x) = p(x)q'(x)$  for all  $x \in S$  except for finitely many roots of  $p$  and  $q$ . Thus the two polynomials  $p'q$  and  $pq'$  are equal for infinitely many choices of value. Thus  $p'(x)q(x) = p(x)q'(x)$ . Dividing by  $q(x)q'(x)$ , we see that  $p'(x)/q'(x) = p(x)/q(x) = f(x)$ . Thus  $f(x)$  can be written as the quotient of two polynomials with rational coefficients. Multiplying up by some integer, it can be written as the quotient of two polynomials with integer coefficients.

Suppose  $f(x) = p''(x)/q''(x)$  where  $p''$  and  $q''$  both have integer coefficients. Then by Euler's division algorithm for polynomials, there exist polynomials  $s$  and  $r$ , both of which have rational coefficients such that  $p''(x) = q''(x)s(x) + r(x)$  and the degree of  $r$  is less than the degree of  $q''$ . Dividing by  $q''(x)$ , we get that  $f(x) = s(x) + r(x)/q''(x)$ . Now there exists an integer  $N$  such that  $Ns(x)$  has integral coefficients. Then  $Nf(x) - Ns(x)$  is an integer for all  $x \in S$ . However, this is equal to the rational function  $Nr/q''$ , which has a higher degree denominator than numerator, so tends to 0 as  $x$  tends to  $\infty$ . Thus for all sufficiently large  $x \in S$ ,  $Nf(x) - Ns(x) = 0$  and hence  $r(x) = 0$ . Thus  $r$  has infinitely many roots, and is 0. Thus  $f(x) = s(x)$ , so  $f$  is a polynomial.

**Problem 5.** Let  $a, b, c, d, e > 0$  be real numbers such that  $a^2 + b^2 + c^2 = d^2 + e^2$  and  $a^4 + b^4 + c^4 = d^4 + e^4$ . Compare the numbers  $a^3 + b^3 + c^3$  and  $d^3 + e^3$ . (20 points)

**Solution.** Without loss of generality  $a \geq b \geq c, d \geq e$ . Let  $c^2 = e^2 + \Delta, \Delta \in \mathbb{R}$ . Then  $d^2 = a^2 + b^2 + \Delta$  and the second equation implies

$$a^4 + b^4 + (e^2 + \Delta)^2 = (a^2 + b^2 + \Delta)^2 + e^4, \quad \Delta = -\frac{a^2b^2}{a^2+b^2-e^2}.$$

(Here  $a^2 + b^2 - e^2 \geq \frac{2}{3}(a^2 + b^2 + c^2) - \frac{1}{2}(d^2 + e^2) = \frac{1}{6}(d^2 + e^2) > 0$ .)

Since  $c^2 = e^2 - \frac{a^2b^2}{a^2+b^2-e^2} = \frac{(a^2-e^2)(e^2-b^2)}{a^2+b^2-e^2} > 0$  then  $a > e > b$ .

Therefore  $d^2 = a^2 + b^2 - \frac{a^2b^2}{a^2+b^2-e^2} < a^2$  and  $a > d \geq e > b \geq c$ .

Consider a function  $f(x) = a^x + b^x + c^x - d^x - e^x, x \in \mathbb{R}$ . We shall prove that  $f(x)$  has only two zeroes  $x = 2$  and  $x = 4$  and changes the sign at these points. Suppose the contrary. Then Rolle's theorem implies that  $f'(x)$  has at least two distinct zeroes. Without loss of generality  $a = 1$ . Then  $f'(x) = \ln b \cdot b^x + \ln c \cdot c^x - \ln d \cdot d^x - \ln e \cdot e^x, x \in \mathbb{R}$ . If  $f'(x_1) = f'(x_2) = 0, x_1 < x_2$ , then

$$\ln b \cdot b^{x_i} + \ln c \cdot c^{x_i} = \ln d \cdot d^{x_i} + \ln e \cdot e^{x_i}, \quad i = 1, 2,$$

but since  $1 > d \geq e > b \geq c$  we have

$$\frac{(-\ln b) \cdot b^{x_2} + (-\ln c) \cdot c^{x_2}}{(-\ln b) \cdot b^{x_1} + (-\ln c) \cdot c^{x_1}} \leq b^{x_2-x_1} < e^{x_2-x_1} \leq \frac{(-\ln d) \cdot d^{x_2} + (-\ln e) \cdot e^{x_2}}{(-\ln d) \cdot d^{x_1} + (-\ln e) \cdot e^{x_1}},$$

a contradiction. Therefore  $f(x)$  has a constant sign at each of the intervals  $(-\infty, 2), (2, 4)$  and  $(4, \infty)$ . Since  $f(0) = 1$  then  $f(x) > 0, x \in (-\infty, 2) \cup (4, \infty)$  and  $f(x) < 0, x \in (2, 4)$ . In particular,  $f(3) = a^3 + b^3 + c^3 - d^3 - e^3 < 0$ .

**Problem 6.** Find all sequences  $a_0, a_1, \dots, a_n$  of real numbers where  $n \geq 1$  and  $a_n \neq 0$ , for which the following statement is true:

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is an  $n$  times differentiable function and  $x_0 < x_1 < \dots < x_n$  are real numbers such that  $f(x_0) = f(x_1) = \dots = f(x_n) = 0$  then there exists an  $h \in (x_0, x_n)$  for which

$$a_0f(h) + a_1f'(h) + \dots + a_n f^{(n)}(h) = 0.$$

(20 points)

**Solution.** Let  $A(x) = a_0 + a_1x + \dots + a_nx^n$ . We prove that sequence  $a_0, \dots, a_n$  satisfies the required property if and only if all zeros of polynomial  $A(x)$  are real.

(a) Assume that all roots of  $A(x)$  are real. Let us use the following notations. Let  $I$  be the identity operator on  $\mathbb{R} \rightarrow \mathbb{R}$  functions and  $D$  be differentiation operator. For an arbitrary polynomial  $P(x) = p_0 + p_1x + \dots + p_nx^n$ , write  $P(D) = p_0I + p_1D + p_2D^2 + \dots + p_nD^n$ . Then the statement can be written as  $(A(D)f)(\xi) = 0$ .

First prove the statement for  $n = 1$ . Consider the function

$$g(x) = e^{\frac{a_0}{a_1}x} f(x).$$

Since  $g(x_0) = g(x_1) = 0$ , by Rolle's theorem there exists a  $\xi \in (x_0, x_1)$  for which

$$g'(\xi) = \frac{a_0}{a_1} e^{\frac{a_0}{a_1}\xi} f(\xi) + e^{\frac{a_0}{a_1}\xi} f'(\xi) = \frac{e^{\frac{a_0}{a_1}\xi}}{a_1} (a_0f(\xi) + a_1f'(\xi)) = 0.$$

Now assume that  $n > 1$  and the statement holds for  $n - 1$ . Let  $A(x) = (x - c)B(x)$  where  $c$  is a real root of polynomial  $A$ . By the  $n = 1$  case, there exist  $y_0 \in (x_0, x_1), y_1 \in (x_1, x_2), \dots, y_{n-1} \in (x_{n-1}, x_n)$  such that  $f'(y_j) - cf(y_j) = 0$  for all  $j = 0, 1, \dots, n - 1$ . Now apply the induction hypothesis for polynomial  $B(x)$ , function  $g = f' - cf$  and points  $y_0, \dots, y_{n-1}$ . The hypothesis says that there exists a  $\xi \in (y_0, y_{n-1}) \subset (x_0, x_n)$  such that

$$(B(D)g)(\xi) = (B(D)(D - cI)f)(\xi) = (A(D)f)(\xi) = 0.$$

(b) Assume that  $u + vi$  is a complex root of polynomial  $A(x)$  such that  $v \neq 0$ . Consider the linear differential equation  $a_n g^{(n)} + \dots + a_1 g' + g = 0$ . A solution of this equation is  $g_1(x) = e^{ux} \sin vx$  which has infinitely many zeros.

Let  $k$  be the smallest index for which  $a_k \neq 0$ . Choose a small  $\varepsilon > 0$  and set  $f(x) = g_1(x) + \varepsilon x^k$ . If  $\varepsilon$  is sufficiently small then  $g$  has the required number of roots but  $a_0f + a_1f' + \dots + a_n f^{(n)} = a_k \varepsilon \neq 0$  everywhere.

13<sup>th</sup> International Mathematics Competition for University Students  
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 Second Day

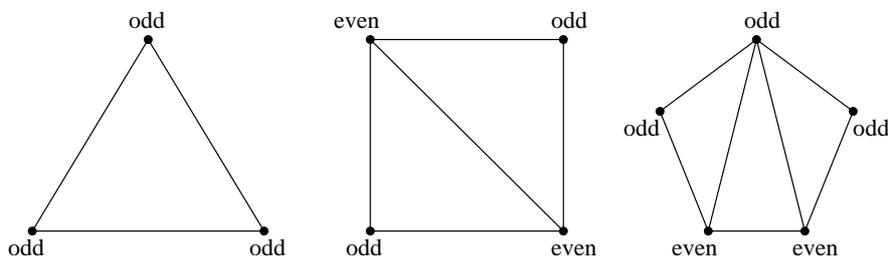
**Problem 1.** Let  $V$  be a convex polygon with  $n$  vertices.

(a) Prove that if  $n$  is divisible by 3 then  $V$  can be triangulated (i.e. dissected into non-overlapping triangles whose vertices are vertices of  $V$ ) so that each vertex of  $V$  is the vertex of an odd number of triangles.

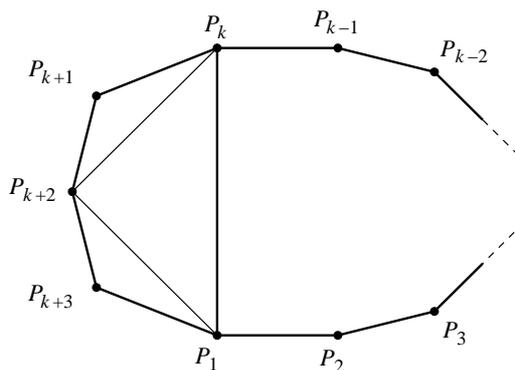
(b) Prove that if  $n$  is not divisible by 3 then  $V$  can be triangulated so that there are exactly two vertices that are the vertices of an even number of the triangles.

(20 points)

**Solution.** Apply induction on  $n$ . For the initial cases  $n = 3, 4, 5$ , chose the triangulations shown in the Figure to prove the statement.



Now assume that the statement is true for some  $n = k$  and consider the case  $n = k + 3$ . Denote the vertices of  $V$  by  $P_1, \dots, P_{k+3}$ . Apply the induction hypothesis on the polygon  $P_1P_2 \dots P_k$ ; in this triangulation each of vertices  $P_1, \dots, P_k$  belong to an odd number of triangles, except two vertices if  $n$  is not divisible by 3. Now add triangles  $P_1P_kP_{k+2}$ ,  $P_kP_{k+1}P_{k+2}$  and  $P_1P_{k+2}P_{k+3}$ . This way we introduce two new triangles at vertices  $P_1$  and  $P_k$  so parity is preserved. The vertices  $P_{k+1}$ ,  $P_{k+2}$  and  $P_{k+3}$  share an odd number of triangles. Therefore, the number of vertices shared by even number of triangles remains the same as in polygon  $P_1P_2 \dots P_k$ .



**Problem 2.** Find all functions  $f : \mathbb{R} \longrightarrow \mathbb{R}$  such that for any real numbers  $a < b$ , the image  $f([a, b])$  is a closed interval of length  $b - a$ .

(20 points)

**Solution.** The functions  $f(x) = x + c$  and  $f(x) = -x + c$  with some constant  $c$  obviously satisfy the condition of the problem. We will prove now that these are the only functions with the desired property.

Let  $f$  be such a function. Then  $f$  clearly satisfies  $|f(x) - f(y)| \leq |x - y|$  for all  $x, y$ ; therefore,  $f$  is continuous. Given  $x, y$  with  $x < y$ , let  $a, b \in [x, y]$  be such that  $f(a)$  is the maximum and  $f(b)$  is the minimum of  $f$  on  $[x, y]$ . Then  $f([x, y]) = [f(b), f(a)]$ ; hence

$$y - x = f(a) - f(b) \leq |a - b| \leq y - x$$

This implies  $\{a, b\} = \{x, y\}$ , and therefore  $f$  is a monotone function. Suppose  $f$  is increasing. Then  $f(x) - f(y) = x - y$  implies  $f(x) - x = f(y) - y$ , which says that  $f(x) = x + c$  for some constant  $c$ . Similarly, the case of a decreasing function  $f$  leads to  $f(x) = -x + c$  for some constant  $c$ .

**Problem 3.** Compare  $\tan(\sin x)$  and  $\sin(\tan x)$  for all  $x \in (0, \frac{\pi}{2})$ .

(20 points)

**Solution.** Let  $f(x) = \tan(\sin x) - \sin(\tan x)$ . Then

$$f'(x) = \frac{\cos x}{\cos^2(\sin x)} - \frac{\cos(\tan x)}{\cos^2 x} = \frac{\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x)}{\cos^2 x \cdot \cos^2(\tan x)}$$

Let  $0 < x < \arctan \frac{\pi}{2}$ . It follows from the concavity of cosine on  $(0, \frac{\pi}{2})$  that

$$\sqrt[3]{\cos(\tan x) \cdot \cos^2(\sin x)} < \frac{1}{3} [\cos(\tan x) + 2 \cos(\sin x)] \leq \cos \left[ \frac{\tan x + 2 \sin x}{3} \right] < \cos x,$$

the last inequality follows from  $\left[ \frac{\tan x + 2 \sin x}{3} \right]' = \frac{1}{3} \left[ \frac{1}{\cos^2 x} + 2 \cos x \right] \geq \sqrt[3]{\frac{1}{\cos^2 x} \cdot \cos x \cdot \cos x} = 1$ . This proves that  $\cos^3 x - \cos(\tan x) \cdot \cos^2(\sin x) > 0$ , so  $f'(x) > 0$ , so  $f$  increases on the interval  $[0, \arctan \frac{\pi}{2}]$ . To end the proof it is enough to notice that (recall that  $4 + \pi^2 < 16$ )

$$\tan \left[ \sin \left( \arctan \frac{\pi}{2} \right) \right] = \tan \frac{\pi/2}{\sqrt{1 + \pi^2/4}} > \tan \frac{\pi}{4} = 1.$$

This implies that if  $x \in [\arctan \frac{\pi}{2}, \frac{\pi}{2}]$  then  $\tan(\sin x) > 1$  and therefore  $f(x) > 0$ .

**Problem 4.** Let  $v_0$  be the zero vector in  $\mathbb{R}^n$  and let  $v_1, v_2, \dots, v_{n+1} \in \mathbb{R}^n$  be such that the Euclidean norm  $|v_i - v_j|$  is rational for every  $0 \leq i, j \leq n + 1$ . Prove that  $v_1, \dots, v_{n+1}$  are linearly dependent over the rationals.

(20 points)

**Solution.** By passing to a subspace we can assume that  $v_1, \dots, v_n$  are linearly independent over the reals. Then there exist  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$  satisfying

$$v_{n+1} = \sum_{j=1}^n \lambda_j v_j$$

We shall prove that  $\lambda_j$  is rational for all  $j$ . From

$$-2 \langle v_i, v_j \rangle = |v_i - v_j|^2 - |v_i|^2 - |v_j|^2$$

we get that  $\langle v_i, v_j \rangle$  is rational for all  $i, j$ . Define  $A$  to be the rational  $n \times n$ -matrix  $A_{ij} = \langle v_i, v_j \rangle$ ,  $w \in \mathbb{Q}^n$  to be the vector  $w_i = \langle v_i, v_{n+1} \rangle$ , and  $\lambda \in \mathbb{R}^n$  to be the vector  $(\lambda_i)_i$ . Then,

$$\langle v_i, v_{n+1} \rangle = \sum_{j=1}^n \lambda_j \langle v_i, v_j \rangle$$

gives  $A\lambda = w$ . Since  $v_1, \dots, v_n$  are linearly independent,  $A$  is invertible. The entries of  $A^{-1}$  are rationals, therefore  $\lambda = A^{-1}w \in \mathbb{Q}^n$ , and we are done.

**Problem 5.** Prove that there exists an infinite number of relatively prime pairs  $(m, n)$  of positive integers such that the equation

$$(x + m)^3 = nx$$

has three distinct integer roots.

(20 points)

**Solution.** Substituting  $y = x + m$ , we can replace the equation by

$$y^3 - ny + mn = 0.$$

Let two roots be  $u$  and  $v$ ; the third one must be  $w = -(u + v)$  since the sum is 0. The roots must also satisfy

$$uv + uw + vw = -(u^2 + uv + v^2) = -n, \quad \text{i.e.} \quad u^2 + uv + v^2 = n$$

and

$$uvw = -uv(u + v) = mn.$$

So we need some integer pairs  $(u, v)$  such that  $uv(u + v)$  is divisible by  $u^2 + uv + v^2$ . Look for such pairs in the form  $u = kp, v = kq$ . Then

$$u^2 + uv + v^2 = k^2(p^2 + pq + q^2),$$

and

$$uv(u + v) = k^3pq(p + q).$$

Choosing  $p, q$  such that they are coprime then setting  $k = p^2 + pq + q^2$  we have  $\frac{uv(u + v)}{u^2 + uv + v^2} = p^2 + pq + q^2$ .

Substituting back to the original quantities, we obtain the family of cases

$$n = (p^2 + pq + q^2)^3, \quad m = p^2q + pq^2,$$

and the three roots are

$$x_1 = p^3, \quad x_2 = q^3, \quad x_3 = -(p + q)^3.$$

**Problem 6.** Let  $A_i, B_i, S_i$  ( $i = 1, 2, 3$ ) be invertible real  $2 \times 2$  matrices such that

(1) not all  $A_i$  have a common real eigenvector;

(2)  $A_i = S_i^{-1}B_iS_i$  for all  $i = 1, 2, 3$ ;

(3)  $A_1A_2A_3 = B_1B_2B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

Prove that there is an invertible real  $2 \times 2$  matrix  $S$  such that  $A_i = S^{-1}B_iS$  for all  $i = 1, 2, 3$ .

(20 points)

**Solution.** We note that the problem is trivial if  $A_j = \lambda I$  for some  $j$ , so suppose this is not the case. Consider then first the situation where *some*  $A_j$ , say  $A_3$ , has two distinct real eigenvalues. We may assume that  $A_3 = B_3 = \begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}$  by conjugating both sides. Let  $A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $B_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ . Then

$$\begin{aligned} a + d = \text{Tr } A_2 &= \text{Tr } B_2 = a' + d' \\ a\lambda + d\mu = \text{Tr}(A_2A_3) = \text{Tr } A_1^{-1} &= \text{Tr } B_1^{-1} = \text{Tr}(B_2B_3) = a'\lambda + d'\mu. \end{aligned}$$

Hence  $a = a'$  and  $d = d'$  and so also  $bc = b'c'$ . Now we cannot have  $c = 0$  or  $b = 0$ , for then  $(1, 0)^\top$  or  $(0, 1)^\top$  would be a common eigenvector of all  $A_j$ . The matrix  $S = \begin{pmatrix} c' & \\ & c \end{pmatrix}$  conjugates  $A_2 = S^{-1}B_2S$ , and as  $S$  commutes with  $A_3 = B_3$ , it follows that  $A_j = S^{-1}B_jS$  for all  $j$ .

If the distinct eigenvalues of  $A_3 = B_3$  are not real, we know from above that  $A_j = S^{-1}B_jS$  for some  $S \in \text{GL}_2\mathbb{C}$  unless all  $A_j$  have a common eigenvector over  $\mathbb{C}$ . Even if they do, say  $A_jv = \lambda_jv$ , by taking the conjugate square root it follows that  $A_j$ 's can be simultaneously diagonalized. If  $A_2 = \begin{pmatrix} a & \\ & d \end{pmatrix}$  and  $B_2 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ , it follows as above that  $a = a'$ ,  $d = d'$  and so  $b'c' = 0$ . Now  $B_2$  and  $B_3$  (and hence  $B_1$  too) have a common eigenvector over  $\mathbb{C}$  so they too can be simultaneously diagonalized. And so  $SA_j = B_jS$  for some  $S \in \text{GL}_2\mathbb{C}$  in either case. Let  $S_0 = \text{Re } S$  and  $S_1 = \text{Im } S$ . By separating the real and imaginary components, we are done if either  $S_0$  or  $S_1$  is invertible. If not,  $S_0$  may be conjugated to some  $T^{-1}S_0T = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$ , with  $(x, y)^\top \neq (0, 0)^\top$ , and it follows that all  $A_j$  have a common eigenvector  $T(0, 1)^\top$ , a contradiction.

We are left with the case when *no*  $A_j$  has distinct eigenvalues; then these eigenvalues by necessity are real. By conjugation and division by scalars we may assume that  $A_3 = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$  and  $b \neq 0$ . By further conjugation by upper-triangular matrices (which preserves the shape of  $A_3$  up to the value of  $b$ ) we can also assume that  $A_2 = \begin{pmatrix} 0 & u \\ & v \end{pmatrix}$ . Here  $v^2 = \text{Tr}^2 A_2 = 4 \det A_2 = -4u$ . Now  $A_1 = A_3^{-1}A_2^{-1} = \begin{pmatrix} -(b+v)/u & 1 \\ & 1/u \end{pmatrix}$ , and hence  $(b+v)^2/u^2 = \text{Tr}^2 A_1 = 4 \det A_1 = -4/u$ . Comparing these two it follows that  $b = -2v$ . What we have done is simultaneously reduced all  $A_j$  to matrices whose all entries depend on  $u$  and  $v$  ( $= -\det A_2$  and  $\text{Tr } A_2$ , respectively) only, but these themselves are invariant under similarity. So  $B_j$ 's can be simultaneously reduced to the very same matrices.

# IMC2007, Blagoevgrad, Bulgaria

Day 1, August 5, 2007

**Problem 1.** Let  $f$  be a polynomial of degree 2 with integer coefficients. Suppose that  $f(k)$  is divisible by 5 for every integer  $k$ . Prove that all coefficients of  $f$  are divisible by 5.

**Solution 1.** Let  $f(x) = ax^2 + bx + c$ . Substituting  $x = 0$ ,  $x = 1$  and  $x = -1$ , we obtain that  $5|f(0) = c$ ,  $5|f(1) = (a + b + c)$  and  $5|f(-1) = (a - b + c)$ . Then  $5|f(1) + f(-1) - 2f(0) = 2a$  and  $5|f(1) - f(-1) = 2b$ . Therefore 5 divides  $2a$ ,  $2b$  and  $c$  and the statement follows.

**Solution 2.** Consider  $f(x)$  as a polynomial over the 5-element field (i.e. modulo 5). The polynomial has 5 roots while its degree is at most 2. Therefore  $f \equiv 0 \pmod{5}$  and all of its coefficients are divisible by 5.

**Problem 2.** Let  $n \geq 2$  be an integer. What is the minimal and maximal possible rank of an  $n \times n$  matrix whose  $n^2$  entries are precisely the numbers  $1, 2, \dots, n^2$ ?

**Solution.** The minimal rank is 2 and the maximal rank is  $n$ . To prove this, we have to show that the rank can be 2 and  $n$  but it cannot be 1.

(i) The rank is at least 2. Consider an arbitrary matrix  $A = [a_{ij}]$  with entries  $1, 2, \dots, n^2$  in some order. Since permuting rows or columns of a matrix does not change its rank, we can assume that  $1 = a_{11} < a_{21} < \dots < a_{n1}$  and  $a_{11} < a_{12} < \dots < a_{1n}$ . Hence  $a_{n1} \geq n$  and  $a_{1n} \geq n$  and at least one of these inequalities is strict. Then  $\det \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} < 1 \cdot n^2 - n \cdot n = 0$  so  $\text{rk}(A) \geq \text{rk} \begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} \geq 2$ .

(ii) The rank can be 2. Let

$$T = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & \ddots & \vdots \\ n^2 - n + 1 & n^2 - n + 2 & \dots & n^2 \end{bmatrix}$$

The  $i^{\text{th}}$  row is  $(1, 2, \dots, n) + n(i-1) \cdot (1, 1, \dots, 1)$  so each row is in the two-dimensional subspace generated by the vectors  $(1, 2, \dots, n)$  and  $(1, 1, \dots, 1)$ . We already proved that the rank is at least 2, so  $\text{rk}(T) = 2$ .

(iii) The rank can be  $n$ , i.e. the matrix can be nonsingular. Put odd numbers into the diagonal, only even numbers above the diagonal and arrange the entries under the diagonal arbitrarily. Then the determinant of the matrix is odd, so the rank is complete.

**Problem 3.** Call a polynomial  $P(x_1, \dots, x_k)$  good if there exist  $2 \times 2$  real matrices  $A_1, \dots, A_k$  such that

$$P(x_1, \dots, x_k) = \det \left( \sum_{i=1}^k x_i A_i \right).$$

Find all values of  $k$  for which all homogeneous polynomials with  $k$  variables of degree 2 are good.

(A polynomial is homogeneous if each term has the same total degree.)

**Solution.** The possible values for  $k$  are 1 and 2.

If  $k = 1$  then  $P(x) = \alpha x^2$  and we can choose  $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ .

If  $k = 2$  then  $P(x, y) = \alpha x^2 + \beta y^2 + \gamma xy$  and we can choose matrices  $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$  and  $A_2 = \begin{pmatrix} 0 & \beta \\ -1 & \gamma \end{pmatrix}$ .

Now let  $k \geq 3$ . We show that the polynomial  $P(x_1, \dots, x_k) = \sum_{i=0}^k x_i^2$  is not good. Suppose that

$P(x_1, \dots, x_k) = \det \left( \sum_{i=0}^k x_i A_i \right)$ . Since the first columns of  $A_1, \dots, A_k$  are linearly dependent, the first

column of some non-trivial linear combination  $y_1A_1 + \dots + y_kA_k$  is zero. Then  $\det(y_1A_1 + \dots + y_kA_k) = 0$  but  $P(y_1, \dots, y_k) \neq 0$ , a contradiction.

**Problem 4.** Let  $G$  be a finite group. For arbitrary sets  $U, V, W \subset G$ , denote by  $N_{UVW}$  the number of triples  $(x, y, z) \in U \times V \times W$  for which  $xyz$  is the unity.

Suppose that  $G$  is partitioned into three sets  $A, B$  and  $C$  (i.e. sets  $A, B, C$  are pairwise disjoint and  $G = A \cup B \cup C$ ). Prove that  $N_{ABC} = N_{CBA}$ .

**Solution.** We start with three preliminary observations.

Let  $U, V$  be two arbitrary subsets of  $G$ . For each  $x \in U$  and  $y \in V$  there is a unique  $z \in G$  for which  $xyz = e$ . Therefore,

$$N_{UVG} = |U \times V| = |U| \cdot |V|. \quad (1)$$

Second, the equation  $xyz = e$  is equivalent to  $yzx = e$  and  $zxy = e$ . For arbitrary sets  $U, V, W \subset G$ , this implies

$$\{(x, y, z) \in U \times V \times W : xyz = e\} = \{(x, y, z) \in U \times V \times W : yzx = e\} = \{(x, y, z) \in U \times V \times W : zxy = e\}$$

and therefore

$$N_{UVW} = N_{VWU} = N_{WUV}. \quad (2)$$

Third, if  $U, V \subset G$  and  $W_1, W_2, W_3$  are disjoint sets and  $W = W_1 \cup W_2 \cup W_3$  then, for arbitrary  $U, V \subset G$ ,

$$\begin{aligned} \{(x, y, z) \in U \times V \times W : xyz = e\} &= \{(x, y, z) \in U \times V \times W_1 : xyz = e\} \cup \\ &\cup \{(x, y, z) \in U \times V \times W_2 : xyz = e\} \cup \{(x, y, z) \in U \times V \times W_3 : xyz = e\} \end{aligned}$$

so

$$N_{UVW} = N_{UVW_1} + N_{UVW_2} + N_{UVW_3}. \quad (3)$$

Applying these observations, the statement follows as

$$\begin{aligned} N_{ABC} &= N_{ABG} - N_{ABA} - N_{ABB} = |A| \cdot |B| - N_{BAA} - N_{BAB} = \\ &= N_{BAG} - N_{BAA} - N_{BAB} = N_{BAC} = N_{CBA}. \end{aligned}$$

**Problem 5.** Let  $n$  be a positive integer and  $a_1, \dots, a_n$  be arbitrary integers. Suppose that a function  $f : \mathbb{Z} \rightarrow \mathbb{R}$  satisfies  $\sum_{i=1}^n f(k + a_i \ell) = 0$  whenever  $k$  and  $\ell$  are integers and  $\ell \neq 0$ . Prove that  $f = 0$ .

**Solution.** Let us define a subset  $\mathcal{I}$  of the polynomial ring  $\mathbb{R}[X]$  as follows:

$$\mathcal{I} = \left\{ P(X) = \sum_{j=0}^m b_j X^j : \sum_{j=0}^m b_j f(k + j\ell) = 0 \text{ for all } k, \ell \in \mathbb{Z}, \ell \neq 0 \right\}.$$

This is a subspace of the real vector space  $\mathbb{R}[X]$ . Furthermore,  $P(X) \in \mathcal{I}$  implies  $X \cdot P(X) \in \mathcal{I}$ . Hence,  $\mathcal{I}$  is an ideal, and it is non-zero, because the polynomial  $R(X) = \sum_{i=1}^n X^{a_i}$  belongs to  $\mathcal{I}$ . Thus,  $\mathcal{I}$  is generated (as an ideal) by some non-zero polynomial  $Q$ .

If  $Q$  is constant then the definition of  $\mathcal{I}$  implies  $f = 0$ , so we can assume that  $Q$  has a complex zero  $c$ . Again, by the definition of  $\mathcal{I}$ , the polynomial  $Q(X^m)$  belongs to  $\mathcal{I}$  for every natural number  $m \geq 1$ ; hence  $Q(X)$  divides  $Q(X^m)$ . This shows that all the complex numbers

$$c, c^2, c^3, c^4, \dots$$

are roots of  $Q$ . Since  $Q$  can have only finitely many roots, we must have  $c^N = 1$  for some  $N \geq 1$ ; in particular,  $Q(1) = 0$ , which implies  $P(1) = 0$  for all  $P \in \mathcal{I}$ . This contradicts the fact that  $R(X) = \sum_{i=1}^n X^{a_i} \in \mathcal{I}$ , and we are done.

**Problem 6.** How many nonzero coefficients can a polynomial  $P(z)$  have if its coefficients are integers and  $|P(z)| \leq 2$  for any complex number  $z$  of unit length?

**Solution.** We show that the number of nonzero coefficients can be 0, 1 and 2. These values are possible, for example the polynomials  $P_0(z) = 0$ ,  $P_1(z) = 1$  and  $P_2(z) = 1 + z$  satisfy the conditions and they have 0, 1 and 2 nonzero terms, respectively.

Now consider an arbitrary polynomial  $P(z) = a_0 + a_1z + \dots + a_nz^n$  satisfying the conditions and assume that it has at least two nonzero coefficients. Dividing the polynomial by a power of  $z$  and optionally replacing  $p(z)$  by  $-p(z)$ , we can achieve  $a_0 > 0$  such that conditions are not changed and the number of nonzero terms is preserved. So, without loss of generality, we can assume that  $a_0 > 0$ .

Let  $Q(z) = a_1z + \dots + a_{n-1}z^{n-1}$ . Our goal is to show that  $Q(z) = 0$ .

Consider those complex numbers  $w_0, w_1, \dots, w_{n-1}$  on the unit circle for which  $a_n w_k^n = |a_n|$ ; namely, let

$$w_k = \begin{cases} e^{2k\pi i/n} & \text{if } a_n > 0 \\ e^{(2k+1)\pi i/n} & \text{if } a_n < 0 \end{cases} \quad (k = 0, 1, \dots, n-1).$$

Notice that

$$\sum_{k=0}^{n-1} Q(w_k) = \sum_{k=0}^{n-1} Q(w_0 e^{2k\pi i/n}) = \sum_{j=1}^{n-1} a_j w_0^j \sum_{k=0}^{n-1} (e^{2j\pi i/n})^k = 0.$$

Taking the average of polynomial  $P(z)$  at the points  $w_k$ , we obtain

$$\frac{1}{n} \sum_{k=0}^{n-1} P(w_k) = \frac{1}{n} \sum_{k=0}^{n-1} (a_0 + Q(w_k) + a_n w_k^n) = a_0 + |a_n|$$

and

$$2 \geq \frac{1}{n} \sum_{k=0}^{n-1} |P(w_k)| \geq \left| \frac{1}{n} \sum_{k=0}^{n-1} P(w_k) \right| = a_0 + |a_n| \geq 2.$$

This obviously implies  $a_0 = |a_n| = 1$  and  $|P(w_k)| = |2 + Q(w_k)| = 2$  for all  $k$ . Therefore, all values of  $Q(w_k)$  must lie on the circle  $|2 + z| = 2$ , while their sum is 0. This is possible only if  $Q(w_k) = 0$  for all  $k$ . Then polynomial  $Q(z)$  has at least  $n$  distinct roots while its degree is at most  $n - 1$ . So  $Q(z) = 0$  and  $P(z) = a_0 + a_n z^n$  has only two nonzero coefficients.

**Remark.** From Parseval's formula (i.e. integrating  $|P(z)|^2 = P(z)\overline{P(z)}$  on the unit circle) it can be obtained that

$$|a_0|^2 + \dots + |a_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^2 dt \leq \frac{1}{2\pi} \int_0^{2\pi} 4 dt = 4. \quad (4)$$

Hence, there cannot be more than four nonzero coefficients, and if there are more than one nonzero term, then their coefficients are  $\pm 1$ .

It is also easy to see that equality in (4) cannot hold two or more nonzero coefficients, so it is sufficient to consider only polynomials of the form  $1 \pm x^m \pm x^n$ . However, we do not know (yet :-)) any simpler argument for these cases than the proof above.

# IMC2007, Blagoevgrad, Bulgaria

Day 2, August 6, 2007

**Problem 1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Suppose that for any  $c > 0$ , the graph of  $f$  can be moved to the graph of  $cf$  using only a translation or a rotation. Does this imply that  $f(x) = ax + b$  for some real numbers  $a$  and  $b$ ?

**Solution.** No. The function  $f(x) = e^x$  also has this property since  $ce^x = e^{x+\log c}$ .

**Problem 2.** Let  $x, y$ , and  $z$  be integers such that  $S = x^4 + y^4 + z^4$  is divisible by 29. Show that  $S$  is divisible by  $29^4$ .

**Solution.** We claim that  $29 \mid x, y, z$ . Then,  $x^4 + y^4 + z^4$  is clearly divisible by  $29^4$ .

Assume, to the contrary, that 29 does not divide all of the numbers  $x, y, z$ . Without loss of generality, we can suppose that  $29 \nmid x$ . Since the residue classes modulo 29 form a field, there is some  $w \in \mathbb{Z}$  such that  $xw \equiv 1 \pmod{29}$ . Then,  $(xw)^4 + (yw)^4 + (zw)^4$  is also divisible by 29. So we can assume that  $x \equiv 1 \pmod{29}$ .

Thus, we need to show that  $y^4 + z^4 \equiv -1 \pmod{29}$ , i.e.  $y^4 \equiv -1 - z^4 \pmod{29}$ , is impossible. There are only eight fourth powers modulo 29,

$$\begin{aligned} 0 &\equiv 0^4, \\ 1 &\equiv 1^4 \equiv 12^4 \equiv 17^4 \equiv 28^4 \pmod{29}, \\ 7 &\equiv 8^4 \equiv 9^4 \equiv 20^4 \equiv 21^4 \pmod{29}, \\ 16 &\equiv 2^4 \equiv 5^4 \equiv 24^4 \equiv 27^4 \pmod{29}, \\ 20 &\equiv 6^4 \equiv 14^4 \equiv 15^4 \equiv 23^4 \pmod{29}, \\ 23 &\equiv 3^4 \equiv 7^4 \equiv 22^4 \equiv 26^4 \pmod{29}, \\ 24 &\equiv 4^4 \equiv 10^4 \equiv 19^4 \equiv 25^4 \pmod{29}, \\ 25 &\equiv 11^4 \equiv 13^4 \equiv 16^4 \equiv 18^4 \pmod{29}. \end{aligned}$$

The differences  $-1 - z^4$  are congruent to 28, 27, 21, 12, 8, 5, 4, and 3. None of these residue classes is listed among the fourth powers.

**Problem 3.** Let  $C$  be a nonempty closed bounded subset of the real line and  $f : C \rightarrow C$  be a nondecreasing continuous function. Show that there exists a point  $p \in C$  such that  $f(p) = p$ .

(A set is closed if its complement is a union of open intervals. A function  $g$  is nondecreasing if  $g(x) \leq g(y)$  for all  $x \leq y$ .)

**Solution.** Suppose  $f(x) \neq x$  for all  $x \in C$ . Let  $[a, b]$  be the smallest closed interval that contains  $C$ . Since  $C$  is closed,  $a, b \in C$ . By our hypothesis  $f(a) > a$  and  $f(b) < b$ . Let  $p = \sup\{x \in C : f(x) > x\}$ . Since  $C$  is closed and  $f$  is continuous,  $f(p) \geq p$ , so  $f(p) > p$ . For all  $x > p$ ,  $x \in C$  we have  $f(x) < x$ . Therefore  $f(f(p)) < f(p)$  contrary to the fact that  $f$  is non-decreasing.

**Problem 4.** Let  $n > 1$  be an odd positive integer and  $A = (a_{ij})_{i,j=1\dots n}$  be the  $n \times n$  matrix with

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i - j \equiv \pm 2 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Find  $\det A$ .



For every element  $X = \{x_1, x_2, \dots, x_r\} \in \mathcal{P}$ , define  $A_X = A_{x_1}A_{x_2} \dots A_{x_r}$  and  $v_X = A_X v$ . Finally, write  $\bar{X} = \{1, 2, \dots, k\} \setminus X$  for the complement of  $X$ .

Now take  $X, Y \in \mathcal{P}$  with  $X \not\supseteq Y$ . Then  $A_{\bar{X}}$  annihilates  $v_Y$ , because  $X \not\supseteq Y$  implies the existence of some  $y \in Y \setminus X = Y \cap \bar{X}$ , and

$$A_{\bar{X}}v_Y = A_{\bar{X} \setminus \{y\}}A_yA_yv_{Y \setminus \{y\}} = 0,$$

since  $A_y^2 = 0$ . So,  $A_{\bar{X}}$  annihilates the span of all the  $v_Y$  with  $X \not\supseteq Y$ . This implies that  $v_X$  does not lie in this span, because  $A_{\bar{X}}v_X = v_{\{1,2,\dots,k\}} \neq 0$ . Therefore, the vectors  $v_X$  (with  $X \in \mathcal{P}$ ) are linearly independent; hence  $n \geq |\mathcal{P}| = 2^k$ .

**Problem 6.** Let  $f \neq 0$  be a polynomial with real coefficients. Define the sequence  $f_0, f_1, f_2, \dots$  of polynomials by  $f_0 = f$  and  $f_{n+1} = f_n + f'_n$  for every  $n \geq 0$ . Prove that there exists a number  $N$  such that for every  $n \geq N$ , all roots of  $f_n$  are real.

**Solution.** For the proof, we need the following

**Lemma 1.** For any polynomial  $g$ , denote by  $d(g)$  the minimum distance of any two of its real zeros ( $d(g) = \infty$  if  $g$  has at most one real zero). Assume that  $g$  and  $g + g'$  both are of degree  $k \geq 2$  and have  $k$  distinct real zeros. Then  $d(g + g') \geq d(g)$ .

*Proof of Lemma 1:* Let  $x_1 < x_2 < \dots < x_k$  be the roots of  $g$ . Suppose  $a, b$  are roots of  $g + g'$  satisfying  $0 < b - a < d(g)$ . Then,  $a, b$  cannot be roots of  $g$ , and

$$\frac{g'(a)}{g(a)} = \frac{g'(b)}{g(b)} = -1. \quad (1)$$

Since  $\frac{g'}{g}$  is strictly decreasing between consecutive zeros of  $g$ , we must have  $a < x_j < b$  for some  $j$ .

For all  $i = 1, 2, \dots, k - 1$  we have  $x_{i+1} - x_i > b - a$ , hence  $a - x_i > b - x_{i+1}$ . If  $i < j$ , both sides of this inequality are negative; if  $i \geq j$ , both sides are positive. In any case,  $\frac{1}{a-x_i} < \frac{1}{b-x_{i+1}}$ , and hence

$$\frac{g'(a)}{g(a)} = \sum_{i=1}^{k-1} \frac{1}{a-x_i} + \underbrace{\frac{1}{a-x_k}}_{<0} < \sum_{i=1}^{k-1} \frac{1}{b-x_{i+1}} + \underbrace{\frac{1}{b-x_1}}_{>0} = \frac{g'(b)}{g(b)}$$

This contradicts (1).

Now we turn to the proof of the stated problem. Denote by  $m$  the degree of  $f$ . We will prove by induction on  $m$  that  $f_n$  has  $m$  distinct real zeros for sufficiently large  $n$ . The cases  $m = 0, 1$  are trivial; so we assume  $m \geq 2$ . Without loss of generality we can assume that  $f$  is monic. By induction, the result holds for  $f'$ , and by ignoring the first few terms we can assume that  $f'_n$  has  $m - 1$  distinct real zeros for all  $n$ . Let us denote these zeros by  $x_1^{(n)} > x_2^{(n)} > \dots > x_{m-1}^{(n)}$ . Then  $f_n$  has minima in  $x_1^{(n)}, x_3^{(n)}, x_5^{(n)}, \dots$ , and maxima in  $x_2^{(n)}, x_4^{(n)}, x_6^{(n)}, \dots$ . Note that in the interval  $(x_{i+1}^{(n)}, x_i^{(n)})$ , the function  $f'_{n+1} = f'_n + f''_n$  must have a zero (this follows by applying Rolle's theorem to the function  $e^x f'_n(x)$ ); the same is true for the interval  $(-\infty, x_{m-1}^{(n)})$ . Hence, in each of these  $m - 1$  intervals,  $f'_{n+1}$  has *exactly* one zero. This shows that

$$x_1^{(n)} > x_1^{(n+1)} > x_2^{(n)} > x_2^{(n+1)} > x_3^{(n)} > x_3^{(n+1)} > \dots \quad (2)$$

**Lemma 2.** We have  $\lim_{n \rightarrow \infty} f_n(x_j^{(n)}) = -\infty$  if  $j$  is odd, and  $\lim_{n \rightarrow \infty} f_n(x_j^{(n)}) = +\infty$  if  $j$  is even.

Lemma 2 immediately implies the result: For sufficiently large  $n$ , the values of all maxima of  $f_n$  are positive, and the values of all minima of  $f_n$  are negative; this implies that  $f_n$  has  $m$  distinct zeros.

*Proof of Lemma 2:* Let  $d = \min\{d(f'), 1\}$ ; then by Lemma 1,  $d(f'_n) \geq d$  for all  $n$ . Define  $\varepsilon = \frac{(m-1)d^{m-1}}{m^{m-1}}$ ; we will show that

$$f_{n+1}(x_j^{(n+1)}) \geq f_n(x_j^{(n)}) + \varepsilon \quad \text{for } j \text{ even.} \quad (3)$$

(The corresponding result for odd  $j$  can be shown similarly.) Do to so, write  $f = f_n$ ,  $b = x_j^{(n)}$ , and choose  $a$  satisfying  $d \leq b - a \leq 1$  such that  $f'$  has no zero inside  $(a, b)$ . Define  $\xi$  by the relation  $b - \xi = \frac{1}{m}(b - a)$ ; then  $\xi \in (a, b)$ . We show that  $f(\xi) + f'(\xi) \geq f(b) + \varepsilon$ .

Notice, that

$$\begin{aligned} \frac{f''(\xi)}{f'(\xi)} &= \sum_{i=1}^{m-1} \frac{1}{\xi - x_i^{(n)}} \\ &= \sum_{i < j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{< \frac{1}{\xi - a}} + \frac{1}{\xi - b} + \sum_{i > j} \underbrace{\frac{1}{\xi - x_i^{(n)}}}_{< 0} \\ &< (m-1) \frac{1}{\xi - a} + \frac{1}{\xi - b} = 0. \end{aligned}$$

The last equality holds by definition of  $\xi$ . Since  $f'$  is positive and  $\frac{f''}{f'}$  is decreasing in  $(a, b)$ , we have that  $f''$  is negative on  $(\xi, b)$ . Therefore,

$$f(b) - f(\xi) = \int_{\xi}^b f'(t) dt \leq \int_{\xi}^b f'(\xi) dt = (b - \xi)f'(\xi)$$

Hence,

$$\begin{aligned} f(\xi) + f'(\xi) &\geq f(b) - (b - \xi)f'(\xi) + f'(\xi) \\ &= f(b) + (1 - (\xi - b))f'(\xi) \\ &= f(b) + (1 - \frac{1}{m}(b - a))f'(\xi) \\ &\geq f(b) + (1 - \frac{1}{m})f'(\xi). \end{aligned}$$

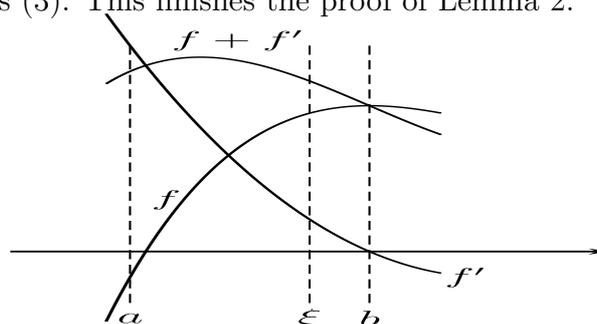
Together with

$$f'(\xi) = |f'(\xi)| = m \prod_{i=1}^{m-1} \underbrace{|\xi - x_i^{(n)}|}_{\geq |\xi - b|} \geq m|\xi - b|^{m-1} \geq \frac{d^{m-1}}{m^{m-2}}$$

we get

$$f(\xi) + f'(\xi) \geq f(b) + \varepsilon.$$

Together with (2) this shows (3). This finishes the proof of Lemma 2.



# IMC2008, Blagoevgrad, Bulgaria

Day 1, July 27, 2008

**Problem 1.** Find all continuous functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x) - f(y)$  is rational for all reals  $x$  and  $y$  such that  $x - y$  is rational.

**Solution.** We prove that  $f(x) = ax + b$  where  $a \in \mathbb{Q}$  and  $b \in \mathbb{R}$ . These functions obviously satisfy the conditions.

Suppose that a function  $f(x)$  fulfills the required properties. For an arbitrary rational  $q$ , consider the function  $g_q(x) = f(x+q) - f(x)$ . This is a continuous function which attains only rational values, therefore  $g_q$  is constant.

Set  $a = f(1) - f(0)$  and  $b = f(0)$ . Let  $n$  be an arbitrary positive integer and let  $r = f(1/n) - f(0)$ . Since  $f(x + 1/n) - f(x) = f(1/n) - f(0) = r$  for all  $x$ , we have

$$f(k/n) - f(0) = (f(1/n) - f(0)) + (f(2/n) - f(1/n)) + \dots + (f(k/n) - f((k-1)/n)) = kr$$

and

$$f(-k/n) - f(0) = -(f(0) - f(-1/n)) - (f(-1/n) - f(-2/n)) - \dots - (f(-(k-1)/n) - f(-k/n)) = -kr$$

for  $k \geq 1$ . In the case  $k = n$  we get  $a = f(1) - f(0) = nr$ , so  $r = a/n$ . Hence,  $f(k/n) - f(0) = kr = ak/n$  and then  $f(k/n) = a \cdot k/n + b$  for all integers  $k$  and  $n > 0$ .

So, we have  $f(x) = ax + b$  for all rational  $x$ . Since the function  $f$  is continuous and the rational numbers form a dense subset of  $\mathbb{R}$ , the same holds for all real  $x$ .

**Problem 2.** Denote by  $V$  the real vector space of all real polynomials in one variable, and let  $P: V \rightarrow \mathbb{R}$  be a linear map. Suppose that for all  $f, g \in V$  with  $P(fg) = 0$  we have  $P(f) = 0$  or  $P(g) = 0$ . Prove that there exist real numbers  $x_0, c$  such that  $P(f) = cf(x_0)$  for all  $f \in V$ .

**Solution.** We can assume that  $P \neq 0$ .

Let  $f \in V$  be such that  $P(f) \neq 0$ . Then  $P(f^2) \neq 0$ , and therefore  $P(f^2) = aP(f)$  for some non-zero real  $a$ . Then  $0 = P(f^2 - af) = P(f(f - a))$  implies  $P(f - a) = 0$ , so we get  $P(a) \neq 0$ . By rescaling, we can assume that  $P(1) = 1$ . Now  $P(X + b) = 0$  for  $b = -P(X)$ . Replacing  $P$  by  $\hat{P}$  given as

$$\hat{P}(f(X)) = P(f(X + b))$$

we can assume that  $P(X) = 0$ .

Now we are going to prove that  $P(X^k) = 0$  for all  $k \geq 1$ . Suppose this is true for all  $k < n$ . We know that  $P(X^n + e) = 0$  for  $e = -P(X^n)$ . From the induction hypothesis we get

$$P((X + e)(X + 1)^{n-1}) = P(X^n + e) = 0,$$

and therefore  $P(X + e) = 0$  (since  $P(X + 1) = 1 \neq 0$ ). Hence  $e = 0$  and  $P(X^n) = 0$ , which completes the inductive step. From  $P(1) = 1$  and  $P(X^k) = 0$  for  $k \geq 1$  we immediately get  $P(f) = f(0)$  for all  $f \in V$ .

**Problem 3.** Let  $p$  be a polynomial with integer coefficients and let  $a_1 < a_2 < \dots < a_k$  be integers.

- a) Prove that there exists  $a \in \mathbb{Z}$  such that  $p(a_i)$  divides  $p(a)$  for all  $i = 1, 2, \dots, k$ .
- b) Does there exist an  $a \in \mathbb{Z}$  such that the product  $p(a_1) \cdot p(a_2) \cdot \dots \cdot p(a_k)$  divides  $p(a)$ ?

**Solution.** The theorem is obvious if  $p(a_i) = 0$  for some  $i$ , so assume that all  $p(a_i)$  are nonzero and pairwise different.

There exist numbers  $s, t$  such that  $s|p(a_1), t|p(a_2), st = \text{lcm}(p(a_1), p(a_2))$  and  $\text{gcd}(s, t) = 1$ .

As  $s, t$  are relatively prime numbers, there exist  $m, n \in \mathbb{Z}$  such that  $a_1 + sn = a_2 + tm =: b_2$ . Obviously  $s|p(a_1 + sn) - p(a_1)$  and  $t|p(a_2 + tm) - p(a_2)$ , so  $st|p(b_2)$ .

Similarly one obtains  $b_3$  such that  $p(a_3)|p(b_3)$  and  $p(b_2)|p(b_3)$  thus also  $p(a_1)|p(b_3)$  and  $p(a_2)|p(b_3)$ .

Reasoning inductively we obtain the existence of  $a = b_k$  as required.

The polynomial  $p(x) = 2x^2 + 2$  shows that the second part of the problem is not true, as  $p(0) = 2, p(1) = 4$  but no value of  $p(a)$  is divisible by 8 for integer  $a$ .

**Remark.** One can assume that the  $p(a_i)$  are nonzero and ask for  $a$  such that  $p(a)$  is a nonzero multiple of all  $p(a_i)$ . In the solution above, it can happen that  $p(a) = 0$ . But every number  $p(a + np(a_1)p(a_2) \dots p(a_k))$  is also divisible by every  $p(a_i)$ , since the polynomial is nonzero, there exists  $n$  such that  $p(a + np(a_1)p(a_2) \dots p(a_k))$  satisfies the modified thesis.

**Problem 4.** We say a triple  $(a_1, a_2, a_3)$  of nonnegative reals is *better* than another triple  $(b_1, b_2, b_3)$  if **two out of the three** following inequalities  $a_1 > b_1, a_2 > b_2, a_3 > b_3$  are satisfied. We call a triple  $(x, y, z)$  *special* if  $x, y, z$  are nonnegative and  $x + y + z = 1$ . Find all natural numbers  $n$  for which there is a set  $S$  of  $n$  *special* triples such that for any given *special* triple we can find at least one *better* triple in  $S$ .

**Solution.** The answer is  $n \geq 4$ .

Consider the following set of special triples:

$$\left(0, \frac{8}{15}, \frac{7}{15}\right), \quad \left(\frac{2}{5}, 0, \frac{3}{5}\right), \quad \left(\frac{3}{5}, \frac{2}{5}, 0\right), \quad \left(\frac{2}{15}, \frac{11}{15}, \frac{2}{15}\right).$$

We will prove that any special triple  $(x, y, z)$  is worse than one of these (triple  $a$  is worse than triple  $b$  if triple  $b$  is better than triple  $a$ ). We suppose that some special triple  $(x, y, z)$  is actually not worse than the first three of the triples from the given set, derive some conditions on  $x, y, z$  and prove that, under these conditions,  $(x, y, z)$  is worse than the fourth triple from the set.

Triple  $(x, y, z)$  is not worse than  $(0, \frac{8}{15}, \frac{7}{15})$  means that  $y \geq \frac{8}{15}$  or  $z \geq \frac{7}{15}$ . Triple  $(x, y, z)$  is not worse than  $(\frac{2}{5}, 0, \frac{3}{5})$  —  $x \geq \frac{2}{5}$  or  $z \geq \frac{3}{5}$ . Triple  $(x, y, z)$  is not worse than  $(\frac{3}{5}, \frac{2}{5}, 0)$  —  $x \geq \frac{3}{5}$  or  $y \geq \frac{2}{5}$ . Since  $x + y + z = 1$ , then it is impossible that all inequalities  $x \geq \frac{2}{5}, y \geq \frac{2}{5}$  and  $z \geq \frac{7}{15}$  are true. Suppose that  $x < \frac{2}{5}$ , then  $y \geq \frac{2}{5}$  and  $z \geq \frac{3}{5}$ . Using  $x + y + z = 1$  and  $x \geq 0$  we get  $x = 0, y = \frac{2}{5}, z = \frac{3}{5}$ . We obtain the triple  $(0, \frac{2}{5}, \frac{3}{5})$  which is worse than  $(\frac{2}{15}, \frac{11}{15}, \frac{2}{15})$ . Suppose that  $y < \frac{2}{5}$ , then  $x \geq \frac{3}{5}$  and  $z \geq \frac{7}{15}$  and this is a contradiction to the admissibility of  $(x, y, z)$ . Suppose that  $z < \frac{7}{15}$ , then  $x \geq \frac{3}{5}$  and  $y \geq \frac{8}{15}$ . We get (by admissibility, again) that  $z \leq \frac{1}{15}$  and  $y \leq \frac{3}{5}$ . The last inequalities imply that  $(\frac{2}{15}, \frac{11}{15}, \frac{2}{15})$  is better than  $(x, y, z)$ .

We will prove that for any given set of three special triples one can find a special triple which is not worse than any triple from the set. Suppose we have a set  $S$  of three special triples

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3).$$

Denote  $a(S) = \min(x_1, x_2, x_3), b(S) = \min(y_1, y_2, y_3), c(S) = \min(z_1, z_2, z_3)$ . It is easy to check that  $S_1$ :

$$\begin{pmatrix} \frac{x_1 - a}{1 - a - b - c}, & \frac{y_1 - b}{1 - a - b - c}, & \frac{z_1 - c}{1 - a - b - c} \\ \frac{x_2 - a}{1 - a - b - c}, & \frac{y_2 - b}{1 - a - b - c}, & \frac{z_2 - c}{1 - a - b - c} \\ \frac{x_3 - a}{1 - a - b - c}, & \frac{y_3 - b}{1 - a - b - c}, & \frac{z_3 - c}{1 - a - b - c} \end{pmatrix}$$

is a set of three special triples also (we may suppose that  $a + b + c < 1$ , because otherwise all three triples are equal and our statement is trivial).

If there is a special triple  $(x, y, z)$  which is not worse than any triple from  $S_1$ , then the triple

$$((1 - a - b - c)x + a, (1 - a - b - c)y + b, (1 - a - b - c)z + c)$$

is special and not worse than any triple from  $S$ . We also have  $a(S_1) = b(S_1) = c(S_1) = 0$ , so we may suppose that the same holds for our starting set  $S$ .

Suppose that one element of  $S$  has two entries equal to 0.

Note that one of the two remaining triples from  $S$  is not worse than the other. This triple is also not worse than all triples from  $S$  because any special triple is not worse than itself and the triple with two zeroes.

So we have  $a = b = c = 0$  but we may suppose that all triples from  $S$  contain at most one zero. By transposing triples and elements in triples (elements in all triples must be transposed simultaneously) we may achieve the following situation  $x_1 = y_2 = z_3 = 0$  and  $x_2 \geq x_3$ . If  $z_2 \geq z_1$ , then the second triple  $(x_2, 0, z_2)$  is not worse than the other two triples from  $S$ . So we may assume that  $z_1 \geq z_2$ . If  $y_1 \geq y_3$ , then the first triple is not worse than the second and the third and we assume  $y_3 \geq y_1$ . Consider the three pairs of numbers  $x_2, y_1$ ;  $z_1, x_3$ ;  $y_3, z_2$ . The sum of all these numbers is three and consequently the sum of the numbers in one of the pairs is less than or equal to one. If it is the first pair then the triple  $(x_2, 1 - x_2, 0)$  is not worse than all triples from  $S$ , for the second we may take  $(1 - z_1, 0, z_1)$  and for the third —  $(0, y_3, 1 - y_3)$ . So we found a desirable special triple for any given  $S$ .

**Problem 5.** Does there exist a finite group  $G$  with a normal subgroup  $H$  such that  $|\text{Aut } H| > |\text{Aut } G|$ ?

**Solution.** Yes. Let  $H$  be the commutative group  $H = \mathbb{F}_2^3$ , where  $\mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z}$  is the field with two elements. The group of automorphisms of  $H$  is the general linear group  $\text{GL}_3\mathbb{F}_2$ ; it has

$$(8 - 1) \cdot (8 - 2) \cdot (8 - 4) = 7 \cdot 6 \cdot 4 = 168$$

elements. One of them is the shift operator  $\phi : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$ .

Now let  $T = \{a^0, a^1, a^2\}$  be a group of order 3 (written multiplicatively); it acts on  $H$  by  $\tau(a) = \phi$ . Let  $G$  be the semidirect product  $G = H \rtimes_\tau T$ . In other words,  $G$  is the group of 24 elements

$$G = \{ba^i : b \in H, i \in (\mathbb{Z}/3\mathbb{Z})\}, \quad ab = \phi(b)a.$$

$G$  has one element  $e$  of order 1 and seven elements  $b, b \in H, b \neq e$  of order 2.

If  $g = ba$ , we find that  $g^2 = baba = b\phi(b)a^2 \neq e$ , and that

$$g^3 = b\phi(b)a^2ba = b\phi(b)a\phi(b)a^2 = b\phi(b)\phi^2(b)a^3 = \psi(b),$$

where the homomorphism  $\psi : H \rightarrow H$  is defined as  $\psi : (x_1, x_2, x_3) \mapsto (x_1 + x_2 + x_3)(1, 1, 1)$ . It is clear that  $g^3 = \psi(b) = e$  for 4 elements  $b \in H$ , while  $g^6 = \psi^2(b) = e$  for all  $b \in H$ .

We see that  $G$  has 8 elements of order 3, namely  $ba$  and  $ba^2$  with  $b \in \text{Ker } \psi$ , and 8 elements of order 6, namely  $ba$  and  $ba^2$  with  $b \notin \text{Ker } \psi$ . That accounts for orders of all elements of  $G$ .

Let  $b_0 \in H \setminus \text{Ker } \psi$  be arbitrary; it is easy to see that  $G$  is generated by  $b_0$  and  $a$ . As every automorphism of  $G$  is fully determined by its action on  $b_0$  and  $a$ , it follows that  $G$  has no more than

$$7 \cdot 8 = 56$$

automorphisms.

**Remark.**  $G$  and  $H$  can be equivalently presented as subgroups of  $S_6$ , namely as  $H = \langle (12), (34), (56) \rangle$  and  $G = \langle (135)(246), (12) \rangle$ .

**Problem 6.** For a permutation  $\sigma = (i_1, i_2, \dots, i_n)$  of  $(1, 2, \dots, n)$  define  $D(\sigma) = \sum_{k=1}^n |i_k - k|$ . Let  $Q(n, d)$  be the number of permutations  $\sigma$  of  $(1, 2, \dots, n)$  with  $d = D(\sigma)$ . Prove that  $Q(n, d)$  is even for  $d \geq 2n$ .

**Solution.** Consider the  $n \times n$  determinant

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-1} \\ x & 1 & \dots & x^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x^{n-1} & x^{n-2} & \dots & 1 \end{vmatrix}$$

where the  $ij$ -th entry is  $x^{|i-j|}$ . From the definition of the determinant we get

$$\Delta(x) = \sum_{(i_1, \dots, i_n) \in S_n} (-1)^{\text{inv}(i_1, \dots, i_n)} x^{D(i_1, \dots, i_n)}$$

where  $S_n$  is the set of all permutations of  $(1, 2, \dots, n)$  and  $\text{inv}(i_1, \dots, i_n)$  denotes the number of inversions in the sequence  $(i_1, \dots, i_n)$ . So  $Q(n, d)$  has the same parity as the coefficient of  $x^d$  in  $\Delta(x)$ .

It remains to evaluate  $\Delta(x)$ . In order to eliminate the entries below the diagonal, subtract the  $(n-1)$ -th row, multiplied by  $x$ , from the  $n$ -th row. Then subtract the  $(n-2)$ -th row, multiplied by  $x$ , from the  $(n-1)$ -th and so on. Finally, subtract the first row, multiplied by  $x$ , from the second row.

$$\Delta(x) = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ x & 1 & \dots & x^{n-3} & x^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & \dots & 1 & x \\ x^{n-1} & x^{n-2} & \dots & x & 1 \end{vmatrix} = \dots = \begin{vmatrix} 1 & x & \dots & x^{n-2} & x^{n-1} \\ 0 & 1-x^2 & \dots & x^{n-3}-x^{n-1} & x^{n-2}-x^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-x^2 & x-x^3 \\ 0 & 0 & \dots & 0 & 1-x^2 \end{vmatrix} = (1-x^2)^{n-1}.$$

For  $d \geq 2n$ , the coefficient of  $x^d$  is 0 so  $Q(n, d)$  is even.

# IMC2008, Blagoevgrad, Bulgaria

## Day 2, July 28, 2008

**Problem 1.** Let  $n, k$  be positive integers and suppose that the polynomial  $x^{2k} - x^k + 1$  divides  $x^{2n} + x^n + 1$ . Prove that  $x^{2k} + x^k + 1$  divides  $x^{2n} + x^n + 1$ .

**Solution.** Let  $f(x) = x^{2n} + x^n + 1$ ,  $g(x) = x^{2k} - x^k + 1$ ,  $h(x) = x^{2k} + x^k + 1$ . The complex number  $x_1 = \cos(\frac{\pi}{3k}) + i \sin(\frac{\pi}{3k})$  is a root of  $g(x)$ .

Let  $\alpha = \frac{\pi n}{3k}$ . Since  $g(x)$  divides  $f(x)$ ,  $f(x_1) = g(x_1) = 0$ . So,  $0 = x_1^{2n} + x_1^n + 1 = (\cos(2\alpha) + i \sin(2\alpha)) + (\cos \alpha + i \sin \alpha) + 1 = 0$ , and  $(2 \cos \alpha + 1)(\cos \alpha + i \sin \alpha) = 0$ . Hence  $2 \cos \alpha + 1 = 0$ , i.e.  $\alpha = \pm \frac{2\pi}{3} + 2\pi c$ , where  $c \in \mathbb{Z}$ .

Let  $x_2$  be a root of the polynomial  $h(x)$ . Since  $h(x) = \frac{x^{3k}-1}{x^k-1}$ , the roots of the polynomial  $h(x)$  are distinct and they are  $x_2 = \cos \frac{2\pi s}{3k} + i \sin \frac{2\pi s}{3k}$ , where  $s = 3a \pm 1, a \in \mathbb{Z}$ . It is enough to prove that  $f(x_2) = 0$ . We have  $f(x_2) = x_2^{2n} + x_2^n + 1 = (\cos(4s\alpha) + \sin(4s\alpha)) + (\cos(2s\alpha) + \sin(2s\alpha)) + 1 = (2 \cos(2s\alpha) + 1)(\cos(2s\alpha) + i \sin(2s\alpha)) = 0$  (since  $2 \cos(2s\alpha) + 1 = 2 \cos(2s(\pm \frac{2\pi}{3} + 2\pi c)) + 1 = 2 \cos(\frac{4\pi s}{3}) + 1 = 2 \cos(\frac{4\pi}{3}(3a \pm 1)) + 1 = 0$ ).

**Problem 2.** Two different ellipses are given. One focus of the first ellipse coincides with one focus of the second ellipse. Prove that the ellipses have at most two points in common.

**Solution.** It is well known that an ellipse might be defined by a focus (a point) and a directrix (a straight line), as a locus of points such that the distance to the focus divided by the distance to directrix is equal to a given number  $e < 1$ . So, if a point  $X$  belongs to both ellipses with the same focus  $F$  and directrices  $l_1, l_2$ , then  $e_1 \cdot l_1X = FX = e_2 \cdot l_2X$  (here we denote by  $l_1X, l_2X$  distances between the corresponding line and the point  $X$ ). The equation  $e_1 \cdot l_1X = e_2 \cdot l_2X$  defines two lines, whose equations are linear combinations with coefficients  $e_1, \pm e_2$  of the normalized equations of lines  $l_1, l_2$  but of those two only one is relevant, since  $X$  and  $F$  should lie on the same side of each directrix. So, we have that all possible points lie on one line. The intersection of a line and an ellipse consists of at most two points.

**Problem 3.** Let  $n$  be a positive integer. Prove that  $2^{n-1}$  divides

$$\sum_{0 \leq k < n/2} \binom{n}{2k+1} 5^k.$$

**Solution.** As is known, the Fibonacci numbers  $F_n$  can be expressed as  $F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right)$ . Expanding this expression, we obtain that  $F_n = \frac{1}{2^{n-1}} \left( \binom{n}{1} + \binom{n}{3} 5 + \dots + \binom{n}{l} 5^{\frac{l-1}{2}} \right)$ , where  $l$  is the greatest odd number such that  $l \leq n$  and  $s = \frac{l-1}{2} \leq \frac{n}{2}$ .

So,  $F_n = \frac{1}{2^{n-1}} \sum_{k=0}^s \binom{n}{2k+1} 5^k$ , which implies that  $2^{n-1}$  divides  $\sum_{0 \leq k < n/2} \binom{n}{2k+1} 5^k$ .

**Problem 4.** Let  $\mathbb{Z}[x]$  be the ring of polynomials with integer coefficients, and let  $f(x), g(x) \in \mathbb{Z}[x]$  be nonconstant polynomials such that  $g(x)$  divides  $f(x)$  in  $\mathbb{Z}[x]$ . Prove that if the polynomial  $f(x) - 2008$  has at least 81 distinct integer roots, then the degree of  $g(x)$  is greater than 5.

**Solution.** Let  $f(x) = g(x)h(x)$  where  $h(x)$  is a polynomial with integer coefficients.

Let  $a_1, \dots, a_{81}$  be distinct integer roots of the polynomial  $f(x) - 2008$ . Then  $f(a_i) = g(a_i)h(a_i) = 2008$  for  $i = 1, \dots, 81$ . Hence,  $g(a_1), \dots, g(a_{81})$  are integer divisors of 2008.

Since  $2008 = 2^3 \cdot 251$  (2, 251 are primes) then 2008 has exactly 16 distinct integer divisors (including the negative divisors as well). By the pigeonhole principle, there are at least 6 equal numbers among  $g(a_1), \dots, g(a_{81})$  (because  $81 > 16 \cdot 5$ ). For example,  $g(a_1) = g(a_2) = \dots = g(a_6) = c$ . So  $g(x) - c$  is

a nonconstant polynomial which has at least 6 distinct roots (namely  $a_1, \dots, a_6$ ). Then the degree of the polynomial  $g(x) - c$  is at least 6.

**Problem 5.** Let  $n$  be a positive integer, and consider the matrix  $A = (a_{ij})_{1 \leq i, j \leq n}$ , where

$$a_{ij} = \begin{cases} 1 & \text{if } i + j \text{ is a prime number,} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that  $|\det A| = k^2$  for some integer  $k$ .

**Solution.** Call a square matrix *of type (B)*, if it is of the form

$$\begin{pmatrix} 0 & b_{12} & 0 & \dots & b_{1,2k-2} & 0 \\ b_{21} & 0 & b_{23} & \dots & 0 & b_{2,2k-1} \\ 0 & b_{32} & 0 & \dots & b_{3,2k-2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{2k-2,1} & 0 & b_{2k-2,3} & \dots & 0 & b_{2k-2,2k-1} \\ 0 & b_{2k-1,2} & 0 & \dots & b_{2k-1,2k-2} & 0 \end{pmatrix}.$$

Note that every matrix of this form has determinant zero, because it has  $k$  columns spanning a vector space of dimension at most  $k - 1$ .

Call a square matrix *of type (C)*, if it is of the form

$$C' = \begin{pmatrix} 0 & c_{11} & 0 & c_{12} & \dots & 0 & c_{1,k} \\ c_{11} & 0 & c_{12} & 0 & \dots & c_{1,k} & 0 \\ 0 & c_{21} & 0 & c_{22} & \dots & 0 & c_{2,k} \\ c_{21} & 0 & c_{22} & 0 & \dots & c_{2,k} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & c_{k,1} & 0 & c_{k,2} & \dots & 0 & c_{k,k} \\ c_{k,1} & 0 & c_{k,2} & 0 & \dots & c_{k,k} & 0 \end{pmatrix}$$

By permutations of rows and columns, we see that

$$|\det C'| = \left| \det \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \right| = |\det C|^2,$$

where  $C$  denotes the  $k \times k$ -matrix with coefficients  $c_{i,j}$ . Therefore, the determinant of any matrix of type (C) is a perfect square (up to a sign).

Now let  $X'$  be the matrix obtained from  $A$  by replacing the first row by  $(1 \ 0 \ 0 \ \dots \ 0)$ , and let  $Y$  be the matrix obtained from  $A$  by replacing the entry  $a_{11}$  by 0. By multi-linearity of the determinant,  $\det(A) = \det(X') + \det(Y)$ . Note that  $X'$  can be written as

$$X' = \begin{pmatrix} 1 & 0 \\ v & X \end{pmatrix}$$

for some  $(n - 1) \times (n - 1)$ -matrix  $X$  and some column vector  $v$ . Then  $\det(A) = \det(X) + \det(Y)$ . Now consider two cases. If  $n$  is odd, then  $X$  is of type (C), and  $Y$  is of type (B). Therefore,  $|\det(A)| = |\det(X)|$  is a perfect square. If  $n$  is even, then  $X$  is of type (B), and  $Y$  is of type (C); hence  $|\det(A)| = |\det(Y)|$  is a perfect square.

The set of primes can be replaced by any subset of  $\{2\} \cup \{3, 5, 7, 9, 11, \dots\}$ .

**Problem 6.** Let  $\mathcal{H}$  be an infinite-dimensional real Hilbert space, let  $d > 0$ , and suppose that  $S$  is a set of points (not necessarily countable) in  $\mathcal{H}$  such that the distance between any two distinct points in  $S$  is equal to  $d$ . Show that there is a point  $y \in \mathcal{H}$  such that

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : x \in S \right\}$$

is an orthonormal system of vectors in  $\mathcal{H}$ .

**Solution.** It is clear that, if  $\mathcal{B}$  is an orthonormal system in a Hilbert space  $\mathcal{H}$ , then  $\{(d/\sqrt{2})e : e \in \mathcal{B}\}$  is a set of points in  $\mathcal{H}$ , any two of which are at distance  $d$  apart. We need to show that every set  $S$  of equidistant points is a translate of such a set.

We begin by noting that, if  $x_1, x_2, x_3, x_4 \in S$  are four distinct points, then

$$\begin{aligned} \langle x_2 - x_1, x_2 - x_1 \rangle &= d^2, \\ \langle x_2 - x_1, x_3 - x_1 \rangle &= \frac{1}{2} (\|x_2 - x_1\|^2 + \|x_3 - x_1\|^2 - \|x_2 - x_3\|^2) = \frac{1}{2}d^2, \\ \langle x_2 - x_1, x_4 - x_3 \rangle &= \langle x_2 - x_1, x_4 - x_1 \rangle - \langle x_2 - x_1, x_3 - x_1 \rangle = \frac{1}{2}d^2 - \frac{1}{2}d^2 = 0. \end{aligned}$$

This shows that scalar products among vectors which are finite linear combinations of the form

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n,$$

where  $x_1, x_2, \dots, x_n$  are distinct points in  $S$  and  $\lambda_1, \lambda_2, \dots, \lambda_n$  are integers with  $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$ , are universal across all such sets  $S$  in all Hilbert spaces  $\mathcal{H}$ ; in particular, we may conveniently evaluate them using examples of our choosing, such as the canonical example above in  $\mathbb{R}^n$ . In fact this property trivially follows also when coefficients  $\lambda_i$  are rational, and hence by continuity any real numbers with sum 0.

If  $S = \{x_1, x_2, \dots, x_n\}$  is a finite set, we form

$$x = \frac{1}{n} (x_1 + x_2 + \cdots + x_n),$$

pick a non-zero vector  $z \in [\text{Span}(x_1 - x, x_2 - x, \dots, x_n - x)]^\perp$  and seek  $y$  in the form  $y = x + \lambda z$  for a suitable  $\lambda \in \mathbb{R}$ . We find that

$$\langle x_1 - y, x_2 - y \rangle = \langle x_1 - x - \lambda z, x_2 - x - \lambda z \rangle = \langle x_1 - x, x_2 - x \rangle + \lambda^2 \|z\|^2.$$

$\langle x_1 - x, x_2 - x \rangle$  may be computed by our remark above as

$$\begin{aligned} \langle x_1 - x, x_2 - x \rangle &= \frac{d^2}{2} \left\langle \left( \frac{1}{n} - 1, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right)^\top, \left( \frac{1}{n}, \frac{1}{n} - 1, \frac{1}{n}, \dots, \frac{1}{n} \right)^\top \right\rangle_{\mathbb{R}^n} \\ &= \frac{d^2}{2} \left( \frac{2}{n} \left( \frac{1}{n} - 1 \right) + \frac{n-2}{n^2} \right) = -\frac{d^2}{2n}. \end{aligned}$$

So the choice  $\lambda = \frac{d}{\sqrt{2n}\|z\|}$  will make all vectors  $\frac{\sqrt{2}}{d}(x_i - y)$  orthogonal to each other; it is easily checked as above that they will also be of length one.

Let now  $S$  be an infinite set. Pick an infinite sequence  $T = \{x_1, x_2, \dots, x_n, \dots\}$  of distinct points in  $S$ . We claim that the sequence

$$y_n = \frac{1}{n} (x_1 + x_2 + \cdots + x_n)$$

is a Cauchy sequence in  $\mathcal{H}$ . (This is the crucial observation.) Indeed, for  $m > n$ , the norm  $\|y_m - y_n\|$  may be computed by the above remark as

$$\begin{aligned}\|y_m - y_n\|^2 &= \frac{d^2}{2} \left\| \left( \frac{1}{m} - \frac{1}{n}, \dots, \frac{1}{m} - \frac{1}{n}, \frac{1}{m}, \dots, \frac{1}{m} \right)^\top \right\|_{\mathbb{R}^m}^2 = \frac{d^2}{2} \left( \frac{n(m-n)^2}{m^2 n^2} + \frac{m-n}{m^2} \right) \\ &= \frac{d^2}{2} \frac{(m-n)(m-n+n)}{m^2 n} = \frac{d^2}{2} \frac{m-n}{mn} = \frac{d^2}{2} \left( \frac{1}{n} - \frac{1}{m} \right) \rightarrow 0, \quad m, n \rightarrow \infty.\end{aligned}$$

By completeness of  $\mathcal{H}$ , it follows that there exists a limit

$$y = \lim_{n \rightarrow \infty} y_n \in \mathcal{H}.$$

We claim that  $y$  satisfies all conditions of the problem. For  $m > n > p$ , with  $n, p$  fixed, we compute

$$\begin{aligned}\|x_n - y_m\|^2 &= \frac{d^2}{2} \left\| \left( -\frac{1}{m}, \dots, -\frac{1}{m}, 1 - \frac{1}{m}, -\frac{1}{m}, \dots, -\frac{1}{m} \right)^\top \right\|_{\mathbb{R}^m}^2 \\ &= \frac{d^2}{2} \left[ \frac{m-1}{m^2} + \frac{(m-1)^2}{m^2} \right] = \frac{d^2}{2} \frac{m-1}{m} \rightarrow \frac{d^2}{2}, \quad m \rightarrow \infty,\end{aligned}$$

showing that  $\|x_n - y\| = d/\sqrt{2}$ , as well as

$$\begin{aligned}\langle x_n - y_m, x_p - y_m \rangle &= \frac{d^2}{2} \left\langle \left( -\frac{1}{m}, \dots, -\frac{1}{m}, \dots, 1 - \frac{1}{m}, \dots, -\frac{1}{m} \right)^\top, \right. \\ &\quad \left. \left( -\frac{1}{m}, \dots, 1 - \frac{1}{m}, \dots, -\frac{1}{m}, \dots, -\frac{1}{m} \right)^\top \right\rangle_{\mathbb{R}^m} \\ &= \frac{d^2}{2} \left[ \frac{m-2}{m^2} - \frac{2}{m} \left( 1 - \frac{1}{m} \right) \right] = -\frac{d^2}{2m} \rightarrow 0, \quad m \rightarrow \infty,\end{aligned}$$

showing that  $\langle x_n - y, x_p - y \rangle = 0$ , so that

$$\left\{ \frac{\sqrt{2}}{d}(x_n - y) : n \in \mathbb{N} \right\}$$

is indeed an orthonormal system of vectors.

This completes the proof in the case when  $T = S$ , which we can always take if  $S$  is countable. If it is not, let  $x', x''$  be any two distinct points in  $S \setminus T$ . Then applying the above procedure to the set

$$T' = \{x', x'', x_1, x_2, \dots, x_n, \dots\}$$

it follows that

$$\lim_{n \rightarrow \infty} \frac{x' + x'' + x_1 + x_2 + \dots + x_n}{n+2} = \lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = y$$

satisfies that

$$\left\{ \frac{\sqrt{2}}{d}(x' - y), \frac{\sqrt{2}}{d}(x'' - y) \right\} \cup \left\{ \frac{\sqrt{2}}{d}(x_n - y) : n \in \mathbb{N} \right\}$$

is still an orthonormal system.

This is true for any distinct  $x', x'' \in S \setminus T$ ; it follows that the entire system

$$\left\{ \frac{\sqrt{2}}{d}(x - y) : x \in S \right\}$$

is an orthonormal system of vectors in  $\mathcal{H}$ , as required.

International Mathematics Competition for University Students  
July 25–30 2009, Budapest, Hungary

Day 1

**Problem 1.**

Suppose that  $f$  and  $g$  are real-valued functions on the real line and  $f(r) \leq g(r)$  for every rational  $r$ . Does this imply that  $f(x) \leq g(x)$  for every real  $x$  if

- a)  $f$  and  $g$  are non-decreasing?
- b)  $f$  and  $g$  are continuous?

**Solution.** a) No. Counter-example:  $f$  and  $g$  can be chosen as the characteristic functions of  $[\sqrt{3}, \infty)$  and  $(\sqrt{3}, \infty)$ , respectively.

b) Yes. By the assumptions  $g - f$  is continuous on the whole real line and nonnegative on the rationals. Since any real number can be obtained as a limit of rational numbers we get that  $g - f$  is nonnegative on the whole real line.

**Problem 2.**

Let  $A$ ,  $B$  and  $C$  be real square matrices of the same size, and suppose that  $A$  is invertible. Prove that if  $(A - B)C = BA^{-1}$ , then  $C(A - B) = A^{-1}B$ .

**Solution.** A straightforward calculation shows that  $(A - B)C = BA^{-1}$  is equivalent to  $AC - BC - BA^{-1} + AA^{-1} = I$ , where  $I$  denotes the identity matrix. This is equivalent to  $(A - B)(C + A^{-1}) = I$ . Hence,  $(A - B)^{-1} = C + A^{-1}$ , meaning that  $(C + A^{-1})(A - B) = I$  also holds. Expansion yields the desired result.

**Problem 3.**

In a town every two residents who are not friends have a friend in common, and no one is a friend of everyone else. Let us number the residents from 1 to  $n$  and let  $a_i$  be the number of friends of the  $i$ -th resident. Suppose that  $\sum_{i=1}^n a_i^2 = n^2 - n$ . Let  $k$  be the smallest number of residents (at least three) who can be seated at a round table in such a way that any two neighbors are friends. Determine all possible values of  $k$ .

**Solution.** Let us define the simple, undirected graph  $G$  so that the vertices of  $G$  are the town's residents and the edges of  $G$  are the friendships between the residents. Let  $V(G) = \{v_1, v_2, \dots, v_n\}$  denote the vertices of  $G$ ;  $a_i$  is degree of  $v_i$  for every  $i$ . Let  $E(G)$  denote the edges of  $G$ . In this terminology, the problem asks us to describe the length  $k$  of the shortest cycle in  $G$ .

Let us count the walks of length 2 in  $G$ , that is, the ordered triples  $(v_i, v_j, v_l)$  of vertices with  $v_i v_j, v_j v_l \in E(G)$  ( $i = l$  being allowed). For a given  $j$  the number is obviously  $a_j^2$ , therefore the total number is  $\sum_{i=1}^n a_i^2 = n^2 - n$ .

Now we show that there is an injection  $f$  from the set of ordered pairs of distinct vertices to the set of these walks. For  $v_i v_j \notin E(G)$ , let  $f(v_i, v_j) = (v_i, v_l, v_j)$  with arbitrary  $l$  such that  $v_i v_l, v_l v_j \in E(G)$ . For  $v_i v_j \in E(G)$ , let  $f(v_i, v_j) = (v_i, v_j, v_i)$ .  $f$  is an injection since for  $i \neq l$ ,  $(v_i, v_j, v_l)$  can only be the image of  $(v_i, v_l)$ , and for  $i = l$ , it can only be the image of  $(v_i, v_j)$ .

Since the number of ordered pairs of distinct vertices is  $n^2 - n$ ,  $\sum_{i=1}^n a_i^2 \geq n^2 - n$ . Equality holds iff  $f$  is surjective, that is, iff there is exactly one  $l$  with  $v_i v_l, v_l v_j \in E(G)$  for every  $i, j$  with  $v_i v_j \notin E(G)$  and there is no such  $l$  for any  $i, j$  with  $v_i v_j \in E(G)$ . In other words, iff  $G$  contains neither  $C_3$  nor  $C_4$  (cycles of length 3 or 4), that is,  $G$  is either a forest (a cycle-free graph) or the length of its shortest cycle is at least 5.

It is easy to check that if every two vertices of a forest are connected by a path of length at most 2, then the forest is a star (one vertex is connected to all others by an edge). But  $G$  has  $n$  vertices, and none of them has degree  $n - 1$ . Hence  $G$  is not forest, so it has cycles. On the other hand, if the length of a cycle  $C$  of  $G$  is at least 6 then it has two vertices such that both arcs of  $C$  connecting them are longer than 2. Hence there is a path connecting them that is shorter than both arcs. Replacing one of the arcs by this path, we have a closed walk shorter than  $C$ . Therefore length of the shortest cycle is 5.

Finally, we must note that there is at least one  $G$  with the prescribed properties – e.g. the cycle  $C_5$  itself satisfies the conditions. Thus 5 is the sole possible value of  $k$ .

**Problem 4.**

Let  $p(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$  be a complex polynomial. Suppose that  $1 = c_0 \geq c_1 \geq \dots \geq c_n \geq 0$  is a sequence of real numbers which is convex (i.e.  $2c_k \leq c_{k-1} + c_{k+1}$  for every  $k = 1, 2, \dots, n-1$ ), and consider the polynomial

$$q(z) = c_0a_0 + c_1a_1z + c_2a_2z^2 + \dots + c_na_nz^n.$$

Prove that

$$\max_{|z| \leq 1} |q(z)| \leq \max_{|z| \leq 1} |p(z)|.$$

**Solution.** The polynomials  $p$  and  $q$  are regular on the complex plane, so by the Maximum Principle,  $\max_{|z| \leq 1} |q(z)| = \max_{|z|=1} |q(z)|$ , and similarly for  $p$ . Let us denote  $M_f = \max_{|z|=1} |f(z)|$  for any regular function  $f$ . Thus it suffices to prove that  $M_q \leq M_p$ .

First, note that we can assume  $c_n = 0$ . Indeed, for  $c_n = 1$ , we get  $p = q$  and the statement is trivial; otherwise,  $q(z) = c_np(z) + (1 - c_n)r(z)$ , where  $r(z) = \sum_{j=0}^n \frac{c_j - c_n}{1 - c_n} a_j z^j$ . The sequence  $c'_j = \frac{c_j - c_n}{1 - c_n}$  also satisfies the prescribed conditions (it is a positive linear transform of the sequence  $c_n$  with  $c'_0 = 1$ ), but  $c'_n = 0$  too, so we get  $M_r \leq M_p$ . This is enough:  $M_q = |q(z_0)| \leq c_n|p(z_0)| + (1 - c_n)|r(z_0)| \leq c_nM_p + (1 - c_n)M_r \leq M_p$ .

Using the Cauchy formulas, we can express the coefficients  $a_j$  of  $p$  from its values taken over the positively oriented circle  $S = \{|z| = 1\}$ :

$$a_j = \frac{1}{2\pi i} \int_S \frac{p(z)}{z^{j+1}} dz = \frac{1}{2\pi} \int_S \frac{p(z)}{z^j} |dz|$$

for  $0 \leq j \leq n$ , otherwise

$$\int_S \frac{p(z)}{z^j} |dz| = 0.$$

Let us use these identities to get a new formula for  $q$ , using only the values of  $p$  over  $S$ :

$$2\pi \cdot q(w) = \sum_{j=0}^n c_j \left( \int_S p(z) z^{-j} |dz| \right) w^j.$$

We can exchange the order of the summation and the integration (sufficient conditions to do this obviously apply):

$$2\pi \cdot q(w) = \int_S \left( \sum_{j=0}^n c_j (w/z)^j \right) p(z) |dz|.$$

It would be nice if the integration kernel (the sum between the brackets) was real. But this is easily arranged – for  $-n \leq j \leq -1$ , we can add the conjugate expressions, because by the above remarks, they are zero anyway:

$$2\pi \cdot q(w) = \sum_{j=0}^n c_j \left( \int_S p(z) z^{-j} |dz| \right) w^j = \sum_{j=-n}^n c_{|j|} \left( \int_S p(z) z^{-j} |dz| \right) w^j,$$

$$2\pi \cdot q(w) = \int_S \left( \sum_{j=-n}^n c_{|j|} (w/z)^j \right) p(z) |dz| = \int_S K(w/z) p(z) |dz|,$$

where

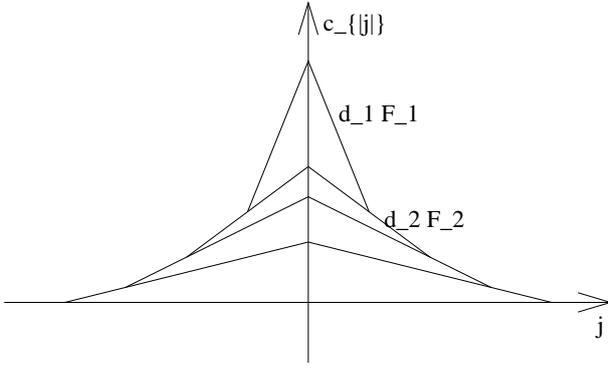
$$K(u) = \sum_{j=-n}^n c_{|j|} u^j = c_0 + 2 \sum_{j=1}^n c_j \Re(u^j)$$

for  $u \in S$ .

Let us examine  $K(u)$ . It is a real-valued function. Again from the Cauchy formulas,  $\int_S K(u) |du| = 2\pi c_0 = 2\pi$ . If  $\int_S |K(u)| |du| = 2\pi$  still holds (taking the absolute value does not increase the integral), then for every  $w$ :

$$2\pi |q(w)| = \left| \int_S K(w/z) p(z) |dz| \right| \leq \int_S |K(w/z)| \cdot |p(z)| |dz| \leq M_p \int_S |K(u)| |du| = 2\pi M_p;$$

this would conclude the proof. So it suffices to prove that  $\int_S |K(u)| |du| = \int_S K(u) |du|$ , which is to say,  $K$  is non-negative.



Now let us decompose  $K$  into a sum using the given conditions for the numbers  $c_j$  (including  $c_n = 0$ ). Let  $d_k = c_{k-1} - 2c_k + c_{k+1}$  for  $k = 1, \dots, n$  (setting  $c_{n+1} = 0$ ); we know that  $d_k \geq 0$ . Let  $F_k(u) = \sum_{j=-k+1}^{k-1} (k - |j|)u^j$ . Then  $K(u) = \sum_{k=1}^n d_k F_k(u)$  by easy induction (or see Figure for a graphical illustration). So it suffices to prove that  $F_k(u)$  is real and  $F_k(u) \geq 0$  for  $u \in S$ . This is reasonably well-known (as  $\frac{F_k}{k}$  is the Fejér kernel), and also very easy:

$$\begin{aligned} F_k(u) &= (1 + u + u^2 + \dots + u^{k-1})(1 + u^{-1} + u^{-2} + \dots + u^{-(k-1)}) = \\ &= (1 + u + u^2 + \dots + u^{k-1})\overline{(1 + u + u^2 + \dots + u^{k-1})} = |1 + u + u^2 + \dots + u^{k-1}|^2 \geq 0 \end{aligned}$$

This completes the proof.

### Problem 5.

Let  $n$  be a positive integer. An  $n$ -simplex in  $\mathbb{R}^n$  is given by  $n + 1$  points  $P_0, P_1, \dots, P_n$ , called its *vertices*, which do not all belong to the same hyperplane. For every  $n$ -simplex  $S$  we denote by  $v(S)$  the volume of  $S$ , and we write  $C(S)$  for the center of the unique sphere containing all the vertices of  $S$ .

Suppose that  $P$  is a point inside an  $n$ -simplex  $S$ . Let  $S_i$  be the  $n$ -simplex obtained from  $S$  by replacing its  $i$ -th vertex by  $P$ . Prove that

$$v(S_0)C(S_0) + v(S_1)C(S_1) + \dots + v(S_n)C(S_n) = v(S)C(S).$$

**Solution 1.** We will prove this by induction on  $n$ , starting with  $n = 1$ . In that case we are given an interval  $[a, b]$  with a point  $p \in (a, b)$ , and we have to verify

$$(b - p)\frac{b + p}{2} + (p - a)\frac{p + a}{2} = (b - a)\frac{b + a}{2},$$

which is true.

Now let assume the result is true for  $n - 1$  and prove it for  $n$ . We have to show that the point

$$X = \sum_{j=0}^n \frac{v(S_j)}{v(S)} O(S_j)$$

has the same distance to all the points  $P_0, P_1, \dots, P_n$ . Let  $i \in \{0, 1, 2, \dots, n\}$  and define the sets  $M_i = \{P_0, P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n\}$ . The set of all points having the same distance to all points in  $M_i$  is a line  $h_i$  orthogonal to the hyperplane  $E_i$  determined by the points in  $M_i$ . We are going to show that  $X$  lies on every  $h_i$ . To do so, fix some index  $i$  and notice that

$$X = \frac{v(S_i)}{v(S)} O(S_i) + \frac{v(S) - v(S_i)}{v(S)} \cdot \underbrace{\sum_{j \neq i} \frac{v(S_j)}{v(S) - v(S_i)} O(S_j)}_Y$$

and  $O(S_i)$  lies on  $h_i$ , so that it is enough to show that  $Y$  lies on  $h_i$ .

A map  $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}^n$  will be called *affine* if there are points  $A, B \in \mathbb{R}^n$  such that  $f(\lambda) = \lambda A + (1 - \lambda)B$ . Consider the ray  $g$  starting in  $P_i$  and passing through  $P$ . For  $\lambda > 0$  let  $P_\lambda = (1 - \lambda)P + \lambda P_i$ , so that  $P_\lambda$  is an affine function describing the points of  $g$ . For every such  $\lambda$  let  $S_j^\lambda$  be the  $n$ -simplex obtained from  $S$  by replacing the  $j$ -th vertex by  $P_\lambda$ . The point  $O(S_j^\lambda)$  is the intersection of the fixed line  $h_j$  with the hyperplane orthogonal to

$g$  and passing through the midpoint of the segment  $\overline{P_i P_\lambda}$  which is given by an affine function. This implies that also  $O(S_j^\lambda)$  is an affine function. We write  $\varphi_j = \frac{v(S_j)}{v(S) - v(S_i)}$ , and then

$$Y_\lambda = \sum_{j \neq i} \varphi_j O(S_j^\lambda)$$

is an affine function. We want to show that  $Y_\lambda \in h_i$  for all  $\lambda$  (then specializing to  $\lambda = 1$  gives the desired result). It is enough to do this for two different values of  $\lambda$ .

Let  $g$  intersect the sphere containing the vertices of  $S$  in a point  $Z$ ; then  $Z = P_\lambda$  for a suitable  $\lambda > 0$ , and we have  $O(S_j^\lambda) = O(S)$  for all  $j$ , so that  $Y_\lambda = O(S) \in h_i$ . Now let  $g$  intersect the hyperplane  $E_i$  in a point  $Q$ ; then  $Q = P_\lambda$  for some  $\lambda > 0$ , and  $Q$  is different from  $Z$ . Define  $T$  to be the  $(n-1)$ -simplex with vertex set  $M_i$ , and let  $T_j$  be the  $(n-1)$ -simplex obtained from  $T$  by replacing the vertex  $P_j$  by  $Q$ . If we write  $v'$  for the volume of  $(n-1)$ -simplices in the hyperplane  $E_i$ , then

$$\begin{aligned} \frac{v'(T_j)}{v'(T)} &= \frac{v(S_j^\lambda)}{v(S)} = \frac{v(S_j)}{\sum_{k \neq i} v(S_k^\lambda)} \\ &= \frac{\lambda v(S_j)}{\sum_{k \neq i} \lambda v(S_k)} = \frac{v(S_j)}{v(S) - v(S_i)} = \varphi_j. \end{aligned}$$

If  $p$  denotes the orthogonal projection onto  $E_i$  then  $p(O(S_j^\lambda)) = O(T_j)$ , so that  $p(Y_\lambda) = \sum_{j \neq i} \varphi_j O(T_j)$  equals  $O(T)$  by induction hypothesis, which implies  $Y_\lambda \in p^{-1}(O(T)) = h_i$ , and we are done.

**Solution 2.** For  $n = 1$ , the statement is checked easily.

Assume  $n \geq 2$ . Denote  $O(S_j) - O(S)$  by  $q_j$  and  $P_j - P$  by  $p_j$ . For all distinct  $j$  and  $k$  in the range  $0, \dots, n$  the point  $O(S_j)$  lies on a hyperplane orthogonal to  $p_k$  and  $P_j$  lies on a hyperplane orthogonal to  $q_k$ . So we have

$$\begin{cases} \langle p_i, q_j - q_k \rangle = 0 \\ \langle q_i, p_j - p_k \rangle = 0 \end{cases}$$

for all  $j \neq i \neq k$ . This means that the value  $\langle p_i, q_j \rangle$  is independent of  $j$  as long as  $j \neq i$ , denote this value by  $\lambda_i$ . Similarly,  $\langle q_i, p_j \rangle = \mu_i$  for some  $\mu_i$ . Since  $n \geq 2$ , these equalities imply that all the  $\lambda_i$  and  $\mu_i$  values are equal, in particular,  $\langle p_i, q_j \rangle = \langle p_j, q_i \rangle$  for any  $i$  and  $j$ .

We claim that for such  $p_i$  and  $q_i$ , the volumes

$$V_j = |\det(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_n)|$$

and

$$W_j = |\det(q_0, \dots, q_{j-1}, q_{j+1}, \dots, q_n)|$$

are proportional. Indeed, first assume that  $p_0, \dots, p_{n-1}$  and  $q_0, \dots, q_{n-1}$  are bases of  $\mathbb{R}^n$ , then we have

$$\begin{aligned} V_j &= \frac{1}{|\det(q_0, \dots, q_{n-1})|} \left| \det \left( (\langle p_k, q_l \rangle)_{\substack{k \neq j \\ l < n}} \right) \right| = \\ &= \frac{1}{|\det(q_0, \dots, q_{n-1})|} \left| \det \left( (\langle p_k, q_l \rangle)_{\substack{l \neq j \\ k < n}} \right) \right| = \left| \frac{\det(p_0, \dots, p_{n-1})}{\det(q_0, \dots, q_{n-1})} \right| W_j. \end{aligned}$$

If our assumption did not hold after any reindexing of the vectors  $p_i$  and  $q_i$ , then both  $p_i$  and  $q_i$  span a subspace of dimension at most  $n-1$  and all the volumes are 0.

Finally, it is clear that  $\sum q_j W_j / \det(q_0, \dots, q_n) = 0$ : the weight of  $p_j$  is the height of 0 over the hyperplane spanned by the rest of the vectors  $q_k$  relative to the height of  $p_j$  over the same hyperplane, so the sum is parallel to all the faces of the simplex spanned by  $q_0, \dots, q_n$ . By the argument above, we can change the weights to the proportional set of weights  $V_j / \det(p_0, \dots, p_n)$  and the sum will still be 0. That is,

$$\begin{aligned} 0 &= \sum q_j \frac{V_j}{\det(p_0, \dots, p_n)} = \sum (O(S_j) - O(S)) \frac{v(S_j)}{v(S)} = \\ &= \frac{1}{v(S)} \left( \sum O(S_j) v(S_j) - O(S) \sum v(S_j) \right) = \frac{1}{v(S)} \left( \sum O(S_j) v(S_j) - O(S) v(S) \right), \end{aligned}$$

q.e.d.

International Mathematics Competition for University Students  
July 25–30 2009, Budapest, Hungary

Day 2

**Problem 1.**

Let  $\ell$  be a line and  $P$  a point in  $\mathbb{R}^3$ . Let  $S$  be the set of points  $X$  such that the distance from  $X$  to  $\ell$  is greater than or equal to two times the distance between  $X$  and  $P$ . If the distance from  $P$  to  $\ell$  is  $d > 0$ , find the volume of  $S$ .

**Solution.** We can choose a coordinate system of the space such that the line  $\ell$  is the  $z$ -axis and the point  $P$  is  $(d, 0, 0)$ . The distance from the point  $(x, y, z)$  to  $\ell$  is  $\sqrt{x^2 + y^2}$ , while the distance from  $P$  to  $X$  is  $|PX| = \sqrt{(x-d)^2 + y^2 + z^2}$ . Square everything to get rid of the square roots. The condition can be reformulated as follows: the square of the distance from  $\ell$  to  $X$  is at least  $4|PX|^2$ .

$$\begin{aligned}x^2 + y^2 &\geq 4((x-d)^2 + y^2 + z^2) \\0 &\geq 3x^2 - 8dx + 4d^2 + 3y^2 + 4z^2 \\ \left(\frac{16}{3} - 4\right)d^2 &\geq 3\left(x - \frac{4}{3}d\right)^2 + 3y^2 + 4z^2\end{aligned}$$

A translation by  $\frac{4}{3}d$  in the  $x$ -direction does not change the volume, so we get

$$\begin{aligned}\frac{4}{3}d^2 &\geq 3x_1^2 + 3y^2 + 4z^2 \\ 1 &\geq \left(\frac{3x_1}{2d}\right)^2 + \left(\frac{3y}{2d}\right)^2 + \left(\frac{\sqrt{3}z}{d}\right)^2,\end{aligned}$$

where  $x_1 = x - \frac{4}{3}d$ . This equation defines a solid ellipsoid in canonical form. To compute its volume, perform a linear transformation: we divide  $x_1$  and  $y$  by  $\frac{2d}{3}$  and  $z$  by  $\frac{d}{\sqrt{3}}$ . This changes the volume by the factor  $\left(\frac{2d}{3}\right)^2 \frac{d}{\sqrt{3}} = \frac{4d^3}{9\sqrt{3}}$  and turns the ellipsoid into the unit ball of volume  $\frac{4}{3}\pi$ . So before the transformation the volume was  $\frac{4d^3}{9\sqrt{3}} \cdot \frac{4}{3}\pi = \frac{16\pi d^3}{27\sqrt{3}}$ .

**Problem 2.**

Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a two times differentiable function satisfying  $f(0) = 1$ ,  $f'(0) = 0$ , and for all  $x \in [0, \infty)$ ,

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that for all  $x \in [0, \infty)$ ,

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

**Solution.** We have  $f''(x) - 2f'(x) - 3(f'(x) - 2f(x)) \geq 0$ ,  $x \in [0, \infty)$ .

Let  $g(x) = f'(x) - 2f(x)$ ,  $x \in [0, \infty)$ . It follows that

$$g'(x) - 3g(x) \geq 0, \quad x \in [0, \infty),$$

hence

$$(g(x)e^{-3x})' \geq 0, \quad x \in [0, \infty),$$

therefore

$$\begin{aligned}g(x)e^{-3x} &\geq g(0) = -2, \quad x \in [0, \infty) \quad \text{or equivalently} \\ f'(x) - 2f(x) &\geq -2e^{3x}, \quad x \in [0, \infty).\end{aligned}$$

Analogously we get

$$\begin{aligned}(f(x)e^{-2x})' &\geq -2e^x, \quad x \in [0, \infty) \quad \text{or equivalently} \\ (f(x)e^{-2x} + 2e^x)' &\geq 0, \quad x \in [0, \infty).\end{aligned}$$

It follows that

$$\begin{aligned}f(x)e^{-2x} + 2e^x &\geq f(0) + 2 = 3, \quad x \in [0, \infty) \quad \text{or equivalently} \\ f(x) &\geq 3e^{2x} - 2e^{3x}, \quad x \in [0, \infty).\end{aligned}$$

**Problem 3.**

Let  $A, B \in M_n(\mathbb{C})$  be two  $n \times n$  matrices such that

$$A^2B + BA^2 = 2ABA.$$

Prove that there exists a positive integer  $k$  such that  $(AB - BA)^k = 0$ .

**Solution 1.** Let us fix the matrix  $A \in M_n(\mathbb{C})$ . For every matrix  $X \in M_n(\mathbb{C})$ , let  $\Delta X := AX - XA$ . We need to prove that the matrix  $\Delta B$  is nilpotent.

Observe that the condition  $A^2B + BA^2 = 2ABA$  is equivalent to

$$\Delta^2 B = \Delta(\Delta B) = 0. \quad (1)$$

$\Delta$  is linear; moreover, it is a derivation, i.e. it satisfies the Leibniz rule:

$$\Delta(XY) = (\Delta X)Y + X(\Delta Y), \quad \forall X, Y \in M_n(\mathbb{C}).$$

Using induction, one can easily generalize the above formula to  $k$  factors:

$$\Delta(X_1 \cdots X_k) = (\Delta X_1)X_2 \cdots X_k + \cdots + X_1 \cdots X_{j-1}(\Delta X_j)X_{j+1} \cdots X_k + X_1 \cdots X_{n-1}\Delta X_k, \quad (2)$$

for any matrices  $X_1, X_2, \dots, X_k \in M_n(\mathbb{C})$ . Using the identities (1) and (2) we obtain the equation for  $\Delta^k(B^k)$ :

$$\Delta^k(B^k) = k!(\Delta B)^k, \quad \forall k \in \mathbb{N}. \quad (3)$$

By the last equation it is enough to show that  $\Delta^n(B^n) = 0$ .

To prove this, first we observe that equation (3) together with the fact that  $\Delta^2 B = 0$  implies that  $\Delta^{k+1}B^k = 0$ , for every  $k \in \mathbb{N}$ . Hence, we have

$$\Delta^k(B^j) = 0, \quad \forall k, j \in \mathbb{N}, j < k. \quad (4)$$

By the Cayley–Hamilton Theorem, there are scalars  $\alpha_0, \alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$  such that

$$B^n = \alpha_0 I + \alpha_1 B + \cdots + \alpha_{n-1} B^{n-1},$$

which together with (4) implies that  $\Delta^n B^n = 0$ .

**Solution 2.** Set  $X = AB - BA$ . The matrix  $X$  commutes with  $A$  because

$$AX - XA = (A^2B - ABA) - (ABA - BA^2) = A^2B + BA^2 - 2ABA = 0.$$

Hence for any  $m \geq 0$  we have

$$X^{m+1} = X^m(AB - BA) = AX^m B - X^m BA.$$

Take the trace of both sides:

$$\text{tr } X^{m+1} = \text{tr } A(X^m B) - \text{tr } (X^m B)A = 0$$

(since for any matrices  $U$  and  $V$ , we have  $\text{tr } UV = \text{tr } VU$ ). As  $\text{tr } X^{m+1}$  is the sum of the  $m+1$ -st powers of the eigenvalues of  $X$ , the values of  $\text{tr } X, \dots, \text{tr } X^n$  determine the eigenvalues of  $X$  uniquely, therefore all of these eigenvalues have to be 0. This implies that  $X$  is nilpotent.

**Problem 4.**

Let  $p$  be a prime number and  $\mathbb{F}_p$  be the field of residues modulo  $p$ . Let  $W$  be the smallest set of polynomials with coefficients in  $\mathbb{F}_p$  such that

- the polynomials  $x + 1$  and  $x^{p-2} + x^{p-3} + \cdots + x^2 + 2x + 1$  are in  $W$ , and
- for any polynomials  $h_1(x)$  and  $h_2(x)$  in  $W$  the polynomial  $r(x)$ , which is the remainder of  $h_1(h_2(x))$  modulo  $x^p - x$ , is also in  $W$ .

How many polynomials are there in  $W$ ?

**Solution.** Note that both of our polynomials are bijective functions on  $\mathbb{F}_p$ :  $f_1(x) = x + 1$  is the cycle  $0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow (p-1) \rightarrow 0$  and  $f_2(x) = x^{p-2} + x^{p-3} + \dots + x^2 + 2x + 1$  is the transposition  $0 \leftrightarrow 1$  (this follows from the formula  $f_2(x) = \frac{x^{p-1}-1}{x-1} + x$  and Fermat's little theorem). So any composition formed from them is also a bijection, and reduction modulo  $x^p - x$  does not change the evaluation in  $\mathbb{F}_p$ . Also note that the transposition and the cycle generate the symmetric group ( $f_1^k \circ f_2 \circ f_1^{p-k}$  is the transposition  $k \leftrightarrow (k+1)$ ), and transpositions of consecutive elements clearly generate  $S_p$ , so we get all  $p!$  permutations of the elements of  $\mathbb{F}_p$ .

The set  $W$  only contains polynomials of degree at most  $p-1$ . This means that two distinct elements of  $W$  cannot represent the same permutation. So  $W$  must contain those polynomials of degree at most  $p-1$  which permute the elements of  $\mathbb{F}_p$ . By minimality,  $W$  has exactly these  $p!$  elements.

**Problem 5.**

Let  $\mathbb{M}$  be the vector space of  $m \times p$  real matrices. For a vector subspace  $S \subseteq \mathbb{M}$ , denote by  $\delta(S)$  the dimension of the vector space generated by all columns of all matrices in  $S$ .

Say that a vector subspace  $T \subseteq \mathbb{M}$  is a *covering matrix space* if

$$\bigcup_{A \in T, A \neq 0} \ker A = \mathbb{R}^p.$$

Such a  $T$  is *minimal* if it does not contain a proper vector subspace  $S \subset T$  which is also a covering matrix space.

(a) (8 points) Let  $T$  be a minimal covering matrix space and let  $n = \dim T$ . Prove that

$$\delta(T) \leq \binom{n}{2}.$$

(b) (2 points) Prove that for every positive integer  $n$  we can find  $m$  and  $p$ , and a minimal covering matrix space  $T$  as above such that  $\dim T = n$  and  $\delta(T) = \binom{n}{2}$ .

**Solution 1.** (a) We will prove the claim by constructing a suitable decomposition  $T = Z_0 \oplus Z_1 \oplus \dots$  and a corresponding decomposition of the space spanned by all columns of  $T$  as  $W_0 \oplus W_1 \oplus \dots$ , such that  $\dim W_0 \leq n-1$ ,  $\dim W_1 \leq n-2$ , etc., from which the bound follows.

We first claim that, in every covering matrix space  $S$ , we can find an  $A \in S$  with  $\text{rk } A \leq \dim S - 1$ . Indeed, let  $S_0 \subseteq S$  be some minimal covering matrix space. Let  $s = \dim S_0$  and fix some subspace  $S' \subset S_0$  of dimension  $s-1$ .  $S'$  is not covering by minimality of  $S_0$ , so that we can find an  $u \in \mathbb{R}^p$  with  $u \notin \cup_{B \in S', B \neq 0} \text{Ker } B$ . Let  $V = S'(u)$ ; by the rank-nullity theorem,  $\dim V = s-1$ . On the other hand, as  $S_0$  is covering, we have that  $Au = 0$  for some  $A \in S_0 \setminus S'$ . We claim that  $\text{Im } A \subset V$  (and therefore  $\text{rk}(A) \leq s-1$ ).

For suppose that  $Av \notin V$  for some  $v \in \mathbb{R}^p$ . For every  $\alpha \in \mathbb{R}$ , consider the map  $f_\alpha : S_0 \rightarrow \mathbb{R}^m$  defined by  $f_\alpha : (\tau + \beta A) \mapsto \tau(u + \alpha v) + \beta Av$ ,  $\tau \in S'$ ,  $\beta \in \mathbb{R}$ . Note that  $f_0$  is of rank  $s = \dim S_0$  by our assumption, so that some  $s \times s$  minor of the matrix of  $f_0$  is non-zero. The corresponding minor of  $f_\alpha$  is thus a nonzero polynomial of  $\alpha$ , so that it follows that  $\text{rk } f_\alpha = s$  for all but finitely many  $\alpha$ . For such an  $\alpha \neq 0$ , we have that  $\text{Ker } f_\alpha = \{0\}$  and thus

$$0 \neq \tau(u + \alpha v) + \beta Av = (\tau + \alpha^{-1}\beta A)(u + \alpha v)$$

for all  $\tau \in S'$ ,  $\beta \in \mathbb{R}$  not both zero, so that  $B(u + \alpha v) \neq 0$  for all nonzero  $B \in S_0$ , a contradiction.

Let now  $T$  be a minimal covering matrix space, and write  $\dim T = n$ . We have shown that we can find an  $A \in T$  such that  $W_0 = \text{Im } A$  satisfies  $w_0 = \dim W_0 \leq n-1$ . Denote  $Z_0 = \{B \in T : \text{Im } B \subset W_0\}$ ; we know that  $t_0 = \dim Z_0 \geq 1$ . If  $T = Z_0$ , then  $\delta(T) \leq n-1$  and we are done. Else, write  $T = Z_0 \oplus T_1$ , also write  $\mathbb{R}^m = W_0 \oplus V_1$  and let  $\pi_1 : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be the projection onto the  $V_1$ -component. We claim that

$$T_1^\sharp = \{\pi_1 \tau_1 : \tau_1 \in T_1\}$$

is also a covering matrix space. Note here that  $\pi_1^\sharp : T_1 \rightarrow T_1^\sharp$ ,  $\tau_1 \mapsto (\pi_1 \tau_1)$  is an isomorphism. In particular we note that  $\delta(T) = w_0 + \delta(T_1^\sharp)$ .

Suppose that  $T_1^\sharp$  is not a covering matrix space, so we can find a  $v_1 \in \mathbb{R}^p$  with  $v_1 \notin \cup_{\tau_1 \in T_1, \tau_1 \neq 0} \text{Ker}(\pi_1 \tau_1)$ . On the other hand, by minimality of  $T$  we can find a  $u_1 \in \mathbb{R}^p$  with  $u_1 \notin \cup_{\tau_0 \in Z_0, \tau_0 \neq 0} \text{Ker } \tau_0$ . The maps  $g_\alpha : Z_0 \rightarrow V$ ,

$\tau_0 \mapsto \tau_0(u_1 + \alpha v_1)$  and  $h_\beta : T_1 \rightarrow V_1$ ,  $\tau_1 \mapsto \pi_1(\tau_1(v_1 + \beta u_1))$  have  $\text{rk } g_0 = t_0$  and  $\text{rk } h_0 = n - t_0$  and thus both  $\text{rk } g_\alpha = t_0$  and  $\text{rk } h_{\alpha^{-1}} = n - t_0$  for all but finitely many  $\alpha \neq 0$  by the same argument as above. Pick such an  $\alpha$  and suppose that

$$(\tau_0 + \tau_1)(u_1 + \alpha v_1) = 0$$

for some  $\tau_0 \in Z_0$ ,  $\tau_1 \in T_1$ . Applying  $\pi_1$  to both sides we see that we can only have  $\tau_1 = 0$ , and then  $\tau_0 = 0$  as well, a contradiction given that  $T$  is a covering matrix space.

In fact, the exact same proof shows that, in general, if  $T$  is a minimal covering matrix space,  $\mathbb{R}^m = V_0 \oplus V_1$ ,  $T_0 = \{\tau \in T : \text{Im } \tau \subset V_0\}$ ,  $T = T_0 \oplus T_1$ ,  $\pi_1 : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is the projection onto the  $V_1$ -component, and  $T_1^\sharp = \{\pi_1 \tau_1 : \tau_1 \in T_1\}$ , then  $T_1^\sharp$  is a covering matrix space.

We can now repeat the process. We choose a  $\pi_1 A_1 \in T_1^\sharp$  such that  $W_1 = (\pi_1 A_1)(\mathbb{R}^p)$  has  $w_1 = \dim W_1 \leq n - t_0 - 1 \leq n - 2$ . We write  $Z_1 = \{\tau_1 \in T_1 : \text{Im}(\pi_1 \tau_1) \subset W_1\}$ ,  $T_1 = Z_1 \oplus T_2$  (and so  $T = (Z_0 \oplus Z_1) \oplus T_2$ ),  $t_1 = \dim Z_1 \geq 1$ ,  $V_1 = W_1 \oplus V_2$  (and so  $\mathbb{R}^m = (W_0 \oplus W_1) \oplus V_2$ ),  $\pi_2 : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is the projection onto the  $V_2$ -component, and  $T_2^\sharp = \{\pi_2 \tau_2 : \tau_2 \in T_2\}$ , so that  $T_2^\sharp$  is also a covering matrix space, etc.

We conclude that

$$\begin{aligned} \delta(T) &= w_0 + \delta(T_1) = w_0 + w_1 + \delta(T_2) = \cdots \\ &\leq (n-1) + (n-2) + \cdots \leq \binom{n}{2}. \end{aligned}$$

(b) We consider  $\binom{n}{2} \times n$  matrices whose rows are indexed by  $\binom{n}{2}$  pairs  $(i, j)$  of integers  $1 \leq i < j \leq n$ . For every  $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ , consider the matrix  $A(u)$  whose entries  $A(u)_{(i,j),k}$  with  $1 \leq i < j \leq n$  and  $1 \leq k \leq n$  are given by

$$(A(u))_{(i,j),k} = \begin{cases} u_i, & k = j, \\ -u_j, & k = i, \\ 0, & \text{otherwise.} \end{cases}$$

It is immediate that  $\text{Ker } A(u) = \mathbb{R} \cdot u$  for every  $u \neq 0$ , so that  $S = \{A(u) : u \in \mathbb{R}^n\}$  is a covering matrix space, and in fact a minimal one.

On the other hand, for any  $1 \leq i < j \leq n$ , we have that  $A(e_i)_{(i,j),j}$  is the  $(i, j)^{\text{th}}$  vector in the standard basis of  $\mathbb{R}^{\binom{n}{2}}$ , where  $e_i$  denotes the  $i^{\text{th}}$  vector in the standard basis of  $\mathbb{R}^n$ . This means that  $\delta(S) = \binom{n}{2}$ , as required.

**Solution 2.** (for part a)

Let us denote  $X = \mathbb{R}^p$ ,  $Y = \mathbb{R}^m$ . For each  $x \in X$ , denote by  $\mu_x : T \rightarrow Y$  the evaluation map  $\tau \mapsto \tau(x)$ . As  $T$  is a covering matrix space,  $\ker \mu_x > 0$  for every  $x \in X$ . Let  $U = \{x \in X : \dim \ker \mu_x = 1\}$ .

Let  $T_1$  be the span of the family of subspaces  $\{\ker \mu_x : x \in U\}$ . We claim that  $T_1 = T$ . For suppose the contrary, and let  $T' \subset T$  be a subspace of  $T$  of dimension  $n - 1$  such that  $T_1 \subseteq T'$ . This implies that  $T'$  is a covering matrix space. Indeed, for  $x \in U$ ,  $(\ker \mu_x) \cap T' = \ker \mu_x \neq 0$ , while for  $x \notin U$  we have  $\dim \mu_x \geq 2$ , so that  $(\ker \mu_x) \cap T' \neq 0$  by computing dimensions. However, this is a contradiction as  $T$  is minimal.

Now we may choose  $x_1, x_2, \dots, x_n \in U$  and  $\tau_1, \tau_2, \dots, \tau_n \in T$  in such a way that  $\ker \mu_{x_i} = \mathbb{R} \tau_i$  and  $\tau_i$  form a basis of  $T$ . Let us complete  $x_1, \dots, x_n$  to a sequence  $x_1, \dots, x_d$  which spans  $X$ . Put  $y_{ij} = \tau_i(x_j)$ . It is clear that  $y_{ij}$  span the vector space generated by the columns of all matrices in  $T$ . We claim that the subset  $\{y_{ij} : i > j\}$  is enough to span this space, which clearly implies that  $\delta(T) \leq \binom{n}{2}$ .

We have  $y_{ii} = 0$ . So it is enough to show that every  $y_{ij}$  with  $i < j$  can be expressed as a linear combination of  $y_{ki}$ ,  $k = 1, \dots, n$ . This follows from the following lemma:

*Lemma.* For every  $x_0 \in U$ ,  $0 \neq \tau_0 \in \ker \mu_{x_0}$  and  $x \in X$ , there exists a  $\tau \in T$  such that  $\tau_0(x) = \tau(x_0)$ .

*Proof.* The operator  $\mu_{x_0}$  has rank  $n - 1$ , which implies that for small  $\varepsilon$  the operator  $\mu_{x_0 + \varepsilon x}$  also has rank  $n - 1$ . Therefore one can produce a rational function  $\tau(\varepsilon)$  with values in  $T$  such that  $m_{x_0 + \varepsilon x}(\tau(\varepsilon)) = 0$ . Taking the derivative at  $\varepsilon = 0$  gives  $\mu_{x_0}(\tau_0) + \mu_x(\tau'(0)) = 0$ . Therefore  $\tau = -\tau'(0)$  satisfies the desired property.

**Remark.** Lemma in solution 2 is the same as the claim  $\text{Im } A \subset V$  at the beginning of solution 1, but the proof given here is different. It can be shown that all minimal covering spaces  $T$  with  $\dim T = \binom{n}{2}$  are essentially the ones described in our example.

# IMC2010, Blagoevgrad, Bulgaria

Day 1, July 26, 2010

**Problem 1.** Let  $0 < a < b$ . Prove that

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geq e^{-a^2} - e^{-b^2}.$$

**Solution 1.** Let  $f(x) = \int_0^x (t^2 + 1)e^{-t^2} dt$  and let  $g(x) = -e^{-x^2}$ ; both functions are increasing. By Cauchy's Mean Value Theorem, there exists a real number  $x \in (a, b)$  such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)} = \frac{(x^2 + 1)e^{-x^2}}{2xe^{-x^2}} = \frac{1}{2} \left( x + \frac{1}{x} \right) \geq \sqrt{x \cdot \frac{1}{x}} = 1.$$

Then

$$\int_a^b (x^2 + 1)e^{-x^2} dx = f(b) - f(a) \geq g(b) - g(a) = e^{-a^2} - e^{-b^2}.$$

**Solution 2.**

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geq \int_a^b 2xe^{-x^2} dx = [-e^{-x^2}]_a^b = e^{-a^2} - e^{-b^2}.$$

**Problem 2.** Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{5 \cdot 6 \cdot 7 \cdot 8} + \dots$$

**Solution 1.** Let

$$F(x) = \sum_{k=0}^{\infty} \frac{x^{4k+4}}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

This power series converges for  $|x| \leq 1$  and our goal is to compute  $F(1)$ .

Differentiating 4 times, we get

$$F^{(IV)}(x) = \sum_{k=0}^{\infty} x^{4k} = \frac{1}{1-x^4}.$$

Since  $F(0) = F'(0) = F''(0) = F'''(0) = 0$  and  $F$  is continuous at  $1 - 0$  by Abel's continuity theorem,

integrating 4 times we get

$$\begin{aligned}
F'''(y) &= F'''(0) + \int_0^y F^{(IV)}(x) dx = \int_0^y \frac{dx}{1-x^4} = \frac{1}{2} \arctan y + \frac{1}{4} \log(1+y) - \frac{1}{4} \log(1-y), \\
F''(z) &= F''(0) + \int_0^z F'''(y) dy = \int_0^z \left( \frac{1}{2} \arctan y + \frac{1}{4} \log(1+y) - \frac{1}{4} \log(1-y) \right) dy = \\
&= \frac{1}{2} \left( z \arctan z - \int_0^z \frac{y}{1+y^2} dy \right) + \frac{1}{4} \left( (1+z) \log(1+z) - \int_0^z dy \right) + \frac{1}{4} \left( (1-z) \log(1-z) + \int_0^z dy \right) = \\
&= \frac{1}{2} z \arctan z - \frac{1}{4} \log(1+z^2) + \frac{1}{4} (1+z) \log(1+z) + \frac{1}{4} (1-z) \log(1-z), \\
F'(t) &= \int_0^t \left( \frac{1}{2} z \arctan z - \frac{1}{4} \log(1+z^2) + \frac{1}{4} (1+z) \log(1+z) + \frac{1}{4} (1-z) \log(1-z) \right) dt = \\
&= \frac{1}{4} \left( (1+t^2) \arctan t - t \right) - \frac{1}{4} \left( t \log(1+t^2) - 2t + 2 \arctan t \right) + \\
&\quad + \frac{1}{8} \left( (1+t)^2 \log(1+t) - t - \frac{1}{2} t^2 \right) - \frac{1}{8} \left( (1-t)^2 \log(1-t) + t - \frac{1}{2} t^2 \right) = \\
&= \frac{1}{4} (-1+t^2) \arctan t - \frac{1}{4} t \log(1+t^2) + \frac{1}{8} (1+t)^2 \log(1+t) - \frac{1}{8} (1-t)^2 \log(1-t), \\
F(1) &= \int_0^1 \left( \frac{1}{4} (-1+t^2) \arctan t - \frac{1}{4} t \log(1+t^2) + \frac{1}{8} (1+t)^2 \log(1+t) - \frac{1}{8} (1-t)^2 \log(1-t) \right) dt = \\
&= \left[ \frac{-3t+t^3}{12} \arctan t + \frac{1-3t^2}{24} \log(1+t^2) + \frac{(1+t)^3}{24} \log(1+t) + \frac{(1-t)^3}{24} \log(1-t) \right]_0^1 = \frac{\ln 2}{4} - \frac{\pi}{24}.
\end{aligned}$$

**Remark.** The computation can be shorter if we change the order of integrations.

$$\begin{aligned}
F(1) &= \int_{t=0}^1 \int_{z=0}^t \int_{y=0}^z \int_{x=0}^y \frac{1}{1-x^4} dx dy dz dt = \int_{x=0}^1 \frac{1}{1-x^4} \int_{y=x}^1 \int_{z=y}^1 \int_{t=z}^1 dt dz dy dx = \\
&= \int_{x=0}^1 \frac{1}{1-x^4} \left( \frac{1}{6} \int_{y=x}^1 \int_{z=x}^1 \int_{t=x}^1 dt dz dy \right) dx = \int_0^1 \frac{1}{1-x^4} \cdot \frac{(1-x)^3}{6} dx = \\
&= \left[ -\frac{1}{6} \arctan x - \frac{1}{12} \log(1+x^2) + \frac{1}{3} \log(1+x) \right]_0^1 = \frac{\ln 2}{4} - \frac{\pi}{24}.
\end{aligned}$$

**Solution 2.** Let

$$\begin{aligned}
A_m &= \sum_{k=0}^m \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \sum_{k=0}^m \left( \frac{1}{6} \cdot \frac{1}{4k+1} - \frac{1}{2} \cdot \frac{1}{4k+2} + \frac{1}{2} \cdot \frac{1}{4k+3} - \frac{1}{6} \cdot \frac{1}{4k+4} \right), \\
B_m &= \sum_{k=0}^m \left( \frac{1}{4k+1} - \frac{1}{4k+3} \right), \\
C_m &= \sum_{k=0}^m \left( \frac{1}{4k+1} - \frac{1}{4k+2} + \frac{1}{4k+3} - \frac{1}{4k+4} \right) \quad \text{and} \\
D_m &= \sum_{k=0}^m \left( \frac{1}{4k+2} - \frac{1}{4k+4} \right).
\end{aligned}$$

It is easy check that

$$A_m = \frac{1}{3} C_m - \frac{1}{6} B_m - \frac{1}{6} D_m.$$

Therefore,

$$\lim A_m = \lim \frac{2C_m - B_m - D_m}{6} = \frac{2 \ln 2 - \frac{\pi}{4} - \frac{1}{2} \ln 2}{6} = \frac{1}{4} \ln 2 - \frac{\pi}{24}.$$

**Problem 3.** Define the sequence  $x_1, x_2, \dots$  inductively by  $x_1 = \sqrt{5}$  and  $x_{n+1} = x_n^2 - 2$  for each  $n \geq 1$ . Compute

$$\lim_{n \rightarrow \infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}.$$

**Solution.** Let  $y_n = x_n^2$ . Then  $y_{n+1} = (y_n - 2)^2$  and  $y_{n+1} - 4 = y_n(y_n - 4)$ . Since  $y_2 = 9 > 5$ , we have  $y_3 = (y_2 - 2)^2 > 5$  and inductively  $y_n > 5, n \geq 2$ . Hence,  $y_{n+1} - y_n = y_n^2 - 5y_n + 4 > 4$  for all  $n \geq 2$ , so  $y_n \rightarrow \infty$ .

By  $y_{n+1} - 4 = y_n(y_n - 4)$ ,

$$\begin{aligned} \left( \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}} \right)^2 &= \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1}} \\ &= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1} - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_{n-1}}{y_n - 4} = \dots \\ &= \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{1}{y_1 - 4} = \frac{y_{n+1} - 4}{y_{n+1}} \rightarrow 1. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}} = 1.$$

**Problem 4.** Let  $a, b$  be two integers and suppose that  $n$  is a positive integer for which the set

$$\mathbb{Z} \setminus \{ax^n + by^n \mid x, y \in \mathbb{Z}\}$$

is finite. Prove that  $n = 1$ .

**Solution.** Assume that  $n > 1$ . Notice that  $n$  may be replaced by any prime divisor  $p$  of  $n$ . Moreover,  $a$  and  $b$  should be coprime, otherwise the numbers not divisible by the greatest common divisor of  $a, b$  cannot be represented as  $ax^n + by^n$ .

If  $p = 2$ , then the number of the form  $ax^2 + by^2$  takes not all possible remainders modulo 8. If, say,  $b$  is even, then  $ax^2$  takes at most three different remainders modulo 8,  $by^2$  takes at most two, hence  $ax^2 + by^2$  takes at most  $3 \times 2 = 6$  different remainders. If both  $a$  and  $b$  are odd, then  $ax^2 + by^2 \equiv x^2 \pm y^2 \pmod{4}$ ; the expression  $x^2 + y^2$  does not take the remainder 3 modulo 4 and  $x^2 - y^2$  does not take the remainder 2 modulo 4.

Consider the case when  $p \geq 3$ . The  $p$ th powers take exactly  $p$  different remainders modulo  $p^2$ . Indeed,  $(x + kp)^p$  and  $x^p$  have the same remainder modulo  $p^2$ , and all numbers  $0^p, 1^p, \dots, (p-1)^p$  are different even modulo  $p$ . So,  $ax^p + by^p$  take at most  $p^2$  different remainders modulo  $p^2$ . If it takes strictly less than  $p^2$  different remainders modulo  $p^2$ , we get infinitely many non-representable numbers. If it takes exactly  $p^2$  remainders, then  $ax^p + by^p$  is divisible by  $p^2$  only if both  $x$  and  $y$  are divisible by  $p$ . Hence if  $ax^p + by^p$  is divisible by  $p^2$ , it is also divisible by  $p^p$ . Again we get infinitely many non-representable numbers, for example the numbers congruent to  $p^2$  modulo  $p^3$  are non-representable.

**Problem 5.** Suppose that  $a, b, c$  are real numbers in the interval  $[-1, 1]$  such that

$$1 + 2abc \geq a^2 + b^2 + c^2.$$

Prove that

$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$$

for all positive integers  $n$ .

**Solution 1.** Consider the symmetric matrix

$$A = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}.$$

By the constraint we have  $\det A \geq 0$  and  $\det \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}, \det \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \geq 0$ . Hence  $A$  is positive semidefinite, and  $A = B^2$  for some symmetric real matrix  $B$ .

Let the rows of  $B$  be  $x, y, z$ . Then  $|x| = |y| = |z| = 1$ ,  $a = x \cdot y$ ,  $b = y \cdot z$  and  $c = z \cdot x$ , where  $|x|$  and  $x \cdot y$  denote the Euclidean norm and scalar product. Denote by  $X = \otimes^n x$ ,  $Y = \otimes^n y$ ,  $Z = \otimes^n z$  the  $n$ th tensor powers, which belong to  $\mathbb{R}^{3^n}$ . Then  $|X| = |Y| = |Z| = 1$ ,  $X \cdot Y = a^n$ ,  $Y \cdot Z = b^n$  and  $Z \cdot X = c^n$ .

So, the matrix  $\begin{pmatrix} 1 & a^n & b^n \\ a^n & 1 & c^n \\ b^n & c^n & 1 \end{pmatrix}$ , being the Gram matrix of three vectors in  $\mathbb{R}^{3^n}$ , is positive semidefinite, and its determinant,  $1 + 2(abc)^n - a^{2n} - b^{2n} - c^{2n}$  is non-negative.

**Solution 2.** The constraint can be written as

$$(a - bc)^2 \leq (1 - b^2)(1 - c^2). \quad (1)$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} (a^{n-1} + a^{n-2}bc + \dots + b^{n-1}c^{n-1})^2 &\leq (|a|^{n-1} + |a|^{n-2}|bc| + \dots + |bc|^{n-1})^2 \leq \\ &\leq (1 + |bc| + \dots + |bc|^{n-1})^2 \leq (1 + |b|^2 + \dots + |b|^{2(n-1)})(1 + |c|^2 + \dots + |c|^{2(n-1)}) \end{aligned}$$

Multiplying by (1), we get

$$\begin{aligned} (a - bc)^2(a^{n-1} + a^{n-2}bc + \dots + b^{n-1}c^{n-1})^2 &\leq \\ &\left( (1 - b^2)(1 + |b|^2 + \dots + |b|^{2(n-1)}) \right) \left( (1 - c^2)(1 + |c|^2 + \dots + |c|^{2(n-1)}) \right), \\ (a^n - b^n c^n)^2 &\leq (1 - b^n)(1 - c^n), \\ 1 + 2(abc)^n &\geq a^{2n} + b^{2n} + c^{2n}. \end{aligned}$$

# IMC2010, Blagoevgrad, Bulgaria

## Day 2, July 27, 2010

**Problem 1.** (a) A sequence  $x_1, x_2, \dots$  of real numbers satisfies

$$x_{n+1} = x_n \cos x_n \quad \text{for all } n \geq 1.$$

Does it follow that this sequence converges for all initial values  $x_1$ ?

(b) A sequence  $y_1, y_2, \dots$  of real numbers satisfies

$$y_{n+1} = y_n \sin y_n \quad \text{for all } n \geq 1.$$

Does it follow that this sequence converges for all initial values  $y_1$ ?

**Solution 1.** (a) NO. For example, for  $x_1 = \pi$  we have  $x_n = (-1)^{n-1}\pi$ , and the sequence is divergent.

(b) YES. Notice that  $|y_n|$  is nonincreasing and hence converges to some number  $a \geq 0$ .

If  $a = 0$ , then  $\lim y_n = 0$  and we are done. If  $a > 0$ , then  $a = \lim |y_{n+1}| = \lim |y_n \sin y_n| = a \cdot |\sin a|$ , so  $\sin a = \pm 1$  and  $a = (k + \frac{1}{2})\pi$  for some nonnegative integer  $k$ .

Since the sequence  $|y_n|$  is nonincreasing, there exists an index  $n_0$  such that  $(k + \frac{1}{2})\pi \leq |y_n| < (k + 1)\pi$  for all  $n > n_0$ . Then all the numbers  $y_{n_0+1}, y_{n_0+2}, \dots$  lie in the union of the intervals  $[(k + \frac{1}{2})\pi, (k + 1)\pi)$  and  $(-(k + 1)\pi, -(k + \frac{1}{2})\pi]$ .

Depending on the parity of  $k$ , in one of the intervals  $[(k + \frac{1}{2})\pi, (k + 1)\pi)$  and  $(-(k + 1)\pi, -(k + \frac{1}{2})\pi]$  the values of the sine function is positive; denote this interval by  $I_+$ . In the other interval the sine function is negative; denote this interval by  $I_-$ . If  $y_n \in I_-$  for some  $n > n_0$  then  $y_n$  and  $y_{n+1} = y_n \sin y_n$  have opposite signs, so  $y_{n+1} \in I_+$ . On the other hand, if  $y_n \in I_+$  for some  $n > n_0$  then  $y_n$  and  $y_{n+1}$  have the same sign, so  $y_{n+1} \in I_+$ . In both cases,  $y_{n+1} \in I_+$ .

We obtained that the numbers  $y_{n_0+2}, y_{n_0+3}, \dots$  lie in  $I_+$ , so they have the same sign. Since  $|y_n|$  is convergent, this implies that the sequence  $(y_n)$  is convergent as well.

**Solution 2 for part (b).** Similarly to the first solution,  $|y_n| \rightarrow a$  for some real number  $a$ .

Notice that  $t \cdot \sin t = (-t) \sin(-t) = |t| \sin |t|$  for all real  $t$ , hence  $y_{n+1} = |y_n| \sin |y_n|$  for all  $n \geq 2$ . Since the function  $t \mapsto t \sin t$  is continuous,  $y_{n+1} = |y_n| \sin |y_n| \rightarrow |a| \sin |a| = a$ .

**Problem 2.** Let  $a_0, a_1, \dots, a_n$  be positive real numbers such that  $a_{k+1} - a_k \geq 1$  for all  $k = 0, 1, \dots, n - 1$ . Prove that

$$1 + \frac{1}{a_0} \left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right) \left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_n}\right).$$

**Solution.** Apply induction on  $n$ . Considering the empty product as 1, we have equality for  $n = 0$ .

Now assume that the statement is true for some  $n$  and prove it for  $n + 1$ . For  $n + 1$ , the statement can be written as the sum of the inequalities

$$1 + \frac{1}{a_0} \left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \leq \left(1 + \frac{1}{a_0}\right) \cdots \left(1 + \frac{1}{a_n}\right)$$

(which is the induction hypothesis) and

$$\frac{1}{a_0} \left(1 + \frac{1}{a_1 - a_0}\right) \cdots \left(1 + \frac{1}{a_n - a_0}\right) \cdot \frac{1}{a_{n+1} - a_0} \leq \left(1 + \frac{1}{a_0}\right) \cdots \left(1 + \frac{1}{a_n}\right) \cdot \frac{1}{a_{n+1}}. \quad (1)$$

Hence, to complete the solution it is sufficient to prove (1).

To prove (1), apply a second induction. For  $n = 0$ , we have to verify

$$\frac{1}{a_0} \cdot \frac{1}{a_1 - a_0} \leq \left(1 + \frac{1}{a_0}\right) \frac{1}{a_1}.$$

Multiplying by  $a_0 a_1 (a_1 - a_0)$ , this is equivalent with

$$\begin{aligned} a_1 &\leq (a_0 + 1)(a_1 - a_0) \\ a_0 &\leq a_0 a_1 - a_0^2 \\ 1 &\leq a_1 - a_0. \end{aligned}$$

For the induction step it is sufficient that

$$\left(1 + \frac{1}{a_{n+1} - a_0}\right) \cdot \frac{a_{n+1} - a_0}{a_{n+2} - a_0} \leq \left(1 + \frac{1}{a_{n+1}}\right) \cdot \frac{a_{n+1}}{a_{n+2}}.$$

Multiplying by  $(a_{n+2} - a_0)a_{n+2}$ ,

$$\begin{aligned} (a_{n+1} - a_0 + 1)a_{n+2} &\leq (a_{n+1} + 1)(a_{n+2} - a_0) \\ a_0 &\leq a_0 a_{n+2} - a_0 a_{n+1} \\ 1 &\leq a_{n+2} - a_{n+1}. \end{aligned}$$

**Remark 1.** It is easy to check from the solution that equality holds if and only if  $a_{k+1} - a_k = 1$  for all  $k$ .

**Remark 2.** The statement of the problem is a direct corollary of the identity

$$1 + \sum_{i=0}^n \left( \frac{1}{x_i} \prod_{j \neq i} \left(1 + \frac{1}{x_j - x_i}\right) \right) = \prod_{i=0}^n \left(1 + \frac{1}{x_i}\right).$$

**Problem 3.** Denote by  $S_n$  the group of permutations of the sequence  $(1, 2, \dots, n)$ . Suppose that  $G$  is a subgroup of  $S_n$ , such that for every  $\pi \in G \setminus \{e\}$  there exists a unique  $k \in \{1, 2, \dots, n\}$  for which  $\pi(k) = k$ . (Here  $e$  is the unit element in the group  $S_n$ .) Show that this  $k$  is the same for all  $\pi \in G \setminus \{e\}$ .

**Solution.** Let us consider the action of  $G$  on the set  $X = \{1, \dots, n\}$ . Let

$$G_x = \{g \in G: g(x) = x\} \quad \text{and} \quad Gx = \{g(x): g \in G\}$$

be the stabilizer and the orbit of  $x \in X$  under this action, respectively. The condition of the problem states that

$$G = \bigcup_{x \in X} G_x \tag{1}$$

and

$$G_x \cap G_y = \{e\} \quad \text{for all} \quad x \neq y. \tag{2}$$

We need to prove that  $G_x = G$  for some  $x \in X$ .

Let  $Gx_1, \dots, Gx_k$  be the distinct orbits of the action of  $G$ . Then one can write (1) as

$$G = \bigcup_{i=1}^k \bigcup_{y \in Gx_i} G_y. \tag{3}$$

It is well known that

$$|Gx| = \frac{|G|}{|G_x|}. \quad (4)$$

Also note that if  $y \in Gx$  then  $Gy = Gx$  and thus  $|Gy| = |Gx|$ . Therefore,

$$|G_x| = \frac{|G|}{|Gx|} = \frac{|G|}{|Gy|} = |G_y| \quad \text{for all } y \in Gx. \quad (5)$$

Combining (3), (2), (4) and (5) we get

$$|G| - 1 = |G \setminus \{e\}| = \left| \bigcup_{i=1}^k \bigcup_{y \in Gx_i} G_y \setminus \{e\} \right| = \sum_{i=1}^k \frac{|G|}{|G_{x_i}|} (|G_{x_i}| - 1),$$

hence

$$1 - \frac{1}{|G|} = \sum_{i=1}^k \left( 1 - \frac{1}{|G_{x_i}|} \right). \quad (6)$$

If for some  $i, j \in \{1, \dots, k\}$   $|G_{x_i}|, |G_{x_j}| \geq 2$  then

$$\sum_{i=1}^k \left( 1 - \frac{1}{|G_{x_i}|} \right) \geq \left( 1 - \frac{1}{2} \right) + \left( 1 - \frac{1}{2} \right) = 1 > 1 - \frac{1}{|G|}$$

which contradicts with (6), thus we can assume that

$$|G_{x_1}| = \dots = |G_{x_{k-1}}| = 1.$$

Then from (6) we get  $|G_{x_k}| = |G|$ , hence  $G_{x_k} = G$ .

**Problem 4.** Let  $A$  be a symmetric  $m \times m$  matrix over the two-element field all of whose diagonal entries are zero. Prove that for every positive integer  $n$  each column of the matrix  $A^n$  has a zero entry.

**Solution.** Denote by  $e_k$  ( $1 \leq k \leq m$ ) the  $m$ -dimensional vector over  $F_2$ , whose  $k$ -th entry is 1 and all the other elements are 0. Furthermore, let  $u$  be the vector whose all entries are 1. The  $k$ -th column of  $A^n$  is  $A^n e_k$ . So the statement can be written as  $A^n e_k \neq u$  for all  $1 \leq k \leq m$  and all  $n \geq 1$ .

For every pair of vectors  $x = (x_1, \dots, x_m)$  and  $y = (y_1, \dots, y_m)$ , define the bilinear form  $(x, y) = x^T y = x_1 y_1 + \dots + x_m y_m$ . The product  $(x, y)$  has all basic properties of scalar products (except the property that  $(x, x) = 0$  implies  $x = 0$ ). Moreover, we have  $(x, x) = (x, u)$  for every vector  $x \in F_2^m$ .

It is also easy to check that  $(w, Aw) = w^T Aw = 0$  for all vectors  $w$ , since  $A$  is symmetric and its diagonal elements are 0.

*Lemma.* Suppose that  $v \in F_2^m$  a vector such that  $A^n v = u$  for some  $n \geq 1$ . Then  $(v, v) = 0$ .

*Proof.* Apply induction on  $n$ . For odd values of  $n$  we prove the lemma directly. Let  $n = 2k + 1$  and  $w = A^k v$ . Then

$$(v, v) = (v, u) = (v, A^n v) = v^T A^n v = v^T A^{2k+1} v = (A^k v, A^{k+1} v) = (w, Aw) = 0.$$

Now suppose that  $n$  is even,  $n = 2k$ , and the lemma is true for all smaller values of  $n$ . Let  $w = A^k v$ ; then  $A^k w = A^n v = u$  and thus we have  $(w, w) = 0$  by the induction hypothesis. Hence,

$$(v, v) = (v, u) = v^T A^n v = v^T A^{2k} v = (A^k v)^T (A^k v) = (A^k v, A^k v) = (w, w) = 0.$$

The lemma is proved.

Now suppose that  $A^n e_k = u$  for some  $1 \leq k \leq m$  and positive integer  $n$ . By the Lemma, we should have  $(e_k, e_k) = 0$ . But this is impossible because  $(e_k, e_k) = 1 \neq 0$ .

**Problem 5.** Suppose that for a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  and real numbers  $a < b$  one has  $f(x) = 0$  for all  $x \in (a, b)$ . Prove that  $f(x) = 0$  for all  $x \in \mathbb{R}$  if

$$\sum_{k=0}^{p-1} f\left(y + \frac{k}{p}\right) = 0$$

for every prime number  $p$  and every real number  $y$ .

**Solution.** Let  $N > 1$  be some integer to be defined later, and consider set of real polynomials

$$\mathcal{J}_N = \left\{ c_0 + c_1x + \dots + c_nx^n \in \mathbb{R}[x] \mid \forall x \in \mathbb{R} \sum_{k=0}^n c_k f\left(x + \frac{k}{N}\right) = 0 \right\}.$$

Notice that  $0 \in \mathcal{J}_N$ , any linear combinations of any elements in  $\mathcal{J}_N$  is in  $\mathcal{J}_N$ , and for every  $P(x) \in \mathcal{J}_N$  we have  $xP(x) \in \mathcal{J}_N$ . Hence,  $\mathcal{J}_N$  is an ideal of the ring  $\mathbb{R}[x]$ .

By the problem's conditions, for every prime divisors of  $N$  we have  $\frac{x^N - 1}{x^{N/p} - 1} \in \mathcal{J}_N$ . Since  $\mathbb{R}[x]$  is a principal ideal domain (due to the Euclidean algorithm), the greatest common divisor of these polynomials is an element of  $\mathcal{J}_N$ . The complex roots of the polynomial  $\frac{x^N - 1}{x^{N/p} - 1}$  are those  $N$ th roots of unity whose order does not divide  $N/p$ . The roots of the greatest common divisor is the intersection of such sets; it can be seen that the intersection consist of the primitive  $N$ th roots of unity. Therefore,

$$\gcd \left\{ \frac{x^N - 1}{x^{N/p} - 1} \mid p|N \right\} = \Phi_N(x)$$

is the  $N$ th cyclotomic polynomial. So  $\Phi_N \in \mathcal{J}_N$ , which polynomial has degree  $\varphi(N)$ .

Now choose  $N$  in such a way that  $\frac{\varphi(N)}{N} < b - a$ . It is well-known that  $\liminf_{N \rightarrow \infty} \frac{\varphi(N)}{N} = 0$ , so there exists such a value for  $N$ . Let  $\Phi_N(x) = a_0 + a_1x + \dots + a_{\varphi(N)}x^{\varphi(N)}$  where  $a_{\varphi(N)} = 1$  and  $|a_0| = 1$ . Then, by the definition of  $\mathcal{J}_N$ , we have  $\sum_{k=0}^{\varphi(N)} a_k f\left(x + \frac{k}{N}\right) = 0$  for all  $x \in \mathbb{R}$ .

If  $x \in [b, b + \frac{1}{N})$ , then

$$f(x) = - \sum_{k=0}^{\varphi(N)-1} a_k f\left(x - \frac{\varphi(N)-k}{N}\right).$$

On the right-hand side, all numbers  $x - \frac{\varphi(N)-k}{N}$  lie in  $(a, b)$ . Therefore the right-hand side is zero, and  $f(x) = 0$  for all  $x \in [b, b + \frac{1}{N})$ . It can be obtained similarly that  $f(x) = 0$  for all  $x \in (a - \frac{1}{N}, a]$  as well. Hence,  $f = 0$  in the interval  $(a - \frac{1}{N}, b + \frac{1}{N})$ . Continuing in this fashion we see that  $f$  must vanish everywhere.

# IMC2011, Blagoevgrad, Bulgaria

Day 1, July 30, 2011

**Problem 1.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. A point  $x$  is called a *shadow point* if there exists a point  $y \in \mathbb{R}$  with  $y > x$  such that  $f(y) > f(x)$ . Let  $a < b$  be real numbers and suppose that

- all the points of the open interval  $I = (a, b)$  are shadow points;
- $a$  and  $b$  are not shadow points.

Prove that

- $f(x) \leq f(b)$  for all  $a < x < b$ ;
- $f(a) = f(b)$ .

(José Luis Díaz-Barrero, Barcelona)

**Solution.** (a) We prove by contradiction. Suppose that exists a point  $c \in (a, b)$  such that  $f(c) > f(b)$ .

By Weierstrass' theorem,  $f$  has a maximal value  $m$  on  $[c, b]$ ; this value is attained at some point  $d \in [c, b]$ . Since  $f(d) = \max_{[c,b]} f \geq f(c) > f(b)$ , we have  $d \neq b$ , so  $d \in [c, b) \subset (a, b)$ . The point  $d$ , lying in  $(a, b)$ , is a shadow point, therefore  $f(y) > f(d)$  for some  $y > d$ . From combining our inequalities we get  $f(y) > f(d) > f(b)$ .

Case 1:  $y > b$ . Then  $f(y) > f(b)$  contradicts the assumption that  $b$  is not a shadow point.

Case 2:  $y \leq b$ . Then  $y \in (d, b] \subset [c, b]$ , therefore  $f(y) > f(d) = m = \max_{[c,b]} f \geq f(y)$ , contradiction again.

(b) Since  $a < b$  and  $a$  is not a shadow point, we have  $f(a) \geq f(b)$ .

By part (a), we already have  $f(x) \leq f(b)$  for all  $x \in (a, b)$ . By the continuity at  $a$  we have

$$f(a) = \lim_{x \rightarrow a+0} f(x) \leq \lim_{x \rightarrow a+0} f(b) = f(b)$$

Hence we have both  $f(a) \geq f(b)$  and  $f(a) \leq f(b)$ , so  $f(a) = f(b)$ .

**Problem 2.** Does there exist a real  $3 \times 3$  matrix  $A$  such that  $\text{tr}(A) = 0$  and  $A^2 + A^t = I$ ? ( $\text{tr}(A)$  denotes the trace of  $A$ ,  $A^t$  is the transpose of  $A$ , and  $I$  is the identity matrix.)

(Moubinool Omarjee, Paris)

**Solution.** The answer is NO.

Suppose that  $\text{tr}(A) = 0$  and  $A^2 + A^t = I$ . Taking the transpose, we have

$$A = I - (A^2)^t = I - (A^t)^2 = I - (I - A^2)^2 = 2A^2 - A^4,$$

$$A^4 - 2A^2 + A = 0.$$

The roots of the polynomial  $x^4 - 2x^2 + x = x(x-1)(x^2+x-1)$  are  $0, 1, \frac{-1 \pm \sqrt{5}}{2}$  so these numbers can be the eigenvalues of  $A$ ; the eigenvalues of  $A^2$  can be  $0, 1, \frac{1 \pm \sqrt{5}}{2}$ .

By  $\text{tr}(A) = 0$ , the sum of the eigenvalues is 0, and by  $\text{tr}(A^2) = \text{tr}(I - A^t) = 3$  the sum of squares of the eigenvalues is 3. It is easy to check that this two conditions cannot be satisfied simultaneously.

**Problem 3.** Let  $p$  be a prime number. Call a positive integer  $n$  *interesting* if

$$x^n - 1 = (x^p - x + 1)f(x) + pg(x)$$

for some polynomials  $f$  and  $g$  with integer coefficients.

- Prove that the number  $p^p - 1$  is interesting.
- For which  $p$  is  $p^p - 1$  the minimal interesting number?

(Eugene Goryachko and Fedor Petrov, St. Petersburg)

**Solution.** (a) Let's reformulate the property of being interesting:  $n$  is interesting if  $x^n - 1$  is divisible by  $x^p - x + 1$  in the ring of polynomials over  $\mathbb{F}_p$  (the field of residues modulo  $p$ ). All further congruences are modulo  $x^p - x + 1$  in this ring. We have  $x^p \equiv x - 1$ , then  $x^{p^2} = (x^p)^p \equiv (x - 1)^p \equiv x^p - 1 \equiv x - 2$ ,  $x^{p^3} = (x^{p^2})^p \equiv (x - 2)^p \equiv x^p - 2^p \equiv x - 2^p - 1 \equiv x - 3$  and so on by Fermat's little theorem, finally  $x^{p^p} \equiv x - p \equiv x$ ,

$$x(x^{p^p-1} - 1) \equiv 0.$$

Since the polynomials  $x^p - x + 1$  and  $x$  are coprime, this implies  $x^{p^p-1} - 1 \equiv 0$ .

(b) We write

$$x^{1+p+p^2+\dots+p^{p-1}} = x \cdot x^p \cdot x^{p^2} \cdot \dots \cdot x^{p^{p-1}} \equiv x(x-1)(x-2)\dots(x-(p-1)) = x^p - x \equiv -1,$$

hence  $x^{2(1+p+p^2+\dots+p^{p-1})} \equiv 1$  and  $a = 2(1+p+p^2+\dots+p^{p-1})$  is an interesting number.

If  $p > 3$ , then  $a = \frac{2}{p-1}(p^p - 1) < p^p - 1$ , so we have an interesting number less than  $p^p - 1$ . On the other hand, we show that  $p = 2$  and  $p = 3$  do satisfy the condition. First notice that by  $\gcd(x^m - 1, x^k - 1) = x^{\gcd(m,k)} - 1$ , for every fixed  $p$  the greatest common divisors of interesting numbers is also an interesting number. Therefore the minimal interesting number divides all interesting numbers. In particular, the minimal interesting number is a divisor of  $p^p - 1$ .

For  $p = 2$  we have  $p^p - 1 = 3$ , so the minimal interesting number is 1 or 3. But  $x^2 - x + 1$  does not divide  $x - 1$ , so 1 is not interesting. Then the minimal interesting number is 3.

For  $p = 3$  we have  $p^p - 1 = 26$  whose divisors are 1, 2, 13, 26. The numbers 1 and 2 are too small and  $x^{13} \equiv -1 \not\equiv +1$  as shown above, so none of 1, 2 and 13 is interesting. So 26 is the minimal interesting number.

Hence,  $p^p - 1$  is the minimal interesting number if and only if  $p = 2$  or  $p = 3$ .

**Problem 4.** Let  $A_1, A_2, \dots, A_n$  be finite, nonempty sets. Define the function

$$f(t) = \sum_{k=1}^n \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}|}.$$

Prove that  $f$  is nondecreasing on  $[0, 1]$ .

( $|A|$  denotes the number of elements in  $A$ .)

(Levon Nurbekyan and Vardan Voskanyan, Yerevan)

**Solution 1.** Let  $\Omega = \bigcup_{i=1}^n A_i$ . Consider a random subset  $X$  of  $\Omega$  which chosen in the following way: for each  $x \in \Omega$ , choose the element  $x$  for the set  $X$  with probability  $t$ , independently from the other elements.

Then for any set  $C \subset \Omega$ , we have

$$P(C \subset X) = t^{|C|}.$$

By the *inclusion-exclusion principle*,

$$\begin{aligned} & P((A_1 \subset X) \text{ or } (A_2 \subset X) \text{ or } \dots \text{ or } (A_n \subset X)) = \\ &= \sum_{k=1}^n \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (-1)^{k-1} P(A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k} \subset X) = \\ &= \sum_{k=1}^n \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \dots \cup A_{i_k}|}. \end{aligned}$$

The probability  $P((A_1 \subset X) \text{ or } \dots \text{ or } (A_n \subset X))$  is a nondecreasing function of the probability  $t$ .

**Problem 5.** Let  $n$  be a positive integer and let  $V$  be a  $(2n - 1)$ -dimensional vector space over the two-element field. Prove that for arbitrary vectors  $v_1, \dots, v_{4n-1} \in V$ , there exists a sequence  $1 \leq i_1 < \dots < i_{2n} \leq 4n - 1$  of indices such that  $v_{i_1} + \dots + v_{i_{2n}} = 0$ .

(Ilya Bogdanov, Moscow and Géza Kós, Budapest)

**Solution.** Let  $V = \text{aff}\{v_1, \dots, v_{4n-1}\}$ . The statement  $v_{i_1} + \dots + v_{i_{2n}} = 0$  is translation-invariant (i.e. replacing the vectors by  $v_1 - a, \dots, v_{4n-1} - a$ ), so we may assume that  $0 \in V$ . Let  $d = \dim V$ .

*Lemma.* The vectors can be permuted in such a way that  $v_1 + v_2, v_3 + v_4, \dots, v_{2d-1} + v_{2d}$  form a basis of  $V$ .

*Proof.* We prove by induction on  $d$ . If  $d = 0$  or  $d = 1$  then the statement is trivial.

First choose the vector  $v_1$  such a way that  $\text{aff}(v_2, v_3, \dots, v_{4n-1}) = V$ ; this is possible since  $V$  is generated by some  $d + 1$  vectors and we have  $d + 1 \leq 2n < 4n - 1$ . Next, choose  $v_2$  such that  $v_2 \neq v_1$ . (By  $d > 0$ , not all vectors are the same.)

Now let  $\ell = \{0, v_1 + v_2\}$  and let  $V' = V/\ell$ . For any  $w \in V$ , let  $\tilde{w} = \ell + w = \{w, w + v_1 + v_2\}$  be the class of the factor space  $V'$  containing  $w$ . Apply the induction hypothesis to the vectors  $\tilde{v}_3, \dots, \tilde{v}_{4n-1}$ . Since  $\dim V' = d - 1$ , the vectors can be permuted in such a way that  $\tilde{v}_3 + \tilde{v}_4, \dots, \tilde{v}_{2d-1} + \tilde{v}_{2d}$  is a basis of  $V'$ . Then  $v_1 + v_2, v_3 + v_4, \dots, v_{2d-1} + v_{2d}$  is a basis of  $V$ .

Now we can assume that  $v_1 + v_2, v_3 + v_4, \dots, v_{2d-1} + v_{2d}$  is a basis of  $V$ . The vector  $w = (v_1 + v_3 + \dots + v_{2d-1}) + (v_{2d+1} + v_{2d+2} + \dots + v_{2n+d})$  is the sum of  $2n$  vectors, so  $w \in V$ . Hence,  $w + \varepsilon_1(v_1 + v_2) + \dots + \varepsilon_d(v_{2d-1} + v_{2d}) = 0$  with some  $\varepsilon_1, \dots, \varepsilon_d \in \mathbb{F}_2$ , therefore

$$\sum_{i=1}^d \left( (1 - \varepsilon_i)v_{2i-1} + \varepsilon_i v_{2i} \right) + \sum_{i=2d+1}^{2n+d} v_i = 0.$$

The left-hand side is the sum of  $2n$  vectors.

# IMC2011, Blagoevgrad, Bulgaria

Day 2, July 31, 2011

**Problem 1.** Let  $(a_n)_{n=0}^{\infty}$  be a sequence with  $\frac{1}{2} < a_n < 1$  for all  $n \geq 0$ . Define the sequence  $(x_n)_{n=0}^{\infty}$  by

$$x_0 = a_0, \quad x_{n+1} = \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} \quad (n \geq 0).$$

What are the possible values of  $\lim_{n \rightarrow \infty} x_n$ ? Can such a sequence diverge?

Johnson Olaleru, Lagos

**Solution 1.** We prove by induction that

$$0 < 1 - x_n < \frac{1}{2^{n+1}}.$$

Then we will have  $(1 - x_n) \rightarrow 0$  and therefore  $x_n \rightarrow 1$ .

The case  $n = 0$  is true since  $\frac{1}{2} < x_0 = a_0 < 1$ .

Supposing that the induction hypothesis holds for  $n$ , from the recurrence relation we get

$$1 - x_{n+1} = 1 - \frac{a_{n+1} + x_n}{1 + a_{n+1}x_n} = \frac{1 - a_{n+1}}{1 + a_{n+1}x_n}(1 - x_n).$$

By

$$0 < \frac{1 - a_{n+1}}{1 + a_{n+1}x_n} < \frac{1 - \frac{1}{2}}{1 + 0} = \frac{1}{2}$$

we obtain

$$0 < 1 - x_{n+1} < \frac{1}{2}(1 - x_n) < \frac{1}{2} \cdot \frac{1}{2^{n+1}} = \frac{1}{2^{n+2}}.$$

Hence, the sequence converges in all cases and  $x_n \rightarrow 1$ .

**Solution 2.** As is well-known,

$$\tanh(u + v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

for all real numbers  $u$  and  $v$ .

Setting  $u_n = \operatorname{ar} \tanh a_n$  we have  $x_n = \tanh(u_0 + u_1 + \dots + u_n)$ . Then  $u_0 + u_1 + \dots + u_n > (n + 1)\operatorname{ar} \tanh \frac{1}{2}$  and  $\lim_{n \rightarrow \infty} x_n = \lim_{u \rightarrow \infty} \tanh u = 1$ .

**Remark.** If the condition  $a_n \in (\frac{1}{2}, 1)$  is replaced by  $a_n \in (0, 1)$  then the sequence remains increasing and bounded, but the limit can be less than 1.

**Problem 2.** An alien race has three genders: male, female, and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual — if  $x$  likes  $y$ , then  $y$  likes  $x$ .

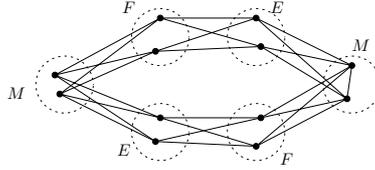
The race is sending an expedition to colonize a planet. The expedition has  $n$  males,  $n$  females, and  $n$  emales. It is known that every expedition member likes at least  $k$  persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

- Show that if  $n$  is even and  $k = \frac{n}{2}$ , then it might be impossible to create even one married triple.
- Show that if  $k \geq \frac{3n}{4}$ , then it is always possible to create  $n$  disjoint married triples, thus marrying all of the expedition members.

Fedor Duzhin and Nick Gravin, Singapore

**Solution.** (a) Let  $M$  be the set of males,  $F$  the set of females, and  $E$  the set of emales. Consider the (tripartite) graph  $G$  with vertices  $M \cup F \cup E$  and edges for likes. A 3-cycle is then a possible family. We'll call  $G$  the *graph of likes*.

First, let  $k = \frac{n}{2}$ . Then  $n$  has to be even and we need to construct a graph of likes with no 3-cycles. We'll do the following: divide each of the sets  $M$ ,  $F$ , and  $E$  into two equal parts and draw all edges between two parts as shown below:



Clearly, there is no 3-cycle.

(b) First divide the the expedition into male-emale-female triples arbitrarily. Let the *unhappiness* of such a subdivision be the number of pairs of aliens that belong to the same triple but don't like each other. We shall show that if unhappiness is positive, then the unhappiness can be decreased by a simple operation. It will follow that after several steps the unhappiness will be reduced to zero, which will lead to the happy marriage of everybody.

Assume that we have an emale which doesn't like at least one member of its triple (the other cases are similar). We perform the following operation: we swap this emale with another emale, so that each of these two emales will like the members of their new triples. Thus the unhappiness related to this emales will decrease, and the other pairs that contribute to the unhappiness remain unchanged, therefore the unhappiness will be decreased.

So, it remains to prove that such an operation is always possible. Enumerate the triples with  $1, 2, \dots, n$  and denote by  $E_i, F_i, M_i$  the emale, female, and male members of the  $i$ th triple, respectively. Without loss of generality we may assume that  $E_1$  doesn't like either  $F_1$  or  $M_1$  or both. We have to find an index  $i > 1$  such that  $E_i$  likes the couple  $F_1, M_1$  and  $E_1$  likes the couple  $F_i, M_i$ ; then we can swap  $E_1$  and  $E_i$ .

There are at most  $n/4$  indices  $i$  for which  $E_1$  dislikes  $F_i$  and at most  $n/4$  indices for which  $E_1$  dislikes  $M_i$ , so there are no more than  $n/2$  indices  $i$  for which  $E_1$  dislikes someone from the couple  $M_i, F_i$ , and the set of these undesirable indexes includes 1. Similarly, there are no more than  $n/2$  indices such that either  $M_1$  or  $F_1$  dislikes  $E_i$ . Since both undesirable sets of indices have at most  $n/2$  elements and both contain 1, their union doesn't cover all indices, so we have some  $i$  which satisfies all conditions. Therefore we can always perform the operation that decreases unhappiness.

**Solution 2 (for part b).** Suppose that  $k \geq \frac{3n}{4}$  and let's show that it's possible to marry all of the colonists. First, we'll prove that there exists a perfect matching between  $M$  and  $F$ . We need to check the condition of Hall's marriage theorem. In other words, for  $A \subset M$ , let  $B \subset F$  be the set of all vertices of  $F$  adjacent to at least one vertex of  $A$ . Then we need to show that  $|A| \leq |B|$ . Let us assume the contrary, that is  $|A| > |B|$ . Clearly,  $|B| \geq k$  if  $A$  is not empty. Let's consider any  $f \in F \setminus B$ . Then  $f$  is not adjacent to any vertex in  $A$ , therefore,  $f$  has degree in  $M$  not more than  $n - |A| < n - |B| \leq n - k \leq \frac{n}{4}$ , a contradiction.

Let's now construct a new bipartite graph, say  $H$ . The set of its vertices is  $P \cup E$ , where  $P$  is the set of pairs male-female from the perfect matching we just found. We will have an edge from  $(m, f) = p \in P$  to  $e \in E$  for each 3-cycle  $(m, f, e)$  of the graph  $G$ , where  $(m, f) \in P$  and  $e \in E$ . Notice that the degree of each vertex of  $P$  in  $H$  is then at least  $2k - n$ .

What remains is to show that  $H$  satisfies the condition of Hall's marriage theorem and hence has a perfect matching. Assume, on the contrary, that the following happens. There is  $A \subset P$  and  $B \subset E$  such that  $|A| = l$ ,  $|B| < l$ , and  $B$  is the set of all vertices of  $E$  adjacent to at least one vertex of  $A$ . Since the degree of each vertex of  $P$  is at least  $2k - n$ , we have  $2k - n \leq |B| < l$ . On the other hand, let  $e \in E \setminus B$ . Then for each pair  $(m, f) = p \in P$ , at most one of the pairs  $(e, m)$  and  $(e, f)$  is joined by an edge and hence the degree of  $e$  in  $G$  is at most  $|M \setminus A| + |F \setminus A| + |A| = 2(n - l) + l = 2n - l$ . But the degree of any vertex of  $G$  is  $2k$  and thus we get  $2k \leq 2n - l$ , that is,  $l \leq 2n - 2k$ .

Finally,  $2k - n < l \leq 2n - 2k$  implies that  $k < \frac{3n}{4}$ . This contradiction concludes the solution.

**Problem 3.** Determine the value of

$$\sum_{n=1}^{\infty} \ln \left( 1 + \frac{1}{n} \right) \cdot \ln \left( 1 + \frac{1}{2n} \right) \cdot \ln \left( 1 + \frac{1}{2n+1} \right).$$

**Solution.** Define  $f(n) = \ln\left(\frac{n+1}{n}\right)$  for  $n \geq 1$ , and observe that  $f(2n) + f(2n+1) = f(n)$ . The well-known inequality  $\ln(1+x) \leq x$  implies  $f(n) \leq 1/n$ . Furthermore introduce

$$g(n) = \sum_{k=n}^{2n-1} f^3(k) < n f^3(n) \leq 1/n^2.$$

Then

$$\begin{aligned} g(n) - g(n+1) &= f^3(n) - f^3(2n) - f^3(2n+1) \\ &= (f(2n) + f(2n+1))^3 - f^3(2n) - f^3(2n+1) \\ &= 3(f(2n) + f(2n+1))f(2n)f(2n+1) \\ &= 3f(n)f(2n)f(2n+1), \end{aligned}$$

therefore

$$\sum_{n=1}^N f(n)f(2n)f(2n+1) = \frac{1}{3} \sum_{n=1}^N g(n) - g(n+1) = \frac{1}{3} (g(1) - g(N+1)).$$

Since  $g(N+1) \rightarrow 0$  as  $N \rightarrow \infty$ , the value of the considered sum hence is

$$\sum_{n=1}^{\infty} f(n)f(2n)f(2n+1) = \frac{1}{3} g(1) = \frac{1}{3} \ln^3(2).$$

**Problem 4.** Let  $f(x)$  be a polynomial with real coefficients of degree  $n$ . Suppose that  $\frac{f(k) - f(m)}{k - m}$  is an integer for all integers  $0 \leq k < m \leq n$ . Prove that  $a - b$  divides  $f(a) - f(b)$  for all pairs of distinct integers  $a$  and  $b$ .

Fedor Petrov, St. Petersburg

**Solution 1.** We need the following

*Lemma.* Denote the least common multiple of  $1, 2, \dots, k$  by  $L(k)$ , and define

$$h_k(x) = L(k) \cdot \binom{x}{k} \quad (k = 1, 2, \dots).$$

Then the polynomial  $h_k(x)$  satisfies the condition, i.e.  $a - b$  divides  $h_k(a) - h_k(b)$  for all pairs of distinct integers  $a, b$ .

*Proof.* It is known that

$$\binom{a}{k} = \sum_{j=0}^k \binom{a-b}{j} \binom{b}{k-j}.$$

(This formula can be proved by comparing the coefficient of  $x^k$  in  $(1+x)^a$  and  $(1+x)^{a-b}(1+x)^b$ .) From here we get

$$h_k(a) - h_k(b) = L(k) \left( \binom{a}{k} - \binom{b}{k} \right) = L(k) \sum_{j=1}^k \binom{a-b}{j} \binom{b}{k-j} = (a-b) \sum_{j=1}^k \frac{L(k)}{j} \binom{a-b-1}{j-1} \binom{b}{k-j}.$$

On the right-hand side all fractions  $\frac{L(k)}{j}$  are integers, so the right-hand side is a multiple of  $(a, b)$ . The lemma is proved.

Expand the polynomial  $f$  in the basis  $1, \binom{x}{1}, \binom{x}{2}, \dots$  as

$$f(x) = A_0 + A_1 \binom{x}{1} + A_2 \binom{x}{2} + \dots + A_n \binom{x}{n}. \quad (1)$$

We prove by induction on  $j$  that  $A_j$  is a multiple of  $L(j)$  for  $1 \leq j \leq n$ . (In particular,  $A_j$  is an integer for  $j \geq 1$ .) Assume that  $L(j)$  divides  $A_j$  for  $1 \leq j \leq m-1$ . Substituting  $m$  and some  $k \in \{0, 1, \dots, m-1\}$  in (1),

$$\frac{f(m) - f(k)}{m - k} = \sum_{j=1}^{m-1} \frac{A_j}{L(j)} \cdot \frac{h_j(m) - h_j(k)}{m - k} + \frac{A_m}{m - k}.$$

Since all other terms are integers, the last term  $\frac{A_m}{m-k}$  is also an integer. This holds for all  $0 \leq k < m$ , so  $A_m$  is an integer that is divisible by  $L(m)$ .

Hence,  $A_j$  is a multiple of  $L(j)$  for every  $1 \leq j \leq n$ . By the lemma this implies the problem statement.

**Solution 2.** The statement of the problem follows immediately from the following claim, applied to the polynomial  $g(x, y) = \frac{f(x)-f(y)}{x-y}$ .

*Claim.* Let  $g(x, y)$  be a real polynomial of two variables with total degree less than  $n$ . Suppose that  $g(k, m)$  is an integer whenever  $0 \leq k < m \leq n$  are integers. Then  $g(k, m)$  is a integer for every pair  $k, m$  of integers.

*Proof.* Apply induction on  $n$ . If  $n = 1$  then  $g$  is a constant. This constant can be read from  $g(0, 1)$  which is an integer, so the claim is true.

Now suppose that  $n \geq 2$  and the claim holds for  $n - 1$ . Consider the polynomials

$$g_1(x, y) = g(x + 1, y + 1) - g(x, y + 1) \quad \text{and} \quad g_2(x, y) = g(x, y + 1) - g(x, y). \quad (1)$$

For every pair  $0 \leq k < m \leq n - 1$  of integers, the numbers  $g(k, m)$ ,  $g(k, m + 1)$  and  $g(k + 1, m + 1)$  are all integers, so  $g_1(k, m)$  and  $g_2(k, m)$  are integers, too. Moreover, in (1) the maximal degree terms of  $g$  cancel out, so  $\deg g_1, \deg g_2 < \deg g$ . Hence, we can apply the induction hypothesis to the polynomials  $g_1$  and  $g_2$  and we thus have  $g_1(k, m), g_2(k, m) \in \mathbb{Z}$  for all  $k, m \in \mathbb{Z}$ .

In view of (1), for all  $k, m \in \mathbb{Z}$ , we have that

- (a)  $g(0, 1) \in \mathbb{Z}$ ;
- (b)  $g(k, m) \in \mathbb{Z}$  if and only if  $g(k + 1, m + 1) \in \mathbb{Z}$ ;
- (c)  $g(k, m) \in \mathbb{Z}$  if and only if  $g(k, m + 1) \in \mathbb{Z}$ .

For arbitrary integers  $k, m$ , apply (b)  $|k|$  times then apply (c)  $|m - k - 1|$  times as

$$g(k, m) \in \mathbb{Z} \Leftrightarrow \dots \Leftrightarrow g(0, m - k) \in \mathbb{Z} \Leftrightarrow \dots \Leftrightarrow g(0, 1) \in \mathbb{Z}.$$

Hence,  $g(k, m) \in \mathbb{Z}$ . The claim has been proved.

**Problem 5.** Let  $F = A_0A_1 \dots A_n$  be a convex polygon in the plane. Define for all  $1 \leq k \leq n - 1$  the operation  $f_k$  which replaces  $F$  with a new polygon

$$f_k(F) = A_0 \dots A_{k-1}A'_kA_{k+1} \dots A_n,$$

where  $A'_k$  is the point symmetric to  $A_k$  with respect to the perpendicular bisector of  $A_{k-1}A_{k+1}$ . Prove that  $(f_1 \circ f_2 \circ \dots \circ f_{n-1})^n(F) = F$ . We suppose that all operations are well-defined on the polygons, to which they are applied, i.e. results are convex polygons again. ( $A_0, A_1, \dots, A_n$  are the vertices of  $F$  in consecutive order.)

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**Solution.** The operations  $f_i$  are rational maps on the  $2(n - 1)$ -dimensional phase space of coordinates of the vertices  $A_1, \dots, A_{n-1}$ . To show that  $(f_1 \circ f_2 \circ \dots \circ f_{n-1})^n$  is the identity, it is sufficient to verify this on some open set. For example, we can choose a neighborhood of the regular polygon, then all intermediate polygons in the proof will be convex.

Consider the operations  $f_i$ . Notice that (i)  $f_i \circ f_i = id$  and (ii)  $f_i \circ f_j = f_j \circ f_i$  for  $|i - j| \geq 2$ . We also show that (iii)  $(f_i \circ f_{i+1})^3 = id$  for  $1 \leq i \leq n - 1$ .

The operations  $f_i$  and  $f_{i+1}$  change the order of side lengths by interchanging two consecutive sides; after performing  $(f_i \circ f_{i+1})^3$ , the side lengths are in the original order. Moreover, the sums of opposite angles in the convex quadrilateral  $A_{i-1}A_iA_{i+1}A_{i+2}$  are preserved in all operations. These quantities uniquely determine the quadrilateral, because with fixed sides, both angles  $\angle A_1A_2A_3$  and  $\angle A_1A_4A_3$  decrease when  $A_1A_3$  increases. Hence, property (iii) is proved.

In the symmetric group  $S_n$ , the transpositions  $\sigma_i = (i, i + 1)$ , which form a generator system, satisfy the same properties (i–iii). It is well-known that  $S_n$  is the maximal group with  $n - 1$  generators, satisfying (i–iii). In  $S_n$  we have  $(\sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_{n-1})^n = (1, 2, 3, \dots, n)^n = id$ , so this implies  $(f_1 \circ f_2 \circ \dots \circ f_{n-1})^n = id$ .

**IMC 2012, Blagoevgrad, Bulgaria**  
**Day 1, July 28, 2012**

**Problem 1.** For every positive integer  $n$ , let  $p(n)$  denote the number of ways to express  $n$  as a sum of positive integers. For instance,  $p(4) = 5$  because

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Also define  $p(0) = 1$ .

Prove that  $p(n) - p(n - 1)$  is the number of ways to express  $n$  as a sum of integers each of which is strictly greater than 1.

(Proposed by Fedor Duzhin, Nanyang Technological University)

**Solution 1.** The statement is true for  $n = 1$ , because  $p(0) = p(1) = 1$  and the only partition of 1 contains the term 1. In the rest of the solution we assume  $n \geq 2$ .

Let  $\mathcal{P}_n = \{(a_1, \dots, a_k) : k \in \mathbb{N}, a_1 \geq \dots \geq a_k, a_1 + \dots + a_k = n\}$  be the set of partitions of  $n$ , and let  $\mathcal{Q}_n = \{(a_1, \dots, a_k) \in \mathcal{P}_n : a_k = 1\}$  the set of those partitions of  $n$  that contain the term 1. The set of those partitions of  $n$  that do not contain 1 as a term, is  $\mathcal{P}_n \setminus \mathcal{Q}_n$ . We have to prove that  $|\mathcal{P}_n \setminus \mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{P}_{n-1}|$ .

Define the map  $\varphi: \mathcal{P}_{n-1} \rightarrow \mathcal{Q}_n$  as

$$\varphi(a_1, \dots, a_k) = (a_1, \dots, a_k, 1).$$

This is a partition of  $n$  containing 1 as a term (so indeed  $\varphi(a_1, \dots, a_k) \in \mathcal{Q}_n$ ). Moreover, each partition  $(a_1, \dots, a_k, 1) \in \mathcal{Q}_n$  uniquely determines  $(a_1, \dots, a_k)$ . Therefore the map  $\varphi$  is a bijection between the sets  $\mathcal{P}_{n-1}$  and  $\mathcal{Q}_n$ . Then  $|\mathcal{P}_{n-1}| = |\mathcal{Q}_n|$ . Since  $\mathcal{Q}_n \subset \mathcal{P}_n$ ,

$$|\mathcal{P}_n \setminus \mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{Q}_n| = |\mathcal{P}_n| - |\mathcal{P}_{n-1}| = p(n) - p(n - 1).$$

**Solution 2 (outline).** Denote by  $q(n)$  the number of partitions of  $n$  not containing 1 as term ( $q(0) = 1$  as the only partition of 0 is the empty sum), and define the generating functions

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n \quad \text{and} \quad G(x) = \sum_{n=0}^{\infty} q(n)x^n.$$

Since  $q(n) \leq p(n) < 2^n$ , these series converge in some interval, say for  $|x| < \frac{1}{2}$ , and the values uniquely determine the coefficients.

According to Euler's argument, we have

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + \dots) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}$$

and

$$G(x) = \sum_{n=0}^{\infty} q(n)x^n = \prod_{k=2}^{\infty} (1 + x^k + x^{2k} + \dots) = \prod_{k=2}^{\infty} \frac{1}{1 - x^k}.$$

Then  $G(x) = (1 - x)F(x)$ . Comparing the coefficient of  $x^n$  in this identity we get  $q(n) = p(n) - p(n - 1)$ .

**Problem 2.** Let  $n$  be a fixed positive integer. Determine the smallest possible rank of an  $n \times n$  matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

(Proposed by Ilya Bogdanov and Grigoriy Chelnokov, MIPT, Moscow)

**Solution.** For  $n = 1$  the only matrix is  $(0)$  with rank 0. For  $n = 2$  the determinant of such a matrix is negative, so the rank is 2. We show that for all  $n \geq 3$  the minimal rank is 3.

Notice that the first three rows are linearly independent. Suppose that some linear combination of them, with coefficients  $c_1, c_2, c_3$ , vanishes. Observe that from the first column one deduces that  $c_2$  and  $c_3$  either have opposite signs or both zero. The same applies to the pairs  $(c_1, c_2)$  and  $(c_1, c_3)$ . Hence they all must be zero.

It remains to give an example of a matrix of rank (at most) 3. For example, the matrix

$$\begin{aligned} & \begin{pmatrix} 0^2 & 1^2 & 2^2 & \dots & (n-1)^2 \\ (-1)^2 & 0^2 & 1^2 & \dots & (n-2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-n+1)^2 & (-n+2)^2 & (-n+3)^2 & \dots & 0^2 \end{pmatrix} = \left( (i-j)^2 \right)_{i,j=1}^n = \\ & = \begin{pmatrix} 1^2 \\ 2^2 \\ \vdots \\ n^2 \end{pmatrix} (1, 1, \dots, 1) - 2 \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} (1, 2, \dots, n) + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (1^2, 2^2, \dots, n^2) \end{aligned}$$

is the sum of three matrices of rank 1, so its rank cannot exceed 3.

**Problem 3.** Given an integer  $n > 1$ , let  $S_n$  be the group of permutations of the numbers  $1, 2, \dots, n$ . Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group  $S_n$ . It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group  $S_n$ . The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

(Proposed by Fedor Petrov, St. Petersburg State University)

**Solution.** Player A can win for  $n = 2$  (by selecting the identity) and for  $n = 3$  (selecting a 3-cycle).

We prove that B has a winning strategy for  $n \geq 4$ . Consider the moment when all permitted moves lose immediately, and let  $H$  be the subgroup generated by the elements selected by the players. Choosing another element from  $H$  would not lose immediately, so all elements of  $H$  must have been selected. Since  $H$  and any other element generate  $S_n$ ,  $H$  must be a maximal subgroup in  $S_n$ .

If  $|H|$  is even, then the next player is A, so B wins. Denote by  $n_i$  the order of the subgroup generated by the first  $i$  selected elements; then  $n_1 | n_2 | n_3 | \dots$ . We show that B can achieve that  $n_2$  is even and  $n_2 < n!$ ; then  $|H|$  will be even and A will be forced to make the final – losing – move.

Denote by  $g$  the element chosen by A on his first move. If the order  $n_1$  of  $g$  is even, then B may choose the identical permutation  $id$  and he will have  $n_2 = n_1$  even and  $n_2 = n_1 < n!$ .

If  $n_1$  is odd, then  $g$  is a product of disjoint odd cycles, so it is an even permutation. Then B can choose the permutation  $h = (1, 2)(3, 4)$  which is another even permutation. Since  $g$  and  $h$  are elements of the alternating group  $A_n$ , they cannot generate the whole  $S_n$ . Since the order of  $h$  is 2, B achieves  $2 | n_2$ .

**Remark.** If  $n \geq 4$ , all subgroups of odd order are subgroups of  $A_n$  which has even order. Hence, all maximal subgroups have even order and B is never forced to lose.

**Problem 4.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a continuously differentiable function that satisfies  $f'(t) > f(f(t))$  for all  $t \in \mathbb{R}$ . Prove that  $f(f(f(t))) \leq 0$  for all  $t \geq 0$ .

**Solution.**

*Lemma 1.* Either  $\lim_{t \rightarrow +\infty} f(t)$  does not exist or  $\lim_{t \rightarrow +\infty} f(t) \neq +\infty$ .

*Proof.* Assume that the limit is  $+\infty$ . Then there exists  $T_1 > 0$  such that for all  $t > T_1$  we have  $f(t) > 2$ . There exists  $T_2 > 0$  such that  $f(t) > T_1$  for all  $t > T_2$ . Hence,  $f'(t) > f(f(t)) > 2$  for  $t > T_2$ . Hence, there exists  $T_3$  such that  $f(t) > t$  for  $t > T_3$ . Then  $f'(t) > f(f(t)) > f(t)$ ,  $f'(t)/f(t) > 1$ , after integration  $\ln f(t) - \ln T_3 > t - T_3$ , i.e.  $f(t) > T_3 e^{t-T_3}$  for all  $t > T_3$ . Then  $f'(t) > f(f(t)) > T_3 e^{f(t)-T_3}$  and  $f'(t)e^{-f(t)} > T_3 e^{-T_3}$ . Integrating from  $T_3$  to  $t$  yields  $e^{-f(t)} - e^{-f(T_3)} > (t-T_3)T_3 e^{-T_3}$ . The right-hand side tends to infinity, but the left-hand side is bounded from above, a contradiction.  $\square$

*Lemma 2.* For all  $t > 0$  we have  $f(t) < t$ .

*Proof.* By Lemma 1, there are some positive real numbers  $t$  with  $f(t) < t$ . Hence, if the statement is false then there is some  $t_0 > 0$  with  $f(t_0) = t_0$ .

Case I: There exist some value  $t \geq t_0$  with  $f(t) < t_0$ . Let  $T = \inf\{t \geq t_0 : f(t) < t_0\}$ . By the continuity of  $f$ ,  $f(T) = t_0$ . Then  $f'(T) > f(f(T)) = f(t_0) = t_0 > 0$ . This implies  $f > f(T) = t_0$  in a right neighbourhood, contradicting the definition of  $T$ .

Case II:  $f(t) \geq t_0$  for all  $t \geq t_0$ . Now we have  $f'(t) > f(f(t)) \geq t_0 > 0$ . So,  $f'$  has a positive lower bound over  $(t_0, \infty)$ , which contradicts Lemma 1.  $\square$

*Lemma 3.* (a) If  $f(s_1) > 0$  and  $f(s_2) \geq s_1$ , then  $f(s) > s_1$  for all  $s > s_2$ .

(b) In particular, if  $s_1 \leq 0$  and  $f(s_1) > 0$ , then  $f(s) > s_1$  for all  $s > s_1$ .

*Proof.* Suppose that there are values  $s > s_2$  with  $f(s) \leq s_1$  and let  $S = \inf\{s > s_2 : f(s) \leq s_1\}$ . By the continuity we have  $f(S) = s_1$ . Similarly to Lemma 2, we have  $f'(S) > f(f(S)) = f(s_1) > 0$ . If  $S > s_2$  then in a left neighbourhood of  $S$  we have  $f < s_1$ , contradicting the definition of  $S$ . Otherwise, if  $S = s_2$  then we have  $f > s_1$  in a right neighbourhood of  $s_2$ , contradiction again.

Part (b) follows if we take  $s_2 = s_1$ .  $\square$

With the help of these lemmas the proof goes as follows. Assume for contradiction that there exists some  $t_0 > 0$  with  $f(f(f(t_0))) > 0$ . Let  $t_1 = f(t_0)$ ,  $t_2 = f(t_1)$  and  $t_3 = f(t_2) > 0$ . We show that  $0 < t_3 < t_2 < t_1 < t_0$ . By lemma 2 it is sufficient to prove that  $t_1$  and  $t_2$  are positive. If  $t_1 < 0$ , then  $f(t_1) \leq 0$  (if  $f(t_1) > 0$  then taking  $s_1 = t_1$  in Lemma 3(b) yields  $f(t_0) > t_1$ , contradiction). If  $t_1 = 0$  then  $f(t_1) \leq 0$  by lemma 2 and the continuity of  $f$ . Hence, if  $t_1 \leq 0$ , then also  $t_2 \leq 0$ . If  $t_2 = 0$  then  $f(t_2) \leq 0$  by lemma 2 and the continuity of  $f$  (contradiction,  $f(t_2) = t_3 > 0$ ). If  $t_2 < 0$ , then by lemma 3(b),  $f(t_0) > t_2$ , so  $t_1 > t_2$ . Applying lemma 3(a) we obtain  $f(t_1) > t_2$ , contradiction. We have proved  $0 < t_3 < t_2 < t_1 < t_0$ .

By lemma 3(a) ( $f(t_1) > 0$ ,  $f(t_0) \geq t_1$ ) we have  $f(t) > t_1$  for all  $t > t_0$  and similarly  $f(t) > t_2$  for all  $t > t_1$ . It follows that for  $t > t_0$  we have  $f'(t) > f(f(t)) > t_2 > 0$ . Hence,  $\lim_{t \rightarrow +\infty} f(t) = +\infty$ , which is a contradiction. This contradiction proves that  $f(f(f(t))) \leq 0$  for all  $t > 0$ . For  $t = 0$  the inequality follows from the continuity of  $f$ .

**Problem 5.** Let  $a$  be a rational number and let  $n$  be a positive integer. Prove that the polynomial  $X^{2^n}(X+a)^{2^n} + 1$  is irreducible in the ring  $\mathbb{Q}[X]$  of polynomials with rational coefficients.

(Proposed by Vincent Jugé, École Polytechnique, Paris)

**Solution.** First let us consider the case  $a = 0$ . The roots of  $X^{2^{n+1}} + 1$  are exactly all primitive roots of unity of order  $2^{n+2}$ , namely  $e^{2\pi i \frac{k}{2^{n+2}}}$  for odd  $k = 1, 3, 5, \dots, 2^{n+2} - 1$ . It is a cyclotomic polynomial, hence irreducible in  $\mathbb{Q}[X]$ .

Let now  $a \neq 0$  and suppose that the polynomial in the question is reducible. Substituting  $X = Y - \frac{a}{2}$  we get a polynomial  $(Y - \frac{a}{2})^{2^n}(Y + \frac{a}{2})^{2^n} + 1 = (Y^2 - \frac{a^2}{4})^{2^n} + 1$ . It is again a cyclotomic polynomial in the variable  $Z = Y^2 - \frac{a^2}{4}$ , and therefore it is not divisible by any polynomial in  $Y^2$  with rational

coefficients. Let us write this polynomial as the product of irreducible monic polynomials in  $Y$  with appropriate multiplicities, i.e.

$$\left(Y^2 - \frac{a^2}{4}\right)^{2^n} + 1 = \prod_{i=1}^r f_i(Y)^{m_i} \quad f_i \text{ monic, irreducible, all different.}$$

Since the left-hand side is a polynomial in  $Y^2$  we must have  $\prod_i f_i(Y)^{m_i} = \prod_i f_i(-Y)^{m_i}$ . By the above argument non of the  $f_i$  is a polynomial in  $Y^2$ , i.e.  $f_i(-Y) \neq f_i(Y)$ . Therefore for every  $i$  there is  $i' \neq i$  such that  $f_i(-Y) = \pm f_{i'}(Y)$ . In particular  $r$  is even and irreducible factors  $f_i$  split into pairs. Let us renumber them so that  $f_1, \dots, f_{\frac{r}{2}}$  belong to different pairs and we have  $f_{i+\frac{r}{2}}(-Y) = \pm f_i(Y)$ . Consider the polynomial  $f(Y) = \prod_{i=1}^{r/2} f_i(Y)^{m_i}$ . This polynomial is monic of degree  $2^n$  and  $(Y^2 - \frac{a^2}{4})^{2^n} + 1 = f(Y)f(-Y)$ . Let us write  $f(Y) = Y^{2^n} + \dots + b$  where  $b \in \mathbb{Q}$  is the constant term, i.e.  $b = f(0)$ . Comparing constant terms we then get  $(\frac{a}{2})^{2^{n+1}} + 1 = b^2$ . Denote  $c = (\frac{a}{2})^{2^{n-1}}$ . This is a nonzero rational number and we have  $c^4 + 1 = b^2$ .

It remains to show that there are no rational solutions  $c, b \in \mathbb{Q}$  to the equation  $c^4 + 1 = b^2$  with  $c \neq 0$  which will contradict our assumption that the polynomial under consideration is reducible. Suppose there is a solution. Without loss of generality we can assume that  $c, b > 0$ . Write  $c = \frac{u}{v}$  with  $u$  and  $v$  coprime positive integers. Then  $u^4 + v^4 = (bv^2)^2$ . Let us denote  $w = bv^2$ , this must be a positive integer too since  $u, v$  are positive integers. Let us show that the set  $\mathcal{T} = \{(u, v, w) \in \mathbb{N}^3 \mid u^4 + v^4 = w^2 \text{ and } u, v, w \geq 1\}$  is empty. Suppose the contrary and consider some triple  $(u, v, w) \in \mathcal{T}$  such that  $w$  is minimal. Without loss of generality, we may assume that  $u$  is odd.  $(u^2, v^2, w)$  is a primitive Pythagorean triple and thus there exist relatively prime integers  $d > e \geq 1$  such that  $u^2 = d^2 - e^2$ ,  $v^2 = 2de$  and  $w = d^2 + e^2$ . In particular, considering the equation  $u^2 = d^2 - e^2$  in  $\mathbb{Z}/4\mathbb{Z}$  proves that  $d$  is odd and  $e$  is even. Therefore, we can write  $d = f^2$  and  $e = 2g^2$ . Moreover, since  $u^2 + e^2 = d^2$ ,  $(u, e, d)$  is also a primitive Pythagorean triple: there exist relatively prime integers  $h > i \geq 1$  such that  $u = h^2 - i^2$ ,  $e = 2hi = 2g^2$  and  $d = h^2 + i^2$ . Once again, we can write  $h = k^2$  and  $i = l^2$ , so that we obtain the relation  $f^2 = d = h^2 + i^2 = k^4 + l^4$  and  $(k, l, f) \in \mathcal{T}$ . Then, the inequality  $w > d^2 = f^4 \geq f$  contradicts the minimality of  $w$ .

**Remark 1.** One can also use Galois theory arguments in order to solve this question. Let us denote the polynomial in the question by  $P(X) = X^{2^n}(X+a)^{2^n} + 1$  and we will also need the cyclotomic polynomial  $T(X) = X^{2^n} + 1$ . As we already said, when  $a = 0$  then  $P(X)$  is itself cyclotomic and hence irreducible. Let now  $a \neq 0$  and  $x$  be any complex root of  $P(x) = 0$ . Then  $\zeta = x(x+a)$  satisfies  $T(\zeta) = 0$ , hence it is a primitive root of unity of order  $2^{n+1}$ . The field  $\mathbb{Q}[x]$  is then an extension of  $\mathbb{Q}[\zeta]$ . The latter field is cyclotomic and its degree over  $\mathbb{Q}$  is  $\dim_{\mathbb{Q}}(\mathbb{Q}[\zeta]) = 2^n$ . Since the polynomial in the question has degree  $2^{n+1}$  we see that it is reducible if and only if the above mentioned extension is trivial, i.e.  $\mathbb{Q}[x] = \mathbb{Q}[\zeta]$ . For the sake of contradiction we will now assume that this is indeed the case. Let  $S(X)$  be the minimal polynomial of  $x$  over  $\mathbb{Q}$ . The degree of  $S$  is then  $2^n$  and we can number its roots by odd numbers in the set  $I = \{1, 3, \dots, 2^{n+1} - 1\}$  so that  $S(X) = \prod_{k \in I} (X - x_k)$  and  $x_k(x_k + a) = \zeta^k$  because Galois automorphisms of  $\mathbb{Q}[\zeta]$  map  $\zeta$  to  $\zeta^k, k \in I$ . Then one has

$$S(X)S(-a - X) = \prod_{k \in I} (X - x_k)(-a - X - x_k) = (-1)^{|I|} \prod_{k \in I} (X(X+a) - \zeta^k) = T(X(X+a)) = P(X).$$

In particular  $P(-\frac{a}{2}) = S(-\frac{a}{2})^2$ , i.e.  $(\frac{a}{2})^{2^{n+1}} + 1 = \left(\left(\frac{a}{2}\right)^{2^n} + 1\right)^2$ . Therefore the rational numbers  $c = (\frac{a}{2})^{2^{n-1}} \neq 0$  and  $b = (\frac{a}{2})^{2^n} + 1$  satisfy  $c^4 + 1 = b^2$  which is a contradiction as it was shown in the first proof.

**Remark 2.** It is well-known that the Diophantine equation  $x^4 + y^4 = z^2$  has only trivial solutions (i.e. with  $x = 0$  or  $y = 0$ ). This implies immediately that  $c^4 + 1 = b^2$  has no rational solution with nonzero  $c$ .

**IMC 2012, Blagoevgrad, Bulgaria**  
**Day 2, July 29, 2012**

**Problem 1.** Consider a polynomial

$$f(x) = x^{2012} + a_{2011}x^{2011} + \dots + a_1x + a_0.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients  $a_0, \dots, a_{2011}$  and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values.

Homer's goal is to make  $f(x)$  divisible by a fixed polynomial  $m(x)$  and Albert's goal is to prevent this.

- (a) Which of the players has a winning strategy if  $m(x) = x - 2012$ ?
- (b) Which of the players has a winning strategy if  $m(x) = x^2 + 1$ ?

(Proposed by Fedor Duzhin, Nanyang Technological University)

**Solution.** We show that Homer has a winning strategy in both part (a) and part (b).

(a) Notice that the last move is Homer's, and only the last move matters. Homer wins if and only if  $f(2012) = 0$ , i.e.

$$2012^{2012} + a_{2011}2012^{2011} + \dots + a_k2012^k + \dots + a_12012 + a_0 = 0. \quad (1)$$

Suppose that all of the coefficients except for  $a_k$  have been assigned values. Then Homer's goal is to establish (1) which is a linear equation on  $a_k$ . Clearly, it has a solution and hence Homer can win.

(b) Define the polynomials

$$g(y) = a_0 + a_2y + a_4y^2 + \dots + a_{2010}y^{1005} + y^{1006} \quad \text{and} \quad h(y) = a_1 + a_3y + a_5y^2 + \dots + a_{2011}y^{1005},$$

so  $f(x) = g(x^2) + h(x^2) \cdot x$ . Homer wins if he can achieve that  $g(y)$  and  $h(y)$  are divisible by  $y + 1$ , i.e.  $g(-1) = h(-1) = 0$ .

Notice that both  $g(y)$  and  $h(y)$  have an even number of undetermined coefficients in the beginning of the game. A possible strategy for Homer is to follow Albert: whenever Albert assigns a value to a coefficient in  $g$  or  $h$ , in the next move Homer chooses the value for a coefficient in the same polynomial. This way Homer defines the last coefficient in  $g$  and he also chooses the last coefficient in  $h$ . Similarly to part (a), Homer can choose these two last coefficients in such a way that both  $g(-1) = 0$  and  $h(-1) = 0$  hold.

**Problem 2.** Define the sequence  $a_0, a_1, \dots$  inductively by  $a_0 = 1$ ,  $a_1 = \frac{1}{2}$  and

$$a_{n+1} = \frac{na_n^2}{1 + (n+1)a_n} \quad \text{for } n \geq 1.$$

Show that the series  $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$  converges and determine its value.

(Proposed by Christophe Debry, KU Leuven, Belgium)

**Solution.** Observe that

$$ka_k = \frac{(1 + (k + 1)a_k)a_{k+1}}{a_k} = \frac{a_{k+1}}{a_k} + (k + 1)a_{k+1} \quad \text{for all } k \geq 1,$$

and hence

$$\sum_{k=0}^n \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^n (ka_k - (k + 1)a_{k+1}) = \frac{1}{2} + 1 \cdot a_1 - (n + 1)a_{n+1} = 1 - (n + 1)a_{n+1} \quad (1)$$

for all  $n \geq 0$ .

By (1) we have  $\sum_{k=0}^n \frac{a_{k+1}}{a_k} < 1$ . Since all terms are positive, this implies that the series  $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$  is convergent. The sequence of terms,  $\frac{a_{k+1}}{a_k}$  must converge to zero. In particular, there is an index  $n_0$  such that  $\frac{a_{k+1}}{a_k} < \frac{1}{2}$  for  $n \geq n_0$ . Then, by induction on  $n$ , we have  $a_n < \frac{C}{2^n}$  with some positive constant  $C$ . From  $na_n < \frac{Cn}{2^n} \rightarrow 0$  we get  $na_n \rightarrow 0$ , and therefore

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{a_{k+1}}{a_k} = \lim_{n \rightarrow \infty} (1 - (n + 1)a_{n+1}) = 1.$$

**Remark.** The inequality  $a_n \leq \frac{1}{2^n}$  can be proved by a direct induction as well.

**Problem 3.** Is the set of positive integers  $n$  such that  $n! + 1$  divides  $(2012n)!$  finite or infinite?  
(Proposed by Fedor Petrov, St. Petersburg State University)

**Solution 1.** Consider a positive integer  $n$  with  $n! + 1 \mid (2012n)!$ . It is well-known that for arbitrary nonnegative integers  $a_1, \dots, a_k$ , the number  $(a_1 + \dots + a_k)!$  is divisible by  $a_1! \cdot \dots \cdot a_k!$ . (The number of sequences consisting of  $a_1$  digits 1,  $\dots$ ,  $a_k$  digits  $k$ , is  $\frac{(a_1 + \dots + a_k)!}{a_1! \cdot \dots \cdot a_k!}$ .) In particular,  $(n!)^{2012}$  divides  $(2012n)!$ .

Since  $n! + 1$  is co-prime with  $(n!)^{2012}$ , their product  $(n! + 1)(n!)^{2012}$  also divides  $(2012n)!$ , and therefore

$$(n! + 1) \cdot (n!)^{2012} \leq (2012n)!.$$

By the known inequalities  $(\frac{n+1}{e})^n < n! \leq n^n$ , we get

$$\begin{aligned} \left(\frac{n}{e}\right)^{2013n} &< (n!)^{2013} < (n! + 1) \cdot (n!)^{2012} \leq (2012n)! < (2012n)^{2012n} \\ n &< 2012^{2012} e^{2013}. \end{aligned}$$

Therefore, there are only finitely many such integers  $n$ .

**Remark.** Instead of the estimate  $(\frac{n+1}{e})^n < n!$ , we may apply the *Multinomial theorem*:

$$(x_1 + \dots + x_\ell)^N = \sum_{k_1 + \dots + k_\ell = N} \frac{N!}{k_1! \cdot \dots \cdot k_\ell!} x_1^{k_1} \cdot \dots \cdot x_\ell^{k_\ell}.$$

Applying this to  $N = 2012n$ ,  $\ell = 2012$  and  $x_1 = \dots = x_\ell = 1$ ,

$$\begin{aligned} \frac{(2012n)!}{(n!)^{2012}} &< \underbrace{(1 + 1 + \dots + 1)}_{2012}^{2012n} = 2012^{2012n}, \\ n! < n! + 1 &\leq \frac{(2012n)!}{(n!)^{2012}} < 2012^{2012n}. \end{aligned}$$

On the right-hand side we have a geometric progression which increases slower than the factorial on the left-hand side, so this is true only for finitely many  $n$ .

**Solution 2.** Assume that  $n > 2012$  is an integer with  $n! + 1 \mid (2012n)!$ . Notice that all prime divisors of  $n! + 1$  are greater than  $n$ , and all prime divisors of  $(2012n)!$  are smaller than  $2012n$ .

Consider a prime  $p$  with  $n < p < 2012n$ . Among  $1, 2, \dots, 2012n$  there are  $\left\lfloor \frac{2012n}{p} \right\rfloor < 2012$  numbers divisible by  $p$ ; by  $p^2 > n^2 > 2012n$ , none of them is divisible by  $p^2$ . Therefore, the exponent of  $p$  in the prime factorization of  $(2012n)!$  is at most 2011. Hence,

$$n! + 1 = \gcd(n! + 1, (2012n)!) < \prod_{n < p < 2012p} p^{2011}.$$

Applying the inequality  $\prod_{p \leq X} p < 4^X$ ,

$$n! < \prod_{n < p < 2012p} p^{2011} < \left( \prod_{p < 2012n} p \right)^{2011} < (4^{2012n})^{2011} = (4^{2012 \cdot 2011})^n. \quad (2)$$

Again, we have a factorial on the left-hand side and a geometric progression on the right-hand side.

**Problem 4.** Let  $n \geq 2$  be an integer. Find all real numbers  $a$  such that there exist real numbers  $x_1, \dots, x_n$  satisfying

$$x_1(1 - x_2) = x_2(1 - x_3) = \dots = x_{n-1}(1 - x_n) = x_n(1 - x_1) = a. \quad (1)$$

(Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

**Solution.** Throughout the solution we will use the notation  $x_{n+1} = x_1$ .

We prove that the set of possible values of  $a$  is

$$\left(-\infty, \frac{1}{4}\right] \cup \left\{ \frac{1}{4 \cos^2 \frac{k\pi}{n}}; k \in \mathbb{N}, 1 \leq k < \frac{n}{2} \right\}.$$

In the case  $a \leq \frac{1}{4}$  we can choose  $x_1$  such that  $x_1(1 - x_1) = a$  and set  $x_1 = x_2 = \dots = x_n$ . Hence we will now suppose that  $a > \frac{1}{4}$ .

The system (1) gives the recurrence formula

$$x_{i+1} = \varphi(x_i) = 1 - \frac{a}{x_i} = \frac{x_i - a}{x_i}, \quad i = 1, \dots, n.$$

The fractional linear transform  $\varphi$  can be interpreted as a projective transform of the real projective line  $\mathbb{R} \cup \{\infty\}$ ; the map  $\varphi$  is an element of the group  $\text{PGL}_2(\mathbb{R})$ , represented by the linear transform  $M = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix}$ . (Note that  $\det M \neq 0$  since  $a \neq 0$ .) The transform  $\varphi^n$  can be represented by  $M^n$ . A point  $[u, v]$  (written in homogenous coordinates) is a fixed point of this transform if and only if  $(u, v)^T$  is an eigenvector of  $M^n$ . Since the entries of  $M^n$  and the coordinates  $u, v$  are real, the corresponding eigenvalue is real, too.

The characteristic polynomial of  $M$  is  $x^2 - x + a$ , which has no real root for  $a > \frac{1}{4}$ . So  $M$  has two conjugate complex eigenvalues  $\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{4a - 1}i)$ . The eigenvalues of  $M^n$  are  $\lambda_{1,2}^n$ , they are real if and only if  $\arg \lambda_{1,2} = \pm \frac{k\pi}{n}$  with some integer  $k$ ; this is equivalent with

$$\pm \sqrt{4a - 1} = \tan \frac{k\pi}{n},$$

$$a = \frac{1}{4} \left( 1 + \tan^2 \frac{k\pi}{n} \right) = \frac{1}{4 \cos^2 \frac{k\pi}{n}}.$$

If  $\arg \lambda_1 = \frac{k\pi}{n}$  then  $\lambda_1^n = \lambda_2^n$ , so the eigenvalues of  $M^n$  are equal. The eigenvalues of  $M$  are distinct, so  $M$  and  $M^n$  have two linearly independent eigenvectors. Hence,  $M^n$  is a multiple of the identity. This means that the projective transform  $\varphi^n$  is the identity; starting from an arbitrary point  $x_1 \in \mathbb{R} \cup \{\infty\}$ , the cycle  $x_1, x_2, \dots, x_n$  closes at  $x_{n+1} = x_1$ . There are only finitely many cycles  $x_1, x_2, \dots, x_n$  containing the point  $\infty$ ; all other cycles are solutions for (1).

**Remark.** If we write  $x_j = P + Q \tan t_j$  where  $P, Q$  and  $t_1, \dots, t_n$  are real numbers, the recurrence relation re-writes as

$$(P + Q \tan t_j)(1 - P - Q \tan t_{j+1}) = a$$

$$(1 - P)Q \tan t_j - PQ \tan t_{j+1} = a + P(P - 1) + Q^2 \tan t_j \tan t_{j+1} \quad (j = 1, 2, \dots, n).$$

In view of the identity  $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$ , it is reasonable to choose  $P = \frac{1}{2}$ , and  $Q = \sqrt{a - \frac{1}{4}}$ . Then the recurrence leads to

$$t_j - t_{j+1} \equiv \arctan \sqrt{4a - 1} \pmod{\pi}.$$

**Problem 5.** Let  $c \geq 1$  be a real number. Let  $G$  be an abelian group and let  $A \subset G$  be a finite set satisfying  $|A + A| \leq c|A|$ , where  $X + Y := \{x + y \mid x \in X, y \in Y\}$  and  $|Z|$  denotes the cardinality of  $Z$ . Prove that

$$|\underbrace{A + A + \dots + A}_{k \text{ times}}| \leq c^k |A|$$

for every positive integer  $k$ . (Plünnecke's inequality)

(Proposed by Przemyslaw Mazur, Jagiellonian University)

**Solution.** Let  $B$  be a nonempty subset of  $A$  for which the value of the expression  $c_1 = \frac{|A+B|}{|B|}$  is the least possible. Note that  $c_1 \leq c$  since  $A$  is one of the possible choices of  $B$ .

*Lemma 1.* For any finite set  $D \subset G$  we have  $|A + B + D| \leq c_1 |B + D|$ .

*Proof.* Apply induction on the cardinality of  $D$ . For  $|D| = 1$  the Lemma is true by the definition of  $c_1$ . Suppose it is true for some  $D$  and let  $x \notin D$ . Let  $B_1 = \{y \in B \mid x + y \in B + D\}$ . Then  $B + (D \cup \{x\})$  decomposes into the union of two disjoint sets:

$$B + (D \cup \{x\}) = (B + D) \cup ((B \setminus B_1) + \{x\})$$

and therefore  $|B + (D \cup \{x\})| = |B + D| + |B| - |B_1|$ . Now we need to deal with the cardinality of the set  $A + B + (D \cup \{x\})$ . Writing  $A + B + (D \cup \{x\}) = (A + B + D) \cup (A + B + \{x\})$  we count some of the elements twice: for example if  $y \in B_1$ , then  $A + \{y\} + \{x\} \subset (A + B + D) \cap (A + B + \{x\})$ . Therefore all the elements of the set  $A + B_1 + \{x\}$  are counted twice and thus

$$|A + B + (D \cup \{x\})| \leq |A + B + D| + |A + B + \{x\}| - |A + B_1 + \{x\}| =$$

$$= |A + B + D| + |A + B| - |A + B_1| \leq c_1(|B + D| - |B| - |B_1|) = c_1 |B + (D \cup \{x\})|,$$

where the last inequality follows from the inductive hypothesis and the observation that  $\frac{|A+B|}{|B|} = c_1 \leq \frac{|A+B_1|}{|B_1|}$  (or  $B_1$  is the empty set).  $\square$

*Lemma 2.* For every  $k \geq 1$  we have  $|\underbrace{A + \dots + A}_{k \text{ times}} + B| \leq c_1^k |B|$ .

*Proof.* Induction on  $k$ . For  $k = 1$  the statement is true by definition of  $c_1$ . For greater  $k$  set  $D = \underbrace{A + \dots + A}_{k-1 \text{ times}}$  in the previous lemma:  $|\underbrace{A + \dots + A}_{k \text{ times}} + B| \leq c_1 |\underbrace{A + \dots + A}_{k-1 \text{ times}} + B| \leq c_1^k |B|$ .  $\square$

Now notice that

$$|\underbrace{A + \dots + A}_{k \text{ times}}| \leq |\underbrace{A + \dots + A}_{k \text{ times}} + B| \leq c_1^k |B| \leq c^k |A|.$$

**Remark.** The proof above due to Giorgios Petridis and can be found at <http://gowers.wordpress.com/2011/02/10/a-new-way-of-proving-sumset-estimates/>

**IMC 2012, Blagoevgrad, Bulgaria**  
**Day 1, July 28, 2012**

**Problem 1.** Let  $A$  and  $B$  be real symmetric matrices with all eigenvalues strictly greater than 1. Let  $\lambda$  be a real eigenvalue of matrix  $AB$ . Prove that  $|\lambda| > 1$ .

(Proposed by Pavel Kozhevnikov, MIPT, Moscow)

**Solution.** The transforms given by  $A$  and  $B$  strictly increase the length of every nonzero vector, this can be seen easily in a basis where the matrix is diagonal with entries greater than 1 in the diagonal. Hence their product  $AB$  also strictly increases the length of any nonzero vector, and therefore its real eigenvalues are all greater than 1 or less than  $-1$ .

**Problem 2.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function. Suppose  $f(0) = 0$ . Prove that there exists  $\xi \in (-\pi/2, \pi/2)$  such that

$$f''(\xi) = f(\xi)(1 + 2 \tan^2 \xi).$$

(Proposed by Karen Keryan, Yerevan State University, Yerevan, Armenia)

**Solution.** Let  $g(x) = f(x) \cos x$ . Since  $g(-\pi/2) = g(0) = g(\pi/2) = 0$ , by Rolle's theorem there exist some  $\xi_1 \in (-\pi/2, 0)$  and  $\xi_2 \in (0, \pi/2)$  such that

$$g'(\xi_1) = g'(\xi_2) = 0.$$

Now consider the function

$$h(x) = \frac{g'(x)}{\cos^2 x} = \frac{f'(x) \cos x - f(x) \sin x}{\cos^2 x}.$$

We have  $h(\xi_1) = h(\xi_2) = 0$ , so by Rolle's theorem there exist  $\xi \in (\xi_1, \xi_2)$  for which

$$\begin{aligned} 0 = h'(\xi) &= \frac{g''(\xi) \cos^2 \xi + 2 \cos \xi \sin \xi g'(\xi)}{\cos^4 \xi} = \\ &= \frac{(f''(\xi) \cos \xi - 2f'(\xi) \sin \xi - f(\xi) \cos \xi) \cos \xi + 2 \sin \xi (f'(\xi) \cos \xi - f(\xi) \sin \xi)}{\cos^3 \xi} = \\ &= \frac{f''(\xi) \cos^2 \xi - f(\xi)(\cos^2 \xi + 2 \sin^2 \xi)}{\cos^3 \xi} = \frac{1}{\cos \xi} (f''(\xi) - f(\xi)(1 + 2 \tan^2 \xi)). \end{aligned}$$

The last yields the desired equality.

**Problem 3.** There are  $2n$  students in a school ( $n \in \mathbb{N}$ ,  $n \geq 2$ ). Each week  $n$  students go on a trip. After several trips the following condition was fulfilled: every two students were together on at least one trip. What is the minimum number of trips needed for this to happen?

(Proposed by Oleksandr Rybak, Kiev, Ukraine)

**Solution.** We prove that for any  $n \geq 2$  the answer is 6.

First we show that less than 6 trips is not sufficient. In that case the total quantity of students in all trips would not exceed  $5n$ . A student meets  $n - 1$  other students in each trip, so he or she takes part on at least 3 excursions to meet all of his or her  $2n - 1$  schoolmates. Hence the total quantity of students during the trips is not less than  $6n$  which is impossible.

Now let's build an example for 6 trips.

If  $n$  is even, we may divide  $2n$  students into equal groups  $A, B, C, D$ . Then we may organize the trips with groups  $(A, B), (C, D), (A, C), (B, D), (A, D)$  and  $(B, C)$ , respectively.

If  $n$  is odd and divisible by 3, we may divide all students into equal groups  $E, F, G, H, I, J$ . Then the members of trips may be the following:  $(E, F, G), (E, F, H), (G, H, I), (G, H, J), (E, I, J), (F, I, J)$ .

In the remaining cases let  $n = 2x + 3y$  be, where  $x$  and  $y$  are natural numbers. Let's form the groups  $A, B, C, D$  of  $x$  students each, and  $E, F, G, H, I, J$  of  $y$  students each. Then we apply the previous cases and organize the following trips:  $(A, B, E, F, G), (C, D, E, F, H), (A, C, G, H, I), (B, D, G, H, J), (A, D, E, I, J), (B, C, F, I, J)$ .

**Problem 4.** Let  $n \geq 3$  and let  $x_1, \dots, x_n$  be nonnegative real numbers. Define  $A = \sum_{i=1}^n x_i, B = \sum_{i=1}^n x_i^2$

and  $C = \sum_{i=1}^n x_i^3$ . Prove that

$$(n + 1)A^2B + (n - 2)B^2 \geq A^4 + (2n - 2)AC.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

**Solution.** Let

$$p(X) = \prod_{i=1}^n (X - x_i) = X^n - AX^{n-1} + \frac{A^2 - B}{2}X^{n-2} - \frac{A^3 - 3AB + 2C}{6}X^{n-3} + \dots$$

The  $(n - 3)$ th derivative of  $p$  has three nonnegative real roots  $0 \leq u \leq v \leq w$ . Hence,

$$\frac{6}{n!}p^{(n-3)}(X) = X^3 - \frac{3A}{n}X^2 + \frac{3(A^2 - B)}{n(n-1)}X - \frac{A^3 - 3AB + 2C}{n(n-1)(n-2)} = (X - u)(X - v)(X - w),$$

so

$$u + v + w = \frac{3A}{n}, \quad uv + vw + wu = \frac{3(A^2 - B)}{n(n-1)} \quad \text{and} \quad uvw = \frac{A^3 - 3AB + 2C}{n(n-1)(n-2)}.$$

From these we can see that

$$\begin{aligned} & \frac{n^2(n-1)^2(n-2)}{9} ((n+1)A^2B + (n-2)B^2 - A^4 - (2n-2)AC) = \dots = \\ & = u^2v^2 + v^2w^2 + w^2u^2 - uvw(u+v+w) = uv(u-w)(v-w) + vw(v-u)(w-u) + wu(w-v)(u-v) \geq \\ & \geq 0 + uw(v-u)(w-v) + wu(w-v)(u-v) = 0. \end{aligned}$$

**Problem 5.** Does there exist a sequence  $(a_n)$  of complex numbers such that for every positive integer  $p$  we have that  $\sum_{n=1}^{\infty} a_n^p$  converges if and only if  $p$  is not a prime?

(Proposed by Tomáš Bárta, Charles University, Prague)

**Solution.** The answer is YES. We prove a more general statement; suppose that  $N = C \cup D$  is an arbitrary decomposition of  $N$  into two disjoint sets. Then there exists a sequence  $(a_n)_{n=1}^{\infty}$  such that  $\sum_{n=1}^{\infty} a_n^p$  is convergent for  $p \in C$  and divergent for  $p \in D$ .

Define  $C_k = C \cap [1, k]$  and  $D_k \cap [1, k]$ .

*Lemma.* For every positive integer  $k$  there exists a positive integer  $N_k$  and a sequence  $X_k = (x_{k,1}, \dots, x_{k,N_k})$  of complex numbers with the following properties:

(a) For  $p \in D_k$ , we have  $\left| \sum_{j=1}^{N_k} x_{k,j}^p \right| \geq 1$ .

(b) For  $p \in C_k$ , we have  $\sum_{j=1}^{N_k} x_{k,j}^p = 0$ ; moreover,  $\left| \sum_{j=1}^m x_{k,j}^p \right| \leq \frac{1}{k}$  holds for  $1 \leq m \leq N_k$ .

*Proof.* First we find some complex numbers  $z_1, \dots, z_k$  with

$$\sum_{j=1}^k z_j^p = \begin{cases} 0 & p \in C_k \\ 1 & p \in D_k \end{cases} \quad (1)$$

As is well-known, this system of equations is equivalent to another system  $\sigma_\nu(z_1, \dots, z_k) = w_\nu$  ( $\nu = 1, 2, \dots, k$ ) where  $\sigma_\nu$  is the  $\nu$ th elementary symmetric polynomial, and the constants  $w_\nu$  are uniquely determined by the Newton-Waring-Girard formulas. Then the numbers  $z_1, \dots, z_k$  are the roots of the polynomial  $z^k - w_1 z^{k-1} + \dots + (-1)^k w_k$  in some order.

Now let

$$M = \left[ \max_{1 \leq m \leq k, p \in C_k} \left| \sum_{j=1}^m z_j^p \right| \right]$$

and let  $N_k = k \cdot (kM)^k$ . We define the numbers  $x_{k,1}, \dots, x_{k,N_k}$  by repeating the sequence  $(\frac{z_1}{kM}, \frac{z_2}{kM}, \dots, \frac{z_k}{kM})$   $(kM)^k$  times, i.e.  $x_{k,\ell} = \frac{z_j}{kM}$  if  $\ell \equiv j \pmod{k}$ . Then we have

$$\sum_{j=1}^{N_k} x_{k,j}^p = (kM)^k \sum_{j=1}^k \left( \frac{z_j}{kM} \right)^p = (kM)^{k-p} \sum_{j=1}^k z_j^p;$$

then from (1) the properties (a) and the first part of (b) follows immediately. For the second part of (b), suppose that  $p \in C_k$  and  $1 \leq m \leq N_k$ ; then  $m = kr + s$  with some integers  $r$  and  $1 \leq s \leq k$  and hence

$$\left| \sum_{j=1}^m x_{k,j}^p \right| = \left| \sum_{j=1}^{kr} + \sum_{j=kr+1}^{kr+s} \right| = \left| \sum_{j=1}^s \left( \frac{z_j}{kM} \right)^p \right| \leq \frac{M}{(kM)^p} \leq \frac{1}{k}.$$

The lemma is proved.

Now let  $S_k = N_1, \dots, N_k$  (we also define  $S_0 = 0$ ). Define the sequence (a) by simply concatenating the sequences  $X_1, X_2, \dots$ :

$$(a_1, a_2, \dots) = (x_{1,1}, \dots, x_{1,N_1}, x_{2,1}, \dots, x_{2,N_2}, \dots, x_{k,1}, \dots, x_{k,N_k}, \dots); \quad (1)$$

$$a_{S_k+j} = x_{k+1,j} \quad (1 \leq j \leq N_{k+1}). \quad (2)$$

If  $p \in D$  and  $k \geq p$  then

$$\left| \sum_{j=S_k+1}^{S_{k+1}} a_j^p \right| = \left| \sum_{j=1}^{N_{k+1}} x_{k+1,j}^p \right| \geq 1;$$

By Cauchy's convergence criterion it follows that  $\sum a_n^p$  is divergent.

If  $p \in C$  and  $S_u < n \leq S_{u+1}$  with some  $u \geq p$  then

$$\left| \sum_{j=S_p+1}^n a_n^p \right| = \left| \sum_{k=p+1}^{u-1} \sum_{j=1}^{N_k} x_{k,j}^p + \sum_{j=1}^{n-S_{u-1}} x_{u,j}^p \right| = \left| \sum_{j=1}^{n-S_{u-1}} x_{u,j}^p \right| \leq \frac{1}{u}.$$

Then it follows that  $\sum_{n=S_p+1}^{\infty} a_n^p = 0$ , and thus  $\sum_{n=1}^{\infty} a_n^p = 0$  is convergent.

**IMC 2013, Blagoevgrad, Bulgaria**  
**Day 2, August 9, 2013**

**Problem 1.** Let  $z$  be a complex number with  $|z + 1| > 2$ . Prove that  $|z^3 + 1| > 1$ .  
 (Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

**Solution.** Since  $z^3 + 1 = (z + 1)(z^2 - z + 1)$ , it suffices to prove that  $|z^2 - z + 1| \geq \frac{1}{2}$ .  
 Assume that  $z + 1 = re^{\varphi i}$ , where  $r = |z + 1| > 2$ , and  $\varphi = \arg(z + 1)$  is some real number. Then

$$z^2 - z + 1 = (re^{\varphi i} - 1)^2 - (re^{\varphi i} - 1) + 1 = r^2 e^{2\varphi i} - 3re^{\varphi i} + 3,$$

and

$$\begin{aligned} |z^2 - z + 1|^2 &= (r^2 e^{2\varphi i} - 3re^{\varphi i} + 3)(r^2 e^{-2\varphi i} - 3re^{-\varphi i} + 3) = \\ &= r^4 + 9r^2 + 9 - (6r^3 + 18r) \cos \varphi + 6r^2 \cos 2\varphi = \\ &= r^4 + 9r^2 + 9 - (6r^3 + 18r) \cos \varphi + 6r^2(2 \cos^2 \varphi - 1) = \\ &= 12 \left( r \cos \varphi - \frac{r^2 + 3}{4} \right)^2 + \frac{1}{4}(r^2 - 3)^2 > 0 + \frac{1}{4} = \frac{1}{4}. \end{aligned}$$

This finishes the proof.

**Problem 2.** Let  $p$  and  $q$  be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1} (-1)^{\lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{q} \rfloor} = \begin{cases} 0 & \text{if } pq \text{ is even,} \\ 1 & \text{if } pq \text{ is odd.} \end{cases} \quad (*)$$

(Here  $\lfloor x \rfloor$  denotes the integer part of  $x$ .)

(Proposed by Alexander Bolbot, State University, Novosibirsk)

**Solution.** Suppose first that  $pq$  is even (which implies that  $p$  and  $q$  have opposite parities), and let  $a_k = (-1)^{\lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{q} \rfloor}$ . We show that  $a_k + a_{pq-1-k} = 0$ , so the terms on the left-and side of  $(*)$  cancel out in pairs.

For every positive integer  $k$  we have  $\left\{ \frac{k}{p} \right\} + \left\{ \frac{pq-1-k}{p} \right\} = \frac{p-1}{p}$ , hence

$$\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{pq-1-k}{p} \right\rfloor = \left( \frac{k}{p} - \left\{ \frac{k}{p} \right\} \right) + \left( \frac{pq-1-k}{p} - \left\{ \frac{pq-1-k}{p} \right\} \right) = \frac{pq-1}{p} - \frac{p-1}{p} = q-1$$

and similarly

$$\left\lfloor \frac{pq-1-k}{q} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor = p-1.$$

Since  $p$  and  $q$  have opposite parities, it follows that  $\lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{q} \rfloor$  and  $\lfloor \frac{pq-1-k}{p} \rfloor + \lfloor \frac{pq-1-k}{q} \rfloor$  have opposite parities and therefore  $a_{pq-1-k} = -a_k$ .

Now suppose that  $pq$  is odd. For every index  $k$ , denote by  $p_k$  and  $q_k$  the remainders of  $k$  modulo  $p$  and  $q$ , respectively. (I.e.,  $0 \leq p_k < p$  and  $0 \leq q_k < q$  such that  $k \equiv p_k \pmod{p}$  and  $k \equiv q_k \pmod{q}$ .)

Notice that

$$\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor \equiv p \left\lfloor \frac{k}{p} \right\rfloor + q \left\lfloor \frac{k}{q} \right\rfloor = (k - p_k) + (k - q_k) \equiv p_k + q_k \pmod{2}.$$

Since  $p$  and  $q$  are co-prime, by the Chinese remainder theorem the map  $k \mapsto (p_k, q_k)$  is a bijection between the sets  $\{0, 1, \dots, pq - 1\}$  and  $\{0, 1, \dots, p - 1\} \times \{0, 1, \dots, q - 1\}$ . Hence

$$\sum_{k=0}^{pq-1} (-1)^{\lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{q} \rfloor} = \sum_{k=0}^{pq-1} (-1)^{p_k + q_k} = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} (-1)^{i+j} = \left( \sum_{i=0}^{p-1} (-1)^i \right) \cdot \left( \sum_{j=0}^{q-1} (-1)^j \right) = 1.$$

**Problem 3.** Suppose that  $v_1, \dots, v_d$  are unit vectors in  $\mathbb{R}^d$ . Prove that there exists a unit vector  $u$  such that

$$|u \cdot v_i| \leq 1/\sqrt{d}$$

for  $i = 1, 2, \dots, d$ .

(Here  $\cdot$  denotes the usual scalar product on  $\mathbb{R}^d$ .)

(Proposed by Tomasz Tkocz, University of Warwick)

**Solution.** If  $v_1, \dots, v_d$  are linearly dependent then we can simply take a unit vector  $u$  perpendicular to  $\text{span}(v_1, \dots, v_d)$ . So assume that  $v_1, \dots, v_d$  are linearly independent. Let  $w_1, \dots, w_d$  be the dual basis of  $(v_1, \dots, v_d)$ , i.e.

$$w_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } 1 \leq i, j \leq d.$$

From  $w_i \cdot v_i = 1$  we have  $|w_i| \geq 1$ .

For every sequence  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{+1, -1\}^d$  of signs define  $u_\varepsilon = \sum_{i=1}^d \varepsilon_i w_i$ . Then we have

$$|u_\varepsilon \cdot v_k| = \left| \sum_{i=1}^d \varepsilon_i (w_i \cdot v_k) \right| = \left| \sum_{i=1}^d \varepsilon_i \delta_{ik} \right| = |\varepsilon_k| = 1 \quad \text{for } k = 1, \dots, d.$$

Now estimate the average of  $|u_\varepsilon|^2$ .

$$\begin{aligned} \frac{1}{2^d} \sum_{\varepsilon \in \{+1, -1\}^d} |u_\varepsilon|^2 &= \frac{1}{2^d} \sum_{\varepsilon \in \{+1, -1\}^d} \left( \sum_{i=1}^d \varepsilon_i w_i \right) \cdot \left( \sum_{j=1}^d \varepsilon_j w_j \right) = \\ &= \sum_{i=1}^d \sum_{j=1}^d (w_i \cdot w_j) \left( \frac{1}{2^d} \sum_{\varepsilon \in \{+1, -1\}^d} \varepsilon_i \varepsilon_j \right) = \sum_{i=1}^d \sum_{j=1}^d (w_i \cdot w_j) \delta_{ij} = \sum_{i=1}^d |w_i|^2 \geq d. \end{aligned}$$

It follows that there is a  $\varepsilon$  such that  $|u_\varepsilon|^2 \geq d$ . For that  $\varepsilon$  the vector  $u = \frac{u_\varepsilon}{|u_\varepsilon|}$  satisfies the conditions.

**Problem 4.** Does there exist an infinite set  $M$  consisting of positive integers such that for any  $a, b \in M$ , with  $a < b$ , the sum  $a + b$  is square-free?

(A positive integer is called square-free if no perfect square greater than 1 divides it.)

(Proposed by Fedor Petrov, St. Petersburg State University)

**Solution.** The answer is yes. We construct an infinite sequence  $1 = n_1 < 2 = n_2 < n_3 < \dots$  so that  $n_i + n_j$  is square-free for all  $i < j$ . Suppose that we already have some numbers  $n_1 < \dots < n_k$  ( $k \geq 2$ ), which satisfy this condition and find a suitable number  $n_{k+1}$  to be the next element of the sequence.

We will choose  $n_{k+1}$  of the form  $n_{k+1} = 1 + Mx$ , with  $M = ((n_1 + \dots + n_k + 2k)!)^2$  and some positive integer  $x$ . For  $i = 1, 2, \dots, k$  we have  $n_i + n_{k+1} = 1 + Mx + n_i = (1 + n_i)m_i$ , where  $m_i$  and  $M$  are co-prime, so any perfect square dividing  $1 + Mx + n_i$  is co-prime with  $M$ .

In order to find a suitable  $x$ , take a large  $N$  and consider the values  $x = 1, 2, \dots, N$ . If a value  $1 \leq x \leq N$  is not suitable, this means that there is an index  $1 \leq i \leq k$  and some prime  $p$  such that  $p^2 | 1 + Mx + n_i$ . For  $p \leq 2k$  this is impossible because  $p | M$ . Moreover, we also have  $p^2 \leq 1 + Mx + n_i < M(N + 1)$ , so  $2k < p < \sqrt{M(N + 1)}$ .

For any fixed  $i$  and  $p$ , the values for  $x$  for which  $p^2 | 1 + Mx + n_i$  form an arithmetic progression with difference  $p^2$ . Therefore, there are at most  $\frac{N}{p^2} + 1$  such values. In total, the number of unsuitable values  $x$  is less than

$$\begin{aligned} \sum_{i=1}^k \sum_{2k < p < \sqrt{M(N+1)}} \left( \frac{N}{p^2} + 1 \right) &< k \cdot \left( N \sum_{p > 2k} \frac{1}{p^2} + \sum_{p < \sqrt{M(N+1)}} 1 \right) < \\ &< kN \sum_{p > 2k} \left( \frac{1}{p-1} - \frac{1}{p} \right) + k\sqrt{M(N+1)} < \frac{N}{2} + k\sqrt{M(N+1)}. \end{aligned}$$

If  $N$  is big enough then this is less than  $N$ , and there exist a suitable choice for  $x$ .

**Problem 5.** Consider a circular necklace with 2013 beads. Each bead can be painted either white or green. A painting of the necklace is called *good*, if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

(Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.)

(Proposed by Vsevolod Bykov and Oleksandr Rybak, Kiev)

**Solution 1.** For  $k = 0, 1, \dots$  denote by  $N_k$  be the number of *good open laces*, consisting of  $k$  (white and green) beads in a row, such that among any 21 successive beads there is at least one green bead. For  $k \leq 21$  all laces have this property, so  $N_k = 2^k$  for  $0 \leq k \leq 20$ ; in particular,  $N_0$  is odd, and  $N_1, \dots, N_{20}$  are even.

For  $k \geq 21$ , there must be a green bead among the last 21 ones. Suppose that the last green bead is at the  $\ell$ th position; then  $\ell \geq k - 20$ . The previous  $\ell - 1$  beads have  $N_{\ell-1}$  good colorings, and every such good coloring provides a good lace of length  $k$ . Hence,

$$N_k = N_{k-1} + N_{k-2} + \dots + N_{k-21} \quad \text{for } k \geq 21. \tag{1}$$

From (1) we can see that  $N_{21} = N_0 + \dots + N_{20}$  is odd, and  $N_{22} = N_1 + \dots + N_{21}$  is also odd.

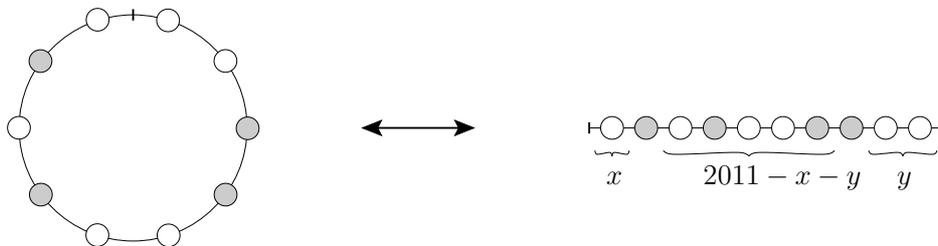
Applying (1) again to the term  $N_{k-1}$ ,

$$N_k = N_{k-1} + \dots + N_{k-21} = \left( N_{k-2} + \dots + N_{k-22} \right) + N_{k-2} + \dots + N_{k-21} \equiv N_{k-22} \pmod{2}$$

so the sequence of parities in  $(N_k)$  is periodic with period 22. We conclude that

- $N_k$  is odd if  $k \equiv 0 \pmod{22}$  or  $k \equiv 21 \pmod{22}$ ;
- $N_k$  is even otherwise.

Now consider the good circular necklaces of 2013 beads. At a fixed point between two beads cut each. The resulting open lace may have some consecutive white beads at the two ends, altogether at most 20. Suppose that there are  $x$  white beads at the beginning and  $y$  white beads at the end; then we have  $x, y \geq 0$  and  $x + y \leq 20$ , and we have a good open lace in the middle, between the first and the last green beads. That middle lace consist of  $2011 - x - y$  beads. So, for any fixed values of  $x$  and  $y$  the number of such cases is  $N_{2011-x-y}$ .



It is easy to see that from such a good open lace we can reconstruct the original circular lace. Therefore, the number of good circular necklaces is

$$\sum_{x+y \leq 20} N_{2011-x-y} = N_{2011} + 2N_{2010} + 3N_{2009} + \dots + 21N_{1991} \equiv N_{2011} + N_{2009} + N_{2007} + \dots + N_{1991} \pmod{2}.$$

By  $91 \cdot 22 - 1 = 2001$  the term  $N_{2001}$  is odd, the other terms are all even, so the number of the good circular necklaces is odd.

**Solution 2 (by Yoav Krauz, Israel).** There is just one good monochromatic necklace. Let us count the parity of good necklaces having both colors.

For each necklace, we define an *adjusted necklace*, so that at position 0 we have a white bead and at position 1 we have a green bead. If the necklace is satisfying the condition, it corresponds to itself; if both beads 0 and 1 are white we rotate it (so that the bead 1 goes to place 0) until bead 1 becomes green; if bead 1 is green, we rotate it in the opposite direction until the bead 0 will be white. This procedure is called *adjusting*, and the place between the white and green bead which are rotated into places 0 and 1 will be called *distinguished place*. The interval consisting of the subsequent green beads after the distinguished place and subsequent white beads before it will be called *distinguished interval*.

For each adjusted necklace we have several necklaces corresponding to it, and the number of them is equal to the length of distinguished interval (the total number of beads in it) minus 1. Since we count only the parity, we can disregard the adjusted necklaces with even distinguished intervals and count once each adjusted necklace with odd distinguished interval.

Now we shall prove that the number of necklaces with odd distinguished intervals is even by grouping them in pairs. The pairing is the following. If the number of white beads with in the distinguished interval is even, we turn the last white bead (at the distinguished place) into green. The white interval remains, since a positive even number minus 1 is still positive. If the number of white beads in the distinguished interval is odd, we turn the green bead next to the distinguished place into white. The green interval remains since it was even; the white interval was odd and at most 19 so it will become even and at most 20, so we still get a good necklace.

This pairing on good necklaces with distinguished intervals of odd length shows, that the number of such necklaces is even; hence the total number of all good necklaces using both colors is even. Therefore, together with monochromatic green necklace, the number of good necklaces is odd.

# IMC 2014, Blagoevgrad, Bulgaria

Day 1, July 31, 2014

**Problem 1.** Determine all pairs  $(a, b)$  of real numbers for which there exists a unique symmetric  $2 \times 2$  matrix  $M$  with real entries satisfying  $\text{trace}(M) = a$  and  $\det(M) = b$ .

(Proposed by Stephan Wagner, Stellenbosch University)

**Solution 1.** Let the matrix be

$$M = \begin{bmatrix} x & z \\ z & y \end{bmatrix}.$$

The two conditions give us  $x + y = a$  and  $xy - z^2 = b$ . Since this is symmetric in  $x$  and  $y$ , the matrix can only be unique if  $x = y$ . Hence  $2x = a$  and  $x^2 - z^2 = b$ . Moreover, if  $(x, y, z)$  solves the system of equations, so does  $(x, y, -z)$ . So  $M$  can only be unique if  $z = 0$ . This means that  $2x = a$  and  $x^2 = b$ , so  $a^2 = 4b$ .

If this is the case, then  $M$  is indeed unique: if  $x + y = a$  and  $xy - z^2 = b$ , then

$$(x - y)^2 + 4z^2 = (x + y)^2 + 4z^2 - 4xy = a^2 - 4b = 0,$$

so we must have  $x = y$  and  $z = 0$ , meaning that

$$M = \begin{bmatrix} a/2 & 0 \\ 0 & a/2 \end{bmatrix}$$

is the only solution.

**Solution 2.** Note that  $\text{trace}(M) = a$  and  $\det(M) = b$  if and only if the two eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $M$  are solutions of  $x^2 - ax + b = 0$ . If  $\lambda_1 \neq \lambda_2$ , then

$$M_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}$$

are two distinct solutions, contradicting uniqueness. Thus  $\lambda_1 = \lambda_2 = \lambda = a/2$ , which implies  $a^2 = 4b$  once again. In this case, we use the fact that  $M$  has to be diagonalisable as it is assumed to be symmetric. Thus there exists a matrix  $T$  such that

$$M = T^{-1} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \cdot T,$$

however this reduces to  $M = \lambda(T^{-1} \cdot I \cdot T) = \lambda I$ , which shows again that  $M$  is unique.

**Problem 2.** Consider the following sequence

$$(a_n)_{n=1}^{\infty} = (1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, \dots).$$

Find all pairs  $(\alpha, \beta)$  of positive real numbers such that  $\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n a_k}{n^\alpha} = \beta$ .

(Proposed by Tomas Barta, Charles University, Prague)

**Solution.** Let  $N_n = \binom{n+1}{2}$  (then  $a_{N_n}$  is the first appearance of number  $n$  in the sequence) and consider limit of the subsequence

$$b_{N_n} := \frac{\sum_{k=1}^{N_n} a_k}{N_n^\alpha} = \frac{\sum_{k=1}^n 1 + \dots + k}{\binom{n+1}{2}^\alpha} = \frac{\sum_{k=1}^n \binom{k+1}{2}}{\binom{n+1}{2}^\alpha} = \frac{\binom{n+2}{3}}{\binom{n+1}{2}^\alpha} = \frac{\frac{1}{6}n^3(1+2/n)(1+1/n)}{(1/2)^\alpha n^{2\alpha}(1+1/n)^\alpha}.$$

We can see that  $\lim_{n \rightarrow \infty} b_{N_n}$  is positive and finite if and only if  $\alpha = 3/2$ . In this case the limit is equal to  $\beta = \frac{\sqrt{2}}{3}$ . So, this pair  $(\alpha, \beta) = (\frac{3}{2}, \frac{\sqrt{2}}{3})$  is the only candidate for solution. We will show convergence of the original sequence for these values of  $\alpha$  and  $\beta$ .

Let  $N$  be a positive integer in  $[N_n + 1, N_{n+1}]$ , i.e.,  $N = N_n + m$  for some  $1 \leq m \leq n+1$ . Then we have

$$b_N = \frac{\binom{n+2}{3} + \binom{m+1}{2}}{(\binom{n+1}{2} + m)^{3/2}}$$

which can be estimated by

$$\frac{\binom{n+2}{3}}{(\binom{n+1}{2} + n)^{3/2}} \leq b_N \leq \frac{\binom{n+2}{3} + \binom{n+1}{2}}{\binom{n+1}{2}^{3/2}}.$$

Since both bounds converge to  $\frac{\sqrt{2}}{3}$ , the sequence  $b_N$  has the same limit and we are done.

**Problem 3.** Let  $n$  be a positive integer. Show that there are positive real numbers  $a_0, a_1, \dots, a_n$  such that for each choice of signs the polynomial

$$\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \dots \pm a_1 x \pm a_0$$

has  $n$  distinct real roots.

(Proposed by Stephan Neupert, TUM, München)

**Solution.** We proceed by induction on  $n$ . The statement is trivial for  $n = 1$ . Thus assume that we have some  $a_n, \dots, a_0$  which satisfy the conditions for some  $n$ . Consider now the polynomials

$$\tilde{P}(x) = \pm a_n x^{n+1} \pm a_{n-1} x^n \pm \dots \pm a_1 x^2 \pm a_0 x$$

By induction hypothesis and  $a_0 \neq 0$ , each of these polynomials has  $n+1$  distinct zeros, including the  $n$  nonzero roots of  $\pm a_n x^n \pm a_{n-1} x^{n-1} \pm \dots \pm a_1 x \pm a_0$  and 0. In particular none of the polynomials has a root which is a local extremum. Hence we can choose some  $\varepsilon > 0$  such that for each such polynomial  $\tilde{P}(x)$  and each of its local extrema  $s$  we have  $|\tilde{P}(s)| > \varepsilon$ . We claim that then each of the polynomials

$$P(x) = \pm a_n x^{n+1} \pm a_{n-1} x^n \pm \dots \pm a_1 x^2 \pm a_0 x \pm \varepsilon$$

has exactly  $n + 1$  distinct zeros as well. As  $\tilde{P}(x)$  has  $n + 1$  distinct zeros, it admits a local extremum at  $n$  points. Call these local extrema  $-\infty = s_0 < s_1 < s_2 < \dots < s_n < s_{n+1} = \infty$ . Then for each  $i \in \{0, 1, \dots, n\}$  the values  $\tilde{P}(s_i)$  and  $\tilde{P}(s_{i+1})$  have opposite signs (with the obvious convention at infinity). By choice of  $\varepsilon$  the same holds true for  $P(s_i)$  and  $P(s_{i+1})$ . Hence there is at least one real zero of  $P(x)$  in each interval  $(s_i, s_{i+1})$ , i.e.  $P(x)$  has at least (and therefore exactly)  $n + 1$  zeros. This shows that we have found a set of positive reals  $a'_{n+1} = a_n, a'_n = a_{n-1}, \dots, a'_1 = a_0, a'_0 = \varepsilon$  with the desired properties.

**Problem 4.** Let  $n > 6$  be a perfect number, and let  $n = p_1^{e_1} \dots p_k^{e_k}$  be its prime factorisation with  $1 < p_1 < \dots < p_k$ . Prove that  $e_1$  is an even number.

A number  $n$  is *perfect* if  $s(n) = 2n$ , where  $s(n)$  is the sum of the divisors of  $n$ .

(Proposed by Javier Rodrigo, Universidad Pontificia Comillas)

**Solution.** Suppose that  $e_1$  is odd, contrary to the statement.

We know that  $s(n) = \prod_{i=1}^k (1 + p_i + p_i^2 + \dots + p_i^{e_i}) = 2n = 2p_1^{e_1} \dots p_k^{e_k}$ . Since  $e_1$  is an odd number,  $p_1 + 1$  divides the first factor  $1 + p_1 + p_1^2 + \dots + p_1^{e_1}$ , so  $p_1 + 1$  divides  $2n$ . Due to  $p_1 + 1 > 2$ , at least one of the primes  $p_1, \dots, p_k$  divides  $p_1 + 1$ . The primes  $p_3, \dots, p_k$  are greater than  $p_1 + 1$  and  $p_1$  cannot divide  $p_1 + 1$ , so  $p_2$  must divide  $p_1 + 1$ . Since  $p_1 + 1 < 2p_2$ , this is possible only if  $p_2 = p_1 + 1$ , therefore  $p_1 = 2$  and  $p_2 = 3$ . Hence,  $6|n$ .

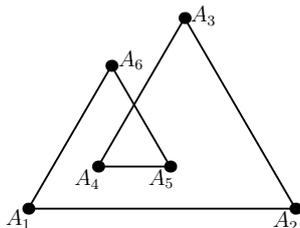
Now  $n, \frac{n}{2}, \frac{n}{3}, \frac{n}{6}$  and 1 are distinct divisors of  $n$ , so

$$s(n) \geq n + \frac{n}{2} + \frac{n}{3} + \frac{n}{6} + 1 = 2n + 1 > 2n,$$

contradiction.

**Remark.** It is well-known that all even perfect numbers have the form  $n = 2^{p-1}(2^p - 1)$  such that  $p$  and  $2^p - 1$  are primes. So if  $e_1$  is odd then  $k = 2$ ,  $p_1 = 2$ ,  $p_2 = 2^p - 1$ ,  $e_1 = p - 1$  and  $e_2 = 1$ . If  $n > 6$  then  $p > 2$  so  $p$  is odd and  $e_1 = p - 1$  should be even.

**Problem 5.** Let  $A_1A_2 \dots A_{3n}$  be a closed broken line consisting of  $3n$  line segments in the Euclidean plane. Suppose that no three of its vertices are collinear, and for each index  $i = 1, 2, \dots, 3n$ , the triangle  $A_iA_{i+1}A_{i+2}$  has counterclockwise orientation and  $\angle A_iA_{i+1}A_{i+2} = 60^\circ$ , using the notation  $A_{3n+1} = A_1$  and  $A_{3n+2} = A_2$ . Prove that the number of self-intersections of the broken line is at most  $\frac{3}{2}n^2 - 2n + 1$ .



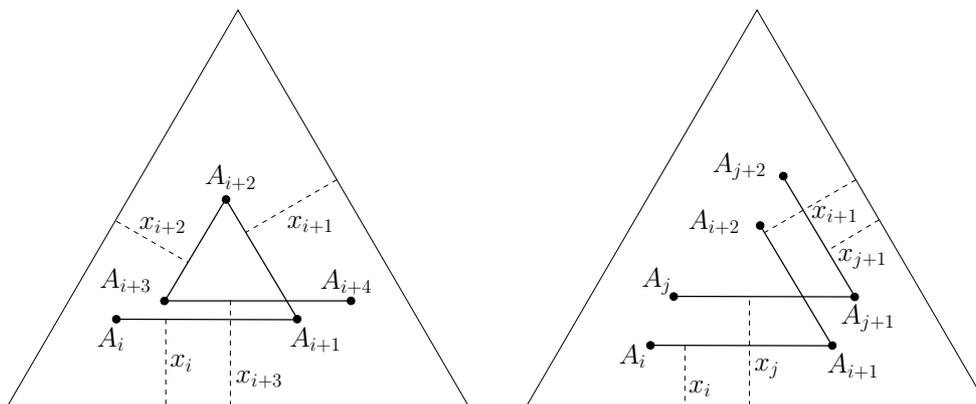
(Proposed by Martin Langer)

**Solution.** Place the broken line inside an equilateral triangle  $T$  such that their sides are parallel to the segments of the broken line. For every  $i = 1, 2, \dots, 3n$ , denote by  $x_i$  the

distance between the segment  $A_i A_{i+1}$  and that side of  $T$  which is parallel to  $A_i A_{i+1}$ . We will use indices modulo  $3n$  everywhere.

It is easy to see that if  $i \equiv j \pmod{3}$  then the polylines  $A_i A_{i+1} A_{i+2}$  and  $A_j A_{j+1} A_{j+2}$  intersect at most once, and this is possible only if either  $x_i < x_{i+1}$  and  $x_j > x_{j+1}$  or  $x_i < x_{i+1}$  and  $x_j > x_{j+1}$ . Moreover, such cases cover all self-intersections. So, the number of self-intersections cannot exceed number of pairs  $(i, j)$  with the property

(\*)  $i \equiv j \pmod{3}$ , and  $(x_i < x_{i+1} \text{ and } x_j > x_{j+1})$  or  $(x_i > x_{i+1} \text{ and } x_j < x_{j+1})$ .



Grouping the indices  $1, 2, \dots, 3n$ , by remainders modulo 3, we have  $n$  indices in each residue class. Altogether there are  $3 \binom{n}{2}$  index pairs  $(i, j)$  with  $i \equiv j \pmod{3}$ . We will show that for every integer  $k$  with  $1 \leq k < \frac{n}{2}$ , there is some index  $i$  such that the pair  $(i, i + 6k)$  does not satisfy (\*). This is already  $\lfloor \frac{n-1}{2} \rfloor$  pair; this will prove that there are at most

$$3 \binom{n}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor \geq \frac{3}{2}n^2 - 2n + 1$$

self-intersections.

Without loss of generality we may assume that  $x_{3n} = x_0$  is the smallest among  $x_1, \dots, x_{3n}$ . Suppose that all of the pairs

$$(-6k, 0), \quad (-6k + 1, 1), \quad (-6k + 2, 2), \quad \dots, \quad (-1, 6k - 1), \quad (0, 6k) \quad (**)$$

satisfy (\*). Since  $x_0$  is minimal, we have  $x_{-6k} > x_0$ . The pair  $(-6k, 0)$  satisfies (\*), so  $x_{-6k+1} < x_1$ . Then we can see that  $x_{-6k+2} > x_2$ , and so on; finally we get  $x_0 > x_{6k}$ . But this contradicts the minimality of  $x_0$ . Therefore, there is a pair in (\*\*) that does not satisfy (\*).

**Remark.** The bound  $3 \binom{n}{2} - \lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{3}{2}n^2 - 2n + 1 \rfloor$  is sharp.

# IMC 2014, Blagoevgrad, Bulgaria

Day 2, August 1, 2014

**Problem 1.** For a positive integer  $x$ , denote its  $n^{\text{th}}$  decimal digit by  $d_n(x)$ , i.e.  $d_n(x) \in \{0, 1, \dots, 9\}$  and  $x = \sum_{n=1}^{\infty} d_n(x)10^{n-1}$ . Suppose that for some sequence  $(a_n)_{n=1}^{\infty}$ , there are only finitely many zeros in the sequence  $(d_n(a_n))_{n=1}^{\infty}$ . Prove that there are infinitely many positive integers that do not occur in the sequence  $(a_n)_{n=1}^{\infty}$ .

(Proposed by Alexander Bolbot, State University, Novosibirsk)

**Solution 1.** By the assumption there is some index  $n_0$  such that  $d_n(a_n) \neq 1$  for  $n \geq n_0$ . We show that

$$a_{n+1}, a_{n+2}, \dots > 10^n \quad \text{for } n \geq n_0. \quad (1)$$

Notice that in the sum  $a_n = \sum_{k=1}^{\infty} d_k(a_n)10^{k-1}$  we have the term  $d_n(a_n)10^{n-1}$  with  $d_n(a_n) \geq 1$ .

Therefore,  $a_n \geq 10^{n-1}$ . Then for  $m > n$  we have  $a_m \geq 10^m > 10^n$ . This proves (1).

From (1) we know that only the first  $n$  elements,  $a_1, a_2, \dots, a_n$  may lie in the interval  $[1, 10^n]$ . Hence, at least  $10^n - n$  integers in this interval do not occur in the sequence at all. As  $\lim(10^n - n) = \infty$ , this shows that there are infinitely many numbers that do not appear among  $a_1, a_2, \dots$ .

**Solution 2.** We will use Cantor's diagonal method to construct infinitely many positive integers that do not occur in the sequence  $(a_n)$

Assume that  $d_n(a_n) \neq 0$  for  $n > n_0$ . Define the sequence of digits

$$g_n = \begin{cases} 2 & d_n(a_n) = 1 \\ 1 & d_n(a_n) \neq 1. \end{cases}$$

Hence  $g_n \neq d_n(a_n)$  for every positive integer  $n$ . Let

$$x_k = \sum_{n=1}^k g_n \cdot 10^{n-1} \quad \text{for } k = 1, 2, \dots$$

As  $x_{k+1} \geq 10^k > x_k$ , the sequence  $(x_k)$  is increasing and so it contains infinitely many distinct positive integers. We show that the numbers  $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots$  do not occur in the sequence  $(a_n)$ ; in other words,  $x_k \neq a_n$  for every pair  $n \geq 1$  and  $k \geq n_0$  of integers.

Indeed, if  $k \geq n$  then  $d_n(x_k) = g_n \neq d_n(a_n)$ , so  $x_k \neq a_n$ .

If  $n > k \geq n_0$  then  $d_n(x_k) = 0 \neq d_n(a_n)$ , so  $x_k \neq a_n$ .

**Problem 2.** Let  $A = (a_{ij})_{i,j=1}^n$  be a symmetric  $n \times n$  matrix with real entries, and let  $\lambda_1, \lambda_2, \dots, \lambda_n$  denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii}a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

(Proposed by Martin Niepel, Comenius University, Bratislava)

**Solution.** Eigenvalues of a real symmetric matrix are real, hence the inequality makes sense. Similarly, for Hermitian matrices diagonal entries as well as eigenvalues have to be real.

Since the trace of a matrix is the sum of its eigenvalues, for  $A$  we have

$$\sum_{i=1}^n a_{ii} = \sum_{i=1}^n \lambda_i,$$

and consequently

$$\sum_{i=1}^n a_{ii}^2 + 2 \sum_{i<j} a_{ii} a_{jj} = \sum_{i=1}^n \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j.$$

Therefore our inequality is equivalent to

$$\sum_{i=1}^n a_{ii}^2 \leq \sum_{i=1}^n \lambda_i^2.$$

Matrix  $A^2$ , which is equal to  $A^T A$  (or  $A^* A$  in Hermitian case), has eigenvalues  $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ . On the other hand, the trace of  $A^T A$  gives the square of the Frobenius norm of  $A$ , so we have

$$\sum_{i=1}^n a_{ii}^2 \leq \sum_{i,j=1}^n |a_{ij}|^2 = \text{tr}(A^T A) = \text{tr}(A^2) = \sum_{i=1}^n \lambda_i^2.$$

The inequality follows, and it is clear that the equality holds for diagonal matrices only.

**Remark.** Same statement is true for Hermitian matrices.

**Problem 3.** Let  $f(x) = \frac{\sin x}{x}$ , for  $x > 0$ , and let  $n$  be a positive integer. Prove that  $|f^{(n)}(x)| < \frac{1}{n+1}$ , where  $f^{(n)}$  denotes the  $n^{\text{th}}$  derivative of  $f$ .

(Proposed by Alexander Bolbot, State University, Novosibirsk)

**Solution 1.** Putting  $f(0) = 1$  we can assume that the function is analytic in  $\mathbb{R}$ . Let  $g(x) = x^{n+1}(f^{(n)}(x) - \frac{1}{n+1})$ . Then  $g(0) = 0$  and

$$g'(x) = (n+1)x^n \left( f^{(n)}(x) - \frac{1}{n+1} \right) + x^{n+1} f^{(n+1)}(x) =$$

$$= x^n \left( (n+1)f^{(n)}(x) + x f^{(n+1)}(x) - 1 \right) = x^n \left( (x f(x))^{(n+1)} - 1 \right) = x^n (\sin^{(n+1)}(x) - 1) \leq 0.$$

Hence  $g(x) \leq 0$  for  $x > 0$ . Taking into account that  $g'(x) < 0$  for  $0 < x < \frac{\pi}{2}$  we obtain the desired (strict) inequality for  $x > 0$ .

**Solution 2.**

$$\left(\frac{\sin x}{x}\right)^{(n)} = \frac{d^n}{dx^n} \int_0^1 -\cos(xt) dt = \int_0^1 \frac{\partial^n}{\partial x^n} (-\cos(xt)) dt = \int_0^1 t^n g_n(xt) dt$$

where the function  $g_n(u)$  can be  $\pm \sin u$  or  $\pm \cos u$ , depending on  $n$ . We only need that  $|g_n| \leq 1$  and equality occurs at finitely many points. So,

$$\left|\left(\frac{\sin x}{x}\right)^{(n)}\right| \leq \int_0^1 t^n |g_n(xt)| dt < \int_0^1 t^n dt = \frac{1}{n+1}.$$

**Problem 4.** We say that a subset of  $\mathbb{R}^n$  is *k-almost contained* by a hyperplane if there are less than  $k$  points in that set which do not belong to the hyperplane. We call a finite set of points *k-generic* if there is no hyperplane that  $k$ -almost contains the set. For each pair of positive integers  $k$  and  $n$ , find the minimal number  $d(k, n)$  such that every finite  $k$ -generic set in  $\mathbb{R}^n$  contains a  $k$ -generic subset with at most  $d(k, n)$  elements.

(Proposed by Shachar Carmeli, Weizmann Inst. and Lev Radzivilovsky, Tel Aviv Univ.)

**Solution.** The answer is:  $d(k, n) = \begin{cases} k \cdot n & k, n > 1 \\ k + n & \text{otherwise} \end{cases}$

Throughout the solution, we shall often say that a hyperplanes **skips** a point to signify that the plane does not contain that point.

For  $n = 1$  the claim is obvious.

For  $k = 1$  we have an arbitrary finite set of points in  $\mathbb{R}^n$  such that neither hyperplane contains it entirely. We can build a subset of  $n + 1$  points step by step: on each step we add a point, not contained in the minimal plane spanned by the previous points. Thus any 1-generic set contains a non-degenerate simplex of  $n + 1$  points, and obviously a non-degenerate simplex of  $n + 1$  points cannot be reduced without losing 1-genericity.

In the case  $k, n > 1$  we shall give an example of  $k \cdot n$  points. On each of the Cartesian axes choose  $k$  distinct points, different from the origin. Let's show that this set is  $k$ -generic. There are two types of planes: containing the origin and skipping it. If a plane contains the origin, it either contains all the chose points of a axis or skips all of them. Since no plane contains all axes, it skips the  $k$  chosen points on one of the axes. If a plane skips the origin, it it contains at most one point of each axis. Therefore it skips at least  $n(k - 1)$  points. It remains to verify a simple inequality  $n(k - 1) \geq k$  which is equivalent to  $(n - 1)(k - 1) \geq 1$  which holds for  $n, k > 1$ .

The example we have shown is minimal by inclusion: if any point is removed, say a point from axis  $i$ , then the hyperplane  $x_i = 0$  skips only  $k - 1$  points, and our set stops being  $k$ -generic. Hence  $d(k, n) \geq kn$ .

It remains to prove that Hence  $d(k, n) \geq kn$  for  $k, n > 1$ , meaning: for each  $k$ -generic finite set of points, it is possible to choose a  $k$ -generic subset of at most  $kn$  points. Let us call a subset of points **minimal** if by taking out any point, we loose  $k$ -genericity. It suffices to prove that any minimal  $k$ -generic subset in  $\mathbb{R}^n$  has at most  $kn$  points. A hyperplane will be called **ample** if it skips precisely  $k$  points. A point cannot be removed from a  $k$ -generic set, if and only if it is skipped by an ample hyperplane. Thus, in a minimal set each point is skipped by an ample hyperplane.

Organise the following process: on each step we choose an ample hyperplane, and paint blue all the points which are skipped by it. Each time we choose an ample hyperplane, which skips one of the unpainted points. The unpainted points at each step (after the beginning) is the intersection of all chosen hyperplanes. The intersection set of chosen hyperplanes is reduced with each step (since at least one point is being painted on each step).

Notice, that on each step we paint at most  $k$  points. So if we start with a minimal set of more than  $nk$  points, we can choose  $n$  planes and still have at least one unpainted points. The intersection of the chosen planes is a point (since on each step the dimension of the intersection plane was reduced), so there are at most  $nk + 1$  points in the set. The last unpainted point will be denoted by  $O$ . The last unpainted line (which was formed on the step before the last) will be denoted by  $\ell_1$ .

This line is an intersection of all the chosen hyperplanes except the last one. If we have more than  $nk$  points, then  $\ell_1$  contains exactly  $k + 1$  points from the set, one of which is  $O$ .

We could have executed the same process with choosing the same hyperplanes, but in different order. Anyway, at each step we would paint at most  $k$  points, and after  $n$  steps only  $O$  would remain unpainted; so it was precisely  $k$  points on each step. On step before the last, we might get a different line, which is intersection of all planes except the last one. The lines obtained in this way will be denoted  $\ell_1, \ell_2, \dots, \ell_n$ , and each contains exactly  $k$  points except  $O$ . Since we have  $O$  and  $k$  points on  $n$  lines, that is the entire set. Notice that the vectors spanning these lines are linearly independent (since for each line we have a hyperplane containing all the other lines except that line). So by removing  $O$  we obtain the example that we've described already, which is  $k$ -generic.

**Remark.** From the proof we see, that the example is unique.

**Problem 5.** For every positive integer  $n$ , denote by  $D_n$  the number of permutations  $(x_1, \dots, x_n)$  of  $(1, 2, \dots, n)$  such that  $x_j \neq j$  for every  $1 \leq j \leq n$ . For  $1 \leq k \leq \frac{n}{2}$ , denote by  $\Delta(n, k)$  the number of permutations  $(x_1, \dots, x_n)$  of  $(1, 2, \dots, n)$  such that  $x_i = k + i$  for every  $1 \leq i \leq k$  and  $x_j \neq j$  for every  $1 \leq j \leq n$ . Prove that

$$\Delta(n, k) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n - (k+i)}.$$

(Proposed by Combinatorics; Ferdowsi University of Mashhad, Iran; Mirzavaziri)

**Solution.** Let  $a_r \in \{i_1, \dots, i_k\} \cap \{a_1, \dots, a_k\}$ . Thus  $a_r = i_s$  for some  $s \neq r$ . Now there are two cases:

Case 1.  $a_s \in \{i_1, \dots, i_k\}$ . Let  $a_s = i_t$ . In this case a derangement  $x = (x_1, \dots, x_n)$  satisfies the condition  $x_{i_j} = a_j$  if and only if the derangement  $x' = (x'_1, \dots, x'_{i_t-1}, x'_{i_t+1}, x'_n)$  of the set  $[n] \setminus \{i_t\}$  satisfies the condition  $x'_{i_j} = a'_j$  for all  $j \neq t$ , where  $a'_j = a_j$  for  $j \neq s$  and  $a'_s = a_t$ . This provides a one to one correspondence between the derangements  $x = (x_1, \dots, x_n)$  of  $[n]$  with  $x_{i_j} = a_j$  for the given sets  $\{i_1, \dots, i_k\}$  and  $\{a_1, \dots, a_k\}$  with  $\ell$  elements in their intersections, and the derangements  $x' = (x'_1, \dots, x'_{i_t-1}, x'_{i_t+1}, x'_n)$  of  $[n] \setminus \{i_t\}$  with  $x_{i_j} = a'_j$  for the given sets  $\{i_1, \dots, i_k\} \setminus \{i_t\}$  and  $\{a'_1, \dots, a'_k\} \setminus \{a'_t\}$  with  $\ell - 1$  elements in their intersections.

Case 2.  $a_s \notin \{i_1, \dots, i_k\}$ . In this case a derangement  $x = (x_1, \dots, x_n)$  satisfies the condition  $x_{i_j} = a_j$  if and only if the derangement  $x' = (x'_1, \dots, x'_{a_s-1}, x'_{a_s+1}, x'_n)$  of the

set  $[n] \setminus \{a_s\}$  satisfies the condition  $x'_{i_j} = a_j$  for all  $j \neq s$ . This provides a one to one correspondence between the derangements  $x = (x_1, \dots, x_n)$  of  $[n]$  with  $x_{i_j} = a_j$  for the given sets  $\{i_1, \dots, i_k\}$  and  $\{a_1, \dots, a_k\}$  with  $\ell$  elements in their intersections, and the derangements  $x' = (x'_1, \dots, x'_{a_s-1}, x'_{a_s+1}, x'_n)$  of  $[n] \setminus \{a_s\}$  with  $x_{i_j} = a_j$  for the given sets  $\{i_1, \dots, i_k\} \setminus \{i_s\}$  and  $\{a_1, \dots, a_k\} \setminus \{a_s\}$  with  $\ell - 1$  elements in their intersections.

These considerations show that  $\Delta(n, k, \ell) = \Delta(n - 1, k - 1, \ell - 1)$ . Iterating this argument we have

$$\Delta(n, k, \ell) = \Delta(n - \ell, k - \ell, 0).$$

We can therefore assume that  $\ell = 0$ . We thus evaluate  $\Delta(n, k, 0)$ , where  $2k \leq n$ . For  $k = 0$ , we obviously have  $\Delta(n, 0, 0) = D_n$ . For  $k \geq 1$ , we claim that

$$\Delta(n, k, 0) = \Delta(n - 1, k - 1, 0) + \Delta(n - 2, k - 1, 0).$$

For a derangement  $x = (x_1, \dots, x_n)$  satisfying  $x_{i_j} = a_j$  there are two cases:  $x_{a_1} = i_1$  or  $x_{a_1} \neq i_1$ .

If the first case occurs then we have to evaluate the number of derangements of the set  $[n] \setminus \{i_1, a_1\}$  for the given sets  $\{i_2, \dots, i_k\}$  and  $\{a_2, \dots, a_k\}$  with 0 elements in their intersections. The number is equal to  $\Delta(n - 2, k - 1, 0)$ .

If the second case occurs then we have to evaluate the number of derangements of the set  $[n] \setminus \{a_1\}$  for the given sets  $\{i_2, \dots, i_k\}$  and  $\{a_2, \dots, a_k\}$  with 0 elements in their intersections. The number is equal to  $\Delta(n - 1, k - 1, 0)$ .

We now use induction on  $k$  to show that

$$\Delta(n, k, 0) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n - (k+i)}, \quad 2 \leq 2k \leq n.$$

For  $k = 1$  we have

$$\Delta(n, 1, 0) = \Delta(n - 1, 0, 0) + \Delta(n - 2, 0, 0) = D_{n-1} + D_{n-2} = \frac{D_n}{n-1}.$$

Now let the result be true for  $k - 1$ . We can write

$$\begin{aligned}
\Delta(n, k, 0) &= \Delta(n - 1, k - 1, 0) + \Delta(n - 2, k - 1, 0) \\
&= \sum_{i=0}^{k-2} \binom{k-2}{i} \frac{D_{n-(k-1+i)}}{(n-1)-(k-1+i)} + \sum_{i=0}^{k-2} \binom{k-2}{i} \frac{D_{(n-1)-(k-1+i)}}{(n-2)-(k-1+i)} \\
&= \sum_{i=0}^{k-2} \binom{k-2}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)} + \sum_{i=1}^{k-1} \binom{k-2}{i-1} \frac{D_{n-(k+i-1)}}{(n-1)-(k+i-1)} \\
&= \frac{D_{(n+1)-k}}{n-k} + \sum_{i=1}^{k-2} \binom{k-2}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)} \\
&\quad + \frac{D_{(n+1)-(2k-1)}}{n-(2k-1)} + \sum_{i=1}^{k-2} \binom{k-2}{i-1} \frac{D_{(n+1)-(k+i)}}{n-(k+i)} \\
&= \frac{D_{(n+1)-k}}{n-k} + \sum_{i=1}^{k-2} \left[ \binom{k-2}{i} + \binom{k-2}{i-1} \right] \frac{D_{(n+1)-(k+i)}}{n-(k+i)} + \frac{D_{(n+1)-(2k-1)}}{n-(2k-1)} \\
&= \frac{D_{(n+1)-k}}{n-k} + \sum_{i=1}^{k-2} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)} + \frac{D_{(n+1)-(2k-1)}}{n-(2k-1)} \\
&= \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{(n+1)-(k+i)}}{n-(k+i)}.
\end{aligned}$$

**Remark.** As a corollary of the above problem, we can solve the first problem. Let  $n = 2k$ ,  $i_j = j$  and  $a_j = k + j$  for  $j = 1, \dots, k$ . Then a derangement  $x = (x_1, \dots, x_n)$  satisfies the condition  $x_{i_j} = a_j$  if and only if  $x' = (x_{k+1}, \dots, x_n)$  is a permutation of  $[k]$ . The number of such permutations  $x'$  is  $k!$ . Thus  $\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{D_{k+1-i}}{k-i} = k!$ .

# IMC 2015, Blagoevgrad, Bulgaria

Day 1, July 29, 2015

**Problem 1.** For any integer  $n \geq 2$  and two  $n \times n$  matrices with real entries  $A, B$  that satisfy the equation

$$A^{-1} + B^{-1} = (A + B)^{-1}$$

prove that  $\det(A) = \det(B)$ .

Does the same conclusion follow for matrices with complex entries?

(10 points)

**Problem 2.** For a positive integer  $n$ , let  $f(n)$  be the number obtained by writing  $n$  in binary and replacing every 0 with 1 and vice versa. For example,  $n = 23$  is 10111 in binary, so  $f(n)$  is 1000 in binary, therefore  $f(23) = 8$ . Prove that

$$\sum_{k=1}^n f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

(10 points)

**Problem 3.** Let  $F(0) = 0$ ,  $F(1) = \frac{3}{2}$ , and  $F(n) = \frac{5}{2}F(n-1) - F(n-2)$  for  $n \geq 2$ .

Determine whether or not  $\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$  is a rational number.

(10 points)

**Problem 4.** Determine whether or not there exist 15 integers  $m_1, \dots, m_{15}$  such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16).$$

(10 points)

**Problem 5.** Let  $n \geq 2$ , let  $A_1, A_2, \dots, A_{n+1}$  be  $n+1$  points in the  $n$ -dimensional Euclidean space, not lying on the same hyperplane, and let  $B$  be a point strictly inside the convex hull of  $A_1, A_2, \dots, A_{n+1}$ . Prove that  $\angle A_i B A_j > 90^\circ$  holds for at least  $n$  pairs  $(i, j)$  with  $1 \leq i < j \leq n+1$ .

(10 points)

# IMC 2015, Blagoevgrad, Bulgaria

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**Problem 1.** For any integer  $n \geq 2$  and two  $n \times n$  matrices with real entries  $A, B$  that satisfy the equation

$$A^{-1} + B^{-1} = (A + B)^{-1}$$

prove that  $\det(A) = \det(B)$ .

Does the same conclusion follow for matrices with complex entries?

(Proposed by Zbigniew Skoczylas, Wrocław University of Technology)

**Solution.** Multiplying the equation by  $(A + B)$  we get

$$\begin{aligned} I &= (A + B)(A + B)^{-1} = (A + B)(A^{-1} + B^{-1}) = \\ &= AA^{-1} + AB^{-1} + BA^{-1} + BB^{-1} = I + AB^{-1} + BA^{-1} + I \\ &AB^{-1} + BA^{-1} + I = 0. \end{aligned}$$

Let  $X = AB^{-1}$ ; then  $A = XB$  and  $BA^{-1} = X^{-1}$ , so we have  $X + X^{-1} + I = 0$ ; multiplying by  $(X - I)X$ ,

$$0 = (X - I)X \cdot (X + X^{-1} + I) = (X - I) \cdot (X^2 + X + I) = X^3 - I.$$

Hence,

$$\begin{aligned} X^3 &= I \\ (\det X)^3 &= \det(X^3) = \det I = 1 \\ \det X &= 1 \\ \det A &= \det(XB) = \det X \cdot \det B = \det B. \end{aligned}$$

In case of complex matrices the statement is false. Let  $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ . Obviously  $\omega \notin \mathbb{R}$  and  $\omega^3 = 1$ , so  $0 = 1 + \omega + \omega^2 = 1 + \omega + \bar{\omega}$ .

Let  $A = I$  and let  $B$  be a diagonal matrix with all entries along the diagonal equal to either  $\omega$  or  $\bar{\omega} = \omega^2$  such a way that  $\det(B) \neq 1$  (if  $n$  is not divisible by 3 then one may set  $B = \omega I$ ). Then  $A^{-1} = I$ ,  $B^{-1} = \bar{B}$ . Obviously  $I + B + \bar{B} = 0$  and

$$(A + B)^{-1} = (-\bar{B})^{-1} = -B = I + \bar{B} = A^{-1} + B^{-1}.$$

By the choice of  $A$  and  $B$ ,  $\det A = 1 \neq \det B$ .

**Problem 2.** For a positive integer  $n$ , let  $f(n)$  be the number obtained by writing  $n$  in binary and replacing every 0 with 1 and vice versa. For example,  $n = 23$  is 10111 in binary, so  $f(n)$  is 1000 in binary, therefore  $f(23) = 8$ . Prove that

$$\sum_{k=1}^n f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

(Proposed by Stephan Wagner, Stellenbosch University)

**Solution.** If  $r$  and  $k$  are positive integers with  $2^{r-1} \leq k < 2^r$  then  $k$  has  $r$  binary digits, so  $k + f(k) = \underbrace{11\dots1}_{r \text{ digits}} = 2^r - 1$ .

Assume that  $2^{s-1} - 1 \leq n \leq 2^s - 1$ . Then

$$\begin{aligned} \frac{n(n+1)}{2} + \sum_{k=1}^n f(k) &= \sum_{k=1}^n (k + f(k)) = \\ &= \sum_{r=1}^{s-1} \sum_{2^{r-1} \leq k < 2^r} (k + f(k)) + \sum_{2^{s-1} \leq k \leq n} (k + f(k)) = \\ &= \sum_{r=1}^{s-1} 2^{r-1} \cdot (2^r - 1) + (n - 2^{s-1} + 1) \cdot (2^s - 1) = \\ &= \sum_{r=1}^{s-1} 2^{2r-1} - \sum_{r=1}^{s-1} 2^{r-1} + (n - 2^{s-1} + 1)(2^s - 1) = \\ &= \frac{2}{3}(4^{s-1} - 1) - (2^{s-1} - 1) + (2^s - 1)n - 2^{2s-1} + 3 \cdot 2^{s-1} - 1 = \\ &= (2^s - 1)n - \frac{1}{3}4^s + 2^s - \frac{2}{3} \end{aligned}$$

and therefore

$$\begin{aligned} \frac{n^2}{4} - \sum_{k=1}^n f(k) &= \frac{n^2}{4} - \left( (2^s - 1)n - \frac{1}{3}4^s + 2^s - \frac{2}{3} - \frac{n(n+1)}{2} \right) = \\ &= \frac{3}{4}n^2 - (2^s - \frac{3}{2})n + \frac{1}{3}4^s - 2^s + \frac{2}{3} = \\ &= \frac{3}{4} \left( n - \frac{2^{s+1} - 2}{3} \right) \left( n - \frac{2^{s+1} - 4}{3} \right). \end{aligned}$$

Notice that the difference of the last two factors is less than 1, and one of them must be an integer:  $\frac{2^{s+1}-2}{3}$  is integer if  $s$  is even, and  $\frac{2^{s+1}-4}{3}$  is integer if  $s$  is odd. Therefore, either one of them is 0, resulting a zero product, or both factors have the same sign, so the product is strictly positive. This solves the problem and shows that equality occurs if  $n = \frac{2^{s+1} - 2}{3}$  ( $s$  is even) or  $n = \frac{2^{s+1} - 4}{3}$  ( $s$  is odd).

**Problem 3.** Let  $F(0) = 0$ ,  $F(1) = \frac{3}{2}$ , and  $F(n) = \frac{5}{2}F(n-1) - F(n-2)$  for  $n \geq 2$ .

Determine whether or not  $\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$  is a rational number.

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

**Solution 1.** The characteristic equation of our linear recurrence is  $x^2 - \frac{5}{2}x + 1 = 0$ , with roots  $x_1 = 2$  and  $x_2 = \frac{1}{2}$ . So  $F(n) = a \cdot 2^n + b \cdot (\frac{1}{2})^n$  with some constants  $a, b$ . By  $F(0) = 0$  and  $F(1) = \frac{3}{2}$ , these constants satisfy  $a + b = 0$  and  $2a + \frac{b}{2} = \frac{3}{2}$ . So  $a = 1$  and  $b = -1$ , and therefore

$$F(n) = 2^n - 2^{-n}.$$

Observe that

$$\frac{1}{F(2^n)} = \frac{2^{2^n}}{(2^{2^n})^2 - 1} = \frac{1}{2^{2^n} - 1} - \frac{1}{(2^{2^n})^2 - 1} = \frac{1}{2^{2^n} - 1} - \frac{1}{2^{2^{n+1}} - 1},$$

so

$$\sum_{n=0}^{\infty} \frac{1}{F(2^n)} = \sum_{n=0}^{\infty} \left( \frac{1}{2^{2^n} - 1} - \frac{1}{2^{2^{n+1}} - 1} \right) = \frac{1}{2^{2^0} - 1} = 1.$$

Hence the sum takes the value 1, which is rational.

**Solution 2.** As in the first solution we find that  $F(n) = 2^n - 2^{-n}$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{F(2^n)} &= \sum_{n=0}^{\infty} \frac{1}{2^{2^n} - 2^{-2^n}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})^{2^n}}{1 - (\frac{1}{2})^{2^{n+1}}} \\ &= \sum_{n=0}^{\infty} (\frac{1}{2})^{2^n} \sum_{k=0}^{\infty} \left( (\frac{1}{2})^{2^{n+1}} \right)^k = \sum_{n=0}^{\infty} (\frac{1}{2})^{2^n} \sum_{k=0}^{\infty} (\frac{1}{2})^{2k \cdot 2^n} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (\frac{1}{2})^{2^n(2k+1)} = \sum_{m=1}^{\infty} (\frac{1}{2})^m = 1. \end{aligned}$$

(Here we used the fact that every positive integer  $m$  has a unique representation  $m = 2^n(2k+1)$  with non-negative integers  $n$  and  $k$ .)

This shows that the series converges to 1.

**Problem 4.** Determine whether or not there exist 15 integers  $m_1, \dots, m_{15}$  such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16). \quad (1)$$

(Proposed by Gerhard Woeginger, Eindhoven University of Technology)

**Solution.** We show that such integers  $m_1, \dots, m_{15}$  do not exist.

Suppose that (1) is satisfied by some integers  $m_1, \dots, m_{15}$ . Then the argument of the complex number  $z_1 = 1 + 16i$  coincides with the argument of the complex number

$$z_2 = (1+i)^{m_1} (1+2i)^{m_2} (1+3i)^{m_3} \dots (1+15i)^{m_{15}}.$$

Therefore the ratio  $R = z_2/z_1$  is real (and not zero). As  $\operatorname{Re} z_1 = 1$  and  $\operatorname{Re} z_2$  is an integer,  $R$  is a nonzero integer.

By considering the squares of the absolute values of  $z_1$  and  $z_2$ , we get

$$(1 + 16^2)R^2 = \prod_{k=1}^{15} (1 + k^2)^{m_k}.$$

Notice that  $p = 1 + 16^2 = 257$  is a prime (the fourth Fermat prime), which yields an easy contradiction through  $p$ -adic valuations: all prime factors in the right hand side are strictly below  $p$  (as  $k < 16$  implies  $1 + k^2 < p$ ). On the other hand, in the left hand side the prime  $p$  occurs with an odd exponent.

**Problem 5.** Let  $n \geq 2$ , let  $A_1, A_2, \dots, A_{n+1}$  be  $n + 1$  points in the  $n$ -dimensional Euclidean space, not lying on the same hyperplane, and let  $B$  be a point strictly inside the convex hull of  $A_1, A_2, \dots, A_{n+1}$ . Prove that  $\angle A_i B A_j > 90^\circ$  holds for at least  $n$  pairs  $(i, j)$  with  $1 \leq i < j \leq n + 1$ .

(Proposed by Géza Kós, Eötvös University, Budapest)

**Solution.** Let  $\mathbf{v}_i = \overrightarrow{BA_i}$ . The condition  $\angle A_i B A_j > 90^\circ$  is equivalent with  $\mathbf{v}_i \cdot \mathbf{v}_j < 0$ . Since  $B$  is an interior point of the simplex, there are some weights  $w_1, \dots, w_{n+1} > 0$  with  $\sum_{i=1}^{n+1} w_i \mathbf{v}_i = \mathbf{0}$ .

Let us build a graph on the vertices  $1, \dots, n + 1$ . Let the vertices  $i$  and  $j$  be connected by an edge if  $\mathbf{v}_i \cdot \mathbf{v}_j < 0$ . We show that this graph is connected. Since every connected graph on  $n + 1$  vertices has at least  $n$  edges, this will prove the problem statement.

Suppose the contrary that the graph is not connected; then the vertices can be split in two disjoint nonempty sets, say  $V$  and  $W$  such that  $V \cup W = \{1, 2, \dots, n + 1\}$ . Since there is no edge between the two vertex sets, we have  $\mathbf{v}_i \cdot \mathbf{v}_j \geq 0$  for all  $i \in V$  and  $j \in W$ .

Consider

$$0 = \left( \sum_{i \in V \cup W} w_i \mathbf{v}_i \right)^2 = \left( \sum_{i \in V} w_i \mathbf{v}_i \right)^2 + \left( \sum_{i \in W} w_i \mathbf{v}_i \right)^2 + 2 \sum_{i \in V} \sum_{i \in W} w_i w_j (\mathbf{v}_i \cdot \mathbf{v}_j).$$

Notice that all terms are nonnegative on the right-hand side. Moreover,  $\sum_{i \in V} w_i \mathbf{v}_i \neq \mathbf{0}$  and

$\sum_{i \in W} w_i \mathbf{v}_i \neq \mathbf{0}$ , so there are at least two strictly nonzero terms, contradiction.

**Remark 1.** The number  $n$  in the statement is sharp; if  $\mathbf{v}_{n+1} = (1, 1, \dots, 1)$  and  $\mathbf{v}_i = (\underbrace{0, \dots, 0}_{i-1}, -1, \underbrace{0, \dots, 0}_{n-i})$  for  $i = 1, \dots, n$  then  $\mathbf{v}_i \cdot \mathbf{v}_j < 0$  holds only when  $i = n + 1$  or  $j = n + 1$ .

**Remark 2.** The origin of the problem is here: <http://math.stackexchange.com/questions/476640/n-simplex-in-an-intersection-of-n-balls/789390>

# IMC 2015, Blagoevgrad, Bulgaria

Day 2, July 30, 2015

**Problem 6.** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2.$$

(10 points)

**Problem 7.** Compute

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx.$$

(10 points)

**Problem 8.** Consider all  $26^{26}$  words of length 26 in the Latin alphabet. Define the *weight* of a word as  $1/(k+1)$ , where  $k$  is the number of letters not used in this word. Prove that the sum of the weights of all words is  $3^{75}$ .

(10 points)

**Problem 9.** An  $n \times n$  complex matrix  $A$  is called *t-normal* if  $AA^t = A^tA$  where  $A^t$  is the transpose of  $A$ . For each  $n$ , determine the maximum dimension of a linear space of complex  $n \times n$  matrices consisting of t-normal matrices.

(10 points)

**Problem 10.** Let  $n$  be a positive integer, and let  $p(x)$  be a polynomial of degree  $n$  with integer coefficients. Prove that

$$\max_{0 \leq x \leq 1} |p(x)| > \frac{1}{e^n}.$$

(10 points)

# IMC 2015, Blagoevgrad, Bulgaria

Day 2, July 30, 2015

**Problem 6.** Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} < 2.$$

(Proposed by Ivan Krijan, University of Zagreb)

**Solution.** We prove that

$$\frac{1}{\sqrt{n(n+1)}} < \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}. \quad (1)$$

Multiplying by  $\sqrt{n(n+1)}$ , the inequality (1) is equivalent with

$$\begin{aligned} 1 &< 2(n+1) - 2\sqrt{n(n+1)} \\ 2\sqrt{n(n+1)} &< n + (n+1) \end{aligned}$$

which is true by the AM-GM inequality.

Applying (1) to the terms in the left-hand side,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} < \sum_{n=1}^{\infty} \left( \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} \right) = 2.$$

**Problem 7.** Compute

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx.$$

(Proposed by Jan Šustek, University of Ostrava)

**Solution 1.** We prove that

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = 1.$$

For  $A > 1$  the integrand is greater than 1, so

$$\frac{1}{A} \int_1^A A^{\frac{1}{x}} dx > \frac{1}{A} \int_1^A 1 dx = \frac{1}{A}(A-1) = 1 - \frac{1}{A}.$$

In order to find a tight upper bound, fix two real numbers,  $\delta > 0$  and  $K > 0$ , and split the interval into three parts at the points  $1 + \delta$  and  $K \log A$ . Notice that for sufficiently large  $A$  (i.e., for  $A > A_0(\delta, K)$  with some  $A_0(\delta, K) > 1$ ) we have  $1 + \delta < K \log A < A$ .) For  $A > 1$  the integrand is decreasing, so we can estimate it by its value at the starting points of the intervals:

$$\begin{aligned} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx &= \frac{1}{A} \left( \int_1^{1+\delta} + \int_{1+\delta}^{K \log A} + \int_{K \log A}^A \right) < \\ &= \frac{1}{A} \left( \delta \cdot A + (K \log A - 1 - \delta) A^{\frac{1}{1+\delta}} + (A - K \log A) A^{\frac{1}{K \log A}} \right) < \\ &< \frac{1}{A} \left( \delta A + K A^{\frac{1}{1+\delta}} \log A + A \cdot A^{\frac{1}{K \log A}} \right) = \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}. \end{aligned}$$

Hence, for  $A > A_0(\delta, K)$  we have

$$1 - \frac{1}{A} < \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx < \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Taking the limit  $A \rightarrow \infty$  we obtain

$$1 \leq \liminf_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq \limsup_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq \delta + e^{\frac{1}{K}}.$$

Now from  $\delta \rightarrow +0$  and  $K \rightarrow \infty$  we get

$$1 \leq \liminf_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq \limsup_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx \leq 1,$$

so  $\liminf_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = \limsup_{A \rightarrow \infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = 1$  and therefore

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = 1.$$

**Solution 2.** We will employ l'Hospital's rule.

Let  $f(A, x) = A^{\frac{1}{x}}$ ,  $g(A, x) = \frac{1}{x} A^{\frac{1}{x}}$ ,  $F(A) = \int_1^A f(A, x) dx$  and  $G(A) = \int_1^A g(A, x) dx$ . Since  $\frac{\partial}{\partial A} f$  and  $\frac{\partial}{\partial A} g$  are continuous, the parametric integrals  $F(A)$  and  $G(A)$  are differentiable with respect to  $A$ , and

$$F'(A) = f(A, A) + \int_1^A \frac{\partial}{\partial A} f(A, x) dx = A^{\frac{1}{A}} + \int_1^A \frac{1}{x} A^{\frac{1}{x}-1} dx = A^{\frac{1}{A}} + \frac{1}{A} G(A),$$

and

$$\begin{aligned} G'(A) &= g(A, A) + \int_1^A \frac{\partial}{\partial A} g(A, x) dx = \frac{A^{\frac{1}{A}}}{A} + \int_1^A \frac{1}{x^2} A^{\frac{1}{x}-1} dx = \\ &= A^{\frac{1}{A}} + \left[ \frac{-1}{\log A} A^{\frac{1}{x}-1} \right]_1^A = \frac{A^{\frac{1}{A}}}{A} - \frac{A^{\frac{1}{A}}}{A \log A} + \frac{1}{\log A}. \end{aligned}$$

Since  $\lim_{A \rightarrow \infty} A^{\frac{1}{A}} = 1$ , we can see that  $\lim_{A \rightarrow \infty} G'(A) = 0$ . Applying l'Hospital's rule to  $\lim_{A \rightarrow \infty} \frac{G(A)}{A}$  we get

$$\lim_{A \rightarrow \infty} \frac{G(A)}{A} = \lim_{A \rightarrow \infty} \frac{G'(A)}{1} = 0,$$

so

$$\lim_{A \rightarrow \infty} F'(A) = \lim_{A \rightarrow \infty} \left( A^{\frac{1}{A}} + \frac{G(A)}{A} \right) = 1 + 0 = 1.$$

Now applying l'Hospital's rule to  $\lim_{A \rightarrow \infty} \frac{F(A)}{A}$  we get

$$\lim_{A \rightarrow +\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} dx = \lim_{A \rightarrow \infty} \frac{F(A)}{A} = \lim_{A \rightarrow \infty} \frac{F'(A)}{1} = 1.$$

**Problem 8.** Consider all  $26^{26}$  words of length 26 in the Latin alphabet. Define the *weight* of a word as  $1/(k+1)$ , where  $k$  is the number of letters not used in this word. Prove that the sum of the weights of all words is  $3^{75}$ .

(Proposed by Fedor Petrov, St. Petersburg State University)

**Solution.** Let  $n = 26$ , then  $3^{75} = (n+1)^{n-1}$ . We use the following well-known

*Lemma.* If  $f(x)$  is a polynomial of degree at most  $n$ , then its  $(n+1)$ -st finite difference vanishes:  $\Delta^{n+1} f(x) := \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} f(x+i) \equiv 0$ .

*Proof.* If  $\Delta$  is the operator which maps  $f(x)$  to  $f(x+1) - f(x)$ , then  $\Delta^{n+1}$  is indeed  $(n+1)$ -st power of  $\Delta$  and the claim follows from the observation that  $\Delta$  decreases the power of a polynomial.

In other words,  $f(x) = \sum_{i=1}^{n+1} (-1)^{i+1} \binom{n+1}{i} f(x+i)$ . Applying this for  $f(x) = (n-x)^n$ , substituting  $x = -1$  and denoting  $i = j+1$  we get

$$(n+1)^n = \sum_{j=0}^n (-1)^j \binom{n+1}{j+1} (n-j)^n = (n+1) \sum_{j=0}^n \binom{n}{j} \cdot \frac{(-1)^j}{j+1} \cdot (n-j)^n.$$

The  $j$ -th summand  $\binom{n}{j} \cdot \frac{(-1)^j}{j+1} \cdot (n-j)^n$  may be interpreted as follows: choose  $j$  letters, consider all  $(n-j)^n$  words without those letters and sum up  $\frac{(-1)^j}{j+1}$  over all those words. Now we change the order of summation, counting at first by words. For any fixed word  $W$  with  $k$  absent letters we get  $\sum_{j=0}^k \binom{k}{j} \cdot \frac{(-1)^j}{j+1} = \frac{1}{k+1} \cdot \sum_{j=0}^k (-1)^j \cdot \binom{k+1}{j+1} = \frac{1}{k+1}$ , since the alternating sum of binomial coefficients  $\sum_{j=-1}^k (-1)^j \cdot \binom{k+1}{j+1}$  vanishes. That is, after changing order of summation we get exactly initial sum, and it equals  $(n+1)^{n-1}$ .

**Problem 9.** An  $n \times n$  complex matrix  $A$  is called *t-normal* if  $AA^t = A^tA$  where  $A^t$  is the transpose of  $A$ . For each  $n$ , determine the maximum dimension of a linear space of complex  $n \times n$  matrices consisting of t-normal matrices.

(Proposed by Shachar Carmeli, Weizmann Institute of Science)

**Solution.**

*Answer:* The maximum dimension of such a space is  $\frac{n(n+1)}{2}$ .

The number  $\frac{n(n+1)}{2}$  can be achieved, for example the symmetric matrices are obviously t-normal and they form a linear space with dimension  $\frac{n(n+1)}{2}$ . We shall show that this is the maximal possible dimension.

Let  $M_n$  denote the space of  $n \times n$  complex matrices, let  $S_n \subset M_n$  be the subspace of all symmetric matrices and let  $A_n \subset M_n$  be the subspace of all anti-symmetric matrices, i.e. matrices  $A$  for which  $A^t = -A$ .

Let  $V \subset M_n$  be a linear subspace consisting of t-normal matrices. We have to show that  $\dim(V) \leq \dim(S_n)$ . Let  $\pi : V \rightarrow S_n$  denote the linear map  $\pi(A) = A + A^t$ . We have

$$\dim(V) = \dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi))$$

so we have to prove that  $\dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi)) \leq \dim(S_n)$ . Notice that  $\text{Ker}(\pi) \subseteq A_n$ .

We claim that for every  $A \in \text{Ker}(\pi)$  and  $B \in V$ ,  $A\pi(B) = \pi(B)A$ . In other words,  $\text{Ker}(\pi)$  and  $\text{Im}(\pi)$  commute. Indeed, if  $A, B \in V$  and  $A = -A^t$  then

$$\begin{aligned} (A+B)(A+B)^t &= (A+B)^t(A+B) \Leftrightarrow \\ \Leftrightarrow AA^t + AB^t + BA^t + BB^t &= A^tA + A^tB + B^tA + B^tB \Leftrightarrow \\ \Leftrightarrow AB^t - BA &= -AB + B^tA \Leftrightarrow A(B+B^t) = (B+B^t)A \Leftrightarrow \\ \Leftrightarrow A\pi(B) &= \pi(B)A. \end{aligned}$$

Our bound on the dimension on  $V$  follows from the following lemma:

*Lemma.* Let  $X \subseteq S_n$  and  $Y \subseteq A_n$  be linear subspaces such that every element of  $X$  commutes with every element of  $Y$ . Then

$$\dim(X) + \dim(Y) \leq \dim(S_n)$$

*Proof.* Without loss of generality we may assume  $X = Z_{S_n}(Y) := \{x \in S_n : xy = yx \quad \forall y \in Y\}$ . Define the bilinear map  $B : S_n \times A_n \rightarrow \mathbb{C}$  by  $B(x, y) = \text{tr}(d[x, y])$  where  $[x, y] = xy - yx$  and  $d = \text{diag}(1, \dots, n)$  is the matrix with diagonal elements  $1, \dots, n$  and zeros off the diagonal. Clearly  $B(X, Y) = \{0\}$ . Furthermore, if  $y \in Y$  satisfies that  $B(x, y) = 0$  for all  $x \in S_n$  then  $\text{tr}(d[x, y]) = -\text{tr}([d, x], y) = 0$  for every  $x \in S_n$ .

We claim that  $\{[d, x] : x \in S_n\} = A_n$ . Let  $E_i^j$  denote the matrix with 1 in the entry  $(i, j)$  and 0 in all other entries. Then a direct computation shows that  $[d, E_i^j] = (j-i)E_i^j$  and therefore  $[d, E_i^j + E_j^i] = (j-i)(E_i^j - E_j^i)$  and the collection  $\{(j-i)(E_i^j - E_j^i)\}_{1 \leq i < j \leq n}$  span  $A_n$  for  $i \neq j$ .

It follows that if  $B(x, y) = 0$  for all  $x \in S_n$  then  $\text{tr}(yz) = 0$  for every  $z \in A_n$ . But then, taking  $z = \bar{y}$ , where  $\bar{y}$  is the entry-wise complex conjugate of  $y$ , we get  $0 = \text{tr}(y\bar{y}) = -\text{tr}(y\bar{y}^t)$  which is the sum of squares of all the entries of  $y$ . This means that  $y = 0$ .

It follows that if  $y_1, \dots, y_k \in Y$  are linearly independent then the equations

$$B(x, y_1) = 0, \quad \dots, \quad B(x, y_k) = 0$$

are linearly independent as linear equations in  $x$ , otherwise there are  $a_1, \dots, a_k$  such that  $B(x, a_1 y_1 + \dots + a_k y_k) = 0$  for every  $x \in S_n$ , a contradiction to the observation above. Since the solution of  $k$  linearly independent linear equations is of codimension  $k$ ,

$$\begin{aligned} \dim(\{x \in S_n : [x, y_i] = 0, \text{ for } i = 1, \dots, k\}) &\leq \\ &\leq \dim(x \in S_n : B(x, y_i) = 0 \text{ for } i = 1, \dots, k) = \dim(S_n) - k. \end{aligned}$$

The lemma follows by taking  $y_1, \dots, y_k$  to be a basis of  $Y$ .

Since  $\text{Ker}(\pi)$  and  $\text{Im}(\pi)$  commute, by the lemma we deduce that

$$\dim(V) = \dim(\text{Ker}(\pi)) + \dim(\text{Im}(\pi)) \leq \dim(S_n) = \frac{n(n+1)}{2}.$$

**Problem 10.** Let  $n$  be a positive integer, and let  $p(x)$  be a polynomial of degree  $n$  with integer coefficients. Prove that

$$\max_{0 \leq x \leq 1} |p(x)| > \frac{1}{e^n}.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

**Solution.** Let

$$M = \max_{0 \leq x \leq 1} |p(x)|.$$

For every positive integer  $k$ , let

$$J_k = \int_0^1 (p(x))^{2k} dx.$$

Obviously  $0 < J_k < M^{2k}$  is a rational number. If  $(p(x))^{2k} = \sum_{i=0}^{2kn} a_{k,i} x^i$  then  $J_k = \sum_{i=0}^{2kn} \frac{a_{k,i}}{i+1}$ . Taking the least common denominator, we can see that  $J_k \geq \frac{1}{\text{lcm}(1, 2, \dots, 2kn+1)}$ .

An equivalent form of the prime number theorem is that  $\log \text{lcm}(1, 2, \dots, N) \sim N$  if  $N \rightarrow \infty$ . Therefore, for every  $\varepsilon > 0$  and sufficiently large  $k$  we have

$$\text{lcm}(1, 2, \dots, 2kn+1) < e^{(1+\varepsilon)(2kn+1)}$$

and therefore

$$\begin{aligned} M^{2k} > J_k &\geq \frac{1}{\text{lcm}(1, 2, \dots, 2kn+1)} > \frac{1}{e^{(1+\varepsilon)(2kn+1)}}, \\ M &> \frac{1}{e^{(1+\varepsilon)(n+\frac{1}{2k})}}. \end{aligned}$$

Taking  $k \rightarrow \infty$  and then  $\varepsilon \rightarrow +0$  we get

$$M \geq \frac{1}{e^n}.$$

Since  $e$  is transcendental, equality is impossible.

**Remark.** The constant  $\frac{1}{e} \approx 0.3679$  is not sharp. It is known that the best constant is between 0.4213 and 0.4232. (See I. E. Pritsker, The Gelfond–Schnirelman method in prime number theory, *Canad. J. Math.* 57 (2005), 1080–1101.)