Internationale Mathematikolympiade 1959-1968

IMO 1959

Problem A1
Prove that $(21n+4)/(14n+3)$ is irreducible for every natural number n.

Solution
$3(14n+3) - 2(21n+4) = 1$.

Problem A2
For what real values of x is $\sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}} = A$, given (a) $A = \sqrt{2}$, (b) $A = 1$, (c) $A = 2$, where only non-negative real numbers are allowed in square roots and the root always denotes the non-negative root?

Answer
(a) any x in the interval $[1/2,1]$; (b) no solutions; (c) $x=3/2$.

Solution
Note that we require $x \geq 1/2$ to avoid a negative sign under the inner square roots. Since $(x-1)^2 \geq 0$, we have $x \geq \sqrt{2x-1}$, so there is no difficulty with $\sqrt{x - \sqrt{2x-1}}$, provided that $x \geq 1/2$.
Squaring gives $2x + 2\sqrt{x^2-2x+1} = A^2$. Note that the square root is $|x-1|$, not simply $(x-1)$. So we get finally $2x + 2|x-1| = A^2$. It is now easy to see that we get the solutions above.

Problem A3
Let a, b, c be real numbers. Given the equation for cos x:

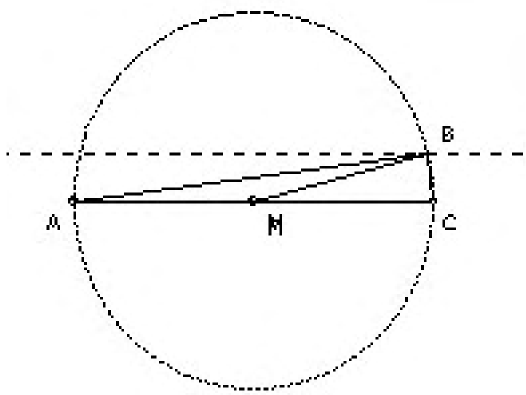
$$a \cos^2 x + b \cos x + c = 0,$$

form a quadratic equation in cos 2x whose roots are the same values of x. Compare the equations in cos x and cos 2x for $a=4$, $b=2$, $c=-1$.

Solution
You need that $\cos 2x = 2 \cos^2 x - 1$. Some easy manipulation then gives:
$a^2\cos^2 2x + (2a^2 + 4ac - 2b^2) \cos 2x + (4c^2 + 4ac - 2b^2 + a^2) = 0$.
The equations are the same for the values of a, b, c given. The angles are $2\pi/5$ (or $8\pi/5$) and $4\pi/5$ (or $6\pi/5$).

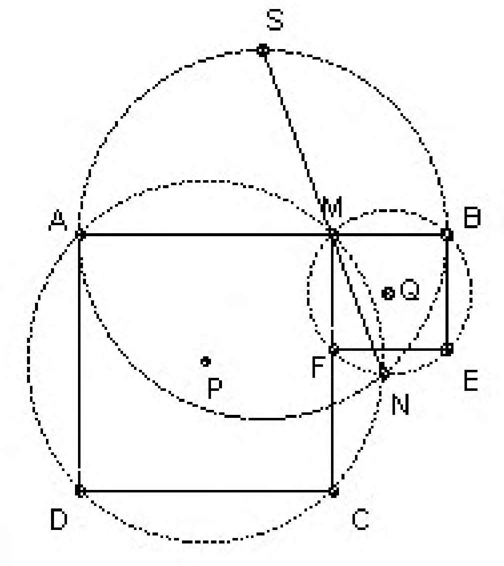
Problem B1
Given the length $|AC|$, construct a triangle ABC with $\angle ABC = 90^o$, and the median BM satisfying $BM^2 = AB \cdot BC$.

Solution
Area $= AB \cdot BC/2$ (because $\angle ABC = 90^o$) $= BM^2/2$ (required) $= AC^2/8$ (because $BM = AM = MC$), so B lies a distance $AC/4$ from AC. Take B as the intersection of a circle diameter AC with a line parallel to AC distance $AC/4$.



Problem B2
An arbitrary point M is taken in the interior of the segment AB. Squares AMCD and MBEF are constructed on the same side of AB. The circles circumscribed about these squares, with centers P and Q, intersect at M and N.
(a) prove that AF and BC intersect at N;
(b) prove that the lines MN pass through a fixed point S (independent of M);
(c) find the locus of the midpoints of the segments PQ as M varies.

Solution



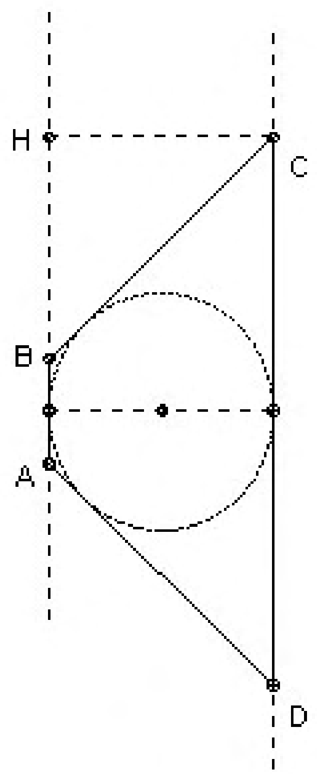
(a) $\angle ANM = \angle ACM = 45^o$. But $\angle FNM = \angle FEM = 45^o$, so A, F, N are collinear. Similarly, $\angle BNM = \angle BEM = 45^o$ and $\angle CNM = 180^o - \angle CAM = 135^o$, so B, N, C are collinear.

(b) Since $\angle ANM = \angle BNM = 45^o$, $\angle ANB = 90^o$, so N lies on the semicircle diameter AB. Let NM meet the circle diameter AB again at S. $\angle ANS = \angle BNS$ implies AS = BS and hence S is a fixed point.

(c) Clearly the distance of the midpoint of PQ from AB is AB/4. Since it varies continuously with M, it must be the interval between the two extreme positions, so the locus is a segment length AB/2 centered over AB.

Problem B3

The planes P and Q are not parallel. The point A lies in P but not Q, and the point C lies in Q but not P. Construct points B in P and D in Q such that the quadrilateral ABCD satisfies the following conditions: (1) it lies in a plane, (2) the vertices are in the order A, B, C, D, (3) it is an isosceles trapezeoid with AB is parallel to CD (meaning that AD = BC, but AD is not parallel to BC unless it is a square), and (4) a circle can be inscribed in ABCD touching the sides.



Solution

Let the planes meet in the line L. Then AB and CD must be parallel to L. Let H be the foot of the perpendicular from C to AB. The fact that a circle can be inscribed implies AB + CD = BC + AD (equal tangents from A, B, C, D to the circle). Also CD = AB ± 2BH. This leads to AH = AD = BC.

The construction is now easy. First construct the point H. Then using the circle center C radius AH, construct B. Using the circle center A radius AH construct D.

Note that if CH > AH then no construction is possible. If CH < AH, then there are two solutions, one with AB > CD, the other with AB < CD. If CH = AH, then there is a single solution, which is a square.

IMO 1960

Problem A1

Determine all 3 digit numbers N which are divisible by 11 and where N/11 is equal to the sum of the squares of the digits of N.

Answer

550, 803.

Solution

So, put N/11 = 10a + b. If $a + b \le 9$, we have $2a^2 + 2ab + 2b^2 = 10a + b$ (*), so b is even. Put b = 2B, then $B = a(a-5) + 2aB + 4B^2$, which is even. So b must be a multiple of 4, so b = 0, 4 or 8. If b = 0, then (*) gives a = 5 and we get the solution 550. If b = 4, then (*) gives $a^2 - a + 14 = 0$, which has no integral solutions. If b = 8, then (since $a + b \le 9$ and $a > 0$) a must be 1, but that does not satisfy (*).

If $a + b > 9$, we have $(a+1)^2 + (a+b-10)^2 + b^2 = 10a + b$, or $2a^2 + 2ab + 2b^2 - 28a - 21b + 101 = 0$ (**), so b is odd. Put b = 2B+1. Then $a^2 + 2aB + 4B^2 - 13a - 17B + 41 = 0$. But a(a-13) is even, so B is odd. Hence b = 3 or 7. If b = 3, then (**) gives $a^2 - 11a + 28 = 0$, so $a = 4$ or 7. But $a + b > 9$, so $a = 7$. That gives the solution 803. If $b = 7$, then (**) gives $a^2 - 7a + 26 = 0$, which has no integral solutions.

Problem A2

For what real values of x does the following inequality hold:

$$4x^2/(1 - \sqrt{(1 + 2x)})^2 < 2x + 9 ?$$

Answer

$- 1/2 \le x < 45/8$.

Solution

We require the first inequality to avoid imaginary numbers. Hence we may set $x = -1/2 + a^2/2$, where $a \ge 0$. The inequality now gives immediately $a < 7/2$ and hence $x < 45/8$. It is a matter of taste whether to avoid $x = 0$. I would allow it because the limit as x tends to 0 of the lhs is 4, and the inequality holds.

Problem A3

In a given right triangle ABC, the hypotenuse BC, length a, is divided into n equal parts with n an odd integer. The central part subtends an angle α at A. h is the perpendicular distance from A to BC. Prove that:

$$\tan \alpha = 4nh/(an^2 - a).$$

Solution

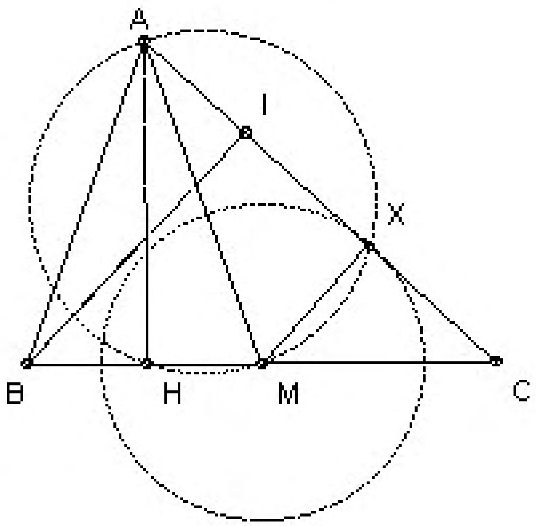
Let M be the midpoint of BC, and P and Q the two points $a/2n$ either side of it, with P nearer B. Then $\alpha = \angle PAQ = \angle QAH - \angle PAH$ (taking angles as negative if P (or Q) lies to the left of H). So $\tan \alpha = (QH - PH)/(AH^2 + QH\cdot PH) = AH\cdot PQ/(AH^2 + (MH - a/2n)(MH + a/2n)) = (ah/n)/(a^2/4 - a^2/(4n^2)) = 4nh/(an^2 - a)$.

Problem B1

Construct a triangle ABC given the lengths of the altitudes from A and B and the length of the median from A.

Solution

Let M be the midpoint of BC, AH the altitude from A, and BI the altitude from B. Start by constructing AHM. Take X on the circle diameter AM with MX = BI/2. Let the lines AX, HM meet at C and take B so that BM = MC. [This works because CMX and CBI are similar with MX = BI/2 and hence CM = CB/2.]

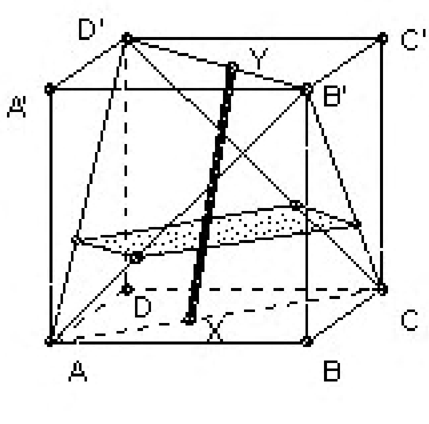


Problem B2

The cube ABCDA'B'C'D' has A above A', B above B' and so on. X is any point of the face diagonal AC and Y is any point of B'D'.

(a) find the locus of the midpoint of XY;

(b) find the locus of the point Z which lies one-third of the way along XY, so that ZY=2·XZ.

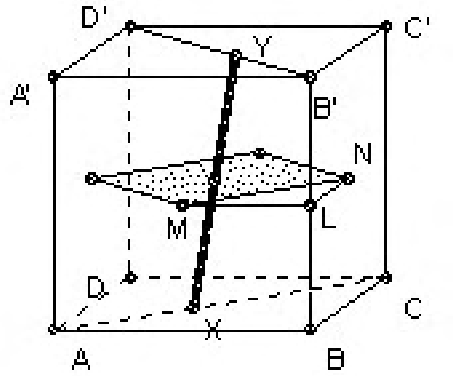


Solution

The key idea is that the midpoint must lie in the plane half-way between ABCD and A'B'C'D'. Similarly, Z must lie in the plane one-third of the way from ABCD to A'B'C'D'.

(a) Regard ABCD as horizontal. Then the locus is the square with vertices the midpoints of the vertical faces (shown shaded in the diagram).

Take Y at B' and let X vary, then we trace out MN. Similarly, we can get the other sides. Now with Y at B', take X in general position, so the midpoint of XY is on MN. Now move Y to D', the midpoint traces out a line parallel to the other two sides of the square, so we can get any point inside the square. But equally, it is clear that any point inside the triangle LMN corresponds to a point Y on the ray D'B' not between B' and D', so it does not lie in the locus. Similarly for the other three triangles. So the locus is the square.



(b) A similar argument shows that the locus is the rectangle shown in the diagram below which is √2/3 x 2√2/3.

Problem B3

A cone of revolution has an inscribed sphere tangent to the base of the cone (and to the sloping surface of the cone). A cylinder is circumscribed about the sphere so that its base lies in the base of the cone. The volume of the cone is V_1 and the volume of the cylinder is V_2.

(a) Prove that $V_1 \neq V_2$;

(b) Find the smallest possible value of V_1/V_2. For this case construct the half angle of the cone.

Solution

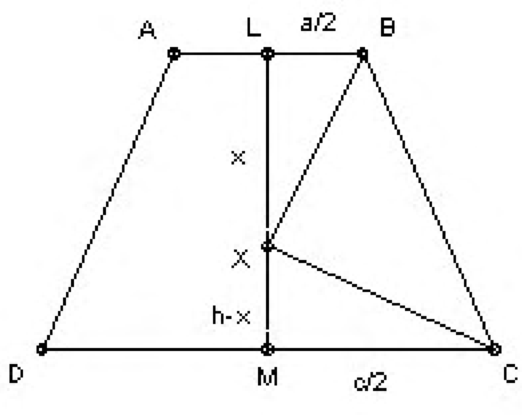
Let the vertex of the cone be V, the center of the sphere be O and the center of the base be X. Let the radius of the sphere be r and the half-angle of the cone θ.

Then the the cone's height is VO + OX = $r(1 + 1/\sin\theta)$, and the radius of its base is $r(1 + 1/\sin\theta)\tan\theta$. Hence $V_1/V_2 = (1/6)(1 + 1/\sin\theta)^3 \tan^2\theta = (1 + s)^3(6s(1 - s^2))$, where $s = \sin\theta$.

We claim that $(1 + s)^3(6s(1 - s^2)) \geq 4/3$. This is equivalent to $1 + 3s + 3s^2 + s^3 \geq 8s - 3s^3$ or $1 - 5s + 3s^2 + 9s^3 >= 0$. But we can factorise the cubic as $(1 - 3s)^2(1 + s)$. So we have $V_1/V_2 \geq 4/3$ with equality iff $s = 1/3$.

Problem B4

In the isosceles trapezoid ABCD (AB parallel to DC, and BC = AD), let AB = a, CD = c and let the perpendicular distance from A to CD be h. Show how to construct all points X on the axis of symmetry such that ∠BXC = ∠AXD = 90°. Find the distance of each such X from AB and from CD. What is the condition for such points to exist?



Solution

Since angle BXC = 90°, X lies on the circle diameter BC. In general this will intersect the axis of symmetry in 0, 1 or 2 points. By symmetry any points of intersection X will also lie on the circle diameter AD and so will have angle AXD = 90° also.

Let L be the midpoint of AB, and M the midpoint of CD. Let X lie on LM a distance x from L. We have LB = a/2, MC = c/2, and XM = h - x. The triangles LBX and MXC are similar, so $2x/a = c/(2(h-x))$. Hence $4x^2 - 4xh + ac = 0$, so $x = h/2 \pm (\sqrt{(h^2 - ac)})/2$.

There are 0, 1, 2 points according as $h^2 <, =, > ac$.

IMO 1961

Problem A1

Solve the following equations for x, y and z:

$$x + y + z = a; \quad x^2 + y^2 + z^2 = b^2; \quad xy = z^2$$

What conditions must a and b satisfy for x, y and z to be distinct positive numbers?

Solution

A routine slog gives $z = (a^2 - b^2)/2a$, x and $y = (a^2 + b^2)/4a \pm \sqrt{(10a^2b^2 - 3a^4 - 3b^4)}/4a$.
A little care is needed with the conditions. Clearly x, y, z positive implies $a > 0$, and then z positive implies $|b| < a$. The expression under the root must be positive. It helps if you notice that it factorizes as $(3a^2 - b^2)(3b^2 - a^2)$. The second factor is positive because $|b| < a$, so the first factor must also be positive and hence $a < \sqrt{3}\,|b|$. These conditions are also sufficient to ensure that x and y are distinct, but then z must also be distinct because $z^2 = xy$.

Problem A2

Let a, b, c be the sides of a triangle and A its area. Prove that:

$$a^2 + b^2 + c^2 \ge 4\sqrt{3}\ A$$

When do we have equality?

Solution

One approach is a routine slog from Heron's formula. The inequality is quickly shown to be equivalent to $a^2b^2 + b^2c^2 + c^2a^2 \le a^4 + b^4 + c^4$, which is true since $a^2b^2 \le (a^4 + b^4)/2$. We get equality iff the triangle is equilateral.
Another approach is to take an altitude lying inside the triangle. If it has length h and divides the base into lengths r and s, then we quickly find that the inequality is equivalent to $(h - (r + s)\sqrt{3}/2)^2 + (r - s)^2 \ge 0$, which is true. We have equality iff $r = s$ and $h = (r + s)\sqrt{3}/2$, which means the triangle is equilateral.

Problem A3

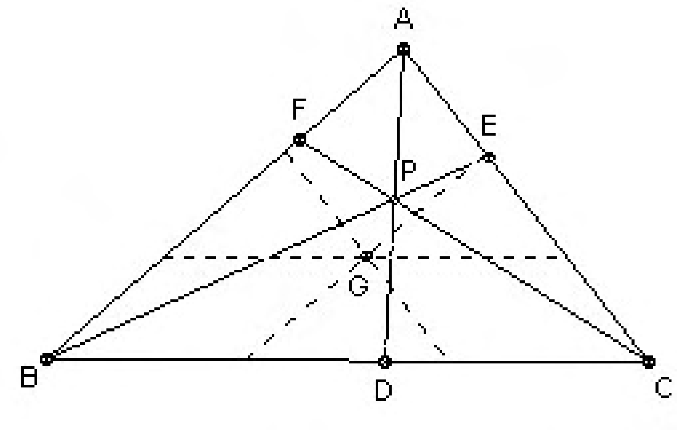
Solve the equation $\cos^n x - \sin^n x = 1$, where n is a natural number.

Solution

Since $\cos^2 x + \sin^2 x = 1$, we cannot have solutions with n not 2 and $0 < |\cos x|, |\sin x| < 1$. Nor can we have solutions with $n=2$, because the sign is wrong. So the only solutions have $\sin x = 0$ or $\cos x = 0$, and these are: x = multiple of π, and n even; x even multiple of π and n odd; x = even multiple of π + 3π/2 and n odd.

Problem B1

P is inside the triangle ABC. PA intersects BC in D, PB intersects AC in E, and PC intersects AB in F. Prove that at least one of AP/PD, BP/PE, CP/PF does not exceed 2, and at least one is not less than 2.



Solution

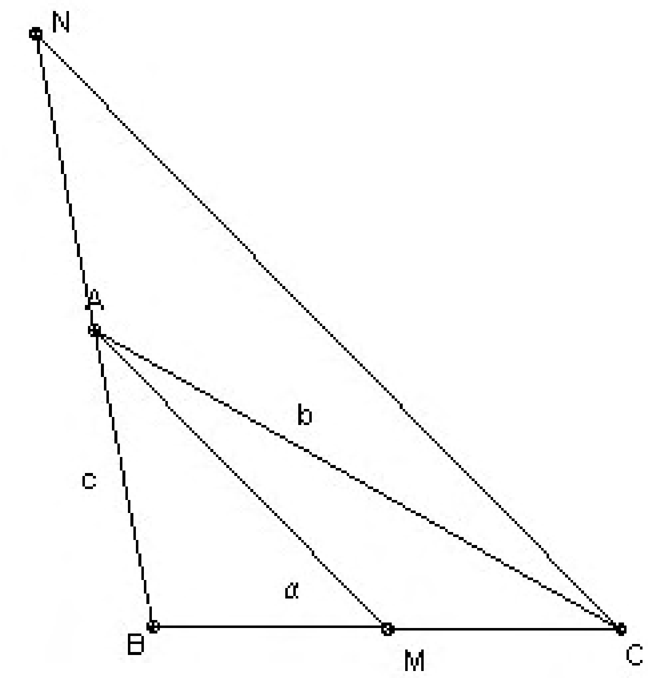
Take lines through the centroid parallel to the sides of the triangle. The result is then obvious.

Problem B2

Construct the triangle ABC, given the lengths AC = b, AB = c and the acute AMB = α, where M is the midpoint of BC. Prove that the construction is possible if and only if

$$b \tan(\alpha/2) \le c < b.$$

When does equality hold?



Answer
Equality holds if ∠BAC = 90^o and ∠ACB = α/2

Solution
The key is to take N so that A is the midpoint of NB, then ∠NCB = α.
The construction is as follows: take BN length 2AB. Take circle through B and N such that the ∠BPN = α for points P on the arc BN. Take A as the midpoint of BN and let the circle center A, radius AC cut the arc BN at C. In general there are two possibilities for C.
Let X be the intersection of the arc BN and the perpendicular to the segment BN through A. For the construction to be possible we require AX ≥ AC > AB. But AB/AX = tan α/2, so we get the condition in the question.
Equality corresponds to C = X and hence to ∠BAC = 90^o and ∠ACB = α/2.

Problem B3
Given 3 non-collinear points A, B, C and a plane p not parallel to ABC and such that A, B, C are all on the same side of p. Take three arbitrary points A', B', C' in p. Let A'', B'', C'' be the midpoints of AA', BB', CC' respectively, and let O be the centroid of A'', B'', C''. What is the locus of O as A', B', C' vary?

Solution
The key is to notice that O is the midpoint of the segment joining the centroids of ABC and A'B'C'. The centroid of ABC is fixed, so the locus is just the plane parallel to p and midway between p and the centroid of ABC.

IMO 1962

Problem A1
Find the smallest natural number with 6 as the last digit, such that if the final 6 is moved to the front of the number it is multiplied by 4.

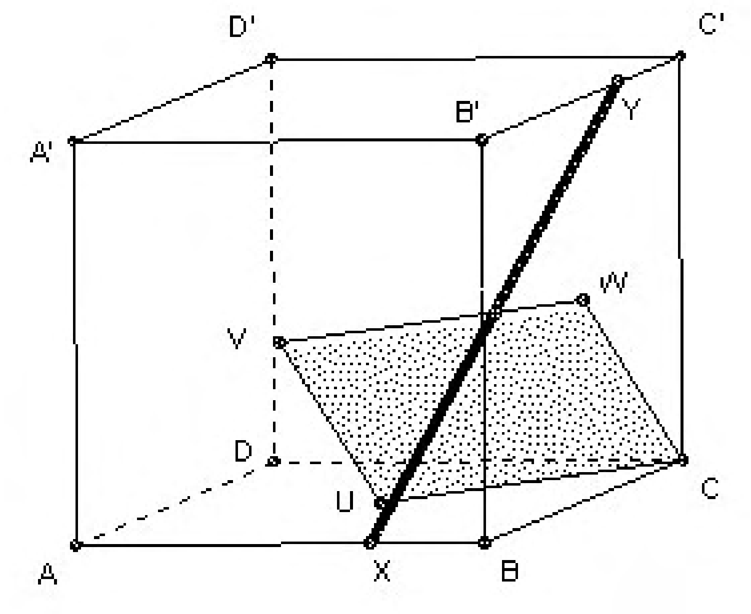
Solution
We have $4(10n+6) = 6 \cdot 10^m + n$, where n has m digits. So $13n + 8 = 2 \cdot 10^m$. Hence n = 2n' and $13n' = 10^m - 4$. Dividing, we quickly find that the smallest n', m satisfying this are: n' = 7692, m = 5. Hence the answer is 153846.

Problem A2
Find all real x satisfying: √(3 - x) - √(x + 1) > 1/2.

Solution
It is easy to show that the inequality implies |x-1| > √31/8, so x > 1 + √31/8, or x < 1 - √31/8. But the converse is not true.
Indeed, we easily see that x > 1 implies the lhs < 0. Also care is needed to ensure that the expressions under the root signs are not negative, which implies -1 ≤ x ≤ 3. Putting this together, suggests the solution is -1 ≤ x < 1 - √31/8, which we can easily check.

Problem A3
The cube ABCDA'B'C'D' has upper face ABCD and lower face A'B'C'D' with A directly above A' and so on. The point x moves at constant speed along the perimeter of ABCD, and the

point Y moves at the same speed along the perimeter of B'C'CB. X leaves A towards B at the same moment as Y leaves B' towards C'. What is the locus of the midpoint of XY?



Solution

Answer: the rhombus CUVW, where U is the center of ABCD, V is the center of ABB'A, and W is the center of BCC'B'.

Take rectangular coordinates with A as (0, 0, 0) and C' as (1, 1, 1). Let M be the midpoint of XY.

Whilst X is on AB and Y on B'C', X is (x, 0, 0) and Y is (1, x, 1), so M is (x/2 + 1/2, x/2, 1/2) = x (1, 1/2, 1/2) + (1-x) (1/2, 0, 1/2) = x W + (1-x) V, so M traces out the line VW.

Whilst X is on BC and Y is on C'C, X is (1, x, 0) and Y is (1, 1, 1-x), so M is (1, x/2+1/2, 1/2 - x/2) = x (1, 1, 0) + (1-x) (1, 1/2, 1/2) = x C + (1-x) W, so M traces out the line WC.

Whilst X is on CD and Y is on CB, X is (1-x, 1, 0) and Y is (1, 1-x, 0), so M is (1-x/2, 1-x/2, 0) = x (1, 1, 0) + (1-x) (1/2, 1/2, 0) = x C + (1-x) U, so M traces out the line CU.

Whilst X is on DA and Y is on BB', X is (0, 1-x, 0) and Y is (1, 0, x), so M is (1/2, 1/2 - x/2, x/2) = x (1/2, 0, 1/2) + (1-x) (1/2, 1/2, 0) = x V + (1-x) U, so M traces out the line UV.

Problem B1

Find all real solutions to $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

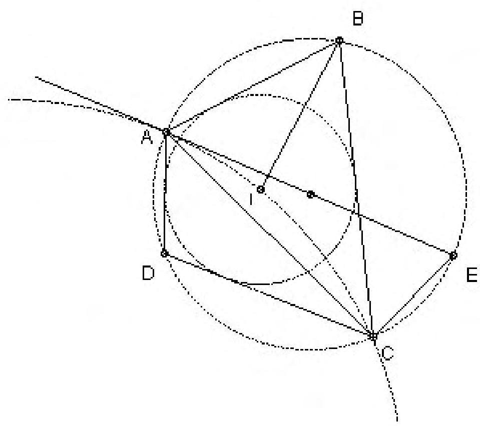
Solution

Put c = cos x, and use $\cos 3x = 4c^3 - 3c$, $\cos 2x = 2c^2 - 1$. We find the equation given is equivalent to $c = 0$, $c^2 = 1/2$ or $c^2 = 3/4$. Hence $x = \pi/2, 3\pi/2, \pi/4, 3\pi/4, \pi/6, 5\pi/6$ or any multiple of π plus one of these.

Problem B2

Given three distinct points A, B, C on a circle K, construct a point D on K, such that a circle can be inscribed in ABCD.

Solution



I be the center of the inscribed circle. Consider the quadrilateral ABCI. $\angle BAI = 1/2 \angle BAD$ and $\angle BCI = 1/2 \angle BCD$, so $\angle BAI + \angle BCI = 90^o$, since ABCD is cyclic. Hence $\angle AIC = 270^o - \angle ABC$. So if we draw a circle through A and C such that for X points on the arc AC $\angle AXC = 90^o + \angle ABC$, then the intersection of the circle with the angle bisector of $\angle ABC$ gives the point I.

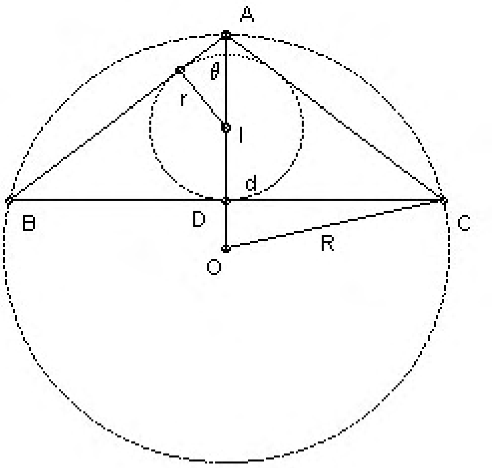
To draw this circle take the diameter AE. Then $\angle CAE = 180^o - \angle ACE - \angle AEC = 90^o - \angle ABC$. So we want AE to be tangent to the circle. Thus the center of the circle is on the perpendicular to AE through A and on the perpendicular bisector of AC.

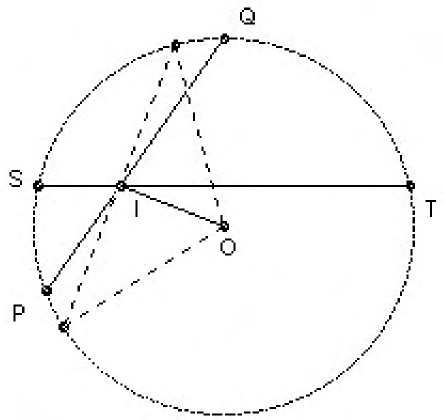
To prove the construction possible we use the fact that a quadrilateral ABCD has an inscribed circle iff AB + CD = BC + AD. For D near C on the circumcircle of ABC we have AB + CD < BC + AD, whilst for D near A we have AB + CD > BC + AD, so as D moves continuously along the circumcircle there must be a point with equality. [Proof that the condition is sufficient: it is clearly necessary (use fact that tangents from a point are of equal length). So take a circle touching AB, BC and AD and let the other tangent from C (not BC) meet AD in D'. Then CD' - CD = AD' - AD, hence D'= D.]

Problem B3

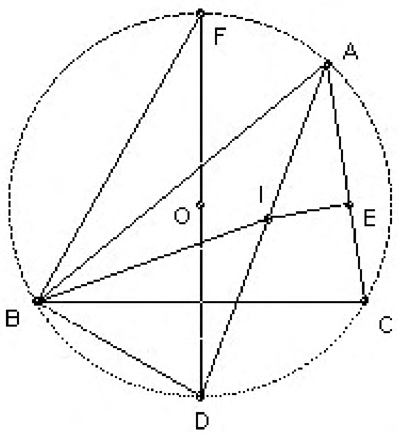
The radius of the circumcircle of an isosceles triangle is R and the radius of its inscribed circle is r. Prove that the distance between the two centers is $\sqrt{(R(R - 2r))}$.

Solution



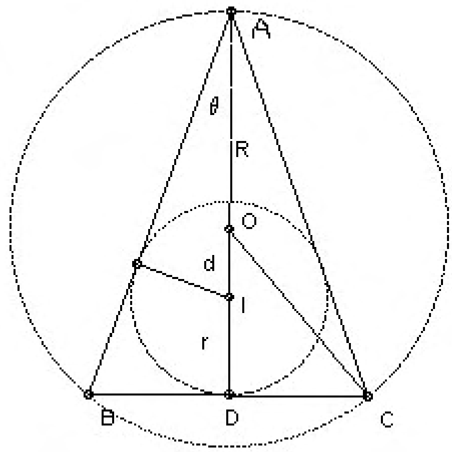


Let the triangle be ABC with AB = AC, let the incenter be I and the circumcenter O. Let the distance IO be d, taking d positive if O is closer to A than I, negative if I is closer. Let the ∠OAB be θ.



Then $r = (R + d) \sin \theta$, and $r + d = R \cos 2\theta$. It helps to draw a figure to check that this remains true for the various possible configurations. Using $\cos 2\theta = 1 - 2 \sin^2\theta$, we find that $(d + R + r)(d^2 - R(R - 2r)) = 0$. But OI < OA, so d is not - R - r. Hence result.

This result is known as Euler's formula and is true for any triangle. Suppose two chords PQ and ST of a circle intersect at I. Then PIS and TIQ are similar, so PI·IQ = SI·IT. Take the special case when ST is perpendicular to OI, where O is the center of the circle, then $SI \cdot IT = SI^2 = R^2 - OI^2$, where R is the radius of the circle, so $PI \cdot IQ = R^2 - OI^2$.



Now let O be the circumcenter, I the incenter of an arbitrary triangle ABC. Extend AI to meet the circumcircle again at D. Then by the above $IO^2 = R^2 - AI \cdot ID$. If E is the foot of the perpendicular from I to AC, then $AI = r/\sin(A/2)$. We show that DI = DB. ∠DBI = ∠DBC + ∠CBI = ∠DAC + ∠DBI = A/2 + B/2. ∠DIB = ∠IAB + ∠IBA = A/2 + B/2. Hence ∠DBI = ∠DIB, so DI = DB, as claimed. Take F on the circle so that DF is a diameter, then ∠DFB = ∠DAB = A/2, so DB = 2R sin A/2. Thus $IO^2 = R^2 - r/\sin(A/2)\ 2R \sin(A/2) = R^2 - 2Rr$.

Problem B4

Prove that a regular tetrahedron has five distinct spheres each tangent to its six extended edges. Conversely, prove that if a tetrahedron has five such spheres then it is regular.

Solution
First part is obvious. The wrong way to do the second part is to start looking for the locus of the center of a sphere which touches three edges. The key is to notice that the tangents to a sphere from a given point have the same length.
Let the tetrahedron be $A_1A_2A_3A_4$. Let S be the sphere inside the tetrahedron, S_1 the tetrahedron opposite A_1, and so on. Let the tangents to S from A_i have length a_i. Then the side A_iA_j has length a_i+a_j. Now consider the tangents to S_1 from A_1. Their lengths are $a_1 + 2a_2$, $a_1 + 2a_3$, and $a_1 + 2a_4$. Hence $a_2 = a_3 = a_4$. Similarly, considering S_2, we have that $a_1 = a_3 = a_4$.

IMO 1963

Problem A1
For which real values of p does the equation
$\sqrt{(x^2 - p)} + 2\sqrt{(x^2 - 1)} = x$ have real roots? What are the roots?

Solution
I must admit to having formed rather a dislike for this type of question which came up in almost every one of the early IMOs. Its sole purpose seems to be to teach you to be careful with one-way implications: the fact that $a^2 = b^2$ does not imply $a = b$.
The lhs is non-negative, so x must be non-negative. Moreover $2\sqrt{(x^2 - 1)} \le x$, so $x \le 2/\sqrt{3}$. Also $\sqrt{(x^2 - p)} \le x$, so $p \ge 0$.
Squaring etc gives that any solution must satisfy $x^2 = (p - 4)^2/(16 - 8p)$. We require $x \le 2/\sqrt{3}$ and hence $(3p - 4)(p + 4) \le 0$, so $p \le 4/3$.
Substituting back in the original equality we get $|3p-4| + 2|p| = |p - 4|$, which is indeed true for any p satisfying $0 \le p \le 4/3$.

Problem A2
Given a point A and a segment BC, determine the locus of all points P in space for which $APX = 90^o$ for some X on the segment BC.

Solution
Take the solid sphere on diameter AB, and the solid sphere on diameter AC. Then the locus is the points in one sphere but not the other (or on the surface of either sphere). Given P, consider the plane through P perpendicular to AP and the parallel planes through the other two points of intersection of AP with the two spheres (apart from A) which pass through B and C.

Problem A3
An n-gon has all angles equal and the lengths of consecutive sides satisfy $a_1 \ge a_2 \ge \dots \ge a_n$. Prove that all the sides are equal.

Solution
For n odd consider the perpendicular distance of the shortest side from the opposite vertex. This is a sum of terms a_i x cosine of some angle. We can go either way round. The angles are the same in both cases, so the inequalities give that $a_1 = a_{n-1}$, and hence $a_1 = a_i$ for all $i < n$. We get $a_1 = a_n$ by repeating the argument for the next shortest side. The case n even is easier, because we take a line through the vertex with sides a_1 and a_n making equal angles with them and look at the perpendicular distance to the opposite vertex. This gives immediately that $a_1 = a_n$.

Problem B1
Find all solutions $x_1, \dots , x_5$ to the five equations $x_i + x_{i+2} = y\, x_{i+1}$ for $i = 1, \dots , 5$, where subscripts are reduced by 5 if necessary.

Solution
Successively eliminate variables to get $x_1(y - 2)(y^2 + y - 1)^2 = 0$. We have the trivial solution $x_i = 0$ for any y. For $y = 2$, we find $x_i = s$ for all i (where s is arbitrary). Care is needed for the case $y^2 + y - 1 = 0$, because after eliminating three variables the two

remaining equations have a factor $y^2 + y - 1$, and so they are automatically satisfied. In this case, we can take any two x_i arbitrary and still get a solution. For example, $x_1 = s$, $x_2 = t$, $x_3 = - s + yt$, $x_4 = - ys - yt$, $x_5 = ys - t$.

Problem B2

Prove that $\cos \pi/7 - \cos 2\pi/7 + \cos 3\pi/7 = 1/2$.

Solution

Consider the roots of $x^7 + 1 = 0$. They are $e^{i\pi/7}$, $e^{i3\pi/7}$, ... , $e^{i13\pi/7}$ and must have sum zero since there is no x^6 term. Hence, in particular, their real parts sum to zero. But $\cos 7\pi/7 = -1$ and the others are equal in pairs, because $\cos(2\pi - x) = \cos x$. So we get $\cos \pi/7 + \cos 3\pi/7 + \cos 5\pi/7 = 1/2$. Finally since $\cos(\pi - x) = - \cos x$, $\cos 5\pi/7 = - \cos 2\pi/7$.

Problem B3

Five students A, B, C, D, E were placed 1 to 5 in a contest with no ties. One prediction was that the result would be the order A, B, C, D, E. But no student finished in the position predicted and to two students predicted to finish consecutively did so. For example, the outcome for C and D was not 1, 2 (respectively), or 2, 3, or 3, 4 or 4, 5. Another prediction was the order D, A, E, C, B. Exactly two students finished in the places predicted and two disjoint pairs predicted to finish consecutively did so. Determine the outcome.

Solution

Start from the second prediction. The disjoint pairs can only be: DA, EC; DC, CB; or AE, CB. The additional requirement of just two correct places means that the only possibilities (in the light of the information about the second prediction) are: DABEC, DACBE, EDACB, AEDCB. The first is ruled out because AB are consecutive. The second is ruled out because C is in the correct place. The fourth is ruled out because A is in the correct place. This leaves EDACB, which is indeed a solution.

IMO 1964

Problem A1

(a) Find all natural numbers n for which 7 divides $2^n - 1$.
(b) Prove that there is no natural number n for which 7 divides $2^n + 1$.

Solution

$2^3 = 1 \pmod 7$. Hence $2^{3m} = 1 \pmod 7$, $2^{3m+1} = 2 \pmod 7$, and $2^{3m+2} = 4 \pmod 7$. Hence we never have 7 dividing $2^n + 1$, and 7 divides $2^n - 1$ iff 3 divides n.

Problem A2

Suppose that a, b, c are the sides of a triangle. Prove that:

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \le 3abc.$$

Solution

The condition that a, b, c be the sides of a triangle, together with the appearance of quantities like $a + b - c$ is misleading. The inequality holds for any $a , b, c \ge 0$.
At most one of $(b+c-a)$, $(c+a-b)$, $(a+b-c)$ can be negative. If one of them is negative, then certainly:

$$abc \ge (b + c - a)(c + a - b)(a + b - c) \quad (*)$$

since the lhs is non-negative and the rhs is non-positive.
(*) is also true if none of them is negative. For then the arithmetic/geometric mean on $b + c - a$, $c + a - b$ gives:

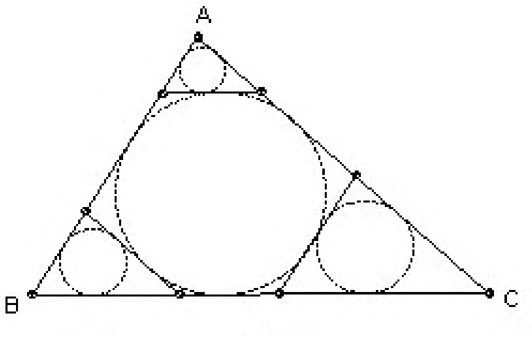
$$c^2 \ge (b + c - a)(c + a - b).$$

Similarly for a^2 and b^2. Multiplying and taking the square root gives (*). Multiplying out easily gives the required result.

Problem A3

Triangle ABC has sides a, b, c. Tangents to the inscribed circle are constructed parallel to the sides. Each tangent forms a triangle with the other two sides of the triangle and a circle is inscribed in each of these three triangles. Find the total area of all four inscribed circles.

Solution



This is easy once you realize that the answer is not nice and the derivation a slog. Use r = 2·area/perimeter and Heron's formula: area k is given by $16k^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c)$.
The small triangles at the vertices are similar to the main triangle and smaller by a factor (h - 2r)/h, where h is the relevant altitude. For the triangle opposite side a: (h - 2r)/h = 1 - 2(2k/p)/(2k/a) = 1 - 2a/p = (b + c - a)/(a + b + c).
Hence the total area is $((a + b + c)^2 + (b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2)/(a + b + c)^2 \text{ pi } r^2 = (a^2 + b^2 + c^2).\text{pi}.(b + c - a)(c + a - b)(a + b - c)/(a + b + c)^3$.

Problem B1
Each pair from 17 people exchange letters on one of three topics. Prove that there are at least 3 people who write to each other on the same topic. [In other words, if we color the edges of the complete graph K_{17} with three colors, then we can find a triangle all the same color.]

Solution
Take any person. He writes to 16 people, so he must write to at least 6 people on the same topic. If any of the 6 write to each other on that topic, then we have a group of three writing to each other on the same topic. So assume they all write to each other on the other two topics. Take any of them, B. He must write to at least 3 of the other 5 on the same topic. If two of these write to each other on this topic, then they form a group of three with B. Otherwise, they must all write to each other on the third topic and so from a group of three.

Problem B2
5 points in a plane are situated so that no two of the lines joining a pair of points are coincident, parallel or perpendicular. Through each point lines are drawn perpendicular to each of the lines through two of the other 4 points. Determine the maximum number of intersections these perpendiculars can have.

Solution
It is not hard to see that the required number is at most 315. But it is not at all obvious how you prove it actually is 315, short of calculating the 315 points intersection for a specific example.
Call the points A, B, C, D, E. Given one of the points, the other 4 points determine 6 lines, so there are 6 perpendiculars through the given point and hence 30 perpendiculars in all. These determine at most 30.29/2 = 435 points of intersection. But some of these necessarily coincide. There are three groups of coincidences. The first is that the 6 perpendiculars through A meet in one point (namely A), not the expected 15. So we lose 5.14 = 70 points. Second, the lines through C, D and E perpendicular to AB are all parallel, and do not give the expected 3 points of intersection, so we lose another 10.3 = 30 points. Third, the line through A perpendicular to BC is an altitude of the triangle ABC, as are the lines through B perpendicular to AC, and the through C perpendicular to AB. So we only get one point of intersection instead of three, thus losing another 10.2 = 20 points. These coincidences are clearly all distinct (the categories do not overlap), so they bring us down to a maximum of 435 - 120 =315.
There is no obvious reason why there should be any further coincidences. But that is not quite the same as proving that there are no more. Indeed, for particular positions of the

points A, B, C, D, E we can certainly arrange for additional coincidences (the constraints given in the problem are not sufficient to prevent additional coincidences). So we have to prove that it is possible to arrange the points so that there are no additional coincidences. I cannot see how to do this, short of exhibiting a particular set of points, which would be extremely tiresome. Apparently the contestants were instructed verbally that they did not have to do it.

Problem B3

ABCD is a tetrahedron and D_0 is the centroid of ABC. Lines parallel to DD_0 are drawn through A, B and C and meet the planes BCD, CAD and ABD in A_0, B_0, and C_0 respectively. Prove that the volume of ABCD is one-third of the volume of $A_0B_0C_0D_0$. Is the result true if D_0 is an arbitrary point inside ABC?

Solution

Yes, indeed it is true for an arbitrary point in the plane of ABC not on any of the lines AB, BC, CA

Take D as the origin. Let A, B, C be the points **a**, **b**, **c** respectively. Then D_0 is p**a** + q**b** + r**c** with p + q + r = 1 and p, q, r > 0. So a point on the line parallel to DD_0 through A is **a** + s(p**a** + q**b** + r**c**. It is also in the plane DBC if s = -1/p, so A_0 is the point - q/p **b** - r/p **c**. Similarly, B_0 is - p/q **a** - r/q **c**, and C_0 is - p/r **a** - q/r **b**.

The volume of ABCD is 1/6 |**a x b.c**| and the volume of $A_0B_0C_0D_0$ is 1/6 |(p**a** + (q + q/p)**b** + (r + r/p)**c**) **x** ((p + p/q)**a** + q**b** + (r + r/q)**c**).((p + p/r)**a** + (q + q/r)**b** + r**c**)|

Thus vol $A_0B_0C_0D_0$/vol ABCD = abs value of the determinant:

$$\begin{vmatrix} p & q + q/p & r + r/p \\ p + p/q & q & r + r/q \\ p + p/r & q + q/r & r \end{vmatrix}$$

which is easily found to be 2 + p + q + r = 3.

IMO 1965

Problem A1

Find all x in the interval [0, 2π] which satisfy:

$$2 \cos x \le |\sqrt{(1 + \sin 2x)} - \sqrt{(1 - \sin 2x)}| \le \sqrt{2}.$$

Solution

Let y = $|\sqrt{(1 + \sin 2x)} - \sqrt{(1 - \sin 2x)}|$. Then $y^2 = 2 - 2|\cos 2x|$. If x belongs to [0, π/4] or [3π/4, 5π/4] or [7π/4], then cos 2x is non-negative, so $y^2 = 2 - 2 \cos 2x = 4 \sin^2 x$, so y = 2|sin x|. We have cos x <= |sin x| except for x in [0, π/4] and [7π/4, 2π]. So that leaves [3π/4, 5π/4] in which we certainly have |sin x| ≤ 1/√2.

If x belongs (π/4, 3π/4) or (5π/5, 7π/4), then cos 2x is negative, so $y^2 = 2 + 2 \cos 2x = 4 \cos^2 x$. So y = 2 |cos x|. So the first inequality certainly holds. The second also holds. Thus the inequalities hold for all x in [π/4, 7π/4].

Problem A2

The coefficients a_{ij} of the following equations

$$a_{11}x_1 + a_{12}\, x_2 + a_{13}\, x_3 = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

satisfy the following: (a) a_{11}, a_{22}, a_{33} are positive, (b) other a_{ij} are negative, (c) the sum of the coefficients in each equation is positive. Prove that the only solution is $x_1 = x_2 = x_3 = 0$.

Solution

The slog solution is to multiply out the determinant and show it is non-zero. A slicker solution is to take the x_i with the largest absolute value. Say $|x_1| \ge |x_2|, |x_3|$. Then looking at the first equation we have an immediate contradiction, since the first term has larger absolute value than the sum of the absolute values of the second two terms.

Problem A3

The tetrahedron ABCD is divided into two parts by a plane parallel to AB and CD. The distance of the plane from AB is k times its distance from CD. Find the ratio of the volumes of the two parts.

Solution
Let the plane meet AD at X, BD at Y, BC at Z and AC at W. Take plane parallel to BCD through WX and let it meet AB in P.
Since the distance of AB from WXYZ is k times the distance of CD, we have that AX = k·XD and hence that AX/AD = k/(k+1). Similarly AP/AB = AW/AC = AX/AD. XY is parallel to AB, so also AX/AD = BY/BD = BZ/BC.
vol ABWXYZ = vol APWX + vol WXPBYZ. APWX is similar to the tetrahedron ABCD. The sides are k/(k+1) times smaller, so vol APWX = $k^3(k+1)^3$ vol ABCD. The base of the prism WXPBYZ is BYZ which is similar to BCD with sides k/(k+1) times smaller and hence area $k^2(k+1)^2$ times smaller. Its height is 1/(k+1) times the height of A above ABCD, so vol prism = 3 $k^2(k+1)^3$ vol ABCD. Thus vol ABWXYZ = $(k^3 + 3k^2)/(k+1)^3$ vol ABCD. We get the vol of the other piece as vol ABCD - vol ABWXYZ and hence the ratio is (after a little manipulation) $k^2(k+3)/(3k+1)$.

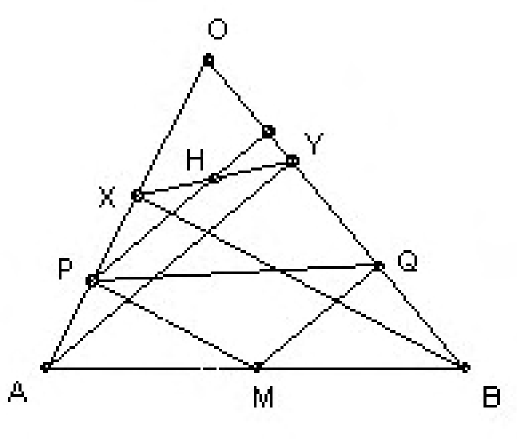
Problem B1
Find all sets of four real numbers such that the sum of any one and the product of the other three is 2.

Answer
1,1,1,1 or 3,-1,-1,-1.

Solution
Let the numbers be $x_1, \dots , x_4$. Let $t = x_1x_2x_3x_4$. Then $x_1 + t/x_1 = 2$. So all the x_i are roots of the quadratic $x^2 - 2x + t = 0$. This has two roots, whose product is t.
If all x_i are equal to x, then $x^3 + x = 2$, and we must have x = 1. If not, then if x_1 and x_2 are unequal roots, we have $x_1x_2 = t$ and $x_1x_2x_3x_4 = t$, so $x_3x_4 = 1$. But x_3 and x_4 are still roots of $x^2 - 2x + t = 0$. They cannot be unequal, otherwise $x_3x_4 = t$, which gives t = 1 and hence all $x_i = 1$. Hence they are equal, and hence both 1 or both -1. Both 1 gives t = 1 and all $x_i = 1$. Both -1 gives t = -3 and hence x_i = 3, -1, -1, -1 (in some order).

Problem B2
The triangle OAB has O acute. M is an arbitrary point on AB. P and Q are the feet of the perpendiculars from M to OA and OB respectively. What is the locus of H, the orthocenter of the triangle OPQ (the point where its altitudes meet)? What is the locus if M is allowed to vary of the interior of OAB?

Solution

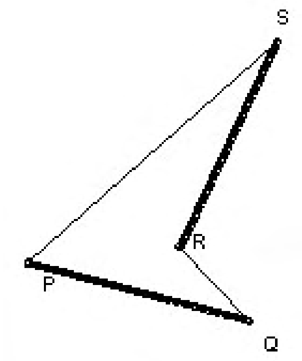


Let X be the foot of the perpendicular from B to OA, and Y the foot of the perpendicular from A to OB. We show that the orthocenter of OPQ lies on XY.
MP is parallel to BX, so AM/MB = AP/PX. Let H be the intersection of XY and the perpendicular from P to OB. PH is parallel to AY, so AP/PX = YH/HX. MQ is parallel to AY, so AM/MB = YQ/BQ. Hence YQ/BQ = YH/HX and so QH is parallel to BX and hence perpendicular to AO, so H is the orthocenter of OPQ as claimed.
If we restrict M to lie on a line A'B' parallel to AB (with A' on OA, B' on OB) then the locus is a line X'Y' parallel to XY, so as M moves over the whole interior, the locus is the interior of the triangle OXY.

Problem B3

Given $n > 2$ points in the plane, prove that at most n pairs of points are the maximum distance apart (of any two points in the set).

Solution



The key is that if two segments length d do not intersect then we can find an endpoint of one which is a distance $> d$ from an endpoint of the other.

Given this, the result follows easily by induction. If false for n, then there is a point A in three pairs AB, AC and AD of length d (the maximum distance). Take AC to lie between AB and AD. Now C cannot be in another pair. Suppose it was in CX. Then CX would have to cut both AB and AD, which is impossible.

To prove the result about the segments, suppose they are PQ and RS. We must have angle PQR less than 90^o, otherwise PR > PQ = d. Similarly, the other angles of the quadrilateral must all be less than 90^o. Contradiction.

IMO 1966

Problem A1

Problems A, B and C were posed in a mathematical contest. 25 competitors solved at least one of the three. Amongst those who did not solve A, twice as many solved B as C. The number solving only A was one more than the number solving A and at least one other. The number solving just A equalled the number solving just B plus the number solving just C. How many solved just B?

Answer

6.

Solution

Let a solve just A, b solve just B, c solve just C, and d solve B and C but not A. Then $25 - a - b - c - d$ solve A and at least one of B or C. The conditions give:

$b + d = 2(c + d)$; $a = 1 + 25 - a - b - c - d$; $a = b + c$.

Eliminating a and d, we get: $4b + c = 26$. But $d = b - 2c \geq 0$, so $b = 6$, $c = 2$.

Problem A2

Prove that if $BC + AC = \tan C/2\ (BC \tan A + AC \tan B)$, then the triangle ABC is isosceles.

Solution

A straight slog works. Multiply up to get $(a + b) \cos A \cos B \cos C/2 = a \sin A \cos B \sin C/2 + b \cos A \sin B \sin C/2$ (where $a = BC$, $b = AC$, as usual). Now use $\cos(A + C/2) = \cos A \cos C/2 - \sin A \sin C/2$ and similar relation for $\cos (B + C/2)$ to get: $a \cos B \cos(A + C/2) + b \cos A \cos (B + C/2) = 0$. Using $C/2 = 90^o - A/2 - B/2$, we find that $\cos(A + C/2) = - \cos(B + C/2)$ (and $= 0$ only if $A = B$). Result follows.

Problem A3

Prove that a point in space has the smallest sum of the distances to the vertices of a regular tetrahedron iff it is the center of the tetrahedron.

Solution

Let the tetrahedron be ABCD and let P be a general point. Let X be the midpoint of CD. Let P' be the foot of the perpendicular from P to the plane ABX. We show that if P does not coincide with P', then PA + PB + PC + PD > P'A + P'B + P'C + P'D.

PA > P'A (because angle PP'A = 90^o) and PB > P'B. P'CD is isosceles and PCD is not but P is the same perpendicular distance from the line CD as P'. It follows that PC + PD > P'C + P'D. The easiest way to see this is to reflect C and D in the line PP' to give C' and D'. Then PC = PC', and PC' + PD > C'D = P'C' + P'D = P'C + P'D.

So if P has the smallest sum, it must lie in the plane ABX and similarly in the plane CDY, where Y is the midpoint of AB, and hence on the line XY. Similarly, it must lie on the line joining the midpoints of another pair of opposite sides and hence must be the center.

Problem B1
Prove that $1/\sin 2x + 1/\sin 4x + \dots + 1/\sin 2^n x = \cot x - \cot 2^n x$ for any natural number n and any real x (with $\sin 2^n x$ non-zero).

Solution
$\cot y - \cot 2y = \cos y/\sin y - (2 \cos^2 y - 1)/(2 \sin y \cos y) = 1/(2 \sin y \cos y) = 1/\sin 2y$.
The result is now easy. Use induction. True for n = 1 (just take y = x). Suppose true for n, then taking $y = 2^n x$, we have $1/\sin 2^{n+1}x = \cot 2^n x - \cot 2^{n+1}x$ and result follows for n + 1.

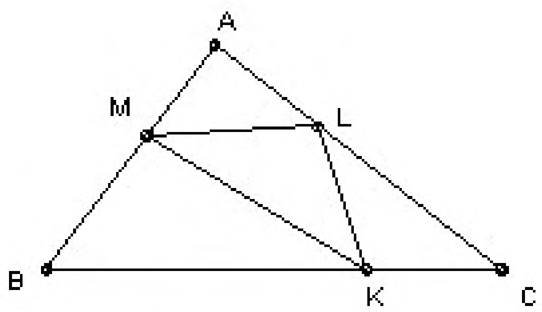
Problem B2
Solve the equations:
$|a_i - a_1|\ x_1 + |a_i - a_2|\ x_2 + |a_i - a_3|\ x_3 + |a_i - a_4|\ x_4 = 1$, i = 1, 2, 3, 4, where a_i are distinct reals.

Answer
$x_1 = 1/(a_1 - a_4)$, $x_2 = x_3 = 0$, $x_4 = 1/(a_1 - a_4)$.

Solution
Take $a_1 > a_2 > a_3 > a_4$. Subtracting the equation for i=2 from that for i=1 and dividing by $(a_1 - a_2)$ we get:
$- x_1 + x_2 + x_3 + x_4 = 0$.
Subtracting the equation for i=4 from that for i=3 and dividing by $(a_3 - a_4)$ we get:
$- x_1 - x_2 - x_3 + x_4 = 0$.
Hence $x_1 = x_4$. Subtracting the equation for i=3 from that for i=2 and dividing by $(a_2 - a_3)$ we get:
$- x_1 - x_2 + x_3 + x_4 = 0$.
Hence $x_2 = x_3 = 0$, and $x_1 = x_4 = 1/(a_1 - a_4)$.

Problem B3
Take any points K, L, M on the sides BC, CA, AB of the triangle ABC. Prove that at least one of the triangles AML, BKM, CLK has area ≤ 1/4 area ABC.

Solution
If not, then considering ALM we have $4\cdot AL\cdot AM\cdot \sin A > AB\cdot AC\cdot \sin A$, so $4\cdot AL\cdot AM > AB\cdot AC = (AM + BM)(AL + CL)$, so $3\cdot AL\cdot AM > AM\cdot CL + BM\cdot AL + BM\cdot CL$. Set $k = BK/CK$, $l = CL/AL$, $m = AM/BM$, and this inequality becomes:
$3 > l + 1/m + l/m$.
Similarly, considering the other two triangles we get: $3 > k + 1/l + k/l$, and $3 > m + 1/k + m/k$.



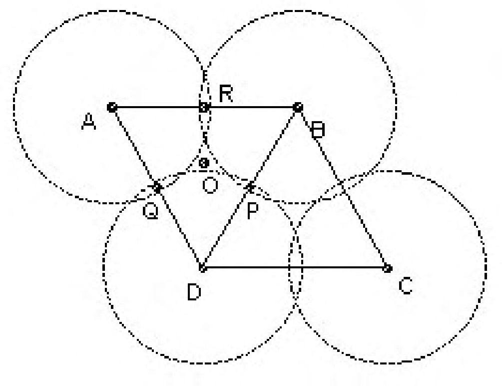
Adding gives: $9 > k + l + m + 1/k + 1/l + 1/m + k/l + l/m + m/k$, which is false by the arithmetic/geometric mean inequality.

IMO 1967

Problem A1
The parallelogram ABCD has AB = a, AD = 1, angle BAD = A, and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram iff
$a \le \cos A + \sqrt{3} \sin A$.

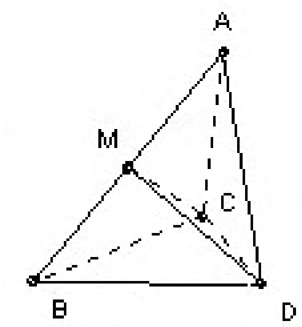
Solution

Evidently the parallelogram is a red herring, since the circles cover it iff and only if the three circles center A, B, D cover the triangle ABD.



The three circles radius x and centers the three vertices cover an acute-angled triangle ABD iff x is at least R, the circumradius. The circumcenter O is a distance R from each vertex, so the condition is clearly necessary. If the midpoints of BD, DA, AB are P, Q, R, then the circle center A, radius R covers the quadrilateral AQOR, the circle center B, radius R covers the quadrilateral BROP, and the circle center D radius R covers the quadrilateral DPOQ, so the condition is also sufficient.
We need an expression for R in terms of a and A. We can express BD two ways: 2R sin A, and $\sqrt{(a^2 + 1 - 2a \cos A)}$. So a necessary and sufficient condition for the covering is $4 \sin^2 A \geq (a^2 + 1 - 2a \cos A)$, which reduces to $a \leq \cos A + \sqrt{3} \sin A$, since $\cos A \leq a$ (the foot of the perpendicular from D onto AB must lie between A and B).

Problem 2
Prove that a tetrahedron with just one edge length greater than 1 has volume at most 1/8.



Solution
Let the tetrahedron be ABCD and assume that all edges except AB have length at most 1. The volume is the 1/3 x area BCD x height of A above BCD. The height is at most the height of A above CD, so we maximise the volume by taking the planes ACD and BCD to be perpendicular. If AC or AD is less than 1, then we can increase the altitude from A to CD whilst keeping BCD fixed by taking AC = AD = 1. A similar argument shows that we must have BC = BD = 1.
But the volume is also the 1/3 x area ABC x height of D above ABC, so we must adjust CD to maximise this height. We want the angle between planes ABC and ABD to be as close as possible to 90^o. The angle increases with increasing CD until it becomes 90^o. CMD is then a right-angled triangle. Now the angle ACB must be less than the angle between the planes ACD and BCD and hence $< 90^o$, so angle ACM $< 45^o$, so CM $> 1/\sqrt{2}$. Similarly DM. Hence when CMD $= 90^o$ we have CD > 1. Thus we maximise the height of D above ABC by taking CD = 1.
So BCD is equilateral with area $(\sqrt{3})/4$. ACD is also equilateral with altitude $(\sqrt{3})/2$. Since the planes ACD and BCD are perpendicular, that is also the height of A above BCD. So the volume is $1/3 \times (\sqrt{3})/4 \times (\sqrt{3})/2 = 1/8$.

Problem A3
Let k, m, n be natural numbers such that m + k + 1 is a prime greater than n + 1. Let $c_s = s(s+1)$. Prove that:

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \dots (c_{m+n} - c_k)$$

is divisible by the product $c_1c_2 \dots c_n$.

Solution
The key is that $c_a - c_b = (a - b)(a + b + 1)$. Hence the product $(c_{m+1} - c_k)(c_{m+2} - c_k) \dots (c_{m+n} - c_k)$ is the product of the n consecutive numbers (m - k + 1), ... , (m - k + n), times the product of the n consecutive numbers (m + k + 2), ... , (m + k + n + 1). The first product is just the binomial coefficient (m-k+n)Cn times n!, so it is divisible by n!. The second product is 1/(m + k + 1) x (m + k + 1)(m + k + 2) ... (m + k + n + 1) = 1/(m + k + 1) x (m+k+n+1)C(n+1) x (n+1)!. But m + k + 1 is a prime greater than n + 1, so it has no factors in common with (n+1)!, hence the second product is divisible by (n+1)!. Finally note that $c_1c_2 \dots c_n$= n! (n+1)!.

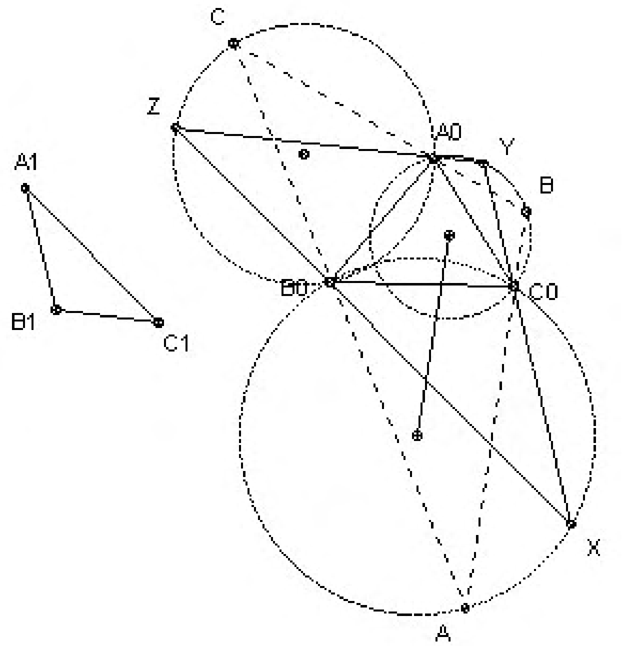
Problem B1

$A_0B_0C_0$ and $A_1B_1C_1$ are acute-angled triangles. Construct the triangle ABC with the largest possible area which is circumscribed about $A_0B_0C_0$ (BC contains A_0, CA contains B_0, and AB contains C_0) and similar to $A_1B_1C_1$.

Solution

Take any triangle similar to $A_1B_1C_1$ and circumscribing $A_0B_0C_0$. For example, take an arbitrary line through A_0 and then lines through B_0 and C_0 at the appropriate angles to the first line. Label the triangle's vertices X, Y, Z so that A_0 lies on YZ, B_0 on ZX, and C_0 on XY. Now any circumscribed ABC (labeled with the same convention) must have C on the circle through A_0, B_0 and Z, because it has $\angle C = \angle Z = \angle C_1$. Similarly it must have B on the circle through C_0, A_0 and Y, and it must have A on the circle through B_0, C_0 and X.

Consider the side AB. It passes through C_0. Its length is twice the projection of the line joining the centers of the two circles onto AB (because each center projects onto the midpoint of the part of AB that is a chord of its circle). But this projection is maximum when it is parallel to the line joining the two centers. The area is maximised when AB is maximised (because all the triangles are similar), so we take AB parallel to the line joining the centers. [Note, in passing, that this proves that the other sides must also be parallel to the lines joining the respective centers and hence that the three centers form a triangle similar to $A_1B_1C_1$.]



Problem B2

$a_1, \dots , a_8$ are reals, not all zero. Let $c_n = a_1^n + a_2^n + \dots + a_8^n$ for n = 1, 2, 3, ... Given that an infinite number of c_n are zero, find all n for which c_n is zero.

Solution

Take $|a_1| \ge |a_2| \ge \dots \ge |a_8|$. Suppose that $|a_1|, \dots , |a_r|$ are all equal and greater than $|a_{r+1}|$. Then for sufficiently large n, we can ensure that $|a_s|^n < 1/8\ |a_1|^n$ for s > r, and hence the sum of $|a_s|^n$ for all s > r is less than $|a_1|^n$. Hence r must be even with half of $a_1, \dots , a_r$ positive and half negative.

If that does not exhaust the a_i, then in a similar way there must be an even number of a_i with the next largest value of $|a_i|$, with half positive and half negative, and so on. Thus we find that $c_n = 0$ for all odd n.

Problem B3

In a sports contest a total of m medals were awarded over n days. On the first day one medal and 1/7 of the remaining medals were awarded. On the second day two medals and 1/7 of the remaining medals were awarded, and so on. On the last day, the remaining n medals were awarded. How many medals were awarded, and over how many days?

Solution

Let the number of medals remaining at the start of day r be m_r. Then $m_1 = m$, and $6(m_k - k)/7 = m_{k+1}$ for k < n with $m_n = n$.

After a little rearrangement, we find that $m = 1 + 2(7/6) + 3(7/6)^2 + \dots + n(7/6)^{n-1}$.
Summing, we get $m = 36(1 - (n + 1)(7/6)^n + n\,(7/6)^{n+1}) = 36 + (n - 6)7^n/6^{n-1}$. 6 and 7 are coprime, so 6^{n-1} must divide n - 6. But $6^{n-1} > n - 6$, so n = 6 and m = 36.

IMO 1968

Problem A1
Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.

Solution
Let the sides be a, a+1, a+2, the angle oppose a be A, the angle opposite a+1 be B, and the angle opposite a+2 be C.
Using the cosine rule, we find cos A = (a+5)/(2a+4), cos B = (a+1)/2a, cos C = (a-3)/2a.
Finally, using $\cos 2x = 2 \cos^2 x - 1$, we find solutions a = 4 for C = 2A, a = 1 for B = 2A, and no solutions for C = 2B.
a = 1 is a degenerate solution (the triangle has the three vertices collinear). The other solution is 4, 5, 6.

Problem A2
Find all natural numbers n the product of whose decimal digits is $n^2 - 10n - 22$.

Solution
Suppose n has m > 1 digits. Let the first digit be d. Then the product of the digits is at most $d.9^{m-1} < d.10^{m-1} <= n$. But $(n^2 - 10n - 22) - n = n(n - 11) - 22 > 0$ for n >= 13. So there are no solutions for $n \geq 13$. But $n^2 - 10n - 22 < 0$ for $n \leq 11$, so the only possible solution is n = 12 and indeed that is a solution.

Problem A3
a, b, c are real with a non-zero. $x_1, x_2, \dots, x_n$ satisfy the n equations:

$$ax_i^2 + bx_i + c = x_{i+1}, \text{ for } 1 \leq i < n$$
$$ax_n^2 + bx_n + c = x_1$$

Prove that the system has zero, 1 or >1 real solutions according as $(b - 1)^2 - 4ac$ is <0, =0 or >0.

Solution
Let $f(x) = ax^2 + bx + c - x$. Then $f(x)/a = (x + (b-1)/2a)^2 + (4ac - (b-1)^2)/4a^2$. Hence if $4ac - (b-1)^2 > 0$, then f(x) has the same sign for all x. But f(x) > 0 means $ax^2 + bx + c > x$, so if $\{x_i\}$ is a solution, then either $x_1 < x_2 < \dots < x_n < x_1$, or $x_1 > x_2 > \dots > x_n > x_1$. Either way we have a contradiction. So if $4ac - (b-1)^2 > 0$ there cannot be any solutions.
If $4ac - (b-1)^2 = 0$, then we can argue in the same way that either $x_1 \leq x_2 \leq \dots \leq x_n \leq x_1$, or $x_1 \geq x_2 \geq \dots \geq x_n \geq x_1$. So we must have all x_i = the single root of f(x) = 0 (which clearly is a solution).
If $4ac - (b-1)^2 < 0$, then f(x) = 0 has two distinct real roots y and z and so we have at least two solutions to the equations: all $x_i = y$, and all $x_i = z$. We may, however, have additional solutions. For example, if a = 1, b = 0, c = -1 and n is even, then we have the additional solution $x_1 = x_3 = x_5 = \dots = 0$, $x_2 = x_4 = \dots = -1$.

Problem B1
Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.

Solution
The trick is to consider the longest side. That avoids getting into lots of different possible cases for which edge is longer than the sum of the other two.
So assume the result is false and let AB be the longest side. Then we have AB > AC + AD and BA > BC + BD. So 2AB > AC + AD + BC + BC. But by the triangle inequality, AB < AC + CB, AB < AD + DB, so 2AB < AC + CB + AD + DB. Contradiction.

Problem B2
Let f be a real-valued function defined for all real numbers, such that for some a > 0 we have

$$f(x + a) = 1/2 + \sqrt{(f(x) - f(x)^2)} \text{ for all } x.$$

Prove that f is periodic, and give an example of such a non-constant f for $a = 1$.

Solution

Directly from the equality given: $f(x+a) \geq 1/2$ for all x, and hence $f(x) \geq 1/2$ for all x.
So $f(x+2a) = 1/2 + \sqrt{f(x+a) - f(x+a)^2} = 1/2 + \sqrt{f(x+a)}\sqrt{1 - f(x+a)} = 1/2 + \sqrt{1/4 - f(x) + f(x)^2} = 1/2 + (f(x) - 1/2) = f(x)$. So f is periodic with period 2a.
We may take f(x) to be arbitrary in the interval [0,1). For example, let $f(x) = 1$ for $0 \leq x < 1$, $f(x) = 1/2$ for $1 \leq x < 2$. Then use $f(x+2) = f(x)$ to define f(x) for all other values of x.

Problem B3

For every natural number n evaluate the sum

$[(n+1)/2] + [(n+2)/4] + [(n+4)/8] + \dots + [(n+2^k)/2^{k+1}] + \dots$, where [x] denotes the greatest integer $\leq x$.

Solution

For any real x we have $[x] = [x/2] + [(x+1]/2]$. For if $x = 2n + 1 + k$, where n is an integer and $0 \leq k < 1$, then lhs = $2n + 1$, and rhs = $n + n + 1$. Similarly, if $x = 2n + k$.
Hence for any integer n, we have: $[n/2^k] - [n/2^{k+1}] = [(n/2^k + 1)/2] = [(n + 2^k)/2^{k+1}]$.
Hence summing over k, and using the fact that $n < 2^k$ for sufficiently large k, so that $[n/2^k] = 0$, we have: $n = [(n + 1)/2] + [(n + 2)/4] + [(n + 4)/8] + \dots$.

Internationale Mathematikolympiade

IMO 1969

Problem A1
Prove that there are infinitely many positive integers m, such that $n^4 + m$ is not prime for any positive integer n.

Solution
$n^4 + 4r^4 = (n^2 + 2rn + 2r^2)(n^2 - 2rn + 2r^2)$. Clearly the first factor is greater than 1, the second factor is $(n - r)^2 + r^2$, which is also greater than 1 for r greater than 1. So we may take $m = 4r^4$ for any r greater than 1.

Problem A2
Let $f(x) = \cos(a_1 + x) + 1/2 \cos(a_2 + x) + 1/4 \cos(a_3 + x) + \dots + 1/2^{n-1} \cos(a_n + x)$, where a_i are real constants and x is a real variable. If $f(x_1) = f(x_2) = 0$, prove that $x_1 - x_2$ is a multiple of π.

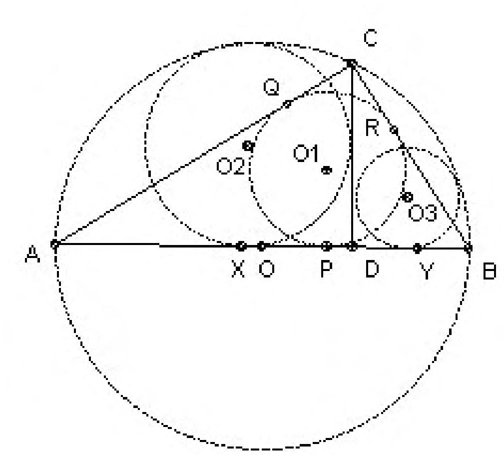
Solution
f is not identically zero, because $f(-a_1) = 1 + 1/2 \cos(a_2 - a_1) + \dots > 1 - 1/2 - 1/4 - \dots - 1/2^{n-1} > 0$.
Using the expression for $\cos(x + y)$ we obtain $f(x) = b \cos x + c \sin x$, where $b = \cos a_1 + 1/2 \cos a_2 + \dots + 1/2^{n-1} \cos a_n$, and $c = -\sin a_1 - 1/2 \sin a_2 - \dots - 1/2^{n-1} \sin a_n$. b and c are not both zero, since f is not identically zero, so $f(x) = \sqrt{(b^2 + c^2)} \cos(d + x)$, where $\cos d = b/\sqrt{(b^2 + c^2)}$, and $\sin d = c/\sqrt{(b^2 + c^2)}$. Hence the roots of $f(x) = 0$ are just $m\pi + \pi/2 - d$.

Problem A3
For each of k = 1, 2, 3, 4, 5 find necessary and sufficient conditions on a > 0 such that there exists a tetrahedron with k edges length a and the remainder length 1.

Solution
A plodding question. Take the tetrahedron to be ABCD.
Take k = 1 and AB to have length a, the other edges length 1. Then we can hinge triangles ACD and BCD about CD to vary AB. The extreme values evidently occur with A, B, C, D coplanar. The least value, 0, when A coincides with B, and the greatest value $\sqrt{3}$, when A and B are on opposite sides of CD. We rule out the extreme values on the grounds that the tetrahedron is degenerate, thus obtaining $0 < a < \sqrt{3}$.
For k = 5, the same argument shows that $0 < 1 < \sqrt{3}\, a$, and hence $a > 1/\sqrt{3}$.
For k = 2, there are two possible configurations: the sides length a adjacent, or not. Consider first the adjacent case. Take the sides length a to be AC and AD. As before, the two extreme cases gave A, B, C, D coplanar. If A and B are on opposite sides of CD, then $a = \sqrt{(2 - \sqrt{3})}$. If they are on the same side, then $a = \sqrt{(2 + \sqrt{3})}$. So this configuration allows any a satisfying $\sqrt{(2 - \sqrt{3})} < a < \sqrt{(2 + \sqrt{3})}$.
The other configuration has AB = CD = a. One extreme case has a = 0. We can increase a until we reach the other extreme case with ADBC a square side 1, giving $a = \sqrt{2}$. So this configuration allows any a satisfying $0 < a < \sqrt{2}$. Together, the two configurations allow any a satisfying: $0 < a < \sqrt{(2 + \sqrt{3})}$.
This also solves the case k = 4, and allows any a satisfying: $a > 1/\sqrt{(2 + \sqrt{3})} = \sqrt{(2 - \sqrt{3})}$.
For k = 3, any value of a > 0 is allowed. For a <= 1, we may take the edges length a to form a triangle. For $a \geq 1$ we may take a triangle with unit edges and the edges joining the vertices to the fourth vertex to have length a.

Problem B1
C is a point on the semicircle diameter AB, between A and B. D is the foot of the perpendicular from C to AB. The circle K_1 is the in-circle of ABC, the circle K_2 touches CD, DA and the semicircle, the circle K_3 touches CD, DB and the semicircle. Prove that K_1, K_2 and K_3 have another common tangent apart from AB.

Solution



Let the three centers be O_1, O_2 and O_3. We show that O_1 is the midpoint of O_2O_3. In fact it is sufficient to show that O_1 lies on O_2O_3, because then we can reflect the known tangent AB in the line O_2O_3.
As usual, let AB = c, BC = a, CA = b. Let the in-circle touch AB at P, AC at Q and BC at R. Then since angle ACB = 90, O_1QCR is a square. Also AQ = AP and BP = BR, so r_1 = b - AP, and r_1 = a - BP = a - (c - AP). Adding: $r_1 = (a + b - c)/2$, and AP = (b + c - a)/2.
Let the circle center O_2 touch AB at X, and the circle center O_3 touch AB at Y. Let O be the midpoint of AB. Now consider the right-angled triangle OXO_2. Since the circle center O_2 touches the semicircle, $OO_2 = c/2 - r_2$. OX = OD + DX = $(c/2 - AD) + r_2$. Also, by similar triangles, AD = b^2/c. So, using Pythagoras: $(c/2 - r_2)^2 = r_2^2 + (c/2 - b^2/c + r_2)^2$. Multiplying out and rearranging: $r_2^2 - 2r_2(c - b^2/c) - (b^2 - b^4/c^2)$. But ABC is right-angled, so $c^2 = a^2 + b^2$, and hence $c - b^2/c = a^2/c$ and $b^2 - b^4/c^2 = a^2b^2/c^2$. So $r_2^2 + 2r_2\, a^2/c - a^2b^2/c^2 = 0$, which has roots $r_2 = a - a^2/c$ (positive) and $- a + a^2/c$ (negative). So $r_2 = a - a^2/c$. Similarly, $r_3 = b - b^2/c$. So $O_2X + O_3Y = XY = r_2 + r_3 = a + b - c = 2\, r_1$.
XP = AP - AX = AP - (AD - DX) = $(b + c - a)/2 - (b^2/c - r_2) = (b + c - a)/2 - (c - a) = (a + b - c)/2 = r_1$. We now have all we need: XP = PY = PO_1, and $XO_2 + YO_3 = 2\, PO_1$.

Problem B2
Given n > 4 points in the plane, no three collinear. Prove that there are at least (n-3)(n-4)/2 convex quadrilaterals with vertices amongst the n points.

Solution
(n-3)(n-4)/2 is a poor lower bound.
Observe first that any 5 points include 4 forming a convex quadrilateral. For take the convex hull. If it consists of more than 3 points, we are done. If not, it must consist of 3 points, A, B and C, with the other 2 points, D and E, inside the triangle ABC. Two vertices of the triangle must lie on the same side of the line DE and they form convex quadrilateral with D and E.
Given n points, we can choose 5 in n(n-1)(n-2)(n-3)(n-4)/120 different ways. Each choice gives us a convex quadrilateral, but any given convex quadrilateral may arise from n-4 different sets of 5 points, so we have at least n(n-1)(n-2)(n-3)/120 different convex quadrilaterals. We now show that n(n-1)(n-2)(n-3)/120 ≥ (n-3)(n-4)/2 for all n ≥ 5.
We wish to prove that n(n-1)(n-2) ≥ 60(n-4), or n(n-1)(n-2) - 60(n-4) ≥ 0. Trial shows equality for n = 5 and 6, so we can factorise and get (n-5)(n-6)(n+8), which is clearly at least 0 for n at least 5.

Problem B3
Given real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, satisfying $x_1 > 0$, $x_2 > 0$, $x_1y_1 > z_1^2$, and $x_2y_2 > z_2^2$, prove that:

$$8/((x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2) \le 1/(x_1y_1 - z_1^2) + 1/(x_2y_2 - z_2^2).$$

Give necessary and sufficient conditions for equality.

Solution
Let $a_1 = x_1y_1 - z_1^2$ and $a_2 = x_2y_2 - z_2^2$. We apply the arithmetic/geometric mean result 3 times:
(1) to a_1^2, a_2^2, giving $2a_1a_2 \le a_1^2 + a_2^2$;
(2) to a_1, a_2, giving $\sqrt{a_1a_2} \le (a_1 + a_2)/2$;
(3) to a_1y_2/y_1, a_2y_1/y_2, giving $\sqrt{a_1a_2} \le (a_1y_2/y_1 + a_2y_1/y_2)/2$;
We also use $(z_1/y_1 - z_2/y_2)^2 \ge 0$. Now $x_1y_1 > z_1^2 \ge 0$, and $x_1 > 0$, so $y_1 > 0$. Similarly, $y_2 > 0$. So:
(4) $y_1y_2(z_1/y_1 - z_2/y_2)^2 \ge 0$, and hence $z_1^2y_2/y_1 + z_2^2y_1/y_2 \ge 2z_1z_2$.

Using (3) and (4) gives $2\sqrt{(a_1a_2)} \le (x_1y_2 + x_2y_1) - (z_1^2y_2/y_1 + z_2^2y_1/y_2) \le (x_1y_2 + x_2y_1 - 2z_1z_2)$.
Multiplying by (2) gives: $4a_1a_2 \le (a_1 + a_2)(x_1y_2 + x_2y_1 - 2z_1z_2)$.
Adding (1) and $2a_1a_2$ gives: $8a_1a_2 \le (a_1 + a_2)^2 + (a_1 + a_2)(x_1y_2 + x_2y_1 - 2z_1z_2) = a(a_1 + a_2)$, where $a = (x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2$. Dividing by a_1a_2a gives the required inequality.
Equality requires $a_1 = a_2$ from (1), $y_1 = y_2$ from (2), $z_1 = z_2$ from (3), and hence $x_1 = x_2$. Conversely, it is easy to see that these conditions are sufficient for equality.

IMO 1970

Problem A1
M is any point on the side AB of the triangle ABC. r, r_1, r_2 are the radii of the circles inscribed in ABC, AMC, BMC. q is the radius of the circle on the opposite side of AB to C, touching the three sides of AB and the extensions of CA and CB. Similarly, q_1 and q_2. Prove that $r_1r_2q = rq_1q_2$.

Solution
We need an expression for r/q. There are two expressions, one in terms of angles and the other in terms of sides. The latter is a poor choice, because it is both harder to derive and less useful. So we derive the angle expression.
Let I be the center of the in-circle for ABC and X the center of the external circle for ABC. I is the intersection of the two angle bisectors from A and B, so $c = r (\cot A/2 + \cot B/2)$. The X lies on the bisector of the external angle, so angle XAB is $90^o - A/2$. Similarly, angle XBA is $90^o - B/2$, so $c = q (\tan A/2 + \tan B/2)$. Hence $r/q = (\tan A/2 + \tan B/2)/(\cot A/2 + \cot B/2) = \tan A/2 \tan B/2$.
Applying this to the other two triangles, we get $r_1/q_1 = \tan A/2 \tan CMA/2$, $r_2/q_2 = \tan B/2 \tan CMB/2$. But $CMB/2 = 90^o - CMA/2$, so $\tan CMB/2 = 1/\tan CMA/2$. Hence result.

Problem A2
We have $0 \le x_i < b$ for $i = 0, 1, \dots , n$ and $x_n > 0$, $x_{n-1} > 0$. If $a > b$, and $x_nx_{n-1}\dots x_0$ represents the number A base a and B base b, whilst $x_{n-1}x_{n-2}\dots x_0$ represents the number A' base a and B' base b, prove that $A'B < AB'$.

Solution
We have $a^nb^m > b^na^m$ for $n > m$. Hence $a^nB' > b^nA'$. Adding a^nb^n to both sides gives $a^nB > b^nA$. Hence $x_na^nB > x_nb^nA$. But $x_na^n = A - A'$ and $x_nb^n = B - B'$, so $(A - A')B > (B - B')A$. Hence result.
Note that the only purpose of requiring $x_{n-1} > 0$ is to prevent A' and B' being zero.

Problem A3
The real numbers $a_0, a_1, a_2, \dots$ satisfy $1 = a_0 <= a_1 \le a_2 <= \dots$. $b_1, b_2, b_3, \dots$ are defined by b_n = sum for k = 1 to n of $(1 - a_{k-1}/a_k)/\sqrt{a_k}$.
(a) Prove that $0 \le b_n < 2$.
(b) Given c satisfying $0 \le c < 2$, prove that we can find a_n so that $b_n > c$ for all sufficiently large n.

Solution
(a) Each term of the sum is non-negative, so b_n is non-negative. Let $c_k = \sqrt{a_k}$. Then the kth term $= (1 - a_{k-1}/a_k)/\sqrt{a_k} = c_{k-1}^2/c_k (1/a_{k-1} - 1/a_k) = c_{k-1}^2/c_k (1/c_{k-1} + 1/c_k)(1/c_{k-1} - 1/c_k)$. But $c_{k-1}^2/c_k (1/c_{k-1} + 1/c_k) \le 2$, so the kth term $\le 2(1/c_{k-1} - 1/c_k)$. Hence $b_n <= 2 - 2/c_n < 2$.
(b) Let $c_k = d^k$, where d is a constant > 1, which we will choose later. Then the kth term is $(1 - 1/d^2)1/d^k$, so $b_n = (1 - 1/d^2)(1 - 1/d^{n+1})/(1 - 1/d) = (1 + 1/d)(1 - 1/d^{n+1})$. Now take d sufficiently close to 1 that $1 + 1/d > c$, and then take n sufficiently large so that $(1 + 1/d)(1 - 1/d^{n+1}) > c$.

Problem B1
Find all positive integers n such that the set {n, n+1, n+2, n+3, n+4, n+5} can be partitioned into two subsets so that the product of the numbers in each subset is equal.

Solution
The only primes dividing numbers in the set can be 2, 3 or 5, because if any larger prime was a factor, then it would only divide one number in the set and hence only one product. Three of the numbers must be odd. At most one of the odd numbers can be a multiple of 3 and at most one can be a multiple of 5. The other odd number cannot have any prime factors. The only such number is 1, so the set must be {1, 2, 3, 4, 5, 6}, but that does not work because only one of the numbers is a multiple of 5. So there are no such sets.

Problem B2
In the tetrahedron ABCD, angle BDC = 90^o and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:
$(AB + BC + CA)^2 \le 6(AD^2 + BD^2 + CD^2)$.
When do we have equality?

Solution
The first step is to show that angles ADB and ADC are also 90^o. Let H be the intersection of the altitudes of ABC and let CH meet AB at X. Planes CED and ABC are perpendicular and AB is perpendicular to the line of intersection CE. Hence AB is perpendicular to the plane CDE and hence to ED. So $BD^2 = DE^2 + BE^2$. Also $CB^2 = CE^2 + BE^2$. Subtracting: $CB^2 - BD^2 = CE^2 - DE^2$. But $CB^2 - BD^2 = CD^2$, so $CE^2 = CD^2 + DE^2$, so angle CDE = 90^o. But angle CDB = 90^o, so CD is perpendicular to the plane DAB, and hence angle CDA = 90^o. Similarly, angle ADB = 90^o.
Hence $AB^2 + BC^2 + CA^2 = 2(DA^2 + DB^2 + DC^2)$. But now we are done, because Cauchy's inequality gives $(AB + BC + CA)^2 \le 3(AB^2 + BC^2 + CA^2)$.
We have equality iff we have equality in Cauchy's inequality, which means AB = BC = CA.

Problem B3
Given 100 coplanar points, no 3 collinear, prove that at most 70% of the triangles formed by the points have all angles acute.

Solution
At most 3 of the triangles formed by 4 points can be acute. It follows that at most 7 out of the 10 triangles formed by any 5 points can be acute. For given 10 points, the maximum no. of acute triangles is: the no. of subsets of 4 points x 3/the no. of subsets of 4 points containing 3 given points. The total no. of triangles is the same expression with the first 3 replaced by 4. Hence at most 3/4 of the 10, or 7.5, can be acute, and hence at most 7 can be acute.
The same argument now extends the result to 100 points. The maximum number of acute triangles formed by 100 points is: the no. of subsets of 5 points x 7/the no. of subsets of 5 points containing 3 given points. The total no. of triangles is the same expression with 7 replaced by 10. Hence at most 7/10 of the triangles are acute.

IMO 1971

Problem A1
Let $E_n = (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n) + \dots + (a_n - a_1)(a_n - a_2) \dots (a_n - a_{n-1})$. Let S_n be the proposition that $E_n \ge 0$ for all real a_i.
Prove that S_n is true for n = 3 and 5, but for no other n > 2.

Solution
Take $a_1 < 0$, and the remaining $a_i = 0$. Then $E_n = a_1^{n-1} < 0$ for n even, so the proposition is false for even n.
Suppose n ≥ 7 and odd. Take any c > a > b, and let $a_1 = a$, $a_2 = a_3 = a_4 = b$, and $a_5 = a_6 = \dots = a_n = c$. Then $E_n = (a - b)^3(a - c)^{n-4} < 0$. So the proposition is false for odd n ≥ 7.
Assume $a_1 \ge a_2 \ge a_3$. Then in E_3 the sum of the first two terms is non-negative, because $(a_1 - a_3) \ge (a_2 - a_3)$. The last term is also non-negative. Hence $E_3 \ge 0$, and the proposition is true for n = 3.
It remains to prove S_5. Suppose $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5$. Then the sum of the first two terms in E_5 is $(a_1 - a_2)\{(a_1 - a_3)(a_1 - a_4)(a_1 - a_5) - (a_2 - a_3)(a_2 - a_4)(a_2 - a_5)\} \ge 0$. The third term is

non-negative (the first two factors are non-positive and the last two non-negative). The sum of the last two terms is: $(a_4 - a_5)\{(a_1 - a_5)(a_2 - a_5)(a_3 - a_5) - (a_1 - a_4)(a_2 - a_4)(a_3 - a_4)\} \ge 0$. Hence $E_5 \ge 0$.

Problem A2
Let P_1 be a convex polyhedron with vertices $A_1, A_2, \dots , A_9$. Let P_i be the polyhedron obtained from P_1 by a translation that moves A_1 to A_i. Prove that at least two of the polyhedra $P_1, P_2, \dots , P_9$ have an interior point in common.

Solution
The result is false for 8 vertices - for example, the cube. We get 8 cubes, with only faces in common, forming a cube 8 times as large.
This suggests a trick. Each P_i is contained in D, the polyhedron formed from P_1 by doubling the scale. Take A_1 as the origin and take the vertex B_i to have twice the coordinates of A_i. Given a point X inside P_1, the midpoint of P_iX must lie in P_1 by convexity. Hence the point with doubled coordinates, which is obtained by adding the coordinates of A_i to the coordinates of X, lies in D. In other words every point of P_i lies in D. But the volume of D is 8 times the volume of P_1, which is less than the sum of the volumes of $P_1, \dots , P_9$.

Problem A3
Prove that we can find an infinite set of positive integers of the form $2^n - 3$ (where n is a positive integer) every pair of which are relatively prime.

Solution
We show how to enlarge a set of r such integers to a set of r+1. So suppose $2^{n_1} - 3, \dots , 2^{n_r} - 3$ are all relatively prime. The idea is to find $2^n - 1$ divisible by $m = (2^{n_1} - 3) \dots (2^{n_r} - 3)$, because then $2^n - 3$ must be relatively prime to all of the factors of m. At least two of $2^0, 2^1, \dots , 2^m$ must be congruent mod m. So suppose $m_1 > m_2$ and $2^{m_1} = 2^{m_2} \pmod m$, then we must have $2^{m_1 - m_2} - 1 = 0 \pmod m$, since m is odd. So we may take n_{r+1} to be $m_1 - m_2$.

Problem B1
All faces of the tetrahedron ABCD are acute-angled. Take a point X in the interior of the segment AB, and similarly Y in BC, Z in CD and T in AD.
(a) If $\angle DAB + \angle BCD \ne \angle CDA + \angle ABC$, then prove that none of the closed paths XYZTX has minimal length;
(b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest paths XYZTX, each with length 2 AC sin k, where $2k = \angle BAC + \angle CAD + \angle DAB$.

Solution
The key is to pretend the tetrahedron is made of cardboard, cut it along three edges and unfold it. Suppose we do this to get the hexagon CAC'BDB'. Now the path is a line joining Y on B'C to Y' on the opposite side BC' of the hexagon. Clearly this line must be straight for a minimal path. If B'C and BC' are parallel, then we can take Y anywhere on the side and the minimal path length is the expression given.
But if they are not parallel, then the minimal path will come from an extreme position. Suppose CC' < BB'. If the interior angle CAC' is less than 180^o, then the minimal path is obtained by taking Y at C. But this does not meet the requirement that Y be an interior point of the edge, so there is no minimal path in the permitted set. If the interior angle CAC' is greater than 180, then the minimal path is obtained by taking X and T at A. Again this is not permitted.
The problem therefore reduces to finding the condition for B'C and BC' to be parallel. This is evidently angles BCD + DCA + CAD + BAD + BAC + ACB = 360^o. But DCA + CAD = 180^o - ADC, and BAC + ACB = 180^o - ABC, so we obtain the condition given.

Problem B2
Prove that for every positive integer m we can find a finite set S of points in the plane, such that given any point A of S, there are exactly m points in S at unit distance from A.

Solution

Take $a_1, a_2, \dots, a_m$ to be points a distance 1/2 from the origin O. Form the set of 2^m points $\pm a_1 \pm a_2 \pm \dots \pm a_m$. Given such a point, it is at unit distance from the m points with just one coefficient different. So we are home, provided that we can choose the a_i to avoid any other pairs of points being at unit distance, and to avoid any degeneracy (where some of the 2^m points coincide).
The distance between two points in the set is $|c_1a_1 + c_2a_2 + \dots + c_ma_m|$, where $c_i = 0$, 2 or -2. So let us choose the a_i inductively. Suppose we have already chosen up to m. The constraints on a_{m+1} are that we do not have $|c_1a_1 + c_2a_2 + \dots + c_ma_m + 2a_{m+1}|$ equal to 0 or 1 for any $c_i = 0$, 2 or -2, apart from the trivial cases of all $c_i = 0$. Each | | = 0 rules out a single point and each | | = 1 rules out a circle which intersects the circle radius 1/2 about the origin at 2 points and hence rules out two points. So the effect of the constraints is to rule out a finite number of points, whereas we have uncountably many to choose from.

Problem B3

Let $A = (a_{ij})$, where i, j = 1, 2, ... , n, be a square matrix with all a_{ij} non-negative integers. For each i, j such that $a_{ij} = 0$, the sum of the elements in the ith row and the jth column is at least n. Prove that the sum of all the elements in the matrix is at least $n^2/2$.

Solution

Let x be the smallest row or column sum. If x >= n/2, then we are done, so assume x < n/2. Suppose it is a row. (If not, interchange rows and columns.) The number of non-zero elements in the row, y, must also satisfy y < n/2, since each non-zero element is at least 1. Now move across this row summing the columns. The y columns with a non-zero element have sum at least x (by the definition of x). The n - y columns with a zero have sum at least n - x. Hence the total sum is at least $xy + (n - x)(n - y) = n^2/2 + (n - 2x)(n - 2y)/2 > n^2/2$.
The result is evidently best possible, because we can fill the matrix alternately with zeros and ones (so that $a_{ij} = 1$ if i and j are both odd or both even, 0 otherwise). For n even, every row and column has n/2 1s, so the condition is certainly satisfied and the total sum is $n^2/2$. For n odd, odd numbered rows have (n+1)/2 1s and even numbered one less. But the only zeros are in positions which have either the row or the column odd-numbered, so the sum in such cases is n as required. The total sum is $n^2/2 + 1/2$. Alternatively, for n even, we could place n/2 down the main diagonal.

IMO 1972

Problem A1

Given any set of ten distinct numbers in the range 10, 11, ... , 99, prove that we can always find two disjoint subsets with the same sum.

Solution

The number of non-empty subsets is $2^{10} - 1 = 1023$. The sum of each subset is at most 90 + ... + 99 = 945, so there must be two distinct subsets A and B with the same sum. A \ B and B \ A are disjoint subsets, also with the same sum.

Problem A2

Given n > 4, prove that every cyclic quadrilateral can be dissected into n cyclic quadrilaterals.

Solution

A little tinkering soon shows that it is easy to divide a cyclic quadrilateral ABCD into 4 cyclic quadrilaterals. Take a point P inside the quadrilateral and take an arbitrary line PK joining it to AB. Now take L on BC so that $\angle KPL = 180^o - \angle B$ (thus ensuring that KPLB is cyclic), then M on CD so that $\angle LPM = 180^o - \angle C$, then N on AD so that $\angle MPN = 180^o - \angle D$. Then $\angle NPK = 180^o - \angle A$. We may need to impose some restrictions on P and K to ensure that we can obtain the necessary angles. It is not clear, however, what to do next.
The trick is to notice that the problem is easy if two sides are parallel. For then we may take arbitrarily many lines parallel to the parallel sides and divide the original quadrilateral into any number of parts.

So we need to arrange our choice of P and K so that one of the new quadrilaterals has parallel sides. But that is easy, since K is arbitrary. So take PK parallel to AD, then we must also have PL parallel to CD.
It remains to consider how we ensure that the points lie on the correct sides. Consider first K and L. K cannot lie on AD since PK is parallel to AD, and we can avoid it lying on BC by taking P sufficiently close to D. Similarly, taking P sufficiently close to D ensures that L lies on BC. Now suppose that M and N are both on AD. Then if we keep K fixed and move P closer to CD, N will move on to CD, leaving M on AD.

Problem A3
Prove that (2m)!(2n)! is a multiple of m!n!(m+n)! for any non-negative integers m and n.

Solution
The trick is to find a recurrence relation for f(m,n) = (2m)!(2n)!/(m!n!(m+n)!). In fact, f(m,n) = 4 f(m,n-1) - f(m+1,n-1), which is sufficient to generate all the f(m,n), given that f(m,0) = (2m)!/(m!m!), which we know to be integeral.

Problem B1
Find all positive real solutions to:

$$(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) \le 0$$
$$(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) \le 0$$
$$(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) \le 0$$
$$(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) \le 0$$
$$(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) \le 0$$

Solution
Answer: $x_1 = x_2 = x_3 = x_4 = x_5$.
The difficulty with this problem is that it has more information than we need. There is a neat solution in Greitzer which shows that all we need is the sum of the 5 inequalities, because one can rewrite that as $(x_1x_2 - x_1x_4)^2 + (x_2x_3 - x_2x_5)^2 + \dots + (x_5x_1 - x_5x_3)^2 + (x_1x_3 - x_1x_5)^2 + \dots + (x_5x_2 - x_5x_4)^2 \le 0$. The difficulty is how one ever dreams up such an idea!
The more plodding solution is to break the symmetry by taking x_1 as the largest. If the second largest is x_2, then the first inequality tells us that x_1^2 or $x_2^2 = x_3x_5$. But if x_3 and x_5 are unequal, then the larger would exceed x_1 or x_2. Contradiction. Hence $x_3 = x_5$ and also equals x_2 or x_1. If they equal x_1, then they would also equal x_2 (by definition of x_2), so in any case they must equal x_2. Now the second inequality gives $x_2 = x_1x_4$. So either all the numbers are equal, or $x_1 > x_2 = x_3 = x_5 > x_4$. But in the second case the last inequality is violated. So the only solution is all numbers equal.
If the second largest is x_5, then we can use the last inequality to deduce that $x_2 = x_4 = x_5$ and proceed as before.
If the second largest is x_3, then the fourth inequality gives that $x_1 = x_3 = x_5$ or $x_1 = x_3 = x_4$. In the first case, x_5 is the second largest and we are home already. In the second case, the third inequality gives $x_3^2 = x_2x_5$ and hence $x_3 = x_2 = x_5$ (or one of x_2, x_5 would be larger than the second largest). So x_5 is the second largest and we are home.
Finally, if the second largest is x_4, then the second inequality gives $x_1 = x_2 = x_4$ or $x_1 = x_3 = x_4$. Either way, we have a case already covered and so the numbers are all equal.

Problem B2
f and g are real-valued functions defined on the real line. For all x and y, $f(x + y) + f(x - y) = 2f(x)g(y)$. f is not identically zero and $|f(x)| \le 1$ for all x. Prove that $|g(x)| \le 1$ for all x.

Solution
Let k be the least upper bound for $|f(x)|$. Suppose $|g(y)| > 1$. Take any x with $|f(x)| > 0$, then $2k \ge |f(x+y)| + |f(x-y)| \ge |f(x+y) + f(x-y)| = 2|g(y)||f(x)|$, so $|f(x)| < k/|g(y)|$. In other words, $k/|g(y)|$ is an upper bound for $|f(x)|$ which is less than k. Contradiction.

Problem B3
Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

Solution

Intuitively, we can place A and B on the two outer planes with AB perpendicular to the planes. Then tilt AB in one direction until we bring C onto one of the middle planes (keeping A and B on the outer planes), then tilt AB the other way (keeping A, B, C on their respective planes) until D gets onto the last plane.

Take A as the origin. Let the vectors AB, AC, AD be **b**, **c**, **d**. Take p as one of the outer planes. Let the distances to the other planes be e, f, g. Now we find a vector **n** satisfying: **n.b** = e, **n.c** = f, **n.d** = g. This is a system of three equations in three unknowns with non-zero determinant (because **b.c x d** is non-zero), so it has a solution **n**. Scale the tetrahedron by |**n**|, orient p perpendicular to **n**/|**n**|, then B, C, D will be on the other planes as required.

IMO 1973

Problem A1

$OP_1, OP_2, \dots , OP_{2n+1}$ are unit vectors in a plane. $P_1, P_2, \dots , P_{2n+1}$ all lie on the same side of a line through O. Prove that $|OP_1 + \dots + OP_{2n+1}| \ge 1$.

Solution

We proceed by induction on n. It is clearly true for n = 1. Assume it is true for 2n-1. Given OP_i for 2n+1, reorder them so that all OP_i lie between OP_{2n} and OP_{2n+1}. Then $u = OP_{2n} + OP_{2n+1}$ lies along the angle bisector of angle $P_{2n}OP_{2n+1}$ and hence makes an angle less than 90^o with $v = OP_1 + OP_2 + \dots + OP_{2n-1}$ (which must lie between OP_1 and OP_{2n-1} and hence between OP_{2n} and OP_{2n+1}. By induction $|v| \ge 1$. But $|u + v| \ge |v|$ (use the cosine formula). Hence the result is true for 2n+1.

It is clearly best possible: take $OP_1 = \dots = OP_n = -OP_{n+1} = \dots = -OP_{2n}$, and OP_{2n+1} in an arbitrary direction.

Problem A2

Can we find a finite set of non-coplanar points, such that given any two points, A and B, there are two others, C and D, with the lines AB and CD parallel and distinct?

Solution

To warm up, we may notice that a regular hexagon is a planar set satisfying the condition. Take two regular hexagons with a common long diagonal and their planes perpendicular. Now if we take A, B in the same hexagon, then we can find C, D in the same hexagon. If we take A in one and B in the other, then we may take C at the opposite end of a long diagonal from A, and D at the opposite end of a long diagonal from B.

Problem A3

a and b are real numbers for which the equation $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has at least one real solution. Find the least possible value of $a^2 + b^2$.

Solution

Put $y = x + 1/x$ and the equation becomes $y^2 + ay + b - 2 = 0$, which has solutions $y = -a/2 \pm\sqrt{(a^2 + 8 - 2b)}/2$. We require $|y| \ge 2$ for the original equation to have a real root and hence we need $|a| + \sqrt{(a^2 + 8 - 4b)} \ge 4$. Squaring gives $2|a| - b \ge 2$. Hence $a^2 + b^2 \ge a^2 + (2 - 2|a|)^2 = 5a^2 - 8|a| + 4 = 5(|a| - 4/5)^2 + 4/5$. So the least possible value of $a^2 + b^2$ is 4/5, achieved when a = 4/5, b = -2/5. In this case, the original equation is $x^4 + 4/5\ x^3 - 2/5\ x^2 + 4/5\ x + 1 = (x + 1)^2(x^2 - 6/5\ x + 1)$.

Problem B1

A soldier needs to sweep a region with the shape of an equilateral triangle for mines. The detector has an effective radius equal to half the altitude of the triangle. He starts at a vertex of the triangle. What path should he follow in order to travel the least distance and still sweep the whole region?

Solution

In particular he must sweep the other two vertices. Let us take the triangle to be ABC, with side 1 and assume the soldier starts at A. So the path must intersect the circles radius $\sqrt{3}/4$ centered on the other two vertices. Let us look for the shortest path of this type. Suppose it intersects the circle center B at X and the circle center C at Y, and goes first to X and then to Y. Clearly the path from A to X must be a straight line and the path from X to Y must be a straight line. Moreover the shortest path from X to the circle center C follows the line XC and has length AX + XC - $\sqrt{3}/4$. So we are looking for the point X which minimises AX + XC.
Consider the point P where the altitude intersects the circle. By the usual reflection argument the distance AP + PC is shorter than the distance AP' + P'C for any other point P' on the line perpendicular to the altitude through P. Moreover for any point X on the circle, take AX to cut the line at P'. Then AX + XC > AP' + P'C > AP + PC.
It remains to check that the three circles center A, X, Y cover the triangle. In fact the circle center X covers the whole triangle except for a small portion near A and a small portion near C, which are covered by the triangles center A and Y.

Problem B2
G is a set of non-constant functions f. Each f is defined on the real line and has the form $f(x) = ax + b$ for some real a, b. If f and g are in G, then so is fg, where fg is defined by $fg(x) = f(g(x))$. If f is in G, then so is the inverse f^{-1}. If $f(x) = ax + b$, then $f^{-1}(x) = x/a - b/a$. Every f in G has a fixed point (in other words we can find x_f such that $f(x_f) = x_f$. Prove that all the functions in G have a common fixed point.

Solution
$f(x) = ax + b$ has fixed point $b/(1-a)$. If $a = 1$, then b must be 0, and any point is a fixed point. So suppose $f(x) = ax + b$ and $g(x) = ax + b'$ are in G. Then h the inverse of f is given by $h(x) = x/a - b/a$, and $hg(x) = x + b'/a - b/a$. This is in G, so we must have $b' = b$. Suppose $f(x) = ax + b$, and $g(x) = cx + d$ are in G. Then $fg(x) = acx + (ad + b)$, and $gf(x) = acx + (bc + d)$. We must have $ad + b = bc + d$ and hence $b/(1-a) = c/(1-d)$, in other words f and g have the same fixed point.

Problem B3
$a_1, a_2, \dots , a_n$ are positive reals, and q satisfies $0 < q < 1$. Find $b_1, b_2, \dots , b_n$ such that:
(a) $a_i < b_i$ for $i = 1, 2, \dots , n$,
(b) $q < b_{i+1}/b_i < 1/q$ for $i = 1, 2, \dots , n-1$,
(c) $b_1 + b_2 + \dots + b_n < (a_1 + a_2 + \dots + a_n)(1 + q)/(1 - q)$.

Solution
We notice that the constraints are linear, in the sense that if b_i is a solution for a_i, q, and b_i' is a solution for a_i', q, then for any k, k' > 0 a solution for $ka_i + k'a_i'$, q is $kb_i + k'b_i'$. Also a "near" solution for $a_h = 1$, other $a_i = 0$ is $b_1 = q^{h-1}$, $b_2 = q^{h-2}$, ... , $b_{h-1} = q$, $b_h = 1$, $b_{h+1} = q$, ... , $b_n = q^{n-h}$. "Near" because the inequalities in (a) and (b) are not strict.
However, we might reasonably hope that the inequalities would become strict in the linear combination, and indeed that is true. Define $b_r = q^{r-1}a_1 + q^{r-2}a_2 + \dots + qa_{r-1} + a_r + qa_{r+1} + \dots + q^{n-r}a_n$. Then we may easily verify that (a) - (c) hold.

IMO 1974

Problem A1
Three players play the following game. There are three cards each with a different positive integer. In each round the cards are randomly dealt to the players and each receives the number of counters on his card. After two or more rounds, one player has received 20, another 10 and the third 9 counters. In the last round the player with 10 received the largest number of counters. Who received the middle number on the first round?

Solution
The player with 9 counters.
The total of the scores, 39, must equal the number of rounds times the total of the cards. But 39 has no factors except 1, 3, 13 and 39, the total of the cards must be at least 1 + 2

+ 3 = 6, and the number of rounds is at least 2. Hence there were 3 rounds and the cards total 13.
The highest score was 20, so the highest card is at least 7. The score of 10 included at least one highest card, so the highest card is at most 8. The lowest card is at most 2, because if it was higher then the highest card would be at most 13 - 3 - 4 = 6, whereas we know it is at least 7. Thus the possibilities for the cards are: 2, 3, 8; 2, 4, 7; 1, 4, 8; 1, 5, 7. But the only one of these that allows a score of 20 is 1, 4, 8. Thus the scores were made up: 8 + 8 + 4 = 20, 8 + 1 + 1 = 10, 4 + 4 + 1 = 9. The last round must have been 4 to the player with 20, 8 to the player with 10 and 1 to the player with 9. Hence on each of the other two rounds the cards must have been 8 to the player with 20, 1 to the player with 10 and 4 to the player with 9.

Problem A2
Prove that there is a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB if and only if sin A sin B ≤ $\sin^2(C/2)$

Solution
Extend CD to meet the circumcircle of ABC at E. Then CD·DE = AD·DB, so CD is the geometric mean of AD and DB iff CD = DE. So we can find such a point iff the distance of C from AB is less than the distance of AB from the furthest point of the arc AB on the opposite side of AB to C. The furthest point F is evidently the midpoint of the arc AB. F lies on the angle bisector of C. So ∠FAB = ∠FAC = ∠C/2. Hence distance of F from AB is c/2 tan C/2 (as usual we set c = AB, b = CA, a = BC). The distance of C from AB is a sin B. So a necessary and sufficient condition is c/2 tan C/2 ≥ a sin B. But by the sine rule, a = c sin A/sin C, so the condition becomes (sin C/2 sin C)/(2 cos C/2) ≥ sin A sin B. But sin C = 2 sin C/2 cos C/2, so we obtain the condition quoted in the question.

Problem A3
Prove that the sum from k = 0 to n of (2n+1)C(2k+1) 2^{3k} is not divisible by 5 for any non-negative integer n. [rCs denotes the binomial coefficient r!/(s!(r-s)!) .]

Solution
Let k = √8. Then $(1 + k)^{2n+1}$ = a + bk, where b is the sum given in the question. Similarly, $(1 - k)^{2n+1}$ = a - bk. This looks like a dead end, because eliminating a gives an unhelpful expression for b. The trick is to multiply the two expressions to get $7^{2n+1} = 8b^2 - a^2$. This still looks unhelpful, but happens to work, because we soon find that $7^{2n+1} \neq \pm 2 \pmod 5$. So if b was a multiple of 5 then we would have a square congruent to ±2 (mod 5) which is impossible.

Problem B1
An 8 x 8 chessboard is divided into p disjoint rectangles (along the lines between squares), so that each rectangle has the same number of white squares as black squares, and each rectangle has a different number of squares. Find the maximum possible value of p and all possible sets of rectangle sizes.

Solution
The requirement that the number of black and white squares be equal is equivalent to requiring that the each rectangle has an even number of squares. 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 72 > 64, so p < 8. There are 5 possible divisions of 64 into 7 unequal even numbers: 2 + 4 + 6 + 8 + 10 + 12 + 22; 2 + 4 + 6 + 8 + 10 + 16 + 18; 2 + 4 + 6 + 8 + 12 + 14 + 18; 2 + 4 + 6 + 10 + 12 + 14 + 16. The first is ruled out because a rectangle with 22 squares would have more than 8 squares on its longest side. The others are all possible.

2 2 2 2 2 2 2 4   2 2 2 2 2 2 2 2

2 2 2 2 2 2 2 4   2 2 2 2 2 2 2 2

1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5

1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5

1 1 1 1 1 5 5 4   1 1 1 1 1 1 5 5

1 1 1 1 1 6 6 4   3 3 3 3 3 7 6 6

3 3 3 3 3 6 6 4   3 3 3 3 3 7 6 6

3 3 3 3 3 7 7 4   4 4 4 4 4 4 4 4

2 2 2 2 2 2 2 7   1 1 1 1 1 1 1 1

2 2 2 2 2 2 2 7   1 1 1 1 1 1 1 1

1 1 1 1 1 1 4 4   2 2 2 2 2 2 2 7

1 1 1 1 1 1 4 4   2 2 2 2 2 2 2 7

1 1 1 1 1 1 4 4   3 3 3 3 3 3 6 6

3 3 3 3 3 3 4 4   3 3 3 3 3 3 6 6

3 3 3 3 3 3 6 6   4 4 4 4 4 5 5 5

5 5 5 5 5 5 6 6   4 4 4 4 4 5 5 5

Problem B2
Determine all possible values of a/(a+b+d) + b/(a+b+c) + c/(b+c+d) + d/(a+c+d) for positive reals a, b, c, d.

Solution
We show first that the sum must lie between 1 and 2. If we replace each denominator by a+b+c+d then we reduce each term and get 1. Hence the sum is more than 1. Suppose a is the largest of the four reals. Then the first term is less than 1. The second and fourth terms have denominators greater than b+c+d, so the terms are increased if we replace the denominators by b+c+d. But then the last three terms sum to 1. Thus the sum of the last three terms is less than 1. Hence the sum is less than 2.
If we set a = c = 1 and make b and d small, then the first and third terms can be made arbitarily close to 1 and the other two terms arbitarily close to 0, so we can make the sum arbitarily close to 2. If we set a = 1, c = d and make b and c/b arbitarily small, then the first term is arbitarily close to 1 and the last three terms are all arbitarily small, so we can make the sum arbitarily close to 1. Hence, by continuity, we can achieve any value in the open interval (1,2).

Problem B3
Let P(x) be a polynomial with integer coefficients of degree d > 0. Let n be the number of distinct integer roots to P(x) = 1 or -1. Prove that n ≤ d + 2.

Solution
Suppose that A(x) and B(x) are two polynomials with integer coefficients which are identical except for their constant terms, which differ by 2. Suppose A(r) = 0, and B(s) =0 with r and s integers. Then subtracting we get 2 plus a sum of terms $a(r^i - s^i)$. Each of these terms is divisible by (r - s), so 2 must be divisible by (r - s). Hence r and s differ by 0, 1 or 2.
Now let r be the smallest root of P(x) = 1 and P(x) = -1. The polynomial with r as a root can have at most d distinct roots and hence at most d distinct integer roots. If s is a root of

the other equation then s must differ from r by 0, 1, or 2. But $s \geq r$, so $s = r$, $r+1$ or $r+2$. Hence the other equation adds at most 2 distinct integer roots.

IMO 1975

Problem A1
Let $x_1 \geq x_2 \geq \ldots \geq x_n$, and $y_1 \geq y_2 \geq \ldots \geq y_n$ be real numbers. Prove that if z_i is any permutation of the y_i, then:

$$\sum_1^n (x_i - y_i)^2 \leq \sum_1^n (x_i - z_i)^2.$$

Solution
If $x \geq x'$ and $y \geq y'$, then $(x - y)^2 + (x' - y')^2 \leq (x - y')^2 + (x' - y)^2$. Hence if $i < j$, but $z_i \leq z_j$, then swapping z_i and z_j reduces the sum of the squares. But we can return the order of the z_i to y_i by a sequence of swaps of this type: first swap 1 to the 1st place, then 2 to the 2nd place and so on.

Problem A2
Let $a_1 < a_2 < a_3 < \ldots$ be positive integers. Prove that for every $i >= 1$, there are infinitely many a_n that can be written in the form $a_n = ra_i + sa_j$, with r, s positive integers and $j > i$.

Solution
We must be able to find a set S of infinitely many a_n in some residue class mod a_i. Take a_j to be a member of S. Then for any a_n in S satisfying $a_n > a_j$, we have $a_n = a_j$ + a multiple of a_i.

Problem A3
Given any triangle ABC, construct external triangles ABR, BCP, CAQ on the sides, so that $\angle PBC = 45^o$, $\angle PCB = 30^o$, $\angle QAC = 45^o$, $\angle QCA = 30^o$, $\angle RAB = 15^o$, $\angle RBA = 15^o$. Prove that $\angle QRP = 90^o$ and QR = RP.

Solution
Trigonometry provides a routine solution. Let BC = a, CA = b, AB = c. Then, by the sine rule applied to AQC, $AQ = b/(2 \sin 105^o) = b/(2 \cos 15^o)$. Similarly, $PB = a/(2 \cos 15)$. Also $AR = RB = c/(2 \cos 15^o)$. So by the cosine rule $RP^2 = (a^2 + c^2 - 2ac \cos(B+60^o))/(4 \cos^2 15^o)$, and $RQ^2 = (b^2 + c^2 - 2bc \cos(A+60^o))/(4 \cos^2 15^o)$. So RP = RQ is equivalent to: $a^2 - 2ac \cos(60^o+B) = b^2 - 2bc \cos(60^o+A)$ and hence to $a^2 - ac \cos B + \sqrt{3}\, ac \sin B = b^2 - bc \cos A + \sqrt{3}\, bc \sin A$. By the sine rule, the sine terms cancel. Also $b - b \cos A = a \cos C$, and $a - c \cos B = b \cos C$, so the last equality is true and hence RP = RQ. We get an exactly similar expression for PQ^2 and show that it equals $2\, RP^2$ in the same way.
A more elegant solution is to construct S on the outside of AB so that ABS is equilateral. Then we find that CAS and QAR are similar and that CBS and PBR are similar. So QR/CS = PR/CS. The ratio of the sides is the same in each case (CA/QA = CB/PB since CQA and CPB are similar), so QR = PR. Also there is a 45^o rotation between QAR and CAS and another 45^o rotation between CBS and PBR, hence QR and PR are at 90^o.

Problem B1
Let A be the sum of the decimal digits of 4444^{4444}, and B be the sum of the decimal digits of A. Find the sum of the decimal digits of B.

Solution
Let $X = 4444^{4444}$.Then X has less than 4.4444 = 17776 digits, so A is at most 9.17776 = 159984. Hence B is at most 6.9 = 54. But all these numbers are congruent mod 9. 4444 = -2 (mod 9), so $X = (-2)^{4444}$ (mod 9). But $(-2)^3 = 1$ (mod 9), and 4444 = 1 (mod 3), so X = -2 = 7 (mod 9). But any number less than 55 and congruent to 7 has digit sum 7 (possibilities are 7, 16, 25, 34, 43, 52). Hence the answer is 7.

Problem B2
Find 1975 points on the circumference of a unit circle such that the distance between each pair is rational, or prove it impossible.

Solution

Let x be the angle $\cos^{-1}4/5$, so that $\cos x = 4/5$, $\sin x = 3/5$. Take points on the unit circle at angles 2nx for n integral. Then the distance between the points at angles 2nx and 2mx is $2 \sin(n - m)x$. The usual formula, giving $\sin(n - m)x$ in terms of sin x and cos x, shows that $\sin(n - m)x$ is rational. So it only remains to show that this process generates arbitarily many distinct points, in other words that x is not a rational multiple of π.

This is quite hard. There is an elegant argument in sections 5 and 8 of Hadwiger et al, Combinatorial geometry in the Plane. But we can avoid it by observing that there are only finitely many numbers with are nth roots of unity for $n \le 2 \times 1975$, whereas there are infinitely many Pythagorean triples, so we simply pick a triple which is not such a root of unity.

Problem B3

Find all polynomials P(x, y) in two variables such that:

(1) $P(tx, ty) = t^nP(x, y)$ for some positive integer n and all real t, x, y;

(2) for all real x, y, z: $P(y + z, x) + P(z + x, y) + P(x + y, z) = 0$;

(3) $P(1, 0) = 1$.

Solution

(1) means that P is homogeneous of degree n for some n. Experimenting with low n, shows that the only solutions for n = 1, 2, 3 are $(x - 2y)$, $(x + y)(x - 2y)$, $(x + y)^2(x - 2y)$. It then obvious by inspection that $(x + y)^n(x - 2y)$ is a solution for any n. Taking x = y = z in (2) shows that $P(2x,x) = 0$, so $(x - 2y)$ is always a factor. Taking x = y = 1, z = -2 gives $P(1,-1)(2^n - 2) = 0$, so $(x + y)$ is a factor for n > 1. All this suggests (but does not prove) that the general solution is $(x + y)^n(x - 2y)$.

Take y = 1 - x, z = 0 in (2) and we get: $P(x, 1-x) = -1 - P(1-x, x)$. In particular, $P(0,1) = -2$. Now take z = 1 - x - y and we get: $P(1-x, x) + P(1-y, y) + P(x+y, 1-x-y) = 0$ and hence $f(x+y) = f(x) + f(y)$, where $f(x) = P(1-x, x) - 1$. By induction we conclude that, for any integer m and real x, $f(mx) = mf(x)$. Hence $f(1/s) = 1/s\, f(1)$ and $f(r/s) = r/s\, f(1)$ for any integers r, s. But $P(0,1) = -2$, so $f(1) = -3$. So $f(x) = -3x$ for all rational x. But f is continuous, so $f(x) = -3x$ for all x. So set $x = b/(a+b)$, where a and b are arbitrary reals (with a+b non-zero). Then $P(a,b) = (a+b)^nP(1-x, x) = (a+b)^n(-3b/(a+b) + 1) = (a+b)^{n-1}(a-2b)$, as claimed. [For a+b = 0, we appeal to continuity, or use the already derived fact that for n > 1, P(a,b) = 0.]

IMO 1976

Problem A1

A plane convex quadrilateral has area 32, and the sum of two opposite sides and a diagonal is 16. Determine all possible lengths for the other diagonal.

Solution

At first sight, the length of the other diagonal appears unlikely to be significantly constrained. However, a little experimentation shows that it is hard to get such a low value as 16. This suggests that 16 may be the smallest possible value.

If the diagonal which is part of the 16 has length x, then the area is the sum of the areas of two triangles base x, which is xy/2, where y is the sum of the altitudes of the two triangles. y must be at most (16 - x), with equality only if the two triangles are right-angled. But $x(16 - x)/2 = (64 - (x - 8)^2)/2 \le 32$ with equality only iff x = 8. Thus the only way we can achieve the values given is with one diagonal length 8 and two sides perpendicular to this diagonal with lengths totalling 8. But in this case the other diagonal has length $8\sqrt{2}$.

Problem A2

Let $P_1(x) = x^2 - 2$, and $P_{i+1} = P_1(P_i(x))$ for i = 1, 2, 3, ... Show that the roots of $P_n(x) = x$ are real and distinct for all n.

Solution

We show that the graph of P_n can be divided into 2^n lines each joining the top and bottom edges of the square side 4 centered on the origin (vertices (2,2), (-2,2), (-2,-2), (-2,2)).

We are then home because the upward sloping diagonal of the square, which represents the graph of $y = x$, must cut each of these lines and hence give 2^n distinct real roots of $P_n(x) = x$ in the range [-2,2]. But P_n is a polynomial of degree 2^n, so it has exactly 2^n roots. Hence all its roots are real and distinct.
We prove the result about the graph by induction. It is true for n = 1: the first line is the graph from x = -2 to 0, and the second line is the graph from 0 to 2. So suppose it is true for n. Then P_1 turns each of the 2^n lines for P_n into two lines for P_{n+1}, so the result is true for n+1.

Problem A3
A rectangular box can be completely filled with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, with their edges parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of the box.

Solution
Answer: 2 x 3 x 5 or 2 x 5 x 6.
This is somewhat messy. The basic idea is that the sides cannot be too long, because then the ratio becomes too big. Let k denote the (real) cube root of 2. Given any integer n, let n' denote the least integer such that n'k <= n. Let the sides of the box be $a \le b \le c$. So we require 5a'b'c' = abc (*).
It is useful to derive n' for small n: 1' = 0, 2' = 1, 3' = 2, 4' = 3, 5' = 3, 6' = 4, 7' = 5, 8' = 6, 9' = 7, 10' = 7.
Clearly $n'k \ge n-2$. But $6^3 > 0.4\ 8^3$, and hence $(n'k)^3 \ge (n - 2)^3 > 0.4\ n^3$ for all $n \ge 8$. We can check directly that $(n'k)^3 > 0.4\ n^3$ for n = 3, 4, 5, 6, 7. So we must have a = 2 (we cannot have a = 1, because 1' = 0).
From (*) we require b or c to be divisible by 5. Suppose we take it to be 5. Then since 5' = 3, the third side n must satisfy: n' = 2/3 n. We can easily check that 2k/3 < 6/7 and hence (2/3 nk + 1) < n for $n \ge 7$, so n' > 2/3 n for $n \ge 7$. This just leaves the values n = 3 and n = 6 to check (since n' = 2/3 n is integral so n must be a multiple of 3). Referring to the values above, both these work. So this gives us two possible boxes: 2 x 3 x 5 and 2 x 5 x 6.
The only remaining possibility is that the multiple of 5 is at least 10. But then it is easy to check that if it is m then $m'/m \ge 7/10$. It follows from (*) that the third side r must satisfy r'/r <= 4/7. But using the limit above and referring to the small values above, this implies that r must be 2. So a = b = 2. But now c must satisfy c' = 4/5 c. However, that is impossible because 4/5 k > 1.

Problem B1
Determine the largest number which is the product of positive integers with sum 1976.

Solution
Answer: $2 \cdot 3^{658}$.
There cannot be any integers larger than 4 in the maximal product, because for n > 4, we can replace n by 3 and n - 3 to get a larger product. There cannot be any 1s, because there must be an integer r > 1 (otherwise the product would be 1) and r + 1 > 1.r. We can also replace any 4s by two 2s leaving the product unchanged. Finally, there cannot be more than two 2s, because we can replace three 2s by two 3s to get a larger product. Thus the product must consist of 3s, and either zero, one or two 2s. The number of 2s is determined by the remainder on dividing the number 1976 by 3.
$1976 = 3 \cdot 658 + 2$, so there must be just one 2, giving the product $2 \cdot 3^{658}$.

Problem B2
n is a positive integer and m = 2n. a_{ij} = 0, 1 or -1 for $1 \le i \le n$, $1 \le j \le m$. The m unknowns $x_1, x_2, \dots , x_m$ satisfy the n equations:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m = 0,$$

for i = 1, 2, ... , n. Prove that the system has a solution in integers of absolute value at most m, not all zero.

Solution

We use a counting argument. If the modulus of each x_i is at most n, then each of the linear combinations has a value between $-2n^2$ and $2n^2$, so there are at most $(4n^2 + 1)$ possible values for each linear combination and at most $(2n^2 + 1)^n$ possible sets of values. But there are $2n+1$ values for each x_i with modulus at most n, and hence $(2n+1)^{2n} = (4n^2+4n+1)^n$ sets of values. So two distinct sets must give the same set of values for the linear combinations. But now if these sets are x_i and x_i', then the values x_i-x_i' give zero for each linear combination, and have modulus at most 2n. Moreover they are not all zero, since the two sets of values were distinct.

Problem B3

The sequence $u_0, u_1, u_2, \dots$ is defined by: $u_0= 2$, $u_1 = 5/2$, $u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$ for n = 1, 2, ... Prove that $[u_n] = 2^{(2n - (-1)n)/3}$, where [x] denotes the greatest integer less than or equal to x.

Solution

Experience with recurrence relations suggests that the solution is probably the value given for $[u_n]$ plus its inverse. It is straightforward to verify this guess by induction.
Squaring u_{n-1} gives the sum of positive power of 2, its inverse and 2. So $u_{n-1} - 2$ = the sum of a positive power of 2 and its inverse. Multiplying this by u_n gives a positive power of 2 + its inverse + 2 + 1/2, and we can check that the power of 2 is correct for u_{n+1}.

IMO 1977

Problem A1

Construct equilateral triangles ABK, BCL, CDM, DAN on the inside of the square ABCD. Show that the midpoints of KL, LM, MN, NK and the midpoints of AK, BK, BL, CL, CM, DM, DN, AN form a regular dodecahedron.

Solution

The most straightforward approach is to use coordinates. Take A, B, C, D to be (1,1), (-1,1), (-1,-1), (1,-1). Then K, L, M, N are (0, -2k), (2k, 0), (0, 2k), (-2k, 0), where $k = (\sqrt{3} - 1)/2$. The midpoints of KL, LM, MN, NK are (k, -k), (k, k), (-k, k), (-k, -k). These are all a distance $k\sqrt{2}$ from the origin, at angles 315, 45, 135, 225 respectively. The midpoints of AK, BK, BL, CL, CM, DM, DN, AN are (h, j), (-h, j), (-j, h), (-j, -h), (-h, -j), (h, -j), (j, -h), (j, h), where h = 1/2, $j = (1 - 1/2 \sqrt{3})$. These are also at a distance $k\sqrt{2}$ from the origin, at angles 15, 165, 105, 255, 195, 345, 285, 75 respectively. For this we need to consider the right-angled triangle sides k, h, j. The angle x between h and k has sin x = j/k and cos x = h/k. So $\sin 2x = 2 \sin x \cos x = 2hj/k^2 = 1/2$. Hence x = 15.
So the 12 points are all at the same distance from the origin and at angles 15 + 30n, for n = 0, 1, 2, ... , 11. Hence they form a regular dodecagon.

Problem A2

In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

Solution

Answer: 16. $x_1 + \dots + x_7 < 0$, $x_8 + \dots + x_{14} < 0$, so $x_1 + \dots + x_{14} < 0$. But $x_4 + \dots + x_{14} > 0$, so $x_1 + x_2 + x_3 < 0$. Also $x_5 + \dots + x_{11} < 0$ and $x_1 + \dots + x_{11} > 0$, so $x_4 > 0$. If there are 17 or more elements then the same argument shows that $x_5, x_6, x_7 > 0$. But $x_1 + \dots + x_7 < 0$, and $x_5 + \dots + x_{11} < 0$, whereas $x_1 + \dots + x_{11} > 0$, so $x_5 + x_6 + x_7 < 0$. Contradiction.
If we assume that there is a solution for n = 16 and that the sum of 7 consecutive terms is -1 and that the sum of 11 consecutive terms is 1, then we can easily solve the equations to get: 5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5 and we can check that this works for 16.

Problem A3

Given an integer n > 2, let V_n be the set of integers 1 + kn for k a positive integer. A number m in V_n is called indecomposable if it cannot be expressed as the product of two members of V_n. Prove that there is a number in V_n which can be expressed as the product of indecomposable members of V_n in more than one way (decompositions which differ solely in the order of factors are not regarded as different).

Solution

Take a, b, c, d = -1 (mod n). The idea is to take abcd which factorizes as ab.cd or ac.bd. The hope is that ab, cd, ac, bd will not factorize in V_n. But a little care is needed, since this is not necessarily true.

Try taking a = b = n - 1, c = d = 2n -1. a^2 must be indecomposable because it is less than the square of the smallest element in V_n. If ac = $2n^2 - 3n + 1$ is decomposable, then we have kk'n + k + k' = 2n - 3 for some k, k' >= 1. But neither of k or k' can be 2 or more, because then the lhs is too big, and k = k' = 1 does not work unless n = 5. Similarly, if c^2 is decomposable, then we have kk'n + k + k' = 4n - 4. k = k' = 1 only works for n = 2, but we are told n > 2. k = 1, k' = 2 does not work (it would require n = 7/2). k = 1, k' = 3 only works for n = 8. Other possibilities make the lhs too big.

So if n is not 5 or 8, then we can take the number to be $(n - 1)^2(2n - 1)^2$, which factors as $(n - 1)^2 \times (2n - 1)^2$ or as (n - 1)(2n - 1) x (n - 1)(2n - 1). This does not work for 5 or 8: 16·81 = 36·36, but 36 decomposes as 6·6; 49·225 = 105·105, but 225 decomposes as 9·25. For n = 5, we can use 3136 = 16·196 = 56·56. For n = 8, we can use 25921 = 49·529 = 161·161.

Problem B1

Define f(x) = 1 - a cos x - b sin x - A cos 2x - B sin 2x, where a, b, A, B are real constants. Suppose that f(x) ≥ 0 for all real x. Prove that $a^2 + b^2 \le 2$ and $A^2 + B^2 \le 1$.

Solution

Take y so that cos y = $a/\sqrt{(a^2 + b^2)}$, sin y = $b/\sqrt{(a^2 + b^2)}$, and z so that cos 2z = $A/\sqrt{(A^2 + B^2)}$, sin 2z = $B/\sqrt{(A^2 + B^2)}$. Then f(x) = 1 - c cos(x - y) - C cos2(x - z), where c = $\sqrt{(a^2 + b^2)}$, C = $\sqrt{(A^2 + B^2)}$.

f(z) + f(π + z) ≥ 0 gives C ≤ 1. f(y + π/4) + f(y - π/4) ≥ 0 gives c ≤ √2.

Problem B2

Let a and b be positive integers. When $a^2 + b^2$ is divided by a + b, the quotient is q and the remainder is r. Find all pairs a, b such that $q^2 + r = 1977$.

Solution

$a^2 + b^2 >= (a + b)^2/2$, so q ≥ (a + b)/2. Hence r < 2q. The largest square less than 1977 is 1936 = 44^2. 1977 = $44^2 + 41$. The next largest gives 1977 = $43^2 + 128$. But 128 > 2.43. So we must have q = 44, r = 41. Hence $a^2 + b^2 = 44(a + b) + 41$, so $(a - 22)^2 + (b - 22)^2 = 1009$. By trial, we find that the only squares with sum 1009 are 28^2 and 15^2. This gives two solutions 50, 37 or 50, 7.

Problem B3

The function f is defined on the set of positive integers and its values are positive integers. Given that f(n+1) > f(f(n)) for all n, prove that f(n) = n for all n.

Solution

The first step is to show that f(1) < f(2) < f(3) < ... . We do this by induction on n. We take S_n to be the statement that f(n) is the unique smallest element of { f(n), f(n+1), f(n+2), ... }. For m > 1, f(m) > f(s) where s = f(m-1), so f(m) is not the smallest member of the set {f(1), f(2), f(3), ... }. But the set is bounded below by zero, so it must have a smallest member. Hence the unique smallest member is f(1). So S_1 is true.

Suppose S_n is true. Take m > n+1. Then m-1 > n, so by S_n, f(m-1) > f(n). But S_n also tells us that f(n) > f(n-1) > ... > f(1), so f(n) ≥ n - 1 + f(1) ≥ n. Hence f(m-1) ≥ n+1. So f(m-1) belongs to { n+1, n+2, n+3, .. }. But we are given that f(m) > f(f(m-1)), so f(m) is not the smallest element of { f(n+1), f(n+2), f(n+3), ... }. But there must be a smallest element, so f(n+1) must be the unique smallest member, which establishes S_{n+1}. So, S_n is true for all n.

So $n \le m$ implies $f(n) <= f(m)$. Suppose for some m, $f(m) \ge m+1$, then $f(f(m)) \ge f(m+1)$. Contradiction. Hence $f(m) \le m$ for all m. But since $f(1) \ge 1$ and $f(m) > f(m-1) > \dots > f(1)$, we also have $f(m) \ge m$. Hence $f(m) = m$ for all m.

IMO 1978

Problem A1

m and n are positive integers with $m < n$. The last three decimal digits of 1978^m are the same as the last three decimal digits of 1978^n. Find m and n such that $m + n$ has the least possible value.

Solution

We require $1978^m(1978^{n-m} - 1)$ to be a multiple of $1000=8\cdot125$. So we must have 8 divides 1978^m, and hence $m \ge 3$, and 125 divides $1978^{n-m} - 1$.
By Euler's theorem, $1978^{\varphi(125)} = 1 \pmod{125}$. $\varphi(125) = 125 - 25 = 100$, so $1978^{100} = 1 \pmod{125}$. Hence the smallest r such that $1978^r = 1 \pmod{125}$ must be a divisor of 100 (because if it was not, then the remainder on dividing it into 100 would give a smaller r). That leaves 9 possibilities to check: 1, 2, 4, 5, 10, 20, 25, 50, 100. To reduce the work we quickly find that the smallest s such that $1978^s = 1 \pmod 5$ is 4 and hence r must be a multiple of 4. That leaves 4, 20, 100 to examine.
We find $978^2 = 109 \pmod{125}$, and hence $978^4 = 6 \pmod{125}$. Hence $978^{20} = 6^5 = 36\cdot91 = 26 \pmod{125}$. So the smallest r is 100 and hence the solution to the problem is 3, 103.

Problem A2

P is a point inside a sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V and W. Q denotes the vertex diagonally opposite P in the parallelepiped determined by PU, PV, PW. Find the locus of Q for all possible sets of such rays from P.

Solution

Suppose ABCD is a rectangle and X any point inside, then $XA^2 + XC^2 = XB^2 + XD^2$. This is most easily proved using coordinates. Take the origin O as the center of the rectangle and take OA to be the vector $\underline{a}$, and OB to be $\underline{b}$. Since it is a rectangle, $|\underline{a}| = |\underline{b}|$. Then OC is $-\underline{a}$ and OD is $-\underline{b}$. Let OX be $\underline{c}$. Then $XA^2 + XC^2 = (\underline{a} - \underline{c})^2 + (\underline{a} + \underline{c})^2 = 2\underline{a}^2 + 2\underline{c}^2 = 2\underline{b}^2 + 2\underline{c}^2 = XB^2 + XD^2$.
Let us fix U. Then the plane k perpendicular to PU through P cuts the sphere in a circle center C. V and W must lie on this circle. Take R so that PVRW is a rectangle. By the result just proved $CR^2 = 2CV^2 - CP^2$. OC is also perpendicular to the plane k. Extend it to X, so that CX = PU. Then extend XU to Y so that YR is perpendicular to k. Now $OY^2 = OX^2 + XY^2 = OX^2 + CR^2 = OX^2 + 2CV^2 - CP^2 = OU^2 - UX^2 + 2CV^2 - CP^2 = OU^2 - CP^2 + 2(OV^2 - OC^2) - CP^2 = 3OU^2 - 2OP^2$. Thus the locus of Y is a sphere.

Problem A3

The set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), f(3), \dots \}$, $\{g(1), g(2), g(3), \dots \}$, where $f(1) < f(2) < f(3) < \dots$, and $g(1) < g(2) < g(3) < \dots$, and $g(n) = f(f(n)) + 1$ for n = 1, 2, 3, ... Determine f(240).

Solution

Let $F = \{f(1), f(2), f(3), \dots \}$, $G = \{g(1), g(2), g(3), \dots \}$, $N_n = \{1, 2, 3, \dots , n\}$. $f(1) \ge 1$, so $f(f(1)) \ge 1$ and hence $g(1) \ge 2$. So 1 is not in G, and hence must be in F. It must be the smallest element of F and so f(1) = 1. Hence g(1) = 2. We can never have two successive integers n and n+1 in G, because if g(m) = n+1, then f(something) = n and so n is in F and G. Contradiction. In particular, 3 must be in F, and so f(2) = 3.
Suppose f(n) = k. Then g(n) = f(k) + 1. So $|N_{f(k)+1} \quad G| = n$. But $|N_{f(k)+1} \quad F| = k$, so $n + k = f(k) + 1$, or $f(k) = n + k - 1$. Hence $g(n) = n + k$. So $n + k + 1$ must be in F and hence $f(k+1) = n + k + 1$. This so given the value of f for n we can find it for k and k+1.
Using k+1 each time, we get, successively, f(2) = 3, f(4) = 6, f(7) = 11, f(12) = 19, f(20) = 32, f(33) = 53, f(54) = 87, f(88) = 142, f(143) = 231, f(232) = 375, which is not much help. Trying again with k, we get: f(3) = 4, f(4) = 6, f(6) = 9, f(9) = 14, f(14) = 22, f(22) = 35, f(35) = 56, f(56) = 90, f(90) = 145, f(145) = 234. Still not right, but we can try

backing up slightly and using k+1: f(146) = 236. Still not right, we need to back up further: f(91) = 147, f(148) = 239, f(240) = 388.

Problem B1

In the triangle ABC, AB = AC. A circle is tangent internally to the circumcircle of the triangle and also to AB, AC at P, Q respectively. Prove that the midpoint of PQ is the center of the incircle of the triangle.

Solution

It is not a good idea to get bogged down in complicated formulae for the various radii. The solution is actually simple.

By symmetry the midpoint, M, is already on the angle bisector of A, so it is sufficient to show it is on the angle bisector of B. Let the angle bisector of A meet the circumcircle again at R. AP is a tangent to the circle touching AB at P, so $\angle$ PRQ = $\angle$ APQ = $\angle$ ABC. Now the quadrilateral PBRM is cyclic because the angles PBR, PMR are both 90^o. Hence $\angle$ PBM = $\angle$ PRM = ($\angle$ PRQ)/2, so BM does indeed bisect angle B as claimed.

Problem B2

$\{a_k\}$ is a sequence of distinct positive integers. Prove that for all positive integers n, $\sum_1^n a_k/k^2 \ge \sum_1^n 1/k$.

Solution

We use the general rearrangement result: given $b_1 \ge b_2 \ge \dots \ge b_n$, and $c_1 \le c_2 \le \dots \le c_n$, if $\{a_i\}$ is a permutation of $\{c_i\}$, then $\sum a_ib_i \ge \sum c_ib_i$. To prove it, suppose that i < j, but $a_i > a_j$. Then interchanging a_i and a_j does not increase the sum, because $(a_i - a_j)(b_i - b_j) \ge 0$, and hence $a_ib_i + a_jb_j \ge a_jb_i + a_ib_j$. By a series of such interchanges we transform $\{a_i\}$ into $\{c_i\}$ (for example, first swap c_1 into first place, then c_2 into second place and so on).

Hence we do not increase the sum by permuting $\{a_i\}$ so that it is in increasing order. But now we have $a_i > i$, so we do not increase the sum by replacing a_i by i and that gives the sum from 1 to n of 1/k.

Problem B3

An international society has its members from six different countries. The list of members has 1978 names, numbered 1, 2, ... , 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice the number of a member from his own country.

Solution

The trick is to use differences.

At least 6.329 = 1974, so at least 330 members come from the same country, call it C1. Let their numbers be $a_1 < a_2 < \dots < a_{330}$. Now take the 329 differences $a_2 - a_1$, $a_3 - a_1$, ... , $a_{330} - a_1$. If any of them are in C1, then we are home, so suppose they are all in the other five countries.

At least 66 must come from the same country, call it C2. Write the 66 as $b_1 < b_2 < \dots < b_{66}$. Now form the 65 differences $b_2 - b_1$, $b_3 - b_1$, ... , $b_{66} - b_1$. If any of them are in C2, then we are home. But each difference equals the difference of two of the original a_is, so if it is in C1 we are also home.

So suppose they are all in the other four countries. At least 17 must come from the same country, call it C3. Write the 17 as $c_1 < c_2 < \dots < c_{17}$. Now form the 16 differences $c_2 - c_1$, $c_3 - c_1$, ... , $c_{17} - c_1$. If any of them are in C3, we are home. Each difference equals the difference of two b_is, so if any of them are in C2 we are home. [For example, consider $c_i - c_1$. Suppose $c_i = b_n - b_1$ and $c_1 = b_m - b_1$, then $c_i - c_1 = b_n - b_m$, as claimed.]. Each difference also equals the difference of two a_is, so if any of them are in C1, we are also home. [For example, consider $c_i - c_1$, as before. Suppose $b_n = a_j - a_1$, $b_m = a_k - a_1$, then $c_i - c_1 = b_n - b_m = a_j - a_k$, as claimed.]

So suppose they are all in the other three countries. At least 6 must come from the same country, call it C4. We look at the 5 differences and conclude in the same way that at least 3 must come from C5. Now the 2 differences must both be in C6 and their difference must be in one of the C1, ... , C6 giving us the required sum.

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IMO 1979

Problem A1

Let m and n be positive integers such that:

$m/m = 1 - 1/2 + 1/3 - 1/4 + \ldots - 1/1318 + 1/1319.$

Prove that m is divisible by 1979.

Solution

This is difficult.

The obvious step of combining adjacent terms to give $1/(n(n+1)$ is unhelpful. The trick is to separate out the negative terms:

$1 - 1/2 + 1/3 - 1/4 + \ldots - 1/1318 + 1/1319 = 1 + 1/2 + 1/3 + \ldots + 1/1319 - 2(1/2 + 1/4 + \ldots + 1/1318) = 1/660 + 1/661 + \ldots + 1/1319.$

and to notice that $660 + 1319 = 1979$. Combine terms in pairs from the outside:

$1/660 + 1/1319 = 1979/(660.1319)$; $1/661 + 1/1318 = 1979/(661.1318)$ etc.

There are an even number of terms, so this gives us a sum of terms 1979/m with m not divisible by 1979 (since 1979 is prime and so does not divide any product of smaller numbers). Hence the sum of the 1/m gives a rational number with denominator not divisible by 1979 and we are done.

Problem A2

A prism with pentagons $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$ as the top and bottom faces is given. Each side of the two pentagons and each of the 25 segments A_iB_j is colored red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Prove that all 10 sides of the top and bottom faces have the same color.

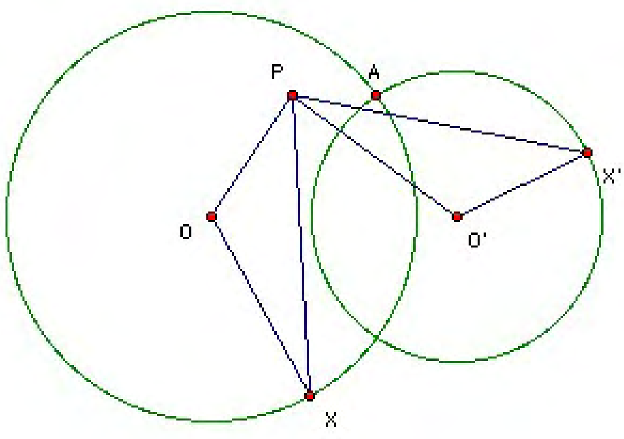
Solution

We show first that the A_i are all the same color. If not then, there is a vertex, call it A_1, with edges A_1A_2, A_1A_5 of opposite color. Now consider the five edges A_1B_i. At least three of them must be the same color. Suppose it is green and that A_1A_2 is also green. Take the three edges to be A_1B_i, A_1B_j, A_1B_k. Then considering the triangles $A_1A_2B_i$, $A_1A_2B_j$, $A_1A_2B_k$, the three edges A_2B_i, A_2B_j, A_2B_k must all be red. Two of B_i, B_j, B_k must be adjacent, but if the resulting edge is red then we have an all red triangle with A_2, whilst if it is green we have an all green triangle with A_1. Contradiction. So the A_i are all the same color. Similarly, the B_i are all the same color. It remains to show that they are the same color. Suppose otherwise, so that the A_i are green and the B_i are red.

Now we argue as before that 3 of the 5 edges A_1B_i must be the same color. If it is red, then as before 2 of the 3 B_i must be adjacent and that gives an all red triangle with A_1. So 3 of the 5 edges A_1B_i must be green. Similarly, 3 of the 5 edges A_2B_i must be green. But there must be a B_i featuring in both sets and it forms an all green triangle with A_1 and A_2. Contradiction. So the A_i and the B_i are all the same color.

Problem A3

Two circles in a plane intersect. A is one of the points of intersection. Starting simultaneously from A two points move with constant speed, each traveling along its own circle in the same sense. The two points return to A simultaneously after one revolution. Prove that there is a fixed point P in the plane such that the two points are always equidistant from P.



Solution

Let the circles have centers O, O' and let the moving points by X, X. Let P be the reflection of A in the perpendicular bisector of OO'. We show that triangles POX, X'O'P are congruent. We have OX = OA (pts on circle) = O'P (reflection). Also OP = O'A (reflection) = O'X' (pts on circle). Also $\angle$ AOX = $\angle$ AO'X' (X and X' circle at same rate), and $\angle$ AOP = $\angle$ AO'P (reflection), so $\angle$ POX = $\angle$ PO'X'. So the triangles are congruent. Hence PX = PX'.

Problem B1
Given a plane k, a point P in the plane and a point Q not in the plane, find all points R in k such that the ratio (QP + PR)/QR is a maximum.

Solution
Consider the points R on a circle center P. Let X be the foot of the perpendicular from Q to k. Assume P is distinct from X, then we minimise QR (and hence maximise (QP + PR)/QR) for points R on the circle by taking R on the line PX. Moreover, R must lie on the same side of P as X. Hence if we allow R to vary over k, the points maximising (QP + PR)/QR must lie on the ray PX. Take S on the line PX on the opposite side of P from X so that PS = PQ. Then for points R on the ray PX we have (QP + PR)/QR = SR/QR = sin RQS/sin QSR. But sin QSR is fixed for points on the ray, so we maximise the ratio by taking $\angle$ RQS = 90^o. Thus there is a single point maximising the ratio.
If P = X, then we still require $\angle$ RQS = 90^o, but R is no longer restricted to a line, so it can be anywhere on a circle center P.

Problem B2
Find all real numbers a for which there exist non-negative real numbers x_1, x_2, x_3, x_4, x_5 satisfying:

$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = a$,
$x_1 + 2^3x_2 + 3^3x_3 + 4^3x_4 + 5^3x_5 = a^2$,
$x_1 + 2^5x_2 + 3^5x_3 + 4^5x_4 + 5^5x_5 = a^3$.

Solution
Take a^2 x 1st equ - 2a x 2nd equ + 3rd equ. The rhs is 0. On the lhs the coefficient of x_n is $a^2n - 2an^3 + n^5 = n(a - n^2)^2$. So the lhs is a sum of non-negative terms. Hence each term must be zero separately, so for each n either $x_n = 0$ or $a = n^2$. So there are just 5 solutions, corresponding to a = 1, 4, 9, 16, 25. We can check that each of these gives a solution. [For $a = n^2$, $x_n = n$ and the other x_i are zero.]

Problem B3
Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that:

$a_{2n-1} = 0$
$a_{2n} = (2 + \sqrt{2})^{n-1}/\sqrt{2} - (2 - \sqrt{2})^{n-1}/\sqrt{2}$.

Solution
Each jump changes the parity of the shortest distance to E. The parity is initially even, so an odd number of jumps cannot end at E. Hence $a_{2n-1} = 0$.
We derive a recurrence relation for a_{2n}. This is not easy to do directly, so we introduce b_n which is the number of paths length n from C to E. Then we have immediately:

$a_{2n} = 2a_{2n-2} + 2b_{2n-2}$ for n > 1
$b_{2n} = 2b_{2n-2} + a_{2n-2}$ for n > 1

Hence, using the first equation: $a_{2n} - 2a_{2n-2} = 2a_{2n-2} - 4a_{2n-4} + 2b_{2n-2} - 4b_{2n-4}$ for n > 2. Using the second equation, this leads to: $a_{2n} = 4a_{2n-2} - 2a_{2n-4}$ for n > 2. This is a linear recurrence relation with the general solution: $a_{2n} = a(2 + \sqrt{2})^{n-1} + b(2 - \sqrt{2})^{n-1}$. But we easily see directly that $a_4 = 2$, $a_6 = 8$ and we can now solve for the coefficients to get the solution given.

IMO 1981

Problem A1

P is a point inside the triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P which minimise:

BC/PD + CA/PE + AB/PF.

Solution

We have PD.BC + PE.CA + PF.AB = 2 area of triangle. Now use Cauchy's inequality with $x_1 = \sqrt{(PD \cdot BC)}$, $x_2 = \sqrt{(PE \cdot CA)}$, $x_3 = \sqrt{(PF \cdot AB)}$, and $y_1 = \sqrt{(BC/PD)}$, $y_2 = \sqrt{(CA/PE)}$, $y_3 = \sqrt{(AB/PF)}$. We get that $(BC + CA + AB)^2 < 2$ x area of triangle x (BC/PD + CA/PE + AB/PF) with equality only if x_i/y_i = const, ie PD = PE = PF. So the unique minimum position for P is the incenter.

Problem A2

Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set {1, 2, ... , n}. Each subset has a smallest element. Let F(n,r) be the arithmetic mean of these smallest elements. Prove that:

F(n,r) = (n+1)/(r+1).

Solution

Denote the binomial coefficient n!/(r!(n-r)!) by nCr.

Evidently nCr F(n,r) = 1 (n-1)C(r-1) + 2 (n-2)C(r-1) + ... + (n-r+1) (r-1)C(r-1). [The first term denotes the contribution from subsets with smallest element 1, the second term smallest element 2 and so on.]

Let the rhs be g(n,r). Then, using the relation (n-i)C(r-1) - (n-i-1)C(r-2) = (n-i-1)C(r-1), we find that g(n,r) - g(n-1,r-1) = g(n-1,r), and we can extend this relation to r=1 by taking g(n,0) = n+1 = (n+1)C1. But g(n,1) = 1 + 2 + ... + n = n(n+1)/2 = (n+1)C2. So it now follows by an easy induction that g(n,r) = (n+1)C(r+1) = nCr (n+1)/(r+1). Hence F(n,r) = (n+1)/(r+1).

Problem A3

Determine the maximum value of $m^2 + n^2$, where m and n are integers in the range 1, 2, ... , 1981 satisfying $(n^2 - mn - m^2)^2 = 1$.

Solution

Experimenting with small values suggests that the solutions of $n^2 - mn - m^2 = 1$ or -1 are successive Fibonacci numbers. So suppose n > m is a solution. This suggests trying m+n, n: $(m+n)^2 - (m+n)n - n^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2) = 1$ or -1. So if n > m is a solution, then m+n, n is another solution. Running this forward from 2,1 gives 3,2; 5,3; 8,5; 13,8; 21,13; 34,21; 55,34; 89,55; 144,89; 233,144; 377,233; 610,377; 987,610; 1597,987; 2584,1597.

But how do we know that there are no other solutions? The trick is to run the recurrence the other way. For suppose n > m is a solution, then try m, n-m: $m^2 - m(n-m) - (n-m)^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2) = 1$ or -1, so that also satisfies the equation. Also if m > 1, then m > n-m (for if not, then n >= 2m, so n(n - m) >= $2m^2$, so $n^2 - nm - m^2 >= m^2 > 1$). So given a solution n > m with m > 1, we have a smaller solution m > n-m. This process must eventually terminate, so it must finish at a solution n, 1 with n > 1. But the only such solution is 2, 1. Hence the starting solution must have been in the forward sequence from 2, 1.

Hence the solution to the problem stated is $1597^2 + 987^2$.

Problem B1

(a) For which n > 2 is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining n - 1 numbers?

(b) For which n > 2 is there exactly one set having this property?

Solution

(a) n = 3 is not possible. For suppose x was the largest number in the set. Then x cannot be divisible by 3 or any larger prime, so it must be a power of 2. But it cannot be a power of 2, because 2^m - 1 is odd and 2^m - 2 is not a positive integer divisible by 2^m.
For k ≥ 2, the set 2k-1, 2k , ... , 4k-2 gives n = 2k. For k ≥ 3, so does the set 2k-5, 2k-4, ... , 4k-6. For k ≥ 2, the set 2k-2, 2k-3, ... , 4k-2 gives n = 2k+1. For k ≥ 4 so does the set 2k-6,2k-5, ... , 4k-6. So we have at least one set for every n ≥ 4, which answers (a).
(b) We also have at least two sets for every n ≥ 4 except possibly n = 4, 5, 7. For 5 we may take as a second set: 8, 9, 10, 11, 12, and for 7 we may take 6, 7, 8, 9 ,10, 11, 12. That leaves n = 4. Suppose x is the largest number in a set with n =4. x cannot be divisible by 5 or any larger prime, because x-1, x-2, x-3 will not be. Moreover, x cannot be divisible by 4, because then x-1 and x-3 will be odd, and x-2 only divisible by 2 (not 4). Similarly, it cannot be divisible by 9. So the only possibilities are 1, 2, 3, 6. But we also require x ≥ 4, which eliminates the first three. So the only solution for n = 4 is the one we have already found: 3, 4, 5, 6.

Problem B2
Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

Solution
Let the triangle be ABC. Let the center of the circle touching AB and AC be D, the center of the circle touching AB and BC be E, and the center of the circle touching AC and BC be F. Because the circles center D and E have the same radius the perpendiculars from D and E to AB have the same length, so DE is parallel to AB. Similarly EF is parallel to BC and FD is parallel to CA. Hence DEF is similar and similarly oriented to ABC. Moreover D must lie on the angle bisector of A since the circle center D touches AB and AC. Similarly E lies on the angle bisector of B and F lies on the angle bisector of C. Hence the incenter I of ABC is also the incenter of DEF and acts as a center of symmetry so that corresponding points P of ABC and P' of DEF lie on a line through I with PI/P'I having a fixed ratio. But OD = OE = OF since the three circles have equal radii, so O is the circumcenter of DEF. Hence it lies on a line with I and the circumcenter of ABC.

Problem B3
The function f(x,y) satisfies: f(0,y) = y + 1, f(x+1,0) = f(x,1), f(x+1,y+1) = f(x,f(x+1,y)) for all non-negative integers x, y. Find f(4, 1981).

Solution
f(1,n) = f(0,f(1,n-1)) = 1 + f(1,n-1). So f(1,n) = n + f(1,0) = n + f(0,1) = n + 2.
f(2,n) = f(1,f(2,n-1)) = f(2,n-1) + 2. So f(2,n) = 2n + f(2,0) = 2n + f(1,1) = 2n + 3.
f(3,n) = f(2,f(3,n-1)) = 2f(3,n-1) + 3. Let u_n = f(3,n) + 3, then u_n = $2u_{n-1}$. Also u_0 = f(3,0) + 3 = f(2,1) + 3 = 8. So u_n = 2^{n+3}, and f(3,n) = 2^{n+3} - 3.
f(4,n) = f(3,f(4,n-1)) = $2^{f(4,n-1)+3}$ - 3. f(4,0) = f(3,1) = 2^4 - 3 = 13. We calculate two more terms to see the pattern: f(4,1) = 2^{24} - 3, f(4,2) = 2^{224} - 3. In fact it looks neater if we replace 4 by 2^2, so that f(4,n) is a tower of n+3 2s less 3.

IMO 1982

Problem A1
The function f(n) is defined on the positive integers and takes non-negative integer values. f(2) = 0, f(3) > 0, f(9999) = 3333 and for all m, n:
f(m+n) - f(m) - f(n) = 0 or 1.
Determine f(1982).

Solution
We show that f(n) = [n/3] for n <= 9999, where [ ] denotes the integral part.
We show first that f(3) = 1. f(1) must be 0, otherwise f(2) - f(1) - f(1) would be negative. Hence f(3) = f(2) + f(1) + 0 or 1 = 0 or 1. But we are told f(3) > 0, so f(3) = 1. It follows by induction that f(3n) ≥ n. For f(3n+3) = f(3) + f(3n) + 0 or 1 = f(3n) + 1 or 2. Moreover

if we ever get $f(3n) > n$, then the same argument shows that $f(3m) > m$ for all $m > n$. But $f(3.3333) = 3333$, so $f(3n) = n$ for all $n <= 3333$.
Now $f(3n+1) = f(3n) + f(1) + 0$ or $1 = n$ or $n + 1$. But $3n+1 = f(9n+3) \geq f(6n+2) + f(3n+1) \geq 3f(3n+1)$, so $f(3n+1) < n+1$. Hence $f(3n+1) = n$. Similarly, $f(3n+2) = n$. In particular $f(1982) = 660$.

Problem A2
A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with a_i opposite A_i. M_i is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of $\angle A_i$. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

Solution
Let B_i be the point of intersection of the interior angle bisector of the angle at A_i with the opposite side. The first step is to figure out which side of $B_i T_i$ lies. Let A_1 be the largest angle, followed by A_2. Then T_2 lies between A_1 and B_2, T_3 lies between A_1 and B_3, and T_1 lies between A_2 and B_1. For $\angle OB_2A_1 = 180^o - A_1 - A_2/2 = A_3 + A_2/2$. But $A_3 + A_2/2 < A_1 + A_2/2$ and their sum is 180^o, so $A_3 + A_2/2 < 90^o$. Hence T_2 lies between A_1 and B_2. Similarly for the others.
Let O be the center of the incircle. Then $\angle T_1OS_2 = \angle T_1OT_2 - 2 \angle T_2OB_2 = 180^o - A_3 - 2(90^o - \angle OB_2T_2) = 2(A_3 + A_2/2) - A_3 = A_2 + A_3$. A similar argument shows $\angle T_1OS_3 = A_2 + A_3$. Hence S_2S_3 is parallel to A_2A_3.
Now $\angle T_3OS_2 = 360^o - \angle T_3OT_1 - \angle T_1OS_2 = 360^o - (180^o - A_2) - (A_2 + A_3) = 180^o - A_3 = A_1 + A_2$. $\angle T_3OS_1 = \angle T_3OT_1 + 2 \angle T_1OB_1 = (180^o - A_2) + 2(90^o - \angle OB_1T_1) = 360^o - A_2 - 2(A_3 + A_1/2) = 2(A_1 + A_2 + A_3) - A_2 - 2A_3 - A_1 = A_1 + A_2 = \angle T_3OS_2$. So S_1S_2 is parallel to A_1A_2. Similarly we can show that S_1S_3 is parallel to A_1A_3.
So $S_1S_2S_3$ is similar to $A_1A_2A_3$ and turned through 180^o. But $M_1M_2M_3$ is also similar to $A_1A_2A_3$ and turned through 180^o. So $S_1S_2S_3$ and $M_1M_2M_3$ are similar and similarly oriented. Hence the lines through corresponding vertices are concurrent.

Problem A3
Consider infinite sequences $\{x_n\}$ of positive reals such that $x_0 = 1$ and $x_0 \geq= x_1 \geq x_2 \geq \dots$.
(a) Prove that for every such sequence there is an $n \geq 1$ such that:
$x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n \geq 3.999$.
(b) Find such a sequence for which:
$x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n < 4$ for all n.

Solution
(a) It is sufficient to show that the sum of the (infinite) sequence is at least 4. Let k be the greatest lower bound of the limits of all such sequences. Clearly $k \geq 1$. Given any $\varepsilon > 0$, we can find a sequence $\{x_n\}$ with sum less than $k + \varepsilon$. But we may write the sum as:
$x_0^2/x_1 + x_1((x_1/x_1)^2/(x_2/x_1) + (x_2/x_1)^2/(x_3/x_1) + \dots + (x_n/x_1)^2/(x_{n+1}/x_1) + \dots)$.
The term in brackets is another sum of the same type, so it is at least k. Hence $k + \varepsilon > 1/x_1 + x_1k$. This holds for all $\varepsilon > 0$, and so $k \geq 1/x_1 + x_1k$. But $1/x_1 + x_1k \geq 2\sqrt{k}$, so $k \geq 4$.
(b) Let $x_n = 1/2^n$. Then $x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n = 2 + 1 + 1/2 + \dots + 1/2^{n-2} = 4 - 1/2^{n-2} < 4$.

Problem B1
Prove that if n is a positive integer such that the equation
$x^3 - 3xy^2 + y^3 = n$
has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

Solution
If x, y is a solution then so is y-x, -x. Hence also -y, x-y. If the first two are the same, then $y = -x$, and $x = y-x = -2x$, so $x = y = 0$, which is impossible, since $n > 0$. Similarly, if any other pair are the same.
$2891 = 2 \pmod 9$ and there is no solution to $x^3 - 3xy^2 + y^3 = 2 \pmod 9$. The two cubes are each -1, 0 or 1, and the other term is 0, 3 or 6, so the only solution is to have the cubes congruent to 1 and -1 and the other term congruent to 0. But the other term cannot

be congruent to 0, unless one of x, y is a multiple of 3, in which case its cube is congruent to 0, not 1 or -1.

Problem B2

The diagonals AC and CE of the regular hexagon ABCDEF are divided by inner points M and N respectively, so that: AM/AC = CN/CE = r.
Determine r if B, M and N are collinear.

Solution

For an inelegant solution one can use coordinates. The advantage of this type of approach is that it is quick and guaranteed to work! Take A as $(0,\sqrt{3})$, B as $(1,\sqrt{3})$, C as $(3/2,\sqrt{3}/2$, D as (1,0). Take the point X, coordinates (x,0), on ED. We find where the line BX cuts AC and CE. The general point on BX is $(k + (1-k)x, k\sqrt{3})$. If this is also the point M with AM/AC = r then we have: $k + (1-k)x = 3r/2$, $k\sqrt{3} = (1-r)\sqrt{3} + r\sqrt{3}/2$. Hence $k = 1 - r/2$, $r = 2/(4-x)$. Similarly, if it is the point N with CN/CE = r, then $k + (1-k)x = 3(1-r)/2$, $k\sqrt{3} = (1-r)\sqrt{3}/2$. Hence $k = (1-r)/2$ and $r = (2-x)/(2+x)$. Hence for the ratios to be equal we require $2/(4-x) = (2-x)/(2+x)$, so $x^2 - 8x + 4 = 0$. We also have $x < 1$, so $x = 4 - \sqrt{12}$. This gives $r = 1/\sqrt{3}$.
A more elegant solution uses the ratio theorem for the triangle EBC. We have CM/MX XB/BE EN/NC = -1. Hence $(1-r)/(r - 1/2)\ (-1/4)\ (1-r)/r = -1$. So $r = 1/\sqrt{3}$.

Problem B3

Let S be a square with sides length 100. Let L be a path within S which does not meet itself and which is composed of line segments A_0A_1, A_1A_2, A_2A_3, ... , $A_{n-1}A_n$ with $A_0 = A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P no greater than 1/2. Prove that there are two points X and Y of L such that the distance between X and Y is not greater than 1 and the length of the part of L which lies between X and Y is not smaller than 198.

Solution

Let the square be A'B'C'D'. The idea is to find points of L close to a particular point of A'D' but either side of an excursion to B'.
We say L approaches a point P' on the boundary of the square if there is a point P on L with $PP' \le 1/2$. We say L approaches P' before Q' if there is a point P on L which is nearer to A_0 (the starting point of L) than any point Q with $QQ' \le 1/2$.
Let A' be the first vertex of the square approached by L. L must subsequently approach both B' and D'. Suppose it approaches B' first. Let B be the first point on L with $BB' \le 1/2$. We can now divide L into two parts L_1, the path from A_0 to B, and L_2, the path from B to A_n. Take X' to be the point on A'D' closest to D' which is approached by L_1. Let X be the corresponding point on L_1. Now every point on X'D' must be approached by L_2 (and X'D' is non-empty, because we know that D' is approached by L but not by L_1). So by compactness X' itself must be approached by L_2. Take Y to be the corresponding point on L_2. $XY \le XX' + X'Y \le 1/2 + 1/2 = 1$. Also $BB' \le 1/2$, so $XB \ge X'B' - XX' - BB' \ge X'B' - 1 \ge A'B' - 1 = 99$. Similarly $YB \ge 99$, so the path $XY \ge 198$.

IMO 1983

Problem A1

Find all functions f defined on the set of positive reals which take positive real values and satisfy:
$f(x(f(y)) = yf(x)$ for all x, y; and $f(x) \to 0$ as $x \to \infty$.

Solution

If $f(k) = 1$, then $f(x) = f(xf(k)) = kf(x)$, so $k =1$. Let $y = 1/f(x)$ and set $k = xf(y)$, then $f(k) = f(xf(y)) = yf(x) = 1$. Hence $f(1) = 1$ and $f(1/f(x)) = 1/x$. Also $f(f(y)) = f(1f(y)) = y$. Hence $f(1/x) = 1/f(x)$. Finally, let $z = f(y)$, so that $f(z) = y$. Then $f(xy) = f(xf(z)) = zf(x) = f(x)f(y)$.
Now notice that $f(xf(x)) = xf(x)$. Let $k = xf(x)$. We show that $k = 1$. $f(k^2) = f(k)f(k) = k^2$ and by a simple induction $f(k^n) = k^n$, so we cannot have $k > 1$, or f(x) would not tend to 0

as x tends to infinity. But f(1/k) = 1/k and the same argument shows that we cannot have 1/k > 1. Hence k = 1.
So the only such function f is f(x) = 1/x.

Problem A2
Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

Solution
Let P_1P_2 and O_1O_2 meet at O. Let OA meet C_2 again at A_2. O is the center of similitude for C_1 and C_2 so $\angle M_1AO_1 = \angle M_2A_2O_2$. Hence if we can show that $\angle M_2AO_2 = \angle M_2A_2O_2$, then we are home.
Let X be the other point of intersection of the two circles. The key is to show that A_2, M_2 and X are collinear, for then $\angle M_2AO_2 = \angle M_2XO_2$ (by reflection) and O_2A_2X is isosceles.
But since O is the center of similitude, M_2A_2 is parallel to M_1A, and by reflection $\angle XM_2O = \angle AM_2O$, so we need to show that triangle AM_1M_2 is isosceles. Extend XA to meet P_1P_2 at Y. Then $YP_1^2 = YA.YX = YP_2^2$, so YX is the perpendicular bisector of M_1M_2, and hence $AM_1 = AM_2$ as required.

Problem A3
Let a , b and c be positive integers, no two of which have a common divisor greater than 1. Show that 2abc - ab - bc - ca is the largest integer which cannot be expressed in the form xbc + yca + zab, where x, y, z are non-negative integers.

Solution
We start with the lemma that bc - b - c is the largest number which cannot be written as mb + nc with m and n non-negative. [Proof: 0, c, 2c, ... , (b-1)c is a complete set of residues mod b. If r > (b-1)c - b, then r = nc (mod b) for some 0 ≤ n ≤ b-1. But r > nc - b, so r = nc + mb for some m ≥ 0. That proves that every number larger than bc - b - c can be written as mb + nc with m and n non-negative. Now consider bc - b - c. It is (b-1)c (mod b), and not congruent to any nc with 0 ≤ n < b-1. So if bc - b - c = mb + nc, then n ≥ b-1. Hence mb + nc ≥ nc ≥ (b-1)c > bc - b - c. Contradiction.]
0, bc, 2bc, ... , (a-1)bc is a complete set of residues mod a. So given N > 2abc - ab - bc - ca we may take xbc = N (mod a) with 0 <= x < a. But N - xbc > 2abc - ab - bc - ca - (a-1)bc = abc - ab - ca = a(bc - b - c). So N - xbc = ka, with k > bc - b - c. Hence we can find non-negative y, z so that k = zb + yc. Hence N = xbc + yca + zab.
Finally, we show that for N = 2abc - ab - bc - ca we cannot find non-negative x, y, z so that N = xbc + yca + zab. N = -bc (mod a), so we must have x = -1 (mod a) and hence x ≥ a-1. Similarly, y ≥ b-1, and z ≥ c-1. Hence xbc + yca + zab ≥ 3abc - ab - bc - ca > N. Contradiction.

Problem B1
Let ABC be an equilateral triangle and E the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

Solution
It does.
Suppose otherwise, that E is the disjoint union of e and e' with no right-angled triangles in either set. Take points X, Y, Z two-thirds of the way along BC, CA, AB respectively (so that BX/BC = 2/3 etc). Then two of X, Y, Z must be in the same set. Suppose X and Y are in e. Now YX is perpendicular to BC, so all points of BC apart from X must be in e'. Take W to be the foot of the perpendicular from Z to BC. Then B and W are in e', so Z must be in e. ZY is perpendicular to AC, so all points of AC apart from Y must be in e'. e' is now far too big. For example let M be the midpoint of BC, then AMC is in e' and right-angled.

Problem B2

Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

Solution

We may notice that an efficient way to build up a set with no APs length 3 is as follows. Having found 2^n numbers in $\{1, 2, \dots , u_n\}$ we add the same pattern starting at $2u_n$, thus giving 2^{n+1} numbers in $\{1, 2, \dots , 3u_n-1\}$. If x is in the first part and y, z in the second part, then 2y is at least $4u_n$, whereas x + z is less than $4u_n$, so x, y, z cannot be an AP length 3. If x and y are in the first part, and z in the second part, then 2y is at most $2u_n$, but x + z is more than $2u_n$, so x, y, z cannot be an AP length 3. To start the process off, we have the 4 numbers 1, 2, 4, 5 in {1, 2, 3, 4, 5}. So $u_2 = 5$. This gives $u_{11} = 88574$, in other words we can find 2048 numbers in the first 88574 with no AP length 3.

If we are lucky, we may notice that if we reduce each number in the set we have constructed by 1 we get the numbers which have no 2 when written base 3. This provides a neater approach. Take x, y, z with no 2 when written in base 3. Then 2y has only 0s and 2s when written base 3. But x + z only has no 1s if x = z. So x, y, z cannot form an AP length 3. Also there are $2^{11} = 2048$ numbers of this type with 11 digits or less and hence ≤ $11111111111_3 = 88573$.

Problem B3

Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \ge 0.$$

Determine when equality occurs.

Solution

Put a = y + z, b = z + x, c = x + y. Then the triangle condition becomes simply x, y, z > 0. The inequality becomes (after some manipulation):

$$xy^3 + yz^3 + zx^3 \ge xyz(x + y + z).$$

Applying Cauchy's inequality we get $(xy^3 + yz^3 + zx^3)(z + x + y) \ge xyz(y + z + x)^2$ with equality iff $xy^3/z = yz^3/x = zx^3/y$. So the inequality holds with equality iff x = y = z. Thus the original inequality holds with equality iff the triangle is equilateral.

IMO 1984

Problem A1

Prove that $0 \le yz + zx + xy - 2xyz \le 7/27$, where x, y and z are non-negative real numbers satisfying x + y + z = 1.

Solution

$(1 - 2x)(1 - 2y)(1 - 2z) = 1 - 2(x + y + z) + 4(yz + zx + xy) - 8xyz = 4(yz + zx + xy) - 8xyz - 1$. Hence $yz + zx + xy - 2xyz = 1/4\ (1 - 2x)(1 - 2y)(1 - 2z) + 1/4$. By the arithmetic/geometric mean theorem $(1 - 2x)(1 - 2y)(1 - 2z) \le ((1 - 2x + 1 - 2y + 1 - 2z)/3)^3 = 1/27$. So $yz + zx + xy - 2xyz \le 1/4\ 28/27 = 7/27$.

Problem A2

Find one pair of positive integers a, b such that ab(a+b) is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

Solution

We find that $(a + b)^7 - a^7 - b^7 = 7ab(a + b)(a^2 + ab + b^2)^2$. So we must find a, b such that $a^2 + ab + b^2$ is divisible by 7^3.

At this point I found a = 18, b = 1 by trial and error.

A more systematic argument turns on noticing that $a^2 + ab + b^2 = (a^3 - b^3)/(a - b)$, so we are looking for a, b with $a^3 = b^3 \pmod{7^3}$. We now remember that $a^{\varphi(m)} = 1 \pmod m$. But $\varphi(7^3) = 2\cdot3\cdot49$, so $a^3 = 1 \pmod{343}$ if $a = n^{98}$. We find $2^{98} = 18\ (343)$, which gives the solution 18, 1.

This approach does not give a flood of solutions. n^{98} = 0, 1, 18, or 324. So the only solutions we get are 1, 18; 18, 324; 1, 324.

Problem A3
Given points O and A in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point X in the plane, the circle C(X) has center O and radius OX + (∠ AOX)/OX, where ∠ AOX is measured in radians in the range [0, 2π). Prove that we can find a point X, not on OA, such that its color appears on the circumference of the circle C(X).

Solution
Suppose the result is false. Let C^1 be any circle center O. Then the locus of points X such that C(X) = C_1 is a spiral from O to the point of intersection of OA and C_1. Every point of this spiral must be a different color from all points of the circle C_1. Hence every circle center O with radius smaller than C_1 must include a point of different color to those on C_1. Suppose there are n colors. Then by taking successively smaller circles C_2, C_3, ... , C_{n+1} we reach a contradiction, since each circle includes a point of different color to those on any of the larger circles.

Problem B1
Let ABCD be a convex quadrilateral with the line CD tangent to the circle on diameter AB. Prove that the line AB is tangent to the circle on diameter CD if and only if BC and AD are parallel.

Solution
If AB and CD are parallel, then AB is tangent to the circle on diameter CD if and only if AB = CD and hence if and only if ABCD is a parallelogram. So the result is true.
Suppose then that AB and DC meet at O. Let M be the midpoint of AB and N the midpoint of CD. Let S be the foot of the perpendicular from N to AB, and T the foot of the perpendicular fromM to CD. We are given that MT = MA. OMT, ONS are similar, so OM/MT = ON/NS and hence OB/OA = (ON - NS)/(ON + NS). So AB is tangent to the circle on diameter CD if and only if OB/OA = OC/OD which is the condition for BC to be parallel to AD.

Problem B2
Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n > 3 vertices. Let p be its perimeter. Prove that:
$n - 3 < 2d/p < [n/2] [(n+1)/2] - 2$, where [x] denotes the greatest integer not exceeding x.

Solution
Given any diagonal AX, let B be the next vertex counterclockwise from A, and Y the next vertex counterclockwise from X. Then the diagonals AX and BY intersect at K. AK + KB > AB and XK + KY > XY, so AX + BY > AB + XY. Keeping A fixed and summing over X gives n - 3 cases. So if we then sum over A we get every diagonal appearing four times on the lhs and every side appearing 2(n-3) times on the rhs, giving 4d > 2(n-3)p.
Denote the vertices as A_0, ... , A_{n-1} and take subscripts mod n. The ends of a diagonal AX are connected by r sides and n-r sides. The idea of the upper limit is that its length is less than the sum of the shorter number of sides. Evaluating it is slightly awkward.
We consider n odd and n even separately. Let n = 2m+1. For the diagonal A_iA_{i+r} with r ≤ m, we have $A_iA_{i+r} \le A_iA_{i+2} + \dots + A_iA_{i+r}$. Summing from r = 2 to m gives for the rhs $(m-1)A_iA_{i+1} + (m-1)A_{i+1}A_{i+2} + (m-2)A_{i+2}A_{i+3} + (m-3)A_{i+3}A_{i+4} + \dots + 1.A_{i+m-1}A_{i+m}$. Now summing over i gives d for the lhs and $p((m-1) + (1 + 2 + \dots + m-1)) = p((m^2 + m - 2)/2)$ for the rhs. So we get $2d/p \le m^2 + m - 2 = [n/2] [(n+1)/2] - 2$.
Let n = 2m. As before we have $A_iA_{i+r} <= A_iA_{i+2} + \dots + A_iA_{i+r}$ for 2 ≤ r ≤ m-1. We may also take $A_iA_{i+m} \le p/2$. Summing as in the even case we get $2d/p = m^2 - 2 = [n/2] [(n+1)/2] - 2$.

Problem B3
Let a, b, c, d be odd integers such that 0 < a < b < c < d and ad = bc. Prove that if a + d = 2^k and b + c = 2^m for some integers k and m, then a = 1.

Solution
$a < c$, so $a(d - c) < c(d - c)$ and hence $bc - ac < c(d - c)$. So $b - a < d - c$, or $a + d > b + c$, so $k > m$.
$bc = ad$, so $b(2^m - b) = a(2^k - a)$. Hence $b^2 - a^2 = 2^m(b - 2^{k-m}a)$. But $b^2 - a^2 = (b + a)(b - a)$, and $(b + a)$ and $(b - a)$ cannot both be divisible by 4 (since a and b are odd), so 2^{m-1} must divide $b + a$ or $b - a$. But if it divides $b - a$, then $b - a \ge 2^{m-1}$, so b and $c > 2^{m-1}$ and $b + c > 2^m$. Contradiction. Hence 2^{m-1} divides $b + a$. If $b + a \ge 2^m = b + c$, then $a \ge c$. Contradiction. Hence $b + a = 2^{m-1}$.
So we have $b = 2^{m-1} - a$, $c = 2^{m-1} + a$, $d = 2^k - a$. Now using $bc = ad$ gives: $2^k a = 2^{2m-2}$. But a is odd, so $a = 1$.

IMO 1985

Problem A1
A circle has center on the side AB of the cyclic quadrilateral ABCD. The other three sides are tangent to the circle. Prove that AD + BC = AB.

Solution
Let the circle touch AD, CD, BC at L, M, N respectively. Take X on the line AD on the same side of A as D, so that AX = AO, where O is the center of the circle. Now the triangles OLX and OMC are congruent: OL = OM = radius of circle, $\angle OLX = \angle OMC = 90^o$, and $\angle OXL = 90^o - A/2 = (180^o - A)/2 = C/2$ (since ABCD is cyclic) $= \angle OCM$. Hence LX = MC. So OA = AL + MC. Similarly, OB = BN + MD. But MC = CN and MD = DL (tangents have equal length), so AB = OA + OB = AL + LD + CN + NB = AD + BC.

Problem A2
Let n and k be relatively prime positive integers with $k < n$. Each number in the set $M = \{1, 2, 3, \dots, n-1\}$ is colored either blue or white. For each i in M, both i and n-i have the same color. For each i in M not equal to k, both i and |i-k| have the same color. Prove that all numbers in M must have the same color.

Solution
n and k are relatively prime, so 0, k, 2k, ... , (n-1)k form a complete set of residues mod n. So k, 2k, ... , (n-1)k are congruent to the numbers 1, 2, ... , n-1 in some order. Suppose ik is congruent to r and (i+1)k is congruent to s. Then either $s = r + k$, or $s = r + k - n$. If $s = r + k$, then we have immediately that $r = s - k$ and s have the same color. If $s = r + k - n$, then $r = n - (k - s)$, so r has the same color as $k - s$, and $k - s$ has the same color as s. So in any case r and s have the same color. By giving i values from 1 to n-2 this establishes that all the numbers have the same color.

Problem A3
For any polynomial $P(x) = a_0 + a_1x + \dots + a_kx^k$ with integer coefficients, the number of odd coefficients is denoted by $o(P)$. For i = 0, 1, 2, ... let $Q_i(x) = (1 + x)^i$. Prove that if $i_1, i_2, \dots, i_n$ are integers satisfying $0 \le i_1 < i_2 < \dots < i_n$, then:
$o(Q_{i1} + Q_{i2} + \dots + Q_{in}) \ge o(Q_{i1})$.

Solution
If i is a power of 2, then all coefficients of Q_i are even except the first and last. [There are various ways to prove this. Let iCr denote the rth coefficient, so iCr = i!/(r!(i-r)!). Suppose $0 < r < i$. Then iCr = i-1Cr-1 i/r, but i-1Cr-1 is an integer and i is divisible by a higher power of 2 than r, hence iCr is even.]
Let $Q = Q_{i1} + \dots + Q_{in}$. We use induction on i_n. If $i_n = 1$, then we must have $n = 2$, $i_1 = 0$, and $i_2 = 1$, so $Q = 2 + x$, which has the same number of odd coefficients as $Q_{i1} = 1$. So suppose it is true for smaller values of i_n. Take m a power of 2 so that $m \le i_n < 2m$. We consider two cases $i_1 \ge m$ and $i_1 < m$.
Consider first $i_1 \ge m$. Then $Q_{i1} = (1 + x)^m A$, $Q = (1 + x)^m B$, where A and B have degree less than m. Moreover, A and B are of the same form as Q_{i1} and Q, (all the i_js are reduced by m, so we have $o(A) \le o(B)$ by induction. Also $o(Q_{i1}) = o((1 + x)^m A) = o(A + x^m A) = 2o(A) \le 2o(B) = o(B + x^m B) = o((1 + x)^m B) = o(Q)$, which establishes the result for i_n.

It remains to consider the case $i_1 < m$. Take r so that $i_r < m$, $i_{r+1} > m$. Set $A = Q_{i1} + \dots + Q_{ir}$, $(1 + x)^m B = Q_{ir+1} + \dots + Q_{in}$, so that A and B have degree < m. Then $o(Q) = o(A + (1 + x)^m B) = o(A + B + x^m B) = o(A + B) + o(B)$. Now $o(A - B) + o(B) >= o(A - B + B) = o(A)$, because a coefficient of A is only odd if just one of the corresponding coefficients of A - B and B is odd. But $o(A - B) = o(A + B)$, because corresponding coefficients of A - B and A + B are either equal or of the same parity. Hence $o(A + B) + o(B) \ge o(A)$. But $o(A) \ge o(Q_{ii})$ by induction. So we have established the result for i_n.

Problem B1

Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that M contains a subset of 4 elements whose product is the 4th power of an integer.

Solution

Suppose we have a set of at least 3.2^n+1 numbers whose prime divisors are all taken from a set of n. So each number can be written as $p_1^{r_1} \dots p_n^{r_n}$ for some non-negative integers r_i, where p_i is the set of prime factors common to all the numbers. We classify each r_i as even or odd. That gives 2^n possibilities. But there are more than $2^n + 1$ numbers, so two numbers have the same classification and hence their product is a square. Remove those two and look at the remaining numbers. There are still more than $2^n + 1$, so we can find another pair. We may repeat to find $2^n + 1$ pairs with a square product. [After removing 2^n pairs, there are still $2^n + 1$ numbers left, which is just enough to find the final pair.] But we may now classify these pairs according to whether each exponent in the square root of their product is odd or even. We must find two pairs with the same classification. The product of these four numbers is now a fourth power.

Applying this to the case given, there are 9 primes less than or equal to 23 (2, 3, 5, 7, 11, 13, 17, 19, 23), so we need at least 3.512 + 1 = 1537 numbers for the argument to work (and we have 1985).

The key is to find the 4th power in two stages, by first finding lots of squares. If we try to go directly to a 4th power, this type of argument does not work (we certainly need more than 5 numbers to be sure of finding four which sum to 0 mod 4, and 5^9 is far too big).

Problem B2

A circle center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. The circumcircles of ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

Solution

The three radical axes of the three circles taken in pairs, BM, NK and AC are concurrent. Let X be the point of intersection. [They cannot all be parallel or B and M would coincide.] The first step is to show that XMNC is cyclic. The argument depends slightly on how the points are arranged. We may have: $\angle XMN = 180^o - \angle BMN = \angle BKN = 180^o - \angle AKN = \angle ACN = 180^o - \angle XCN$, or we may have $\angle XMN = 180^o - \angle BMN = 180^o - \angle BKN = \angle AKN = 180^o - \angle ACN = 180^o - \angle XCN$.

Now $XM.XB = XK.XN = XO^2 - ON^2$. $BM \cdot BX = BN \cdot BC = BO^2 - ON^2$, so $XM \cdot XB - BM \cdot BX = XO^2 - BO^2$. But $XM \cdot XB - BM \cdot BX = XB(XM - BM) = (XM + BM)(XM - BM) = XM^2 - BM^2$. So $XO^2 - BO^2 = XM^2 - BM^2$. Hence OM is perpendicular to XB, or $\angle OMB = 90^o$.

Problem B3

For every real number x_1, construct the sequence $x_1, x_2, \dots$ by setting:

$$x_{n+1} = x_n(x_n + 1/n).$$

Prove that there exists exactly one value of x_1 which gives $0 < x_n < x_{n+1} < 1$ for all n.

Solution

Define $S_0(x) = x$, $S_n(x) = S_{n-1}(x) (S_{n-1}(x) + 1/n)$. The motivation for this is that $x_n = S_{n-1}(x_1)$.

$S_n(0) = 0$ and $S_n(1) > 1$ for all $n > 1$. Also $S_n(x)$ has non-negative coefficients, so it is strictly increasing in the range [0,1]. Hence we can find (unique) solutions a_n, b_n to $S_n(a_n) = 1 - 1/n$, $S_n(b_n) = 1$.
$S_{n+1}(a_n) = S_n(a_n)\ (S_n(a_n) + 1/n) = 1 - 1/n > 1 - 1/(n+1)$, so $a_n < a_{n+1}$. Similarly, $S_{n+1}(b_n) = S_n(b_n)\ (S_n(b_n) + 1/n) = 1 + 1/n > 1$, so $b_n > b_{n+1}$. Thus a_n is an increasing sequence and b_n is a decreasing sequence with all a_n less than all b_n. So we can certainly find at least one point x_1 which is greater than all the a_n and less than all the b_n. Hence $1 - 1/n < S_n(x_1) < 1$ for all n. But $S_n(x_1) = x_{n+1}$. So $x_{n+1} < 1$ for all n. Also $x_n > 1 - 1/n$ implies that $x_{n+1} = x_n(x_n + 1/n) > x_n$. Finally, we obviously have $x_n > 0$. So the resulting series x_n satisfies all the required conditions.
It remains to consider uniqueness. Suppose that there is an x_1 satisfying the conditions given. Then we must have $S_n(x_1)$ lying in the range $1 - 1/n$, 1 for all n. [The lower limit follows from $x_{n+1} = x_n(x_n + 1/n)$.] Hence we must have $a_n < x_1 < b_n$ for all n. We show uniqueness by showing that $b_n - a_n$ tends to zero as n tends to infinity. Since all the coefficients of $S_n(x)$ are non-negative, it is has increasing derivative. $S_n(0) = 0$ and $S_n(b_n) = 1$, so for any x in the range 0, b_n we have $S_n(x) \le x/b_n$. In particular, $1 - 1/n < a_n/b_n$. Hence $b_n - a_n \le b_n - b_n(1 - 1/n) = b_n/n < 1/n$, which tends to zero.

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Problem A1
Let d be any positive integer not equal to 2, 5 or 13. Show that one can find distinct a, b in the set {2, 5, 13, d} such that ab - 1 is not a perfect square.

Solution
Consider residues mod 16. A perfect square must be 0, 1, 4 or 9 (mod 16). d must be 1, 5, 9, or 13 for 2d - 1 to have one of these values. However, if d is 1 or 13, then 13d - 1 is not one of these values. If d is 5 or 9, then 5d - 1 is not one of these values. So we cannot have all three of 2d - 1, 5d - 1, 13d - 1 perfect squares.

Problem A2
Given a point P_0 in the plane of the triangle $A_1A_2A_3$. Define $A_s = A_{s-3}$ for all $s \ge 4$. Construct a set of points $P_1, P_2, P_3, \ldots$ such that P_{k+1} is the image of P_k under a rotation center A_{k+1} through 120^o clockwise for $k = 0, 1, 2, \ldots$. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

Solution
The product of three successive rotations about the three vertices of a triangle must be a translation (see below). But that means that P_{1986} (which is the result of 662 such operations, since 1986 = 3 x 662) can only be P_0 if it is the identity, for a translation by a non-zero amount would keep moving the point further away. It is now easy to show that it can only be the identity if the triangle is equilateral. Take a circle center A_1, radius A_1A_2 and take P on the circle so that a 120^o clockwise rotation about A_1 brings P to A_2. Take a circle center A_3, radius A_3A_2 and take Q on the circle so that a 120^o clockwise rotation about A_3 takes A_2 to Q. Then successive 120^o clockwise rotations about A_1, A_2, A_3 take P to Q. So if these three are equivalent to the identity we must have P = Q. Hence $A_1A_2A_3 = A_1A_2P + A_3A_2P = 30^o + 30^o = 60^o$. Also $A_2P = 2A_1A_2\cos 30^o$ and $= 2A_2A_3\cos 30^o$. Hence $A_1A_2 = A_2A_3$. So $A_1A_2A_2$ is equilateral. Note in passing that it is not sufficient for the triangle to be equilateral. We also have to take the rotations in the right order. If we move around the vertices the opposite way, then we get a net translation.
It remains to show that the three rotations give a translation. Define rectangular coordinates (x, y) by taking A_1 to be the origin and A_2 to be (a, b). Let A_3 be (c, d). A clockwise rotation through 120 degrees about the origin takes (x, y) to $(-x/2 + y\sqrt{3}/2, -x\sqrt{3}/2 - y/2)$. A clockwise rotation through 120 degrees about some other point (e, f) is obtained by subtracting (e, f) to get (x - e, y - f), the coordinates relative to (e, f), then rotating, then adding (e, f) to get the coordinates relative to (0, 0). Thus after the three rotations we will end up with a linear combination of x's, y's, a's, b's, c's and d's for each coordinate. But the linear combination of x's and y's must be just x for the x-coordinate and y for the y-coordinate, since three successive 120 degree rotations about the same

point is the identity. Hence we end up with simply (x + constant, y + constant), in other words, a translation.
[Of course, there is nothing to stop you actually carrying out the computation. It makes things slightly easier to take the triangle to be (0, 0), (1, 0), (a, b). The net result turns out to be (x, y) goes to $(x + 3a/2 - b\sqrt{3}/2, y - \sqrt{3} + a\sqrt{3}/2 + 3b/2)$. For this to be the identity requires $a = 1/2$, $b = \sqrt{3}/2$. So the third vertex must make the triangle equilateral (and it must be on the correct side of the line joining the other two). This approach avoids the need for the argument in the first paragraph above, but is rather harder work.]

Problem A3
To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

Solution
Let S be the sum of the absolute value of each set of adjacent vertices, so if the integers are a, b, c, d, e, then $S = |a| + |b| + |c| + |d| + |e| + |a + b| + |b + c| + |c + d| + |d + e| + |e + a| + |a + b + c| + |b + c + d| + |c + d + e| + |d + e + a| + |e + a + b| + |a + b + c + d| + |b + c + d + e| + |c + d + e + a| + |d + e + a + b| + |e + a + b + c| + |a + b + c + d + e|$. Then the operation reduces S, but S is a greater than zero, so the process must terminate in a finite number of steps. So see that S is reduced, we can simply write out all the terms. Suppose the integers are a, b, c, d, e before the operation, and a+b, -b, b+c, d, e after it. We find that we mostly get the same terms before and after (although not in the same order), so that the sum S' after the operation is $S - |a + c + d + e| + |a + 2b + c + d + e|$. Certainly $a + c + d + e > a + 2b + c + d + e$ since b is negative, and $a + c + d + e > -(a + 2b + c + d + e)$ because $a + b + c + d + e > 0$.

Problem B1
Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) with center O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, with X remaining inside the polygon. Find the locus of X.

Solution
Take AB = 2 and let M be the midpoint of AB. Take coordinates with origin at A, x-axis as AB and y-axis directed inside the n-gon. Let Z move along AB from B towards A. Let YZA be t. Let the coordinates of X be (x, y). YZX = $\pi/2 - \pi/n$, so $XZ = 1/\sin \pi/n$ and $y = XZ \sin(t + \pi/2 - \pi/n) = \sin t + \cot \pi/n \cos t$.
$BY \sin 2\pi/n = YZ \sin t = 2 \sin t$. $MX = \cot \pi/n$. So $x = MY \cos t - BY \cos 2\pi/n + MX \sin t = \cos t + (\cot \pi/n - 2 \cot 2\pi/n) \sin t = \cos t + \tan \pi/n \sin t = y \tan \pi/n$. Thus the locus of X is a star formed of n lines segments emanating from O. X moves out from O to the tip of a line segement and then back to O, then out along the next segment and so on. $x^2 + y^2 = (1/\sin^2\pi/n + 1/\cos^2\pi/n) \cos^2(t + \pi/n)$. Thus the length of each segment is $(1 - \cos \pi/n)/(\sin \pi/n \cos \pi/n)$.

Problem B2
Find all functions f defined on the non-negative reals and taking non-negative real values such that: $f(2) = 0$, $f(x) \neq 0$ for $0 \leq x < 2$, and $f(xf(y)) f(y) = f(x + y)$ for all x, y.

Solution
$f(x+2) = f(xf(2)) f(2) = 0$. So $f(x) = 0$ for all $x \geq 2$.
$f(y) f((2-y)f(y)) = f(2) = 0$. So if $y < 2$, then $f((2-y) f(y)) = 0$ and hence $(2 - y) f(y) \geq 2$, or $f(y) \geq 2/(2 - y)$.
Suppose that for some y_0 we have $f(y_0) > 2/(2 - y_0)$, then we can find $y_1 > y_0$ (and $y_1 < 2$) so that $f(y_0) = 2/(2 - y_1)$. Now let $x_1 = 2 - y_1$. Then $f(x_1f(y_0)) = f(2) = 0$, so $f(x_1 + y_0) = 0$. But $x_1 + y_0 < 2$. Contradiction. So we must have $f(x) = 2/(2 - x)$ for all $x < 2$.

We have thus established that if a function f meets the conditions then it must be defined as above. It remains to prove that with this definition f does meet the conditions. Clearly $f(2) = 0$ and $f(x)$ is non-zero for $0 \le x < 2$. $f(xf(y)) = f(2x/(2 - y))$. If $2x/(2 - y) \ge 2$, then $f(xf(y)) = 0$. But it also follows that $x + y \ge 2$, and so $f(x + y) = 0$ and hence $f(xf(y)) f(y) = f(x + y)$ as required. If $2x/(2 - y) < 2$, then $f(xf(y)) f(y) = 2/(2 - 2x/(2-y)) 2/(2 - y) = 2/(2 - x - y) = f(x + y)$. So the unique function satisfying the conditions is:

$f(x) = 0$ for $x \ge 2$, and $2/(2 - x)$ for $0 \le x < 2$.

Problem B3

Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line L parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on L is not greater than 1?

Solution

Answer: yes.

We prove the result by induction on the number n of points. It is clearly true for n = 1. Suppose it is true for all numbers less than n. Pick an arbitrary point P and color it red. Now take a point in the same row and color it white. Take a point in the same column as the new point and color it red. Continue until either you run out of eligible points or you pick a point in the same column as P. The process must terminate because there are only finitely many points. Suppose the last point picked is Q. Let S be the set of points picked.

If Q is in the same column as P, then it is colored white (because the "same row" points are all white, and the "same column" points are all red). Now every row and column contains an equal number of red points of S and of white points of S. By induction we can color the points excluding those in S, then the difference between the numbers of red and white points in each row and column will be unaffected by adding the points in S and so we will have a coloring for the whole set. This completes the induction for the case where Q is in the same column as P.

If it is not, then continue the path backwards from P. In other words, pick a point in the same column as P and color it white. Then pick a point in the same row as the new point and color it red and so on. Continue until either you run out of eligible points or you pick a point to pair with Q. If Q was picked as being in the same row as its predecessor, this means a point in the same column as Q; if Q was picked as being in the same column as its predecessor, this means a point in the same row as Q. Again the process must terminate. Suppose the last point picked is R. Let S be the set of all points picked.

If R pairs with Q, then we can complete the coloring by induction as before. Suppose S does not pair with Q. Then there is a line (meaning a row or column) containing Q and no uncolored points. There is also a line containing R and no uncolored points. These two lines have an excess of one red or one white. All other lines contain equal number of red and white points of S. Now color the points outside S by induction. This gives a coloring for the whole set, because no line with a color excess in S has any points outside S. So we have completed the induction.

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Problem A1

1. Let $p_n(k)$ be the number of permutations of the set {1, 2, 3, ... , n} which have exactly k fixed points. Prove $\sum_0^n (k\, p_n(k)) = n!$.

Solution

We show first that the number of permutations of n objects with no fixed points is $n!(1/0! - 1/1! + 1/2! - \dots + (-1)^n/n!)$. This follows immediately from the law of inclusion and exclusion: let N_i be the number which fix i, N_{ij} the number which fix i and j, and so on. Then N_0, the number with no fixed points, is $n!$ - all N_i + all $N_{ij} - \dots + (-1)^n N_{1\dots n}$. But $N_i = (n-1)!$, $N_{ij} = (n-2)!$ and so on. So $N_0 = n! (1 - 1/1! + \dots + (-1)^r(n-r)!/(r! (n-r)!) + \dots + (-1)^n/n!) = n! (1/0! - 1/1! + \dots + (-1)^n/n!)$.

Hence the number of permutations of n objects with exactly r fixed points = no. of ways of choosing the r fixed points x no. of perms of the remaining n - r points with no fixed points

= $n!/(r! (n-r)!) \times (n-r)! (1/0! - 1/1! + \dots + (-1)^{n-r}/(n-r)!)$. Thus we wish to prove that the sum from r = 1 to n of $1/(r-1)! (1/0! - 1/1! + \dots + (-1)^{n-r}/(n-r)!)$ is 1. We use induction on n. It is true for n = 1. Suppose it is true for n. Then the sum for n+1 less the sum for n is: $1/0! (-1)^n/n! + 1/1! (-1)^{n-1}/(n-1)! + \dots + 1/n! 1/0! = 1/n! (1 - 1)^n = 0$. Hence it is true for n + 1, and hence for all n.

Problem A2
In an acute-angled triangle ABC the interior bisector of angle A meets BC at L and meets the circumcircle of ABC again at N. From L perpendiculars are drawn to AB and AC, with feet K and M respectively. Prove that the quadrilateral AKNM and the triangle ABC have equal areas.

Solution
AKL and AML are congruent, so KM is perpendicular to AN and area AKNM = KM.AN/2.
AKLM is cyclic (2 opposite right angles), so angle AKM = angle ALM and hence KM/sin BAC = AM/sin AKM (sine rule) = AM/sin ALM = AL.
ABL and ANC are similar, so AB.AC = AN.AL. Hence area ABC = 1/2 AB.AC sin BAC = 1/2 AN.AL sin BAC = 1/2 AN.KM = area AKNM.

Problem A3
Let $x_1, x_2, \dots , x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \dots + x_n^2 = 1$. Prove that for every integer $k \ge 2$ there are integers $a_1, a_2, \dots , a_n$, not all zero, such that $|a_i| \le k - 1$ for all i, and $|a_1x_1 + a_2x_2 + \dots + a_nx_n| \le (k - 1)\sqrt{n}/(k^n - 1)$.

Solution
This is an application of the pigeon-hole principle.
Assume first that all x_i are non-negative. Observe that the sum of the x_i is at most $\sqrt{n}$. Consider the k^n possible values of $\sum_{1\le i\le n} b_ix_i$, where each b_i is an integer in the range [0,k-1]. Each value must lie in the interval $[0, k-1 \sqrt{n}]$. Divide this into k^n-1 equal subintervals. Two values must lie in the same subinterval. Take their difference. Its coefficients are the required a_i. Finally, if any x_i are negative, solve for the absolute values and then flip signs in the a_i.

Problem B1
Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for all n.

Solution
We prove that if $f(f(n)) = n + k$ for all n, where k is a fixed positive integer, then k must be even. If k = 2h, then we may take $f(n) = n + h$.
Suppose $f(m) = n$ with $m = n \pmod k$. Then by an easy induction on r we find $f(m + kr) = n + kr$, $f(n + kr) = m + k(r+1)$. We show this leads to a contradiction. Suppose $m < n$, so $n = m + ks$ for some $s > 0$. Then $f(n) = f(m + ks) = n + ks$. But $f(n) = m + k$, so $m = n + k(s - 1) \ge n$. Contradiction. So we must have $m \ge n$, so $m = n + ks$ for some $s \ge 0$. But now $f(m + k) = f(n + k(s+1)) = m + k(s + 2)$. But $f(m + k) = n + k$, so $n = m + k(s + 1) > n$. Contradiction.
So if $f(m) = n$, then m and n have different residues mod k. Suppose they have r_1 and r_2 respectively. Then the same induction shows that all sufficiently large $s = r_1 \pmod k$ have $f(s) = r_2 \pmod k$, and that all sufficiently large $s = r_2 \pmod k$ have $f(s) = r_1 \pmod k$. Hence if m has a different residue r mod k, then f(m) cannot have residue r_1 or r_2. For if f(m) had residue r_1, then the same argument would show that all sufficiently large numbers with residue r_1 had $f(m) = r \pmod k$. Thus the residues form pairs, so that if a number is congruent to a particular residue, then f of the number is congruent to the pair of the residue. But this is impossible for k odd.

Problem B2
Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of 3 points determines a non-degenerate triangle with rational area.

Solution
Let x_n be the point with coordinates (n, n^2) for $n = 1, 2, 3, \dots$. We show that the distance between any two points is irrational and that the triangle determined by any 3 points has non-zero rational area.
Take $n > m$. $|x_n - x_m|$ is the hypoteneuse of a triangle with sides $n - m$ and $n^2 - m^2 = (n - m)(n + m)$. So $|x_n - x_m| = (n - m)\sqrt{(1 + (n+m)^2)}$. Now $(n + m)^2 < (n + m)^2 + 1 < (n + m + 1)^2 = (n + m)^2 + 1 + 2(n + m)$, so $(n + m)^2 + 1$ is not a perfect square. Hence its square root is irrational. [For this we may use the classical argument. Let N' be a non-square and suppose $\sqrt{N'}$ is rational. Since N' is a non-square we must be able to find a prime p such that p^{2a+1} divides N' but p^{2a+2} does not divide N' for some $a \geq 0$. Define $N = N'/p^{2a}$. Then $\sqrt{N} = (\sqrt{N'})/p^a$, which is also rational. So we have a prime p such that p divides N, but p^2 does not divide N. Take $\sqrt{N} = r/s$ with r and s relatively prime. So $s^2N = r^2$. Now p must divide r, hence p^2 divides r^2 and so p divides s^2. Hence p divides s. So r and s have a common factor. Contradiction. Hence non-squares have irrational square roots.]
Now take $a < b < c$. Let B be the point (b, a^2), C the point (c, a^2), and D the point (c, b^2). Area $x_ax_bx_c$ = area x_ax_cC - area x_ax_bB - area x_bx_cD - area $x_bDCB = (c - a)(c^2 - a^2)/2 - (b - a)(b^2 - a^2)/2 - (c - b)(c^2 - b^2)/2 - (c - b)(b^2 - a^2)$ which is rational.

Problem B3
Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{(n/3)}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

Solution
First observe that if m is relatively prime to $b + 1, b + 2, \dots , 2b - 1, 2b$, then it is not divisible by any number less than 2b. For if $c <= b$, then take the largest $j \geq 0$ such that $2^jc \leq b$. Then $2^{j+1}c$ lies in the range $b + 1, \dots , 2b$, so it is relatively prime to m. Hence c is also. If we also have that $(2b + 1)^2 > m$, then we can conclude that m must be prime, since if it were composite it would have a factor $\leq \sqrt{m}$.
Let $n = 3r^2 + h$, where $0 \leq h < 6r + 3$, so that r is the greatest integer less than or equal to $\sqrt{(n/3)}$. We also take $r \geq 1$. That excludes the value $n = 2$, but for $n = 2$, the result is vacuous, so nothing is lost.
Assume that $n + k(k+1)$ is prime for $k = 0, 1, \dots , r$. We show by induction that $N = n + (r + s)(r + s + 1)$ is prime for $s = 1, 2, \dots , n - r - 2$. By the observation above, it is sufficient to show that $(2r + 2s + 1)^2 > N$, and that N is relatively prime to all of $r + s + 1, r + s + 2, \dots , 2r + 2s$. We have $(2r + 2s + 1)^2 = 4r^2 + 8rs + 4s^2 + 4r + 4s + 1$. Since $r, s \geq 1$, we have $4s + 1 > s + 2$, $4s^2 > s^2$, and $6rs > 3r$. Hence $(2r + 2s + 1)^2 > 4r^2 + 2rs + s^2 + 7r + s + 2 = 3r^2 + 6r + 2 + (r + s)(r + s + 1) >= N$.
Now if N has a factor which divides $2r - i$ with i in the range $-2s$ to $r - s - 1$, then so does $N - (i + 2s + 1)(2r - i) = n + (r - i - s - 1)(r - i - s)$ which has the form $n + s'(s'+1)$ with s' in the range 0 to $r + s - 1$. But $n + s'(s' + 1)$ is prime by induction (or absolutely for $s = 1$), so the only way it can have a factor in common with $2r - i$ is if it divides $2r - i$. But $2r - i \leq 2r + 2s \leq 2n - 4 < 2n$ and $n + s'(s' + 1) \geq n$, so if $n + s'(s' + 1)$ has a factor in common with $2r - i$, then it equals $2r - i = s + r + 1 + s'$. Hence $s'^2 = s - (n - r - 1) < 0$, which is not possible. So we can conclude that N is relatively prime to all of $r + s + 1, \dots , 2r + 2s$ and hence prime.

IMO 1988

Problem A1
Consider two coplanar circles of radii $R > r$ with the same center. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular to BP at P meets the smaller circle again at A (if it is tangent to the circle at P, then $A = P$).
(i) Find the set of values of $AB^2 + BC^2 + CA^2$.
(ii) Find the locus of the midpoint of BC.

Solution

(i) Let M be the midpoint of BC. Let PM = x. Let BC meet the small circle again at Q. Let O be the center of the circles. Since angle APQ = 90 degrees, AQ is a diameter of the small circle, so its length is 2r. Hence $AP^2 = 4r^2 - 4x^2$. $BM^2 = R^2 - OM^2 = R^2 - (r^2 - x^2)$. That is essentially all we need, because we now have: $AB^2 + AC^2 + BC^2 = (AP^2 + (BM - x)^2) + (AP^2 + (BM + x)^2) + 4BM^2 = 2AP^2 + 6BM^2 + 2x^2 = 2(4r^2 - 4x^2) + 6(R^2 - r^2 + x^2) + 2x^2 = 6R^2 + 2r^2$, which is independent of x.
(ii) M is the midpoint of BC and PQ since the circles have a common center. If we shrink the small circle by a factor 2 with P as center, then Q moves to M, and hence the locus of M is the circle diameter OP.

Problem A2
Let n be a positive integer and let $A_1, A_2, \dots , A_{2n+1}$ be subsets of a set B. Suppose that:
(i) Each A_i has exactly 2n elements,
(ii) The intersection of every two distinct A_i contains exactly one element, and
(iii) Every element of B belongs to at least two of the A_i.
For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that A_i has 0 assigned to exactly n of its elements?

Solution
Answer: n even.
Each of the 2n elements of A_i belongs to at least one other A_j because of (iii). But given another A_j it cannot contain more than one element of A_i because of (ii). There are just 2n other A_j available, so each must contain exactly one element of A_i. Hence we can strengthen (iii) to every element of B belongs to exactly two of the As.
This shows that the arrangement is essentially unique. We may call the element of B which belongs to A_i and A_j (i,j). Then A_i contains the 2n elements (i, j) with j not i.
|B| = 1/2 x no. of As x size of each A = n(2n+1). If the labeling with 0s and 1s is possible, then if we list all the elements in each A, n(2n+1) out of the 2n(2n+1) elements have value 0. But each element appears twice in this list, so n(2n+1) must be even. Hence n must be even.

Problem A3
A function f is defined on the positive integers by: f(1) = 1; f(3) = 3; f(2n) = f(n), f(4n + 1) = 2f(2n + 1) - f(n), and f(4n + 3) = 3f(2n + 1) - 2f(n) for all positive integers n. Determine the number of positive integers n less than or equal to 1988 for which f(n) = n.

Solution
Answer: 92.
f(n) is always odd. If $n = b_{r+1}b_r\dots b_2b_1b_0$ in binary and n is odd, so that $b_{r+1} = b_0 = 1$, then $f(n) = b_{r+1}b_1b_2\dots b_rb_0$. If n has r+2 binary digits with r > 0, then there are $2^{[(r+1)/2]}$ numbers with the central r digits symmetrical, so that f(n) = n (because we can choose the central digit and those lying before it arbitarily, the rest are then determined). Also there is one number with 1 digit (1) and one number with two digits (3) satisfying f(n) = 1. So we find a total of 1 + 1 + 2 + 2 + 4 + 4 + 8 + 8 + 16 + 16 = 62 numbers in the range 1 to 1023 with f(n) = n. 1988 = 11111000011. So we also have all 32 numbers in the range 1023 to 2047 except for 11111111111 and 11111011111, giving another 30, or 92 in total.
It remains to prove the assertions above. f(n) odd follows by an easy induction. Next we show that if $2^m < 2n+1 < 2^{m+1}$, then $f(2n+1) = f(n) + 2^m$. Again we use induction. It is true for m = 1 (f(3) = f(1) + 2). So suppose it is true for 1, 2, ... , m. Take 4n+1 so that $2^{m+1} < 4n+1 < 2^{m+2}$, then $f(4n+1) = 2f(2n+1) - f(n) = 2(f(n) + 2^m) - f(n) = f(n) + 2^{m+1} = f(2n) + 2^{m+1}$, so it is true for 4n+1. Similarly, if 4n+3 satisfies, $2^{m+1} < 4n+3 < 2^{m+2}$, then $f(4n+3) = 3f(2n+1) - 2f(n) = f(2n+1) + 2(f(n) + 2^m) - 2f(n) = f(2n+1) + 2^{m+1}$, so it is true for 4n+3 and hence for m+1.
Finally, we prove the formula for f(2n+1). Let $2n+1 = b_{r+1}b_r\dots b_2b_1b_0$ with $b_0 = b_{r+1} = 1$. We use induction on r. So assume it is true for smaller values. Say $b_1 = \dots = b_s = 0$ and $b_{s+1} = 1$ (we may have s = 0, so that we have simply $b_1 = 1$). Then $n = b_{r+1} \dots b_1$ and $f(n) = b_{r+1}b_{s+2}b_{s+3}\dots b_rb_{s+1}$ by induction. So $f(n) + 2^{r+1} = b_{r+1}0\dots0b_{r+1}b_{s+2}\dots b_rb_{s+1}$, where there are s zeros. But we may write this as $b_{r+1}b_1\dots b_sb_{s+1}\dots b_rb_{r+1}$, since $b_1 = \dots = b_s = 0$, and $b_{s+1} = b_{r+1} = 1$. But that is the formula for f(2n+1), so we have completed the induction.

Problem B1

Show that the set of real numbers x which satisfy the inequality:

$1/(x - 1) + 2/(x - 2) + 3/(x - 3) + \dots + 70/(x - 70) \geq 5/4$

is a union of disjoint intervals, the sum of whose lengths is 1988.

Solution

Let $f(x) = 1/(x - 1) + 2/(x - 2) + 3/(x - 3) + \dots + 70/(x - 70)$. For any integer n, $n/(x - n)$ is strictly monotonically decreasing except at x = n, where it is discontinuous. Hence f(x) is strictly monotonically decreasing except at x = 1, 2, ... , 70. For n = any of 1, 2, ... , 70, $n/(x - n)$ tends to plus infinity as x tends to n from above, whilst the other terms $m/(x - m)$ remain bounded. Hence f(x) tends to plus infinity as x tends to n from above. Similarly, f(x) tends to minus infinity as x tends to n from below. Thus in each of the intervals (n, n+1) for n = 1, ... , 69, f(x) decreases monotonically from plus infinity to minus infinity and hence f(x) = 5/4 has a single foot x_n. Also $f(x) \geq 5/4$ for x in $(n, x_n]$ and $f(x) < 5/4$ for x in $(x_n, n+1)$. If x < 0, then every term is negative and hence f(x) < 0 < 5/4. Finally, as x tends to infinity, every term tends to zero, so f(x) tends to zero. Hence f(x) decreases monotonically from plus infinity to zero over the range [70, infinity]. Hence f(x) = 5/4 has a single root x_{70} in this range and f(x) >= 5/4 for x in $(70, x_{70}]$ and $f(x) < 5/4$ for $x > x_{70}$. Thus we have established that $f(x) \geq 5/4$ for x in any of the disjoint intervals $(1, x_1]$, $(2, x_2]$, ... , $(70, x_{70}]$ and f(x) < 5/4 elsewhere.
The total length of these intervals is $(x_1 - 1) + \dots + (x_{70} - 70) = (x_1 + \dots + x_{70}) - (1 + \dots + 70)$. The x_i are the roots of the 70th order polynomial obtained from $1/(x - 1) + 2/(x - 2) + 3/(x - 3) + \dots + 70/(x - 70) = 5/4$ by multiplying both sides by $(x - 1) \dots (x - 70)$. The sum of the roots is minus the coefficient of x^{69} divided by the coefficient of x^{70}. The coefficient of x^{70} is simply k, and the coefficient of x^{69} is $- (1 + 2 + \dots + 70)k - (1 + \dots + 70)$. Hence the sum of the roots is $(1 + \dots + 70)(1 + k)/k$ and the total length of the intervals is $(1 + \dots + 70)/k = 1/2\ 70\cdot71\ 4/5 = 28\cdot71 = 1988$.

Problem B2

ABC is a triangle, right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD and ACD intersects the sides AB, AC at K, L respectively. Show that the area of the triangle ABC is at least twice the area of the triangle AKL.

Solution

The key is to show that AK = AL = AD. We do this indirectly. Take K' on AB and L' on AC so that AK' = AL' = AD. Let the perpendicular to AB at K' meet the line AD at X. Then the triangles AK'X and ADB are congruent. Let J be the incenter of ADB and let r be the in-radius of ADB. Then J lies on the angle bisector of angle BAD a distance r from the line AD. Hence it is also the incenter of AK'X. Hence JK' bisects the right angle AK'X, so AK'J = 45^o and so J lies on K'L'. An exactly similar argument shows that I, the incenter of ADC, also lies on K'L'. Hence we can identify K and K', and L and L'.
The area of AKL is $AK\cdot AL/2 = AD^2/2$, and the area of ABC is $BC\cdot AD/2$, so we wish to show that $2AD \leq BC$. Let M be the midpoint of BC. Then AM is the hypoteneuse of AMD, so $AM \geq AD$ with equality if and only if D = M. Hence $2AD \leq 2AM = BC$ with equality if and only if AB = AC.

Problem B3

Let a and b be positive integers such that ab + 1 divides $a^2 + b^2$. Show that $(a^2 + b^2)/(ab + 1)$ is a perfect square.

Solution

A little experimentation reveals the following solutions: a, a^3 giving a^2; a^3, $a^5 - a$ giving a^2; and the recursive $a_1 = 2$, $b_1 = 8$, $a_{n+1} = b_n$, $b_{n+1} = 4b_n - a_n$ giving 4. The latter may lead us to: if $a^2 + b^2 = k(ab + 1)$, then take A = b, B = kb - a, and then $A^2 + B^2 = k(AB + 1)$. Finally, we may notice that this can be used to go down as well as up.
So starting again suppose that a, b, k is a solution in positive integers to $a^2 + b^2 = k(ab + 1)$. If a = b, then $2a^2 = k(a^2 + 1)$. So a^2 must divide k. But that implies that a = b = k = 1. Let us assume we do not have this trivial solution, so we may take a < b. We also show

that $a^3 > b$. For $(b/a - 1/a)(ab + 1) = b^2 + b/a - b - 1/a < b^2 < a^2 + b^2$. So $k > b/a - 1/a$. But if $a^3 < b$, then $b/a\ (ab + 1) > b^2 + a^2$, so $k < b/a$. But now $b > ak$ and $< ak + 1$, which is impossible. It follows that $k \geq b/a$.
Now define $A = ka - b$, $B = a$. Then we can easily verify that A, B, k also satisfies $a^2 + b^2 = k(ab + 1)$, and B and k are positive integers. Also $a < b$ implies $a^2 + b^2 < ab + b^2 < ab + b^2 + 1 + b/a = (ab + 1)(1 + b/a)$, and hence $k < 1 + b/a$, so $ka - b < a$. Finally, since $k > b/a$, $ka - b \geq 0$. If $ka - b > 0$, then we have another smaller solution, in which case we can repeat the process. But we cannot have an infinite sequence of decreasing numbers all greater than zero, so we must eventually get $A = ka - b = 0$. But now $A^2 + B^2 = k(AB + 1)$, so $k = B^2$. k was unchanged during the descent, so k is a perfect square.

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IMO 1989

Problem A1

Prove that the set {1, 2, ... , 1989} can be expressed as the disjoint union of subsets A_1, A_2, ... , A_{117} in such a way that each A_i contains 17 elements and the sum of the elements in each A_i is the same.

Solution

We construct 116 sets of three numbers. Each set sums to 3 x 995 = 2985. The 348 numbers involved form 174 pairs {r, 1990 - r}. At this point we are essentially done. We take a 117th set which has one {r, 1990 - r} pair and 995. The original 1989 numbers comprise 995 and 994 {r, 1990 - r} pairs. We have used up 995 and 175 pairs, leaving just 819 pairs. We now add 7 pairs to each of our 117 sets, bringing the total of each set up to 2985 + 7.1990 = 1990 x 17/2.
It remains to exhibit the 116 sets. There are many possibilities. We start with:
301, 801, 1883 and the "complementary" set 1990 - 301 = 1689, 1990 - 801 = 1189, 1990 - 1883 = 107. We then add one to each of the first two numbers to get:
302, 802, 1881 and 1688, 1188, 109, and so on:
303, 803, 1879 and 1687, 1187, 111,
...
358, 858, 1769 and 1632, 1132, 221.
We can immediately see that these triples are all disjoint. So the construction is complete.

Problem A2

In an acute-angled triangle ABC, the internal bisector of angle A meets the circumcircle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$ and at least four times the area of the triangle ABC.

Solution

Let I be the point of intersection of AA_0, BB_0, CC_0 (the in-center). BIC = 180 - 1/2 ABC - 1/2 BCA = 180 - 1/2 (180 - CAB) = 90 + 1/2 CAB. Hence CA_1B = 180 - CAB [BA_1CA is cyclic] = 2(180 - BIC) = $2CA_0B$. But $A_1B = A_1C$, so A_1 is the center of the circumcircle of BCA_0. But I lies on this circumcircle ($IBA_0 = ICA_0 = 90$), and hence $A_1A_0 = A_1I$.
Hence area IBA_1 = area A_0BA_1 and area ICA_1 = area A_0CA_1. Hence area IBA_0C = 2 area IBA_1C. Similarly, area ICB_0A = 2 area ICB_1A and area IAC_0B = 2 area IAC_1B. Hence area $A_0B_0C_0$ = 2 area hexagon $AB_1CA_1BC_1$.
Let H be the orthocentre of ABC. Let H_1 be the reflection of H in BC, so H_1 lies on the circumcircle. So area BCH = area BCH_1 <= area BCA_1. Adding to the two similar inequalities gives area ABC <= area hexagon - area ABC.

Problem A3

Let n and k be positive integers, and let S be a set of n points in the plane such that no three points of S are collinear, and for any point P of S there are at least k points of S equidistant from P. Prove that $k < 1/2 + \sqrt{(2n)}$.

Solution

Consider the pairs P, {A, B}, where P, A, B are points of S, and P lies on the perpendicular bisector of AB. There are at least n k(k - 1)/2 such pairs, because for each point P, there are at least k points equidistant from P and hence at least k(k - 1)/2 pairs of points equidistant from P.
If $k \geq 1/2 + \sqrt{(2n)}$, then $k(k - 1) \geq 2n - 1/4 > 2(n - 1)$, and so there are more than n(n - 1) pairs P, {A, B}. But there are only n(n - 1)/2 possible pairs {A, B}, so for some {A_0, B_0} we must be able to find at least 3 points P on the perpendicular bisector of A_0B_0. But these points are collinear, contradicting the assumption in the question.

Problem B1

Let ABCD be a convex quadrilateral such that the sides AB, AD, BC satisfy AB = AD + BC. There exists a point P inside the quadrilateral at a distance h from the line CD such that AP = h + AD and BP = h + BC. Show that:

$1/\sqrt{h} \geq 1/\sqrt{AD} + 1/\sqrt{BC}$.

Solution

Let C_A be the circle center A, radius AD, and C_B the circle center B, radius BC. The circles touch on AB. Let C_P the the circle center P, radius h. C_P touches C_A and C_B and CD. Let t be the common tangent to C_A and C_B whose two points of contact are on the same side of AB as C and D. Then C_P is confined inside the curvilinear triangle whose sides are segments of t, C_A and C_B. Evidently h attains its maximum value, for given lengths AB, AD, BC, when C_P touches t, in which case D must be the point at which t touches C_A, and C the point at which it touches C_B. Suppose E is the point at which t touches C_P.

Angles ADC and BCD are right angles, so $CD^2 = AB^2 - (AD - BC)^2 = 4\ AD\ BC$. Similarly, $DE^2 = 4\ h\ AD$, and $CE^2 = 4\ h\ BC$. But CD = DE + CE, so $1/\sqrt{h} = 1/\sqrt{AD} + 1/\sqrt{BC}$. This gives the maximum value of h, so in general we have the inequality stated.

Problem B2

Prove that for each positive integer n there exist n consecutive positive integers none of which is a prime or a prime power.

Solution

Consider $(N!)^2+2$, $(N!)^2+3$, ... , $(N!)^2+N$. $(N!)^2+r$ is divisible by r, but $((N!)^2+r)/r = N!(N!/r) + 1$, which is greater than one, but relatively prime to r since $N!(N!/r)$ is divisible by r. For each r we may take a prime p_r dividing r, so $(N!)^2+r$ is divisible by p_r, but is not a power of p_r. Hence it is not a prime or a prime power. Taking N = n+1 gives n consecutive numbers as required.

Problem B3

A permutation $\{x_1, x_2, \dots , x_m\}$ of the set $\{1, 2, \dots , 2n\}$ where n is a positive integer is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \dots , 2n-1\}$. Show that for each n there are more permutations with property P than without.

Solution

Let A_k be the set of permutations with k and k+n in neighboring positions, and let A be the set of permutations with property P, so that A is the union of the A_k.

Then $|A| = \text{Sum}_k |A_k| - \text{Sum}_{k<l} |A_k \;\; A_l| + \text{Sum}_{k<l<m} |A_k \;\; A_l \;\; A_m| - \dots$. But this is an alternating sequence of monotonically decreasing terms, hence $|A| \geq \sum_k |A_k| - \text{Sum}_{k<l} |A_k \;\; A_l|$.

But $|A_k| = 2\ (2n - 1)!$ (two orders for k, k+n and then (2n - 1)! ways of arranging the 2n - 1 items, treating k, k+n as a single item). Similarly, $|A_k \;\; A_l| = 4\ (2n - 2)!$ So $|A| \geq (2n - 2)!\ [n.2(2n -1) - n(n - 1)/2\ 4] = 2n^2\ (2n - 2)! > (2n)!/2$.

IMO 1990

Problem A1

Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent at E to the circle through D, E and M intersects the lines BC and AC at F and G respectively. Find EF/EG in terms of t = AM/AB.

Solution

$\angle$ECF = $\angle$DCB (same angle) = $\angle$DAB (ACBD is cyclic) = $\angle$MAD (same angle). Also $\angle$CEF = $\angle$EMD (GE tangent to circle EMD) = $\angle$AMD (same angle). So triangles CEF and AMD are similar.

$\angle$CEG = 180^o - $\angle$CEF = 180^o - $\angle$EMD = $\angle$BMD. Also $\angle$ECG = $\angle$ACD (same angle) = $\angle$ABD (BCAD is cyclic) = $\angle$MBD (same angle). So triangles CEG and BMD are similar.

Hence EF/CE = MD/AM, EG/CE = MD/BM, and so dividing, EF/EG = BM/AM = (1- t)/t.

Problem A2
Take $n \geq 3$ and consider a set E of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

Solution
Answer: n for n = 0 or 1 (mod 3), n - 1 for n = 2 (mod 3).
Label the points 1 to 2n - 1. Two points have exactly n points between them if their difference (mod 2n - 1) is n - 2 or n + 1. We consider separately the three cases n = 3m, 3m + 1 and 3m + 2.
Let n = 3m. First, we exhibit a bad coloring with n - 1 black points. Take the black points to be 1, 4, 7, ... , 6m - 2 (2m points) and 2, 5, 8, ... , 3m - 4 (m - 1 points). It is easy to check that this is bad. The two points which could pair with r to give n points between are r + 3m - 2 and r + 3m + 1. Considering the first of these, 1, 4, 7, ... , 6m - 2 would pair with 3m - 1, 3m + 2, 3m + 5, ... , 6m - 1, 3, 6, ... , 3m - 6, none of which are black. Considering the second, they would pair with 3m + 2, 3m + 5, ... , 6m - 1, 3, ... , 3m - 3, none of which are black. Similarly, 2, 5, 8, ... , 3m - 4 would pair with 3m, 3m + 3, ... , 6m - 3, none of which are black. So the set is bad.
Now if we start with 1 and keep adding 3m - 2, reducing by 6m - 1 when necessary to keep the result in the range 1, ... , 6m - 1, we eventually get back to 1: 1, 3m - 1, 6m - 3, 3m - 4, 6m - 6, ... , 2, 3m, 6m - 2, 3m - 3, 6m - 5, ... , 3, 3m + 1, 6m - 1, ... , 4, 3m + 2, 1. The sequence includes all 6m - 1 numbers. Moreover a bad coloring cannot have any two consecutive numbers colored black. But this means that at most n - 1 out of the 2n - 1 numbers in the sequence can be black. This establishes the result for n = 3m.
Take n = 3m + 1. A bad coloring with n - 1 black points has the following black points: 1, 4, 7, ... , 3m - 2 (m points) and 2, 5, 8, ... , 6m - 1 (2m points). As before we add n - 2 repeatedly starting with 1 to get: 1, 3m, 6m - 1, 3m - 3, 6m - 4, ... , 3, 3m + 2, 6m + 1, 3m - 1, ... , 2, 3m + 1, 6m, 3m - 2, ... , 1. No two consecutive numbers can be black in a bad set, so a bad set can have at most n - 1 points.
Finally, take n = 3m + 2. A bad coloring with n - 2 points is 1, 2, ... , n - 2. This time when we add n - 2 = 3m repeatedly starting with 1, we get back to 1 after including only one-third of the numbers: 1, 3m + 1, 6m + 1, 3m - 2, ... , 4, 3m + 4, 1. The usual argument shows that at most m of these 2m + 1 numbers can be colored black in a bad set.
Similarly, we may add 3m repeatedly starting with 2 to get another 2m + 1 numbers: 2, 3m + 2, 6m + 2, 3m - 1, ... , 3m + 5, 2. At most m of these can be black in a bad set.
Similarly at most m of the 2m + 1 numbers: 3, 3m + 3, 6m + 3, 3m, ... , 3m + 6, 3 can be black. So in total at most 3m = n - 2 can be black in a bad set.

Problem A3
Determine all integers greater than 1 such that $(2^n + 1)/n^2$ is an integer.

Solution
Answer: n = 3.
Since $2^n + 1$ is odd, n must also be odd. Let p be its smallest prime divisor. Let x be the smallest positive integer such that $2^x = -1 \pmod p$, and let y be the smallest positive integer such that $2^y = 1 \pmod p$. y certainly exists and indeed $y < p$, since $2^{p-1} = 1 \pmod p$. x exists since $2^n = -1 \pmod p$. Write $n = ys + r$, with $0 \leq r < y$. Then $-1 = 2^n = (2^y)^s 2^r = 2^r \pmod p$, so $x \leq r < y$ (r cannot be 0, since - 1 is not 1 (mod p)).
Now write $n = hx + k$, with $0 \leq k < x$. Then $-1 = 2^n = (-1)^h 2^k \pmod p$. Suppose $k > 0$. Then if h is odd we contradict the minimality of y, and if h is even we contradict the minimality of x. So k = 0 and x divides n. But x < p and p is the smallest prime dividing n, so x = 1. Hence 2 = -1 (mod p) and so p = 3.
Now suppose that 3^m is the largest power of 3 dividing n. We show that m must be 1. Expand $(3 - 1)^n + 1$ by the binomial theorem, to get (since n is odd): $1 - 1 + n.3 - 1/2\, n(n - 1)\, 3^2 + \ldots = 3n - (n - 1)/2\, n\, 3^2 + \ldots$. Evidently 3n is divisible by 3^{m+1}, but not 3^{m+2}. We show that the remaining terms are all divisible by 3^{m+2}. It follows that 3^{m+1} is the

highest power 3 dividing $2^n + 1$. But $2^n + 1$ is divisible by n^2 and hence by 3^{2m}, so m must be 1.
The general term is $(3^m a)Cb\ 3^b$, for $b \geq 3$. The binomial coefficients are integral, so the term is certainly divisible by 3^{m+2} for $b \geq m+2$. We may write the binomial coefficient as $(3^m a/b)\ (3^m - 1)/1\ (3^m - 2)/2\ (3^m - 3)/3 \dots (3^m - (b-1)) / (b - 1)$. For b not a multiple of 3, the first term has the form 3^m c/d, where 3 does not divide c or d, and the remaining terms have the form c/d, where 3 does not divide c or d. So if b is not a multiple of 3, then the binomial coefficient is divisible by 3^m, since b > 3, this means that the whole term is divisible by at least 3^{m+3}. Similarly, for b a multiple of 3, the whole term has the same maximum power of 3 dividing it as $3^m\ 3^b/b$. But b is at least 3, so $3^b/b$ is divisible by at least 9, and hence the whole term is divisible by at least 3^{m+2}.
We may check that n = 3 is a solution. If n > 3, let n = 3 t and let q be the smallest prime divisor of t. Let w be the smallest positive integer for which $2^w = -1 \pmod q$, and v the smallest positive integer for which $2^v = 1 \pmod q$. v certainly exists and < q since $2^{q-1} = 1 \pmod q$. $2^n = -1 \pmod q$, so w exists and, as before, w < v. Also as before, we conclude that w divides n. But w < q, the smallest prime divisor of n, except 3. So w = 1 or 3. These do not work, because then $2 = -1 \pmod q$ and so q = 3, or $2^3 = -1 \pmod q$ and again q =3, whereas we know that q > 3.

Problem B1
Construct a function from the set of positive rational numbers into itself such that f(x f(y)) = f(x)/y for all x, y.

Solution
We show first that f(1) = 1. Taking x = y = 1, we have f(f(1)) = f(1). Hence f(1) = f(f(1)) = f(1 f(f(1))) = f(1)/f(1) = 1.
Next we show that f(xy) = f(x)f(y). For any y we have 1 = f(1) = f(1/f(y) f(y)) = f(1/f(y))/y, so if z = 1/f(y) then f(z) = y. Hence f(xy) = f(xf(z)) = f(x)/z = f(x) f(y).
Finally, f(f(x)) = f(1 f(x)) = f(1)/x = 1/x.
We are not required to find *all* functions, just one. So divide the primes into two infinite sets $S = \{p_1, p_2, \dots \}$ and $T= \{q_1, q_2, \dots \}$. Define $f(p_n) = q_n$, and $f(q_n) = 1/p_n$. We extend this definition to all rationals using f(xy) = f(x) f(y): $f(p_{i1}p_{i2}\dots q_{j1}q_{j2}\dots/(p_{k1}\dots q_{m1}\dots)) = p_{m1}\dots q_{i1}\dots/(p_{j1}\dots q_{k1}\dots)$. It is now trivial to verify that f(x f(y)) = f(x)/y.

Problem B2
Given an initial integer $n_0 > 1$, two players A and B choose integers $n_1, n_2, n_3, \dots$ alternately according to the following rules:
Knowing n_{2k}, A chooses any integer n_{2k+1} such that $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$.
Knowing n_{2k+1}, B chooses any integer n_{2k+2} such that $n_{2k+1}/n_{2k+2} = p^r$ for some prime p and integer $r \geq 1$.
Player A wins the game by choosing the number 1990; player B wins by choosing the number 1. For which n_0 does
(a) A have a winning strategy?
(b) B have a winning strategy?
(c) Neither player have a winning strategy?

Solution
Answer: if n_0 = 2, 3, 4 or 5 then A loses; if $n_0 \geq 8$, then A wins; if n_0 = 6 or 7 , then it is a draw.
A's strategy given a number n is as follows:
(1) if $n \in [8, 11]$, pick 60
(2) if $n \in [12, 16]$, pick 140
(3) if $n \in [17, 22]$, pick 280
(4) if $n \in [23, 44]$, pick 504
(5) if $n \in [45, 1990]$, pick 1990
(6) if n = 1991 = 11.181 (181 is prime), pick 1991
(7) if $n \in [11^r 181 + 1, 11^{r+1} 181]$ for some r > 0, pick $11^{r+1}181$.
Clearly (5) wins immediately for A. After (4) B has 7.8.9 so must pick 56, 63, 72 or 168, which gives A an immediate win by (5). After (3) B must pick 35, 40, 56, 70 or 140, so A

wins by (4) and (5). After (2) B must pick 20, 28, 35 or 70, so A wins by (3) - (5). After (1) B must pick 12, 15, 20 or 30, so A wins by (2) - (5).
If B is given $11^{r+1}181$, then B must pick 181, 11.181, ... , 11^r.181 or 11^{r+1}, all of which are ≤ 11^r.181. So if A is given a number n in (6) or (7) then after a turn each A is given a number < n (and >= 11), so after a finite number of turns A wins.
If B gets a number less than 6, then he can pick 1 and win. Hence if A is given 2, he loses, because he must pick a number less than 5. Now if B gets a number of 11 or less, he wins by picking 1 or 2. Hence if A is given 3, he loses, because he must pick a number less than 10. Now if B gets a number of 19 or less, he can win by picking 1, 2 or 3. So if A is given 4 he loses. Now if B is given 29 or less, he can pick 1, 2, 3 or 4 and win. So if A is given 5 he loses.
We now have to consider what happens if A gets 6 or 7. He must pick 30 or more, or B wins. If he picks 31, 32, 33, 34, 35 or 36, then B wins by picking (for example) 1, 1, 3, 2, 5, 4 respectively. So his only hope given 6 is to pick 30. B also wins given any of 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48, 49 (winning moves, for example, 37, 1; 38, 2; 39, 3; 40, 5; 41, 1; 43, 1; 44, 4; 45, 5; 46, 3; 47, 1; 48, 3]. So A's only hope given 7 is to pick 30 or 42.
If B is faced with 30=2·3·5, then he has a choice of 6, 10, 15. We have already established that 10 and 15 will lose, so he must pick 6. Thus 6 is a draw: A must pick 30 or lose, and then B must pick 6 or lose.
If B is faced with 42=2·3·7, then he has a choice of 6, 14 or 21. We have already established that 14 and 21 lose, so he must pick 6. Thus 7 is also a draw: A must pick 30 or 42, and then B must pick 6.

Problem B3
Prove that there exists a convex 1990-gon such that all its angles are equal and the lengths of the sides are the numbers 1^2, 2^2, ... , 1990^2 in some order.

Solution
In the complex plane we can represent the sides as $p_n{}^2w^n$, where p_n is a permutation of (1, 2, ... , 1990) and w is a primitive 1990th root of unity.
The critical point is that 1990 is a product of more than 2 distinct primes: 1990 = 2·5·199. So we can write w = -1·a·b, where -1 is primitive 2nd root of unity, a is a primitive 5th root of unity, and b is a primitive 199th root of unity.
Now given one of the 1990th roots we may write it as $(-1)^ia^jb^k$, where 0 < i < 2, 0 < j < 5, 0 < k < 199 and hence associate it with the integer r(i,j,k) = 1 + 995i + 199j + k. This is a bijection onto (1, 2, ... , 1990). We have to show that the sum of $r(i,j,k)^2$ $(-1)^ia^jb^k$ is zero. We sum first over i. This gives -995^2 x sum of a^jb^k which is zero, and - 1990 x sum s(j,k) a^jb^k, where s(j,k) = 1 + 199j + k. So it is sufficient to show that the sum of s(j,k) a^jb^k is zero. We now sum over j. The 1 + k part of s(j,k) immediately gives zero. The 199j part gives a constant times b^k, which gives zero when summed over k.

IMO 1991

Problem A1
Given a triangle ABC, let I be the incenter. The internal bisectors of angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that:
  1/4 < AI·BI·CI/(AA'·BB'·CC') ≤ 8/27.

Solution
Consider the areas of the three triangles ABI, BCI, CAI. Taking base BC we conclude that (area ABI + area CAI)/area ABC = AI/AA'. On the other hand, if r is the radius of the in-circle, then area ABI = AB.r/2 and similarly for the other two triangles. Hence AI/AA' = (CA + AB)/p, where p is the perimeter. Similarly BI/BB' = (AB + BC)/p and CI/CC' = (BC + CA)/p. But the arithmetic mean of (CA + AB)/p, (AB + BC)/p and (BC + CA)/p is 2/3. Hence their product is at most $(2/3)^3$ = 8/27.
Let AB + BC - CA = 2z, BC + CA - AB = 2x, CA + AB - BC = 2y. Then x, y, z are all positive and we have AB = y + z, BC = z + x, CA = x + y. Hence (AI/AA')(BI/BB')(CI/CC') = (1/2 + y/p)(1/2 + z/p)(1/2 + x/p) > 1/8 + (x+y+z)/(4p) = 1/8 + 1/8 = 1/4.

Problem A2

Let $n > 6$ be an integer and let $a_1, a_2, \ldots, a_k$ be all the positive integers less than n and relatively prime to n. If $a_2 - a_1 = a_3 - a_2 = \ldots = a_k - a_{k-1} > 0$, prove that n must be either a prime number or a power of 2.

Solution

If n is odd, then 1 and 2 are prime to n, so all integers < n are prime to n, and hence is prime.
If n = 4k, then 2k-1 and 2k+1 are prime to n, so all odd integers < n are prime to n, and hence n must be a power of 2.
If n = 4k+2, then 2k+1 divides n, but 2k+3 and 2k+5 are prime to n. But if n > 6, then 2k+5 < n, so this cannot be a solution.

Problem A3

Let S = {1, 2, 3, ... 280}. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

Solution

Answer: 217.
Let A be the subset of all multiples of 2, 3, 5 or 7. Then A has 216 members and every 5-subset has 2 members with a common factor. [To show that |A| = 216, let a_n be the number of multiples of n in S. Then $a_2 = 140$, $a_3 = 93$, $a_5 = 56$, $a_6 = 46$, $a_{10} = 28$, $a_{15} = 18$, $a_{30} = 9$. Hence the number of multiples of 2, 3 or 5 = $a_2 + a_3 + a_5 - a_6 - a_{10} - a_{15} + a_{30} = 206$. There are ten additional multiples of 7: 7, 49, 77, 91, 119, 133, 161, 203, 217, 259.]
Let P be the set consisting of 1 and all the primes < 280. Define:
A1 = {2·41, 3·37, 5·31, 7·29, 11·23, 13·19}
A2 = {2·37, 3·31, 5·29, 7·23, 11·19, 13·17}
A3 = {2··31, 3·29, 5·23, 7·19, 11·17, 13·13}
B1 = {2·29, 3·23, 5·19, 7·17, 11·13}
B2 = {2·23, 3·19, 5·17, 7·13, 11·11}
Note that these 6 sets are disjoint subsets of S and the members of any one set are relatively prime in pairs. But P has 60 members, the three As have 6 each, and the two Bs have 5 each, a total of 88. So any subset T of S with 217 elements must have at least 25 elements in common with their union. But 6·4 = 24 < 25, so T must have at least 5 elements in common with one of them. Those 5 elements are the required subset of elements relatively prime in pairs.

Problem B1

Suppose G is a connected graph with k edges. Prove that it is possible to label the edges 1, 2, ... , k in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is 1.
[A graph is a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of edges belongs to at most one edge. The graph is connected if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, \ldots, v_m = y$, such that each pair v_i, v_{i+1} $(0 \le i < m)$ is joined by an edge.]

Solution

The basic idea is that consecutive numbers are relatively prime.
We construct a labeling as follows. Pick any vertex A and take a path from A along unlabeled edges. Label the edges consecutively 1, 2, 3, ... as the path is constructed. Continue the path until it reaches a vertex with no unlabeled edges. Let B be the endpoint of the path. A is now guaranteed to have the gcd (= greatest common divisor) of its edges 1, because one of its edges is labeled 1. All the vertices between A and B are guaranteed to have gcd 1 because they have at least one pair of edges with consecutive numbers. Finally, either B has only one edge, in which case its gcd does not matter, or it is also one of the vertices between A and B, in which case its gcd is 1.
Now take any vertex C with an unlabeled edge and repeat the process. The same argument shows that all the new vertices on the new path have gcd 1. The endpoint is fine, because

either it has only one edge (in which case its gcd does not matter) or it has already got gcd 1. Repeat until all the edges are labeled.

Problem B2

Let ABC be a triangle and X an interior point of ABC. Show that at least one of the angles XAB, XBC, XCA is less than or equal to 30^o.

Solution

Let P, Q, R be the feet of the perpendiculars from X to BC, CA, AB respectively. Use A, B, C to denote the interior angles of the triangle (BAC, CBA, ACB). We have PX = BX sin XBC = CX sin(C - XCA), QX = CX sin XCA = AX sin(A - XAB), RX = AX sin XAB = BX sin(B - XBC). Multiplying: sin(A - XAB) sin(B - XBC) sin(C - XCA) = sin A sin B sin C.

Now observe that sin(A - x)/sin x = sin A cot x - cos A is a strictly decreasing function of x (over the range 0 to π), so if XAB, XBC and XCA are all greater than 30, then sin(A - 30) $\sin(B - 30^o) \sin(C - 30^o) > \sin^3 30^o = 1/8$.

But $\sin(A - 30^o) \sin(B - 30^o) = (\cos(A - B) - \cos(A + B - 60^o))/2 \le (1 - \cos(A + B - 60^o))/2 = (1 - \sin(C - 30^o))/2$, since $(A - 30^o) + (B - 30^o) + (C - 30^o) = 90^o$. Hence $\sin(A - 30^o) \sin(B - 30^o) \sin(C - 30^o) \le 1/2\ (1 - \sin(C - 30^o)) \sin(C - 30^o) = 1/2\ (1/4 - (\sin(C - 30^o) - 1/2)^2) \le 1/8$. So XAB, XBC, XCA cannot all be greater than 30^o.

Problem B3

Given any real number a > 1 construct a bounded infinite sequence $x_0, x_1, x_2, \dots$ such that $|x_n - x_m|\ |n - m|^a \ge 1$ for every pair of distinct n, m.

[An infinite sequence $x_0, x_1, x_2, \dots$ of real numbers is bounded if there is a constant C such that $|x_n| < C$ for all n.]

Solution

Let $t = 1/2^a$. Define c = 1 - t/(1 - t). Since a > 1, c > 0. Now given any integer n > 0, take the binary expansion $n = \sum_i b_i 2^i$, and define $x_n = 1/c \sum_{bi>0} t^i$. For example, taking $n = 21 = 2^4 + 2^2 + 2^0$, we have $x_{21} = (t^4 + t^2 + t^0)/c$. We show that for any unequal n, m, $|x_n - x_m|\ |n - m|^a \ge 1$. This solves the problem, since the x_n are all positive and bounded by $(\sum t^n)/c = 1/(1 - 2t)$.

Take k to be the highest power of 2 dividing both n and m. Then $|n - m| \ge 2^k$. Also, in the binary expansions for n and m, the coefficients of $2^0, 2^1, \dots, 2^{k-1}$ agree, but the coefficients for 2^k are different. Hence $c\ |x_n - x_m| = t^k + \sum_{i>k} y_i$, where $y_i = 0$, t^i or $-t^i$. Certainly $\sum_{i>k} y_i > -\sum_{i>k} t^i = t^{k+1}/(1 - t)$, so $c\ |x_n - x_m| > t^k(1 - t/(1 - t)) = c\ t^k$. Hence $|x_n - x_m|\ |n - m|^a > t^k 2^{ak} = 1$.

IMO 1992

Problem A1

Find all integers a, b, c satisfying 1 < a < b < c such that (a - 1)(b -1)(c - 1) is a divisor of abc - 1.

Solution

Answer: a = 2, b = 4, c = 8; or a = 3, b = 5, c = 15.

Let $k = 2^{1/3}$. If a ≥ 5, then k(a - 1) > a. [Check: $(k(a - 1))^3 - a^3 = a^3 - 6a^2 + 6a - 2$. For a ≥ 6, $a^3 \ge 6a^2$ and 6a > 2, so we only need to check a = 5: 125 - 150 + 30 - 2 = 3.] We know that c > b > a, so if a ≥ 5, then 2(a - 1)(b - 1)(c - 1) > abc > abc - 1. So we must have a = 2, 3 or 4.

Suppose abc - 1 = n(a - 1)(b - 1)(c - 1). We consider separately the cases n = 1, n = 2 and n ≥ 3. If n = 1, then a + b + c = ab + bc + ca. But that is impossible, because a, b, c are all greater than 1 and so a < ab, b < bc and c < ca.

Suppose n = 2. Then abc - 1 is even, so all a, b, c are odd. In particular, a = 3. So we have 4(b - 1)(c - 1) = 3bc - 1, and hence bc + 5 = 4b + 4c. If b >= 9, then bc >= 9c > 4c + 4b. So we must have b = 5 or 7. If b = 5, then we find c = 15, which gives a solution. If b = 7, then we find c = 23/3 which is not a solution.

The remaining case is n >= 3. If a = 2, we have n(bc - b - c + 1) = 2bc - 1, or (n - 2)bc + (n + 1) = nb + nc. But b ≥ 3, so (n - 2)bc ≥ (3n - 6)c ≥ 2nc for n ≥ 6, so we must have n

= 3, 4 or 5. If n = 3, then bc + 4 = 3b + 3c. If b >= 6, then bc ≥ 6c > 3b + 3c, so b = 3, 4 or 5. Checking we find only b = 4 gives a solution: a = 2, b = 4, c = 8. If n = 4, then (n - 2)bc, nb and nc are all even, but (n + 1) is odd, so there is no solution. If n = 5, then 3bc + 6 = 5b + 5c. b = 3 gives c = 9/4, which is not a solution. b >= 4 gives 3bc > 10c > 5b + 5c, so there are no solutions.
If a = 3, we have 2n(bc - b - c + 1) = 3bc - 1, or (2n - 3)bc + (2n + 1) = 2nb + 2nc. But b ≥ 4, so (2n - 3)bc ≥ (8n - 12)c ≥ 4nc > 2nc + 2nb. So there are no solutions. Similarly, if a = 4, we have (3n - 4)bc + (3n + 1) = 3nb + 3nc. But b ≥ 4, so (3n - 4)bc ≥ (12n - 16)c > 6nc > 3nb + 3nc, so there are no solutions.

Problem A2
Find all functions f defined on the set of all real numbers with real values, such that $f(x^2 + f(y)) = y + f(x)^2$ for all x, y.

Solution
The first step is to establish that f(0) = 0. Putting x = y = 0, and f(0) = t, we get $f(t) = t^2$. Also, $f(x^2+t) = f(x)^2$, and $f(f(x)) = x + t^2$. We now evaluate $f(t^2+f(1)^2)$ two ways. First, it is $f(f(1)^2 + f(t)) = t + f(f(1))^2 = t + (1 + t^2)^2 = 1 + t + 2t^2 + t^4$. Second, it is $f(t^2 + f(1 + t)) = 1 + t + f(t)^2 = 1 + t + t^4$. So t = 0, as required.
It follows immediately that f(f(x)) = x, and $f(x^2) = f(x)^2$. Given any y, let z = f(y). Then y = f(z), so $f(x^2 + y) = z + f(x)^2 = f(y) + f(x)^2$. Now given any positive x, take z so that $x = z^2$. Then $f(x + y) = f(z^2 + y) = f(y) + f(z)^2 = f(y) + f(z^2) = f(x) + f(y)$. Putting y = -x, we get 0 = f(0) = f(x + -x) = f(x) + f(-x). Hence f(-x) = - f(x). It follows that f(x + y) = f(x) + f(y) and f(x - y) = f(x) - f(y) hold for all x, y.
Take any x. Let f(x) = y. If y > x, then let z = y - x. f(z) = f(y - x) = f(y) - f(x) = x - y = -z. If y < x, then let z = x - y and f(z) = f(x - y) = f(x) - f(y) = y - x. In either case we get some z > 0 with f(z) = -z < 0. But now take w so that $w^2 = z$, then $f(z) = f(w^2) = f(w)^2 >= 0$. Contradiction. So we must have f(x) = x.

Problem A3
Consider 9 points in space, no 4 coplanar. Each pair of points is joined by a line segment which is colored either blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

Solution
We show that for n = 32 we can find a coloring without a monochrome triangle. Take two squares $R_1R_2R_3R_4$ and $B_1B_2B_3B_4$. Leave the diagonals of each square uncolored, color the remaining edges of R red and the remaining edges of B blue. Color blue all the edges from the ninth point X to the red square, and red all the edges from X to the blue square. Color R_iB_j red if i and j have the same parity and blue otherwise.
Clearly X is not the vertex of a monochrome square, because if XY and XZ are the same color then, YZ is either uncolored or the opposite color. There is no triangle within the red square or the blue square, and hence no monochrome triangle. It remains to consider triangles of the form $R_iR_jB_k$ and $B_iB_jR_k$. But if i and j have the same parity, then R_iR_j is uncolored (and similarly B_iB_j), whereas if they have opposite parity, then R_iB_k and R_jB_k have opposite colors (and similarly B_iR_k and B_jR_k).
It remains to show that for n = 33 we can always find a monochrome triangle. There are three uncolored edges. Take a point on each of the uncolored edges. The edges between the remaining 6 points must all be colored. Take one of these, X. At least 3 of the 5 edges to X, say XA, XB, XC must be the same color (say red). If AB is also red, then XAB is monochrome. Similarly, for BC and CA. But if AB, BC and CA are all blue, then ABC is monochrome.

Problem B1
L is a tangent to the circle C and M is a point on L. Find the locus of all points P such that there exist points Q and R on L equidistant from M with C the incircle of the triangle PQR.

Solution

Answer: Let X be the point where C meets L, let O be the center of C, let XO cut C gain at Z, and take Y on QR so that M be the midpoint of XY. Let L' be the line YZ. The locus is the open ray from Z along L' on the opposite side to Y.
Let C' be the circle on the other side of QR to C which also touches the segment QR and the lines PQ and QR. Let C' touch QR at Y'. If we take an expansion (technically, homothecy) center P, factor PY'/PZ, then C goes to C', the tangent to C at Z goes to the line QR, and hence Z goes to Y'. But it is easy to show that QX = RY'.
We focus on the QORO'. Evidently X,Y' are the feet of the perpendiculars from O, O' respectively to QR. Also, OQO' = ORO' = 90. So QY'O' and OXQ are similar, and hence QY'/Y'O' = OX/XQ. Also RXO and O'Y'R are similar, so RX/XO = O'Y'/Y'R. Hence QY'·XQ = OX·O'Y' = RX·Y'R. Hence QX/RX = QX/(QR - QX) = RY'/(QR - RY') = RY'/QY'. Hence QX = RY'.
But QX = RY by construction (M is the midpoint of XY and QR), so Y = Y'. Hence P lies on the open ray as claimed. Conversely, if we take P on this ray, then by the same argument QX = RY. But M is the midpoint of XY, so M must also be the midpoint of QR, so the locus is the entire (open) ray.

Problem B2
Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that:

$|S|^2 <= |S_x|\ |S_y|\ |S_z|$, where |A| denotes the number of points in the set A.

Solution
Induction on the number of different z-coordinates in S.
For 1, it is sufficient to note that $S = S_z$ and $|S| \le |S_x|\ |S_y|$ (at most $|S_x|$ points of S project onto each of the points of S_y).
In the general case, take a horizontal (constant z) plane dividing S into two non-empty parts T and U. Clearly, $|S| = |T| + |U|$, $|S_x| = |T_x| + |U_x|$, and $|S_y| = |T_y| + |U_y|$.
By induction, $|S| = |T| + |U| \le (|T_x|\ |T_y|\ |T_z|)^{1/2} + (|U_x|\ |U_y|\ |U_z|)^{1/2}$. But $|T_z|, |U_z| \le |S_z|$, and for any positive a, b, c, d we have $(a\ b)^{1/2} + (c\ d)^{1/2} \le (\ (a + c)\ (b + d)\ )^{1/2}$ (square!).
Hence $|S| \le |S_z|^{1/2}(\ (\ |T_x| + |U_x|\ )\ (\ |T_y| + |U_y|\ )\ )^{1/2} = (\ |S_x|\ |S_y|\ |S_z|\ )^{1/2}$.

Problem B3
For each positive integer n, S(n) is defined as the greatest integer such that for every positive integer $k \le S(n)$, n^2 can be written as the sum of k positive squares.
(a) Prove that $S(n) <= n^2 - 14$ for each $n \ge 4$.
(b) Find an integer n such that $S(n) = n^2 - 14$.
(c) Prove that there are infinitely many integers n such that $S(n) = n^2 - 14$.

Solution
(a) Let $N = n^2$. Suppose we could express N as a sum of N - 13 squares. Let the number of 4s be a, the number of 9s be b and so on. Then we have 13 = 3a + 8b + 15c + ... Hence c, d, ... must all be zero. But neither 13 nor 8 is a multiple of 3, so there are no solutions. Hence $S(n) \le N - 14$.
A little experimentation shows that the problem is getting started. Most squares cannot be expressed as a sum of two squares. For $N = 13^2 = 169$, we find: 169 = 9 + 4 + 4 + 152 1s, a sum of 155 = N - 14 squares. By grouping four 1s into a 4 repeatedly, we obtain all multiples of 3 plus 2 down to 41 (169 = 9 + 40 4s). Then grouping four 4s into a 16 gives us 38, 35, ... , 11 (169 = 10 16s + 9). Grouping four 16s into a 64 gives us 8 and 5. We obtain the last number congruent to 2 mod 3 by the decomposition: $169 = 12^2 + 5^2$.
For the numbers congruent to 1 mod 3, we start with N - 15 = 154 squares: 169 = 5 4s + 149 1s. Grouping as before gives us all 3m + 1 down to 7: 169 = 64 + 64 + 16 + 16 + 4 + 4 + 1. We may use $169 = 10^2 + 8^2 + 2^2 + 1^2$ for 4.
For multiples of 3, we start with N - 16 = 153 squares: 169 = 9 + 9 + 151 1s. Grouping as before gives us all multiples of 3 down to 9: 169 = 64 + 64 + 16 + 9 + 9 + 4 + 1 + 1 + 1. Finally, we may take $169 = 12^2 + 4^2 + 3^2$ for 3 and split the 4^2 to get $169 = 12^2 + 3^2 + 2^2 + 2^2 + 2^2 + 2^2$ for 6. That completes the demonstration that we can write 13^2 as a sum of k positive squares for all $k <= S(13) = 13^2 - 14$.

We now show how to use the expressions for 13^2 to derive further N. For any N, the grouping technique gives us the high k. Simply grouping 1s into 4s takes us down: from 9 + 4 + 4 + (N-17) 1s to (N-14)/4 + 6 < N/2 or below; from 4 + 4 + 4 + 4 + 4 + (N-20) 1s to (N-23)/4 + 8 < N/2 or below; from 9 + 9 + (N-18) 1s to (N-21)/4 + 5 < N/2 or below. So we can certainly get all k in the range (N/2 to N-14) by this approach. Now suppose that we already have a complete set of expressions for N_1 and for N_2 (where we may have $N_1 = N_2$). Consider $N_3 = N_1N_2$. Writing $N_3 = N_1$(an expression for N_2 as a sum of k squares) gives N_3 as a sum of 1 thru k_2 squares, where $k_2 = N_2 - 14$ squares (since N_1 is a square). Now express N_1 as a sum of two squares: $n_1^2 + n_2^2$. We have $N_3 = n_1^2$(a sum of k_2 squares) + n_2^2(a sum of k squares). This gives N_3 as a sum of $k_2 + 1$ thru $2k_2$ squares. Continuing in this way gives N_3 as a sum of 1 thru k_1k_2 squares. But $k_i = N_i - 14 > 2/3\, N_i$, so $k_1k_2 > N_3/2$. So when combined with the top down grouping we get a complete set of expressions for N_3. This shows that there are infinitely many squares N with a complete set of expressions, for example we may take N = the squares of 13, 13^2, 13^3, ...

IMO 1993

Problem A1

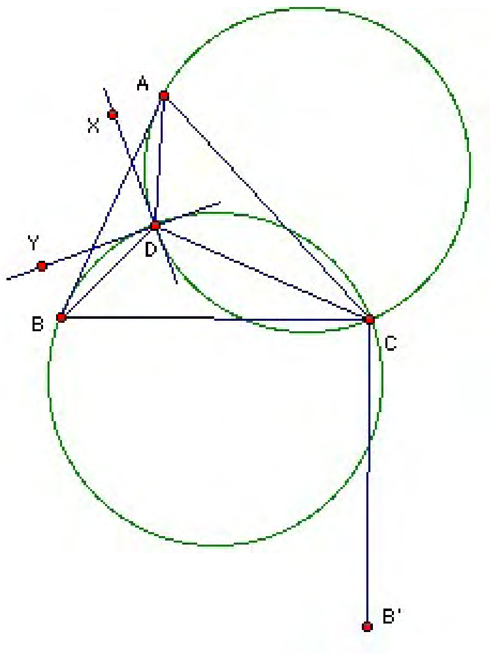
Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that f(x) cannot be expressed as the product of two non-constant polynomials with integer coefficients.

Solution

Suppose $f(x) = (x^r + a_{r-1}x^{r-1} + ... + a_1x \pm 3)(x^s + b_{s-1}x^{s-1} + ... + b_1x \pm 1)$. We show that all the a's are divisible by 3 and use that to establish a contradiction.
First, r and s must be greater than 1. For if r = 1, then ± 3 is a root, so if n is even, we would have $0 = 3^n \pm 5\cdot 3^{n-1} + 3 = 3^{n-1}(3 \pm 5) + 3$, which is false since 3 ± 5 = 8 or -2. Similarly if n is odd we would have $0 = 3^{n-1}(\pm 3 + 5) + 3$, which is false since ±3 + 5 = 8 or 2. If s = 1, then ±1 is a root and we obtain a contradiction in the same way.
So $r \le n - 2$, and hence the coefficients of x, x^2, ... , x^r are all zero. Since the coefficient of x is zero, we have: $a_1 \pm 3b_1 = 0$, so a_1 is divisible by 3. We can now proceed by induction. Assume $a_1, ... , a_t$ are all divisible by 3. Then consider the coefficient of x^{t+1}. If $s-1 \ge t+1$, then a_{t+1} = linear combination of $a_1, ... , a_t \pm 3b_{t+1}$. If $s-1 < t+1$, then a_{t+1} = linear combination of some or all of $a_1, ... , a_t$. Either way, a_{t+1} is divisible by 3. So considering the coefficients of x, x^2, ... , x^{r-1} gives us that all the a's are multiples of 3. Now consider the coefficient of x^r, which is also zero. It is a sum of terms which are multiples of 3 plus ±1, so it is not zero. Contradiction. Hence the factorization is not possible.

Problem A2

Let D be a point inside the acute-angled triangle ABC such that ADB = ACB + 90°, and AC·BD = AD·BC.
(a) Calculate the ratio AB·CD/(AC·BD).
(b) Prove that the tangents at C to the circumcircles of ACD and BCD are perpendicular.



Solution

Take B' so that CB = CB', ∠BCB' = 90° and B' is on the opposite side of BC to A. It is easy to check that ADB, ACB' are similar and DAC, BAB' are similar. Hence AB/BD = AB'/B'C and CD/AC = BB'/AB'. It follows that the ratio given is BB'/B'C which is √2. Take XD the tangent to the circumcircle of ADC at D, so that XD is in the ∠ADB. Similarly, take YD the tangent to the circumcircle BDC at D. Then ∠ADX = ∠ACD, ∠BDY = ∠BCD, so ∠ADX + ∠BDY = ∠ACB and hence ∠XDY = ∠ADB - (∠ADX + ∠BDY) = ∠ADB - ∠ACB = 90°. In other words the tangents to the circumcircles at D are perpendicular. Hence, by symmetry (reflecting in the line of centers) the

tangents at C are perpendicular.
Theo Koupelis, University of Wisconsin, Marathon provided a similar solution (about 10 minutes later!) taking the point B' so that BDB' = 90^o, BD = B'D and ∠B'DA = ∠ACB. DAC, B'AB are similar; and ABC, AB'D are similar.
Marcin Mazur, University of Illinois at Urbana-Champaign provided the first solution I received (about 10 minutes earlier!) using the generalized Ptolemy's equality (as opposed to the easier equality), but I do not know of a slick proof of this, so I prefer the proof above.

Problem A3
On an infinite chessboard a game is played as follows. At the start n^2 pieces are arranged in an n x n block of adjoining squares, one piece on each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of n for which the game can end with only one piece remaining on the board.
Solution
We show first that the game can end with only one piece if n is not a multiple of 3. Note first that the result is true for n = 2 or 4.
n=2
```
XX  ..X  ..X  ...
XX  XX   ..X  ...
               X
```

n = 4
```
      X    X    X    X    X
XXXX XX.X XX.X ..XX ..XX .X..
XXXX XX.X ..XX ..XX X.XX X.XX
XXXX XXXX XXXX XXXX .XXX .XXX
XXXX XXXX XXXX XXXX .XXX .XXX

 X    X    X

.X.. .X.. .XX. .X.. .X.. ....
XX.. ..X. .... ..X. .XX. ..X.
.XXX .XXX .X.X .X.X ...X .X.X
.XXX .XXX .XXX .XXX ..XX ..XX

.... .... .... ....
..XX .X.. .... ....
.X.. .X.. .... ....
..X. ..X. .XX. ...X
```

The key technique is the following three moves which can be used to wipe out three adjacent pieces on the border provided there are pieces behind them:

```
XXX  XX.  XX.  XXX
XXX  XX.  ..X  ...
      X    X
```

We can use this technique to reduce (r + 3) x s rectangle to an r x s rectangle. There is a slight wrinkle for the last two rows of three:

```
XXXX  XX.X  ..XX  ..XX  ...X  ...X
XXXX  XX.X  ..XX  ..XX  ...X  ...X
...X  ..XX  ..XX  .X..  .XX.  ...X
```

Thus we can reduce a square side 3n+2 to a 2 x (3n+2) rectangle. We now show how to wipe out the rectangle. First, we change the 2 x 2 rectangle at one end into a single piece alongside the (now) 2 x 3n rectangle:

```
 X X    . .     . .
 X X    . .     . .
      X X       X
```

Then we use the following technique to shorten the rectangle by 3:

```
X X X    X . X    X . X    . . .    . . .
X X X    X . X    X . X    . . .    . . .
X        X X    X .      X X . X       . X
```

That completes the case of the square side 3n+2. For the square side 3n+1 we can use the technique for removing 3 x r rectangles to reduce it to a 4 x 4 square and then use the technique above for the 4 x 4 rectangle.
Finally, we use a parity argument to show that if n is a multiple of 3, then the square side n cannot be reduced to a single piece. Color the board with 3 colors, red, white and blue:

```
R W B R W B R W B ...
W B R W B R W B R ...
B R W B R W B R W ...
R W B R W B R W B ...
...
```

Let suppose that the single piece is on a red square. Let A be the number of moves onto a red square, B the number of moves onto a white square and C the number of moves onto a blue square. A move onto a red square increases the number of pieces on red squares by 1, reduces the number of pieces on white squares by 1, and reduces the number of pieces on blue squares by 1. Let n = 3m. Then there are initially m pieces on red squares, m on white and m on blue. Thus we have:
$- A + B + C = m-1; \quad A - B + C = m; \quad A + B - C = m.$
Solving, we get A = m, B = m - 1/2, C = m - 1/2. But the number of moves of each type must be integral, so it is not possible to reduce the number of pieces to one if n is a multiple of 3.

Problem B1
For three points P, Q, R in the plane define m(PQR) as the minimum length of the three altitudes of the triangle PQR (or zero if the points are collinear). Prove that for any points A, B, C, X:
$m(ABC) \le m(ABX) + m(AXC) + m(XBC).$

Solution
The length of an altitude is twice the area divided by the length of the corresponding side. Suppose that BC is the longest side of the triangle ABC. Then m(ABC) = area ABC/BC. [If A = B = C, so that BC = 0, then the result is trivially true.]
Consider first the case of X inside ABC. Then area ABC = area ABX + area AXC + area XBC, so m(ABC)/2 = area ABX/BC + area AXC/BC + area XBC/BC. We now claim that the longest side of ABX is at most BC, and similarly for AXC and XBC. It then follows at once that area ABX/BC ≤ area ABX/longest side of ABX = m(ABX)/2 and the result follows (for points X inside ABC).
The claim follows from the following lemma. If Y lies between D and E, then FY is less than the greater than FD and FE. Proof: let H be the foot of the perpendicular from F to DE. One of D and E must lie on the opposite side of Y to H. Suppose it is D. Then FD = FH/cos HFD > FH/cos HFY = FY. Returning to ABCX, let CX meet AB at Y. Consider the three sides of ABX. By definition AB ≤ BC. By the lemma AX is smaller than the larger of AC and AY, both of which do not exceed BC. Hence AX ≤ BC. Similarly BX ≤ BC.
It remains to consider X outside ABC. Let AX meet AC at O. We show that the sum of the smallest altitudes of ABY and BCY is at least the sum of the smallest altitudes of ABO and ACO. The result then follows, since we already have the result for X = O. The altitude from A in ABX is the same as the altitude from A in ABO. The altitude from X in ABX is clearly longer than the altitude from O in ABO (let the altitudes meet the line AB at Q and R

respectively, then triangles BOR and BXQ are similar, so XQ = OR·BX/BO > OR). Finally, let k be the line through A parallel to BX, then the altitude from B in ABX either crosses k before it meets AX, or crosses AC before it crosses AX. If the former, then it is longer than the perpendicular from B to k, which equals the altitude from A to BO. If the latter, then it is longer than the altitude from B to AO. Thus each of the altitudes in ABX is longer than an altitude in ABO, so m(ABX) > m(ABO).

Problem B2

Does there exist a function f from the positive integers to the positive integers such that f(1) = 2, f(f(n)) = f(n) + n for all n, and f(n) < f(n+1) for all n?

Answer

Yes: f(n) = [g*n + ½], where g = (1 + √5)/2 = 1.618 ... .

Solution

Let g(n) = [g*n + ½]. Obviously g(1) = 2. Also g(n+1) = g(n) + 1 or g(n) + 2, so certainly g(n+1) > g(n).
Consider d(n) = g* [g*n + ½] + ½ - ([g*n + ½] + n). We show that it is between 0 and 1. It follows immediately that g(g(n)) = g(n) + n, as required.
Certainly, [g*n + ½] > g*n - ½, so d(n) > 1 - g/2 > 0 (the n term has coefficient g^2 - g - 1 which is zero). Similarly, [g*n + ½] ≤ g*n + ½, so d(n) ≤ g/2 < 1, which completes the proof.

Problem B3

There are n > 1 lamps L_0, L_1, ... , L_{n-1} in a circle. We use L_{n+k} to mean L_k. A lamp is at all times either on or off. Initially they are all on. Perform steps s_0, s_1, ... as follows: at step s_i, if L_{i-1} is lit, then switch L_i from on to off or vice versa, otherwise do nothing. Show that:
(a) There is a positive integer M(n) such that after M(n) steps all the lamps are on again;
(b) If n = 2^k, then we can take M(n) = n^2 - 1.
(c) If n = 2^k + 1, then we can take M(n) = n^2 - n + 1.

Solution

(a) The process cannot terminate, because before the last move a single lamp would have been on. But the last move could not have turned it off, because the adjacent lamp was off. There are only finitely many states (each lamp is on or off and the next move can be at one of finitely many lamps), hence the process must repeat. The outcome of each step is uniquely determined by the state, so either the process moves around a single large loop, or there is an initial sequence of steps as far as state k and then the process goes around a loop back to k. However, the latter is not possible because then state k would have had two different precursors. But a state has only one possible precursor which can be found by toggling the lamp at the current position if the previous lamp is on and then moving the position back one. Hence the process must move around a single large loop, and hence it must return to the initial state.
(b) Represent a lamp by X when on, by - when not. For 4 lamps the starting situation and the situation after 4, 8, 12, 16 steps is as follows:
X X X X
- X - X
X - - X
- - - X
X X X -

On its first move lamp n-2 is switched off and then remains off until each lamp has had n-1 moves. Hence for each of its first n-1 moves lamp n-1 is not toggled and it retains its initial state. After each lamp has had n-1 moves, all of lamps 1 to n-2 are off. Finally over the next n-1 moves, lamps 1 to n-2 are turned on, so that all the lamps are on. We show by induction on k that these statements are all true for n = 2^k. By inspection, they are true for k = 2. So suppose they are true for k and consider 2n = 2^{k+1} lamps. For the first n-1 moves of each lamp the n left-hand and the n right-hand lamps are effectively insulated. Lamps n-1 and 2n-1 remain on. Lamp 2n-1 being on means that lamps 0 to n-2 are in just the same

situation that they would be with a set of only n lamps. Similarly, lamp n-1 being on means that lamps n to 2n-2 are in the same situation that they would be with a set of only n lamps. Hence after each lamp has had n-1 moves, all the lamps are off except for n-1 and 2n-1. In the next n moves lamps 1 to n-2 are turned on, lamp n-1 is turned off, lamps n to 2n-2 remain off, and lamp 2n-1 remains on. For the next n-1 moves for each lamp, lamp n-1 is not toggled, so it remains off. Hence all of n to 2n-2 also remain off and 2n-1 remains on. Lamps 0 to n-2 go through the same sequence as for a set of n lamps. Hence after these n-1 moves for each lamp, all the lamps are off, except for 2n-1. Finally, over the next 2n-1 moves, lamps 0 to 2n-2 are turned on. This completes the induction. Counting moves, we see that there are n-1 sets of n moves, followed by n-1 moves, a total of $n^2 - 1$.
(c) We show by induction on the number of moves that for $n = 2^k + 1$ lamps after each lamp has had m moves, for $i = 0, 1, \dots , 2^k - 2$, lamp i+2 is in the same state as lamp i is after each lamp has had m moves in a set of $n - 1 = 2^k$ lamps (we refer to this as lamp i in the *reduced* case). Lamp 0 is off and lamp 1 is on. It is easy to see that this is true for m = 1 (in both cases odd numbered lamps are on and even numbered lamps are off). Suppose it is true for m. Lamp 2 has the same state as lamp 0 in the reduced case and both toggle since their predecessor lamps are on. Hence lamps 3 to n - 1 behave the same as lamps 1 to n - 3 in the reduced case. That means that lamp n - 1 remains off. Hence lamp 0 does not toggle on its m+1th move and remains off. Hence lamp 1 does not toggle on its m+1th move and remains on. The induction stops working when lamp n - 2 toggles on its nth move in the reduced case, but it works up to and including m = n - 2. So after n - 2 moves for each lamp all lamps are off except lamp 1. In the next two moves nothing happens, then in the following n - 1 moves lamps 2 to n - 1 and lamp 0 are turned on. So all the lamps are on after a total of $(n - 2)n + n + 1 = n^2 + n + 1$ moves.

IMO 1994

Problem A1

Let m and n be positive integers. Let $a_1, a_2, \dots , a_m$ be distinct elements of $\{1, 2, \dots , n\}$ such that whenever $a_i + a_j \le n$ for some i, j (possibly the same) we have $a_i + a_j = a_k$ for some k. Prove that:

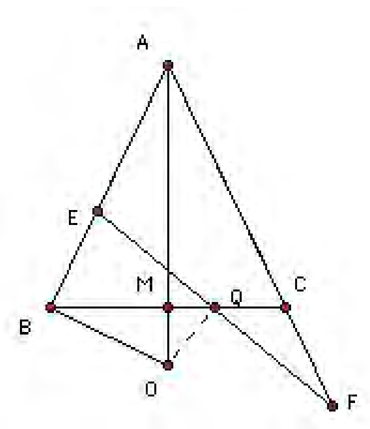
$(a_1 + \dots + a_m)/m \ge (n + 1)/2$.

Solution

Take $a_1 < a_2 < \dots < a_m$. Take $k \le (m+1)/2$. We show that $a_k + a_{m-k+1} \ge n + 1$. If not, then the k distinct numbers $a_1 + a_{m-k+1}, a_2 + a_{m-k+1}, \dots , a_k + a_{m-k+1}$ are all $\le n$ and hence equal to some a_i. But they are all greater than a_{m-k+1}, so each i satisfies $m-k+2 \le i \le m$, which is impossible since there are only k-1 available numbers in the range.

Problem A2

ABC is an isosceles triangle with AB = AC. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB. Q is an arbitrary point on BC different from B and C. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear. Prove that OQ is perpendicular to EF if and only if QE = QF.



Solution

Assume OQ is perpendicular to EF. Then ∠EBO = ∠EQO = 90°, so EBOQ is cyclic. Hence ∠OEQ = ∠OBQ. Also ∠OQF = ∠OCF = 90°, so OQCF is cyclic. Hence ∠OFQ = ∠OCQ. But ∠OCQ = ∠OBQ since ABC is isosceles. Hence ∠OEQ = ∠OFQ, so OE = OF, so triangles OEQ and OFQ are congruent and QE = QF.
Assume QE = QF. If OQ is not perpendicular to EF, then take E'F' through Q perpendicular to OQ with E' on AB and F' on AC. Then QE' = QF', so triangles QEE' and QFF' are congruent. Hence ∠QEE' = ∠QFF'. So CA and AB make the same angles with EF and hence are parallel. Contradiction. So OQ is perpendicular to EF.

Problem A3

For any positive integer k, let f(k) be the number of elements in the set {k+1, k+2, ... , 2k} which have exactly three 1s when written in base 2. Prove that for each positive integer m, there is at least one k with f(k) = m, and determine all m for which there is exactly one k.

Answer

2, 4, ... , n(n-1)/2 + 1, ...

Solution

To get a feel, we calculate the first few values of f explicitly:
f(2) = 0, f(3) = 0
f(4) = f(5) = 1, [7 = 111]
f(6) = 2, [7 = 111, 11 = 1011]
f(7) = f(8) = f(9) = 3 [11 = 1011, 13 = 1101, 14 = 1110]
f(10) = 4 [11, 13, 14, 19 = 10011]
f(11) = f(12) = 5 [13, 14, 19, 21 = 10101, 22 = 10110]
f(13) = 6 [14, 19, 21, 22, 25 = 11001, 26 = 11010]
We show that f(k+1) = f(k) or f(k) + 1. The set for k+1 has the additional elements 2k+1 and 2k+2 and it loses the element k+1. But the binary expression for 2k+2 is the same as that for k+1 with the addition of a zero at the end, so 2k+2 and k+1 have the same number of 1s. So if 2k+1 has three 1s, then f(k+1) = f(k) + 1, otherwise f(k+1) = f(k). Now clearly an infinite number of numbers 2k+1 have three 1s, (all numbers $2^r + 2^s + 1$ for r > s > 0). So f(k) increases without limit, and since it only moves up in increments of 1, it never skips a number. In other words, given any positive integer m we can find k so that f(k) = m.
From the analysis in the last paragraph we can only have a single k with f(k) = m if both 2k-1 and 2k+1 have three 1s, or in other words if both k-1 and k have two 1s. Evidently this happens when k-1 has the form $2^n + 1$. This determines the k, namely $2^n + 2$, but we need to determine the corresponding m = f(k). It is the number of elements of $\{2^n+3, 2^n+4, \dots , 2^{n+1}+4\}$ which have three 1s. Elements with three 1s are either $2^n+2^r+2^s$ with $0 \le r < s < n$, or $2^{n+1}+3$. So there are m= n(n-1)/2 + 1 of them. As a check, for n = 2, we have $k = 2^2+2 = 6$, m = 2, and for n = 3, we have $k = 2^3+2 = 10$, m = 4, which agrees with the f(6) = 2, f(10) = 4 found earlier.

Problem B1

Determine all ordered pairs (m, n) of positive integers for which $(n^3 + 1)/(mn - 1)$ is an integer.

Answer

(1, 2), (1, 3), (2, 1), (2, 2), (2, 5), (3, 1), (3, 5), (5, 2), (5, 3).

Solution

We start by checking small values of n. n = 1 gives $n^3 + 1 = 2$, so m = 2 or 3, giving the solutions (2, 1) and (3, 1). Similarly, n = 2 gives $n^3 + 1 = 9$, so 2m-1 = 1, 3 or 9, giving the solutions (1, 2), (2, 2), (5, 2). Similarly, n = 3 gives $n^3 + 1 = 28$, so 3m - 1 = 2, 14, giving the solutions (1, 3), (5, 3). So we assume hereafter that n > 3.
Let $n^3 + 1 = (mn - 1)h$. Then we must have h = -1 (mod n). Put h = kn - 1. Then $n^3 + 1 = mkn^2 - (m + k)n + 1$. Hence $n^2 = mkn - (m + k)$. (*)   Hence n divides m + k. If $m + k \ge 3n$, then since n > 3 we have at least one of m, k ≥ n + 2. But then $(mn - 1)(kn - 1) \ge (n^2 + 2n - 1)(n - 1) = n^3 + n^2 - 3n + 1 = (n^3 + 1) + n(n - 3) > n^3 + 1$. So we must have m + k = n or 2n.
Consider first m + k = n. We may take m ≥ k (provided that we remember that if m is a solution, then so is n - m). So (*) gives n = m(n - m) - 1. Clearly m = n - 1 is not a solution. If m = n - 2, then n = 2(n - 2) - 1, so n = 5. This gives the two solutions (m, n) = (2, 5) and (3, 5). If m < n - 2 then n - m ≥ 3 and so m(n - m) - 1 ≥ 3m - 1 ≥ 3n/2 - 1 > n for n > 3.

Finally, take m + k = 2n. So (*) gives n + 2 = m(2n - m). Again we may take m ≥ k. m = 2n - 1 is not a solution (we are assuming n > 3). So 2n - m ≥ 2, and hence m(2n - m) ≥ 2m ≥ 2n > n + 2.

Problem B2
Let S be the set of all real numbers greater than -1. Find all functions f :S   S such that f(x + f(y) + xf(y)) = y + f(x) + yf(x) for all x and y, and f(x)/x is strictly increasing on each of the intervals -1 < x < 0 and 0 < x.

Answer
f(x) = -x/(x+1).

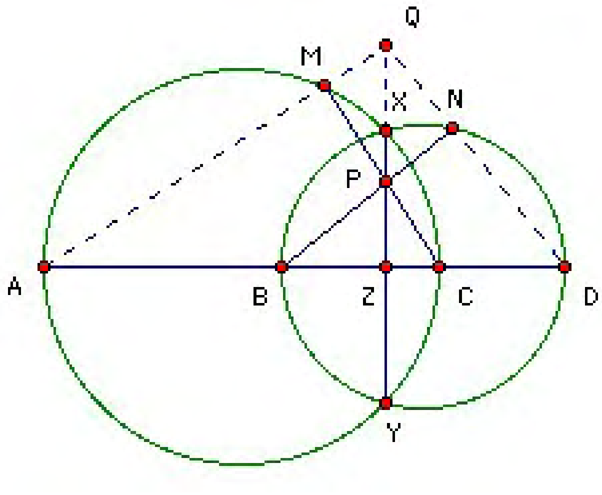
Solution
Suppose f(a) = a. Then putting x = y = a in the relation given, we get f(b) = b, where b = 2a + a^2. If -1 < a < 0, then -1 < b < a. But f(a)/a = f(b)/b. Contradiction. Similarly, if a > 0, then b > a, but f(a)/a = f(b)/b. Contradiction. So we must have a = 0.
But putting x = y in the relation given we get f(k) = k for k = x + f(x) + xf(x). Hence for any x we have x + f(x) + xf(x) = 0 and hence f(x) = -x/(x+1).
Finally, it is straightforward to check that f(x) = -x(x+1) satisfies the two conditions.

Problem B3
Show that there exists a set A of positive integers with the following property: for any infinite set S of primes, there exist two positive integers m in A and n not in A, each of which is a product of k distinct elements of S for some k ≥ 2.

Solution
Let the primes be $p_1 < p_2 < p_3 < \dots$. Let A consists of all products of n distinct primes such that the smallest is greater than p_n. For example: all primes except 2 are in A; 21 is not in A because it is a product of two distinct primes and the smallest is greater than 3. Now let S = $\{p_{i1}, p_{i2}, \dots \}$ be any infinite set of primes. Assume that $p_{i1} < p_{i2} < \dots$. Let n = i_1. Then $p_{i1}p_{i2} \dots p_{in}$ is not in A because it is a product of n distinct primes, but the smallest is not greater than p_n. But $p_{i2}p_{i3} \dots p_{in+1}$ is in A, because it is a product of n distinct primes and the smallest is greater than p_n. But both numbers are products of n distinct elements of S.

IMO 1995

Problem A1
Let A, B, C, D be four distinct points on a line, in that order. The circles with diameter AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.



Solution
Let DN meet XY at Q. Angle QDZ = 90^o - angle NBD = angle BPZ. So triangles QDZ and BPZ are similar. Hence QZ/DZ = BZ/PZ, or QZ = BZ·DZ/PZ. Let AM meet XY at Q'. Then the same argument shows that Q'Z = AZ·CZ/PZ. But BZ·DZ = XZ·YZ = AZ·CZ, so QZ = Q'Z. Hence Q and Q' coincide.

Problem A2
Let a, b, c be positive real numbers with abc = 1. Prove that:

$1/(a^3(b + c)) + 1/(b^3(c + a)) + 1/(c^3(a + b)) \geq 3/2$.

Solution

Put $a = 1/x$, $b = 1/y$, $c = 1/z$. Then $1/(a^3(b+c)) = x^3yz/(y+z) = x^2/(y+z)$. Let the expression given be E. Then by Cauchy's inequality we have $(y+z + z+x + x+y)E \ge (x + y + z)^2$, so $E \ge (x + y + z)/2$. But applying the arithmetic/geometric mean result to x, y, z gives $(x + y + z) \ge 3$. Hence result.

Problem A3

Determine all integers $n > 3$ for which there exist n points $A_1, \dots , A_n$ in the plane, no three collinear, and real numbers $r_1, \dots , r_n$ such that for any distinct i, j, k, the area of the triangle $A_iA_jA_k$ is $r_i + r_j + r_k$.

Answer

$n = 4$.

Solution

The first point to notice is that if no arrangement is possible for n, then no arrangement is possible for any higher integer. Clearly the four points of a square work for n = 4, so we focus on n = 5.

If the 5 points form a convex pentagon, then considering the quadrilateral $A_1A_2A_3A_4$ as made up of two triangles in two ways, we have that $r_1 + r_3 = r_2 + r_4$. Similarly, $A_5A_1A_2A_3$ gives $r_1 + r_3 = r_2 + r_5$, so $r_4 = r_5$.

We show that we cannot have two r's equal (whether or not the 4 points form a convex pentagon). For suppose $r_4 = r_5$. Then $A_1A_2A_4$ and $A_1A_2A_5$ have equal area. If A_4 and A_5 are on the same side of the line A_1A_2, then since they must be equal distances from it, A_4A_5 is parallel to A_1A_2. If they are on opposite sides, then the midpoint of A_4A_5 must lie on A_1A_2. The same argument can be applied to A_1 and A_3, and to A_2 and A_3. But we cannot have two of A_1A_2, A_1A_3 and A_2A_3 parallel to A_4A_5, because then A_1, A_2 and A_3 would be collinear. We also cannot have the midpoint of A_4A_5 lying on two of A_1A_2, A_1A_3 and A_2A_3 for the same reason. So we have established a contradiction. hence no two of the r's can be equal. In particular, this shows that the 5 points cannot form a convex pentagon.

Suppose the convex hull is a quadrilateral. Without loss of generality, we may take it to be $A_1A_2A_3A_4$. A_5 must lie inside one of $A_1A_2A_4$ and $A_2A_3A_4$. Again without loss of generality we may take it to be the latter, so that $A_1A_2A_5A_4$ is also a convex quadrilateral. Then $r_2 + r_4 = r_1 + r_3$ and also $= r_1 + r_5$. So $r_3 = r_5$, giving a contradiction as before.

The final case is the convex hull a triangle, which we may suppose to be $A_1A_2A_3$. Each of the other two points divides its area into three triangles, so we have: $(r_1 + r_2 + r_4) + (r_2 + r_3 + r_4) + (r_3 + r_1 + r_4) = (r_1 + r_2 + r_5) + (r_2 + r_3 + r_5) + (r_3 + r_1 + r_5)$ and hence $r_4 = r_5$, giving a contradiction.

So the arrangement is not possible for 5 and hence not for any $n > 5$.

Problem B1

Find the maximum value of x_0 for which there exists a sequence $x_0, x_1, \dots , x_{1995}$ of positive reals with $x_0 = x_{1995}$ such that for $i = 1, \dots , 1995$:

$$x_{i-1} + 2/x_{i-1} = 2x_i + 1/x_i.$$

Answer

2^{997}.

Solution

The relation given is a quadratic in x_i, so it has two solutions, and by inspection these are $x_i = 1/x_{i-1}$ and $x_{i-1}/2$. For an even number of moves we can start with an arbitrary x_0 and get back to it. Use n-1 halvings, then take the inverse, that gets to $2^{n-1}/x_0$ after n moves. Repeating brings you back to x_0 after 2n moves. However, 1995 is odd!

The sequence given above brings us back to x_0 after n moves, provided that $x_0 = 2^{(n-1)/2}$. We show that this is the largest possible x_0. Suppose we have a halvings followed by an inverse followed by b halvings followed by an inverse. Then if the number of inverses is odd we end up with $2^{a-b+c- \cdots}/x_0$, and if it is even we end up with $x_0/2^{a-b+c- \cdots}$. In the first case, since the final number is x_0 we must have $x_0 = 2^{(a-b+-\dots)/2}$. All the a, b, ... are non-negative and sum to the number of moves less the number of inverses, so we clearly maximise x_0 by taking a single inverse and a = n-1. In the second case, we must have $2^{a-b+c- \dots} = 1$ and hence a - b + c - ... = 0. But that implies that a + b + c + ... is even and hence the total number of moves is even, which it is not. So we must have an odd number of inverses and the maximum value of x_0 is $2^{(n-1)/2}$.

Problem B2
Let ABCDEF be convex hexagon with AB = BC = CD and DE = EF = FA, such that ∠BCD = ∠EFA = 60°. Suppose that G and H are points in the interior of the hexagon such that ∠AGB = ∠DHE = 120°. Prove that AG + GB + GH + DH + HE ≥ CF.

Solution
BCD is an equilateral triangle and AEF is an equilateral triangle. The presence of equilateral triangles and quadrilaterals suggests using Ptolemy's inequality. From CBGD, we get CG·BD ≤ BG·CD + GD·CB, so CG ≤ BG + GD. Similarly from HAFE we get HF ≤ HA + HE. Also CF is shorter than the indirect path C to G to H to F, so CF ≤ CG + GH + HF. But we do not get quite what we want.
However, a slight modification of the argument does work. BAED is symmetrical about BE (because BA = BD and EA = ED). So we may take C' the reflection of C in the line BE and F' the reflection of F. Now C'AB and F'ED are still equilateral, so the same argument gives C'G ≥ AG + GB and HF' ≤ DH + HE. So CF = C'F' ≤ C'G + GH + HF' ≤ AG + GB + GH + DH + HE.

Problem B3
Let p be an odd prime number. How many p-element subsets A of {1, 2, ... , 2p} are there, the sum of whose elements is divisible by p?

Answer
2 + (2pCp - 2)/p, where 2pCp is the binomial coefficient (2p)!/(p!p!).

Solution
Let A be a subset other than {1, 2, ... , p} and {p+1, p+2, ... , 2p}. Consider the elements of A in {1, 2, ... , p}. The number r satisfies 0 < r < p. We can change these elements to another set of r elements of {1, 2, ... , p} by adding 1 to each element (and reducing mod p if necessary). We can repeat this process and get p sets in all. For example, if p = 7 and the original subset of {1, 2, ... , 7} was {3 , 5}, we get:
{3 , 5}, {4, 6}, {5, 7}, {6, 1}, {7, 2}, {1, 3}, {2, 4}.
The sum of the elements in the set is increased by r each time. So, since p is prime, the sums must form a complete set of residues mod p. In particular, they must all be distinct and hence all the subsets must be different.
Now consider the sets A which have a given intersection with {p+1, ... , n}. Suppose the elements in this intersection sum to k mod p. The sets can be partitioned into groups of p by the process described above, so that exactly one member of each group will have the sum -k mod p for its elements in {1, 2, ... , p}. In other words, exactly one member of each group will have the sum of all its elements divisible by p.
There are 2pCp subsets of {1, 2, ... , 2p} of size p. Excluding {1, 2, ... , p} and {p+1, ... , 2p} leaves (2pCp - 2). We have just shown that (2pCp - 2)/p of these have sum divisible by p. The two excluded subsets also have sum divisible by p, so there are 2 + (2pCp - 2)/p subsets in all having sum divisible by p.

IMO 1996

Problem A1
We are given a positive integer r and a rectangular board divided into 20 x 12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is √r. The task is to find a sequence of moves leading between two adjacent corners of the board which lie on the long side.

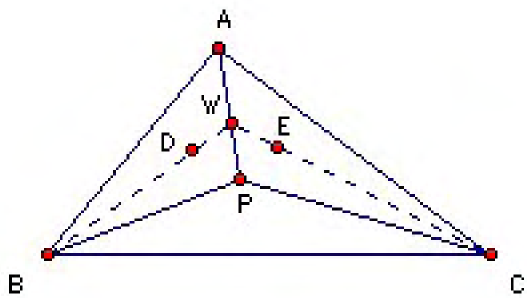
(a) Show that the task cannot be done if r is divisible by 2 or 3.
(b) Prove that the task is possible for r = 73.
(c) Can the task be done for r = 97?

Answer
No.

Solution
(a) Suppose the move is a units in one direction and b in the orthogonal direction. So $a^2 + b^2 = r$. If r is divisible by 2, then a and b are both even or both odd. But that means that we can only access the black squares or the white squares (assuming the rectangle is colored like a chessboard). The two corners are of opposite color, so the task cannot be done. All squares are congruent to 0 or 1 mod 3, so if r is divisible by 3, then a and b must both be multiples of 3. That means that if the starting square has coordinates (0,0), we can only move to squares of the form (3m,3n). The required destination is (19,0) which is not of this form, so the task cannot be done.
(b) If r = 73, then we must have a = 8, b = 3 (or vice versa). There are 4 types of move:
A: (x,y) to (x+8,y+3)
B: (x,y) to (x+3,y+8)
C: (x,y) to (x+8,y-3)
D: (x,y) to (x+3,y-8)
We regard (x,y) to (x-8,y-3) as a negative move of type A, and so on. Then if we have a moves of type A, b of type B and so on, then we require:
$8(a + c) + 3(b + d) = 19;\ 3(a - c) + 8(b - d) = 0.$
A simple solution is a = 5, b = -1, c = -3, d = 2, so we start by looking for solutions of this type. After some fiddling we find:
(0,0) to (8,3) to (16,6) to (8,9) to (11,1) to (19,4) to (11,7) to (19,10) to (16,2) to (8,5) to (16,8) to (19,0).
(c) If r = 97, then we must have a = 9, b = 4. As before, assume we start at (0,0). A good deal of fiddling around fails to find a solution, so we look for reasons why one is impossible. Call moves which change y by 4 "toggle" moves. Consider the central strip y = 4, 5, 6 or 7. Toggle moves must toggle us in and out of the strip. Non-toggle moves cannot be made if we are in the strip and keep us out of it if we are out of it. Toggle moves also change the parity of the x-coordinate, whereas non-toggle moves do not. Now we start and finish out of the strip, so we need an even number of toggle moves. On the other hand, we start with even x and end with odd x, so we need an odd number of toggle moves. Hence the task is impossible.

Problem A2
Let P be a point inside the triangle ABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD, CE meet at a point.

Solution
We need two general results: the angle bisector theorem; and the result about the feet of the perpendiculars from a general point inside a triangle. The second is not so well-known.
Let P be a general point in the triangle ABC with X, Y, Z the feet of the perpendiculars to BC, CA, AB. Then PA = YZ/sin A and $\angle APB - \angle C = \angle XZY$. To prove the first part: AP = AY/sin APY = AY/sin AZY (since AYPZ is cyclic) = YZ/sin A (sine rule). To prove the second part: $\angle XZY = \angle XZP + \angle YZP = \angle XBP + \angle YAP = 90^o - \angle XPB + 90^o - \angle YPA = 180^o - (360^o - \angle APB - \angle XPY) = -180^o + \angle APB + (180^o - \angle C) = \angle APB - \angle C$.



So, returning to the problem, $\angle APB - \angle C = \angle XZY$ and $\angle APC - \angle B = \angle XYZ$. Hence XYZ is isosceles: XY = XZ. Hence PC sin C = PB sin B. But AC sin C = AB sin B, so AB/PB = AC/PC. Let the angle bisector BD meet AP at W. Then, by the angle bisector theorem, AB/PB = AW/WP. Hence AW/WP = AC/PC, so, by the angle bisector theorem, CW is the bisector of angle ACP, as required.

Problem A3
Let S be the set of non-negative integers. Find all functions f: $S \to S$ such that $f(m + f(n)) = f(f(m)) + f(n)$ for all m, n.

Solution
Setting m = n = 0, the given relation becomes: $f(f(0)) = f(f(0)) + f(0)$. Hence $f(0) = 0$. Hence also $f(f(0)) = 0$. Setting m = 0, now gives $f(f(n)) = f(n)$, so we may write the original relation as $f(m + f(n)) = f(m) + f(n)$.
So f(n) is a fixed point. Let k be the smallest non-zero fixed point. If k does not exist, then f(n) is zero for all n, which is a possible solution. If k does exist, then an easy induction shows that $f(qk) = qk$ for all non-negative integers q. Now if n is another fixed point, write $n = kq + r$, with $0 \le r < k$. Then $f(n) = f(r + f(kq)) = f(r) + f(kq) = kq + f(r)$. Hence $f(r) = r$, so r must be zero. Hence the fixed points are precisely the multiples of k.
But f(n) is a fixed point for any n, so f(n) is a multiple of k for any n. Let us take $n_1, n_2, \dots , n_{k-1}$ to be arbitrary non-negative integers and set $n_0 = 0$. Then the most general function satisfying the conditions we have established so far is:

$$f(qk + r) = qk + n_r k \text{ for } 0 \le r < k.$$

We can check that this satisfies the functional equation. Let $m = ak + r$, $n = bk + s$, with $0 \le r, s < k$. Then $f(f(m)) = f(m) = ak + n_r k$, and $f(n) = bk + n_s k$, so $f(m + f(n)) = ak + bk + n_r k + n_s k$, and $f(f(m)) + f(n) = ak + bk + n_r k + n_s k$. So this is a solution and hence the most general solution.

Problem B1
The positive integers a, b are such that 15a + 16b and 16a - 15b are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

Answer
481^2.

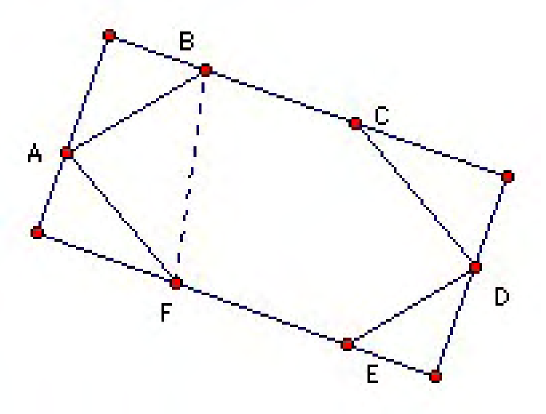
Solution
Put $15a \pm 16b = m^2$, $16a - 15b = n^2$. Then $15m^2 + 16n^2 = 481a = 13 \cdot 37a$. The quadratic residues mod 13 are 0, ±1, ±3, ±4, so the residues of $15m^2$ are 0, ±2, ±5, ±6, and the residues of $16n^2$ are 0, ±1, ±3, ±4. Hence m and n must both be divisible by 13. Similarly, the quadratic residues of 37 are 0, ±1, ±3, ±4, ±7, ±9, ±10, ±11, ±12, ±16, so the residues of $15m^2$ are 0, ±2, ±5, ±6, ±8, ±13, ±14, ±15, ±17, ±18, and the residues of $16n^2$ are 0, ±1, ±3, ±4, ±7, ±9, ±10, ±11, ±12, ±16. Hence m and n must both be divisible by 37. Put $m = 481m'$, $n = 481n'$ and we get: $a = 481(15m'^2 + 16n'^2)$. We also have $481b = 16m^2 - 15n^2$ and hence $b = 481(16m'^2 - 15n'^2)$. The smallest possible solution would come from putting $m' = n' = 1$ and indeed that gives a solution.
This solution is straightforward, but something of a slog - all the residues have to be calculated. A more elegant variant is to notice that $m^4 + n^4 = 481(a^2 + b^2)$. Now if m and n are not divisible by 13 we have $m^4 + n^4 = 0 \pmod{13}$. Take k so that $km = 1 \pmod{13}$, then $(nk)^4 = -(mk)^4 = -1 \pmod{13}$. But that is impossible because then $(nk)^{12} = -1 \pmod{13}$, but $x^{12} = 1 \pmod{13}$ for all non-zero residues. Hence m and n are both divisible by 13. The same argument shows that m and n are both divisible by 37.

Problem B2
Let ABCDEF be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF respectively, and let p denote the perimeter of the hexagon. Prove that:

$$R_A + R_C + R_E \ge p/2.$$

Solution
The starting point is the formula for the circumradius R of a triangle ABC: 2R = a/sin A = b/sin B = c/sin C. [Proof: the side a subtends an angle 2A at the center, so a = 2R sin A.]

This gives that $2R_A = BF/\sin A$, $2R_C = BD/\sin C$, $2R_E = FD/\sin E$. It is clearly not true in general that $BF/\sin A > BA + AF$, although it is true if angle $FAB \ge 120^o$, so we need some argument that involves the hexagon as a whole.



Extend sides BC and FE and take lines perpendicular to them through A and D, thus forming a rectangle. Then BF is greater than or equal to the side through A and the side through D. We may find the length of the side through A by taking the projections of BA and AF giving AB sin B + AF sin F. Similarly the side through D is CD sin C + DE sin E. Hence:

$2BF \ge AB \sin B + AF \sin F + CD \sin C + DE \sin E$. Similarly:

$2BD \ge BC \sin B + CD \sin D + AF \sin A + EF \sin E$, and

$2FD \ge AB \sin A + BC \sin C + DE \sin D + EF \sin F$.

Hence $2BF/\sin A + 2BD/\sin C + 2FD/\sin E \ge AB(\sin A/\sin E + \sin B/\sin A) + BC(\sin B/\sin C + \sin C/\sin E) + CD(\sin C/\sin A + \sin D/\sin C) + DE(\sin E/\sin A + \sin D/\sin E) + EF(\sin E/\sin C + \sin F/\sin E) + AF(\sin F/\sin A + \sin A/\sin C)$.

We now use the fact that opposite sides are parallel, which implies that opposite angles are equal: A = D, B = E, C = F. Each of the factors multiplying the sides in the last expression now has the form $x + 1/x$ which has minimum value 2 when $x = 1$. Hence $2(BF/\sin A + BD/\sin C + FD/\sin E) \ge 2p$ and the result is proved.

Problem B3

Let p, q, n be three positive integers with $p + q < n$. Let $x_0, x_1, \dots, x_n$ be integers such that $x_0 = x_n = 0$, and for each $1 \le i \le n$, $x_i - x_{i-1} = p$ or $-q$. Show that there exist indices $i < j$ with (i, j) not $(0, n)$ such that $x_i = x_j$.

Solution

Let $x_i - x_{i-1} = p$ occur r times and $x_i - x_{i-1} = -q$ occur s times. Then $r + s = n$ and $pr = qs$. If p and q have a common factor d, the $y_i = x_i/d$ form a similar set with p/d and q/d. If the result is true for the y_i then it must also be true for the x_i. So we can assume that p and q are relatively prime. Hence p divides s. Let $s = kp$. If $k = 1$, then $p = s$ and $q = r$, so $p + q = r + s = n$. But we are given $p + q < n$. Hence $k > 1$. Let $p + q = n/k = h$.

Up to this point everything is fairly obvious and the result looks as though it should be easy, but I did not find it so. Some fiddling around with examples suggested that we seem to get $x_i = x_j$ for $j = i + h$. We observe first that $x_{i+h} - x_i$ must be a multiple of h. For suppose e differences are p, and hence h-e are -q. Then $x_{i+h} - x_i = ep - (h - e)q = (e - q)h$. The next step is not obvious. Let $d_i = x_{i+h} - x_i$. We know that all d_is are multiples of h. We wish to show that at least one is zero. Now $d_{i+1} - d_i = (x_{i+h+1} - x_{i+h}) - (x_{i+1} - x_i) = (p \text{ or } -q) - (p \text{ or } -q) = 0$, h or -h. So if neither of d_i nor d_{i+1} are zero, then either both are positive or both are negative (a jump from positive to negative would require a difference of at least 2h). Hence if none of the d_is are zero, then all of them are positive, or all of them are negative. But $d_0 + d_h + \dots + d_{kh}$ is a concertina sum with value $x_n - x_0 = 0$. So this subset of the d_is cannot all be positive or all negative. Hence at least one d_i is zero.

IMO 1997

Problem A1

In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white as on a chessboard. For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along the edges of the squares. Let S_1 be the total area of the black part of the triangle, and S_2 be the total area of the white part. Let $f(m, n) = |S_1 - S_2|$.

(a) Calculate f(m, n) for all positive integers m and n which are either both even or both odd.

(b) Prove that $f(m, n) \le \max(m, n)/2$ for all m, n.
(c) Show that there is no constant C such that $f(m, n) < C$ for all m, n.

Solution

(a) If m and n are both even, then $f(m,n) = 0$. Let M be the midpoint of the hypoteneuse. The critical point is that M is a lattice point. If we rotate the triangle through 180 to give the other half of the rectangle, we find that its coloring is the same. Hence S_1 and S_2 for the triangle are each half their values for the rectangle. But the values for the rectangle are equal, so they must also be equal for the triangle and hence $f(m,n) = 0$.
If m and n are both odd, then the midpoint of the hypoteneuse is the center of a square and we may still find that the coloring of the two halves of the rectangle is the same. This time S_1 and S_2 differ by one for the rectangle, so $f(m,n) = 1/2$.
(b) The result is immediate from (a) for m and n of the same parity. The argument in (a) fails for m and n with opposite parity, because the two halves of the rectangle are oppositely colored. Let m be the odd side. Then if we extend the side length m by 1 we form a new triangle which contains the original triangle. But it has both sides even and hence $S_1 = S_2$. The area added is a triangle base 1 and height n, so area n/2. The worst case would be that all this area was the same color, in which case we would get $f(m,n) = n/2$. But $n <= \max(m,n)$, so this establishes the result.
(c) Intuitively, it is clear that if the hypoteneuse runs along the diagonal of a series of black squares, and we then extend one side, the extra area taken in will be mainly black. We need to make this rigorous. For the diagonal to run along the diagonal of black squares we must have n = m. It is easier to work out the white area added by extending a side. The white area takes the form of a series of triangles each similar to the new n+1 x n triangle. The biggest has sides 1 and $n/(n+1)$. The next biggest has sides $(n-1)/n$ and $(n-1)/(n+1)$, the next biggest $(n-2)/n$ and $(n-2)/(n+1)$ and so on, down to the smallest which is $1/n$ by $1/(n+1)$. Hence the additional white area is $1/2\ (n/(n+1) + (n-1)^2/(n(n+1)) + (n-2)^2/(n(n+1)) + \dots + 1/(n(n+1))\ ) = 1/(2n(n+1))\ (n^2 + \dots + 1^2) = (2n+1)/12$. Hence the additional black area is $n/2 - (2n+1)/12 = n/3 - 1/12$ and the black excess in the additional area is $n/6 - 1/6$. If n is even, then $f(n,n) = 0$ for the original area, so for the new triangle $f(n+1,n) = (n-1)/6$ which is unbounded.

Problem A2

The angle at A is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that AU = TB + TC.

Solution

Extend BV to meet the circle again at X, and extend CW to meet the circle again at Y. Then by symmetry (since the perpendicular bisectors pass through the center of the circle) AU = BX and AU = CY. Also arc AX = arc BU, and arc AY = arc UC. Hence arc XY = arc BC and so angle BYC = angle XBY and hence TY = TB. So AU = CY = CT + TY = CT + TB.

Problem A3

Let $x_1, x_2, \dots, x_n$ be real numbers satisfying $|x_1 + x_2 + \dots + x_n| = 1$ and $|x_i| \le (n+1)/2$ for all i. Show that there exists a permutation y_i of x_i such that $|y_1 + 2y_2 + \dots + ny_n| \le (n+1)/2$.

Solution

Without loss of generality we may assume $x_1 + \dots + x_n = +1$. [If not just reverse the sign of every x_i.] For any given arrangement x_i we use *sum* to mean $x_1 + 2x_2 + 3x_3 + \dots + nx_n$. Now if we add together the sums for $x_1, x_2, \dots, x_n$ and the *reverse* $x_n, x_{n-1}, \dots, x_1$, we get $(n+1)(x_1 + \dots + x_n) = n+1$. So either we are home with the original arrangement or its reverse, or they have sums of opposite sign, one greater than $(n+1)/2$ and one less than $-(n+1)/2$.
A transposition changes the sum from $ka + (k+1)b$ + other terms to $kb + (k+1)a$ + other terms. Hence it changes the sum by $|a - b|$ (where a, b are two of the x_i) which does not exceed n+1. Now we can get from the original arrangement to its reverse by a sequence of

transpositions. Hence at some point in this sequence the sum must fall in the interval $[-(n+1)/2, (n+1)/2]$ (because to get from a point below it to a point above it in a single step requires a jump of more than n+1). That point gives us the required permutation.

Problem B1

An n x n matrix whose entries come from the set S = {1, 2, ... , 2n-1} is called silver matrix if, for each i = 1, 2, ... , n, the ith row and the ith column together contain all elements of S. Show that:

(a) there is no silver matrix for n = 1997;

(b) silver matrices exist for infinitely many values of n.

Solution

(a) If we list all the elements in the rows followed by all the elements in the columns, then we have listed every element in the array twice, so each number in S must appear an even number of times. But considering the ith row with the ith column, we have also given n complete copies of S together with an additional copy of the numbers on the diagonal. If n is odd, then each of the 2n-1 numbers appears an odd number of times in the n complete copies, and at most n numbers can have this converted to an even number by an appearance on the diagonal. So there are no silver matrices for n odd. In particular, there is no silver matrix for n = 1997.

(b) Let $A_{i,j}$ be an n x n silver matrix with 1s down the main diagonal. Define the 2n x 2n matrix $B_{i,j}$ with 1s down the main diagonal as follows: $B_{i,j} = A_{i,j}$; $B_{i+n,j+n} = A_{i,j}$; $B_{i,j+n} = 2n + A_{i,j}$; $B_{i+n,j} = 2n + A_{i,j}$ for i not equal j and $B_{i+n,i} = 2n$. We show that $B_{i,j}$ is silver. Suppose $i \le n$. Then the first half of the ith row is the ith row of $A_{i,j}$, and the top half of the ith column is the ith column of $A_{i,j}$, so between them those two parts comprise the numbers from 1 to 2n - 1. The second half of the ith row is the ith row of $A_{i,j}$ with each element increased by 2n, and the bottom half of the ith column is the ith column of $A_{i,j}$ with each element increased by 2n, so between them they give the numbers from 2n + 1 to 4n - 1. The only exception is that $A_{i+n,i} = 2n$ instead of $2n + A_{i,i}$. We still get $2n + A_{i,i}$ because it was in the second half of the ith row (these two parts do not have an element in common). The 2n fills the gap so that in all we get all the numbers from 1 to 4n - 1.

An exactly similar argument works for i > n. This time the second half of the row and the second half of the column (which overlap by one element) give us the numbers from 1 to 2n - 1, and the first halves (which do not overlap) give us 2n to 4n - 1. So $B_{i,j}$ is silver. Hence there are an infinite number of silver matrices.

Problem B2

Find all pairs (a, b) of positive integers that satisfy $a^{b2} = b^a$.

Answer

(1,1), (16,2), (27,3).

Solution

Notice first that if we have $a^m = b^n$, then we must have $a = c^e$, $b = c^f$, for some c, where m=fd, n=ed and d is the greatest common divisor of m and n. [Proof: express a and b as products of primes in the usual way.]

In this case let d be the greatest common divisor of a and b^2, and put a = de, b^2 = df. Then for some c, $a = c^e$, $b = c^f$. Hence $f c^e = e c^{2f}$. We cannot have e = 2f, for then the c's cancel to give e = f. Contradiction. Suppose 2f > e, then $f = e c^{2f-e}$. Hence e = 1 and $f = c^{2f-1}$. If c = 1, then f = 1 and we have the solution a = b = 1. If $c \ge 2$, then $c^{2f-1} \ge 2^f > f$, so there are no solutions.

Finally, suppose 2f < e. Then $e = f c^{e-2f}$. Hence f = 1 and $e = c^{e-2}$. $c^{e-2} \ge 2^{e-2} \ge e$ for $e \ge 5$, so we must have e = 3 or 4 (e > 2f = 2). e = 3 gives the solution a = 27, b = 3. e = 4 gives the solution a = 16, b = 2.

Problem 6

For each positive integer n, let f(n) denote the number of ways of representing n as a sum of powers of 2 with non-negative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For example, f(4) = 4, because 4 can be represented as 4, 2 + 2, 2 + 1 + 1 or 1 + 1 + 1 + 1. Prove that for any integer $n \geq 3$, $2^{n2/4} < f(2^n) < 2^{n2/2}$.

Solution

The key is to derive a recurrence relation for f(n) [not for $f(2^n)$]. If n is odd, then the sum must have a 1. In fact, there is a one-to-one correspondence between sums for n and sums for n-1. So:

$$f(2n+1) = f(2n)$$

Now consider n even. The same argument shows that there is a one-to-one correspondence between sums for n-1 and sums for n which have a 1. Sums which do not have a 1 are in one-to-one correspondence with sums for n/2 (just halve each term). So:

$$f(2n) = f(2n - 1) + f(n) = f(2n - 2) + f(n).$$

The upper limit is now almost immediate. First, the recurrence relations show that f is monotonic increasing. Now apply the second relation repeatedly to $f(2^{n+1)}$ to get:

$f(2^{n+1}) = f(2^{n+1} - 2^n) + f(2^n - 2^{n-1} + 1) + \dots + f(2^n - 1) + f(2^n) = f(2^n) + f(2^n - 1) + \dots + f(2^{n-1} + 1) + f(2^n)$ (*)

and hence $f(2^{n+1}) \geq (2^{n-1} + 1)f(2^n)$

We can now establish the upper limit by induction. It is false for n = 1 and 2, but almost true for n = 2, in that: $f(2^2) = 2^{22/2}$. Now if $f(2^n) \leq 2^{n2/2}$, then the inequality just established shows that $f(2^{n+1}) < 2^n 2^{n2/2} < 2^{(n2+2n+1)/2} = 2^{(n+1)2/2}$, so it is true for n + 1. Hence it is true for all n > 2.

Applying the same idea to the lower limit does not work. We need something stronger. We may continue (*) inductively to obtain $f(2^{n+1}) = f(2^n) + f(2^n - 1) + \dots + f(3) + f(2) + f(1) + 1$. (**) We now use the following lemma:

$f(1) + f(2) + \dots + f(2r) \geq 2r\, f(r)$

We group the terms on the lhs into pairs and claim that $f(1) + f(2r) \geq f(2) + f(2r-1) \geq f(3) + f(2r-2) \geq \dots \geq f(r) + f(r+1)$. If k is even, then $f(k) = f(k+1)$ and $f(2r-k) = f(2r+1-k)$, so $f(k) + f(2r+1-k) = f(k+1) + f(2r-k)$. If k is odd, then $f(k+1) = f(k) + f((k+1)/2)$ and $f(2r+1-k) = f(2r-k) + f((2r-k+1)/2)$, but f is monotone so $f((k+1)/2) \leq f((2r+1-k)/2)$ and hence $f(k) + f(2r+1-k) \geq f(k+1) + f(2r-k)$, as required.

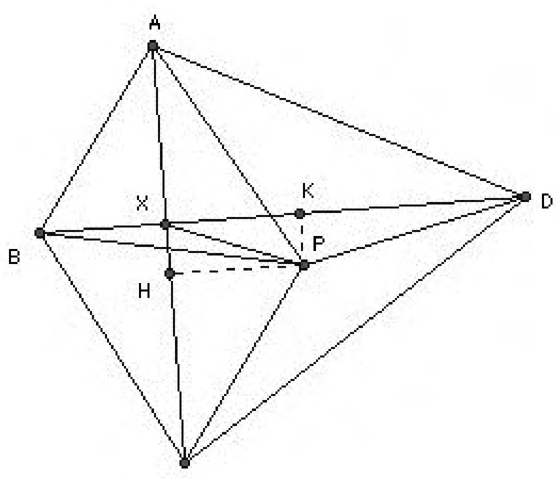
Applying the lemma to (**) gives $f(2^{n+1)} > 2^{n+1}f(2^{n-1})$. This is sufficient to prove the lower limit by induction. It is true for n = 1. Suppose it is true for n. Then $f(2^{n+1}) > 2^{n+1}2^{(n-1)2/4} = 2^{(n2-2n+1+4n+4)/4} > 2^{(n+1)2/4}$, so it is true for n+1.

IMO 1998

Problem A1

In the convex quadrilateral ABCD, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. The point P, where the perpendicular bisectors of AB and DC meet, is inside ABCD. Prove that ABCD is cyclic if and only if the triangles ABP and CDP have equal areas.

Solution



Let AC and BD meet at X. Let H, K be the feet of the perpendiculars from P to AC, BD respectively. We wish to express the areas of ABP and CDP in terms of more tractable triangles. There are essentially two different configurations possible. In the first, we have area PAB = area ABX + area PAX + area PBX, and area PCD = area CDX - area PCX - area PDX. So if the areas being equal is equivalent to: area ABX - area CDX + area PAX + area PCX + area PBX + area PDX = 0. ABX and CDX are right-angled, so we may write their areas as $AX \cdot BX/2$ and $CX \cdot DX/2$. We may also put AX = AH - HX = AH - PK, BX = BK - PH, CX = CH + PK, DX = DK + PH. The other triangles combine in pairs to give area ACP + area BDP = $(AC \cdot PH + BD \cdot PK)/2$. This leads,

after some cancellation to AH·BK = CH·DK. There is a similar configuration with the roles of AB and CD reversed.
The second configuration is area PAB = area ABX + area PAX - PBX, area PCD = area CDX + area PDX - area PCX. In this case AX = AH + PK, BX = BK - PH, CX = CH - PK, DX = DK + PH. But we end up with the same result: AH·BK = CH·DK.
Now if ABCD is cyclic, then it follows immediately that P is the center of the circumcircle and AH = CH, BK = DK. Hence the areas of PAB and PCD are equal.
Conversely, suppose the areas are equal. If PA > PC, then AH > CH. But since PA = PB and PC = PD (by construction), PB > PD, so BK > DK. So AH·BK > CH·DK. Contradiction. So PA is not greater than PC. Similarly it cannot be less. Hence PA = PC. But that implies PA = PB = PC = PD, so ABCD is cyclic.

Problem A2
In a competition there are a contestants and b judges, where b ≥ 3 is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges their ratings coincide for at most k contestants. Prove k/a ≥ (b-1)/2b.

Solution
Let us count the number N of triples (judge, judge, contestant) for which the two judges are distinct and rate the contestant the same. There are b(b-1)/2 pairs of judges in total and each pair rates at most k contestants the same, so N ≤ kb(b-1)/2.
Now consider a fixed contestant X and count the number of pairs of judges rating X the same. Suppose x judges pass X, then there are x(x-1)/2 pairs who pass X and (b-x)(b-x-1)/2 who fail X, so a total of (x(x-1) + (b-x)(b-x-1))/2 pairs rate X the same. But (x(x-1) + (b-x)(b-x-1))/2 = $(2x^2 - 2bx + b^2 - b)/2 = (x - b/2)^2 + b^2/4 - b/2 \ge b^2/4 - b/2 = (b - 1)^2/4 - 1/4$. But $(b - 1)^2/4$ is an integer (since b is odd), so the number of pairs rating X the same is at least $(b - 1)^2/4$. Hence $N \ge a (b - 1)^2/4$. Putting the two inequalities together gives k/a ≥ (b - 1)/2b.

Problem A3
For any positive integer n, let d(n) denote the number of positive divisors of n (including 1 and n). Determine all positive integers k such that $d(n^2) = k\, d(n)$ for some n.

Solution
Let $n = p_1^{a_1} \dots p_r^{a_r}$. Then $d(n) = (a_1 + 1)(a_2 + 1) \dots (a_r + 1)$, and $d(n^2) = (2a_1 + 1)(2a_2 + 1) \dots (2a_r + 1)$. So the a_i must be chosen so that $(2a_1 + 1)(2a_2 + 1) \dots (2a_r + 1) = k (a_1 + 1)(a_2 + 1) \dots (a_r + 1)$. Since all $(2a_i + 1)$ are odd, this clearly implies that k must be odd. We show that conversely, given any odd k, we can find a_i.
We use a form of induction on k. First, it is true for k = 1 (take n = 1). Second, we show that if it is true for k, then it is true for $2^m k - 1$. That is sufficient, since any odd number has the form $2^m k - 1$ for some smaller odd number k. Take $a_i = 2^i((2^m - 1)k - 1)$ for i = 0, 1, ... , m-1. Then $2a_i + 1 = 2^{i+1}(2^m - 1)k - (2^{i+1} - 1)$ and $a_i + 1 = 2^i(2^m - 1)k - (2^i - 1)$. So the product of the $(2a_i + 1)$'s divided by the product of the $(a_i + 1)$'s is $2^m(2^m - 1)k - (2^m - 1)$ divided by $(2^m - 1)k$, or $(2^m k - 1)/k$. Thus if we take these a_is together with those giving k, we get $2^m k - 1$, which completes the induction.

Problem B1
Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Answer (a, b) = (11, 1), (49, 1) or $(7k^2, 7k)$.

Solution
If a < b, then b ≥ a + 1, so $ab^2 + b + 7 > ab^2 + b \ge (a + 1)(ab + 1) = a^2b + a + ab \ge a^2b + a + b$. So there can be no solutions with a < b. Assume then that a ≥ b.
Let k = the integer $(a^2b + a + b)/(ab^2 + b + 7)$. We have $(a/b + 1/b)(ab^2 + b + 7) = ab^2 + a + ab + 7a/b + 7/b + 1 > ab^2 + a + b$. So k < a/b + 1/b. Now if b ≥ 3, then (b - 7/b) > 0 and hence $(a/b - 1/b)(ab^2 + b + 7) = ab^2 + a - a(b - 7/b) - 1 - 7/b < ab^2 + a < ab^2 + a + b$. Hence either b = 1 or 2 or k > a/b - 1/b.

If $a/b - 1/b < k < a/b + 1/b$, then $a - 1 < kb < a + 1$. Hence $a = kb$. This gives the solution $(a, b) = (7k^2, 7k)$.
It remains to consider $b = 1$ and 2. If $b = 1$, then $a + 8$ divides $a^2 + a + 1$ and hence also $a(a + 8) - (a^2 + a + 1) = 7a - 1$, and hence also $7(a + 8) - (7a - 1) = 57$. The only factors bigger than 8 are 19 and 57, so $a = 11$ or 49. It is easy to check that $(a, b) = (11, 1)$ and $(49, 1)$ are indeed solutions.
If $b = 2$, then $4a + 9$ divides $2a^2 + a + 2$, and hence also $a(4a + 9) - 2(2a^2 + a + 2) = 7a - 4$, and hence also $7(4a + 9) - 4(7a - 4) = 79$. The only factor greater than 9 is 79, but that gives $a = 35/2$ which is not integral. Hence there are no solutions for $b = 2$.

Problem B2
Let I be the incenter of the triangle ABC. Let the incircle of ABC touch the sides BC, CA, AB at K, L, M respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that the angle RIS is acute.

Solution
We show that $RI^2 + SI^2 - RS^2 > 0$. The result then follows from the cosine rule.
BI is perpendicular to MK and hence also to RS. So $IR^2 = BR^2 + BI^2$ and $IS^2 = BI^2 + BS^2$. Obviously $RS = RB + BS$, so $RS^2 = BR^2 + BS^2 + 2\, BR{\cdot}BS$. Hence $RI^2 + SI^2 - RS^2 = 2\, BI^2 - 2\, BR{\cdot}BS$. Consider the triangle BRS. The angles at B and M are $90 - B/2$ and $90 - A/2$, so the angle at R is $90 - C/2$. Hence $BR/BM = \cos A/2/\cos C/2$ (using the sine rule). Similarly, considering the triangle BKS, $BS/BK = \cos C/2/\cos A/2$. So $BR{\cdot}BS = BM{\cdot}BK = BK^2$. Hence $RI^2 + SI^2 - RS^2 = 2(BI^2 - BK^2) = 2\, IK^2 > 0$.

Problem B3
Consider all functions f from the set of all positive integers into itself satisfying $f(t^2f(s)) = s\, f(t)^2$ for all s and t. Determine the least possible value of $f(1998)$.

Answer
120

Solution
Let $f(1) = k$. Then $f(kt^2) = f(t)^2$ and $f(f(t)) = k^2t$. Also $f(kt)^2 = 1{\cdot}f(kt)^2 = f(k^3t^2) = f(1^2f(f(kt^2))) = k^2f(kt^2) = k^2f(t)^2$. Hence $f(kt) = k\, f(t)$.
By an easy induction $k^nf(t^{n+1}) = f(t)^{n+1}$. But this implies that k divides $f(t)$. For suppose the highest power of a prime p dividing k is $a > b$, the highest power of p dividing $f(t)$. Then $a > b(1 + 1/n)$ for some integer n. But then $na > (n + 1)b$, so k^n does not divide $f(t)^{n+1}$. Contradiction.
Let $g(t) = f(t)/k$. Then $f(t^2f(s)) = f(t^2kg(s)) = k\, f(t^2g(s) = k^2g(t^2g(s))$, whilst $s\, f(t)^2 = k^2s\, f(t)^2$. So $g(t^2g(s)) = s\, g(t)^2$. Hence g is also a function satisfying the conditions which evidently has smaller values than f (for $k > 1$). It also satisfies $g(1) = 1$. Since we want the smallest possible value of $f(1998)$ we may restrict attention to functions f satisfying $f(1) = 1$. Thus we have $f(f(t) = t$ and $f(t^2) = f(t)^2$. Hence $f(st)^2 = f(s^2t^2) = f(s^2f(f(t^2))) = f(s)^2f(t^2) = f(s)^2f(t)^2$. So $f(st) = f(s)\, f(t)$.
Suppose p is a prime and $f(p) = m{\cdot}n$. Then $f(m)f(n) = f(mn) = f(f(p)) = p$, so one of $f(m)$, $f(n) = 1$. But if $f(m) = 1$, then $m = f(f(m)) = f(1) = 1$. So $f(p)$ is prime. If $f(p) = q$, then $f(q) = p$.
Now we may define f arbitarily on the primes subject only to the conditions that each f(prime) is prime and that if $f(p) = q$, then $f(q) = p$. For suppose that $s = p_1^{a_1}\dots p_r^{a_r}$ and that $f(p_i) = q_i$. If t has any additional prime factors not included in the q_i, then we may add additional p's to the expression for s so that they are included (taking the additional a's to be zero). So suppose $t = q_1^{b_1}\dots q_r^{b_r}$.Then $t^2f(s) = q_1^{2b_1+a_1} \dots q_r^{2b_r+a_r}$ and hence $f(t^2f(s) = p_1^{2b_1+a_1} \dots p_r^{2b_r+a_r} = s\, f(t)^2$.
We want the minimum possible value of $f(1998)$. Now $1998 = 2.3^3.37$, so we achieve the minimum value by taking $f(2) = 3$, $f(3) = 2$, $f(37) = 5$ (and $f(37) = 5$). This gives $f(1998) = 3{\cdot}2^35 = 120$.

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IMO 1999

Problem A1
Find all finite sets S of at least three points in the plane such that for all distinct points A, B in S, the perpendicular bisector of AB is an axis of symmetry for S.

Solution
The possible sets are just the regular n-gons ($n > 2$).
Let $A_1, A_2, \dots, A_k$ denote the vertices of the convex hull of S (and take indices mod k as necessary). We show first that these form a regular k-gon. A_{i+1} must lie on the perpendicular bisector of A_i and A_{i+2} (otherwise its reflection would lie outside the hull). Hence the sides are all equal. Similarly, A_{i+1} and A_{i+2} must be reflections in the perpendicular bisector of A_i and A_{i+3} (otherwise one of the reflections would lie outside the hull). Hence all the angles are equal.
Any axis of symmetry for S must also be an axis of symmetry for the A_i, and hence must pass through the center C of the regular k-gon. Suppose X is a point of S in the interior of k-gon. Then it must lie inside or on some triangle $A_iA_{i+1}C$. C must be the circumcenter of $A_iA_{i+1}X$ (since it lies on the three perpendicular bisectors, which must all be axes of symmetry of S), so X must lie on the circle center C, through A_i and A_{i+1}. But all points of the triangle $A_iA_{i+1}X$ lie strictly inside this circle, except $A_i$and A_{i+1}, so X cannot be in the interior of the k-gon.

Problem A2
Let n >= 2 by a fixed integer. Find the smallest constant C such that for all non-negative reals $x_1, \dots, x_n$:
$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) <= C (\sum x_i)^4$.
Determine when equality occurs.

Answer $C = 1/8$. Equality iff two x_i are equal and the rest zero.

Solution
$(\sum x_i)^4 = (\sum x_i^2 + 2 \sum_{i<j} x_i x_j)^2 \geq 4 (\sum x_i^2) (2 \sum_{i<j} x_i x_j) = 8 \sum_{i<j} (x_i x_j \sum x_k^2) \geq 8 \sum_{i<j} x_i x_j (x_i^2 + x_j^2)$.
The second inequality is an equality only if n - 2 of the x_i are zero. So assume that $x_3 = x_4 = \dots = x_n = 0$. Then for the first inequality to be an equality we require that $(x_1^2 + x_2^2) = 2 x_1 x_2$ and hence that $x_1 = x_2$. However, that is clearly also sufficient for equality.

Problem A3

Given an n x n square board, with n even. Two distinct squares of the board are said to be adjacent if they share a common side, but a square is not adjacent to itself. Find the minimum number of squares that can be marked so that every square (marked or not) is adjacent to at least one marked square.

Answer $n/2\ (n/2 + 1) = n(n + 2)/4$.

Solution
Let $n = 2m$. Color alternate squares black and white (like a chess board). It is sufficient to show that $m(m+1)/2$ white squares are necessary and sufficient to deal with all the black squares.
This is almost obvious if we look at the diagonals.
Look first at the odd-length white diagonals. In every other such diagonal, mark alternate squares (starting from the border each time, so that r+1 squares are marked in a diagonal length 2r+1). Now each black diagonal is adjacent to a picked white diagonal and hence each black square on it is adjacent to a marked white square. In all $1 + 3 + 5 + \dots + m-1 + m + m-2 + \dots + 4 + 2 = 1 + 2 + 3 + \dots + m = m(m+1)/2$ white squares are marked. This proves sufficiency.

For necessity consider the alternate odd-length black diagonals. Rearranging, these have lengths 1, 3, 5, ... , 2m-1. A white square is only adjacent to squares in one of these alternate diagonals and is adjacent to at most 2 squares in it. So we need at least 1 + 2 + 3 + ... + m = m(m+1)/2 white squares.

Problem B1
Find all pairs (n, p) of positive integers, such that: p is prime; $n \le 2p$; and $(p - 1)^n + 1$ is divisible by n^{p-1}.

Answer
(1, p) for any prime p; (2, 2); (3, 3).

Solution
Answer: (1, p) for any prime p; (2, 2); (3, 3).
Evidently (1, p) is a solution for every prime p. Assume n > 1 and take q to be the smallest prime divisor of n. We show first that q = p.
Let x be the smallest positive integer for which $(p - 1)^x = - 1 \pmod q$, and y the smallest positive integer for which $(p - 1)^y = 1 \pmod q$. Certainly y exists and indeed y < q, since $(p - 1)^{q-1} = 1 \pmod q$. We know that $(p - 1)^n = -1 \pmod q$, so x exists also. Writing n = sy + r, with $0 \le r < y$, we conclude that $(p - 1)^r = -1 \pmod q$, and hence $x \le r < y$ (r cannot be zero, since 1 is not -1 (mod q)).
Now write n = hx + k with $0 \le k < x$. Then $-1 = (p - 1)^n = (-1)^h(p - 1)^k \pmod q$. h cannot be even, because then $(p - 1)^k = -1 \pmod q$, contradicting the minimality of x. So h is odd and hence $(p - 1)^k = 1 \pmod q$ with $0 \le k < x < y$. This contradicts the minimality of y unless k = 0, so n = hx. But x < q, so x = 1. So (p - 1) = -1 (mod q). p and q are primes, so q = p, as claimed.
So p is the smallest prime divisor of n. We are also given that $n \le 2p$. So either p = n, or p = 2, n = 4. The latter does not work, so we have shown that n = p. Evidently n = p = 2 and n = p = 3 work. Assume now that p > 3. We show that there are no solutions of this type.
Expand $(p - 1)^p + 1$ by the binomial theorem, to get (since $(-1)^p = -1$): $1 + -1 + p^2 - 1/2\, p(p - 1)p^2 + p(p - 1)(p - 2)/6\, p^3 - \dots$
The terms of the form (bin coeff) p^i with i >= 3 are obviously divisible by p^3, since the binomial coefficients are all integral. Hence the sum is p^2 + a multiple of p^3. So the sum is not divisible by p^3. But for p > 3, p^{p-1} is divisible by p^3, so it cannot divide $(p - 1)^p + 1$, and there are no more solutions.

Problem B2
The circles C_1 and C_2 lie inside the circle C, and are tangent to it at M and N, respectively. C_1 passes through the center of C_2. The common chord of C_1 and C_2, when extended, meets C at A and B. The lines MA and MB meet C_1 again at E and F. Prove that the line EF is tangent to C_2.

Solution
Let O, O_1, O_2 and r, r_1, r_2 be the centers and radii of C, C_1, C_2 respectively. Let EF meet the line O_1O_2 at W, and let $O_2W = x$. We need to prove that $x = r_2$.
Take rectangular coordinates with origin O_2, x-axis O_2O_1, and let O have coordinates (a, b). Notice that O and M do not, in general, lie on O_1O_2. Let AB meet the line O_1O_2 at V.
We observe first that $O_2V = r_2^2/(2 r_1)$. [For example, let X be a point of intersection of C_1 and C_2 and let Y be the midpoint of O_2X. Then O_1YO_2 and XVO_2 are similar. Hence, $O_2V/O_2X = O_2Y/O_2O_1$.]
An expansion (or, to be technical, a *homothecy*) center M, factor r/r_1 takes O_1 to O and EF to AB. Hence EF is perpendicular to O_1O_2. Also the distance of O_1 from EF is r_1/r times the distance of O from AB, so $(r_1 - x) = r_1/r\, (a - r_2^2/(2 r_1))$ (*).
We now need to find a. We can get two equations for a and b by looking at the distances of O from O_1 and O_2. We have:
$(r - r_1)^2 = (r_1 - a)^2 + b^2$, and
$(r - r_2)^2 = a^2 + b^2$.

Subtracting to eliminate b, we get $a = r_2^2/(2 r_1) + r - r r_2/r_1$. Substituting back in (*), we get $x = r_2$, as required.

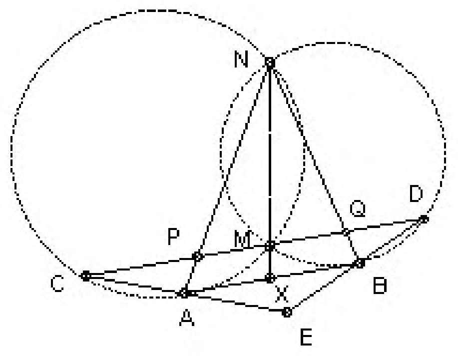
Problem 6
Determine all functions f: R -> R such that f(x - f(y)) = f(f(y)) + x f(y) + f(x) - 1 for all x, y in R. [R is the reals.]

Solution
Let c = f(0) and A be the image f(R). If a is in A, then it is straightforward to find f(a): putting a = f(y) and x = a, we get $f(a - a) = f(a) + a^2 + f(a) - 1$, so $f(a) = (1 + c)/2 - a^2/2$ (*).
The next step is to show that A - A = R. Note first that c cannot be zero, for if it were, then putting y = 0, we get: f(x - c) = f(c) + xc + f(x) - 1 (**) and hence f(0) = f(c) = 1. Contradiction. But (**) also shows that f(x - c) - f(x) = xc + (f(c) - 1). Here x is free to vary over R, so xc + (f(c) - 1) can take any value in R.
Thus given any x in R, we may find a, b in A such that x = a - b. Hence f(x) = f(a - b) = f(b) + ab + f(a) - 1. So, using (*): $f(x) = c - b^2/2 + ab - a^2/2 = c - x^2/2$.
In particular, this is true for x in A. Comparing with (*) we deduce that c = 1. So for all x in R we must have $f(x) = 1 - x^2/2$. Finally, it is easy to check that this satisfies the original relation and hence is the unique solution.

IMO 2000

Problem A1
AB is tangent to the circles CAMN and NMBD. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that PE = QE.

Solution
Angle EBA = angle BDM (because CD is parallel to AB) = angle ABM (because AB is tangent at B). So AB bisects EBM. Similarly, BA bisects angle EAM. Hence E is the reflection of M in AB. So EM is perpendicular to AB and hence to CD. So it suffices to show that MP = MQ.
Let the ray NM meet AB at X. XA is a tangent so $XA^2 = XM \cdot XN$. Similarly, XB is a tangent, so $XB^2 = XM \cdot XN$. Hence XA = XB. But AB and PQ are parallel, so MP = MQ.



Problem A2
A, B, C are positive reals with product 1. Prove that (A - 1 + 1/B)(B - 1 + 1/C)(C - 1 + 1/A) ≤ 1.

Solution
(B - 1 + 1/C) = B(1 - 1/B + 1/(BC)) = B(1 + A - 1/B). Hence, (A - 1 + 1/B)(B - 1 + 1/C) = $B(A^2 - (1 - 1/B)^2) \le B A^2$. So the square of the product of all three $\le B A^2 C B^2 A C^2 = 1$.
Actually, that is not quite true. The last sentence would not follow if we had some negative left hand sides, because then we could not multiply the inequalities. But it is easy to deal separately with the case where (A - 1 + 1/B), (B - 1 + 1/C), (C - 1 + 1/A) are not all positive. If one of the three terms is negative, then the other two must be positive. For example, if A - 1 + 1/B < 0, then A < 1, so C - 1 + 1/A > 0, and B > 1, so B - 1 + 1/C > 0. But if one term is negative and two are positive, then their product is negative and hence less than 1.

Problem A3
k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right

of A such that AB' = k BA. For what values of k can we move the points arbitrarily far to the right by repeated moves?

Answer $k \geq 1/(N-1)$.

Solution
Suppose $k < 1/(N-1)$, so that $k_0 = 1/k - (N - 1) > 0$. Let X be the sum of the distances of the points from the rightmost point. If a move does not change the rightmost point, then it reduces X. If it moves the rightmost point a distance z to the right, then it reduces X by at least $z/k - (N-1)z = k_0 z$. X cannot be reduced below nil. So the total distance moved by the rightmost point is at most X_0/k_0, where X_0 is the initial value of X.
Conversely, suppose $k \geq 1/(N-1)$, so that $k_1 = (N-1) - 1/k \geq 0$. We always move the leftmost point. This has the effect of moving the rightmost point $z > 0$ and increasing X by $(N-1)z - z/k = k_1 z \geq 0$. So X is never decreased. But $z \geq k X/(N-1) \geq k X_0/(N-1) > 0$. So we can move the rightmost point arbitrarily far to the right (and hence all the points, since another N-1 moves will move the other points to the right of the rightmost point).

Problem B1
100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

Answer
12. Place 1, 2, 3 in different boxes (6 possibilities) and then place n in the same box as its residue mod 3. Or place 1 and 100 in different boxes and 2 - 99 in the third box (6 possibilities).

Solution
Let H_n be the corresponding result that for cards numbered 1 to n the only solutions are by residue mod 3, or 1 and n in separate boxes and 2 to n - 1 in the third box. It is easy to check that they *are* solutions. H_n is the assertion that there are no others. H_3 is obviously true (although the two cases coincide). We now use induction on n. So suppose that the result is true for n and consider the case n + 1.
Suppose n + 1 is alone in its box. If 1 is not also alone, then let N be the sum of the largest cards in each of the boxes not containing n + 1. Since $n + 2 \leq N \leq n + (n - 1) = 2n - 1$, we can achieve the same sum N as from a different pair of boxes as $(n + 1) + (N - n - 1)$. Contradiction. So 1 must be alone and we have one of the solutions envisaged in H_{n+1}.
If n + 1 is not alone, then if we remove it, we must have a solution for n. But that solution cannot be the n, 1, 2 to n - 1 solution. For we can easily check that none of the three boxes will then accomodate n + 1. So it must be the mod 3 solution. We can easily check that in this case n + 1 must go in the box with matching residue, which makes the (n + 1) solution the other solution envisaged by H_{n+1}. That completes the induction.

Problem B2
Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]

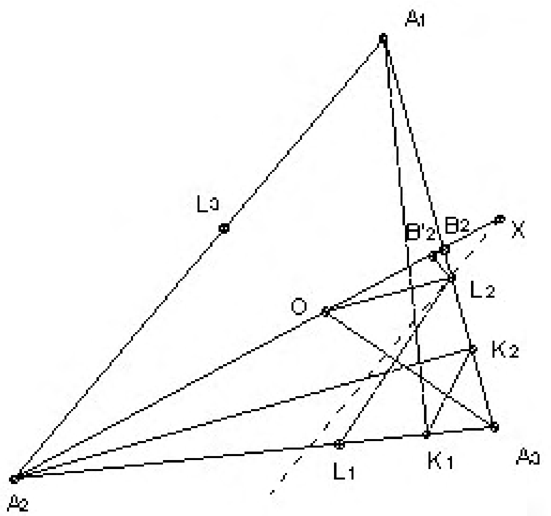
Answer Yes

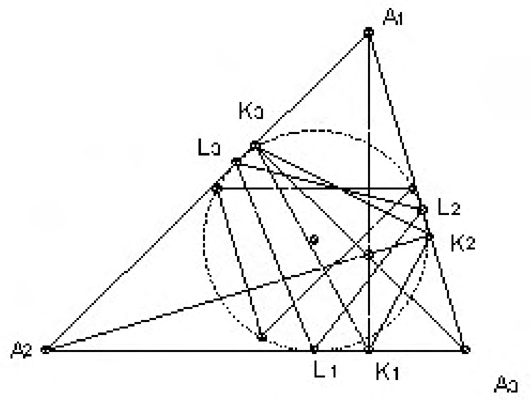
Solution
Note that for b odd we have $2^{ab} + 1 = (2^a + 1)(2^{a(b-1)} - 2^{a(b-2)} + \dots + 1)$, and so $2^a + 1$ is a factor of $2^{ab} + 1$. It is sufficient therefore to find m such that (1) m has only a few distinct prime factors, (2) $2^m + 1$ has a large number of distinct prime factors, (3) m divides $2^m + 1$. For then we can take k, a product of enough distinct primes dividing $2^m + 1$ (but not m), so that km has exactly 2000 factors. Then km still divides $2^m + 1$ and hence $2^{km} + 1$.
The simplest case is where m has only one distinct prime factor p, in other words it is a power of p. But if p is a prime, then p divides $2^p - 2$, so the only p for which p divides $2^p +$

1 is 3. So the questions are whether $a_h = 2^m + 1$ is (1) divisible by $m = 3^h$ and (2) has a large number of distinct prime factors.
$a_{h+1} = a_h(2^{2m} - 2^m + 1)$, where $m = 3^h$. But $2^m = (a_h - 1)$, so $a_{h+1} = a_h(a_h^2 - 3 a_h + 3)$. Now $a_1 = 9$, so an easy induction shows that 3^{h+1} divides a_h, which answers (1) affirmatively. Also, since a_h is a factor of a_{h+1}, any prime dividing a_h also divides a_{h+1}. Put $a_h = 3^{h+1}b_h$. Then $b_{h+1} = b_h(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1)$. Now $(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1) > 1$, so it must have some prime factor $p > 1$. But p cannot be 3 or divide b_h (since $(3^{2h+1}b_h^2 - 3^{h+2}b_h + 1)$ is a multiple of $3b_h$ plus 1), so b_{h+1} has at least one prime factor $p > 3$ which does not divide b_h. So b_{h+1} has at least h distinct prime factors greater than 3, which answers (2) affirmatively. But that is all we need. We can take m in the first paragraph above to be 3^{2000}: (1) m has only one distinct prime factor, (2) $2^m + 1 = 3^{2001}\ b_{2000}$ has at least 1999 distinct prime factors other than 3, (3) m divides $2^m + 1$. Take k to be a product of 1999 distinct prime factors dividing b_{2000}. Then N = km is the required number with exactly 2000 distinct prime factors which divides $2^N + 1$.

Problem B3

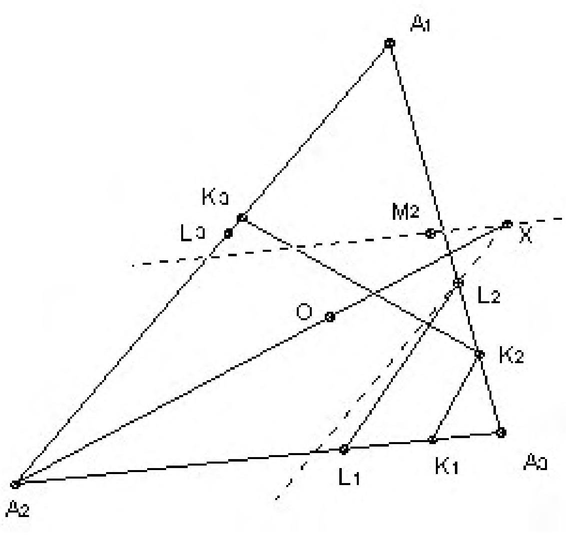
$A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.





Solution

Let O be the centre of the incircle. Let the line parallel to A_1A_2 through L_2 meet the line A_2O at X. We will show that X is the reflection of K_2 in L_2L_3. Let A_1A_3 meet the line A_2O at B_2. Now A_2K_2 is perpendicular to K_2B_2 and OL_2 is perpendicular to L_2B_2, so $A_2K_2B_2$ and OL_2B_2 are similar. Hence $K_2L_2/L_2B_2 = A_2O/OB_2$. But OA_3 is the angle bisector in the triangle $A_2A_3B_2$, so $A_2O/OB_2 = A_2A_3/B_2A_3$.



Take B'_2 on the line A_2O such that $L_2B_2 = L_2B'_2$ (B'_2 is distinct from B_2 unless L_2B_2 is perpendicular to the line). Then angle $L_2B'_2X$ = angle $A_3B_2A_2$. Also, since L_2X is parallel to A_2A_1, angle $L_2XB'_2$ = angle $A_3A_2B_2$. So the triangles $L_2XB'_2$ and $A_3A_2B_2$ are similar. Hence $A_2A_3/B_2A_3 = XL_2/B_2'L_2 = XL_2/B_2L_2$ (since $B'_2L_2 = B_2L_2$).
Thus we have shown that $K_2L_2/L_2B_2 = XL_2/B_2L_2$ and hence that $K_2L_2 = XL_2$. L_2X is parallel to A_2A_1 so angle $A_2A_1A_3$ = angle A_1L_2X = angle L_2XK_2 + angle L_2K_2X = 2 angle L_2XK_2 (isosceles). So angle L_2XK_2 = 1/2 angle $A_2A_1A_3$ = angle A_2A_1O. L_2X and A_2A_1 are parallel, so K_2X and OA_1 are parallel. But OA_1 is perpendicular to L_2L_3, so K_2X is also perpendicular to L_2L_3 and hence

X is the reflection of K_2 in L_2L_3.
Now the angle $K_3K_2A_1$ = angle $A_1A_2A_3$, because it is 90^o - angle $K_3K_2A_2$ = 90^o - angle $K_3A_3A_2$ ($A_2A_3K_2K_3$ is cyclic with A_2A_3 a diameter) = angle $A_1A_2A_3$. So the reflection of K_2K_3 in L_2L_3 is a line through X making an angle $A_1A_2A_3$ with L_2X, in other words, it is the line through X parallel to A_2A_3.
Let M_i be the reflection of L_i in A_iO. The angle M_2XL_2 = 2 angle OXL_2 = 2 angle A_1A_2O (since A_1A_2 is parallel to L_2X) = angle $A_1A_2A_3$, which is the angle betwee L_2X and A_2A_3. So M_2X is parallel to A_2A_3, in other words, M_2 lies on the reflection of K_2K_3 in L_2L_3.
If follows similarly that M_3 lies on the reflection. Similarly, the line M_1M_3 is the reflection of K_1K_3 in L_1L_3, and the line M_1M_2 is the reflection of K_1K_2 in L_1L_2 and hence the triangle formed by the intersections of the three reflections is just $M_1M_2M_3$.

IMO 2001

Problem A1

ABC is acute-angled. O is its circumcenter. X is the foot of the perpendicular from A to BC. Angle C ≥ angle B + 30^o. Prove that angle A + angle COX < 90^o

Solution

Take D on the circumcircle with AD parallel to BC. Angle CBD = angle BCA, so angle ABD ≥ 30^o. Hence angle AOD ≥ 60^o. Let Z be the midpoint of AD and Y the midpoint of BC. Then AZ ≥ R/2, where R is the radius of the circumcircle. But AZ = YX (since AZYX is a rectangle).
Now O cannot coincide with Y (otherwise angle A would be 90^o and the triangle would not be acute-angled). So OX > YX ≥ R/2. But XC = YC - YX < R - YX ≤ R/2. So OX > XC. Hence angle COX < angle OCX. Let CE be a diameter of the circle, so that angle OCX = angle ECB. But angle ECB = angle EAB and angle EAB + angle BAC = angle EAC = 90^o, since EC is a diameter. Hence angle COX + angle BAC < 90^o.

Problem A2

a, b, c are positive reals. Let a' = $\sqrt{}(a^2 + 8bc)$, b' = $\sqrt{}(b^2 + 8ca)$, c' = $\sqrt{}(c^2 + 8ab)$. Prove that a/a' + b/b' + c/c' >= 1.

Solution

A not particularly elegant, but fairly easy, solution is to use Cauchy: $(\sum xy)^2 \le \sum x^2 \sum y^2$.
To get the inequality the right way around we need to take x^2 = a/a' [to be precise, we are taking x_1^2 = a/a', x_2^2 = b/b', x_3^2 = c/c'.]. Take y^2 = a a', so that xy = a. Then we get ∑ a/a' >= $(\sum a)^2/\sum a a'$.
Evidently we need to apply Cauchy again to deal with ∑ a a'. This time we want ∑ a a' ≤ something. The obvious X=a, Y=a' does not work, but if we put $X=a^{1/2}$, $Y=a^{1/2}a'$, then we have $\sum a a' \le (\sum a)^{1/2} (\sum a a'^2)^{1/2}$. So we get the required inequality provided that $(\sum a)^{3/2} \ge (\sum a a'^2)^{1/2}$ or $(\sum a)^3 \ge \sum a a'^2$.
Multiplying out, this is equivalent to: $3(ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2) \ge 18abc$, or $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 \ge 0$, which is clearly true.

Problem A3

Integers are placed in each of the 441 cells of a 21 x 21 array. Each row and each column has at most 6 different integers in it. Prove that some integer is in at least 3 rows and at least 3 columns.

Solution

Notice first that the result is not true for a 20 x 20 array. Make 20 rectangles each 2 x 10, labelled 1, 2, ... , 20. Divide the 20 x 20 array into four quadrants (each 10 x 10). In each of the top left and bottom right quadrants, place 5 rectangles horizontally. In each of the other two quadrants, place 5 rectangles vertically. Now each row intersects 5 vertical rectangles and 1 horizontal. In other words, it contains just 6 different numbers. Similarly each column. But any given number is in either 10 rows and 2 columns or vice versa, so no number is in 3 rows and 3 columns. [None of this is necessary for the solution, but it helps to show what is going on.]

Returning to the 21 x 21 array, assume that an arrangement is possible with no integer in at least 3 rows and at least 3 columns. Color a cell white if its integer appears in 3 or more rows and black if its integer appears in only 1 or 2 rows. We count the white and black squares.
Each row has 21 cells and at most 6 different integers. 6 x 2 < 21, so every row includes an integer which appears 3 or more times and hence in at most 2 rows. Thus at most 5 different integers in the row appear in 3 or more rows. Each such integer can appear at most 2 times in the row, so there are at most 5 x 2 = 10 white cells in the row. This is true for every row, so there are at most 210 white cells in total.
Similarly, any given column has at most 6 different integers and hence at least one appears 3 or more times. So at most 5 different integers appear in 2 rows or less. Each such integer can occupy at most 2 cells in the column, so there are at most 5 x 2 = 10 black cells in the column. This is true for every column, so there are at most 210 black cells in total.
This gives a contradiction since 210 + 210 < 441.

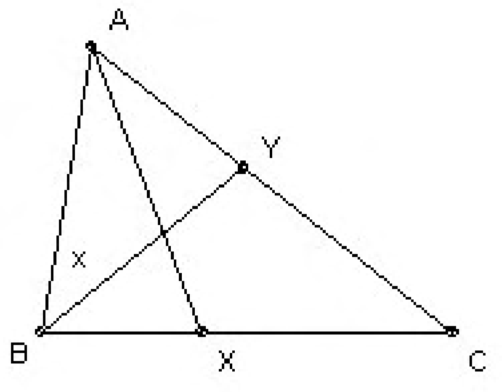
Problem B1
Let $n_1, n_2, \dots , n_m$ be integers, where m is odd. Let $x = (x_1, \dots , x_m)$ denote a permutation of the integers 1, 2, ... , m. Let $f(x) = x_1n_1 + x_2n_2 + \dots + x_mn_m$. Show that for some distinct permutations a, b the difference f(a) - f(b) is a multiple of m!.

Solution
This is a simple application of the pigeon hole principle.
The sum of all m! distinct residues mod m! is not divisible by m! because m! is even (since m > 1). [The residues come in pairs a and m! - a, except for m!/2.].
However, the sum of all f(x) as x ranges over all m! permutations is $1/2\ (m+1)!\ \sum n_i$, which is divisible by m! (since m+1 is even). So at least one residue must occur more than once among the f(x).

Problem B2
ABC is a triangle. X lies on BC and AX bisects angle A. Y lies on CA and BY bisects angle B. Angle A is 60^o. AB + BX = AY + YB. Find all possible values for angle B.

Answer 80^o.

Solution



This is an inelegant solution, but I did get it fast! Without loss of generality we can take length AB = 1. Take angle ABY = x. Note that we can now solve the two triangles AXB and AYB. In particular, using the sine rule, $BX = \sin 30^o/\sin(150^o-2x)$, $AY = \sin x/\sin(120^o-x)$, $YB = \sin 60^o/\sin(120^o-x)$. So we have an equation for x.
Using the usual formula for sin(a + b) etc, and writing s = sin x, c = cos x, we get: $2\sqrt{3}\, s^2c - 4sc - 2\sqrt{3}\, c^3 + 2\sqrt{3}\, c^2 + 6sc - 2s - \sqrt{3} = 0$ or $-\sqrt{3}\ (4c^3 - 2c^2 - 2c + 1) = 2s(2c^2 -3c + 1)$. This has a common factor 2c - 1. So c = 1/2 or $-\sqrt{3}\ (2c^2 - 1) = 2s(c - 1)$ (*).
c = 1/2 means $x = 60^o$ or angle $B = 120^o$. But in that case the sides opposite A and B are parallel and the triangle is degenerate (a case we assume is disallowed). So squaring (*) and using $s^2 = 1 - c^2$, we get: $16c^4 - 8c^3 - 12c^2 + 8c - 1 = 0$. This has another factor 2c - 1. Dividing that out we get: $8c^3 - 6c + 1 = 0$. But we remember that $4c^3 - 3c = \cos 3x$, so we conclude that cos 3x = -1/2. That gives $x = 40^o, 80^o, 160^o, 200^o, 280^o, 320^o$. But we require that $x < 60^o$ to avoid degeneracy. Hence the angle $B = 2x = 80^o$.

Problem B3
K > L > M > N are positive integers such that KM + LN = (K + L - M + N)(-K + L + M + N). Prove that KL + MN is composite.

Solution

Note first that KL+MN > KM+LN > KN+LM, because (KL+MN) - (KM+LN) = (K - N)(L - M) > 0 and (KM+LN) - (KN+LM) = (K - L)(M - N) > 0.

Multiplying out and rearranging, the relation in the question gives K^2 - KM + M^2 = L^2 + LN + N^2. Hence (KM + LN)(L^2 + LN + N^2) = KM(L^2 + LN + N^2) + LN(K^2 - KM + M^2) = KML^2 + KMN^2 + K^2LN + LM^2N = (KL + MN)(KN + LM). In other words (KM + LN) divides (KL + MN)(KN + LM).

Now suppose KL + MN is prime. Since it greater than KM + LN, it can have no common factors with KM + LN. Hence KM + LN must divide the smaller integer KN + LM. Contradiction.

IMO 2002

Problem A1

S is the set of all (h, k) with h, k non-negative integers such that h + k < n. Each element of S is colored red or blue, so that if (h, k) is red and h' ≤ h, k' ≤ k, then (h', k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

Solution

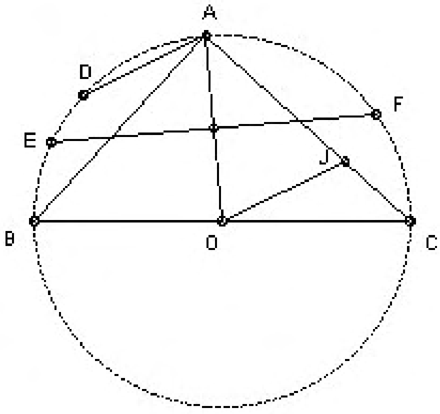
Let a_i be the number of blue members (h, k) in S with h = i, and let b_i be the number of blue members (h, k) with k = i. It is sufficient to show that $b_0, b_1, \dots , b_{n-1}$ is a rearrangement of $a_0, a_1, \dots , a_{n-1}$ (because the number of type 1 subsets is the product of the a_i and the number of type 2 subsets is the product of the b_i).

Let c_i be the largest k such that (i, k) is red. If (i, k) is blue for all k then we put $c_i = -1$. Note that if i < j, then $c_i \ge c_j$, since if (j, c_i) is red, then so is (i, c_i). Note also that (i, k) is red for k ≤ c_i, so the sequence $c_0, c_1, \dots , c_{n-1}$ completely defines the coloring of S.

Let S_i be the set with the sequence $c_0, c_1, \dots , c_i, -1, \dots , -1$, so that $S_{n-1} = S$. We also take S_{-1} as the set with the sequence -1, -1, ... , -1, so that all its members are blue. We show that the rearrangement result is true for S_{-1} and that if it is true for S_i then it is true for S_{i+1}. It is obvious for S_{-1}, because both a_i and b_i are n, n-1, ... , 2, 1. So suppose it is true for S_i (where i < n-1). The only difference between the a_j for S_i and for S_{i+1} is that a_{i+1} = n-i-1 for S_i and (n-i-1)-(c_{i+1}+1) for S_{i+1}. In other words, the number n-i-1 is replaced by the number n-i-c-2, where c = c_{i+1}. The difference in the b_j is that 1 is deducted from each of $b_0, b_1, \dots , b_c$. But these numbers are just n-i-1, n-i-1, n-i-2, ... , n-i-c-1. So the effect of deducting 1 from each is to replace n-i-1 by n-i-c-2, which is the same change as was made to the a_j. So the rearrangement result also holds for S_{i+1}. Hence it holds for S.

Problem A2

BC is a diameter of a circle center O. A is any point on the circle with angle AOC > 60°. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.



Solution

F is equidistant from A and O. But OF = OA, so OFA is equilateral and hence angle AOF = 60°. Since angle AOC > 60°, F lies between A and C. Hence the ray CJ lies between CE and CF.

D is the midpoint of the arc AB, so angle DOB = ½ angle AOB = angle ACB. Hence DO is parallel to AC. But OJ is parallel to AD, so AJOD is a parallelogram. Hence AJ = OD. So AJ = AE = AF, so J lies on the opposite side of EF to A and hence on the same side as C. So J must lie inside the triangle CEF.

Also, since EF is the perpendicular bisector of AO, we have AE = AF = OE, so A is the center of the circle

through E, F and J. Hence angle EFJ = ½ angle EAJ. But angle EAJ = angle EAC (same angle) = angle EFC. Hence J lies on the bisector of angle EFC.
Since EF is perpendicular to AO, A is the midpoint of the arc EF. Hence angle ACE = angle ACF, so J lies on the bisector of angle ECF. Hence J is the incenter.

Problem A3
Find all pairs of integers $m > 2$, $n > 2$ such that there are infinitely many positive integers k for which $(k^n + k^2 - 1)$ divides $(k^m + k - 1)$.

Solution
Answer: $m = 5$, $n = 3$.
Obviously $m > n$. Take polynomials $q(x)$, $r(x)$ with integer coefficients and with degree $r(x) < n$ such that $x^m + x - 1 = q(x)\,(x^n + x^2 - 1) + r(x)$. Then $x^n + x^2 - 1$ divides $r(x)$ for infinitely many positive integers x. But for sufficiently large x, $x^n + x^2 - 1 > r(x)$ since $r(x)$ has smaller degree. So $r(x)$ must be zero. So $x^m + x - 1$ factorises as $q(x)\,(x^n + x^2 - 1)$, where $q(x) = x^{m-n} + a_{m-n-1}x^{m-n-1} + \dots + a_0$.
We have $(x^m + x - 1) = x^{m-n}(x^n + x^2 - 1) + (1 - x)(x^{m-n+1} + x^{m-n} - 1)$, so $(x^n + x^2 - 1)$ must divide $(x^{m-n+1} + x^{m-n} - 1)$. So, in particular, $m \ge 2n-1$. Also $(x^n + x^2 - 1)$ must divide $(x^{m-n+1} + x^{m-n} - 1) - x^{m-2n+1}(x^n + x^2 - 1) = x^{m-n} - x^{m-2n+3} + x^{m-2n+1} - 1$ (*).
(*) can be written as $x^{m-2n+3}(x^{n-3} - 1) + (x^{m-(2n-1)} - 1)$ which is < 0 for all x in (0, 1) unless $n - 3 = 0$ and $m - (2n - 1) = 0$. So unless $n = 3$, $m = 5$, it is has no roots in (0, 1). But $x^n + x^2 - 1$ (which divides it) has at least one becaause it is -1 at x = 0 and +1 at x = 1. So we must have $n = 3$, $m = 5$. It is easy to check that in this case we have an identity.
If $m = 2n-1$, (*) is $x^{n-1} - x^2$. If $n = 3$, this is 0 and indeed we find $m = 5$, $n = 3$ gives an identity. If $n > 3$, then it is $x^2(x^{n-3} - 1)$. But this has no roots in the interval (0, 1), whereas $x^n + x^2 - 1$ has at least one (because it is -1 at x = 0 and +1 at x = 1), so $x^n + x^2 - 1$ cannot be a factor.
If $m > 2n-1$, then (*) has four terms and factorises as $(x - 1)(x^{m-n-1} + x^{m-n-2} + \dots + x^{m-2n+3} + x^{m-2n} + x^{m-2n-1} + \dots + 1)$. Again, this has no roots in the interval (0, 1), whereas $x^n + x^2 - 1$ has at least one, so $x^n + x^2 - 1$ cannot be a factor.

Problem B1
The positive divisors of the integer $n > 1$ are $d_1 < d_2 < \dots < d_k$, so that $d_1 = 1$, $d_k = n$. Let $d = d_1d_2 + d_2d_3 + \dots + d_{k-1}d_k$. Show that $d < n^2$ and find all n for which d divides n^2.

Solution
$d_{k+1-m} <= n/m$. So $d < n^2(1/(1.2) + 1/(2.3) + 1/(3.4) + \dots)$. The inequality is certainly strict because d has only finitely many terms. But $1/(1.2) + 1/(2.3) + 1/(3.4) + \dots = (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \dots = 1$. So $d < n^2$.
Obviously d divides n^2 for n prime. Suppose n is composite. Let p be the smallest prime dividing n. Then $d > n^2/p$. But the smallest divisor of n^2 apart from 1 is p, so if d divides n^2, then $d \le n^2/p$. So d cannot divide n^2 for n composite.

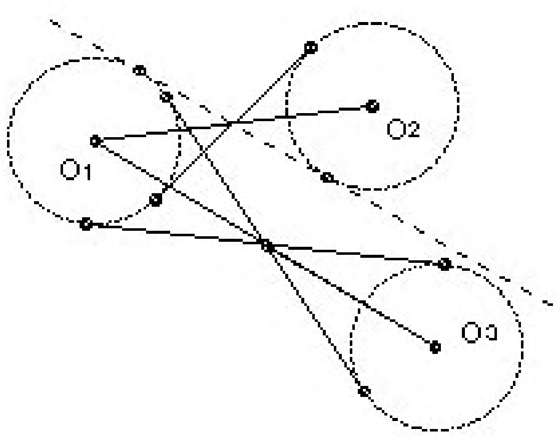
Problem B2
Find all real-valued functions f on the reals such that $(f(x) + f(y))\,(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$ for all x, y, u, v.

Solution
Answer: there are three possible functions: (1) $f(x) = 0$ for all x; (2) $f(x) = 1/2$ for all x; or (3) $f(x) = x^2$.
Put $x = y = 0$, $u = v$, then $4\,f(0)\,f(u) = 2\,f(0)$. So either $f(u) = 1/2$ for all u, or $f(0) = 0$. $f(u) = 1/2$ for all u is certainly a solution. So assume $f(0) = 0$.
Putting $y = v = 0$, $f(x)\,f(u) = f(xu)$ (*). In particular, taking $x = u = 1$, $f(1)^2 = f(1)$. So $f(1) = 0$ or 1. Suppose $f(1) = 0$. Putting $x = y = 1$, $v = 0$, we get $0 = 2f(u)$, so $f(x) = 0$ or all x. That is certainly a solution. So assume $f(1) = 1$.
Putting $x = 0$, $u = v = 1$ we get $2\,f(y) = f(y) + f(-y)$, so $f(-y) = f(y)$. So we need only consider f(x) for x positive. We show next that $f(r) = r^2$ for r rational. The first step is to show that $f(n) = n^2$ for n an integer. We use induction on n. It is true for n = 0 and 1. Suppose it is true for n-1 and n. Then putting $x = n$, $y = u = v = 1$, we get $2f(n) + 2 = f(n-$

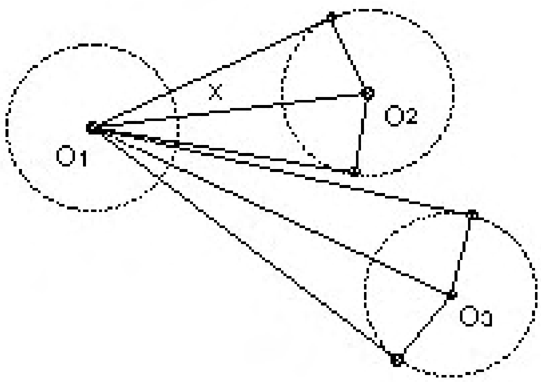
1) + f(n+1), so $f(n+1) = 2n^2 + 2 - (n-1)^2 = (n+1)^2$ and it is true for n+1. Now (*) implies that $f(n)\ f(m/n) = f(m)$, so $f(m/n) = m^2/n^2$ for integers m, n. So we have established $f(r) = r^2$ for all rational r.
From (*) above, we have $f(x^2) = f(x)^2 \geq 0$, so f(x) is always non-negative for positive x and hence for all x. Putting u = y, v = x, we get $(\ f(x) + f(y)\ )^2 = f(x^2 + y^2)$, so $f(x^2 + y^2) = f(x)^2 + 2f(x)f(y) + f(y)^2 \geq f(x)^2 = f(x^2)$. For any u > v > 0, we may put $u = x^2 + y^2$, $v = x^2$ and hence $f(u) \geq f(v)$. In other words, f is an increasing function.
So for any x we may take a sequence of rationals r_n all less than x we converge to x and another sequence of rationals s_n all greater than x which converge to x. Then $r_n^2 = f(r_n) \leq f(x) \leq f(s_n) = s_n^2$ for all x and hence $f(x) = x^2$.

Problem B3
2 circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1, O_2, \dots , O_n$. Show that $\sum_{i<j} 1/O_iO_j \leq (n-1)\pi/4$.

Solution



Denote the circle center O_i by C_i. The tangents from O_1 to C_i contain an angle 2x where $\sin x = 1/O_1O_i$. So $2x > 2/O_1O_i$. These double sectors cannot overlap, so $\sum 2/O_1O_i < \pi$. Adding the equations derived from $O_2, O_3, \dots$ we get $4 \sum O_iO_j < n\pi$, so $\sum O_iO_j < n\pi/4$, which is not quite good enough.



There are two key observations. The first is that it is better to consider the angle $O_iO_1O_j$ than the angle between the tangents to a single circle. It is not hard to show that this angle must exceed both $2/O_1O_i$ and $2/O_1O_j$. For consider the two common tangents to C_1 and C_i which intersect at the midpoint of O_1O_i. The angle between the center line and one of the tangents is at least $2/O_1O_i$. No part of the circle C_j can cross this line, so its center O_j cannot cross the line parallel to the tangent through O_1. In other word, angle $O_iO_1O_j$ is at least $2/O_1O_i$. A similar argument establishes it is at least $2/O_1O_j$.
Now consider the convex hull of the n points O_i. $m \leq n$ of these points form the convex hull and the angles in the convex m-gon sum to $(m-2)\pi$. That is the second key observation. That gains us not one but two amounts $\pi/4$. However, we lose one back. Suppose O_1 is a vertex of the convex hull and that its angle is θ_1. Suppose for convenience that the rays $O_1O_2, O_1O_3, \dots , O_1O_n$ occur in that order with O_2 and O_n adjacent vertices to O_1 in the convex hull. We have that the n-2 angles between adjacent rays sum to θ_1. So we have $\sum 2/O_1O_i < \theta_1$, where the sum is taken over only n-2 of the i, not all n-1. But we can choose which i to drop, because of our freedom to choose either distance for each angle. So we drop the longest distance O_1O_i. [If O_1O_k is the longest, then we work outwards from that ray. Angle $O_{k-1}O_1O_k > 2/O_1O_{k-1}$, and angle $O_kO_1O_{k+1} > 2/O_1O_{k+1}$ and so on.]
We now sum over all the vertices in the convex hull. For any centers O_i inside the hull we use the $\sum_j 2/O_iO_j < \pi$ which we established in the first paragraph, where the sum has all n-1 terms. Thus we get $\sum_{i,j} 2/O_iO_j < (n-2)\pi$, where for vertices i for which O_i is a vertex of the convex hull the sum is only over n-2 values of j and excludes $2/O_iO_{max\ i}$ where $O_{max\ i}$denotes the furthest center from O_i.
Now for O_i a vertex of the convex hull we have that the sum over all j, $\sum 2/O_iO_j$, is the sum Σ' over all but j = max i plus at most $1/(n-2)\ \Sigma'$. In other words we must increase the sum

by at most a factor (n-1)/(n-2) to include the missing term. For O_i not a vertex of the hull, obviously no increase is needed. Thus the full sum $\sum_{i,j} 2/O_iO_j < (n-1)\pi$. Hence $\sum_{i<j} 1/O_iO_j < (n-1)\pi/4$ as required.

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Problem A1
S is the set {1, 2, 3, ... , 1000000}. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets x_i + A are all pairwise disjoint. [Note that x_i + A is the set {a + x_i | a is in A}].

Solution
Having found $x_1, x_2, \dots, x_k$ there are k·101·100 forbidden values for x_{k+1} of the form $x_i + a_m - a_n$ with m and n unequal and another k forbidden values with m = n. Since 99·101·100 + 99 = 10^6 - 1, we can successively choose 100 distinct x_i.

Problem A2
Find all pairs (m, n) of positive integers such that $m^2/(2mn^2 - n^3 + 1)$ is a positive integer.

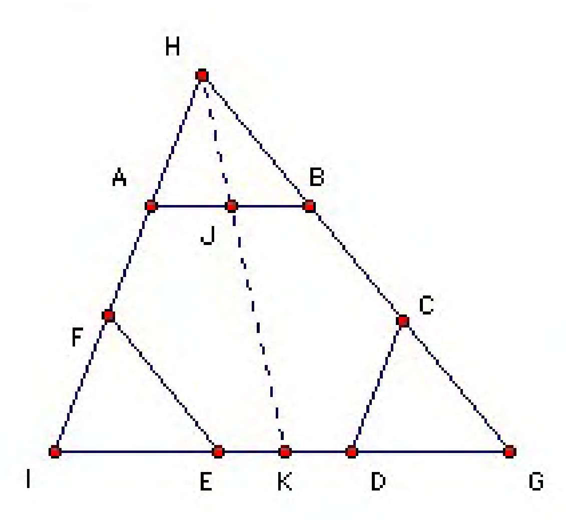
Answer
(m, n) = (2k, 1), (k, 2k) or $(8k^4 - k, 2k)$

Solution
The denominator is $2mn^2 - n^3 + 1 = n^2(2m - n) + 1$, so 2m >= n > 0. If n = 1, then m must be even, in other words, we have the solution (m, n) = (2k, 1).
So assume n > 1. Put $h = m^2/(2mn^2 - n^3 + 1)$. Then we have a quadratic equation for m, namely $m^2 - 2hn^2m + (n^3 - 1)h = 0$. This has solutions hn^2 +- N, where N is the positive square root of $h^2n^4 - hn^3 + h$. Since n > 1, h ≥ 1, N is certainly real. But the sum and product of the roots are both positive, so both roots must be positive. The sum is an integer, so if one root is a positive integer, then so is the other.
The larger root $hn^2 + N$ is greater than hn^2, so the smaller root $< h(n^3 - 1)/(hn^2) < n$. But note that if 2m - n > 0, then since h > 0, we must have the denominator $(2m - n)n^2 + 1$ smaller than the numerator and hence m > n. So for the smaller root we cannot have 2m - n > 0. But 2m - n must be non-negative (since h is positive), so 2m - n = 0 for the smaller root. Hence $hn^2 - N = n/2$. Now $N^2 = (hn^2 - n/2)^2 = h^2n^4 - hn^3 + h$, so $h = n^2/4$. Thus n must be even. Put n = 2k and we get the solutions (m, n) = (k, 2k) and $(8k^4 - k, 2k)$.
We have shown that any solution must be of one of the three forms given, but it is trivial to check that they are all indeed solutions.

Problem A3
A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is ½ √3 times the sum of their lengths. Show that all the hexagon's angles are equal.

Solution
We use bold to denote vectors, so **AB** means the vector from A to B. We take some arbitrary origin and write the vector **OA** as **A** for short. Note that the vector to the midpoint of AB is (**A** + **B**)/2, so the vector from the midpoint of DE to the midpoint of AB is (**A** + **B** - **D** - **E**)/2. So the starting point is |**A** + **B** - **D** - **E**| ≥ √3 (|**A** - **B**| + |**D** - **E**|) and two similar equations. The key is to notice that by the triangle inequality we have |**A** - **B**| + |**D** - **E**| ≥ |**A** - **B** - **D** + **E**| with equality iff the opposite sides AB and DE are parallel. Thus we get |**DA** + **EB**| ≥ √3 |**DA** - **EB**|. Note that DA and EB are diagonals. Squaring, we get $\mathbf{DA}^2 + 2\,\mathbf{DA.EB} + \mathbf{EB}^2 \ge 3(\mathbf{DA}^2 - 2\,\mathbf{DA.EB} + \mathbf{EB}^2)$, or $\mathbf{DA}^2 + \mathbf{EB}^2 \le 4\,\mathbf{DA.EB}$. Similarly, we get $\mathbf{EB}^2 + \mathbf{FC}^2 \le 4\,\mathbf{EB.FC}$ and $\mathbf{FC}^2 + \mathbf{AD}^2 \le 4\,\mathbf{FC.AD} = -4\,\mathbf{FC.DA}$. Adding the three equations gives $2(\mathbf{DA} - \mathbf{EB} + \mathbf{FC})^2 \le 0$. So it must be zero, and hence **DA** - **EB** + **FC** = 0 and opposite sides of the hexagon are parallel.
Note that **DA** - **EB** + **FC** = **A** - **D** - **B** + **E** + **C** - **F** = **BA** + **DC** + **FE**. So **BA** + **DC** + **FE** = 0. In other words, the three vectors can form a triangle.

Since EF is parallel to BC, if we translate EF along the vector **ED** we get CG, an extension of BC. Similarly, if we translate AB along the vector **BC** we get an extension of ED. Since **BA**, **DC** and **FE** form a triangle, AB must translate to DG. Thus HAB and CDG are congruent. Similarly, if we take AF and DE to intersect at I, the triangle FIE is also congruent (and similarly oriented) to HAB and CDG. Take J, K as the midpoints of AB, ED. HIG and HAB are equiangular and hence similar. IE = DG and K is the midpoint of ED, so K is also the midpoint of IG. Hence HJ is parallel to HK, so H, J, K are collinear.



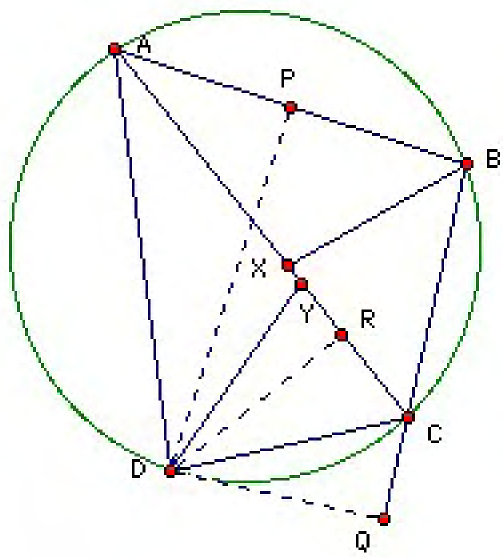
Hence HJ/AB = HK/IG = (HK - HJ)/(IG - AB) = JK/(AB + ED) = ½ √3. Similarly, each of the medians of the triangle HAB is ½ √3 times the corresponding side. We will show that this implies it is equilateral. The required result then follows immediately.

Suppose a triangle has side lengths a, b, c and the length of the median to the midpoint of side length c is m. Then applying the cosine rule twice we get $m^2 = a^2/2 + b^2/2 - c^2/4$. So if $m^2 = ¾\ c^2$, it follows that $a^2 + b^2 = 2c^2$. Similarly, $b^2 + c^2 = 2a^2$. Subtracting, a = c. Similarly for the other pairs of sides.

Problem B1

ABCD is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff RP = RQ.

Solution



APRD is cyclic with diameter AD (because angle APD = angle ARD = 90^o. Suppose its center is O and its radius r. Angle PAR = ½ angle POR, so PR = 2r sin ½POR = AD sin PAR. Similarly, RQ = CD sin RCQ. (Note that it makes no difference if R, P are on the same or opposite sides of the line AD.) But sin PAR = sin BAC, sin RCQ = sin ACB, so applying the sine rule to the triangle ABC, sin RCQ/sin PAR = AB/BC. Thus we have AD/CD = (PR/RQ) (AB/BC). Suppose the angle bisectors of B, D meet AD at X, Y. Then we have AB/BC = AX/CX and AD/CD = AY/CY. Hence (AY/CY)/(AX/CX) = PR/RQ. So PR = RQ iff X = Y, which is the required result.

Problem B2

Given n > 2 and reals $x_1 <= x_2 <= \dots <= x_n$, show that $(\sum_{i,j} |x_i - x_j|)^2 \le (2/3)(n^2 - 1) \sum_{i,j} (x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

Solution

Notice first that if we restrict the sums to i < j, then they are halved. The lhs sum is squared and the rhs sum is not, so the the desired inequality with sums restricted to i < j has (1/3) on the rhs instead of (2/3).

Consider the sum of all $|x_i - x_j|$ with i < j. x_1 occurs in (n-1) terms with a negative sign. x_2 occurs in one term with a positive sign and (n-2) terms with a negative sign, and so on. So we get $-(n-1)x_1 - (n-3)x_2 - (n-5)x_3 - \dots + (n-1)x_n = \sum (2i-1-n)x_i$.

We can now apply Cauchy-Schwartz. The square of this sum is just $\sum x_i^2 \sum (2i-1-n)^2$.

Looking at the other side of the desired inequality, we see immediately that it is $n \sum x_i^2 - (\sum x_i)^2$. We would like to get rid of the second term, but that is easy because if we add h to every x_i the sums in the desired inequality are unaffected (since they use only differences of x_i), so we can choose h so that $\sum x_i$ is zero. Thus we are home if we can show that $\sum (2i-1-n)^2 \le n(n^2 - 1)/3$. That is easy: lhs = $4 \sum i^2 - 4(n+1) \sum i + n(n+1)^2 = (2/3)n(n+1)(2n+1) - 2n(n+1) + n(n+1)^2 = (1/3)n(n+1)(2(2n+1) - 6 + 3(n+1)) = (1/3)n(n^2 - 1)$ = rhs. That establishes the required inequality.
We have equality iff we have equality at the Cauchy-Schwartz step and hence iff x_i is proportional to (2i-1-n). That implies that $x_{i+1} - x_i$ is constant. So equality implies that the sequence is an AP. But if the sequence is an AP with difference d (so $x_{i+1} = x_i + d$) and we take $x_1 = -(d/2)(n-1)$, then we get $x_i = (d/2)(2i-1-n)$ and $\sum x_i = 0$, so we have equality.

Problem B3
Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

Solution
If p = 2, then we can take q = 3, since squares cannot be 2 mod 3. So suppose p is odd. Consider $N = 1 + p + p^2 + \dots + p^{p-1}$. There are p terms. Since p is odd, that means an odd number of odd terms, so N is odd. Also N = p + 1 mod p^2, which is not 1 mod p^2, so N must have a prime factor q which is not 1 mod p^2. We will show that q has the required property.
Since N = 1 mod p, p does not divide N, so q cannot be p. If p = 1 mod q, then N = 1 + 1 + ... + 1 = p mod q. Since N = 0 mod q, that implies q divides p. Contradiction. So q does not divide p - 1.
Now suppose $n^p = p$ mod q (*). We have just shown that n cannot be 1 mod q. We have also shown that q is not p, so n cannot be a multiple of q. So assume n is not 0 or 1 mod q. Take the pth power of both sides of (*). Since $(p - 1)N = p^p - 1$, we have $p^p = 1$ mod q. So n to the power of p^2 is 1 mod q. But $n^{q-1} = 1$ mod q (the well-known Fermat little theorem). So if d = gcd(q-1, p^2), then $n^d = 1$ mod q. We chose q so that q-1 is not divisible by p^2, so d must be 1 or p. But we are assuming n is not 1 mod q, so d cannot be 1. So it must be p. In other words, $n^p = 1$ mod q. But $n^p = p$ mod q, so p = 1 mod q. Contradiction (we showed above that q does not divide p - 1).