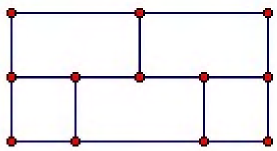
All Soviet Union Math Competitions

1st ASU 1961 problems

Problem 1

Given 12 vertices and 16 edges arranged as follows:



Draw any curve which does not pass through any vertex. Prove that the curve cannot intersect each edge just once. Intersection means that the curve crosses the edge from one side to the other. For example, a circle which had one of the edges as tangent would not intersect that edge.

Solution

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A-----B-----C
|     |     |
D--E--F--G--H
|  |     |  |
I--J-----K--L
```

If a curve intersects the boundary of a region R (such as ABFED), then it moves from inside R to outside or vice versa. Hence if R has an odd number of edges (like ABFED) then a curve intersecting all of them just once must have one endpoint inside R. But there are four such regions (ABFED, BCHGF, EFGKJ and the outside of ABCHLKJID) and only two endpoints. Note that we can easily intersect all edges but one. For example, start above AB, then cross successively AB, AD, DI, DE, EF, EJ, IJ, JK, GK, KL, HL, GH, CH, BC, FG.

Problem 2

Given a rectangle ABCD with AC length e and four circles centers A, B, C, D and radii a, b, c, d respectively, satisfying a+c=b+d<e. Prove you can inscribe a circle inside the quadrilateral whose sides are the two outer common tangents to the circles center A and C, and the two outer common tangents to the circles center B and D.

Solution

Let O be the center of the rectangle. Let r = (a+c)/2 = (b+d)/2. The required circle has center O, radius r. Let an outer common tangent touch the circle center A at W, and the circle center C at X. Let P be the midpoint of WX, then OP is parallel to AW and CX and has length r, hence the circle center O touches AW at P. Similarly for the other common tangents.

Problem 3

Prove that any 39 successive natural numbers include at least one whose digit sum is divisible by 11.

Solution

Let n be the smallest number in the sequence and m the smallest with last digit 0. m and m+10 have different digit sums unless (possibly) the penultimate digit of m is 9, but in that case m+10 and m+20 have different digit sums. So two of m, m+10, m+20 are sure to have different digit sums. Hence at least one has a digit sum not congruent to 1 mod 11. Adding the appropriate final digit gives a number whose digit sum is divisible by 11. This number lies in the range m to m+29 and m<=n+9. Hence the result. n=999981 shows it is best possible.

Problem 4

(a) Arrange 7 stars in the 16 places of a 4 x 4 array, so that no 2 rows and 2 columns contain all the stars.
(b) Prove this is not possible for <7 stars.

Solution

(a)

```
**..
*.*.
.**.
...*
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Pick any two rows. The unpicked stars lie in different columns.

(b) If there is a row with at least 3 stars, pick it. That leaves at most 3 stars, pick the row for one and the columns for the others. Now assume no row has more than 2 stars. 6 stars in <6 rows, so we can pick a row with 2 stars. That leaves 4 stars in 3 rows, so we can pick another row with 2 stars. That leaves 2 stars. Pick their columns. [This glosses over the case of <6 stars. In this case we can add extra stars to make the number up to 6. Now the procedure above deals with the original stars and the extra stars, and in particular with the original stars.]

Problem 5

(a) Given a quadruple (a, b, c, d) of positive reals, transform to the new quadruple (ab, bc, cd, da). Repeat arbitarily many times. Prove that you can never return to the original quadruple unless a=b=c=d=1.
(b) Given n a power of 2, and an n-tuple (a_1, a_2, ... , a_n) transform to a new n-tuple (a_1a_2, a_2a_3, ... , $a_{n-1}a_n$, a_na_1). If all the members of the original n-tuple are 1 or -1, prove that with sufficiently many repetitions you obtain all 1s.

Solution

(a) Let Q_0 be the original quadruple (a, b, c, d) and Q_n the quadruple after n transformations. If abcd>1, then the products form a strictly increasing sequence, so return is impossible. Similarly if abcd<1. So we must have abcd=1.
Let the largest of the four values of a quadruple Q be M(Q). If a member of Q_1 is not 1, then $M(Q_1)>1$. Q_3 consists of the elements of Q_1 squared and permuted, so $M(Q_3) = M(Q_1)^2$. Hence the sequence $M(Q_1)$, $M(Q_3)$, $M(Q_5)$, ... increases without limit. This means no return is possible, because a return would lead to the values cycling.

(b) After r<n transformations, the first number of the n-tuple is the product $a_1^{(r|0)}a_2^{(r|1)} \dots a_{r+1}^{(r|r)}$, where (r|i) denotes the binomial coefficient. [This is an easy induction.] Hence after $n=2^k$ transformations it is a_1^2 times the product $a_2^{(n|1)} \dots a_n^{(n|n-1)}$. So it is sufficient to prove that (n|i) is even for n a power of 2 and 0<i<n. But observe that (n|i) = (n-1|i) n/(n-i) and n is divisible by a higher power of 2 than n-i.

Problem 6

(a) A and B move clockwise with equal angular speed along circles center P and Q respectively. C moves continuously so that AB=BC=CA. Establish C's locus and speed.
*(b) ABC is an equilateral triangle and P satisfies AP=2, BP=3. Establish the maximum possible value of CP.

Solution

(a) Represent A, B as complex numbers $z_1 + w_1e^{it}$, $z_2 + w_2e^{it}$. Then C is $(z_1 + w_1e^{it}) + (z_2 + w_2e^{it} - z_1 - w_1e^{it})\, e^{i\,\pi/3}$, which is also of the form $z + w\, e^{it}$.

However, there is one subtlety. There are actually two circles possible for C depending on which side of AB we place it. The continuity requirement means that C is normally confined to one of the circles. However, if A and B ever coincide then C may be able to switch to the other circle.

If we regard "moves continuously" as allowing a discontinuous velocity, then a switch is always possible (provided A and B coincide).

(b) Answer: 5.

P must be the opposite side of AB to C (or we could increase CP, whilst keeping AP and BP the same, by reflecting in AB). Similarly it must be on the same side of AC as B, and on the same side of BC as A. For any P in this region the quadrilateral APBC is convex and hence satisfies Ptolemy's inequality CP· AB ≤ AP· BC + BP· AC, with equality iff APBC is cyclic. But AB = BC = CA, so we have CP ≤ AP + BP = 5 with equality iff P lies on the arc AB of the circle ABC. Note that there is just one such point, because the locus of P such that BP=1.5 AP is a circle which cuts the arc just once.

Ptolemy's inequality for 4 points A, B, C, D: AB· CD + BC· AD ≥ AC· BD with equality iff ABCD is a cyclic quadrilateral (meaning A, B, C, D lie on a circle in that order).

Proof
Take E inside ABCD such that ∠DAE = ∠CAB and ∠ADE = ∠ACB. Then ADE and ACB are similar, so DE/CB = AD/AC and hence BC· AD = AC· DE. It also follows that AE/AB = AD/AC. But we also have ∠EAB = ∠DAC and hence AEB and ADC are also similar. So EB/AB = DC/AC, and hence AB· CD = AC· EB. Adding, we have: AB· CD + BC· AD = AC(BE + ED) ≥ AC· BD with equality iff E lies on BD, or equivalently ABCD is cyclic.

This glosses over one point. It only follows that ∠EAB = ∠DAC if ABCD is convex. For the convex case, we have that ∠EAB = ∠CAB + ∠EAC and ∠DAC = ∠DAE + ∠EAC, or ∠EAB = ∠CAB - ∠EAC and ∠DAC = ∠DAE - ∠EAC. Either way ∠EAB = ∠DAC. But in the non-convex case, we can have ∠EAB = ∠CAB + ∠EAC and ∠DAC = ∠DAE - ∠EAC (or - ... +) and hence the angles ∠EAB and ∠DAC are not necessarily equal.

Problem 7

Given an m x n array of real numbers. You may change the sign of all numbers in a row or of all numbers in a column. Prove that by repeated changes you can obtain an array with all row and column sums non-negative.

Solution

The array has mn entries. Call an array that can be obtained by repeated changes a reachable array. A reachable array differs from the original only in that some or all of the signs of its mn entries may be different. There are at most 2 possibilities for each sign and hence at most 2^{mn} different reachable arrays. For each reachable array calculate the sum of all its entries. Take the reachable array with the largest such sum. It must have non-negative row and column sums, because if any such sum was negative, changing the sign of that row or column would give another reachable array with strictly greater total sum.

Problem 8

Given n<1 points, some pairs joined by an edge (an edge never joins a point to itself). Given any two distinct points you can reach one from the other in just one way by moving along edges. Prove that there are n-1 edges.

Solution

Every point must have at least one edge. We show that there is a point with just one edge. Suppose the contrary, that every point has at least two edges. We now construct a path in which the same edge or point never appears twice. Starting from any point b, move along an edge to c. c is not already on the path, because otherwise the edge would join b to itself. Now suppose we have reached a point x not previously on the path. x has at least two edges, so it must have another one besides the one we used to reach it. Suppose this joins x to y. If y is already on the path, then we have two distinct ways of moving along edges from x to y: directly, or by backtracking along the path from x to y. But this is impossible, so y is not already on the path and we may extend the path to it. But this procedure allows us to construct a path containing more than the n distinct points available. Contradiction.

The result is now easy. Induction on n. Take a point with just one edge. Remove it and the edge. Then the remaining n-1 points satisfy the premise and hence have just n-2 edges.

Problem 9

Given any natural numbers m, n and k. Prove that we can always find relatively prime natural numbers r and s such that rm+sn is a multiple of k.

Solution

Care is needed. Although easy, this is more awkward than it looks.

Let d=(m,n), the greatest common divisor of m and n. Let r=n/d, s=nhk - m/d, where h is any integer sufficiently large to ensure that s>0. Now rm+sn = mn/d + nnhk - mn/d = nnhk, which is a multiple of k. If e divides r, then it also divides rdhk = nhk. So if e divides r and s, then it also divides s - nhk = -m/d. But n/d and m/d are relatively prime, so e must be 1. Hence r and s are relatively prime.

Problem 10

A and B play the following game with N counters. A divides the counters into 2 piles, each with at least 2 counters. Then B divides each pile into 2 piles, each with at least one counter. B then takes 2 piles according to a rule which both of them know, and A takes the remaining 2 piles. Both A and B make their choices in order to end up with as many counters as possible. There are 3 possibilities for the rule:
R1 B takes the biggest heap (or one of them if there is more than one) and the smallest heap (or one of them if there is more than one).
R2 B takes the two middling heaps (the two heaps that A would take under *R1*).
R3 B has the choice of taking either the biggest and smallest, or the two middling heaps.
For each rule, how many counters will A get if both players play optimally?

Solution

Answers: [N/2], [(N+1)/2], [N/2].

Suppose A leaves piles n, m with $n \le m$.

Under *R1*, B can certainly secure m by dividing the larger pile into 1 and m-1. He cannot do better, because if b is the biggest of the 4 piles, then the smallest is at most m-b. Hence A's best strategy is to leave [N/2], [(N+1)/2].

Under *R2*, if A leaves a=2, b=N-2, then B cannot do better than [N/2], because if he divides the larger pile into a,b with $a \le b$, then he takes a+1. A cannot do better, because if he leaves a,b with $3 \le a \le b$, then B can divide to leave 1, a-1, [b/2], [(b+1)/2]. Now if $a-1 \ge [(b+1)/2]$, then B takes $b \ge [(N+1)/2]$. If $a-1 < [(b+1)/2]$, then B takes a-1+[b/2]. But $a-1 \ge 2$ and $[b/2] \ge [(b+1)/2]-1$, so $a-1+[b/2] \ge 1+[(b+1)/2]$, or B takes at least as many as A, so B takes at least [(N+1)/2].

Under *R3*, A's best strategy is to divide into [N/2],[(N+1)/2]. We have already shown that B can secure [(N+1)/2] and no more by following *R1*. He cannot do better under R2, for if he divides so that the biggest pile comes from [N/2], then the smallest does too and so he gets [(N+1)/2]. If he divides so that the biggest and smallest piles come from [(N+1)/2], then he gets only [N/2]. But one of these must apply, because if he divided so that the smaller from [N/2] was smaller than the smaller from [(N+1)/2], and the bigger from [N/2] was smaller than the bigger from [(N+1)/2], then [N/2] would be at least 2 less than [(N+1)/2] (which it is not).

Problem 11

Given three arbitary infinite sequences of natural numbers, prove that we can find unequal natural numbers m, n such that for each sequence the mth member is not less than the nth member.

Solution

Given any infinite sequence of natural numbers we can find a non-decreasing subsequence (proof below). So suppose the three sequences are a_i, b_i, and c_i. Take a non-decreasing subsequence of a_i. Suppose it is a_{i1}, a_{i2}, a_{i3}, ... Now consider the infinite sequence b_{i1}, b_{i2}, ... It must have a non-decreasing subsequence. Suppose it is b_{j1}, b_{j2}, ... Now consider the infinite sequence c_{j1}, c_{j2}, ... It must have a non-decreasing subsequence c_{k1}, c_{k2}, ... Each of the three sub-sequences a_{k1}, a_{k2}, ... , b_{k1}, b_{k2}, ... , c_{k1}, c_{k2}, ... is non-decreasing. So we may take, for example, m=k2 and n=k1.

[Proof that any infinite sequence of natural numbers has a non-decreasing subsequence: if the original sequence is unbounded, then we can take a strictly increasing subsequence. If not, then since there are only finitely many possible numbers not exceeding the bound, at least one of them must occur infinitely often.]

Problem 12

120 unit squares are arbitarily arranged in a 20 x 25 rectangle (both position and orientation is arbitary). Prove that it is always possible to place a circle of unit diameter inside the rectangle without intersecting any of the squares.

Solution

If a circle with unit diameter intersects a unit square, then its center must lie inside an area $3+\pi/4$, namely an oval centered on the square and comprising: the original square, area 1; four 1 x 1/2 rectangles on the sides, total area 2; and four quarter circles at the corners, total area $\pi/4$. So if it does not intersect any of the 120 unit squares, then it must avoid ovals with a total area of $120 \times (3+\pi/4) = 454.2$. Of course, for many arrangements of the squares, these ovals might overlap substantially, but the worst case would be no overlap.

The circle is also required to lie inside the rectangle, so its center must lie outside a strip 1/2 wide around the edge, and hence inside an inner 19 x 24 rectangle, area 456. The total area of ovals is less, so they cannot cover it completely and it must be possible to place a circle as required.

2nd ASU 1962 problems

Problem 1

ABCD is any convex quadrilateral. Construct a new quadrilateral as follows. Take A' so that A is the midpoint of DA'; similarly, B' so that B is the midpoint of AB'; C' so that C is the midpoint of BC'; and D' so that D is the midpoint of CD'. Show that the area of A'B'C'D' is five times the area of ABCD.

Solution

Compare the triangles A'B'A and ADB. The base of A'B'A can be taken as A'A, which is the same length as AD. The height of A'B'A is AB' times *sin* B'AA', which is twice AB times *sin* BAD. So area A'B'A = 2 area ADB. Similarly, area B'C'B = 2 area BAC, area C'D'C = 2 area CBD, and area D'A'D = 2 area DCA. So adding, the area A'B'A + area C'D'C = 2 area ABCD, and area B'C'B + area D'A'D = 2 area ABCD. But ABCD = A'B'A + B'C'B + C'D'C + D'A'D + ABCD. Hence result.

Problem 2

Given a fixed circle C and a line L throught the center O of C. Take a variable point P on L and let K be the circle center P through O. Let T be the point where a common tangent to C and K meets K. What is the locus of T?

Solution

Let the common tangent meet C at S. Let X be the intersection of C and OP lying between O and P. PT = PO, hence $\angle$POT = $\angle$PTO, so $\angle$OPT = 180^o - 2 $\angle$POT. But PT and OS are parallel, because both are perpendicular to the common tangent. Hence $\angle$POS = 2 $\angle$POT, so $\angle$SOT = $\angle$XOT. Hence TX is tangent to C, in other words T lies on the (fixed) tangent to C at X. Conversely, it is easy to see that any such point can be obtained (just take P such that PO = PT). Thus the required locus is the pair of tangents to C which are perpendicular to L.

Problem 3

Given integers $a_0, a_1, \dots , a_{100}$, satisfying $a_1>a_0$, $a_1>0$, and $a_{r+2}=3 a_{r+1} - 2 a_r$ for r=0, 1, ... , 98. Prove $a_{100} > 2^{99}$.

Solution

An easy induction gives $a_r = (2^r - 1)a_1 - (2^r - 2)a_0$ for r = 2, 3, ... , 100. Hence, in particular, $a_{100} = (2^{100} - 2)(a_1 - a_0) + a_1$. But a_1 and $(a_1 - a_0)$ are both at least 1. Hence result.

Problem 4

Prove that there are no integers a, b, c, d such that the polynomial ax^3+bx^2+cx+d equals 1 at x=19 and 2 at x=62.

Solution

If there were such values, then subtract the equation with $x = 19$ from the equation with $x = 62$ to get: $a(62^3 - 19^3) + b(62^2 - 19^2) + c(62 - 19) = 1$. But the left hand side is divisible by $62 - 19 = 43$, contradiction.

Problem 5

Given an n x n array of numbers. n is odd and each number in the array is 1 or -1. Prove that the number of rows and columns containing an odd number of -1s cannot total n.

Solution

If we change a -1 to 1, we affect the total number of rows and columns (containing an odd number of -1s) by 0, 2 or -2. After changing all the -1s we have total of 0. Hence the starting total must be even. So it cannot be n.

Problem 6

Given the lengths AB and BC and the fact that the medians to those two sides are perpendicular, construct the triangle ABC.

Solution

Let M be the midpoint of AB and X the midpoint of MB. Construct the circle center B, radius BC/2 and the circle diameter AX. If they do not intersect (so BC<AB/2 or BC>AB) then the construction is not possible. If they intersect at N, then take C so that N is the midpoint of BC. Let CM meet AN at O. Then AO/AN = AM/AX =2/3, so the triangles AOM and ANX are similar. Hence $\angle AOM = \angle ANX = 90^o$.

Problem 7

Given four positive real numbers a, b, c, d such that abcd=1, prove that $a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd >= 10$.

Solution

Applying the arithmetic/geometric mean result to the 10 numbers gives the result immediately.

Problem 8

Given a fixed regular pentagon ABCDE with side 1. Let M be an arbitary point inside or on it. Let the distance from M to the closest vertex be r_1, to the next closest be r_2 and so on, so that the distances from M to the five vertices satisfy $r_1 \le r_2 \le r_3 \le r_4 \le r_5$. Find (a) the locus of M which gives r_3 the minimum possible value, and (b) the locus of M which gives r_3 the maximum possible value.

Solution

Let X be the midpoint of AB and O the center of ABCDE. Suppose M lies inside AXO. Then $ME=r_3$. So we maximise r_3 by taking M at X, with distance 1.5590, and we minimise r_3 by taking M as the intersection of AO and EB with distance 0.8090. AXO is one of 10 congruent areas, so the required loci are (a) the 5 midpoints of the diagonals, and (b) the 5 midpoints of the sides.

Problem 9

Given a number with 1998 digits which is divisible by 9. Let x be the sum of its digits, let y be the sum of the digits of x, and z the sum of the digits of y. Find z.

Solution

$x \le 9 \cdot 1998 = 17982$. Hence $y \le$ the greater of 1+7+9+9+9=35 and 9+9+9+9=36. But 9 divides the original number and hence also x, y and z. Hence z=9.

Problem 10

AB=BC and M is the midpoint of AC. H is chosen on BC so that MH is perpendicular to BC. P is the midpoint of MH. Prove that AH is perpendicular to BP.

Solution

Take X on AH so that BX is perpendicular to AH. Extend to meet HM at P'. Let N be the midpoint of AB. A, B, M and X are on the circle center N radius NA (because angles AMB and AXB are 90). Also MN is parallel to BC (because AMN, ACB are similar), so NM is perpendicular to MH, in other words HM is a tangent to the circle. hence $P'M \cdot P'M = P'X \cdot P'B$. Triangles P'XH and P'HB are similar (angles at P' same and both have a right angle), so $P'H/P'X = P'B/P'H$, so $P'H \cdot P'H = P'X \cdot P'B$. Hence P'H = P'M and P' coincides with P.

Problem 11

The triangle ABC satisfies $0 \le AB \le 1 \le BC \le 2 \le CA \le 3$. What is the maximum area it can have?

Solution

If we ignore the restrictions of CA, then the maximum area is 1, achieved when AB is perpendicular to BC. But in this case CA satisfies the restrictions.

Problem 12

Given unequal integers x, y, z prove that $(x-y)^5 + (y-z)^5 + (z-x)^5$ is divisible by 5(x-y)(y-z)(z-x).

Solution

Put x-y=r, y-z=s. Then z-x = -(r+s), and $(x-y)^5 + (y-z)^5 + (z-x)^5 = r^5 + s^5 - (r+s)^5 = -5r^4s - 10r^3s^2 - 10r^2s^3 - 5rs^4 = -5rs(r+s)(r^2 + rs + s^2)$.

Problem 13

Given $a_0, a_1, \dots , a_n$, satisfying $a_0 = a_n = 0$, and and $a_{k-1} - 2a_k + a_{k+1} \ge 0$ for k=0, 1, ... , n-1. Prove that all the numbers are negative or zero.

Solution

The essential point is that if we plot the values a_r against r, then the curve formed by joining the points is cup shaped. Its two endpoints are on the axis, so the other points cannot be above it.There are many ways of turning this insight into a formal proof. Barry Paul's was neater than mine: $a_{r+1}-a_r \geq a_r-a_{r-1}$. Hence (easy induction) if $a_s - a_{s-1} > 0$, then $a_n > a_s$. Take a_s to be the first positive, then certainly $a_s > a_{s-1}$, so $a_n > 0$. Contradiction.

Problem 14

Given two sets of positive numbers with the same sum. The first set has m numbers and the second n. Prove that you can find a set of less than m+n positive numbers which can be arranged to part fill an m x n array, so that the row and column sums are the two given sets.

Example: row sums 1, 5, 3; column sums 2, 7. Array is:

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x5
x1
21
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Solution

Induction on m+n. Trivial for m+n=2.

Let x be the largest number in the two given sets. Suppose it is a row total; let y be the largest column total. If y<x, then replace x by x-y in the set of row totals and remove y from the col totals. By induction find <=m+n-2 positive numbers in an m x (n-1) array with the new totals. Adding a col empty except for y in the row totalling x-y gives the required original set.

If y=x, then drop x from the row totals and y from the col totals and argue as before.

If x was a col total we interchange rows and cols in the argument.

3rd ASU 1963

Problem 1

Given 5 circles. Every 4 have a common point. Prove that there is a point common to all 5.

Solution

Let the circles be a, b, c, d, e. Let A be a point common to b, c, d, e, let B be a point common to a, c, d, e and so on. If any two of A, B, C, D, E coincide then the coincident point is on all 5 circles. Suppose they are all distinct. Then A, B, C are on d and e. Hence d and e coincide (3 points determine a circle). Hence D is on all 5 circles.

Problem 2

8 players compete in a tournament. Everyone plays everyone else just once. The winner of a game gets 1, the loser 0, or each gets 1/2 if the game is drawn. The final result is that everyone gets a different score and the player placing second gets the same as the total of the four bottom players. What was the result of the game between the player placing third and the player placing seventh?

Solution

The bottom 4 played 6 games amongst themselves, so their scores must total at least 6. Hence the number 2 player scored at least 6. The maximum score possible is 7, so if the number 2 player scored more than 6, then he must have scored 6 1/2 and the top player 7. But then the top player must have won all his games, and hence the number 2 player lost at least one game and could not have scored 6 1/2. Hence the number 2 player scored exactly 6, and the bottom 4 players lost all their games with the top 4 players. In particular, the number 3 player won against the number 7 player.

Problem 3

(a) The two diagonals of a quadrilateral each divide it into two parts of equal area. Prove it is a parallelogram.

(b) The three main diagonals of a hexagon each divide it into two parts of equal area. Prove they have a common point. [If ABCDEF is a hexagon, then the main diagonals are AD, BE and CF.]

Solution

(a) Let the quadrilateral be ABCD and let the diagonals AC, BD meet at E. Then area ABC = AC.EB.*sin* CEB/2, and area ADC = AC.ED.*sin* CEB/2, so E is the midpoint of BD. Similarly, it is the midpoint of AC. Hence the triangles AEB and CED are congruent, so angle CDE = angle ABE, and hence AB is parallel to CD. Similarly, AD is parallel to BC.

(b) Let the hexagon be ABCDEF. Let BE, CF meet at J, let AD, CF meet at K, and let AD, BE meet at L. Let AK=a, BJ=b, CJ=c, DL=d, EL=e, FK=f. Also let KL=x, JL=y and JK=z. Consider the pair of diagonals AD, BE. They divide the hexagon into 4 parts: the triangles

ALB and DLE, and the quadrilaterals AFEL and BCDL. Since area ALB + area AFEL = area DLE + area BCDL, and area ALB + area BCDL = area DLE + area AFEL, the two triangles must have the same area (add the two inequalities). But area ALB = 1/2 AL.BL.*sin* ALB, and area DLE = 1/2 DL.EL.*sin* DLE = 1/2 DL.EL.*sin* ALB, so AL.BL = DL.EL or de = (a+x)(b+y). Similarly, considering the other two paris of diagonals, we get bc = (e+y)(f+z) and af = (c+z)(d+x). Multiplying the three inequalities gives: abcdef = (a+f)(b+y)(c+z)(d+x)(e+y)(f+z). But x, y, z are non-negative, so they must be zero and hence the three diagonals pass through a common point.

Problem 4

The natural numbers m and n are relatively prime. Prove that the greatest common divisor of m+n and m^2+n^2 is either 1 or 2.

Solution

If d divides m+n and m^2+n^2, then it also divides $(m+n)^2 - (m^2+n^2) = 2mn$ and hence also $2m(m+n) - 2mn = 2m^2$ and $2n(m+n) - 2mn = 2n^2$. But m and n are relatively prime, so m^2 and n^2 are also. Hence d must divide 2.

Problem 5

Given a circle c and two fixed points A, B on it. M is another point on c, and K is the midpoint of BM. P is the foot of the perpendicular from K to AM.

(a) prove that KP passes through a fixed point (as M varies);
(b) find the locus of P.

Solution

(a) Take Y on the circle so that angle ABY=90. Then AY is a diameter and so angle AMY=90. Take X as the midpoint of BY. Then triangles BXK and BYM are similar, so XK is parallel to YM. Hence XK is perpendicular to AM, and so P is the intersection of XK and AM. In other words, KP always passes through X.
(b) P must lie on the circle diameter AX, and indeed all such points can be obtained (given a point P on the circle, take M as the intersection of AP and the original circle). So the locus of P is the circle diameter AX.

Problem 6

Find the smallest value x such that, given any point inside an equilateral triangle of side 1, we can always choose two points on the sides of the triangle, collinear with the given point and a distance x apart.

Solution

Answer: 2/3.

Let O be the center of ABC. Let AO meet BC at D, let BO meet CA at E, and let CO meet AB at F. Given any point X inside ABC, it lies in one of the quadrilaterals AEOF, CDOE, BFOD. Without loss of generality, it lies in AEOF. Take the line through X parallel to BC. It meets

AB in P and AC in Q. Then PQ is shorter than the parallel line MON with M on AB and N on AC, which has length 2/3. If we twist the segment PXQ so that it continues to pass through X, and P remains on AB and Q on AC, then its length will change continuously. Eventually, one end will reach a vertex, whilst the other will be on the opposite side and hence the length of the segment will be at least that of an altitude, which is greater than 2/3. So at some intermediate position its length will be 2/3.

To show that no value smaller than 2/3 is possible, it is sufficient to show that any segment POQ with P and Q on the sides of the triangle has length at least 2/3. Take P on MB and Q on AN with P, O, Q collinear. Then PQ cos POM = MN - QN cospi/3 + PM cos pi/3. But PM>QN (using the sine rule, PM = OM sinPOM/sinOPM and QN = ON sinQON/sinOQN, but OM=ON, angle POM = angle QON, and angle OQN = angle OPM + pi/3 > angle OPM), and hence PQ > MN sec POM > MN.

Problem 7

(a) A 6 x 6 board is tiled with 2 x 1 dominos. Prove that we can always divide the board into two rectangles each of which is tiled separately (with no domino crossing the dividing line).

#(b) Is this true for an 8 x 8 board?

Solution

(a) We say a domino bridges two columns if half the domino is in each column. We show that for $0<n<6$ the number of dominoes bridging columns n and n+1 must be at least 2 and even.

Consider first n=1. There cannot be 3 dominoes entirely in column 1, or it would be separately tiled. So there must be at least one domino bridging columns 1 and 2. The number must be even, because it must equal the number of squares in column 1 (even) less twice the number of dominoes (entirely) in column 1.

Now suppose it is true for $n<5$ and consider column n+1. There must be at least one domino bridging columns n+1 and n+2, or columns 1 thru n+1 would be separately tiled. The number must be even, because it must equal the number of squares in column n+1 (even) less the number bridging n and n+1 (even) less twice the number entirely in the column.

So in total there are at least 5 x 2 = 10 dominoes bridging columns. By the same argument there are at least another 10 bridging rows, but there are only 18 dominoes in total.

(b) No. For example:

```
1 2 3 3 1 1 2 2
1 2 1 2 2 3 3 1
3 3 1 3 1 2 4 1
1 2 2 3 1 2 4 3
1 3 3 2 2 1 2 3
3 2 1 1 4 1 2 1
3 2 3 2 4 3 3 1
1 1 3 2 1 1 2 2
```

Problem 8

Given a set of n different positive reals $\{a_1, a_2, \dots, a_n\}$. Take all possible non-empty subsets and form their sums. Prove we get at least n(n+1)/2 different sums.

Solution

Assume $a_1 < a_2 < \dots < a_n$. We have the following collection of increasing sums:

$a_1 < a_2 < \dots < a_n$	n sums
$a_1+a_n < a_2+a_n < \dots < a_{n-1}+a_n$	n-1 sums
$a_1+a_{n-1}+a_n < a_2+a_{n-1}+a_n < \dots < a_{n-2}+a_{n-1}+a_n$	n-2 sums
...	
$a_1 + a_2 + \dots + a_n$	1 sum

A total of $1+2+ \dots +n = n(n+1)/2$.

Problem 9

Given a triangle ABC. Let the line through C parallel to the angle bisector of B meet the angle bisector of A at D, and let the line through C parallel to the angle bisector of A meet the angle bisector of B at E. Prove that if DE is parallel to AB, then CA=CB.

Solution

The idea is to find an expression for the perpendicular distance h from D to AB. Let $\gamma = \angle ACB$, $\alpha = \frac{1}{2}\angle CAB$, and $\beta = \frac{1}{2}\angle ABC$. We have $h = AP \sin \alpha$.

Using the sine rule on APC, we have $AP = AC \sin(\gamma+\beta)/\sin(\alpha+\beta)$, so $h = AC \sin \alpha \sin(\gamma+\beta)/\sin(\alpha+\beta)$. Similarly, the perpendicular distance k from E to AB is $BC \sin \beta \sin(\gamma+\alpha)/\sin(\alpha+\beta)$.

We also have that $AC/BC = \sin 2\beta/\sin 2\alpha$, and hence $h/k = \sin 2\beta \sin \alpha \sin(\gamma+\beta)/(\sin 2\alpha \sin \beta \sin(\gamma+\alpha))$. Using the fact that $\sin(\gamma+\beta) = \sin(2\alpha+\beta)$, and the expression for $\sin 2\theta$, we get $h/k = (\sin(2\alpha+2\beta) + \sin 2\alpha)/(\sin(2\alpha+2\beta) + \sin 2\beta)$ and hence h = k iff the triangle is isosceles.

For some reason the geometric solution took me longer to find. Let ED meet BC at X. Then XCD and XBE are isosceles, so BC = BX + XC = DX + XE = DE. Similarly, AC = DE. Hence AC = BC.

Problem 10

An infinite arithmetic progression contains a square. Prove it contains infinitely many squares.

Solution

Let the square be a^2 and the difference d, so that all numbers of the form a^2+nd belong to the arithmetic progression (for n a natural number). Take n to be $2ar + dr^2$, then $a^2+nd = (a+dr)^2$.

Problem 11

Can we label each vertex of a 45-gon with one of the digits 0, 1, ... , 9 so that for each pair of distinct digits i, j one of the 45 sides has vertices labeled i, j?

Solution

10 x 5 > 45, so some digit i_0 must appear less than 5 times. But each occurrence can give at most 2 edges i_0, j, so there are at most 8 edges i_0, j, which is one too few.

Problem 12

Find all real p, q, a, b such that we have $(2x-1)^{20} - (ax+b)^{20} = (x^2+px+q)^{10}$ for all x.

Solution

Comparing coefficients of x^{20}, we must have $a = (2^{20} - 1)^{1/20}$ (note that we allow either the positive or the negative root).

Set x=1/2. Then we must have $(ax + b)^{20} = 0 = (x^2+px+q)^{10}$, and hence ax + b =0 and $x^2+px+q = 0$. So b = -a/2, and 1/4 + p/2 + q = 0.

Set x=0. Then we get $q^{10} = 1 - b^{20} = 1/2^{20}$, so q = 1/4 or -1/4, and p= -1 or 0 respectively. Comparing the coefficients of x^{19}, we must have p = -1 and q = 1/4. So, if there is a solution, then it must be: $a = (2^{20} - 1)^{1/20}$, b = -a/2, p = -1, q = 1/4. This is indeed a solution because with these values, the lhs = $2^{20}(x - 1/2)^{20} - (x - 1/2)^{20}a^{20} = (x - 1/2)^{20} = (x^2 - x + 1/4)^{10}$ = rhs.

Problem 13

We place labeled points on a circle as follows. At step 1, take two points at opposite ends of a diameter and label them both 1. At step n>1, place a point at the midpoint of each arc created at step n-1 and label it with the sum of the labels at the two adjacent points. What is the total sum of the labels after step n?

For example, after step 4 we have: 1, 4, 3, 5, 2, 5, 3, 4, 1, 4, 3, 5, 2, 5, 3, 4.

Solution

Answer: 2.3^{n-1}.

True for n=1. The new points added at step n+1 have twice the sum of the points after step n, because each old point contributes to two new points. hence the total after step n+1 is three times the total after step n.

Problem 14

Given an isosceles triangle, find the locus of the point P inside the triangle such that the distance from P to the base equals the geometric mean of the distances to the sides.

Solution

Let the triangle be ABC, with AB=AC. Take the circle through B and C which has AB and AC as tangents. The required locus is the arc BC.

Suppose P lies on the arc. Let the perpendiculars from P meet BC in L, AB in N and AC in M. Join PB and PC. The triangles PNB and PLC are similar (PNB and PLC are both 90, and NBP = LCP because NB is tangent to the circle). Hence PN/PL = PB/PC. Similarly, triangles PMC and PLB are similar and hence PM/PL = PC/PB. Multiplying gives the required result PL^2 = PM.PN.

If P is inside the circle and not on it, take P' as the intersection of the line AP and the arc. We have PL<P'L, but PM>P'M and PN>P'N, hence PL^2<PM.PN. Similarly, if P is outside the circle and not on it, then PL^2>PM.PN.

4th ASU 1964

Problem 1

In the triangle ABC, the length of the altitude from A is not less than BC, and the length of the altitude from B is not less than AC. Find the angles.

Solution

Let k be twice the area of the triangle. Then $k \ge BC^2$, $k \ge AC^2$ and $k \le AC.BC$, with equality in the last case only if AC is perpendicular to BC. Hence AC and BC have equal lengths and are perpendicular. So the angles are 90, 45, 45.

Problem 2

If m, k, n are natural numbers and n>1, prove that we cannot have $m(m+1) = k^n$.

Solution

m and m+1 have no common divisors, so each must separately be an nth power. But the difference betwee the two nth powers is greater than 1 (for n>1).

Problem 3

Reduce each of the first billion natural numbers (billion = 10^9) to a single digit by taking its digit sum repeatedly. Do we get more 1s than 2s?

Solution

Taking digit sums repeatedly gives the remainder after dividing the number by 9, or 9 if the number is exactly divisible by 9. $10^9 - 1 = 9n$, and for any r>=0 the nine consecutive numbers 9r+1, 9r+2, ... , 9r+9 include just one number giving remainder 1 and one number giving remainder 2. Hence the numbers up to $10^9 - 1$ give equal numbers of 1s and 2s. 10^9 itself gives 1, so there is just one more of the 1s than the 2s.

Problem 4

Given n odd and a set of integers $a_1, a_2, \dots , a_n$, derive a new set $(a_1 + a_2)/2, (a_2 + a_3)/2, \dots , (a_{n-1} + a_n)/2, (a_n + a_1)/2$. However many times we repeat this process for a particular starting set we always get integers. Prove that all the numbers in the starting set are equal.

For example, if we started with 5, 9, 1, we would get 7, 5, 3, and then 6, 4, 5, and then 5, 4.5, 5.5. The last set does not consist entirely of integers.

Solution

Let the smallest value be s and suppose it occurs m times (with m<n). Then the values in the next stage are all at least s, and at most m-1 equal s. So after at most m iterations the smallest value is increased.

We can never reach a stage where all the values are equal, because if $(a_1+a_2)/2 = (a_2+a_3)/2 = ... = (a_{n-1}+a_n)/2 = (a_n+a_1)/2$, then $a_1+a_2 = a_2+a_3$ and hence $a_1 = a_3$. Similarly, $a_3 = a_5$, and so $a_1 = a_3 = a_5 = ... = a_n$ (n odd). Similarly, $a_2 = a_4 = ... = a_{n-1}$. But we also have $a_n + a_1 = a_1 + a_2$ and so $a_2 = a_n$, so that all a_i are equal. In other words, if all the values are equal at a particular stage, then they must have been equal at the previous stage, and hence at every stage.

Thus if the values do not start out all equal, then the smallest value increases indefinitely. But that is impossible, because the sum of the values is the same at each stage, and hence the smallest value can never exceed $(a_1 + ... + a_n)/n$.

Note that for n even the argument breaks down because a set of unequal numbers can iterate into a set of equal numbers. For example: 1, 3, 1, 3, ... , 1, 3.

Problem 5

The convex hexagon ABCDEF has all angles equal. Prove that AB - DE = EF - BC = CD - FA.

(b) Given six lengths $a_1, ... , a_6$ satisfying $a_1 - a_4 = a_5 - a_2 = a_3 - a_6$, show that you can construct a hexagon with sides $a_1, ... , a_6$ and equal angles.

Solution

(a) Extend AB, CD, EF. We get an equilateral triangle with sides AF + AB + BC, BC + CD + DE, ED + EF + FA. Hence AB - DE = CD - FA = EF - BC, as required.

(b) Take an equilateral triangle with sides s, t, u lengths $a_2 + a_3 + a_4$, $a_4 + a_5 + a_6$, and $a_6 + a_1 + a_2$ respectively. Construct BC length a_2 parallel to t with B on u and C on s. Construct DE length a_4 parallel to u with D on s and E on t. Construct FA length a_6 parallel to s with F on t and A on u. Then ABCDEF is the required hexagon, with AB = a_1, BC = a_2 etc.

Problem 6

Find all possible integer solutions for sqrt(x + sqrt(x ... (x + sqrt(x)) ...)) = y, where there are 1998 square roots.

Solution

Let s_1 = sqrt(x), s_2 = sqrt(x + s_1), s_3 = sqrt(x + s_2) and so on. So the equation given is $y = s_{1998}$. We show first that all s_n must be integral for $1 <= n <= 1998$. y is integral, so s_{1998} is integral. Now suppose s_n is integral. Then $s_{n-1} = s_n^2 - x$ is integral, proving the claim.

So in particular s_1 and s_2 are integers and $s_2^2 = s_1^2 + s_1$. But if $s_1 > 0$, then $s_1^2 < s_1^2 + s_1 < (s_1 + 1)^2$, which is impossible. Similarly $s_1 < 0$ is impossible. So the only possible solution is $s_1 = 0$ and hence x = 0 and y = 0.

Problem 7

ABCD is a convex quadrilateral. A' is the foot of the perpendicular from A to the diagonal BD, B' is the foot of the perpendicular from B to the diagonal AC, and so on. Prove that A'B'C'D' is similar to ABCD.

Solution

Let the diagonals meet at O. Then CC'O is similar to AA'O (because CC'O = AA'O = 90, and COC', AOA' are opposite angles), so A'O/C'O = AO/CO. Similarly, B'O/D'O = BO/DO. AA'O is also similar to BB'O, so A'O/B'O = AO/BO. Thus OA':OB':OC':OD' = OA:OB:OC:OD. Hence triangles OA'B' and OAB are similar. Likewise OB'C' and OBC, OC'D' and OCD, and OD'A' and ODA. Hence result.

Problem 8

Find all natural numbers n such that n^2 does not divide n!.

Solution

Answer: n = 4 or prime.

If n = rs, with 1 < r < s, then r < s < n, and hence rsn = n^2 divides n!. Similarly, if n = r^2 with r > 2, then r < 2r < n, and hence n^2 divides n!. This covers all possibilities except n = 4 or n = prime, and it is easy to see that in these cases n^2 does not divide n!.

Problem 9

Given a lattice of regular hexagons. A bug crawls from vertex A to vertex B along edges of the hexagons, taking the shortest possible path (or one of them). Prove that it travels a distance at least AB/2 in one direction. If it travels exactly AB/2 in one direction, how many edges does it traverse?

Solution

```
                                      1      2
    ./ \./ \.           directions    |     /
    |   |   |                         *    *     *
  ./°\./°\./°\.                                    \
  |   |   |   |                                    3
./°\./°\./°\./
|   |   |   |
*\./°\./°\./°
```

Suppose vertex A is that marked * at the bottom left. Without loss of generality, B is in a 60 degree sector as shown. Assume the edges have unit length. The vertices can be partitioned into two sets (marked ° and . in the diagram). Each set forms a skewed lattice with axes at 60 degrees. Any path must alternate between the two lattices.

If B is on the same lattice as A, then we can give B coordinates (m,n) relative to A and the shortest path from A to B must move m units east and n units east of north. The shortest path between a lattice point and the next lattice point east is evidently one edge in direction 3 followed by one edge in direction 2. Similarly, the shortest path between a lattice point and the next lattice point east of north is one edge in direction 1, followed by one edge in direction 2. So a shortest path from A to B must have m+n edges in direction 2.

B is a distance $\sqrt{3}(m+n/2)$ east of A and a distance 3n/2 north of A, so $AB^2 = (3m^2+3mn+3n^2) < (4m^2+8mn+4n^2) = 4(m+n)^2$. So in this case the bug must travel more than AB/2 in direction 2.

Now suppose B is on the other lattice. Let C be the lattice point immediately north of A and D the lattice point in direction 3 from A. Then a shortest path from A to B must either be A to C and then a shortest path from C to B, or A to D and then a shortest path from D to B. Take B to have coordinates (m, n) relative to C or D.

In the first case, $AB^2 = (\sqrt{3}(m+n/2))^2 + (3n/2 + 1)^2 = (3m^2 + 3mn + 3n^2) + 3n + 1$ and a shortest path has m+n units in direction 2. But $4(m + n)^2 > (3m^2 + 3mn + 3n^2) + 3n + 1$, if $m^2 + n^2 + 5mn > 3n + 1$, which is true for m, n at least 1. If m=0 and n=1, then a shortest path has 2 units in direction 1 and $AB = \sqrt{7} < 4$. If m=1 and n=0, then AB=2 and a shortest path has 1 unit in each direction. So in this case (the only one so far) we have equality.

It remains to consider the case where the path starts out towards D. In this case $AB^2 = (\sqrt{3}(m+n/2) + \sqrt{3}/2)^2 + (3n/2 - 1/2)^2 = (3m^2+3mn+3n^2) + 3m + 1$ and a path has m + n units in direction 2. But $4(m + n)^2 > (3m^2 + 3mn + 3n^2) + 3m + 1$ for $m^2+n^2 + 5mn > 3m + 1$, which is true for m, n at least 1. If m=1, n=0, then a shortest path has 2 units in direction 3 and $AB = \sqrt{7} < 4$. Finally, if m=0 and n=1, then a shortest path has 1 unit in each direction and AB = 2.

Thus the answer to the final question is 3, because the only cases where the bug travels exactly AB/2 in one direction are where it goes to the opposite vertex of a hexagon it is on.

Problem 10

A circle center O is inscribed in ABCD (touching every side). Prove that angle AOB + angle COD equals 180 degrees.

Solution

Let AB touch the circle at W, BC at X, CD at Y, and DA at Z. Then AO bisects angle ZOW and BO bisects angle XOW. So angle AOB is half angle ZOX. Similarly angle COD is half angle XOZ and hence angle AOB + angle COD equals 180.

Problem 11

The natural numbers a, b, n are such that for every natural number k not equal to b, b - k divides $a - k^n$. Prove that $a = b^n$.

Solution

We have $k^n - a = b^n - a \pmod{b - k}$. Hence $b^n - a = 0 \pmod{b - k}$ for every k not equal to b. But if b^n does not equal a, then by taking $k - b > b^n - a$ we could render the equation false.

Problem 12

How many (algebraically) different expressions can we obtain by placing parentheses in $a_1/a_2/ \dots /a_n$?

Solution

Answer 2^{n-2}. a_1 must be in the numerator, and a_2 must be in the denominator, but the other symbols can be in either. This is easily proved by induction.

Problem 13

What is the smallest number of tetrahedrons into which a cube can be partitioned?

Solution

Answer: 5.

Tetrahedral faces are triangular, so each cube face requires at least two tetrahedral faces. So at least 12 tetrahedral faces are needed in all. At most three faces of a tetrahedron can be mutually orthogonal (and no two faces can be parallel), so at most 3 faces from each tetrahedron can contribute towards these 12. So we require at least 4 tetrahedra to provide the cube faces. But these tetrahedra each have volume at most 1/6 (1/3 x face area x 1, and face area is at most 1/2). So if we have only 4 tetrahedra in total then their total volume is less than the cube's volume. Contradiction. Hence we need at least 5 tetrahedra.

It can be done with 5: lop off 4 non-adjacent corners to leave a tetrahedron. More precisely, take the cube as ABCDA'B'C'D' with ABCD horizontal, A' directly under A, B' directly under B and so on. Then the five tetrahedra are AA'BD, CC'BC, DD'A'C', BB'A'C', BDA'C'.

Problem 14

a) Find the smallest square with last digit not 0 which becomes another square (not zero) by the deletion of its last two digits. #(b) Find all squares, not containing the digits 0 or 5, such that if the second digit is deleted the resulting number divides the original one.

Solution

(a) This one must have slipped through: 121!

(b) Answer: 16,36,121,484. Suppose the number has more than 2 digits. Write it as $(10m + n)10^r + s$, where $1 <= m <= 9$, $0 <= n <= 9$, $0 <= s < 10^r$. Then we have $k(m.10^r + s) = (10m +n)10^r + s$, for some $k > 1$.

s does not contain the digits 0 or 5, so 5 does not divide s. Hence 5 divides k-1, and so k must be 6, 11, or 16 (if k was 21 or more, then the rhs would be negative). Since 25 does not divide k-1, we must have r=1 and s is a single digit.

We look at each possibility for k in turn. k = 6 gives no solutions. k = 11 gives about two dozen multiples of 11 from 121 to 891. By inspection the only squares are 121 and 484. k = 16 gives 192, which is not a square.

In addition, there is the possibility of 2 digit solutions, which I had overlooked. It is easiest to check each of the 2 digit squares, thus finding the additional solutions 16, 36.

Problem 15

A circle is inscribed in ABCD. AB is parallel to CD, and BC = AD. The diagonals AC, BD meet at E. The circles inscribed in ABE, BCE, CDE, DAE have radius r_1, r_2, r_3, r_4 respectively. Prove that $1/r_1 + 1/r_3 = 1/r_2 + 1/r_4$.

Solution

A necessary and sufficient condition for ABCD to have an inscribed circle is AB + CD = BC + AD. So we have AB + CD = 2AD, which we use repeatedly. Extend DC to X so that BX is parallel to EC. Then DX = AB + CD = 2AD and the triangles DEC, AEB, DBX are similar. Let h be the perpendicular distance from AB to CD. The similar triangles give us the heights of DEC and AEB in terms of h.

$1/r_1$ = perimeter ABE/(2 area ABE) = (AB + 2EB)/(AB.height) = (AB + 2.BD.AB/(AB+CD))/(AB.h.AB/(AB+CD)) = 2(AD + BD)/(AB.h). Similarly, $1/r_3$ = 2(AD + BD)/(CD.h).

The area of AED = area ABD - area ABE = 1/2 AB.h.CD/(2AD), so $1/r_2 = 1/r_4$ = perimeter ADE/(2 area ADE) = (AD + BD)/(h.AB.CD/2AD), and $1/r_2 + 1/r_4$ = 2(AD + BD)/h 2AD/(AB.CD) = 2(AB + BD)/h (AB + CD)/(AB.CD) = $1/r_1 + 1/r_3$.

5th ASU 1965

Problem 1

(a) Each of $x_1, ... , x_n$ is -1, 0 or 1. What is the minimal possible value of the sum of all x_ix_j with $1 <= i < j <= n$? (b) Is the answer the same if the x_i are real numbers satisfying $0 <= |x_i| <= 1$ for $1 <= i <= n$?

Solution

(a) Answer: -[n/2].

Let $A = (x_1 + ... + x_n)^2$, $B = x_1^2 + ... + x_n^2$. Then we must minimize A - B. For n even, we separately minimize A and maximize B by taking half the x's to be +1 and half to be -1. For n odd we can take [n/2] x's to be +1, [n/2] to be -1, and one to be 0. That minimizes A and gives B one less than its maximum. That is the best we can do if we fix A = 0, since A = 0 requires an even number of x's to be non-zero and hence at least one to be zero. If we do not minimize A, then since its value must be an integer, its value will be at least 1. In that case, even if B is maximized we will not get a lower total.

(b) Answer: -[n/2]. For n even, the same argument works. For n odd we can clearly get -[n/2], so it remains to prove that we cannot get a smaller sum. Suppose otherwise, so that x_i is a minimal sum with sum less than -[n/2]. Let $x_n = x$, then the sum is $x(x_1 + ... + x_{n-1})$ + sum of terms x_ix_j with $1 <= i, j < n$. But this is less than the sum for n-1, so $x(x_1 + ... + x_{n-1})$ must be negative, and since it is minimal we must have $|x| = 1$. But the same argument shows that all the terms have modulus 1. We now have a contradiction since we know that the minimum in this case is -[n/2].

Problem 2

Two players have a 3 x 3 board. 9 cards, each with a different number, are placed face up in front of the players. Each player in turn takes a card and places it on the board until all the cards have been played. The first player wins if the sum of the numbers in the first and third rows is greater than the sum in the first and third columns, loses if it is less, and draws if the sums are equal. Which player wins and what is the winning strategy?

Solution

The first player always wins.

Let the board be:

```
. F .
S . S
. F .
```

We call the squares marked F the F-squares, the squares marked S the S-squares, and the remaining squares the neutral squares. The first player wins if the sum of the two cards on the F-squares exceeds the sum of the two cards on the S-squares. We also call the first player F and the second player S.

Let the cards be $a_1>a_2> \dots >a_9$. Let $t_1 = a_1 + a_9$, $t_2 = a_2 + a_8$, $t_3 = a_3 + a_7$, $t_4 = a_4 + a_6$.

If $t_1 > t_2$, or $t_1 = t_2 > t_3$, or $t_1 = t_2 = t_3 >= t_4$ (*), then F's strategy is to get a total of t_1 or better on the F-squares and to force S to a lower score on the S-squares. If (*) does not hold, then F's strategy is to force S to t_1 or lower, and to get a higher score.

If (*) holds, then F starts by playing a_1 to an F-square. S must play to the remaining F-square, otherwise F will play a_3 or better to it on his next move and win. So S must play a_9 to the remaining F-square, giving F a total of t_1.

Now if $t_1 > t_2$, then F forces S to t_2 or worse by playing a_8 to an S-square.

If $t_1 = t_2 > t_3$, then F forces S to t_3 or worse by playing a_2 to a neutral square. If S plays to an S-square, then he cannot do better than $a_3 + a_8$, which loses. So he plays a_8 to a neutral square. But now F plays a_3 to an S-square, and S cannot do better than t_3.

If $t_1 = t_2 = t_3 > t_4$, then F forces S to t_4 or worse. He starts by playing a_2 to a neutral square. If does not prevent F playing a_8 to an S-square on his next move, then he cannot do better than $a_3 + a_8$, which loses. So he must play a_8 to a neutral square. Now F plays a_3 to a neutral square. If S does not prevent F playing a_7 to an S-square on the following move, then he cannot do better than $a_4 + a_7$ which loses, so he plays a_7 to a neutral square. F now plays a_4 to an S-square. S cannot now do better than t_4, which loses.

Finally, if $t_1 = t_2 = t_3 = t_4$, then F proceeds as in the last case except that at the end he plays a_4 to the last neutral square instead of to an S-square. S now gets $a_5 + a_6$ on the S-squares, which loses.

If (*) does not hold, then F starts by playing a_9 to an S-square. If S does not play to the other S-square, then F will play a_7 or a_8 there on his next move and S will lose. So S must play a_1 to the other square, and gets a total of t_1. F now plays to get t_2, t_3 or t_4 on the F-squares.

If $t_1 < t_2$, then F plays a_2 to an F-square and so gets at least t_2 and wins.

If $t_1 = t_2 < t_3$, then F plays a_8 to a neutral square. If S does not prevent F playing $a_2$to an F-square on his next move, then F will get at least $a_2 + a_7$ and win. So S must play a_2 to a neutral square. Now F plays a_3 to an F-square and so gets at least t_3 on the F-squares and wins.

Finally, if $t_1 = t_2 = t_3 < t_4$, then F plays as in the previous case, except that at the end he plays a_7 to a neutral square instead of a_3 to an F-square. S must prevent F playing a_3 to an F-square the following move, or F gets at least $a_3 + a_6$ and wins. So S plays a_3 to a neutral square. F now plays a_4 to an F square and so must get at least t_4, which wins.

Problem 3

A circle is circumscribed about the triangle ABC. X is the midpoint of the arc BC (on the opposite side of BC to A), Y is the midpoint of the arc AC, and Z is the midpoint of the arc AB. YZ meets AB at D and YX meets BC at E. Prove that DE is parallel to AC and that DE passes through the center of the inscribed circle of ABC.

Solution

ZY bisects the angle AYB, so AD/BD = AY/BY. Similarly, XY bisects angle BYC, so CE/BE = CY/BY. But AY = CY. Hence AD/BD = CE/BE. Hence triangles BDE and BAC are similar and DE is parallel to AC.

Let BY intersect AC at W and AX at I. I is the in-center. AI bisects angle BAW, so WI/IB = AW/AB. Now consider the triangles AYW, BYA. Clearly angle AYW = angle BYA. Also angle WAY = angle CAY = angle ABY. Hence the triangles are similar and AW/AY = AB/BY. So AW/AB = AY/BY. Hence WI/IB = AY/BY = AD/BD. So triangles BDI and BAW are similar and DI is parallel to AW and hence to DE. So DE passes through I.

Problem 4

Bus numbers have 6 digits, and leading zeros are allowed. A number is considered lucky if the sum of the first three digits equals the sum of the last three digits. Prove that the sum of all lucky numbers is divisible by 13.

Solution

The total is made up of numbers of the form abcabc, and pairs of numbers abcxyz, xyzabc. The former is abc.1001 and the sum of the pair is 1001(abc + xyz). So the total is divisible by 1001 and hence by 13.

Problem 5

The beam of a lighthouse on a small rock penetrates to a fixed distance d. As the beam rotates the extremity of the beam moves with velocity v. Prove that a ship with speed at most v/8 cannot reach the rock without being illuminated.

Solution

Let the lighthouse be at L. Take time t = 0 at the moment the boat starts its run, so that at t = 0 it is at S a distance d from L, and thereafter it is at a distance less than d. Take A and B a distance d from L so that ALBS is a semicircle with diameter AB and S the midpoint of the arc AB. During the period to t = 2.5 pi.d/v the boat has traveled a distance less than d, so it cannot reach AB. But it is a distance less than d from L, so it must be inside the semicircle. But during this period the beam sweeps across from LA to LB and so it must illuminate the boat.

Problem 6

A group of 100 people is formed to patrol the local streets. Every evening 3 people are on duty. Prove that you cannot arrange for every pair to meet just once on duty.

Solution

Every time a person is on duty he is paired with two other people, so if the arrangement were possible the number of pairs involving a particular person would have to be even. But it is 99.

Problem 7

A tangent to the inscribed circle of a triangle drawn parallel to one of the sides meets the other two sides at X and Y. What is the maximum length XY, if the triangle has perimeter p?

Solution

Let BC be the side parallel to XY, h the length of the altitude from A, and r the radius of the in-circle. Then XY/BC = (h - 2r)/h. But r.p = h.BC. So XY = (p - 2BC)BC/p = (p^2/8 - 2(BC - p/4)^2)/p. So the maximum occurs when BC = p/4 and has value p/8.

Problem 8

The n^2 numbers x_{ij} satisfy the n^3 equations: $x_{ij} + x_{jk} + x_{ki} = 0$. Prove that we can find numbers $a_1, \dots , a_n$ such that $x_{ij} = a_i - a_j$.

Solution

Taking i = j = k, we have that $x_{ii} = 0$. Now taking j=k, we have that $x_{ij} = - x_{ji}$. Define $a_i = x_{i1}$. Then we have $x_{i1} + x_{1j} + x_{ji} = 0$. Hence $x_{ij} = a_i - a_j$.

Problem 9

Can 1965 points be arranged inside a square with side 15 so that any rectangle of unit area placed inside the square with sides parallel to its sides must contain at least one of the points?

Solution

Yes. Place a grid of 900 points in 30 equally spaced rows and columns, so that each point is a distance 15/31 from its nearest neighbours (or 15/31 from the edge). This blocks all rectangles except those slimmer than 1/2. Those slimmer than 1/2 must have length at least 2, so we can block them with a smaller set of rows and columns containing more finely spaced points.

Label the rows 1-30. In each of the 7 rows 3, 7, 11, 15, 19, 23, 27 place an additional 31 points, so that each of these rows has 61 equally spaced points at a spacing of 15/62. Similarly for the columns. So in total we are placing an additional 2.7.31 = 434 points. Any rectangle of length >2 must encounter one of these rows (or columns) and hence must have width less than 1/4. This blocks any rectangle except those with width < 1/4.

In each of the 3 rows 7, 15, 23 place an additional 62 points, so that each of these rows has 123 equally spaced points at a spacing of 15/124. Similarly for the columns. So in total we are placing an additional 2.3.62 = 372 points. Any rectangle of length >4 must encounter one of these rows (or columns) and hence must have width less than 1/8. This blocks any rectangle except those with width < 1/8 and hence length > 8.

In row 15 place an additional 124 points, so that it has a total of 247 equally spaced points at a spacing of 15/247. Similarly for column 15. This requires an additional 248 points. Any rectangle which can fit through these gaps has area at most 15 x 15/247 < 1. So we have blocked all rectangles with area 1 or more and used 900 + 434 + 372 + 248 = 1954 points.

Ilan Mayer, who seems to solve these problems effortlessly, came up with a neater arrangement of points. He used narrowly spaced points along widely spaced diagonals: (k/15, k/15) for k = 1,2,...,224; ((28*n+k)/15, k/15) for n = 1,2,...,7, k = 1,2,...,224-28*n; (k/15, (28*n+k)/15) for n = 1,2,...,7, k = 1,2,...,224-28*n. The diagonals are spaced 28/15 apart, so the biggest rectangle that can be fitted between two diagonals has sides 15/15 less epsilon and 15/15 less epsilon. For example, take the vertices as (14/15 + e, e), (29/15 - e, e), (14/15 + e, 15/15 - e), (29/15 - e, 15/15 - e). If one allows a rectangle to touch points (in other words if one took the rectangles to exclude their boundaries) then this does not work - many 15 x 1/15 rectangles will fit. But one can add an additional point on each of the 15 lines, keeping the points on each line evenly spaced. That blocks rectangles without boundary and still has only 1821 points.

Problem 10

Given n real numbers $a_1, a_2, \dots , a_n$, prove that you can find n integers $b_1, b_2, \dots , b_n$, such that $|a_i - b_i| < 1$ and the sum of any subset of the original numbers differs from the sum of the corresponding b_i by at most (n + 1)/4.

Solution

We can take all a_i to lie in the range (0,1) and all b_i to be 0 or 1. The largest positive value of the sum of $(a_i - b_i)$ for any subset is achieved by taking the subset of those i for which $b_i = 0$. Similarly, the largest negative value is achieved by taking those i for which $b_i = 1$. So the worst subset will be one of those two.

If $a_i < a_j$, then we cannot have $b_i = 1$ and $b_j = 0$ if the set of b_is is to minimise the maximum sum, because swapping them would reduce the sum of a's with b = 0 and the sum of (1 - a)'s with b = 1. So if we order the a's so that $a_1 <= a_2 <= \dots <= a_n$, then a best set of b's is $b_i = 0$ for i <= some k, and $b_i = 1$ for i > k. [If some of the a_i are equal, then we can find equally good sets of b's do not have this form, but we cannot get a lower maximum sum by departing from this form.]

Let $L_i = a_1 + a_2 + \dots + a_i$, and $R_i = a_{i+1} + a_{i+2} + \dots + a_n$. As we increase i the sums L_i increase and the sums R_i decrease, so for some k we must have $L_k < R_k$, $L_{k+1} >= R_{k+1}$. Either k or k+1 must correspond to the optimum choice of b's to minimise the maximum sum.

Now assume that the a's form a maximal set, in other words they are chosen so that the minimum is as large as possible. We show first that in this case $L_{k+1} = R_k$. Suppose $L_{k+1} < R_k$. Then we could increase each of $a_{k+1}, a_{k+2}, \dots , a_n$ by epsilon. This would leave L_k unaffected, but slightly increase L_{k+1} and slightly reduce R_k. For small epsilon this does not change the value of k, but increases the smaller of L_{k+1} and R_k, thus increasing the minimum and contradicting the maximality of the original a's. Similarly, if $L_{k+1} > R_k$, we could decrease each of $a_1, a_2, \dots , a_{k+1}$ by epsilon, thus slightly increasing R_k and reducing L_{k+1}.

Suppose not all of $a_1, a_2, \dots , a_{k+1}$ are equal. Take i so that $a_i < a_{i+1}$. Now increase each of $a_1, a_2, \dots , a_i$ by epsilon and reduce each of $a_{i+1}, a_{i+2}, \dots , a_{k+1}$ by epsilon', with epsilon and epsilon' sufficiently small that we do not upset the ordering or change the value of k, and with their relative sizes chosen so that L_{k+1} is increased. R_k is also increased, so we contradict the maximality of the a's. Hence all $a_1, a_2, \dots , a_{k+1}$ are equal. Similarly, we show that all of $a_{k+1}, \dots , a_n$ are equal. For if not we can increase slightly $a_{k+1}, \dots , a_j$ and reduce slightly $a_{j+1}, \dots , a_n$ to get a contradiction.

So we have established that all the a's must be equal. Suppose n is odd = 2m+1 and that all the a's equal x. Then for the optimum k we have (k+1)x = (2m+1-k)(1-x), hence k+1 = (2m+2)(1-

x) and the maximum difference is (k+1)x = (2m+2)(1-x)x. This is maximised by taking x = 1/2, k = m, and is (m+1)/2 = (n+1)/4. If n is even = 2m, then for the optimum k we have (k+1)x = (2m-k)(1-x), so k+1 = (2m+1)(1-x), and the maximum difference is (k+1)x = (2m+1)(1-x)x. However, in this case we cannot take x = 1/2, because that would give k = m - 1/2 which is non-integral, so we take k = m-1 or m, both of which give a maximum difference of m(m+1)/(2m+1) = n(n+2)/(4n+4) < (n+1)/4.

Problem 11

A tourist arrives in Moscow by train and wanders randomly through the streets on foot. After supper he decides to return to the station along sections of street that he has traversed an odd number of times. Prove that this is always possible. [In other words, given a path over a graph from A to B, find a path from B to A consisting of edges that are used an odd number of times in the first path.]

Solution

Disregard all edges except those used in the path from A to B, and for each of those let the multiplicity be the number of times it was traversed. Let the degree of a vertex be the sum of the multiplicities of its edges. The key is to notice that the degree of every vertex except A and B must be even. For as we traverse the path from A to B we increase the degree by 2 each time we pass through a vertex. But at the start of the path, as we leave A, we only increase its degree by 1. Similarly as we arrive at B for the last time.

Now construct a path from B as follows. Since B has odd degree it must have an edge of odd multiplicity. Suppose the edge connects B to C. Follow that edge and reduce its multiplicity by one, so that B's degree and C's degree are each reduced by one. Now C has odd degree, so it must have an edge of odd multiplicity. Repeat. Since there are only finitely many edges we must eventually be unable to continue the path. But the only way that can happen is if we reach A.

Problem 12

(a) A committee has met 40 times, with 10 members at every meeting. No two people have met more than once at committee meetings. Prove that there are more than 60 people on the committee.

(b) Prove that you cannot make more than 30 subcommittees of 5 members from a committee of 25 members with no two subcommittees having more than one common member.

Solution

(a) Each meeting involves 10.9/2 = 45 pairs. So after 40 meetings, there have been 1800 pairs. We are told that these are all distinct. But if there are N people on the committee, then there are only N(N-1)/2 pairs available. For N=60, this is only 1770.

(b) A subcommittee of 5 has 5.4/2 = 10 pairs. So 31 subcommittees have 310 pairs, and these are all distinct, since no two people are on more than one subcommittee. But a committee of 25 only has 25.24/2 = 300 pairs available.

Problem 13

Given two relatively prime natural numbers r and s, call an integer *good* if it can be represented as mr + ns with m, n non-negative integers and *bad* otherwise. Prove that we can find an integer c, such that just one of k, c - k is good for any k. How many bad numbers are there?

Solution

Notice that 0 is good and all negative numbers are bad. Take c = rs - r - s. First c, is bad. For suppose otherwise: c = mr + ns. Then mr + ns = (s-1)r - s. Hence (s-1-m)r = (n+1)s, so r divides n+1. Say n+1=kr, and then s-1-m=ks, so m = (1-k)s - 1. But n+1 is positive, so k>=1, and hence m is negative. Contradiction.

If k is good, then c-k must be bad (otherwise c would be good). Suppose k is bad. Since r and s are relatively prime we can find integers a and b with ar + bs = 1 and hence integers m and n with mr + ns = k. Adding a multiple of sr - rs to both sides if necessary, this gives a pair m, n with mr + ns = k and m non-negative. Now take the pair with the smallest possible non-negative m. Then m <= s-1 (for otherwise m' = m-s, n' = n+r would be a pair with smaller non-negative m). Also n <= -1, otherwise k would be good. Now c - k = (s - 1 - m)r + (-n - 1)s and the coefficients s - 1 - m and -n - 1 are both non-negative, so c - k is good.

So exactly (rs - r - s + 1)/2 integers are bad.

Problem 14

A spy-plane circles point A at a distance 10 km with speed 1000 km/h. A missile is fired towards the plane from A at the same speed and moves so that it is always on the line between A and the plane. How long does it take to hit?

Solution

Answer: 18pi sec.

Let C be the position of the spy-plane at the moment the missile is fired. Let B be the point a quarter of the way around the circle from C (in the direction the spy-plane is moving). Then the missile moves along the semi-circle on diameter AB and hits the plane at B.

To see this take a point P on the quarter circle and let the line AX meet the semi-circle at Q. Let O be the center of the semicircle. The angle BOQ is twice the angle BAQ, so the arc BP is the same length as the arc BQ. Hence also the arc AQ is the same length as the arc CP.

Problem 15

Prove that the sum of the lengths of the edges of a polyhedron is at least 3 times the greatest distance between two points of the polyhedron.

Solution

If A and B are at the greatest distance, then they must be vertices. For suppose A is not a vertex. Then there is a segment XY entirely contained in the polyhedron with A as an interior point. But now at least one of angles BAX, BAY must be at least 90. Suppose it is BAX. Then BX is longer than BA. Contradiction.

Take a plane through A perpendicular to the line AB. Then the polyhedron must lie entirely on one side of the plane, for if Z lay on the opposite side to B, then BZ would be longer than BA. Now move the plane slightly towards B keeping it perpendicular to AB. The intersection of the plane and the polyhedron must be a small polygon. The polygon must have at least 3 vertices, each of which must lie on an edge of the polyhedron starting at A. Select three of these edges.

As the plane is moved further towards B, the selected vertices may sometimes split into multiple vertices or they may sometimes coalesce. In the former case, just choose one of the daughter vertices. In the latter case, let O be the point of intersection of the plane and AB. Let O' be the point of intersection at the last coalescence (or A if there was none). Then we have three paths along edges, with no edges in common, each of which projects onto O'O and hence has length at least O'O. Now select one or more new vertices to replace any lost through coalescence and repeat.

Problem 16

An alien moves on the surface of a planet with speed not exceeding u. A spaceship searches for the alien with speed v. Prove the spaceship can always find the alien if v>10u.

Solution

The spacecraft flies at a constant height, so that it can see a circular spot on the surface. It starts at the north pole and spirals down to the south pole, overlapping its previous track on each circuit. The alien cannot move fast enough to cross the track before the next circuit, so it is trapped inside a reducing area surrounding the south pole.

The value of 10 is not critical, so we do not have to optimise the details. Take the height above the surface to be half the radius. Then a diameter of the spot subtends an angle 2 $\cos^{-1}(1/1.5)$ at the center of the planet. $1/1.5 < 1/\sqrt{2}$, so the angle is more than 90 degrees. The critical case is evidently when the spacecraft is circling the equator. Using suitable units, we may take the radius of the planet to be 1 and the spaceship speed to be 1. Then the diameter of the spot is pi/2. We take the overlap to be 2/3, so that each revolution the track advances pi/6. If the planet flew in a circle above the equator, the distance for a revolution would be 2pi 1.5 = 3pi. The helical distance must be less than 3pi + pi/6 = 19pi/6. So the alien can travel a distance 19pi/60 < 2/3 pi/2 and is thus trapped as claimed.

6th ASU 1966

Problem 1

There are an odd number of soldiers on an exercise. The distance between every pair of soldiers is different. Each soldier watches his nearest neighbour. Prove that at least one soldier is not being watched.

Solution

The key is to notice that no loops of size greater than two are possible. For suppose we have $A_1, A_2, \dots , A_n$ with A_i watching A_{i+1} for $0 < i < n$, and A_n watching A_1. Then the distance $A_{i-1}A_i$ is greater than the distance A_iA_{i+1} for $1 < i < n$, and the distance A_1A_n is less than the distance A_1A_2. Hence the distance A_1A_n is less than the distance $A_{n-1}A_n$ and so A_{n-1} is closer to A_n than A_1. Contradiction.

Pick any soldier. Now pick the soldier he is watching, and so on. The total number of soldiers is finite so this process must terminate with some soldier watching his predecessor. If the process terminates after more than two soldiers have been picked, then the penultimate soldier is watched by more than one soldier. But in that case there must be another soldier who is unwatched, because the number of soldiers equals the number of soldiers watching.

If the process terminates after just two soldiers, then we have a pair of soldiers watching each other. Now repeat on the remaining soldiers. Either we find a soldier watched twice (in which case some other soldier must be unwatched) or all the soldiers pair off, except one, since the total number is odd. But that soldier must be unwatched.

Problem 2

(a) B and C are on the segment AD with AB = CD. Prove that for any point P in the plane: $PA + PD \geq PB + PC$.

(b) Given four points A, B, C, D on the plane such that for any point P on the plane we have $PA + PD \geq PB + PC$. Prove that B and C are on the segment AD with AB = CD.

Solution

(a) Suppose the points lie in the order A, B, C, D. If P lies on AD, then the result is trivial, and we have equality if P lies outside the segment AD. So suppose P does not lie on AD.

Let M be the midpoint of AD. Take P' so that P, M, P' are collinear and PM = MP'. Then we wish to prove that $PA + AP' > PB + BP'$. Extend P'B to meet PA at Q. Then $P'A + AQ > P'Q$, so $P'A + AP > P'Q + QP$. But $QP + BQ > PB$, so $QP + QP' > PB + PB'$. Hence result.

(b) Let the foot of the perpendicular from B, C onto AD be X, Y respectively. Suppose that N, the midpoint of XY, is on the same side of M, the midpoint of AD, as D. Then take P to be a remote point on the line AD, the opposite side of A to D, so that A, D, M and N are all on the same side of the line PAD from P. Then $PA + PD = 2PM < 2PN \leq PB + PC$. Contradiction. So we must have N coincide with M. But we still have $PA + PD = 2PM = 2PN < PB + PC$, unless both B and C are on the line AD. So we must have B and C on the line AD and AB = CD. It remains to show that B and C are between A and D. Take P = B. Then if C is not between A and D, we have PC > PD (or PA), contradiction.

Problem 3

Can both $x^2 + y$ and $x + y^2$ be squares for x and y natural numbers?

Solution

No. The smallest square greater than x^2 is $(x+1)^2$, so we must have y > 2x. Similarly x > 2y. Contradiction.

Problem 4

A group of children are arranged into two equal rows. Every child in the back row is taller than the child standing in front of him in the other row. Prove that this remains true if each row is rearranged so that the children increase in height from left to right.

Solution

Rearrange the children in the back row into order, and rearrange the front row in the same way, so that each child stays in front of the same child in the back row. Denote heights in the back row by a_i and heights in the front row by b_i. So we have $a_1 \le a_2 \le ... \le a_n$, and $a_i > b_i$ for i = 1, 2, ... , n.

Now if i < j, but $b_i > b_j$, then we may swap b_i and b_j and still have each child taller than the child in front of him. For $b_i < a_i <= a_j$, and $b_j < b_i < a_i$. By repeated swaps we can get the front row into height order. [For example, identify the shortest child and swap him to the first position, then the next shortest and so on.]

Problem 5

A rectangle ABCD is drawn on squared paper with its vertices at lattice points and its sides lying along the gridlines. AD = k AB with k an integer. Prove that the number of shortest paths from A to C starting out along AD is k times the number starting out along AB.

Solution

Let ABCD have n lattice points along the side AB. Then it has kn lattice points along the side AD. Let X be the first lattice point along AB after leaving A. A shortest path from X to C must involve a total of kn + n - 1 moves between lattice points, n - 1 in the direction AB and kn in the direction BC. Hence the total number of such paths is (kn + n - 1)!/((kn)! (n - 1)!). Similarly, the number of paths starting out along AD is (kn + n - 1)!/((kn - 1)! n!). Let m = (kn + n - 1)!/((kn - 1)! (n - 1)!). Then the number starting along AB is m/(kn) and the number starting along AD is m/n, which is k times larger, as required.

Problem 6

Given non-negative real numbers $a_1, a_2, ... , a_n$, such that $a_{i-1} <= a_i <= 2a_{i-1}$ for i = 2, 3, ... , n. Show that you can form a sum $s = b_1a_1 + ... + b_na_n$ with each b_i +1 or -1, so that $0 <= s <= a_1$.

Solution

We show that you can pick $b_n, b_{n-1}, \dots, b_r$ so that $s_r = b_n a_n + b_{n-1}a_{n-1} + \dots + b_r a_r$ satisfies $0 \le s_r \le a_r$. Induction on r. Trivial for r = n. Suppose true for r. Then $-a_{r-1} \le s_r - a_{r-1} \le a_r - a_{r-1} \le a_{r-1}$. So with $b_{r-1} = -1$ we have $|s_{r-1}| \le a_{r-1}$. If necessary, we change the sign of all $b_n, b_{n-1}, \dots, b_{r-1}$ and obtain s_{r-1} as required. So the result is true for all $r \ge 1$ and hence for r = 1.

Problem 7

Prove that you can always draw a circle radius A/P inside a convex polygon with area A and perimeter P.

Solution

Draw a rectangle width A/P on the inside of each side. The rectangles at each vertex must overlap since the angle at the vertex is less than 180. The total area of the rectangles is A, so the area covered must be less than A. Hence we can find a point not in any of the rectangles. But this point must be a distance more than A/P from each side, so we can use it as the center of the required circle.

Problem 8

A graph has at least three vertices. Given any three vertices A, B, C of the graph we can find a path from A to B which does not go through C. Prove that we can find two disjoint paths from A to B.

[A graph is a finite set of vertices such that each pair of distinct vertices has either zero or one edges joining the vertices. A path from A to B is a sequence of vertices $A_1, A_2, \dots, A_n$ such that $A=A_1$, $B=A_n$ and there is an edge between A_i and A_{i+1} for i = 1, 2, ... , n-1. Two paths from A to B are disjoint if the only vertices they have in common are A and B.]

Solution

Take any path from A to B. Suppose it is $A=A_0, A_1, \dots, A_n=B$. We show by induction on r that we can find two disjoint paths from A to A_r. If r = 1, then take any vertex C distinct from A and A_1. Take any path from A_1 to C which does not go through A. Now take any path from C to A which does not go through A_1. Joining these two paths together gives a path p from A to A_1 which does not involve the edge AA_1. Then p and the edge AA_1 are the required disjoint paths.

Suppose now we have two disjoint paths $A, B_1, B_2, \dots, B_s, A_r$ and $A, B_t, B_{t-1}, \dots, B_{s+1}, A_r$ and we wish to find two disjoint paths joining A and A_{r+1}. Take a path between A and A_{r+1} which does not include A_r. If it also avoids all of $B_1, \dots, B_t$, then we are home, because it is disjoint from the alternative path $A, B_1, B_2, \dots, B_s, A_r, A_{r+1}$. If not, let B_i be the first of the B's on the path as we move from A_{r+1} to A. This allows us to construct two disjoint paths from A to A_{r+1}. One path goes from A to B_i and then from B_i to A_{r+1}. The other path goes around the other way to A_r and then along the edge to A_{r+1}. [Explicitly, if $i \le s$, then the paths are $A, B_1, B_2, \dots, B_i$, ... (new path) ... A_{r+1} and $A, B_t, B_{t-1}, \dots, A_r, A_{r+1}$. If $i > s$, then the paths are $A, B_t, B_{t-1}, \dots, B_i$, ... (new path) ... A_{r+1} and $A, B_1, \dots, B_s, A_r, A_{r+1}$.] Hence, by induction, there are two disjoint paths from A to B.

Problem 9

Given a triangle ABC. Suppose the point P in space is such that PH is the smallest of the four altitudes of the tetrahedron PABC. What is the locus of H for all possible P?

Solution

Answer: the triangle DEF with FAE parallel to BC, DBF parallel to CA and DCE parallel to AB.

Let α be the angle between planes ABC and PBC. Let h be the perpendicular distance from H to the line BC, and let h_A be the perpendicular distance from A to the line BC. Then PH = h tan α, and the altitude from A to PBC is $h_A \sin \alpha$. Hence if PH is shorter than the altitude from A we require that $h < h_A \cos \alpha < h_A$. Similar arguments apply for B and C. So if PH is the shortest then H lies within triangle DEF.

If H does lie within DEF, then if we make α sufficiently small we will have $h < h_A \cos \alpha$ and hence PH will be shorter than the altitude from A. Similarly we can make PH sufficiently short that PH is less than the altitudes from B and C. Hence the inside of DEF is the required locus.

Problem 10

Given 100 points on the plane. Prove that you can cover them with a collection of circles whose diameters total less than 100 and the distance between any two of which is more than 1. [The distance between circles radii r and s with centers a distance d apart is the greater of 0 and d - r - s.]

Solution

If we have two circles diameters d and d', the distance between which is less than 1, then they are contained in a circle diameter d+d'+1. [If the line through the centers cuts the circles in A, B, A', B', then take a circle diameter AB'.] So start with 100 circles of diameter 1/1000 each. If any pair is a distance <= 1 apart, then replace them by a single circle, increasing the total diameter by 1. Repeat until all the circles are a distance > 1 apart. We must end up with at least one circle, so the total increase is at most 99. Hence the final total diameter is at most 99 1/10.

Problem 11

The distance from A to B is d kilometers. A plane P is flying with constant speed, height and direction from A to B. Over a period of 1 second the angle PAB changes by α degrees and the angle PBA by β degrees. What is the minimal speed of the plane?

Solution

Answer: $20\pi d\sqrt{(\alpha\beta)}$ kilometers per hour.

Let the plane be at height h and a (horizontal) distance y from A. Let the angle PAB be $\theta+\alpha$ and the angle PBA be φ. After 1 second, the angle PAB is θ and the angle PBA is $\varphi+\beta$. We have immediately that:

$h/y = \tan(\theta+\alpha)$, $h/(d-y) = \tan \varphi$, $h/(y+x) = \tan \theta$, $h/(d-y-x) = \tan(\varphi+\beta)$.

Eliminating θ, we obtain: $h/y = (\tan \alpha + \tan \theta)/(1 - \tan \alpha \tan \theta) = (a(y+x) + h)/(y+x-ah)$, where $a = \tan \alpha$. Hence $x = a(h^2 + y^2)/(h - ay)$. Similarly, eliminating φ, we obtain $x = b(h^2 + (d-y)^2)/(h + (d-y)b)$.

At this point I do not see how to make further progress without approximating. But approximating seems reasonable, since α and β, are certainly small, at least when expressed in radians. For example, typical values might be 10,000 ft for h and more than 10 miles for y or d-y and 500 mph for the aircraft speed. That gives $x = 0.14$ miles, so $x/y = 0.014$ and $x/h = 0.07$. So, let us neglect a/h, b/h, a/y etc. Then we get the simplified expressions: $x = a(h^2 + y^2)/h = b(h^2 + (d-y)^2)/h$.

If $a = b$, then we quickly obtain $y = d/2$, $h = d/2$, $x = ad$. Assume $a > b$. Then we can solve for h, substitute back in and obtain an expression for x in terms of y. It is convenient to divide through by d and to write $X = x/d$, $Y = y/d$. Note that since we are assuming $a > b$, we require $Y < 1/(1 + \sqrt{(a/b)})$. After some manipulation we obtain: $X = ab(1 - 2Y)/\sqrt{((a-b)(b(1-Y)^2 - aY^2)}$. Differentiating, we find that there is a minimum at $Y = b/(a+b)$, which is in the allowed range, and that the minimum value of X is $\sqrt{(ab)}$. By symmetry, we obtain the same result for $a < b$ and we notice that it is also true for $a = b$. So in all cases we have that the minimum value of x is $d\sqrt{(ab)}$.

We are assuming α and β are small, so we may take $a = \alpha$, $b = \beta$. However, the question specified that α and β were measured in degrees, so to obtain the final answer we must convert, giving: $x = d\ (\pi/180)\ \sqrt{(\alpha\beta)}$, and hence speed $= 20\pi d\sqrt{(\alpha\beta)}$ kilometers per hour.

Problem 12

Two players alternately choose the sign for one of the numbers 1, 2, ... , 20. Once a sign has been chosen it cannot be changed. The first player tries to minimize the final absolute value of the total and the second player to maximize it. What is the outcome (assuming both players play perfectly)?

Example: the players might play successively: 1, 20, -19, 18, -17, 16, -15, 14, -13, 12, -11, 10, -9, 8, -7, 6, -5, 4, -3, 2. Then the outcome is 12. However, in this example the second player played badly!

Solution

Answer: 30.

The second player can play the following strategy: (1) if the first player plays $2n-1$ for $1 <= n <= 9$, then he replies $2n$ with the opposite sign; (2) if the first player plays $2n$ for $1 <= n <= 9$, then he replies $2n-1$ with the opposite sign; (3) if the first player plays 19 or 20, then he plays the other with the same sign. This secures a score of at least 39 (from (3)) less 9 x 1 (from (1) and (2)). So he can ensure a score of at least 30.

The first player can play the following strategy: (1) he opens with 1; (2) if the second player plays $2n$ for $1 \le n <= 9$, then he replies with $2n+1$ with the opposite sign; (3) if the second player plays $2n+1$ for $1 <= n <= 9$, then he replies with $2n$ with the opposite sign; (4) if any of these replies are impossible, or if the second player plays 20, then he replies with the highest number available with the opposite sign. If the second player does not play 20 until the last move, then this strategy ensures a score of at most 1 (from (1)) + 9 x 1 (from (2) and (3)) + 20 = 30. Now suppose that the second player plays 20, a_1, a_2, ... , a_n (where $1 \leq n \leq 9$) which require a reply under (4). The reason a_1 required a move under (4) was that a_1-1 or a_1+1 was the 1st player's response to 20. Similarly, the reason a_2 required a move under (4) was that a_2-1 or a_2+1 was the 1st player's response to a_2, and so on. Thus the increment to the absolute value from these moves is at most $|20-a_1+1| + |a_1-a_2+1| + ... + |a_{n-1}-a_n+1| + |a_n| = 20 + n$. The increment from the moves under (2) and (3) is (9 - n) x 1, and the increment from the move under (1) is 1. Hence the maximum absolute value is 30.

Since the 1st player has a strategy to do no worse than 30 and the 2nd player has a strategy to do no worse than 30, these strategies must actually be optimal.

1st ASU 1967

Problem 1

In the acute-angled triangle ABC, AH is the longest altitude (H lies on BC), M is the midpoint of AC, and CD is an angle bisector (with D on AB).

(a) If AH <= BM, prove that the angle ABC <= 60.

(b) If AH = BM = CD, prove that ABC is equilateral.

Solution

As usual let a, b, c be the lengths of BC, CA, AB respectively and let A, B, C denote the angles BAC, ABC, BCA respectively. We use trigonometry and try to express the quantities of interest in terms of a, b and C.

(a) Since AH is the longest altitude, BC must be the shortest side (use area = side x altitude/2). So $b^2 >= a^2$, and $c^2 >= a^2$. Using the formula $c^2 = a^2 + b^2 - 2ab \cos C$, we deduce that $b^2 >= 2ab \cos C$. Hence $2b^2 >= a^2 + 2ab \cos C$. After a little manipulation this gives: $a^2 + b^2 - 2ab \cos C >= 4/3 (a^2 + b^2/4 - ab \cos C)$ or $c^2 >= 4/3\ BM^2$. But we are given that BM >= AH = b sin C, so $(b^2\sin^2C)/c^2 <= 3/4$. But the sine formula gives sin B = (b sin C)/c, so $\sin^2C <= 3/4$. The triangle is acute-angled, hence B <= 60 degrees.

(b) The angle bisector theorem gives AD/BD = b/a, hence AD/AB = b/(a+b), so AD = bc/(a+b). Hence, using the sine formula, CD/sin A = AD/(sin C/2). So CD = bc sin A/((a+b) sin C/2) = ba sin C/((a+b) sin C/2), using the sine formula again. But we are given that CD ≥ AH = b sin C, so a/((a+b) sin C/2) ≥ 1. But a is the shortest side, so a/(a+b) ≤ 1/2 and hence sin C/2 < 1/2. The triangle is acute-angled, so C/2 ≤ 30 degrees, and C ≤ 60 degrees. BC is the shortest side, so A is the smallest angle and hence A ≤ 60 degrees. Also since AH ≤ BM, B ≤ 60 degrees. But the angles sum to 180 degrees, so they must all be 60 degrees and hence the triangle is equilateral.

Problem 2

(a) The digits of a natural number are rearranged and the resultant number is added to the original number. Prove that the answer cannot be 99 ... 9 (1999 nines).

(b) The digits of a natural number are rearranged and the resultant number is added to the original number to give 10^{10}. Prove that the original number was divisible by 10.

Solution

(a) Let the digits of the original number be a_1, a_2, ... and the rearranged digits be b_1, b_2, ... Suppose that in the addition there is a carry, in other words $a_i+b_i > 9$ for some i. Take the largest such i. Then the resulting digit in that position cannot be a 9. Contradiction. So there cannot be any carries. Hence each pair $a_i+b_i= 9$. Let n be the total number of digits 0, 1, 2, 3, and 4 in the number. Then each of these must be paired with a digit 5, 6, 7, 8 or 9. So the total number of digits 5, 6, 7, 8 and 9 must also be n, and hence the number must have an even number of digits. But we are told that the answer and hence the original number has an odd number of digits.

(b) In the addition the carry can never be 2, because that would require the previous carry to be at least 2, and the first carry cannot be 2. So all carries are 0 or 1. If a carry is 1, then all subsequent carries must also be 1. If the first carry is 0, then the corresponding digits must be 0 and hence the original number is divisible by 10. If it is not, then all carries are 1 and hence after the first carry all the digit pairs sum to 9. But arguing as in (a), this means that there must be an even number of digits, excluding the last (where we have a digit sum 10), and hence an odd number of digits in the original number. But 10^{10} has an odd number of digits and hence the original number had an even number of digits. Contradiction.

Problem 3

Four lighthouses are arbitarily placed in the plane. Each has a stationary lamp which illuminates an angle of 90 degrees. Prove that the lamps can be rotated so that at least one lamp is visible from every point of the plane.

Solution

Take a north direction, arbitary except that no points are aligned north-south or east-west. Take the two most northerly points. Point the lamp for the more easterly of these two in the direction SW (so that it covers directions S to W). Point the lamp for the other in the direction SE. For the other two points, point the lamp for the more easterly in the direction NW, and the lamp for the other in the direction NE.

Clearly the lamps cover all directions, the only possible problem is uncovered strips. However, the two lamps pointing N are below the two lamps pointing S, and the two lamps pointing E are west of the two lamps pointing W, so there are no uncovered strips.

Problem 4

(a) Can you arrange the numbers 0, 1, ... , 9 on the circumference of a circle, so that the difference between every pair of adjacent numbers is 3, 4 or 5? For example, we can arrange the numbers 0, 1, ... , 6 thus: 0, 3, 6, 2, 5, 1, 4.

(b) What about the numbers 0, 1, ... , 13?

Solution

No. Each of the numbers 0, 1, 8, 9 can only be adjacent to 3, 4, 5 or 6. But they can only accomodate 3 numbers, not 4.

0, 3, 7, 10, 13, 9, 12, 8, 11, 6, 2, 5, 1, 4 is a solution for 13.

In passing, there are obviously no solutions for 4 or 5. There is just the one solution for 6 (given in the question). For 7 there are 5 solutions: 0, 3, 6, 1, 5, 2, 7, 4; 0, 3, 6, 1, 4, 7, 2, 5; 0, 3, 6, 2, 7, 4, 1, 5; 0, 3, 7, 4, 1, 6, 2, 5; 0, 4, 1, 6, 3, 7, 1, 5. For 8 there is the solution 0, 3, 7, 2, 6, 1, 5, 8, 4, and maybe others.

Problem 5

Prove that there exists a number divisible by 5^{1000} with no zero digit.

Solution

We first find a multiple of 5^{1000} which has no zeros in the last 1000 digits. Suppose that we have a multiple $n.5^{1000}$ whose last zero is in place r (treating the last place as place 0, the next to last as place 1 and so on). Then $n(10^r + 1)$ has the same digits in places 0 to r-1 and a non-zero digit in place r, and hence no zeros in places 0 to r. So repeating, we find a multiple $n.5^{1000}$ with no zeros in the last 1000 digits.

Now let m be the remainder when n is divided by 2^{1000}, so $n = k.2^{1000} + m$, and hence $m.5^{1000} = n.5^{1000} - k.10^{1000}$. So $m.5^{1000}$ has the same last 1000 digits as $n.5^{1000}$. But it has less than 1001 digits, and hence it has exactly 1000 digits and no zeros.

Problem 6

Find all integers x, y satisfying $x^2 + x = y^4 + y^3 + y^2 + y$.

Solution

The only solutions are x,y = -1,1-; 0,-1; -1,0; 0,0; -6,2; or 5,2.

$(y^2 + y/2 - 1/2)(y^2 + y/2 + 1/2) = y^4 + y^3 + 1/4\ y^2 - 1/4 < y^4 + y^3 + y^2 + y$ except for $-1 <= y <= -1/3$. Also $(y^2 + y/2)(y^2 + y/2 + 1) = y^4 + y^3 + 5/4\ y^2 + y/2$ which is greater than $y^4 + y^3 + y^2 + y$ unless $0 <= y <= 2$.

But no integers are greater than $y^2 + y/2 - 1/2$ and less than $y^2 + y/2$. So the only possible solutions have y in the range -1 to 2. Checking these 4 cases, we find the solutions listed.

Problem 7

What is the maximum possible length of a sequence of natural numbers $x_1, x_2, x_3, \ldots$ such that $x_i \le 1998$ for $i >= 1$, and $x_i = |x_{i-1} - x_{i-2}|$ for $i \geq 3$.

Solution

Answer 2998.

The sequence is completely determined by its first two elements. If the largest element of the sequence is n, then it must occur as one of the first two elements. Because x_3 and x_4 are both smaller than the largest of the first two elements and hence all subsequent elements are too.

Let f(n, m) be the length of the sequence with $x_1 = n$, $x_2 = m$. It is straightforward to verify by induction that f(1,2n) = f(2n-1,2n) = 3n + 1, f(2n,1) = f(2n,2n-1) = 3n, f(2n,2n+1) = 3n + 3, f(1,2n+1) = f(2n+1,1) = 3n + 2, f(2n+1,2n) = 3n + 1. A rather more fiddly induction then shows that these are the best possible lengths. Hence the longest sequence with no element more than 1998 is that starting 1, 1998 which has length 2998.

Problem 8

499 white rooks and a black king are placed on a 1000 x 1000 chess board. The rook and king moves are the same as in ordinary chess, except that taking is not allowed and the king is allowed to remain in check. No matter what the initial situation and no matter how white

moves, the black king can always:
(a) get into check (after some finite number of moves);
(b) move so that apart from some initial moves, it is always in check after its move;
(c) move so that apart from some initial moves, it is always in check (even just after white has moved).
Prove or disprove each of (a) - (c).

Solution

(a) True. Black moves to one end of a main diagonal and then moves along the diagonal to the opposite end. Each of the 499 rooks is in some row. Since black moves through each row, every rook must change row. But each of the rooks is also in some column and so every rook must also change column. A rook cannot change row and column in the same move, so white must make at least 998 moves before black reaches the opposite end of the diagonal. But it cannot start until black is two moves from its starting position, because if it moves a rook into row (or column) one or two earlier, then black is checked or can move into check. So it has only 997 moves available, which is one too few.

(b) False. Suppose the contrary, that after move n, the king is always in check after its move. Let the corners of the board be A, B, C, D. After move n, white moves all its rooks inside a square side 23 at corner A. The king must now be in the 23 rows between A and B or in the 23 columns between A and D. Suppose the latter. Then white moves all its rooks inside a square side 23 at corner B. This should take 499 moves. However, it could take longer if black used his king to obstruct the move. The worst case would be 3 x 23 additional moves (the king can only obstruct one row of 23 rooks, and each rook in the obstructed row could take 4 moves instead of one to reach its destination.). During this period the king must remain in the 23 rows from A to B or the 23 columns from A to D, since it must remain in check. Thus it cannot get to B by the completion of the process. In fact, it must be at least 999 - 46 (the total number of moves required) - (499 + 69) (the number of moves available) = 385 moves behind.

White now moves all the rooks inside a square side 23 at corner C. The king cannot cut across (or it will be unchecked). It must keep within 23 squares of the edge. So it ends up 770 moves behind (more in fact, since it cannot obstruct the move as effectively). Finally, white moves all the rooks inside a square side 23 at corner D. The king cannot get to the side CD by the time this process is completed. So there is then a lag of over two hundred moves before it can get back into check. Note that it does not help black to change direction. Whatever black does, white ends up with all the rooks at a corner and the king a long way from the two checked sides.

(c) False. This follows from (b). But we may also use a simpler argument. Take coordinates x = 1 to 1000, y = 1 to 1000. White gets its pieces onto (2,0), (4,0), ... , (998,0). If the king moves onto (2n,*), then white moves its rook from (2n,0) to (2n-1,0), leaving the king unchecked. If the king moves to (2n-1,*) or (2n+1,*), then white moves its rook back to (2n,0), leaving the king unchecked. If the king stays on the line (2n,*), then white fills in time by toggling one of its endmost rooks to an adjacent square (and the king remains unchecked). The only way the black king can escape this repeated unchecking is by moving up to the line y = 0. If it does so, then white transfers all its rooks to the line y = 1000 and repeats the process. The transfer takes 499 moves. It takes black 1000 moves to follow, so during the 501 moves before black catches up, the king is subject to repeated unchecking.

Problem 9

ABCD is a unit square. One vertex of a rhombus lies on side AB, another on side BC, and a third on side AD. Find the area of the set of all possible locations for the fourth vertex of the rhombus.

Solution

Answer: 2 1/3.

Let the square be ABCD. Let the vertices of the rhombus be P on AB, Q on AD, and R on BC. We require the locus of the fourth vertex S of the rhombus. Suppose P is a distance x from B. We may take $x \le 1/2$, since the locus for $x > 1/2$ is just the reflection of the locus for $x < 1/2$. Then since PR is parallel to QS, S is a distance x from the line AD. Also, by continuity, as Q varies over AD (with P fixed a distance x from B), the locus of S is a line segment.

The two extreme positions for S occur when Q coincides with A and when R coincides with C. When Q coincides with A the rhombus has side 1-x. Hence $BR^2 = (1-x)^2 - x^2 = 1 - 2x$. In this case SR is parallel to AB, so the distance of S from AB is $\sqrt{1-2x}$. When R coincides with C, the rhombus has side $\sqrt{1+x^2}$, so $AQ^2 = 1 + x^2 - (1-x)^2 = 2x$. Hence the distance of S from AB is $1 + \sqrt{2x}$.

Thus the locus of S over all possible rhombi is the interior of a curvilinear quadrilateral with vertices MDNC, where M is the midpoint of AB and N is the reflection of M in CD. Moreover the curve from M to C is just the translate of the curve from D to N, for if we put $y = 1/2 - x$, then $\sqrt{1-2x}$ becomes $\sqrt{2y}$. Thus if L is the midpoint of CD, then the area in the MLC plus the area in DLN is just 1/2, and the total area of the curvilinear quadrilateral is 1.

However, the arrangement of the vertices discussed above is not the only one. The order of vertices above is PQSR. We could also have PQRS or PSQR. In either case QR is a side rather than a diagonal of the rhombus. We consider the case PQRS (the case PSQR is just the reflection in the line MN). As before it is convenient to keep P fixed, but this time we take x to be the distance AP. Take y to be the distance AQ.

As before we find that S must lie on a line parallel to BC a distance x from it (on the other side to AD). Again we find that for fixed P, the locus of S is a segment of this line. If we assume that AQ > BR, then the two extreme positions are (1) QR parallel to AB, giving S on the line AB, (2) Q at D, giving S a distance x from the line AB. So as x varies from 0 to 1 we get a right-angled triangle sides 1, 1 and $\sqrt{2}$ and area 1/2. However, we can also have BR > AQ. This gives points below the line AB. The extreme position is with R at C. Suppose QD = y. Then $1 + y^2 = x^2 + (1 - y)^2$, so $y = x^2/2$. This gives S a distance y below the line AB. This gives an additional area of 1/6 (by calculus - integrate $x^2/2$ from 0 to 1; I do not see how to do it without).

The triangle and the curvilinear triangle together form a curvilinear triangle area 1/2 + 1/6 = 2/3. There is an identical triangle formed by reflection in MN. Thus the total area is 1 + 2/3 + 2/3 = 2 1/3.

Thanks to Robert Hill and John Jones for pointing out that the original solution missed out the two triangles.

Problem 10

A natural number k has the property that if k divides n, then the number obtained from n by reversing the order of its digits is also divisible by k. Prove that k is a divisor of 99.

Solution

Let r(m) denote the number obtained from m by reversing the digits.

We show first that k cannot be divisible by 2 or 5. It cannot be divisible by both, for then it ends in a zero and hence r(k) < k and so is not divisible by k (contradiction). So if 5 divides k, then the last digit of k must be 5. Since r(k) is divisible by 5 its last digit must also be 5, so the first digit of k is 5. But now 3k has first digit 1 (3.5 > 10 and 3.6 < 20), so r(3k) has last digit 1 and cannot be divisible by 5. Contradiction. If 2 divides k, then every multiple of k must be even. So the last digit of r(k) must be even and hence the first digit of k must be 2, 4, 6, or 8. If 2, then 5k has first digit 1, so r(2k) is odd. Contradiction. Similarly, if the first digit is 4, 3k has first digit 1; if 6, then 5k has first digit 3; if 8, then 2k has first digit 1. Contradiction. So k is not divisible by 2 or 5.

Suppose $k = 10^n a_n + ... + a_0$. k divides r(k), so $a_0 >= 1$. Hence $(10^{n+1} - 1)k = 10^{2n+1}a_n + ... + 10^{n+1}a_0 - (10^n a_n + ... + a_0) = 10^{2n+1}a_n + ... + 10^{n+1}(a_0-1) + 10^n c_n + ... + 10c_1 + (c_0+1)$, where $c_i = 9 - a_i$. The reverse of this, $10^{2n+1}(c_0+1) + 10^{2n}c_1 + ... + 10^{n+1}c_n + 10^n(a_0-1) + ... + a_n$, is also divisible by k. So is the reverse of k, $10^n a_0 + ... + a_n$ and hence also their difference: $10^n(10^{n+1}(c_0+1) + 10^n c_1 + ... + 10c_n - 1)$. k has no factors 2 or 5, so k must divide $10^{n+1}(c_0+1) + 10^n c_1 + ... + 10c_n - 1$. Adding 10k, we find that k also divides $10^{n+2} + 10^n 9 + ... + 10.9 - 1 = 1099...989$ (n - 2 consecutive 9s) $= 11(10^{n+1} - 1)$.

We can now carry out exactly the same argument starting with $(10^{n+2} - 1)k$. This leads to k dividing $10^{n+2}(c_0+1) + ... + 10^2 c_0 + 10.9 - 1$ and hence also $10^{n+3} + 10^{n+1}9 + ... + 10^2 9 + 10.8 + 9 = 11(10^{n+2} - 1)$. Subtracting 10 times this from the previous number we conclude that k must divide $11(10^{n+1} - 1) - 11(10^{n+1} - 10) = 99$.

Finally, we note that any factor of 99 has the required property. For 3 and 9 divide a number if and only if they divide its digit sum. So if m is divisible by 3 or 9, then the number formed by any rearrangement of its digits is also divisble by 3 or 9. m is divisible by 11 if and only if the difference between the sums of alternate digits is divisible by 11, so if m is divisible by 11, then so is its reverse.

2nd ASU 1968 problems

Problem 1

An octagon has equal angles. The lengths of the sides are all integers. Prove that the opposite sides are equal in pairs.

Solution

Extend the sides to form two rectangles. Let the sides of the octagon have length a, b, c, d, e, f, g, h. Then we can find the rectangle sides. For example, one of the rectangles has opposite sides $a + (b + h)/\sqrt{2}$ and $e + (d + f)/\sqrt{2}$. Hence either $a = e$ or $\sqrt{2} = (b + h - d - f)/(a - e)$. The root is irrational, so we must have $a = e$. Similarly for the other pairs of opposite sides.

Problem 2

Which is greater: 31^{11} or 17^{14}? [No calculators allowed!]

Solution

$17^2 = 289 > 9.31$. So $17^{14} > 9^7\ 31^7$. But $3^7 = 2187 > 31^2$. Hence $17^{14} > 31^{11}$.

Problem 3

A circle radius 100 is drawn on squared paper with unit squares. It does not touch any of the grid lines or pass through any of the lattice points. What is the maximum number of squares can it pass through?

Solution

Take compass directions aligned with the grid. Let N, E, S, W be the most northerly, easterly, southerly and westerly points on the circle. The arc from N to E must cross 100 north-south grid lines and 100 east-west grid lines. Each time it crosses a grid line it changes square (and it never crosses two grid lines at once, because it does not pass through any lattice points), so the arc N to E must pass through 200 in addition to the starting square. Similarly for the other 4 arcs. So the circle passes through a total of 800 squares (we count the starting square in the last 200).

Problem 4

In a group of students, 50 speak English, 50 speak French and 50 speak Spanish. Some students speak more than one language. Prove it is possible to divide the students into 5 groups (not necessarily equal), so that in each group 10 speak English, 10 speak French and 10 speak Spanish.

Solution

Let EF denote the number of students speaking English and French. Similarly define ES, FS, E, F, S, EFS. Then ES + EF + E + EFS = 50, EF + FS + F + EFS = 50. Subtracting: ES - F = FS - E. Similarly, ES - F = EF - S.

Pair off members of FS with members of E. Similarly, members of ES with F,and members of EF with S. The resulting pairs have one person speaking each language. If ES = F, then the only remaining students are those in EFS, who speak all three languages. We thus have a collection of units (pairs or individuals) each containing one speaker of each language.

If ES < F, then after the pairing off we are left with equal numbers of members of E, F, and S. These may be formed into triplets, with each triplet containing one speaker of each language. As before we also have the students in EFS. Again, we have partitioned the student body into units with each unit containing one speaker of each language.

If ES > F, then after the pairing off, we are left with an equal number of members of ES, FS and EF. These may be formed into triplets, with each triplet containing two speakers of each language. So, in this case we partition the student body into units with each unit containing either one speaker of each language, or two speakers of each language.

Finally, we may divide the units into 5 groups with 10 speakers of each language in each group.

Problem 5

Prove that:

$$2/(x^2 - 1) + 4/(x^2 - 4) + 6/(x^2 - 9) + ... + 20/(x^2 - 100) =$$

$$11/((x - 1)(x + 10)) + 11/((x - 2)(x + 9)) + ... + 11/((x - 10)(x + 1)).$$

Solution

lhs $= 1/(x - 1) - 1/(x + 1) + 1/(x - 2) - 1/(x + 2) + ... + 1/(x + 10) - 1/(x - 10) =$

$1/(x - 1) - 1/(x + 10) + 1/(x - 2) - 1/(x + 9) + ... + 1/(x - 10) - 1/(x + 1) =$ rhs.

Problem 6

The difference between the longest and shortest diagonals of the regular n-gon equals its side. Find all possible n.

Solution

Answer: $n = 9$.

For $n < 6$, there is at most one length of diagonal. For $n = 6, 7$ the longest and shortest, and a side of the n-gon form a triangle, so the difference between the longest and shortest is less than the side.

For $n > 7$ the side has length $2R \sin \pi/n$, the shortest diagonal has length $2R \sin 2\pi/n$, and the longest diagonal has length $2R$ for n even and $2R \cos \pi/2n$ for n odd (where R is the radius of the circumcircle). Thus we require:

$$\sin 2\pi/n + \sin \pi/n = 1 \text{ and n even, or}$$

$\sin 2\pi/n + \sin \pi/n = \cos \pi/2n$ and n odd.

Evidently the lhs is a strictly decreasing function of n and the rhs is an increasing function of n, so there can be at most one solution of each equation. The second equation is satisfied by n = 9, although it is easier to see that there is a quadrilateral with the longest diagonal and shortest diagonals as one pair of opposite sides, and 9-gon sides as the other pair of opposite sides. The angle between the longest side and an adjacent side is 60, so that its length is the length of the shortest diagonal plus 2 x 9-gon side x cos 60. Hence that is the only solution for n odd.

For n = 8 we have the same quadrilateral as for the 9-gon except that the angle is 67.5 and hence the difference is less than 1. For n = 10, $\sin 2\pi/10 + \sin \pi/10 = \sin \pi/10 (2 \cos \pi/10 + 1) < 3 \sin \pi/10 < 3 \pi/10 < 1$. So there are no solutions for n even ≥ 10, and hence no solutions for n even.

Problem 7

The sequence a_n is defined as follows: $a_1 = 1$, $a_{n+1} = a_n + 1/a_n$ for $n \geq 1$. Prove that $a_{100} > 14$.

Solution

First we must notice that for $1 \leq a, b$ we have $a < b$, then $a + 1/a < b + 1/b$. This is basic to any estimation.

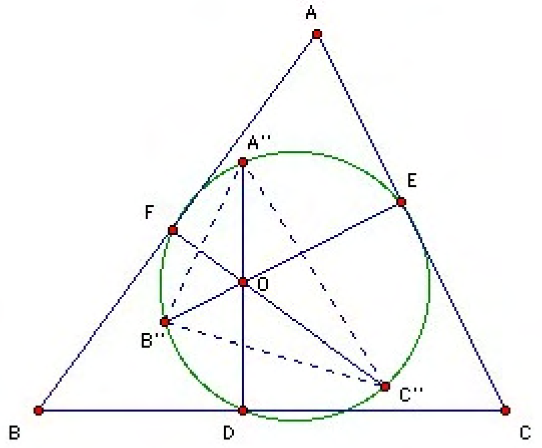
The obvious approach is to notice that if $a_i \leq n$, then $a_{i+1} \geq a_i + 1/n$. Hence it takes at most n steps to get from n - 1 to n. Unfortunately, this does not quite work: we need 2 + 3 + ... + 14 = 104 steps to get from 1 to 14.

The trick is to notice that $a_{n+1}^2 > a_n^2 + 2$. But $a_2 = 2$, so $a_n^2 > 2n$. That gives $a_{100}^2 > 200 > 14^2$.

Problem 8

Given point O inside the acute-angled triangle ABC, and point O' inside the acute-angled triangle A'B'C'. D, E, F are the feet of the perpendiculars from O to BC, CA, AB respectively, and D', E', F' are the feet of the perpendiculars from O' to B'C', C'A', A'B' respectively. OD is parallel to O'A', OE is parallel to O'B' and OF is parallel to O'C'. Also OD· O'A' = OE· O'B' = OF· O'C'. Prove that O'D' is parallel to OA, O'E' to OB and O'F' to OC, and that O'D'· OA = O'E'· OB = O'F'· OC.

Solution



Let Γ be the circumcircle of DEF. Let OD, OE, OF meet it again at A", B", C" respectively. Then the figure O'A'B'C' must be similar to OA"B"C". So to prove that OD is parallel to O'A', we have to prove that AO is perpendicular to B"C".

So AO meets B"C" at D". Now since ∠AFC" = 90^o and ∠AD"C" = 90^o, both F and D" lie on the circle diameter AC". Hence AO· OD" = OF· OC". Similarly, BO meets C"A" at E", and CO meets A"B" at F", and BO· OE" = OD· OA" and CO· OF" = OE· OB". Hence OD"· OA = OE"· OB = OF"· OC. So using the similarity, O'D'· OA = O'E'· OB = O'F'· OC.

Problem 9

Prove that any positive integer not exceeding n! can be written as a sum of at most n distinct factors of n!.

Solution

Given m ≤ n! write m = nq + r (*), with 0 ≤ q, 0 ≤ r < m. Then q ≤ (n-1)!, so q is a sum of at most n-1 distinct factors of (n-1)!. r is itself a factor of n! and is not divisible by n, so (*) expresses m as a sum of at most n distinct factors of n!.

Problem 10

Given a triangle ABC, and D on the segment AB, E on the segment AC, such that AD = DE = AC, BD = AE, and DE is parallel to BC. Prove that BD equals the side of a regular 10-gon inscribed in a circle with radius AC.

Solution

DA = DE, so DAE is isosceles. DE is parallel to BC, so ABC is isosceles, so BA = AC/(2 cos A). Hence BD = AC/(2 cos A) - AC. But AE = 2 AC cos A, so we have an equation for c=cos A: $4 c^2 + 2c - 1 = 0$.

2π/5, 4π/5, 6π/5, 8π/5 and 10π/5 are the roots of: real part of $(\cos θ + i \sin θ)^5 = 1$. Expanding this gives that cos 2π/5, cos 4π/5, cos 6π/5, cos 8π/5 and 1 are the roots of $16c^5 - 20c^3 + 5c - 1 = 0$. Dividing by (c - 1) gives $16c^4 + 16c^3 - 4c^2 - 4c + 1 = (4c^2 + 2c - 1)^2$. So cos 2π/5 (= cos 8π/5) and cos 4π/5 (= cos 6π/5) are the roots of $4c^2 + 2c - 1 = 0$.

We know that $A < 90^o$ (since $A = C$ and their sum is less than 180^o). Hence $A = 2\pi/5$. So $BD = 2\ AC \cos 2\pi/5 = 2\ AC \sin \pi/10$, which is the side length for a regular 10-gon inscribed in a circle radius AC.

Problem 11

Given a regular tetrahedron ABCD, prove that it is contained in the three spheres on diameters AB, BC and AD. Is this true for any tetrahedron?

Solution

Let the tetrahedron have side 1. Then the center O is a distance $1/\sqrt{8}$ from the center of each of the spheres, so it is contained in each of the spheres. We now use convexity.

Two circles with diameters two of the sides of a triangle cover the triangle (consider the foot of the altitude to the third side), so faces ABC and ABD are certainly contained in the spheres. Consider face ACD. The sphere on BC passes through the midpoints of AC and CD, and through C, so it contains the triangle formed by these three points (by convexity). But the rest of ACD is contained in the sphere on AD. Similarly for the face BCD. Hence all the faces are contained in the spheres. But now take any point P inside the tetrahedron. Extend OP to meet a face at X. X lies in one of the spheres, but O also lies in the sphere and hence all points on OX, including P (by convexity).

False in general. Take ABCD to be a plane square, then no points on CD are in the spheres except C and D (and we can obviously distort this slightly to make it less degenerate).

Problem 12

(a) Given a 4 x 4 array with + signs in each place except for one non-corner square on the perimeter which has a - sign. You can change all the signs in any row, column or diagonal. A diagonal can be of any length down to 1. Prove that it is not possible by repeated changes to arrive at all + signs.

(b) What about an 8 x 8 array?

Solution

(a) Let S be the set of the 8 positions on the perimeter not at a corner. Any move changes the sign of either 2 or 0 of the members of S. We start with an odd number of members of S with a minus sign, so we must always have an odd number of members of S with a minus sign and hence cannot get all plus signs.

(b) Also impossible. The same argument works.

Problem 13

The medians divide a triangle into 6 smaller triangles. 4 of the circles inscribed in the smaller triangles have equal radii. Prove that the original triangle is equilateral.

Solution

Denote the side lengths by a, b, c and the corresponding median lengths by m_a, m_b, m_c. The six small triangles all have equal area. [Let the areas be $t_1, \dots , t_6$. It is obvious that the adjacent pairs have equal height and equal base, so we have $t_1 = t_2$, $t_3 = t_4$, $t_5 = t_6$. The three on each side of a median sum to the same area, so $t_1 + t_2 + t_3 = t_4 + t_5 + t_6$, $t_1 + t_5 + t_6 = t_2 + t_3 + t_4$. Subtracting gives $t_1 = t_4$. Similarly, $t_2 = t_5$ and we are home.] So by the usual result that the twice area of a triangle equals its perimeter times its in-radius, we conclude that the perimeters of four of the small triangles are equal.

Two of them must share a side of the original triangle. Suppose it is a. Then we have: $a/2 + m_a/3 + 2m_b/3 = a/2 + m_a/3 + 2m_c/3$. So $m_b = m_c$. That implies that b = c. [Because the triangle formed by the centroid and side a is isosceles, so the median is perpendicular to the side, so the main triangle is isosceles.]

Using the facts that b = c and $m_b = m_c$, we see that two of the remaining small triangles have perimeter $b/2 + m_b$ and two have perimeter $b/2 + m_b/3 + 2m_a/3$. So there are two cases to consider. In the first case $a/2 - m_a/3 = b/2 - m_b/3$. That implies a = b, since if a < b, $m_a > m_b$ (consider the triangle formed by the centroid and the side c). So the triangle is equilateral.

The second case is harder. We have: $a/2 + m_a/3 + 2m_b/3 = b/2 + m_b$, and hence $a/2 + m_a/3 = b/2 + m_b/3$ (*). Take the angle between a and b to be θ. Then $m_a = b \sin \theta$, $a =2b \cos \theta$, and $m_b^2 = b^2/4 + a^2 - ab \cos \theta = b^2/4 + 2b^2 \cos^2\theta$. We can now use (*) to get an equation for θ. First we square (*) to get: $m_b^2 = (3a/2 - 3b/2 + m_a)^2$. We divide out the factor b^2 to get: $1/4 + 2 \cos^2\theta = 3\ 1/4 + 8 \cos^2\theta - 9 \cos \theta + 3 \sin \theta(2 \cos \theta - 1)$. Squaring, so that we can use $\sin^2\theta = 1 - \cos^2\theta$, and writing $c = \cos \theta$, we get: $(1 - c^2)(4c^2 - 4c + 1) = 4c^4 - 12c^3 + 13c^2 - 6c + 1$. Hence $8c^4 - 16c^3 + 10c^2 - 2c = 0$. Factorizing: $c(c - 1)(2c - 1)^2 = 0$. c = 0 and c = 1 give degenerate triangles, so we must have c = 1/2 and hence the triangle is equilateral.

Problem 14

Prove that we can find positive integers x, y satisfying $x^2 + x + 1 = py$ for an infinite number of primes p.

Solution

This is a trivial variant on the proof that there are an infinite number of primes. Suppose that we can only find x, y for a finite number of primes $p_1, p_2, \dots , p_n$. Set $x = p_1p_2 \dots p_n$. Then none of the p_i can divide $x(x+1) + 1$. But it must have prime factors. Contradiction.

Problem 15

9 judges each award 20 competitors a rank from 1 to 20. The competitor's score is the sum of the ranks from the 9 judges, and the winner is the competitor with the lowest score. For each competitor the difference between the highest and lowest ranking (from different judges) is at most 3. What is the highest score the winner could have obtained?

Solution

Answer: 24.

At most 4 competitors can receive a rank 1. For a competitor with a rank 1 can only receive ranks 1, 2, 3 or 4. There are only 36 such ranks available and each competitor with a rank 1 needs 9 of them.

If only one competitor receives a rank 1, then his score is 9. If only 2 competitors receive a rank 1, then one of them must receive at least five rank 1s. His maximum score is then 5.1 + 4.4 = 21. If 4 competitors receive a rank 1, then they must use all the 36 ranks 1, 2, 3, and 4. The total score available is thus 9(1 + 2 + 3 + 4) = 90, so at least one competitor must receive 22 or less. Thus the winner's maximum score is at most 22. If 3 competitors receive a rank 1, then the winner's score is maximised by giving all three competitors the same score and letting them share the 27 ranks 1, 3 and 4. That gives a winner's score of 9(1 + 3 + 4)/3 = 24. That can be achieved in several ways, for example: each competitor gets 3 1s, 3 3s and 3 4s, or one competitor gets 4 1s and 5 4s, another gets 3 1s, 3 3s and 3 4s, another gets 2 1s 6 3s and one 4. Note that it is trivial to arrange ranks for the remaining 17 competitors. For example: give one 5 2s and 4 5s total 30, one 4 2s and 5 5s total 33, and then one 9 6s, one 9 7s and so on.

Thus the answer is 24, with three joint winners. If there is required to be a single winner, then the answer is 23.

Problem 16

$\{a_i\}$ and $\{b_i\}$ are permutations of $\{1/1, 1/2, \dots , 1/n\}$. $a_1 + b_1 \ge a_2 + b_2 \ge \dots \ge a_n + b_n$. Prove that for every m ($1 \le m \le n$) $a_m + a_n \ge 4/m$.

Problem 17

There is a set of scales on the table and a collection of weights. Each weight is on one of the two pans. Each weight has the name of one or more pupils written on it. All the pupils are outside the room. If a pupil enters the room then he moves the weights with his name on them to the other pan. Show that you can let in a subset of pupils one at a time, so that the scales change position after the last pupil has moved his weights.

Problem 18

The streets in a city are on a rectangular grid with m east-west streets and n north-south streets. It is known that a car will leave some (unknown) junction and move along the streets at an unknown and possibly variable speed, eventually returning to its starting point without ever moving along the same block twice. Detectors can be positioned anywhere except at a junction to record the time at which the car passes and it direction of travel. What is the minimum number of detectors needed to ensure that the car's route can be reconstructed?

Problem 19

The circle inscribed in the triangle ABC touches the side AC at K. Prove that the line joining the midpoint of AC with the center of the circle bisects the segment BK.

Problem 20

The sequence $a_1, a_2, \dots , a_n$ satisfies the following conditions: $a_1 = 0$, $|a_i| = |a_{i-1} + 1|$ for i = 2, 3, ... , n. Prove that $(a_1 + a_2 + \dots + a_n)/n \ge -1/2$.

Problem 21

The sides and diagonals of ABCD have rational lengths. The diagonals meet at O. Prove that the length AO is also rational.

Solution

AB = AO cos OAB + BO cos OBA. We can derive a rational expression for cos OAB using the cosine rule for triangle ABC. Similarly for cos OBA using the cosine rule for triangle DAB. So OA = $r_1 + r_2$OB, where r_i denotes a rational number. Similarly, OB = $r_3 + r_4$OC, so OA = $r_5 + r_6$OC. But OA + OC = AC = r_7. Hence OA is rational.

3rd ASU 1969

Problem 1

In the quadrilateral ABCD, BC is parallel to AD. The point E lies on the segment AD and the perimeters of ABE, BCE and CDE are equal. Prove that BC = AD/2.

Solution

Take E_1 on the line AD so that AE_1CB is a parallelogram. Then $AE_1 = BC$, $AB = CE_1$, so triangles ABE_1 and BCE_1 have equal perimeters. Moreover, E_1 is the only point on the line for which this is true. For if we move E a distance x from E_1, then we change AE_1 by x, and CE_1 by less than x. AB and BC are unchanged. So $AB + BE_1$ and $BC + CE_1$ are changed by different amounts. Hence the perimeters of ABE_1 and BCE_1 are no longer equal.

Similarly, let E_2 be the point on the line AD so that $BCDE_2$ is a parallelogram. Then E_2 is the unique point such that BCE_2 and CDE_2 have equal perimeters. So if all three triangles have equal perimeters, then E_1 and E_2 must coincide and hence BC = AE = DE, so BC = AD/2.

Problem 2

A wolf is in the center of a square field and there is a dog at each corner. The wolf can run anywhere in the field, but the dogs can only run along the sides. The dogs' speed is 3/2 times the wolf's speed. The wolf can kill a single dog, but two dogs together can kill the wolf. Prove that the dogs can prevent the wolf escaping.

Problem 3

A finite sequence of 0s and 1s has the following properties: (1) for any $i < j$, the sequences of length 5 beginning at position i and position j are different; (2) if you add an additional digit at either the start or end of the sequence, then (1) no longer holds. Prove that the first 4 digits of the sequence are the same as the last 4 digits.

Solution

Let the last 4 digits be abcd. Then the 5 digit sequences abcd0 and abcd1 must occur somewhere. If neither of them are at the beginning then there are three 5 digit sequences xabcd, two of which must therefore be the same, contradicting (1). Hence abcd are the first 4 digits.

Problem 4

Given positive numbers a, b, c, d prove that at least one of the inequalities does not hold: $a + b < c + d$; $(a + b)(c + d) < ab + cd$; $(a + b)cd < ab(c + d)$.

Solution

From the first and second inequalities we have $ab + cd > a(c + d) + b(a + b)$, so $cd > ad$, and hence $c > a$. We also have $ab + cd > a(a + b) + b(c + d)$, so $cd > bc$, and hence $d > b$. So $1/a + 1/b > 1/c + 1/d$, which contradicts the third inequality.

Problem 5

What is the smallest positive integer a such that we can find integers b and c so that $ax^2 + bx + c$ has two distinct positive roots less than 1?

Solution

$4x^2 - 4x + 1 = (2x - 1)^2$, which has the double root 1/2. So it remains to consider a = 1, 2, 3.

-b/a is the sum of the roots, so b is negative. c/a is the product of the roots, so c is positive. If a = 1, then the product of the roots is c, which is at least 1, so both roots cannot lie strictly between 0 and 1. If a = 2, then the sum of the roots is less than 2, so b must be -1, -2, or -3. The roots are real so $b^2 > 4ac = 8c$. Hence b = -3 and c = 1. But $2x^2 - 3x + 1 = (2x - 1)(x - 1)$ and one root is not less than 1. If a = 3, then b must be -1, -2, ... , or -5. But $b^2 > 4ac = 12c$, so (b, c) = (-4, 1), (-5, 1) or (-5, 2). In the first and last case, the equation has a root 1. In the middle case it has a root $5/6 + \sqrt{13}/6 = 1.434 > 1$. Thus there are no solutions for a = 1, 2, 3 and so the smallest value of a is 4.

Problem 6

n is an integer. Prove that the sum of all fractions 1/rs, where r and s are relatively prime integers satisfying $0 < r < s \le n$, $r + s > n$, is 1/2.

Solution

We use induction on n. If n = 2, then the only such fraction is r = 1, s = 2, giving 1/rs = 1/2, so the result holds. Suppose it holds for n-1. As we move to n, we lose the fractions with r+s=n. The other fractions 1/rs which satisfy the conditions for n-1 also satisfy the conditions for n. We also gain the fractions with s=n. These have sum = 1/n (sum 1/r for all r satisfying 0 < r < n and r relatively prime to n). But if r is relatively prime to n, then so is n - r, and n - r does not equal r (otherwise r divides n). The pair 1/r, 1/(n-r) has sum n/(r(n-r)). So the fractions with s=n have sum equal to the sum of all 1/(r(n-r) with 0 < r < n and r relatively prime to n. But that is exactly the sum of the fractions lost. Thus the total is unchanged as we move from n-1 to n.

Problem 7

Given n points in space such that the triangle formed from any three of the points has an angle greater than 120 degrees. Prove that the points can be labeled 1, 2, 3, ... , n so that the angle defined by i, i+1, i+2 is greater than 120 degrees for i = 1, 2, ... , n-2.

Problem 8

Find four different three-digit numbers (in base 10) starting with the same digit, such that their sum is divisible by three of the numbers.

Solution

Answer: 108, 117, 135, 180. Sum $540 = 108 \cdot 5 = 135 \cdot 4 = 180 \cdot 3$.

Try looking for a number of the form $3 \cdot 4 \cdot 5 \cdot n$. We want 12n, 15n and 20n to have the same first digit. If the first digit is 1, this requires n = 9. We must now check that the fourth number which must be 60n - 12n - 15n - 20n = 13n also has three digits starting with 1. It does, so we are home. [In fact, in this case the first digit must be 1, since 20n > 3/2 12n.]

Problem 9

Every city in a certain state is directly connected by air with at most three other cities in the state, but one can get from any city to any other city with at most one change of plane. What is the maximum possible number of cities?

Solution

Answer: 10.

Take a particular city X. At most 3 cities are directly connected to X. Each of those is directly connected to at most 2 other cities (apart from X). So X is connected with at most one change to at most 9 other cities. Thus the maximum number is at most 10.

We can achieve 10 as follows. Label the cities 1, ... , 10. Make direct connections as follows: 1: 2, 3, 4;   2: 1, 5, 6;   3: 1, 7, 8;   4: 1, 9, 10;   5: 2, 7, 9;   6: 2, 8, 10;   7: 3, 5, 10;   8: 3, 6, 9; 9: 4, 5, 8;   10: 4, 6, 7.

Problem 10

Given a pentagon with equal sides.

(a) Prove that there is a point X on the longest diagonal such that every side subtends an angle at most 90 degrees at X.

(b) Prove that the five circles with diameter one of the pentagon's sides do not cover the pentagon.

Problem 11

Given the equation $x^3 + ax^2 + bx + c = 0$, the first player gives one of a, b, c an integral value. Then the second player gives one of the remaining coefficients an integral value, and finally the first player gives the remaining coefficient an integral value. The first player's objective is to ensure that the equation has three integral roots (not necessarily distinct). The second player's objective is to prevent this. Who wins?

Solution

Answer: the first player.

The first player starts by choosing c = 0. Now if the second player selects a, then he can take b = a - 1. Then the polynomial factorizes as: x(x+1)(x+a-1) with integral roots, 0, -1, 1-a. If the second player selects b, then he can take a = b + 1. Then the polynomial factorizes as x(x+1)(x+b) with integral roots 0, -1, -b.

Problem 12

20 teams compete in a competition. What is the smallest number of games that must be played to ensure that given any three teams at least two play each other?

Problem 13

A regular n-gon is inscribed in a circle radius R. The distance from the center of the circle to the center of a side is h_n. Prove that $(n+1)h_{n+1} - nh_n > R$.

Problem 14

Prove that for any positive numbers $a_1, a_2, \dots, a_n$ we have:

$$a_1/(a_2+a_3) + a_2/(a_3+a_4) + \dots + a_{n-1}/(a_n+a_1) + a_n/(a_1+a_2) > n/4.$$

4th ASU 1970

Problem 1

Given a circle, diameter AB and a point C on AB, show how to construct two points X and Y on the circle such that (1) Y is the reflection of X in the line AB, (2) YC is perpendicular to XA.

Solution

AB is a diameter, so BX is perpendicular to AX and hence parallel to YC. YX is perpendicular to BC, so YC = YB. Hence X and Y lie on the perpendicular bisector of BC.

Problem 2

The product of three positive numbers is 1, their sum is greater than the sum of their inverses. Prove that just one of the numbers is greater than 1.

Solution

The product of the numbers is 1, so they cannot all be greater than 1 or all less than 1. If all equalled 1, then the sum would not be greater than the sum of the inverses. So we must have either one or two greater than 1. Thus it is sufficient to show that we cannot have two of the numbers greater than 1.

Suppose that $a, b > 1$. Then since $a + b + c > 1/a + 1/b + 1/c$, we have $a + b + 1/ab > 1/a + 1/b + ab$, and hence $(1 - 1/a)(1 - 1/b) > (a - 1)(b - 1)$. Dividing by $(a - 1)(b -1)$ gives $ab < 1$. Contradiction.

Problem 3

What is the greatest number of sides of a convex polygon that can equal its longest diagonal?

Solution

Answer: 2, except for the equilateral triangle.

It is easy to find two. Take the two sides to be AB and AC with angle BAC =60 deg, and take the other vertices on the minor arc of the circle center A radius AB between B and C.

Let the longest diagonal have length k. Suppose there are three sides with length k. Extend them (if necessary) so they meet at A, B, C. Suppose $\angle A > 60^o$. Take the vertices on side AB to be P, Q (where we may have P = A, or Q = B, or both). Take the vertices on side AC to be R, S (where we may have R = A, or S = C, or both). Then $AQ \ge k$, $AS \ge k$, so $QS > k$. Contradiction. Hence angle $A \le 60^o$. The same is true for $\angle B$ and $\angle C$. Hence $\angle A = \angle B = \angle C = 60^o$. But now $QS > k$ unless A = P = R. Similarly, B and C must be vertices of the convex polygon, so that it is just an equilateral triangle.

Problem 4

n is a 17 digit number. m is derived from n by taking its decimal digits in the reverse order. Show that at least one digit of n + m is even.

Solution

Let the number be n with digits $d_1d_2 \ldots d_{17}$, so that the reversed number n' has digits $d_{17}d_{16} \ldots d_1$. Let the digits of n + n' be $a_0a_1 \ldots a_{17}$, where a_0 may be zero. Let the carry forward when adding digits to get a_i be c_{i-1}, so that, in general, $c_i + d_i + d_{18-i} = a_i + 10 c_{i-1}$. Obviously c_i is 0 or 1.

Suppose all the digits a_i are odd (except that a_0 may be zero). Now $c_9 + 2d_9 = a_9 + 10 c_8$. Since a_9 is odd, c_9 must be 1. But if we consider $c_{10} + d_{10} + d_8 = a_{10} + 10 c_9$, we see that since a_{10} is odd it is at least 1 and hence $d_8 + d_{10}$ is at least 10. Hence there must be a non-zero carry c_9 in $c_{10} + d_{10} + d_8 = a_{10} + 10 c_9$ irrespective of the value of c_{10}.

We can now iterate and conclude successively that c_{12}, c_{14}, c_{16} must be non-zero.

Problem 5

A room is an equilateral triangle side 100 meters. It is subdivided into 100 rooms, all equilateral triangles with side 10 meters. Each interior wall between two rooms has a door. If you start inside one of the rooms and can only pass through each door once, show that you cannot visit more than 91 rooms. Suppose now the large triangle has side k and is divided into k^2 small triangles by lines parallel to its sides. A chain is a sequence of triangles, such that a triangle can only be included once and consecutive triangles have a common side. What is the largest possible number of triangles in a chain?

Problem 6

Given 5 segments such that any 3 can be used to form a triangle. Show that at least one of the triangles is acute-angled.

Solution

Let the segments have length $a \le b \le c \le d \le e$. Then if all triangles are obtuse we have $e^2 > c^2 + d^2$, $d^2 > b^2 + c^2$, $c^2 > a^2 + b^2$. Adding $e^2 > a^2 + 2b^2 + c^2 > a^2 + 3b^2$. But $e \le a + b$, so $e^2 \le a^2 + 2ab + b^2 \le a^2 + 3b^2$. Contradiction.

Problem 7

ABC is an acute-angled triangle. The angle bisector AD, the median BM and the altitude CH are concurrent. Prove that angle A is more than 45 degrees.

Solution

We use Ceva's theorem. Since AD, BM, CH are concurrent, we have (BD/DC).(CM/MA).(AH/BH) = 1. But CM = MA and since AD is the angle bisector BD/DC = AB/AC, so (AB/AC).(AH/BH) = 1. Hence AH/AC = BH/AB < 1. So angle HAC > angle HCA. But angle AHC = 90 deg, so angle A > 45 deg.

Problem 8

Five n-digit binary numbers have the property that every two numbers have the same digits in just m places, but no place has the same digit in all five numbers. Show that $2/5 \le m/n \le 3/5$.

Problem 9

Show that given 200 integers you can always choose 100 with sum a multiple of 100.

Problem 10

ABC is a triangle with incenter I. M is the midpoint of BC. IM meets the altitude AH at E. Show that AE = r, the radius of the inscribed circle.

Problem 11

Given any positive integer n, show that we can find infinitely many integers m such that m has no zeros (when written as a decimal number) and the sum of the digits of m and mn is the same.

Problem 12

Two congruent rectangles of area A intersect in eight points. Show that the area of the intersection is more than A/2.

Problem 13

If the numbers from 11111 to 99999 are arranged in an arbitrary order show that the resulting 444445 digit number is not a power of 2.

Solution

Let the set of numbers be S. Define the function f on S as follows. Replace each digit i in n by 9-i for $0 < i < 9$. This gives f(n). Then f(f(n)) = n, so f is a bijection. The fixed points have only the digits 0 and 9 and so are all divisible by 9. The other points divide into pairs (n, f(n)) and the sum of each pair is divisible by 9. Hence the sum of all the numbers in S is divisible by 9.

Problem 14

S is the set of all positive integers with n decimal digits or less and with an even digit sum. T is the set of all positive integers with n decimal digits or less and an odd digit sum. Show that the sum of the kth powers of the members of S equals the sum for T if $1 \le k < n$.

Problem 15

The vertices of a regular n-gon are colored (each vertex has only one color). Each color is applied to at least three vertices. The vertices of any given color form a regular polygon. Show that there are two colors which are applied to the same number of vertices.

5th ASU 1971

Problem 1

Prove that we can find a number divisible by 2^n whose decimal representation uses only the digits 1 and 2.

Solution

Induction on n. We claim that we can find N with n digits, all 1 or 2, so that N is divisible by 2^n. True for n = 1: take N = 2. Suppose it is true for n. If 2^{n+1} divides N, then since 2^{n+1} divides 2×10^n it also divides N' obtained from N by placing a 2 in front of it. If 2^{n+1} does not divide N, then $N = 2^n \times$ odd and $10^n = 2^n \times$ odd, so $N + 10^n$ (in other words the n+1 digit number obtained by placing a 1 in front of N) is divisible by 2^{n+1}.

Problem 2

(1) $A_1A_2A_3$ is a triangle. Points B_1, B_2, B_3 are chosen on A_1A_2, A_2A_3, A_3A_1 respectively and points D_1, D_2 D_3 on A_3A_1, A_1A_2, A_2A_3 respectively, so that if parallelograms $A_iB_iC_iD_i$ are formed, then the lines A_iC_i concur. Show that $A_1B_1\cdot A_2B_2\cdot A_3B_3 = A_1D_1\cdot A_2D_2\cdot A_3D_3$.

(2) $A_1A_2 \dots A_n$ is a convex polygon. Points B_i are chosen on A_iA_{i+1} (where we take A_{n+1} to mean A_1), and points D_i on $A_{i-1}A_i$ (where we take A_0 to mean A_n) such that if parallelograms $A_iB_iC_iD_i$ are formed, then the n lines A_iC_i concur. Show that $\prod A_iB_i = \prod A_iD_i$.

Problem 3

(1) Player A writes down two rows of 10 positive integers, one under the other. The numbers must be chosen so that if a is under b and c is under d, then a + d = b + c. Player B is allowed to ask for the identity of the number in row i, column j. How many questions must he ask to be sure of determining all the numbers?

(2) An m x n array of positive integers is written on the blackboard. It has the property that for any four numbers a, b, c, d with a and b in the same row, c and d in the same row, a above c (in the same column) and b above d (in the same column) we have a + d = b + c. If some numbers are wiped off, how many must be left for the table to be accurately restored?

Solution

(1) is trivial. We can write the condition as b - a = d - c, so the 10 numbers in the first row and 1 in the second row can all be chosen arbitrarily. Hence at least 11 questions are needed. But they are also sufficient. Having determined those numbers, the others immediately follow.

(2). The m+n-1 numbers in the first row and first column can all be chosen arbitrarily, but are sufficient to determine all the numbers. Hence at least m+n-1 numbers must survive.

Problem 4

Circles, each with radius less than R, are drawn inside a square side 1000R. There are no points on different circles a distance R apart. Show that the total area covered by the circles does not exceed 340,000 R^2.

Problem 5

You are given three positive integers. A move consists of replacing $m \leq n$ by $2m$, $n-m$. Show that you can always make a series of moves which results in one of the integers becoming zero. [For example, if you start with 4, 5, 10, then you could get 8, 5, 6, then 3, 10, 6, then 6, 7, 6, then 0, 7, 12.]

Problem 6

The real numbers a, b, A, B satisfy $(B - b)^2 < (A - a)(Ba - Ab)$. Show that the quadratics $x^2 + ax + b = 0$ and $x^2 + Ax + B = 0$ have real roots and between the roots of each there is a root of the other.

Problem 7

The projections of a body on two planes are circles. Show that the circles have the same radius.

Problem 8

An integer is written at each vertex of a regular n-gon. A move is to find four adjacent vertices with numbers a, b, c, d (in that order), so that $(a - d)(b - c) < 0$, and then to interchange b and c. Show that only finitely many moves are possible. For example, a possible sequence of moves is shown below:

```
1  7  2  3  5  4
1  2  7  3  5  4
1  2  3  7  5  4
1  2  3  5  7  4
2  1  3  5  7  4
```

Problem 9

A polygon P has an inscribed circle center O. If a line divides P into two polygons with equal areas and equal perimeters, show that it must pass through O.

Problem 10

Given any set S of 25 positive integers, show that you can always find two such that none of the other numbers equals their sum or difference.

Problem 11

A and B are adjacent vertices of a 12-gon. Vertex A is marked - and the other vertices are marked +. You are allowed to change the sign of any n adjacent vertices. Show that by a succession of moves of this type with n = 6 you cannot get B marked - and the other vertices marked +. Show that the same is true if all moves have n = 3 or if all moves have n = 4.

Problem 12

Equally spaced perpendicular lines divide a large piece of paper into unit squares. N squares are colored black. Show that you can always cut out a set of disjoint square pieces of paper, so that all the black squares are removed and the black area of each piece is between 1/5 and 4/5 of its total area.

Problem 13

n is a positive integer. S is the set of all triples (a, b, c) such that $1 \le a, b, c, \le n$. What is the smallest subset X of triples such that for every member of S one can find a member of X which differs in only one position. [For example, for n = 2, one could take X = { (1, 1, 1), (2, 2, 2) }.]

Problem 14

Let $f(x, y) = x^2 + xy + y^2$. Show that given any real x, y one can always find integers m, n such that $f(x-m, y-n) <= 1/3$. What is the corresponding result if $f(x, y) = x^2 + axy + y^2$ with $0 \le a \le 2$?

Problem 15

A switch has two inputs 1, 2 and two outputs 1, 2. It either connects 1 to 1 and 2 to 2, or 1 to 2 and 2 to 1. If you have three inputs 1, 2, 3 and three outputs 1, 2, 3, then you can use three switches, the first across 1 and 2, then the second across 2 and 3, and finally the third across 1 and 2. It is easy to check that this allows the output to be any permutation of the inputs and that at least three switches are required to achieve this. What is the minimum number of switches required for 4 inputs, so that by suitably setting the switches the output can be any permutation of the inputs?

6th ASU 1972 problems

Problem 1

ABCD is a rectangle. M is the midpoint of AD and N is the midpoint of BC. P is a point on the ray CD on the opposite side of D to C. The ray PM intersects AC at Q. Show that MN bisects the angle PNQ.

Problem 2

Given 50 segments on a line show that you can always find either 8 segments which are disjoint or 8 segments with a common point.

Problem 3

Find the largest integer n such that $4^{27} + 4^{1000} + 4^{n}$ is a square.

Problem 4

a, m, n are positive integers and a > 1. Show that if $a^m + 1$ divides $a^n + 1$, then m divides n. The positive integer b is relatively prime to a, show that if $a^m + b^m$ divides $a^n + b^n$ then m divides n.

Problem 5

A sequence of finite sets of positive integers is defined as follows. $S_0 = \{m\}$, where m > 1. Then given S_n you derive S_{n+1} by taking k^2 and k+1 for each element k of S_n. For example, if $S_0 = \{5\}$, then $S_2 = \{7, 26, 36, 625\}$. Show that S_n always has 2^n distinct elements.

Problem 6

Prove that a collection of squares with total area 1 can always be arranged inside a square of area 2 without overlapping.

Problem 7

O is the point of intersection of the diagonals of the convex quadrilateral ABCD. Prove that the line joining the centroids of ABO and CDO is perpendicular to the line joining the orthocenters of BCO and ADO.

Problem 8

9 lines each divide a square into two quadrilaterals with areas 2/5 and 3/5 that of the square. Show that 3 of the lines meet in a point.

Problem 9

A 7-gon is inscribed in a circle. The center of the circle lies inside the 7-gon. A, B, C are adjacent vertices of the 7-gon show that the sum of the angles at A, B, C is less than 450 degrees.

Problem 10

Two players play the following game. At each turn the first player chooses a decimal digit, then the second player substitutes it for one of the stars in the subtraction | **** - **** |. The first player tries to end up with the largest possible result, the second player tries to end up with the smallest possible result. Show that the first player can always play so that the result is at least 4000 and that the second player can always play so that the result is at most 4000.

Problem 11

For positive reals x, y let f(x, y) be the smallest of x, 1/y, y + 1/x. What is the maximum value of f(x, y)? What are the corresponding x, y?

Problem 12

P is a convex polygon and X is an interior point such that for every pair of vertices A, B, the triangle XAB is isosceles. Prove that all the vertices of P lie on some circle center X.

Problem 13

Is it possible to place the digits 0, 1, 2 into unit squares of 100 x 100 cross-lined paper such that every 3 x 4 (and every 4 x 3) rectangle contains three 0s, four 1s and five 2s?

Problem 14

$x_1, x_2, \ldots, x_n$ are positive reals with sum 1. Let s be the largest of $x_1/(1 + x_1)$, $x_2/(1 + x_1 + x_2)$, ... , $x_n/(1 + x_1 + \ldots + x_n)$. What is the smallest possible value of s? What are the corresponding x_i?

Problem 15

n teams compete in a tournament. Each team plays every other team once. In each game a team gets 2 points for a win, 1 for a draw and 0 for a loss. Given any subset S of teams, one can find a team (possibly in S) whose total score in the games with teams in S was odd. Prove that n is even.

7th ASU 1973

Problem 1

You are given 14 coins. It is known that genuine coins all have the same weight and that fake coins all have the same weight, but weigh less than genuine coins. You suspect that 7 particular coins are genuine and the other 7 fake. Given a balance, how can you prove this in three weighings (assuming that you turn out to be correct)?

Solution

Let the coins you suspect to be genuine be $G_1, G_2, \dots , G_7$, and the suspected fakes by $F_1, F_2, \dots , F_7$. First weigh F_1 against G_1. Assuming F_1 weighs less, you have proved that F_1 is fake and G_1 genuine. Second, weigh F_1, G_2, G_3 against G_1, F_2, F_3. Assuming the first three weigh more you have proved that they include more genuine coins than the second three. But the second three includes one genuine coin (G_1) and the first three includes one fake (F_1), so you have proved that G_2 and G_3 are genuine and F_2 and F_3 fake. Finally, weigh $F_1, F_2, F_3, G_4, G_5, G_6, G_7$ against $F_4, F_5, F_6, F_7, G_1, G_2, G_3$. Assuming the first group weighs more it must include more genuine coins and hence just four genuine coins. Similarly, the second group must include four fakes. So you have proved the identity of the remaining coins.

Problem 2

Prove that a 9 digit decimal number whose digits are all different, which does not end with 5 and or contain a 0, cannot be a square.

Problem 3

Given $n > 4$ points, show that you can place an arrow between each pair of points, so that given any point you can reach any other point by travelling along either one or two arrows in the direction of the arrow.

Problem 4

OA and OB are tangent to a circle at A and B. The line parallel to OB through A meets the circle again at C. The line OC meets the circle again at E. The ray AE meets the line OB at K. Prove that K is the midpoint of OB.

Problem 5

$p(x) = ax^2 + bx + c$ is a real quadratic such that $|p(x)| \le 1$ for all $|x| \le 1$. Prove that $|cx^2 + bx + a| \le 2$ for $|x| \le 1$.

Problem 6

Players numbered 1 to 1024 play in a knock-out tournament. There ar no draws, the winner of a match goes through to the next round and the loser is knocked-out, so that there are 512 matches in the first round, 256 in the second and so on. If m plays n and $m < n-2$ then m always wins. What is the largest possible number for the winner?

Problem 7

Define $p(x) = ax^2 + bx + c$. If $p(x) = x$ has no real roots, prove that $p(p(x)) = 0$ has no real roots.

Problem 8

At time 1, n unit squares of an infinite sheet of paper ruled in squares are painted black, the rest remain white. At time k+1, the color of each square is changed to the color held at time k by a majority of the following three squares: the square itself, its northern neighbour and its eastern neighbour. Prove that all the squares are white at time n+1.

Problem 9

ABC is an acute-angled triangle. D is the reflection of A in BC, E is the reflection of B in AC, and F is the reflection of C in AB. Show that the circumcircles of DBC, ECA, FAB meet at a point and that the lines AD, BE, CF meet at a point.

Problem 10

n people are all strangers. Show that you can always introduce some of them to each other, so that afterwards each person has met a different number of the others. [problem: this is false as stated. Each person must have 0, 1, ... or n-1 meetings,so all these numbers must be used. But if one person has met no one, then another cannot have met everyone.]

Problem 11

A king moves on an 8 x 8 chessboard. He can move one square at a time, diagonally or orthogonally (so away from the borders he can move to any of eight squares). He makes a complete circuit of the board, starting and finishing on the same square and visiting every other square just once. His trajectory is drawn by joining the center of the squares he moves to and from for each move. The trajectory does not intersect itself. Show that he makes at least 28 moves parallel to the sides of the board (the others being diagonal) and that a circuit is possible with exactly 28 moves parallel to the sides of the board. If the board has side length 8, what is the maximum and minimum possible length for such a trajectory.

Problem 12

A triangle has area 1, and sides $a \geq b \geq c$. Prove that $b^2 \geq 2$.

Problem 13

A convex n-gon has no two sides parallel. Given a point P inside the n-gon show that there are at most n lines through P which bisect the area of the n-gon.

Problem 14
a, b, c, d, e are positive reals. Show that $(a + b + c + d + e)^2 \geq 4(ab + bc + cd + de + ea)$.

Problem 15
Given 4 points which do not lie in a plane, how many parallelepipeds have all 4 points as vertices?

8th ASU 1974

Problem 1

A collection of n cards is numbered from 1 to n. Each card has either 1 or -1 on the back. You are allowed to ask for the product of the numbers on the back of any three cards. What is the smallest number of questions which will allow you to determine the numbers on the backs of all the cards if n is (1) 30, (2) 31, (3) 32? If 50 cards are arranged in a circle and you are only allowed to ask for the product of the numbers on the backs of three adjacent cards, how many questions are needed to determine the product of the numbers on the backs of all 50 cards?

Problem 2

Find the smallest positive integer which can be represented as $36^m - 5^n$.

Answer

11

Solution

Obviously $11 = 36^1 - 5^2$, and we guess that this is the best possible. We cannot have $36^m - 5^n = k$, where 2, 3 or 5 divides k (because then 2 would divide 5^n and similarly in the other cases). So the only possible values of $k < 11$ are 1, 7.

We have $36^m - 5^n = 1 \mod 5$, so $k \neq 7$. Similarly, $36^m - 5^n = 3 \mod 4$, so $k \neq 1$.

Problem 3

Each side of a convex hexagon is longer than 1. Is there always a diagonal longer than 2? If each of the main diagonals of a hexagon is longer than 2, is there always a side longer than 1?

Problem 4

Circles radius r and R touch externally. AD is parallel to BC. AB and CD touch both circles. AD touches the circle radius r, but not the circle radius R, and BC touches the circle radius R, but not the circle radius r. What is the smallest possible length for AB?

Problem 5

Given n unit vectors in the plane whose sum has length less than one. Show that you can arrange them so that the sum of the first k has length less than 2 for every $1 < k < n$.

Problem 6

Find all real a, b, c such that $|ax + by + cz| + |bx + cy + az| + |cx + ay + bz| = |x + y + z|$ for all real x, y, z.

Problem 7

ABCD is a square. P is on the segment AB and Q is on the segment BC such that BP = BQ. H lies on PC such that BHC is a right angle. Show that DHQ is a right angle.

Problem 8

The n points of a graph are each colored red or blue. At each move we select a point which differs in color from more than half of the points to which it it is joined and we change its color. Prove that this process must finish after a finite number of moves.

Problem 9

Find all positive integers m, n such that n^n has m decimal digits and m^m has n decimal digits.

Problem 10

In the triangle ABC, angle C is 90 deg and AC = BC. Take points D on CA and E on CB such that CD = CE. Let the perpendiculars from D and C to AE meet AB at K and L respectively. Show that KL = LB.

Problem 11

One rat and two cats are placed on a chess-board. The rat is placed first and then the two cats choose positions on the border squares. The rat moves first. Then the cats and the rat move alternately. The rat can move one square to an adjacent square (but not diagonally). If it is on a border square, then it can also move off the board. On a cat move, both cats move one square. Each must move to an adjacent square, and not diagonally. The cats win if one of them moves onto the same square as the rat. The rat wins if it moves off the board. Who wins? Suppose there are three cats (and all three cats move when it is the cats' turn), but that the rat gets an extra initial turn. Prove that the rat wins.

Problem 12

Arrange the numbers 1, 2, ... , 32 in a sequence such that the arithmetic mean of two numbers does not lie between them. (For example, ... 3, 4, 5, 2, 1, ... is invalid, because 2 lies between 1 and 3.) Can you arrange the numbers 1, 2, ... , 100 in the same way?

Problem 13

Find all three digit decimal numbers $a_1a_2a_3$ which equal the mean of the six numbers $a_1a_2a_3$, $a_1a_3a_2$, $a_2a_1a_3$, $a_2a_3a_1$, $a_3a_1a_2$, $a_3a_2a_1$.

Problem 14

No triangle of area 1 can be fitted inside a convex polygon. Show that the polygon can be fitted inside a triangle of area 4.

Problem 15

f is a function on the closed interval [0, 1] with non-negative real values. f(1) = 1 and $f(x + y) \geq f(x) + f(y)$ for all x, y. Show that $f(x) \leq 2x$ for all x. Is it necessarily true that $f(x) \leq 1.9x$ for all x.

Answer

no

Solution

We have $f(x) = f(1) - f(1-x) \le f(1) = 1$. So for $x \ge 1/2$, $f(x) \le 1 \le 2x$. If $x < 1/2$, then for some n we have $1/2^{n+1} \le x < 1/2^n$. Hence by a trivial induction $f(2^n x) \ge 2^n f(x)$. But $f(2^n x) \le 1$, so $f(x) \le 1/2^n \le 2x$.

Note that $f(x) = 0$ for $x \le 1/2$ and 1 for $x > 1/2$ satisfies the conditions. But $f(0.51) = 1 > (1.9)(0.51)$.

Problem 16

The triangle ABC has area 1. D, E, F are the midpoints of the sides BC, CA, AB. P lies in the segment BF, Q lies in the segment CD, R lies in the segment AE. What is the smallest possible area for the intersection of triangles DEF and PQR?

9th ASU 1975 problems

Problem 1

(1) O is the circumcenter of the triangle ABC. The triangle is rotated about O to give a new triangle A'B'C'. The lines AB and A'B' intersect at C'', BC and B'C' intersect at A'', and CA and C'A' intersect at B''. Show that A''B''C'' is similar to ABC.

(2) O is the center of the circle through ABCD. ABCD is rotated about O to give the quadrilateral A'B'C'D'. Prove that the intersection points of corresponding sides form a parallelogram.

Problem 2

A triangle ABC has unit area. The first player chooses a point X on side AB, then the second player chooses a point Y on side BC, and finally the first player chooses a point Z on side CA. The first player tries to arrange for the area of XYZ to be as large as possible, the second player tries to arrange for the area to be as small as possible. What is the optimum strategy for the first player and what is the best he can do (assuming the second player plays optimally)?

Problem 3

What is the smallest perimeter for a convex 32-gon whose vertices are all lattice points?

Problem 4

Given a 7 x 7 square subdivided into 49 unit squares, mark the center of n unit squares, so that no four marks form a rectangle with sides parallel to the square. What is the largest n for which this is possible? What about a 13 x 13 square?

Problem 5

Given a convex hexagon, take the midpoint of each of the six diagonals joining vertices which are separated by a single vertex (so if the vertices are in order A, B, C, D, E, F, then the diagonals are AC, BD, CE, DF, EA, FB). Show that the midpoints form a convex hexagon with a quarter the area of the original.

Problem 6

Show that there are 2^{n+1} numbers each with 2^n digits, all 1 or 2, so that every two numbers differ in at least half their digits.

Problem 7

There are finitely many polygons in the plane. Every two have a common point. Prove that there is a straight line intersecting all the polygons.

Problem 8

a, b, c are positive reals. Show that $a^3 + b^3 + c^3 + 3abc \geq ab(a + b) + bc(b + c) + ca(c + a)$.

Solution

The inequality is homogeneous, so we can take a = 1 and put b = 1+x, c = 1+y, where x, y ≥ 0. Then after some reduction the inequality is equivalent to $x^3+y^3+x^2+y^2-x^2y-xy^2-xy \geq 0$, or (after factorising x^3+y^3) to $(x+y+1)(x-y)^2 + xy \geq 0$, which is obviously true.

Problem 9

Three flies crawl along the perimeter of a triangle. At least one fly makes a complete circuit of the perimeter. For the entire period the center of mass of the flies remains fixed. Show that it must be at the centroid of the triangle. [You may not assume, without proof, that the flies have the same mass, or that they crawl at the same speed, or that any fly crawls at a constant speed.]

Problem 10

The finite sequence a_n has each member 0, 1 or 2. A *move* involves replacing any two unequal members of the sequence by a single member different from either. A series of moves results in a single number. Prove that no series of moves can terminate in a (single) different number.

Solution

Suppose we start with a 0s, b 1s and c 2s. Each move changes the parity of all of a, b, c. Each moves reduces the length of the sequence by 1, so there must be a+b+c-1 moves in all. Hence the total number of 0s, the total number of 1s and the total number of 2s all have their parity changed a+b+c-1 times. So if we end up with just one 0, then a must have the opposite parity to b and c. In that case, we cannot end up with just one 1, or just one 2. Similarly in the other cases.

Problem 11

S is a horizontal strip in the plane. n lines are drawn so that no three are collinear and every pair intersects within the strip. A *path* starts at the bottom border of the strip and consists of a sequence of segments from the n lines. The path must change line at each intersection and must always move upwards. Show that: (1) there are at least n/2 disjoint paths; (2) there is a path of at least n segments; (3) there is a path involving not more than n/2 + 1 of the lines; and (4) there is a path that involves segments from all n lines.

Problem 12

For what n can we color the unit cubes in an n x n x n cube red or green so that every red unit cube has just two red neighbouring cubes (sharing a face) and every green unit cube has just two green neighbouring cubes.

Problem 13

p(x) is a polynomial with integral coefficients. f(n) = the sum of the (decimal) digits in the value p(n). Show that f(n) some value m infinitely many times.

Problem 14

20 teams each play one game with every other team. Each game results in a win or loss (no draws). k of the teams are European. A separate trophy is awarded for the best European team on the basis of the k(k-1)/2 games in which both teams are European. This trophy is won by a single team. The same team comes last in the overall competition (winning fewer games than any other team). What is the largest possible value of k? If draws are allowed and a team scores 2 for a win and 1 for a draw, what is the largest possible value of k?

Problem 15

Given real numbers a_i, b_i and positive reals c_i, d_i, let $e_{ij} = (a_i+b_j)/(c_i+d_j)$. Let $M_i = \max_{0\le j\le n} e_{ij}$, $m_j = \min_{1\le i\le n} e_{ij}$. Show that we can find an e_{ij} with $1 \le i, j \le n$ such that $e_{ij} = M_i = m_j$.

10th ASU 1976 problems

Problem 1

50 watches, all keeping perfect time, lie on a table. Show that there is a moment when the sum of the distances from the center of the table to the center of each dial equals the sum of the distances from the center of the table to the tip of each minute hand.

Problem 2

1000 numbers are written in line 1, then further lines are constructed as follows. If the number m occurs in line n, then we write under it in line n+1, each time it occurs, the number of times that m occurs in line n. Show that lines 11 and 12 are identical. Show that we can choose numbers in line 1, so that lines 10 and 11 are not identical.

Problem 3

(1) The circles C_1, C_2, C_3 with equal radius all pass through the point X. C_i and C_j also intersect at the point Y_{ij}. Show that angle XO_1Y_{12} + angle XO_2Y_{23} + angle XO_3Y_{31} = 180 deg, where O_i is the center of circle C_i.

Problem 4

a_1 and a_2 are positive integers less than 1000. Define $a_n = \min\{ |a_i - a_j| : 0 < i < j < n\}$. Show that $a_{21} = 0$.

Problem 5

Can you label each vertex of a cube with a different three digit binary number so that the numbers at any two adjacent vertices differ in at least two digits?

Problem 6

a, **b**, **c**, **d** are vectors in the plane such that $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$. Show that $|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| \geq |\mathbf{a} + \mathbf{d}| + |\mathbf{b} + \mathbf{d}| + |\mathbf{c} + \mathbf{d}|$.

Problem 7

S is a set of 1976 points which form a regular 1976-gon. T is the set of all points which are the midpoint of at least one pair of points in S. What is the greatest number of points of T which lie on a single circle?

Problem 8

n rectangles are drawn on a rectangular sheet of paper. Each rectangle has its sides parallel to the sides of the paper. No pair of rectangles has an interior point in common. If the rectangles were removed show that the rest of the sheet would be in at most n+1 parts.

Problem 9

There are three straight roads. On each road a man is walking at constant speed. At time $t = 0$, the three men are not collinear. Prove that they will be collinear for $t > 0$ at most twice.

Problem 10

Initially, there is one beetle on each square in the set S. Suddenly each beetle flies to a new square, subject to the following conditions: (1) the new square may be the same as the old or different; (2) more than one beetle may choose the same new square; (3) if two beetles are initially in squares with a common vertex, then after the flight they are either in the same square or in squares with a common vertex. Suppose S is the set of all squares in the middle row and column of a 99 x 99 chess board, is it true that there must always be a beetle whose new square shares a vertex with its old square (or is identical with it)? What if S also includes all the border squares (so S is rows 1, 50 and 99 and columns 1, 50 and 99)? What if S is all squares of the board?

Problem 11

Call a triangle *big* if each side is longer than 1. Show that we can draw 100 big triangles inside an equilateral triangle with side length 5 so that all the triangles are disjoint. Show that you can draw 100 big triangles with every vertex inside or on an equilateral triangle with side 3, so that they cover the equilateral triangle, and any two big triangles either (1) are disjoint, or (2) have as intersection a common vertex, or (3) have as intersection a common side.

Problem 12

n is a positive integer. A universal sequence of length m is a sequence of m integers each between 1 and n such that one can obtain any permutation of 1, 2, ... , n by deleting suitable members of the sequence. For example, 1, 2, 3, 1, 2, 1, 3 is a universal sequence of length 7 for $n = 3$. But 1, 2, 3, 2, 1, 3, 1 is not universal, because one cannot obtain the permutation 3, 1, 2. Show that one can always obtain a universal sequence for n of length $n^2 - n + 1$. Show that a universal sequence for n must have length at least $n(n + 1)/2$. Show that the shortest sequence for $n = 4$ has 12 members. [You are told, but do not have to prove, that there is a universal sequence for n of length $n^2 - 2n + 4$.]

Problem 13

n real numbers are written around a circle. One of the numbers is 1 and the sum of the numbers is 0. Show that there are two adjacent numbers whose difference is at least $n/4$. Show that there is a number which differs from the arithmetic mean of its two neighbours by at least $8/n^2$. Improve this result to some k/n^2 with $k > 8$. Show that for $n = 30$, we can take $k = 1800/113$. Give an example of 30 numbers such that no number differs from the arithmetic mean of its two neighbours by more than $2/113$.

Problem 14

You are given a regular n-gon. Each vertex is marked +1 or -1. A move consists of changing the sign of all the vertices which form a regular k-gon for some $1 < k \le n$. [A regular 2-gon means two vertices which have the center of the n-gon as their midpoint.]. For example, if we label the vertices of a regular 6-gon 1, 2, 3, 4, 5, 6, then you can change the sign of {1, 4}, {2, 5}, {3, 6}, {1, 3, 5}, {2, 4, 6} or {1, 2, 3, 4, 5, 6}. Show that for (1) $n = 15$, (2) $n = 30$, (3) any $n > 2$, we can find some initial marking which cannot be changed to all +1 by any series of moves. Let f(n) be the largest number of markings, so that no one can be obtained from any other by any series of moves. Show that $f(200) = 2^{80}$.

Problem 15

S is a sphere with unit radius. P is a plane through the center. For any point x on the sphere f(x) is the perpendicular distance from x to P. Show that if x, y, z are the ends of three mutually perpendicular radii, then $f(x)^2 + f(y)^2 + f(z)^2 = 1$ (*). Now let g(x) be any function on the points of S taking non-negative real values and satisfying (*). Regard the intersection of P and S as the equator, the poles as the points with f(x) = 1 and lines of longitude as semicircles through both poles. (1) If x and y have the same longitude and both lie on the same side of the equator with x closer to the pole, show that g(x) > g(y). (2) Show that for any x, y on the same side of the equator with x closer to the pole than y we have g(x) > g(y). (3) Show that if x and y are the same distance from the pole then g(x) = g(y). (4) Show that g(x) = f(x) for all x.

11th ASU 1977 problems

Problem 1

P is a polygon. Its sides do not intersect except at its vertices, and no three vertices lie on a line. The pair of sides AB, PQ is called *special* if (1) AB and PQ do not share a vertex and (2) either the *line* AB intersects the *segment* PQ or the line PQ intersects the segment AB. Show that the number of special pairs is even.

Problem 2

n points lie in the plane, not all on a single line. A real number is assigned to each point. The sum of the numbers is zero for all the points lying on any line. Show that all the assigned numbers must be zero.

Problem 3

(1) The triangle ABC is inscribed in a circle. D is the midpoint of the arc BC (not containing A), similarly E and F. Show that the hexagon formed by the intersection of ABC and DEF has its main diagonals parallel to the sides of ABC and intersecting in a single point.

(2) EF meets AB at X and AC at Y. Prove that AXIY is a rhombus, where I is the center of the circle inscribed in ABC.

Problem 4

Black and white tokens are placed around a circle. First all the black tokens with one or two white neighbors are removed. Then all white tokens with one or two black neighbors are removed. Then all black tokens with one or two white neighbors and so on until all the tokens have the same color. Is it possible to arrange 40 tokens so that only one remains after 4 moves? What is the minimum possible number of moves to go from 1000 tokens to one?

Problem 5

a_n is an infinite sequence such that $(a_{n+1} - a_n)/2$ tends to zero. Show that a_n tends to zero.

Problem 6

There are direct routes between every two cities in a country. The fare between each pair of cities is the same in both directions. Two travellers decide to visit all the cities. The first traveller starts at a city and travels to the city with the most expensive fare (or if there are several such, any one of them). He then repeats this process, never visiting a city twice, until he has been to all the cities (so he ends up in a different city from the one he starts from). The second traveller has a similar plan, except that he always chooses the cheapest fare, and does not necessarily start at the same city. Show that the first traveller spends at least as much on fares as the second.

Problem 7

Each vertex of a convex polyhedron has three edges. Each face is a cyclic polygon. Show that its vertices all lie on a sphere.

Problem 8

Given a polynomial $x_{10} + a_9x^9 + ... + a_1x + 1$. Two players alternately choose one of the coefficients a_1 to a_9 (which has not been chosen before) and assign a real value to it. The first player wins iff the resulting polynomial has no real roots. Who wins?

Problem 9

Seven elves sit at a table. Each elf has a cup. In total the cups contain 3 liters of milk. Each elf in turn gives all his milk to the others in equal shares. At the end of the process each elf has the same amount of milk as at the start. What was that?

Problem 10

We call a number *doubly square* if (1) it is a square with an even number 2n of (decimal) digits, (2) its first n digits form a square, (3) its last n digits form a non-zero square. For example, 1681 is doubly square, but 2500 is not. (1) find all 2-digit and 4-digit doubly square numbers. (2) Is there a 6-digit doubly square number? (3) Show that there is a 20-digit doubly square number. (4) Show that there are at least ten 100-digit doubly square numbers. (5) Show that there is a 30-digit doubly square number.

Problem 11

Given a sequence $a_1, a_2, ... , a_n$ of positive integers. Let S be the set of all sums of one or more members of the sequence. Show that S can be divided into n subsets such that the smallest member of each subset is at least half the largest member.

Problem 12

You have 1000 tickets numbered 000, 001, ... , 999 and 100 boxes numbered 00, 01, ... , 99. You may put each ticket into any box whose number can be obtained from the ticket number by deleting one digit. Show that you can put every ticket into 50 boxes, but not into less than 50. Show that if you have 10000 4-digit tickets and you are allowed to delete two digits, then you can put every ticket into 34 boxes. For n+2 digit tickets, where you delete n digits, what is the minimum number of boxes required?

Problem 13

Given a 100 x 100 square divided into unit squares. Several paths are drawn. Each path is drawn along the sides of the unit squares. Each path has its endpoints on the sides of the big square, but does not contain any other points which are vertices of unit squares and lie on the big square sides. No path intersects itself or any other path. Show that there is a vertex apart from the four corners of the big square that is not on any path.

Problem 14

The positive integers $a_1, a_2, \dots , a_m, b_1, b_2, \dots , b_n$ satisfy: $(a_1 + a_2 + \dots + a_m) = (b_1 + b_2 + \dots + b_n) < mn$. Show that we can delete some (but not all) of the numbers so that the sum of the remaining a's equals to the sum of the remaining b's.

Problem 15

Given 1000 square plates in the plane with their sides parallel to the coordinate axes (but possibly overlapping and possibly of different sizes). Let S be the set of points covered by the plates. Show that you can choose a subset T of plates such that every point of S is covered by at least one and at most four plates in T.

Problem 16

You are given a set of scales and a set of n different weights. R represents the state in which the right pan is heavier, L represents the state in which the left pan is heavier and B represents the state in which the pans balance. Show that given any n-letter string of Rs and Ls you can put the weights onto the scales one at a time so that the string represents the successive states of the scales. For example, if the weights were 1, 2 and 3 and the string was LRL, then you would place 1 in the left pan, then 2 in the right pan, then 3 in the left pan.

Problem 17

A polynomial is *monic* if its leading coefficient is 1. Two polynomials p(x) and q(x) *commute* if $p(q(x)) = q(p(x))$.
(1) Find all monic polynomials of degree 3 or less which commute with $x^2 - k$.
(2) Given a monic polynomial p(x), show that there is at most one monic polynomial of degree n which commutes with $p(x)^2$.
(3) Find the polynomials described in (2) for n = 4 and n = 8.
(4) If q(x) and r(x) are monic polynomials which both commute with $p(x)^2$, show that q(x) and r(x) commute.
(5) Show that there is a sequence of polynomials $p_2(x), p_3(x), \dots$ such that $p_2(x) = x^2 - 2$, $p_n(x)$ has degree n and all polynomials in the sequence commute.

12th ASU 1978 problems

Problem 1

a_n is the nearest integer to $\sqrt{n}$. Find $1/a_1 + 1/a_2 + \dots + 1/a_{1980}$.

Problem 2

ABCD is a quadrilateral. M is a point inside it such that ABMD is a parallelogram. $\angle CBM = \angle CDM$. Show that $\angle ACD = \angle BCM$.

Problem 3

Show that there is no positive integer n for which $1000^n - 1$ divides $1978^n - 1$.

Problem 4

If P, Q are points in space the point [PQ] is the point on the line PQ on the opposite side of Q to P and the same distance from Q. K_0 is a set of points in space. Given K_n we derive K_{n+1} by adjoining all the points [PQ] with P and Q in K_n.

(1) K_0 contains just two points A and B, a distance 1 apart, what is the smallest n for which K_n contains a point whose distance from A is at least 1000?

(2) K_0 consists of three points, each pair a distance 1 apart, find the area of the smallest convex polygon containing K_n.

(3) K_0 consists of four points, forming a regular tetrahedron with volume 1. Let H_n be the smallest convex polyhedron containing K_n. How many faces does H_1 have? What is the volume of H_n?

Problem 5

Two players play a game. There is a heap of m tokens and a heap of $n < m$ tokens. Each player in turn takes one or more tokens from the heap which is larger. The number he takes must be a multiple of the number in the smaller heap. For example, if the heaps are 15 and 4, the first player may take 4, 8 or 12 from the larger heap. The first player to clear a heap wins. Show that if $m > 2n$, then the first player can always win. Find all k such that if $m > kn$, then the first player can always win.

Problem 6

Show that there is an infinite sequence of reals $x_1, x_2, x_3, \dots$ such that $|x_n|$ is bounded and for any $m > n$, we have $|x_m - x_n| > 1/(m - n)$.

Problem 7

Let $p(x) = x^2 + x + 1$. Show that for every positive integer n, the numbers n, p(n), p(p(n)), p(p(p(n))), ... are relatively prime.

Problem 8

Show that for some k, you can find 1978 different sizes of square with all its vertices on the graph of the function y = k sin x.

Problem 9

The set S_0 has the single member (5, 19). We derive the set S_{n+1} from S_n by adjoining a pair to S_n. If S_n contains the pair (2a, 2b), then we may adjoin the pair (a, b). If S contains the pair (a, b) we may adjoin (a+1, b+1). If S contains (a, b) and (b, c), then we may adjoin (a, c). Can we obtain (1, 50)? (1, 100)? If We start with (a, b), with $a < b$, instead of (5, 19), for which n can we obtain (1, n)?

Problem 10

An n-gon area A is inscribed in a circle radius R. We take a point on each side of the polygon to form another n-gon. Show that it has perimeter at least 2A/R.

Problem 11

Two players play a game by moving a piece on an n x n chessboard. The piece is initially in a corner square. Each player may move the piece to any adjacent square (which shares a side with its current square), except that the piece may never occupy the same square twice. The first player who is unable to move loses. Show that for even n the first player can always win, and for odd n the second player can always win. Who wins if the piece is initially on a square adjacent to the corner?

Problem 12

Given a set of n non-intersecting segments in the plane. No two segments lie on the same line. Can we successively add n-1 additional segments so that we end up with a single non-intersecting path? Each segment we add must have as its endpoints two existing segment endpoints.

Problem 13

a and b are positive real numbers. x_i are real numbers lying between a and b. Show that $(x_1 + x_2 + ... + x_n)(1/x_1 + 1/x_2 + ... + 1/x_n) \leq n^2(a + b)^2/4ab$.

Problem 14

$n > 3$ is an integer. Let S be the set of lattice points (a, b) with $0 \leq a, b < n$. Show that we can choose n points of S so that no three chosen points are collinear and no four chosen points from a parallelogram.

Problem 15

Given any tetrahedron, show that we can find two planes such that the areas of the projections of the tetrahedron onto the two planes have ratio at least $\sqrt{2}$.

Problem 16

$a_1, a_2, \ldots, a_n$ are real numbers. Let $b_k = (a_1 + a_2 + \ldots + a_k)/k$ for $k = 1, 2, \ldots, n$. Let $C = (a_1 - b_1)^2 + (a_2 - b_2)^2 + \ldots + (a_n - b_n)^2$, and $D = (a_1 - b_n)^2 + (a_2 - b_n)^2 + \ldots + (a_n - b_n)^2$. Show that $C \le D \le 2C$.

Problem 17

Let $x_n = (1 + \sqrt{2} + \sqrt{3})^n$. We may write $x_n = a_n + b_n\sqrt{2} + c_n\sqrt{3} + d_n\sqrt{6}$, where a_n, b_n, c_n, d_n are integers. Find the limit as n tends to infinity of b_n/a_n, c_n/a_n, d_n/a_n.

13th ASU 1979 problems

Problem 1

T is an isosceles triangle. Another isosceles triangle T' has one vertex on each side of T. What is the smallest possible value of area T'/area T?

Problem 2

A grasshopper hops about in the first quadrant ($x, y >= 0$). From (x, y) it can hop to $(x+1, y-1)$ or to $(x-5, y+7)$, but it can never leave the first quadrant. Find the set of points (x, y) from which it can never get further than a distance 1000 from the origin.

Problem 3

In a group of people every person has less than 4 enemies. Assume that A is B's enemy iff B is A's enemy. Show that we can divide the group into two parts, so that each person has at most one enemy in his part.

Problem 4

Let S be the set $\{0, 1\}$. Given any subset of S we may add its arithmetic mean to S (provided it is not already included - S never includes duplicates). Show that by repeating this process we can include the number 1/5 in S. Show that we can eventually include any rational number between 0 and 1.

Problem 5

The real sequence $x_1 \ge x_2 \ge x_3 \ge \dots$ satisfies $x_1 + x_4/2 + x_9/3 + x_{16}/4 + \dots + x_N/n \le 1$ for every square $N = n^2$. Show that it also satisfies $x_1 + x_2/2 + x_3 /3 + \dots + x_n/n \le 3$.

Problem 6

Given a finite set X of points in the plane. S is a set of vectors **AB** where (A, B) are some pairs of points in X. For every point A the number of vectors **AB** (starting at A) in S equals the number of vectors **CA** (ending at A) in S. Show that the sum of the vectors in S is zero.

Problem 7

What is the smallest number of pieces that can be placed on an 8 x 8 chessboard so that every row, column and diagonal has at least one piece? [A diagonal is any line of squares parallel to one of the two main diagonals, so there are 30 diagonals in all.] What is the smallest number for an n x n board?

Problem 8

a and b are real numbers. Find real x and y satisfying: $(x - y (x^2 - y^2)^{1/2} = a(1 - x^2 + y^2)^{1/2}$ and $(y - x (x^2 - y^2)^{1/2} = b(1 - x^2 + y^2)^{1/2}$.

Problem 9

A set of square carpets have total area 4. Show that they can cover a unit square.

Problem 10

x_i are real numbers between 0 and 1. Show that $(x_1 + x_2 + ... + x_n + 1)^2 \geq 4(x_1^2 + x_2^2 + ... + x_n^2)$.

Problem 11

m and n are relatively prime positive integers. The interval [0, 1] is divided into m + n equal subintervals. Show that each part except those at each end contains just one of the numbers 1/m, 2/m, 3/m, ... , (m-1)/m, 1/n, 2/n, ... , (n-1)/n.

Problem 12

Given a point P in space and 1979 lines $L_1, L_2, ... , L_{1979}$ containing it. No two lines are perpendicular. P_1 is a point on L_1. Show that we can find a point A_n on L_n (for n = 2, 3, ... , 1979) such that the following 1979 pairs of lines are all perpendicular: $A_{n-1}A_{n+1}$ and L_n for n = 1, ... , 1979. [We regard A_{-1} as A_{1979} and A_{1980} as A_1.]

Problem 13

Find a sequence $a_1, a_2, ... , a_{25}$ of 0s and 1s such that the following sums are all odd:

$a_1a_1 + a_2a_2 + ... + a_{25}a_{25}$

$a_1a_2 + a_2a_3 + ... + a_{24}a_{25}$

$a_1a_3 + a_2a_4 + ... + a_{23}a_{25}$

...

$a_1a_{24} + a_2a_{25}$

a_1a_{25}

Show that we can find a similar sequence of n terms for some n > 1000.

Problem 14

A convex quadrilateral is divided by its diagonals into four triangles. The incircles of each of the four are equal. Show that the quadrilateral has all its sides equal.

14th ASU 1980

Problem 1

All two digit numbers from 19 to 80 inclusive are written down one after the other as a single number N = 192021...7980. Is N divisible by 1980?

Answer

Yes

Solution

$1980 = 2^2 3^2 5 \cdot 11$. N is obviously divisible by 2^2 and 5. The digits in odd position are 9 + (0+1+2+...+9) + (0+1+2+...+9) + ... + (0+1+2+...+9) + 0 = 9 + 6· 45 = 279. The digits in even position are 1 + (2+2+...+2) + (3+3+...+3) + ... + (7+...+7) + 8 = 9 + 10(2+3+..+7) = 279. So the sum of the digits of N is 2· 279 which is divisible by 9. Hence N is divisible by 9. The difference between the odd and even sums is 0, which is divisible by 11, so N is divisible by 11.

Problem 2

A square is divided into n parallel strips (parallel to the bottom side of the square). The width of each strip is integral. The total width of the strips with odd width equals the total width of the strips with even width. A diagonal of the square is drawn which divides each strip into a left part and a right part. Show that the sum of the areas of the left parts of the odd strips equals the sum of the areas of the right parts of the even strips.

Solution

Let LO be the total area of the left parts of the odd strips, LE the total area of the left parts of the even strips, and RE the total area of the right parts of the even trips. Since the diagonal bisects the square, LO + LE = A/2, where A is the area of the square. Also RE + LE = A/2 (because the total width of the even strips equals the total width of the odd strips). Subtracting, LO = RE, as required.

Problem 3

35 containers of total weight 18 must be taken to a space station. One flight can take any collection of containers weighing 3 or less. It is possible to take any subset of 34 containers in 7 flights. Show that it must be possible to take all 35 containers in 7 flights.

Solution

Let the lightest container be S weighing s. The other containers can be taken in 7 flights. The smallest load on these flights must be $\le (18-s)/7$, and so that flight has spare capacity of at least $3 - (18-s)/7 = (3+s)/7$. Thus it can accomodate S provided that $(3+s)/7 \ge s$, or $s \le 1/2$.

At least one of the 7 flights takes ≤ 4 containers. These weigh at most the weight of the 4 heaviest. Since the 31 lightest weigh at least 31s, the 4 heaviest weigh at most 18-31s. Thus this flight has spare capacity of at least 3 - (18-31s) = 31s-15 and can accomodate S provided that $31s-15 \geq s$, or $s \geq 1/2$.

Problem 4

ABCD is a convex quadrilateral. M is the midpoint of BC and N is the midpoint of CD. If k = AM + AN show that the area of ABCD is less than $k^2/2$.

Problem 5

Are there any solutions in positive integers to $a^4 = b^3 + c^2$?

Solution

We have $b^3 = (a^2 - c)(a^2 + c)$, so one possibility is that a2 ± c are both cubes. So we want two cubes whose sum is twice a square. Looking at the small cubes, we soon find $8 + 64 = 2 \cdot 36$ giving $6^4 = 28^2 + 8^3$. Multiplying through by k^{12} gives an infinite family of solutions. Note that the question does not ask for all solutions.

Problem 6

Given a point P on the diameter AC of the circle K, find the chord BD through P which maximises the area of ABCD.

Problem 7

There are several settlements around Big Lake. Some pairs of settlements are directly connected by a regular shipping service. For all A ≠ B, settlement A is directly connected to X iff B is not directly connected to Y, where B is the next settlement to A counterclockwise and Y is the next settlement to X counterclockwise. Show that you can move between any two settlements with at most 3 trips.

Solution

Suppose there are n settlements $A_1, A_2, \dots, A_n$ in counterclockwise order around the lake. We will use cyclic indices, so that A_{n+1} means A_1, and so on. wlog A_1 has a direct service to A_2. Then it follows that A_2 does not have a direct service to A_3, so A_3 does have a direct service to A_4, and so on. So A_i has direct service to A_{i+1} iff i is odd. But $A_1 = A_{n+1}$ has direct service to A_2, so n must be even.

Now suppose we want to get from A_i to A_j. If there is direct service we are done. So suppose not. Then there must be direct service from A_{i+1} to A_{j+1}. If i and j are both odd, then there is direct service from A_i to A_{i+1} and from A_j to A_{j+1}, so we can make the journey in 3 trips. If i and j are both even, then we can go A_i to A_{i-1} to A_{j-1} to A_j. So suppose i and j have opposite parity. wlog i is odd and j is even. If there is direct service from A_i to A_{j-1}, then we can make the journey in two trips: A_i to A_{j-1} to A_j. If not, then we can go A_i to A_{i+1} to A_j.

Problem 8

A six digit (decimal) number has six different digits, none of them 0, and is divisible by 37. Show that you can obtain at least 23 other numbers which are divisible by 37 by permuting the digits.

Solution

Suppose the digits are a_1, a_2, a_3, a_4, a_5, a_6 and that $a_1a_2a_3a_4a_5a_6$ is divisible by 37. We claim that $a_2a_3a_4a_5a_6a_1$ is also divisible by 37. Put $n = a_2a_3a_4a_5a_6$ and $m = a_1$. The original number is $10^5m + n$ and the derived number is $10n + m$. But 37 divides $10^3 - 1$ and hence also $10^6 - 1$ and $m(10^6 - 1)$. So it also divides $10(10^5m + n) - m(10^6 - 1) = 10n + m$, which proves the claim.

Iterating, we get the original number and 5 others:
$a_1a_2a_3a_4a_5a_6$
$a_2a_3a_4a_5a_6a_1$
$a_3a_4a_5a_6a_1a_2$
$a_4a_5a_6a_1a_2a_3$
$a_5a_6a_1a_2a_3a_4$
$a_6a_1a_2a_3a_4a_5$

Similarly, we have that $a_1a_2a_3a_4a_5a_6 - a_1a_2a_6a_4a_5a_3 = (a_6\text{-}a_3)999$, so $a_1a_2a_6a_4a_5a_3$ is also divisible by 37. Iterating we get:
$a_1a_2a_6a_4a_5a_3$
$a_2a_6a_4a_5a_3a_1$
$a_6a_4a_5a_3a_1a_2$
$a_4a_5a_3a_1a_2a_6$
$a_5a_3a_1a_2a_6a_4$
$a_3a_1a_2a_6a_4a_5$

Similarly, we could swap the first term and the fourth to get:
$a_4a_2a_3a_1a_5a_6$
$a_2a_3a_1a_5a_6a_4$
$a_3a_1a_5a_6a_4a_2$
$a_1a_5a_6a_4a_2a_3$
$a_5a_6a_4a_2a_3a_1$
$a_6a_4a_2a_3a_1a_5$

or the second and the fifth to get:
$a_1a_5a_3a_4a_2a_6$
$a_5a_3a_4a_2a_6a_1$
$a_3a_4a_2a_6a_1a_5$
$a_4a_2a_6a_1a_5a_3$
$a_2a_6a_1a_5a_3a_4$
$a_6a_1a_5a_3a_4a_2$

Problem 9

Find all real solutions to:
$\sin x + 2 \sin(x+y+z) = 0$
$\sin y + 3 \sin(x+y+z) = 0$
$\sin z + 4 \sin(x+y+z) = 0$

Problem 10

Given 1980 vectors in the plane. The sum of every 1979 vectors is a multiple of the other vector. Not all the vectors are multiples of each other. Show that the sum of all the vectors is zero.

Solution

Let the vectors be $\mathbf{x}_i$ and their sum $\mathbf{s}$. Then we have $\mathbf{s} - \mathbf{x}_i = n_i\mathbf{x}_i$ for some scalar n_i. Hence $(n_i+1)\mathbf{x}_i = \mathbf{s}$. If $\mathbf{s}$ is non-zero, then it follows that every vector is a multiple of $\mathbf{s}$ and hence all the vectors are multiples of each other. But we are told that is not true. Hence $\mathbf{s}$ is zero.

Problem 11

Let f(n) be the sum of n and its digits. For example, f(34) = 41. Is there an integer such that f(n) = 1980? Show that given any positive integer m we can find n such that f(n) = m or m+1.

Answer

f(1962) = 1962 + 18 = 1980

Solution

If the last digit of n is not 9, then f(n+1) = f(n) + 2. If the last digit of n is 9, then f(n+1) < f(n). On the other hand f clearly achieves arbitrarily large values. Also f(1) = 1. Now consider any m > 1. Let M be the smallest integer such that f(M) > m. Then f(M-1) ≤ m. Since f(M) > f(M-1) we must have f(M) = f(M-1) + 2. Hence either f(M) = m+1 or f(M-1) = m.

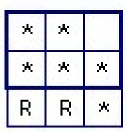
Problem 12

Some unit squares in an infinite sheet of squared paper are colored red so that every 2 x 3 and 3 x 2 rectangle contains exactly two red squares. How many red squares are there in a 9 x 11 rectangle?

Answer

33

Solution



There cannot be two red squares with a common side. For consider as the diagram shows we can immediately conclude that the squares with a * are not red, but now the bold rectangle has at most 1 red square. Contradiction.

*		*
	R	

Consider a red square. One of the two diagonally adjacent squares marked * must be red. But it is now easy to show that all red squares on that diagonal are red and that the other red squares are those on every third parallel diagonal line. Any 9 x 11 rectangle must have just three such diagonals on a 9 cell border row, and hence just 3 red cells in that border row. But the remaining 9 x 10 rectangle can easily be partitioned into fifteen 3 x 2 rectangles, each with 2 red squares.

Problem 13

There is a flu epidemic in elf city. The course of the disease is always the same. An elf is infected one day, he is sick the next, recovered and immune the third, recovered but not immune thereafter. Every day every elf who is not sick visits all his sick friends. If he is not immune he is sure to catch flu if he visits a sick elf. On day 1 no one is immune and one or more elves are infected from some external source. Thereafter there is no further external infection and the epidemic spreads as described above. Show that it is sure to die out (irrespective of the number of elves, the number of friends each has, and the number infected on day 1). Show that if one or more elves is immune on day 1, then it is possible for the epidemic to continue indefinitely.

Solution

This is curiously easy. Write S for sick, N for not sick and not immune, and I for immune. Suppose group A are S on day 1 and group B are N on day 1. Then on day 2, B are S, and A are I. So on day 3 no one is sick, A are N and B are I. Thereafter no one can get sick, so the epidemic has died out.

Suppose on day 1, there is also group C who are I. Then on day 2, B are S, A are I and C are N. So on day 3, C are S, A are N and B are I. On day 4, A are S, B are N and C are I, the same as day 1, so the epidemic continues indefinitely.

Problem 14

Define the sequence a_n of positive integers as follows. $a_1 = m$. $a_{n+1} = a_n$ plus the product of the digits of a_n. For example, if m = 5, we have 5, 10, 10, ... Is there an m for which the sequence is unbounded?

Answer

No.

Solution

Put p(n) for the product of the digits of n. We show that, for sufficiently large n, a sequence starting below it cannot get past the "gap" from 10_n to $10_n + 10_{n-1}$. For suppose N is the last member of the sequence below the gap. Then N has at most n digits, so $p(N) \le 9^n$. But for sufficiently large n (in fact for $n \ge 21$) we have $9^n < 10^{n-1}$. So $N + p(N) < 10^n + 10^{n-1}$. But $N + p(N) > 10^n$ by assumption. Hence N + p(N) is sure to have second digit (from the left) zero.

So all further terms of the sequence are the same. But for any m there is certainly a gap above m, and, as shown, the sequence will not be able to get beyond it. So it is bounded.

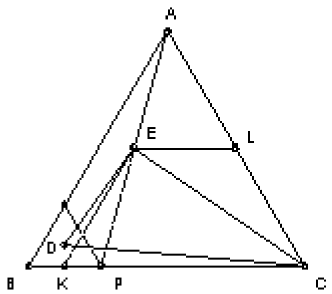
Problem 15

ABC is equilateral. A line parallel to AC meets AB at M and BC at P. D is the center of the equilateral triangle BMP. E is the midpoint of AP. Find the angles of DEC.

Answer

D = 60^o, E = 90^o

Solution



Let K be the midpoint of BP and L the midpoint of AC. EL is parallel to BC, so $\angle ELC = 120^o$. EK is parallel to AB, so $\angle EKC = 60^o$, so ELCK is cyclic. But $\angle DKC = \angle DLC = 90^o$, so DLCK is cyclic. Hence D, K, C, L, E all lie on a circle. Hence $\angle DEC = \angle DLC = 90^o$, and $\angle EDC = \angle EKC = 60^o$.

Problem 16

A rectangular box has sides $x < y < z$. Its perimeter is $p = 4(x + y + z)$, its surface area is $s = 2(xy + yz + zx)$ and its main diagonal has length $d = \sqrt{(x^2 + y^2 + z^2)}$. Show that $3x < (p/4 - \sqrt{(d^2 - s/2)}$ and $3z > (p/4 + \sqrt{(d^2 - s/2)}$.

Solution

We have $3(y-x)(z-x) > 0$, so $3x^2 + 3yz > 3xy + 3xz$. Hence $y^2 + z^2 + 4x^2 + 2yz - 4xy - 4xz > x^2 + y^2 + z^2 - xy - yz - xz$ or $(y + z - 2x)^2 > (d^2 - s/2)$. Hence $(x + y + z) > 3x + \sqrt{(d^2 - s/2)}$. So $3x < p/4 - \sqrt{(d^2 - s/2)}$.

Similarly, $3(z-x)(z-y) > 0$, so $x^2 + y^2 + 4z^2 > x^2 + y^2 + z^2 + 3zx + 3zy - 3xy$, so $(2z - x - y)^2 > x^2 + y^2 + z^2 - xy - yz - zx$ or $(3z - p/4)^2 > (d^2 - s/2)$. Hence $3z > p/4 + \sqrt{(d^2 - s/2)}$.

Problem 17

S is a set of integers. Its smallest element is 1 and its largest element is 100. Every element of S except 1 is the sum of two distinct members of the set or double a member of the set. What is the smallest possible number of integers in S?

Answer

Solution

Let $n = M(n) + m(n)$, where $M(n) \geq m(n)$. Put $M^1(n) = M(n)$, $M^2(n) = M(M(n))$ etc. Then $M(100) \geq 50$, $M^2(100) \geq 25$, $M^3(100) \geq 13$, $M^4(100) \geq 7$, $M^5(100) \geq 4$, $M^6(100) \geq 2$ (and obviously $n > M(n)$), so we need at least 8 numbers. There are several ways of using 9 numbers. For example, {1, 2, 4, 8, 16, 32, 36, 64, 100}, where 36 = 4 + 32, 100 = 36 + 64 and the others are double another number.

Doubling every time does not work: 1, 2, 4, 8, 16, 32, 64, 128. But if we do not double every time, then we cannot get a number larger than 96 with 8 numbers: the best we can do is $1 \cdot 2^6 \cdot (3/2) = 96$ (on the occasion when we do not double the best we can do is to the largest plus the next largest, or 3/2 x the largest). Hence we need at least 9 numbers. [To be more formal, write the elements as $1 = a_1 < a_2 < ... < a_n$, then each a_i must be a sum of preceding elements. The largest possible a_i is $2a_{i-1}$ and the next largest $a_{i-1} + a_{i-2}$ and so on.]

Problem 18

Show that there are infinitely many positive integers n such that $[a^{3/2}] + [b^{3/2}] = n$ has at least 1980 integer solutions.

Solution

Consider all a, b in the range 1, 2, 3, ... , N^2. There are N^4 possible pairs of values. But $[a^{3/2}]$ and $[b^{3/2}]$ are in the range 1, 2, ... , N^3, so their sum is in the range 1, 2, ... , $2N^3$. Hence one of these values has at least N/2 solutions. By taking N sufficiently large we can get $a_1 > 1980$ solutions for some $N_1 \leq 2N^3$. But now by taking N sufficiently large we can get $a_2 > a_1$ solutions for some N_2. Since $a_2 \neq a_1$, we must have $N_2 \neq N_1$. In other words, we have a different n, also with > 1980 solutions. Continuing, we get an infinite sequence of distinct n each with at least 1980 solutions.

Problem 19

ABCD is a tetrahedron. Angles ACB and ADB are 90 deg. Let k be the angle between the lines AC and BD. Show that cos k < CD/AB.

Problem 20

x_0 is a real number in the interval (0, 1) with decimal representation $0.d_1d_2d_3...$. We obtain the sequence x_n as follows. x_{n+1} is obtained from x_n by rearranging the 5 digits d_{n+1}, d_{n+2}, d_{n+3}, d_{n+4}, d_{n+5}. Show that the sequence x_n converges. Can the limit be irrational if x_0 is rational? Find a number x_0 so that every member of the sequence is irrational, no matter how the rearrangements are carried out.

15th ASU 1981 problems

Problem 1

A chess board is placed on top of an identical board and rotated through 45 degrees about its center. What is the area which is black in both boards?

Problem 2

AB is a diameter of the circle C. M and N are any two points on the circle. The chord MA' is perpendicular to the line NA and the chord MB' is perpendicular to the line NB. Show that AA' and BB' are parallel.

Problem 3

Find an example of m and n such that m is the product of n consecutive positive integers and also the product of n+2 consecutive positive integers. Show that we cannot have n = 2.

Problem 4

Write down a row of arbitrary integers (repetitions allowed). Now construct a second row as follows. Suppose the integer n is in column k in the first row. In column k in the second row write down the number of occurrences of n in row 1 in columns 1 to k inclusive. Similarly, construct a third row under the second row (using the values in the second row), and a fourth row. An example follows:

```
7  1  2  1  7  1  1
1  1  1  2  2  3  4
1  2  3  1  2  1  1
1  1  1  2  2  3  4
```

Show that the fourth row is always the same as the second row.

Problem 5

Let S be the set of points (x, y) given by $y \leq -x^2$ and $y \geq x^2 - 2x + a$. Find the area of the rectangle with sides parallel to the axes and the smallest possible area which encloses S.

Problem 6

ABC, CDE, EFG are equilateral triangles (not necessarily the same size). The vertices are counter-clockwise in each case. A, D, G are collinear and AD = DG. Show that BFD is equilateral.

Problem 7

1000 people live in a village. Every evening each person tells his friends all the news he heard during the day. All news eventually becomes known (by this process) to everyone. Show that one can choose 90 people, so that if you give them some news on the same day, then everyone will know in 10 days.

Problem 8

The reals a and b are such that $a \cos x + b \cos 3x > 1$ has no real solutions. Show that $|b| \le 1$.

Problem 9

ABCD is a convex quadrilateral. K is the midpoint of AB and M is the midpoint of CD. L lies on the side BC and N lies on the side AD. KLMN is a rectangle. Show that its area is half that of ABCD.

Problem 10

The sequence a_n of positive integers is such that (1) $a_n \le n^{3/2}$ for all n, and (2) m-n divides $k_m - k_n$ (for all m > n). Find a_n.

Problem 11

Is it possible to color half the cells in a rectangular array white and half black so that in each row and column more than 3/4 of the cells are the same color?

Problem 12

ACPH, AMBE, AHBT, BKXM and CKXP are parallelograms. Show that ABTE is also a parallelogram (vertices are labeled anticlockwise).

Problem 13

Find all solutions (x, y) in positive integers to $x^3 - y^3 = xy + 61$.

Answer

(6,5)

Solution

Put $x = y + a$. Then $(3a-1)y^2 + a(3a-1)y + (a^3-61) = 0$. The first two terms are positive, so the last term must be negative, so a = 1, 2, 3. Trying each case in turn, we get $(y+6)(y-5) = 0$, $5y^2+10y-53 = 0$, $4y^2+12y-17 = 0$. The last two equations have no integers solutions.

Problem 14

Eighteen teams are playing in a tournament. So far, each team has played exactly eight games, each with a different opponent. Show that there are three teams none of which has yet played the other.

Problem 15

ABC is a triangle. A' lies on the side BC with BA'/BC = 1/4. Similarly, B' lies on the side CA with CB'/CA = 1/4, and C' lies on the side AB with AC'/AB = 1/4. Show that the perimeter of A'B'C' is between 1/2 and 3/4 of the perimeter of ABC.

Problem 16

The positive reals x, y satisfy $x^3 + y^3 = x - y$. Show that $x^2 + y^2 < 1$.

Solution

Since x, y are positive, so is $x^3 + y^3$, and hence $x > y$. So $(x^2 + y^2)(x - y) = (x^3 - y^3) - xy(x - y) < x^3 - y^3 = x - y$. Hence $x^2 + y^2 < 1$.

Problem 17

A convex polygon is drawn inside the unit circle. Someone makes a copy by starting with one vertex and then drawing each side successively. He copies the angle between each side and the previous side accurately, but makes an error in the length of each side of up to a factor $1\pm p$. As a result the last side ends up a distance d from the starting point. Show that $d < 4p$.

Problem 18

An integer is initially written at each vertex of a cube. A move is to add 1 to the numbers at two vertices connected by an edge. Is it possible to equalise the numbers by a series of moves in the following cases? (1) The initial numbers are (1) 0, except for one vertex which is 1. (2) The initial numbers are 0, except for two vertices which are 1 and diagonally opposite on a face of the cube. (3) Initially, the numbers going round the base are 1, 2, 3, 4. The corresponding vertices on the top are 6, 7, 4, 5 (with 6 above the 1, 7 above the 2 and so on).

Problem 19

Find 21 consecutive integers, each with a prime factor less than 17.

Problem 20

Each of the numbers from 100 to 999 inclusive is written on a separate card. The cards are arranged in a pile in random order. We take cards off the pile one at a time and stack them into 10 piles according to the last digit. We then put the 1 pile on top of the 0 pile, the 2 pile on top of the 1 pile and so on to get a single pile. We now take them off one at a time and stack them into 10 piles according to the middle digit. We then consolidate the piles as before. We then take them off one at a time and stack them into 10 piles according to the first digit and finally consolidate the piles as before. What can we say about the order in the final pile?

Problem 21

Given 6 points inside a 3 x 4 rectangle, show that we can find two points whose distance does not exceed $\sqrt{5}$.

Problem 22
What is the smallest value of $4 + x^2y^4 + x^4y^2 - 3x^2y^2$ for real x, y? Show that the polynomial cannot be written as a sum of squares. [Note the candidates did not know calculus.]

Problem 23
ABCDEF is a prism. Its base ABC and its top DEF are congruent equilateral triangles. The side edges are AD, BE and CF. Find all points on the base wich are equidistant from the three lines AE, BF and CD.

16th ASU 1982

Problem 1

The circle C has center O and radius r and contains the points A and B. The circle C' touches the rays OA and OB and has center O' and radius r'. Find the area of the quadrilateral OAO'B.

Problem 2

The sequence a_n is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = a_{n+1} + a_n$. The sequence b_n is defined by $b_1 = 2$, $b_2 = 1$, $b_{n+2} = b_{n+1} + b_n$. How many integers belong to both sequences?

Answer

1,2,3 only

Solution

The first few terms are:

n	1	2	3	4	5	6	7
a_n	1	2	3	5	8	13	21
b_n	2	1	3	4	7	11	18

Note that for n = 4, 5 we have $a_{n-1} < b_n < a_n$. So by a trivial induction, the inequality holds for all $n \geq 4$.

Problem 3

N is a sum of n powers of 2. If N is divisible by $2^m - 1$, prove that $n \geq m$. Does there exist a number divisible by 11...1 (m 1s) which has the sum of its digits less than m?

Problem 4

A non-negative real is written at each vertex of a cube. The sum of the eight numbers is 1. Two players choose faces of the cube alternately. A player cannot choose a face already chosen or the one opposite, so the first player plays twice, the second player plays once. Can the first player arrange that the vertex common to all three chosen faces is $\leq 1/6$?

Problem 5

A library is open every day except Wednesday. One day three boys, A, B, C visit the library together for the first time. Thereafter they visit the library many times. A always makes his next visit two days after the previous visit, unless the library is closed on that day, in which case he goes the following day. B always makes his next visit three days after the previous visit (or four if the library is closed). C always makes his next visit four days after the previous visit (or five if the library is closed). For example, if A went first on Monday, his next visit would be Thursday, then Saturday. If B went first on Monday, his next visit would be on Thursday. All three boys are subsequently in the library on a Monday. What day of the week was their first visit?

Problem 6

ABCD is a parallelogram and AB is not equal to BC. M is chosen so that (1) $\angle MAC = \angle DAC$ and M is on the opposite side of AC to D, and (2) $\angle MBD = \angle CBD$ and M is on the opposite side of BD to C. Find AM/BM in terms of k = AC/BD.

Problem 7

3n points divide a circle into 3n arcs. One third of the arcs have length 1, one third have length 2 and one third have length 3. Show that two of the points are at opposite ends of a diameter.

Problem 8

M is a point inside a regular tetrahedron. Show that we can find two vertices A, B of the tetrahedron such that $\cos AMB \le -1/3$.

Problem 9

$0 < x, y, z < \pi/2$. We have $\cos x = x$, $\sin(\cos y) = y$, $\cos(\sin z) = z$. Which of x, y, z is the largest and which the smallest?

Problem 10

P is a polygon with 2n+1 sides. A new polygon is derived by taking as its vertices the midpoints of the sides of P. This process is repeated. Show that we must eventually reach a polygon which is homothetic to P.

Problem 11

$a_1, a_2, \dots , a_{1982}$ is a permutation of 1, 2, ... , 1982. If $a_1 > a_2$, we swap a_1 and a_2. Then if (the new) $a_2 > a_3$ we swap a_2 and a_3. And so on. After 1981 potential swaps we have a new permutation $b_1, b_2, \dots , b_{1982}$. We then compare b_{1982} and b_{1981}. If $b_{1981} > b_{1982}$, we swap them. We then compare b_{1980} and (the new) b_{1981}. So we arrive finally at $c_1, c_2, \dots , c_{1982}$. We find that $a_{100} = c_{100}$. What value is a_{100}?

Problem 12

Cucumber River has parallel banks a distance 1 meter apart. It has some islands with total perimeter 8 meters. It is claimed that it is always possible to cross the river (starting from an arbitrary point) by boat in at most 3 meters. Is the claim always true for any arrangement of islands? [Neglect the current.]

Problem 13

The parabola $y = x^2$ is drawn and then the axes are deleted. Can you restore them using ruler and compasses?

Problem 14

An integer is put in each cell of an n x n array. The difference between the integers in cells which share a side is 0 or 1. Show that some integer occurs at least n times.

Problem 15

x is a positive integer. Put $a = x^{1/12}$, $b = x^{1/4}$, $c = x^{1/6}$. Show that $2^a + 2^b \geq 2^{1+c}$.

Solution

Put $x = r^{12}$. Since x is a positive integer, we have $r \geq 1$. We have to show that $(2^r + 2^{r^3})/2 \geq 2^{r^2}$. But this follows immediately from AM/GM.

Problem 16

What is the largest subset of {1, 2, ... , 1982} with the property that no element is the product of two other distinct elements.

Problem 17

A real number is assigned to each unit square in an infinite sheet of squared paper. Show that some cell contains a number that is less than or equal to at least four of its eight neighbors.

Problem 18

Given a real sequence $a_1, a_2, \ldots , a_n$, show that it is always possible to choose a subsequence such that (1) for each $i \leq n-2$ at least one and at most two of a_i, a_{i+1}, a_{i+2} are chosen and (2) the sum of the absolute values of the numbers in the subsequence is at least $1/6 \sum_1^n |a_i|$.

Problem 19

An n x n array has a cross in n - 1 cells. A move consists of moving a row to a new position or moving a column to a new position. For example, one might move row 2 to row 5, so that row 1 remained in the same position, row 3 became row 2, row 4 became row 3, row 5 became row 4, row 2 became row 5 and the remaining rows remained in the same position. Show that by a series of moves one can end up with all the crosses below the main diagonal.

Problem 20

Let {a} denote the difference between a and the nearest integer. For example {3.8} = 0.2, {-5.4} = 0.4. Show that $|a|\,|a-1|\,|a-2| \ldots |a-n| >= \{a\}\, n!/2^n$.

Problem 21

Do there exist polynomials p(x), q(x), r(x) such that $p(x-y+z)^3 + q(y-z-1)^3 + r(z-2x+1)^3 = 1$ for all x, y, z? Do there exist polynomials p(x), q(x), r(x) such that $p(x-y+z)^3 + q(y-z-1)^3 + r(z-x+1)^3 = 1$ for all x, y, z?

Problem 22

A tetrahedron T' has all its vertices inside the tetrahedron T. Show that the sum of the edge lengths of T' is less than 4/3 times the corresponding sum for T.

17th ASU 1983

Problem 1

A 4 x 4 array of unit cells is made up of a grid of total length 40. Can we divide the grid into 8 paths of length 5? Into 5 paths of length 8?

Problem 2

Three positive integers are written on a blackboard. A move consists of replacing one of the numbers by the sum of the other two less one. For example, if the numbers are 3, 4, 5, then one move could lead to 4, 5, 8 or 3, 5, 7 or 3, 4, 6. After a series of moves the three numbers are 17, 1967 and 1983. Could the initial set have been 2, 2, 2? 3, 3, 3?

Problem 3

C_1, C_2, C_3 are circles, none of which lie inside either of the others. C_1 and C_2 touch at Z, C_2 and C_3 touch at X, and C_3 and C_1 touch at Y. Prove that if the radius of each circle is increased by a factor $2/\sqrt{3}$ without moving their centers, then the enlarged circles cover the triangle XYZ.

Problem 4

Find all real solutions x, y to $y^2 = x^3 - 3x^2 + 2x$, $x^2 = y^3 - 3y^2 + 2y$.

Problem 5

The positive integer k has n digits. It is rounded to the nearest multiple of 10, then to the nearest multiple of 100 and so on (n-1 roundings in all). Numbers midway between are rounded up. For example, 1474 is rounded to 1470, then to 1500, then to 2000. Show that the final number is less than 18k/13.

Problem 6

M is the midpoint of BC. E is any point on the side AC and F is any point on the side AB. Show that area MEF ≤ area BMF + area CME.

Problem 7

a_n is the last digit of $[10^{n/2}]$. Is the sequence a_n periodic? b_n is the last digit of $[2^{n/2}]$. Is the sequence b_n periodic?

Problem 8

A and B are acute angles such that $\sin^2A + \sin^2B = \sin(A + B)$. Show that $A + B = \pi/2$.

Problem 9

The projection of a tetrahedron onto the plane P is ABCD. Can we find a distinct plane P' such that the projection of the tetrahedron onto P' is A'B'C'D' and AA', BB', CC' and DD' are all parallel?

Problem 10

Given a quadratic equation $ax^2 + bx + c$. If it has two real roots $A \le B$, transform the equation to $x^2 + Ax + B$. Show that if we repeat this process we must eventually reach an equation with complex roots. What is the maximum possible number of transformations before we reach such an equation?

Problem 11

a, b, c are positive integers. If a^b divides b^a and c^a divides a^c, show that c^b divides b^c.

Problem 12

A *word* is a finite string of As and Bs. Can we find a set of three 4-letter words, ten 5-letter words, thirty 6-letter words and five 7-letter words such that no word is the beginning of another word. [For example, if ABA was a word, then ABAAB could not be a word.]

Problem 13

Can you place an integer in every square of an infinite sheet of squared paper so that the sum of the integers in every 4 x 6 (or 6 x 4) rectangle is (1) 10, (2) 1?

Problem 14

A point is chosen on each of the three sides of a triangle and joined to the opposite vertex. The resulting lines divide the triangle into four triangles and three quadrilaterals. The four triangles all have area A. Show that the three quadrilaterals have equal area. What is it (in terms of A)?

Problem 15

A group of children form two equal lines side-by-side. Each line contains an equal number of boys and girls. The number of mixed pairs (one boy in one line next to one girl in the other line) equals the number of unmixed pairs (two girls side-by-side or two boys side-by-side). Show that the total number of children in the group is a multiple of 8.

Problem 16

A 1 x k rectangle can be divided by two perpendicular lines parallel to the sides into four rectangles, each with area at least 1 and one with area at least 2. What is the smallest possible k?

Problem 17

O is a point inside the triangle ABC. a = area OBC, b = area OCA, c = area OAB. Show that the vector sum a**OA** + b**OB** + c**OC** is zero.

Problem 18

Show that given any 2m+1 different integers lying between -(2m-1) and 2m-1 (inclusive) we can always find three whose sum is zero.

Problem 19

Interior points D, E, F are chosen on the sides BC, CA, AB (not at the vertices). Let k be the length of the longest side of DEF. Let a, b, c be the lengths of the longest sides of AFE, BDF, CDE respectively. Show that $k \geq \sqrt{3} \min(a, b, c) /2$. When do we have equality?

Problem 20

X is a union of k disjoint intervals of the real line. It has the property that for any $h < 1$ we can find two points of X which are a distance h apart. Show that the sum of the lengths of the intervals in X is at least 1/k.

Problem 21

x is a real. The decimal representation of x includes all the digits at least once. Let f(n) be the number of distinct n-digit segments in the representation. Show that if for some n we have $f(n) \leq n+8$, then x is rational.

18th ASU 1984

Problem 1

Show that we can find n integers whose sum is 0 and whose product is n iff n is divisible by 4.

Problem 2

Show that $(a + b)^2/2 + (a + b)/4 \geq a\sqrt{b} + b\sqrt{a}$ for all positive a and b.

Answer

By AM/GM $\sqrt{(ab)} \leq (a+b)/2$, so $½(a+b) + \sqrt{(ab)} \leq a + b$. Hence $\sqrt{(2a+2b)} \geq \sqrt{a} + \sqrt{b}$ (*).

By AM/GM $(a + b) \geq 2\sqrt{(ab)}$ and $2(a+b) + 1 \geq 2\sqrt{(2a+2b)}$. Multiplying, $(a+b)(2a+2b+1) \geq 4\sqrt{(ab)}\sqrt{(2a+2b)}$. Then using (*) $\geq 4\sqrt{(ab)}(\sqrt{a} + \sqrt{b})$.

Problem 3

ABC and A'B'C' are equilateral triangles and ABC and A'B'C' have the same sense (both clockwise or both counter-clockwise). Take an arbitrary point O and points P, Q, R so that OP is equal and parallel to AA', OQ is equal and parallel to BB', and OR is equal and parallel to CC'. Show that PQR is equilateral.

Problem 4

Take a large number of unit squares, each with one edge red, one edge blue, one edge green, and one edge yellow. For which m, n can we combine mn squares by placing similarly colored edges together to get an m x n rectangle with one side entirely red, another entirely bue, another entirely green, and the fourth entirely yellow.

Problem 5

Let $A = \cos^2 a$, $B = \sin^2 a$. Show that for all real a and positive x, y we have $x^A y^B < x + y$.

Problem 6

Two players play a game. Each takes it in turn to paint three unpainted edges of a cube. The first player uses red paint and the second blue paint. So each player has two moves. The first player wins if he can paint all edges of some face red. Can the first player always win?

Problem 7

n > 3 positive integers are written in a circle. The sum of the two neighbours of each number divided by the number is an integer. Show that the sum of those integers is at least 2n and less than 3n. For example, if the numbers were 3, 7, 11, 15, 4, 1, 2 (with 2 also adjacent to 3), then the sum would be $14/7 + 22/11 + 15/15 + 16/4 + 6/1 + 4/2 + 9/3 = 20$ and $14 \leq 20 < 21$.

Problem 8

The incircle of the triangle ABC has center I and touches BC, CA, AB at D, E, F respectively. The segments AI, BI, CI intersect the circle at D', E', F' respectively. Show that DD', EE', FF' are collinear.

Problem 9

Find all integers m, n such that $(5 + 3\sqrt{2})^m = (3 + 5\sqrt{2})^n$.

Problem 10

$x_1 < x_2 < x_3 < ... < x_n$. y_i is a permutation of the x_i. We have that $x_1 + y_1 < x_2 + y_2 < ... < x_n + y_n$. Prove that $x_i = y_i$.

Problem 11

ABC is a triangle and P is any point. The lines PA, PB, PC cut the circumcircle of ABC again at A'B'C' respectively. Show that there are at most eight points P such that A'B'C' is congruent to ABC.

Problem 12

The positive reals x, y, z satisfy $x^2 + xy + y^2/3 = 25$, $y^2/3 + z^2 = 9$, $z^2 + zx + x^2 = 16$. Find the value of $xy + 2yz + 3zx$.

Problem 13

Starting with the polynomial $x^2 + 10x + 20$, a move is to change the coefficient of x by 1 or to change the coefficient of x^0 by 1 (but not both). After a series of moves the polynomial is changed to $x^2 + 20x + 10$. Is it true that at some intermediate point the polynomial had integer roots?

Answer

Yes.

Solution

We have $x^2 + (n+1)x + n = (x+n)(x+1)$, so $x^2 + ax + b$ has integer roots if $a = b+1$ (and a and b are integers). But initially a-b is -10 and it ends up as +10. Each move changes a-b by ±1, so it must pass through all values between -10 and +10.

Problem 14

The center of a coin radius r traces out a polygon with perimeter p which has an incircle radius R > r. What is the area of the figure traced out by the coin?

Problem 15

Each weight in a set of n has integral weight and the total weight of the set is 2n. A balance is initially empty. We then place the weights onto a pan of the balance one at a time. Each time we place the heaviest weight not yet placed. If the pans balance, then we place the weight onto the left pan. Otherwise, we place the weight onto the lighter pan. Show that when all the weights have been placed, the scales will balance. [For example, if the weights are 2, 2, 1, 1. Then we must place 2 in the left pan, followed by 2 in the right pan, followed by 1 in the left pan, followed by 1 in the right pan.]

Problem 16

A number is prime however we order its digits. Show that it cannot contain more than three different digits. For example, 337 satisfies the conditions because 337, 373 and 733 are all prime.

Problem 17

Find all pairs of digits (b, c) such that the number b ... b6c ... c4, where there are n bs and n cs is a square for all positive integers n.

Problem 18

A, B, C and D lie on a line in that order. Show that if X does not lie on the line then $|XA| + |XD| + | |AB| - |CD| | > |XB| + |XC|$.

Problem 19

The real sequence x_n is defined by $x_1 = 1$, $x_2 = 1$, $x_{n+2} = x_{n+1}^2 - x_n/2$. Show that the sequence converges and find the limit.

Problem 20

The squares of a 1983 x 1984 chess board are colored alternately black and white in the usual way. Each white square is given the number 1 or the number -1. For each black square the product of the numbers in the neighbouring white squares is 1. Show that all the numbers must be 1.

Problem 21

A 3 x 3 chess board is colored alternately black and white in the usual way with the center square white. Each white square is given the number 1 or the number -1. A move consists of simultaneously changing each number to the product of the adjacent numbers. So the four corner squares are each changed to the number previously in the center square and the center square is changed to the product of the four numbers in the corners. Show that after finitely many moves all numbers are 1.

Problem 22

Is ln 1.01 greater or less than 2/201?

Problem 23

C_1, C_2, C_3 are circles with radii r_1, r_2, r_3 respectively. The circles do not intersect and no circle lies inside any other circle. C_1 is larger than the other two. The two outer common tangents to C_1 and C_2 meet at A ("outer" means that the points where the tangent touches the two circles lie on the same side of the line of centers). The two outer common tangents to C_1 and C_3 intersect at B. The two tangents from A to C_3 and the two tangents from B to C_2 form a quadrangle. Show that it has an inscribed circle and find its radius.

Problem 24

Show that any cross-section of a cube through its center has area not less than the area of a face.

19th ASU 1985 problems

Problem 1

ABC is an acute angled triangle. The midpoints of BC, CA and AB are D, E, F respectively. Perpendiculars are drawn from D to AB and CA, from E to BC and AB, and from F to CA and BC. The perpendiculars form a hexagon. Show that its area is half the area of the triangle.

Problem 2

Is there an integer n such that the sum of the (decimal) digits of n is 1000 and the sum of the squares of the digits is 1000^2?

Problem 3

An 8 x 8 chess-board is colored in the usual way. What is the largest number of pieces can be placed on the black squares (at most one per square), so that each piece can be taken by at least one other? A piece A can take another piece B if they are (diagonally) adjacent and the square adjacent to B and opposite to A is empty.

Problem 4

Call a side or diagonal of a regular n-gon a *segment*. How many colors are required to paint all the segments of a regular n-gon, so that each segment has a single color and every two segments with a vertex in common have different colors.

Problem 5

Given a line L and a point O not on the line, can we move an arbitrary point X to O using only rotations about O and reflections in L?

Problem 6

The quadratic $p(x) = ax^2 + bx + c$ has $a > 100$. What is the maximum possible number of integer values x such that $|p(x)| < 50$?

Problem 7

In the diagram below a, b, c, d, e, f, g, h, i, j are distinct positive integers and each (except a, e, h and j) is the sum of the two numbers to the left and above. For example, b = a + e, f = e + h, i = h + j. What is the smallest possible value of d?

```
         j
      h  i
   e  f  g
a  b  c  d
```

Problem 8

$a_1 < a_2 < ... < a_n < ...$ is an unbounded sequence of positive reals. Show that there exists k such that $a_1/a_2 + a_2/a_3 + ... + a_h/a_{h+1} < h-1$ for all $h > k$. Show that we can also find a k such that $a_1/a_2 + a_2/a_3 + ... + a_h/a_{h+1} < h-1985$ for all $h > k$.

Problem 9

Find all pairs (x, y) such that $|\sin x - \sin y| + \sin x \sin y <= 0$.

Problem 10

ABCDE is a convex pentagon. A' is chosen so that B is the midpoint of AA', B' is chosen so that C is the midpoint of BB' and so on. Given A', B', C', D', E', how do we construct ABCDE using ruler and compasses?

Problem 11

The sequence $a_1, a_2, a_3, \dots$ satisfies $a_{4n+1} = 1$, $a_{4n+3} = 0$, $a_{2n} = a_n$. Show that it is not periodic.

Problem 12

n lines are drawn in the plane. Some of the resulting regions are colored black, no pair of painted regions have a boundary line in common (but they may have a common vertex). Show that at most $(n^2 + n)/3$ regions are black.

Problem 13

Each face of a cube is painted a different color. The same colors are used to paint every face of a cubical box a different color. Show that the cube can always be placed in the box, so that every face is a different color from the box face it is in contact with.

Problem 14

The points A, B, C, D, E, F are equally spaced on the circumference of a circle (in that order) and AF is a diameter. The center is O. OC and OD meet BE at M and N respectively. Show that MN + CD = OA.

Problem 15

A move replaces the real numbers a, b, c, d by a-b, b-c, c-d, d-a. If a, b, c, d are not all equal, show that at least one of the numbers can exceed 1985 after a finite number of moves.

Problem 16

$a_1 < a_2 < \dots < a_n$ and $b_1 > b_2 > \dots > b_n$. Taken together the a_i and b_i constitute the numbers 1, 2, ... , 2n. Show that $|a_1 - b_1| + |a_2 - b_2| + \dots + |a_n - b_n| = n^2$.

Problem 17

An r x s x t cuboid is divided into rst unit cubes. Three faces of the cuboid, having a common vertex, are colored. As a result exactly half the unit cubes have at least one face colored. What is the total number of unit cubes?

Problem 18

ABCD is a parallelogram. A circle through A and B has radius R. A circle through B and D has radius R and meets the first circle again at M. Show that the circumradius of AMD is R.

Problem 19

A regular hexagon is divided into 24 equilateral triangles by lines parallel to its sides. 19 different numbers are assigned to the 19 vertices. Show that at least 7 of the 24 triangles have the property that the numbers assigned to its vertices increase counterclockwise.

Problem 20

x is a real number. Define $x_0 = 1 + \sqrt{(1 + x)}$, $x_1 = 2 + x/x_0$, $x_2 = 2 + x/x_1$, ... , $x_{1985} = 2 + x/x_{1984}$. Find all solutions to $x_{1985} = x$.

Answer

3

Solution

If x = 0, then $x^{1985} = 2 \neq x$. Otherwise we find $x_1 = 2 + x/(1+\sqrt{(1+x)}) = 2 + (\sqrt{(1+x)} - 1) = 1 + \sqrt{(1+x)}$. Hence $x_{1985} = 1 + \sqrt{(1+x)}$. So $x - 1 = \sqrt{(1+x)}$. Squaring, x = 0 or 3. We have already ruled out x = 0. It is easy to check that x = 3 is a solution.

Problem 20

x is a real number. Define $x_0 = 1 + \sqrt{(1 + x)}$, $x_1 = 2 + x/x_0$, $x_2 = 2 + x/x_1$, ... , $x_{1985} = 2 + x/x_{1984}$. Find all solutions to $x_{1985} = x$.

Problem 21

A regular pentagon has side 1. All points whose distance from every vertex is less than 1 are deleted. Find the area remaining.

Problem 22

Given a large sheet of squared paper, show that for n > 12 you can cut along the grid lines to get a rectangle of more than n unit squares such that it is impossible to cut it along the grid lines to get a rectangle of n unit squares from it.

Problem 23

The cube ABCDA'B'C'D' has unit edges. Find the distance between the circle circumscribed about the base ABCD and the circumcircle of AB'C.

20th ASU 1986

Problem 1

The quadratic $x^2 + ax + b + 1$ has roots which are positive integers. Show that $(a^2 + b^2)$ is composite.

Solution

Let the roots be c, d, so $c + d = -a$, $cd = b+1$. Hence $a^2 + b^2 = (c^2 + 1)(d^2 + 1)$.

Problem 2

Two equal squares, one with blue sides and one with red sides, intersect to give an octagon with sides alternately red and blue. Show that the sum of the octagon's red side lengths equals the sum of its blue side lengths.

Problem 3

ABC is acute-angled. What point P on the segment BC gives the minimal area for the intersection of the circumcircles of ABP and ACP?

Problem 4

Given n points can one build n-1 roads, so that each road joins two points, the shortest distance between any two points along the roads belongs to $\{1, 2, 3, ... , n(n-1)/2 \}$, and given any element of $\{1, 2, 3, ... , n(n-1)/2 \}$ one can find two points such that the shortest distance between them along the roads is that element?

Problem 5

Prove that there is no convex quadrilateral with vertices at lattice points so that one diagonal has twice the length of the other and the angle between them is 45 degrees.

Problem 6

Prove that we can find an m x n array of squares so that the sum of each row and the sum of each column is also a square.

Problem 7

Two circles intersect at P and Q. A is a point on one of the circles. The lines AP and AQ meet the other circle at B and C respectively. Show that the circumradius of ABC equals the distance between the centers of the two circles. Find the locus of the circumcircle as A varies.

Problem 8

A regular hexagon has side 1000. Each side is divided into 1000 equal parts. Let S be the set of the vertices and all the subdividing points. All possible lines parallel to the sides and with endpoints in S are drawn, so that the hexagon is divided into equilateral triangles with side 1.

Let X be the set of all vertices of these triangles. We now paint any three unpainted members of X which form an equilateral triangle (of any size). We then repeat until every member of X except one is painted. Show that the unpainted vertex is not a vertex of the original hexagon.

Problem 9

Let d(n) be the number of (positive integral) divisors of n. For example, d(12) = 6. Find all n such that $n = d(n)^2$.

Problem 10

Show that for all positive reals x_i we have $1/x_1 + 1/(x_1 + x_2) + ... + n/(x_1 + ... + x_n) < 4/a_1 + 4/a_2 + ... + 4/a_n$.

Problem 11

ABC is a triangle with AB ≠ AC. Show that for each line through A, there is at most one point X on the line (excluding A, B, C) with ∠ABX = ∠ACX. Which lines contain no such points X?

Problem 12

An n x n x n cube is divided into n^3 unit cubes. Show that we can assign a different integer to each unit cube so that the sum of each of the $3n^2$ rows parallel to an edge is zero.

Problem 13

Find all positive integers a, b, c so that $a^2 + b = c$ and a has n > 1 decimal digits all the same, b has n decimal digits all the same, and c has 2n decimal digits all the same.

Problem 14

Two points A and B are inside a convex 12-gon. Show that if the sum of the distances from A to each vertex is a and the sum of the distances from B to each vertex is b, then |a - b| < 10 |AB|.

Problem 15

There are 30 cups each containing milk. An elf is able to transfer milk from one cup to another so that the amount of milk in the two cups is equalised. Is there an initial distribution of milk so that the elf cannot equalise the amount in all the cups by a finite number of such transfers?

Problem 16

A 99 x 100 chess board is colored in the usual way with alternate squares black and white. What fraction of the main diagonal is black? What if the board is 99 x 101?

Problem 17

$A_1A_2 \ldots A_n$ is a regular n-gon and P is an arbitrary point in the plane. Show that if n is even we can choose signs so that the vector sum $\pm PA_1 \pm PA_2 \pm \ldots \pm PA_n = 0$, but if n is odd, then this is only possible for finitely many points P.

Problem 18

A 1 or a -1 is put into each cell of an n x n array as follows. A -1 is put into each of the cells around the perimeter. An unoccupied cell is then chosen arbitrarily. It is given the product of the four cells which are closest to it in each of the four directions. For example, if the cells below containing a number or letter (except x) are filled and we decide to fill x next, then x gets the product of a, b, c and d.

```
-1  -1  -1  -1  -1
-1   a   1      -1
 c   x       d  -1
-1
-1   b  -1  -1  -1
```

What is the minimum and maximum number of 1s that can be obtained?

Problem 19

Prove that $|\sin 1| + |\sin 2| + \ldots + |\sin 3n| > 8n/5$.

Problem 20

Let S be the set of all numbers which can be written as 1/mn, where m and n are positive integers not exceeding 1986. Show that the sum of the elements of S is not an integer.

Problem 21

The incircle of a triangle has radius 1. It also lies inside a square and touches each side of the square. Show that the area inside both the square and the triangle is at least 3.4. Is it at least 3.5?

Problem 22

How many polynomials p(x) have all coefficients 0, 1, 2 or 3 and take the value n at x = 2?

Problem 23

A and B are fixed points outside a sphere S. X and Y are chosen so that S is inscribed in the tetrahedron ABXY. Show that the sum of the angles AXB, XBY, BYA and YAX is independent of X and Y.

21st ASU 1987 problems

Problem 1

Ten players play in a tournament. Each pair plays one match, which results in a win or loss. If the ith player wins a_i matches and loses b_i matches, show that $\sum a_i^2 = \sum b_i^2$.

Problem 2

Find all sets of 6 weights such that for each of n = 1, 2, 3, ... , 63, there is a subset of weights weighing n.

Problem 3

ABCDEFG is a regular 7-gon. Prove that 1/AB = 1/AC + 1/AD.

Problem 4

Your opponent has chosen a 1 x 4 rectangle on a 7 x 7 board. At each move you are allowed to ask whether a particular square of the board belongs to his rectangle. How many questions do you need to be certain of identifying the rectangle. How many questions are needed for a 2 x 2 rectangle?

Problem 5

Prove that $1^{1987} + 2^{1987} + ... + n^{1987}$ is divisible by n+2.

Problem 6

An L is an arrangement of 3 adjacent unit squares formed by deleting one unit square from a 2 x 2 square. How many Ls can be placed on an 8 x 8 board (with no interior points overlapping)? Show that if any one square is deleted from a 1987 x 1987 board, then the remaining squares can be covered with Ls (with no interior points overlapping).

Problem 7

Squares ABC'C", BCA'A", CAB'B" are constructed on the outside of the sides of the triangle ABC. The line A'A" meets the lines AB and AC at P and P'. Similarly, the line B'B" meets the lines BC and BA at Q and Q', and the line C'C" meets the lines CA and CB at R and R'. Show that P, P', Q, Q', R and R' lie on a circle.

Problem 8

$A_1, A_2, ... , A_{2m+1}$ and $B_1, B_2, ... , B_{2n+1}$ are points in the plane such that the 2m+2n+2 lines $A_1A_2, A_2A_3, ... , A_{2m}A_{2m+1}, A_{2m+1}A_1, B_1B_2, B_2B_3, ... , B_{2n}B_{2n+1}, B_{2n+1}B_1$ are all different and no three of them are concurrent. Show that we can find i and j such that A_iA_{i+1}, B_jB_{j+1} are opposite sides of a convex quadrilateral (if i = 2m+1, then we take A_{i+1} to be A_1. Similarly for j = 2n+1).

Problem 9

Find 5 different relatively prime numbers, so that the sum of any subset of them is composite.

Problem 10

ABCDE is a convex pentagon with ∠ABC = ∠ADE and ∠AEC = ∠ADB. Show that ∠BAC = ∠DAE.

Problem 11

Show that there is a real number x such that all of cos x, cos 2x, cos 4x, ... $\cos(2^n x)$ are negative.

Problem 12

The positive reals a, b, c, x, y, z satisfy a + x = b + y = c + z = k. Show that ax + by + cz ≤ k^2.

Problem 13

A real number with absolute value at most 1 is put in each square of a 1987 x 1987 board. The sum of the numbers in each 2 x 2 square is 0. Show that the sum of all the numbers does not exceed 1987.

Problem 14

AB is a chord of the circle center O. P is a point outside the circle and C is a point on the chord. The angle bisector of APC is perpendicular to AB and a distance d from O. Show that BC = 2d.

Problem 15

Players take turns in choosing numbers from the set {1, 2, 3, ... , n}. Once m has been chosen, no divisor of m may be chosen. The first player unable to choose a number loses. Who has a winning strategy for n = 10? For n = 1000?

Problem 16

What is the smallest number of subsets of S = {1, 2, ... , 33}, such that each subset has size 9 or 10 and each member of S belongs to the same number of subsets?

Problem 17

Some lattice points in the plane are marked. S is a set of non-zero vectors. If you take any one of the marked points P and add place each vector in S with its beginning at P, then more vectors will have their ends on marked points than not. Show that there are an infinite number of points.

Problem 18

A convex pentagon is cut along all its diagonals to give 11 pieces. Show that the pieces cannot all have equal areas.

Problem 19

The set $S_0 = \{1, 2!, 4!, 8!, 16!, ... \}$. The set S_{n+1} consists of all finite sums of distinct elements of S_n. Show that there is a positive integer not in S_{1987}.

Problem 20

If the graph of the function $f = f(x)$ is rotated through 90 degrees about the origin, then it is not changed. Show that there is a unique solution to $f(b) = b$. Give an example of such a function.

Problem 21

A convex polyhedron has all its faces triangles. Show that it is possible to color some edges red and the others blue so that given any two vertices one can always find a path between them along the red edges and another path between them along the blue edges.

Problem 22

Show that $(2n+1)^n \geq (2n)^n + (2n-1)^n$ for every positive integer n.

22nd ASU 1988

Problem 1

A book contains 30 stories. Each story has a different number of pages under 31. The first story starts on page 1 and each story starts on a new page. What is the largest possible number of stories that can begin on odd page numbers?

Solution

Answer: 23.

Call stories with an odd number of pages *odd* stories and stories with an even number of pages *even* stories. There are 15 odd stories and 15 even stories. The odd stories change the parity of the starting page (in the sense that the following story starts on a page of opposite parity), whereas the even stories do not. So the odd stories must start alternately on odd and even pages. Hence 8 of them must start on odd pages and 7 on even pages (irrespective of how the stories are arranged). We can, however, control the even stories. In particular, if we put each of them after an even number of odd stories, then they will all begin on odd pages. For example, we could put them all first (before any of the odd stories).

Problem 2

ABCD is a convex quadrilateral. The midpoints of the diagonals and the midpoints of AB and CD form another convex quadrilateral Q. The midpoints of the diagonals and the midpoints of BC and CA form a third convex quadrilateral Q'. The areas of Q and Q' are equal. Show that either AC or BD divides ABCD into two parts of equal area.

Solution

Note that Q is a parallelogram because each side is formed by joining the midpoints of two sides of a triangle, so it is parallel to and half the length of the base of the triangle. But the triangles corresponding to opposite sides have the same base. Hence opposite sides of Q are parallel and equal. Similarly Q'.

Let the midpoints of the diagonals be X, Y. Take two adjacent side midpoints which are on the same side of the line XY. Suppose they are M, the midpoint of AB, and N, the midpoint of BC. Suppose also that X is the midpoint of BD, and Y the midpoint of AC. If X does not lie on AC, then we may assume it lies on the same side of AC as M and N (if not just consider the other two midpoints instead of M and N). So the line parallel to XY through M cuts the altitude from N of NXY. So XYM has the same base XY as XYN, but smaller height, so it has smaller area. Hence the two parallelograms also have different areas. Contradiction. So X must lie on AC. But AX bisects ABD and CX bisects CBD, so AC bisects ABD and CBD and hence ABCD.

Problem 3

Show that there are infinitely many triples of distinct positive integers a, b, c such that each divides the product of the other two and a + b = c + 1.

Solution

$n(n + 1), n(n^2 + n - 1), (n + 1)(n^2 + n - 1)$.

Problem 4

Given a sequence of 19 positive integers not exceeding 88 and another sequence of 88 positive integers not exceeding 19. Show that we can find two subsequences of consecutive terms, one from each sequence, with the same sum.

Solution

We prove the general case. Let the first sequence be $a_1, a_2, \dots, a_m$ and the second sequence be $b_1, b_2, \dots, b_n$, were $0 < a_i \le n$ and $0 < b_j \le m$. Put $s_k = a_1 + a_2 + \dots + a_k$, $t_k = b_1 + b_2 + \dots + b_k$. Assume $s_m > t_n$ (if they are equal, then we are done).

Let f(i) be the smallest k such that $s_k >= t_i$. If it is equal, we are done, so assume $s_k > t_i$. Now consider the n numbers $s_{f(i)} - t_i$. Each is at least 1 and at most n-1 (if it was n or more then $s_{f(i)-1} \ge s_{f(i)} - n \ge t_i$, contradicting the minimality of f(i)). So there must be two the same. So we have $s_{f(i)} - t_i = s_{f(j)} - t_j$ for some $i > j$ and hence $a_{f(j)+1} + a_{f(j)+2} + \dots + a_{f(i)} = b_{j+1} + b_{j+2} + \dots + b_i$.

Problem 5

The quadrilateral ABCD is inscribed in a fixed circle. It has AB parallel to CD and the length AC is fixed, but it is otherwise allowed to vary. If h is the distance between the midpoints of AC and BD and k is the distance between the midpoints of AB and CD, show that the ratio h/k remains constant.

Solution

Let the center of the circle be O and its radius be R. Let $\angle AOB = 2x$ (variable) and let $\angle AOC = 2y$ (fixed). Then $AC = 2R \sin y$. We find $\angle COD = 180^o - x - 2y$, so $2h = 2R \sin x + 2R \sin(x + 2y)$. Angle $ACD = x + y$, so $k = AB \sin(x + y) = 2R \sin(x + y) \sin y$. Hence the ratio $h/k = (\sin x + \sin(x + 2y))/(2 \sin(x + y) \sin y)$. We have $\sin x = \sin(x + y - y) = \sin(x + y) \cos y - \cos(x + y) \sin y$, and $\sin(x + 2y) = \sin(x + y + y) = \sin(x + y) \cos y + \cos(x + y) \sin y$, so $(\sin x + \sin(x + 2y) = 2 \sin(x + y) \cos y$. Hence $h/k = \cot y$, which is constant.

Problem 6

The numbers 1 and 2 are written on an empty blackboard. Whenever the numbers m and n appear on the blackboard the number m + n + mn may be written. Can we obtain (1) 13121, (2) 12131?

Solution

(1) $13121 = 2 + 4373 + 2\cdot 4373$, $4373 = 2 + 1457 + 2\cdot 1457$, $1457 = 2 + 485 + 2\cdot 485$, $485 = 2 + 161 + 2\cdot 161$, $161 = 2 + 53 + 2\cdot 53$, $53 = 2 + 17 + 2\cdot 17$, $17 = 2 + 5 + 2\cdot 5$, $5 = 2 + 1 + 2\cdot 1$.

Put M = m+1, N = n+1. Then we the number derived from m and n is MN - 1. So if M and N are of the form 2^a3^b then so is MN. Thus we can only ever write up numbers of the form $2^a3^b - 1$. But $12131 = 2^2 3^2 337 - 1$, which is not of the required form.

Note that it makes no difference whether the two numbers m, n are allowed to be the same (which is ambiguous).

Problem 7

If rationals x, y satisfy $x^5 + y^5 = 2 x^2 y^2$ show that 1 - x y is the square of a rational.

Solution

Put y = kx, then $x^5(1 + k^5) = 2k^2x^4$, so $x = 2k^2/(1 + k^5)$, $y = 2k^3/(1 + k^5)$ and $1 - xy = (1 - k^5)^2/(1 + k^5)^2$. x and y are rational, so $(1 - k^5)/(1 + k^5)$ is rational.

Problem 8

There are 21 towns. Each airline runs direct flights between every pair of towns in a group of five. What is the minimum number of airlines needed to ensure that at least one airline runs direct flights between every pair of towns?

Solution

Answer: 21.

There are 210 pairs of towns. Each airline serves 10 pairs, so we certainly need at least 21 airlines. The following arrangement shows that 21 is possible:

```
1   2   3   4   5
1   6   7   8   9
1  10  11  12  13
1  14  15  16  17
1  18  19  20  21
2   6  10  14  18
2   7  11  15  19
2   8  12  16  20
2   9  13  17  21
3   6  11  16  21
3   7  10  17  20
3   8  13  14  19
3   9  12  15  18
4   6  12  17  19
4   7  13  16  18
4   8  10  15  21
4   9  11  14  20
5   6  13  15  20
5   7  12  14  21
5   8  11  17  18
5   9  10  16  19
```

Problem 9

Find all positive integers n satisfying $(1 + 1/n)^{n+1} = (1 + 1/1998)^{1998}$.

Solution

Answer: no solutions.

We have $(1 + 1/n)^{n+1} > e > (1 + 1/n)^n$.

Problem 10

A, B, C are the angles of a triangle. Show that $2(\sin A)/A + 2(\sin B)/B + 2(\sin C)/C \le (1/B + 1/C) \sin A + (1/C + 1/A) \sin B + (1/A + 1/B) \sin C$.

Solution

Assume $A \le B \le C$. Then $\sin A \le \sin B$. Also $A \le C < 180^o - A$, so $\sin A \le \sin C$. Similarly $\sin B \le \sin C$. Hence $(1/A - 1/B)(\sin B - \sin A)$, $(1/B - 1/C)(\sin C - \sin B)$ and $(1/C - 1/A)(\sin A - \sin C)$ are all non-negative. Hence their sum is also non-negative, which gives the result.

Problem 11

Form 10A has 29 students who are listed in order on its duty roster. Form 10B has 32 students who are listed in order on its duty roster. Every day two students are on duty, one from form 10A and one from form 10B. Each day just one of the students on duty changes and is replaced by the following student on the relevant roster (when the last student on a roster is replaced he is replaced by the first). On two particular days the same two students were on duty. Is it possible that starting on the first of these days and ending the day before the second, every pair of students (one from 10A and one from 10B) shared duty exactly once?

Solution

Answer: no.

Suppose such an arrangement is possible. Suppose that it includes m cycles through the form 10A roster and n cycles through the 10B roster. Then the total number of changes is $29m + 32n = 29 \times 32$ (since each pair occurs once). But that means 29 divides n and 32 divides m. Both m and n are at least 1, so that means $n \ge 29$ and $m \ge 32$, but then $29m + 32n > 29 \cdot 32$. Contradiction.

Problem 12

In the triangle ABC, the angle C is obtuse and D is a fixed point on the side BC, different from B and C. For any point M on the side BC, different from D, the ray AM intersects the circumcircle S of ABC at N. The circle through M, D and N meets S again at P, different from N. Find the location of the point M which minimises MP.

Solution

Take A' on the circle S such that AA' is parallel to BC. Let the ray AD meet S again at P'. Then ∠MNP' = ∠ANP' (same angle) = ∠AA'P' (A'AP'N cyclic) = ∠A'DB (BC parallel to AA') = angle MDP (opposite angles). So MDNP' is cyclic, so P must be P'. Since P is a fixed point, independent of M, we minimise MP by taking M as the foot of the perpendicular from P to BC.

Problem 13

Show that there are infinitely many odd composite numbers in the sequence 1^1, $1^1 + 2^2$, $1^1 + 2^2 + 3^3$, $1^1 + 2^2 + 3^3 + 4^4$, ...

Solution

We show that infinitely many odd numbers in the sequence are divisible by 3.

If n = 14 mod 36, then n = 36m + 14 for some m. So there are 18m + 7 odd numbers in the sum and 18m + 7 even numbers. Hence the sum is odd. There are 12m + 5 numbers equal to 1 mod 3, 12m + 5 equal to -1 mod 3 and 12m + 4 equal to 0 mod 3. Any product of numbers equal to 1 mod 3 equals 1 mod 3, so if k = 1 mod 3, then k^k = 1 mod 3. Similarly, if k = 0 mod 3, then k^k = 0 mod 3. If k = -1 mod 3 then k^k = -1 mod 3 if k is odd and 1 mod if k is even. Of the 12m + 5 numbers equal to -1 mod 3, 6m + 3 are even and 6m + 2 are odd. Hence the sum = (12m+5)· 1 + (6m+3)· 1 + (6m+2)· -1 + (12m+4)· 0 = 12m+6 = 0 mod 3. So the sum is divisible by 3.

Problem 14

ABC is an acute-angled triangle. The tangents to the circumcircle at A and C meet the tangent at B at M and N. The altitude from B meets AC at P. Show that BP bisects the angle MPN.

Solution

If the tangent at B is parallel to AC, then angle NBC = angle BCA (parallel lines) and angle NBC = angle BAC (NB tangent), so BCA is isosceles and BC = BA. Hence the figure is symmetrical about the line PB and so BP bisects MPN.

So assume AC is not parallel to the tangent at B. Assume it meets it at L on the same side of B as N. Take A' on the line AC so that MA = MA'. We show that LMA' and LNC are similar. Obviously the angles at L are the same. ∠MA'L = ∠MAL (MA = MA') = ∠ABC (MA tangent) = ∠LCN (NC tangent). So the triangles are similar. Hence LN/NC = LM/MA'. But NC = NB and MA' = MA = MB, so LN/NB = LM/MB and hence LM/LN = MB/NB. So the circle on LB as diameter has all points Q on it satisfying QM/QN = BM/BN. But ∠LPB = 90^o, so P must lie on the circle and hence PM/PN = BM/BN. Hence PB is the angle bisector of MPN.

Problem 15

What is the minimal value of b/(c + d) + c/(a + b) for positive real numbers b and c and non-negative real numbers a and d such that b + c ≥ a + d?

Solution

Answer: √2 - 1/2.

Obviously a + d = b + c at the minimum value, because increasing a or d reduces the value. So we may take d = b + c - a. We also take b >= c (interchanging b and c if necessary). Dividing through by b/2 shows that there is no loss of generality in taking b = 2, so 0 < c <= 2. Thus we have to find the minimum value of 2/(2c - a + 2) + c/(a + 2). We show that it is √2 - 1/2.

This is surprisingly awkward. Note first that $(c - (h - k))^2 >= 0$, so $c^2 + c(2k - 2h) + h^2 - 2hk + k^2 >= 0$. Hence $c^2 + ck + h^2 >= (2h - k)(c + k)$. Hence $c/h^2 + 1/(c+k) >= 2/h - k/h^2$ with equality iff $c = h - k$. Applying this to $c/(a+2) + 1/(c+1-a/2)$ where $h = \sqrt{(a+2)}$, $k = 1-a/2$, we find that $c/(a+2) + 2/(2c+2-a) >= 2/\sqrt{(a+2)} + (a-2)/(2a+4)$.

The allowed range for c is $0 \le c \le 2$ and $0 \le a \le c+2$, hence $0 \le a \le 4$. Put $x = 1/\sqrt{(a+2)}$, so $1/\sqrt{6} <= x <= 1/\sqrt{2}$. Then $2/\sqrt{(a+2)} + (a-2)/(2a+4) = 2x + 1/2 - 2x^2 = 1 - (2x-1)^2/2$. We have $-0.184 = (2/\sqrt{6} - 1) \le 2x-1 \le \sqrt{2} - 1 = 0.414$. Hence $c/(a+2) + 2/(2c+2-a) \ge 1 - (\sqrt{2} - 1)^2/2 = \sqrt{2} - 1/2$.

We can easily check that the minimum is achieved at $b = 2$, $c = \sqrt{2} - 1$, $a = 0$, $d = \sqrt{2} + 1$.

Problem 16

n^2 real numbers are written in a square n x n table so that the sum of the numbers in each row and column equals zero. A move is to add a row to one column and subtract it from another (so if the entries are a_{ij} and we select row i, column h and column k, then column h becomes $a_{1h} + a_{i1}, a_{2h} + a_{i2}, \dots , a_{nh} + a_{in}$, column k becomes $a_{1k} - a_{i1}, a_{2k} - a_{i2}, \dots , a_{nk} - a_{in}$, and the other entries are unchanged). Show that we can make all the entries zero by a series of moves.

Problem 17

In the acute-angled triangle ABC, the altitudes BD and CE are drawn. Let F and G be the points of the line ED such that BF and CG are perpendicular to ED. Prove that EF = DG.

Solution

$\angle BDC = \angle BEC = 90^o$, so BCDE is cyclic, so $\angle BDE = \angle BCE = 90^o - \angle B$. Hence $\angle DCG = 90^o - \angle CDG = \angle BDE = 90^o - B$. So DG = CD sin DCG = BC sin CBD sin DCG = BC cos C cos B. Similarly EF.

Problem 18

Find the minimum value of $xy/z + yz/x + zx/y$ for positive reals x, y, z with $x^2 + y^2 + z^2 = 1$.

Solution

Answer: min $\sqrt{3}$ when all equal.

Let us consider z to be fixed and focus on x and y. Put $f(x, y, z) = xy/z + yz/x + zx/y$. We have $f(x, y, z) = p/z + z(1-z^2)/p = (p + k^2/p)/z$, where $p = xy$, and $k = z\sqrt{(1-z^2)}$. Now p can take any value in the range $0 < p \le (1-z^2)/2$. The upper limit is achieved when $x = y$.

We have $p + k^2/p = (p - k)^2/p$. For $p \le k$, $(p - k)$ and $1/p$ are both decreasing functions of p, so $p + k^2/p$ is a decreasing function of p. Thus if p is restricted to the interval (0, h], then for $k \le h$ the minimum value of $p + k^2/p$ is 2k and occurs at $p = k$. For $k \ge h$ the minimum is $h + k^2/h$ and occurs at $p = h$.

We have $h = (1-z^2)/2$, $k = z\sqrt{(1-z^2)}$. So $k \le h$ iff $z <= 1/\sqrt{5}$. So if $z \le 1/\sqrt{5}$, then $f(x, y, z) \ge 2k/z = 2\sqrt{(1-z^2)} \ge 27\sqrt{(1-1/5)} = 4/\sqrt{5} > \sqrt{3}$.

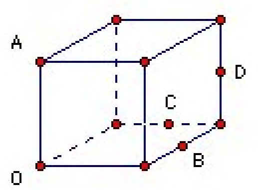
If $z > 1/\sqrt{5}$, then the minimum of $f(x, y, z)$ occurs at $x = y$ and is $x^2/z + z + z = (1-z^2)/(2z) + 2z = 3z/2 + 1/(2z) = (\sqrt{3})/2\ (z\sqrt{3} + 1/(z\sqrt{3})) \geq \sqrt{3}$ with equality at $z = 1/\sqrt{3}$ (and hence $x = y = 1/\sqrt{3}$ also).

Problem 19

A polygonal line connects two opposite vertices of a cube with side 2. Each segment of the line has length 3 and each vertex lies on the faces (or edges) of the cube. What is the smallest number of segments the line can have?

Answer 6

Solution



Suppose one endpoint of a segment length 3 is at A. Evidently the other end could be at the edge midpoints B, C, D. It could also be on the circular arc connecting B and C (with center O and radius $\sqrt{5}$). Similarly, it could be on arcs connecting C and D, or B and D. We claim that if X is a point of one of these arcs other than its endpoints, then the only possible segment length 3 with an endpoint at X (and the other endpoint on the surface of the cube) is AX. wlog we can consider X to be on the arc BC. Take axes with origin O, so that A is (0,0,2). Suppose X is (a,b,0) and that the other endpoint of the segment is Y (x,y,z). Then $XY^2 = (x-a)^2 + (y-b)^2 + z^2 = a^2 + b^2 + z^2 - x(2a-x) - y(2b-y) = 5 + z^2 - x(2a-x) - y(2b-y)$. But $a, b > 1$ since X is not an endpoint of the arc, so $(2a-x)$ and $(2b-y)$ are both positive. Hence $-x(2a-x) - y(2b-y) \leq 0$ with equality iff $x = y = 0$. Similarly, $z^2 \leq 4$ with equality iff $z = 2$. Hence $XY^2 \leq 9$ with equality iff Y = A, which proves the claim.

Thus if the next link of the polygonal line goes from A to anywhere except B, C, D, then it has to go back to A. So a minimal line must go to B, C, or D.

Now from D the line can only go to A or O. For if it goes to Z (x,y,z), then we have $DZ^2 = (x-2)^2 + (y-2)^2 + (z-1)^2 \leq 2^2 + 2^2 + 1^2 = 3^2$ with equality iff $x = 0$, $y = 0$ and $z = 0$ or 2.

So let us take A as the starting point of the polygonal line. wlog the first segment is AD. Then the second segment must be DO (for a minimal line). Thus the best we can do with 2 segments is to move along an edge. It takes three such moves to get to the opposite corner, and hence at least 6 segments. But it is obvious that it can be done with 6 segments.

Problem 20

Let m, n, k be positive integers with $m \geq n$ and $1 + 2 + ... + n = mk$. Prove that the numbers 1, 2, ... , n can be divided into k groups in such a way that the sum of the numbers in each group equals m.

Solution

Induction on n, then m. For n = 1, 2 there is nothing to prove. Assume the result is proved for < n and consider the case n. If n is odd, we have n = n-1 + 1 = n-2 + 2 = ... = (n+1)/2 + (n-1)/2, so the result is true for m = n, k = (n+1)/2. If n is even, we have n + 1 = n-1 + 2 = ... = (n/2 + 1) + (n/2 - 1), so the result is true for m = n+1 and k = n/2. Now suppose it is true for < m.

If 2n > m > n+1, then for m odd we can take the sums m = n + m-n = n-1 + m-n+1 = ... = (m+1)/2 + (m-1)/2. These use up the numbers m-n, m-n+1, ... , n and give some sums of m. By induction the remaining numbers 1, 2, ... , m-n-1 will give the remaining sums of m (obviously m > m-n-1). If m is even, we can take the sums m = n + m-n = n-1 + m-n+1 = ... = (m/2 + 1) + (m/2 - 1). That gives some sums of m and leaves us with the integers 1, 2, ... , m-n-1 and m/2. But since m < 2(n+1), m/2 > m-n-1 and hence we can use the integers 1, 2, ... , m-n-1 to form sums of m/2. With the integer m/2 that gives us sums of m (we know that the parity must come out right because we know that the sum of all the remaining numbers is divisible by m).

Finally, consider m ≥ 2n. In that case we can form k sums of 2n-2k+1: n + (n-2k+1), (n-1) + (n-2k+2), ... , (n-k+1) + (n-k). So we are home provided the remaining integers 1, 2, ... , n-2k can be used to form k sums of m-(2n-2k+1). That follows by induction provided that m-(2n-2k+1) ≥ n-2k, or m+4k-1 ≥ 3n, or m + 2n(n+1)/m - 1 ≥ 3n or (m-2n)(m-n-1) ≥ 0, which is true.

Problem 21
A polygonal line with a finite number of segments has all its vertices on a parabola. Any two adjacent segments make equal angles with the tangent to the parabola at their point of intersection. One end of the polygonal line is also on the axis of the parabola. Show that the other vertices of the polygonal line are all on the same side of the axis.

Problem 22
What is the smallest n for which there is a solution to $\sin x_1 + \sin x_2 + ... + \sin x_n = 0$, $\sin x_1 + 2 \sin x_2 + ... + n \sin x_n = 100$?

Solution

Put $x_1 = x_2 = ... = x_{10} = 3\pi/2$, $x_{11} = x_{12} = ... = x_{20} = \pi/2$. Then $\sin x_1 + \sin x_2 + ... + \sin x_{20} = (-1 - 1 - 1 - ... - 1) + (1 + 1 + ... + 1) = 0$, and $\sin x_1 + 2 \sin x_2 + ... + 20 \sin x_{20} = - (1 + 2 + ... + 10) + (11 + 12 + ... + 20) = 100$. So there is a solution with n = 20. If there is a solution with n < 20, then there must be a solution for n = 19 (put any extra $x_i = 0$). But then $100 = (\sin x_1 + 2 \sin x_2 + ... + 19 \sin x_{19}) - 10 (\sin x_1 + \sin x_2 + ... + \sin x_{19}) = -9 \sin x_1 - 8 \sin x_2 - 7 \sin x_3 - ... - \sin x_9 + \sin x_{11} + 2 \sin x_{12} + ... + 9 \sin x_{19}$. But $|\text{rhs}| \le (9 + 8 + ... + 1) + (1 + 2 + ... + 9) = 90$. Contradiction. So there is no solution for n < 20.

Problem 23
The sequence of integers a_n is given by $a_0 = 0$, $a_n = p(a_{n-1})$, where p(x) is a polynomial whose coefficients are all positive integers. Show that for any two positive integers m, k with greatest common divisor d, the greatest common divisor of a_m and a_k is a_d.

Problem 24
Prove that for any tetrahedron the radius of the inscribed sphere r < ab/(2(a + b)), where a and b are the lengths of any pair of opposite edges.

23rd ASU 1989

Problem 1

7 boys each went to a shop 3 times. Each pair met at the shop. Show that 3 must have been in the shop at the same time.

Problem 2

Can 77 blocks each 3 x 3 x 1 be assembled to form a 7 x 9 x 11 block?

Problem 3

The incircle of ABC touches AB at M. N is any point on the segment BC. Show that the incircles of AMN, BMN, ACN have a common tangent.

Problem 4

A positive integer n has exactly 12 positive divisors $1 = d_1 < d_2 < d_3 < ... < d_{12} = n$. Let $m = d_4 - 1$. We have $d_m = (d_1 + d_2 + d_4) d_8$. Find n.

Problem 5

Eight pawns are placed on a chessboard, so that there is one in each row and column. Show that an even number of the pawns are on black squares.

Problem 6

ABC is a triangle. A', B', C' are points on the segments BC, CA, AB respectively. Angle B'A'C' = angle A and AC'/C'B = BA'/A'C = CB'/B'A. Show that ABC and A'B'C' are similar.

Problem 7

One bird lives in each of n bird-nests in a forest. The birds change nests, so that after the change there is again one bird in each nest. Also for any birds A, B, C, D (not necessarily distinct), if the distance AB < CD before the change, then AB > CD after the change. Find all possible values of n.

Problem 8

Show that the 120 five digit numbers which are permutations of 12345 can be divided into two sets with each set having the same sum of squares.

Problem 9

We are given 1998 normal coins, 1 heavy coin and 1 light coin, which all look the same. We wish to determine whether the average weight of the two abnormal coins is less than, equal to, or greater than the weight of a normal coin. Show how to do this using a balance 4 times or less.

Problem 10

A triangle with perimeter 1 has side lengths a, b, c. Show that $a^2 + b^2 + c^2 + 4abc < 1/2$.

Problem 11

ABCD is a convex quadrilateral. X lies on the segment AB with AX/XB = m/n. Y lies on the segment CD with CY/YD = m/n. AY and DX intersect at P, and BY and CX intersect at Q. Show that area XQYP/area ABCD < $mn/(m^2 + mn + n^2)$.

Problem 12

A 23 x 23 square is tiled with 1 x 1, 2 x 2 and 3 x 3 squares. What is the smallest possible number of 1 x 1 squares?

Problem 13

Do there exist two reals whose sum is rational, but the sum of their nth powers is irrational for all n > 1? Do there exist two reals whose sum is irrational, but the sum of whose nth powers is rational for all n > 1?

Problem 14

An insect is on a square ceiling side 1. The insect can jump to the midpoint of the segment joining it to any of the four corners of the ceiling. Show that in 8 jumps it can get to within 1/100 of any chosen point on the ceiling.

Problem 15

ABCD has AB = CD, but AB not parallel to CD, and AD parallel to BC. The triangle is ABC is rotated about C to A'B'C. Show that the midpoints of BC, B'C and A'D are collinear.

Problem 16

Show that for each integer n > 0, there is a polygon with vertices at lattice points and all sides parallel to the axes, which can be dissected into 1 x 2 (and/or 2 x 1) rectangles in exactly n ways.

Problem 17

Find the smallest positive integer n for which we can find an integer m such that $[10^n/m] = 1989$.

Problem 18

ABC is a triangle. Points D, E, F are chosen on BC, CA, AB such that B is equidistant from D and F, and C is equidistant from D and E. Show that the circumcenter of AEF lies on the bisector of EDF.

Problem 19

S and S' are two intersecting spheres. The line BXB' is parallel to the line of centers, where B is a point on S, B' is a point on S' and X lies on both spheres. A is another point on S, and A' is another point on S' such that the line AA' has a point on both spheres. Show that the segments AB and A'B' have equal projections on the line AA'.

Problem 20

Two walkers are at the same altitude in a range of mountains. The path joining them is piecewise linear with all its vertices above the two walkers. Can they each walk along the path until they have changed places, so that at all times their altitudes are equal?

Problem 21

Find the least possible value of $(x + y)(y + z)$ for positive reals satisfying $(x + y + z) xyz = 1$.

Problem 22

A polyhedron has an even number of edges Show that we can place an arrow on each edge so that each vertex has an even number of arrows pointing towards it (on adjacent edges).

Problem 23

N is the set of positive integers. Does there exist a function $f: N \to N$ such that $f(n+1) = f(f(n)) + f(f(n+2))$ for all n.

Problem 24

A convex polygon is such that any segment dividing the polygon into two parts of equal area which has at least one end at a vertex has length < 1. Show that the area of the polygon is $< \pi/4$.

24th ASU 1990

Problem 1

Show that $x^4 > x - 1/2$ for all real x.

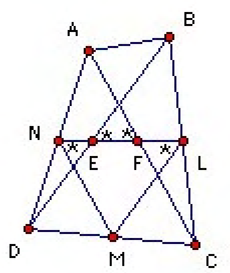
Solution

$x^4 - x + 1/2 = (x^2 - 1/2)^2 + (x - 1/2)^2 \geq 0$. We could only have equality if $x^2 = x = 1/2$, which is impossible, so the inequality is strict.

Problem 2

The line joining the midpoints of two opposite sides of a convex quadrilateral makes equal angles with the diagonals. Show that the diagonals are equal.

Solution



Let L, M, N be the midpoints of BC, CD, DA. Assume that NL makes equal angles with AC and BD, so $\angle NLM = \angle BEL = \angle AFN = \angle LNM$, so LM = MN and hence BD = AC.

Problem 3

A graph has 30 points and each point has 6 edges. Find the total number of triples such that each pair of points is joined or each pair of points is not joined.

Answer

1990

Solution

There are $30\cdot 29\cdot 28/6 = 4060$ triples in all. Let m be the number of triples with 0 or 3 edges, and let n be the number of triples with 1 or 2 edges. So m + n = 4060. Each point is joined to 6 others, so it is in $6\cdot 5/2 = 15$ triples where it is joined to both the other points, and it is in $23\cdot 22/2 = 253$ triples where it is not joined to either of the other points. So the total number of triples (a,b,c), where a is joined to b and c or not joined to b or c is 30(15+253) = 8040. This counts the m triples 3 times each, and the n triples once each, so 3m+n = 8040. Hence m = 1990.

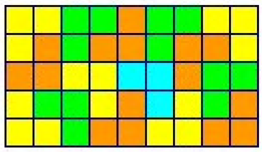
Problem 4

Does there exist a rectangle which can be dissected into 15 congruent polygons which are not rectangles? Can a square be dissected into 15 congruent polygons which are not rectangles?

Answer

yes, yes

Solution

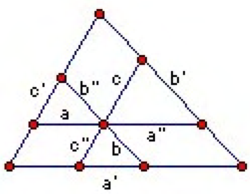


By stretching vertically we get a square.

Problem 5

The point P lies inside the triangle ABC. A line is drawn through P parallel to each side of the triangle. The lines divide AB into three parts length c, c', c" (in that order), and BC into three parts length a, a', a" (in that order), and CA into three parts length b, b', b" (in that order). Show that abc = a'b'c' = a"b"c".

Solution



The three small triangles are similar, so a/a" = c'/c = b"/b' and a/a' = c'/c" = b"/b. Hence (a/a")(b/b") = (c'/c)(c"/c') = c"/c, so abc = a"b"c". Similarly, (a/a')(c/c') = (b"/b)(b'/b") = b'/b, so abc = a'b'c'.

Problem 6

Find three non-zero reals such that all quadratics with those numbers as coefficients have two distinct rational roots.

Answer 1,2,-3

Solution

If a + b + c = 0, then 1 is a root of $ax^2 + bx + c$, and so the other root is -b/a - 1, which is rational.

Problem 7

What is the largest possible value of | ... | $|a_1 - a_2| - a_3| - ... - a_{1990}|$, where $a_1, a_2, ... , a_{1990}$ is a permutation of 1, 2, 3, ... , 1990?

Answer 1989

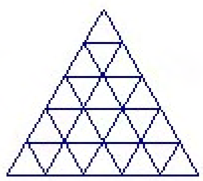
Solution

Since $|a-b| \le \max(a,b)$, a trivial induction shows that the expression does not exceed $\max(a_1, a_2, ... , a_{1990}) = 1990$. But for integers $|a-b|$ has the same parity as a+b, so a trivial induction shows that the expression has the same parity as $a_1 + a_2 + ... + a_{1990} = 1990 \cdot 1991/2$, which is odd. So it cannot exceed 1989. That can be attained by the permutation 2, 4, 5, 3, 6, 8, 9, 7, ... , 4k+2, 4k+4, 4k+5, 4k+3, ... , 1984+2, 1984+4, 1984+5, 1984+3, 1990, 1. Because we get successively 2, 3, 0; 6, 2, 7, 0; 10, 2, 11, 0; ... ; 4k+2, 2, 4k+3, 0; ... ; 1986, 2, 1987, 0; 1990, 1989.

Problem 8

An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1. A path is drawn along the sides of the triangles which passes through each vertex just once. Prove that the path makes an acute angle at at least n vertices.

Solution



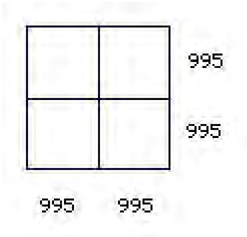
The diagram has 1 + 2 + ... + n = n(n+1)/2 upright triangles and 1 + 2 + ... + n-1 = n(n-1)/2 upside down triangles. It has 1 + 2 + ... + n+1 = (n+1)(n+2)/2 vertices. So the path must be $(n+1)(n+2)/2 - 1 = (n^2+3n)/2$ units long. Each unit length of the path is in just one upright triangle. The path cannot contain all three sides of a small triangle, or it would pass through a vertex more than once. So it must contain two sides of $(n^2+3n)/2 - n(n+1)/2 = n$ triangles. But if it contains two sides of a triangle, then it must make an acute angle at the vertex where they meet.

Problem 9

Can the squares of a 1990 x 1990 chessboard be colored black or white so that half the squares in each row and column are black and cells symmetric with respect to the center are of opposite color?

Answer no

Solution



Suppose it can be done. Divide the board into 4 quadrants. Suppose there are b black and 995^2 - b white squares in the top left quadrant. Then there are 995^2 - b black and b white squares in the bottom right quadrant (by the symmetry property).

If half the squares in each row are black, then half the squares in the first 995 rows are black, so the number of black squares in the top right quadrant is $995 \cdot 1990/2 - b = 995^2 - b$. So if half the squares in each column are black, then half the squares in the right-hand half of the board are black, so $(995^2 - b) + (995^2 - b) = 995^2$, in other words, $b = 995^2/2$, which is impossible.

Problem 10

Let $x_1, x_2, \dots , x_n$ be positive reals with sum 1. Show that $x_1^2/(x_1 + x_2) + x_2^2/(x_2 + x_3) + \dots + x_{n-1}^2/(x_{n-1} + x_n) + x_n^2/(x_n + x_1) \ge 1/2$.

Solution

$\sum x_i^2/(x_i+x_{i+1}) - \sum x_{i+1}^2/(x_i+x_{i+1}) = \sum (x_i - x_{i+1}) = 0$. Hence $\sum x_i^2/(x_i+x_{i+1}) = ½ \sum (x_i^2+x_{i+1}^2)/(x_i+x_{i+1}) \ge ¼ \sum (x_i + x_{i+1}) = ½$.

Problem 11

ABCD is a convex quadrilateral. X is a point on the side AB. AC and DE intersect at Y. Show that the circumcircles of ABC, CDY and BDX have a common point.

Problem 12

Two grasshoppers sit at opposite ends of the interval [0, 1]. A finite number of points (greater than zero) in the interval are marked. A move is for a grasshopper to select a marked point and jump over it to the equidistant point the other side. This point must lie in the interval for the move to be allowed, but it does not have to be marked. What is the smallest n such that if each grasshopper makes n moves or less, then they end up with no marked points between them?

Problem 13

Find all integers n such that $[n/1!] + [n/2!] + \dots + [n/10!] = 1001$.

Problem 14

A, B, C are adjacent vertices of a regular 2n-gon and D is the vertex opposite to B (so that BD passes through the center of the 2n-gon). X is a point on the side AB and Y is a point on the side BC so that angle $XDY = \pi/2n$. Show that DY bisects angle XYC.

Problem 15

A graph has n points and n(n-1)/2 edges. Each edge is colored with one of k colors so that there are no closed monochrome paths. What is the largest possible value of n (given k)?

Problem 16

Given a point X and n vectors $\mathbf{x}_i$ with sum zero in the plane. For each permutation of the vectors we form a set of n points, by starting at X and adding the vectors in order. For example, with the original ordering we get X_1 such that $XX_1 = \mathbf{x}_1$, X_2 such that $X_1X_2 = \mathbf{x}_2$ and so on. Show that for some permutation we can find two points Y, Z with angle YXZ = 60 deg, so that all the points lie inside or on the triangle XYZ.

Problem 17

Two unequal circles intersect at X and Y. Their common tangents intersect at Z. One of the tangents touches the circles at P and Q. Show that ZX is tangent to the circumcircle of PXQ.

Problem 18

Given 1990 piles of stones, containing 1, 2, 3, ... , 1990 stones. A move is to take an equal number of stones from one or more piles. How many moves are needed to take all the stones?

Problem 19

A quadratic polynomial p(x) has positive real coefficients with sum 1. Show that given any positive real numbers with product 1, the product of their values under p is at least 1.

Problem 20

A cube side 100 is divided into a million unit cubes with faces parallel to the large cube. The edges form a lattice. A prong is any three unit edges with a common vertex. Can we decompose the lattice into prongs with no common edges?

Problem 21

For which positive integers n is $3^{2n+1} - 2^{2n+1} - 6^n$ composite?

Answer all $n \neq 1$

Solution

$3^{2n+1} - 2^{2n+1} - 6^n = (3^n - 2^n)(3^{n+1} + 2^{n+1})$, so it is certainly composite for n > 1. For n = 1, it is 27-8-6 = 13, which is prime.

Problem 22

If every altitude of a tetrahedron is at least 1, show that the shortest distance between each pair of opposite edges is more than 2.

Problem 23

A game is played in three moves. The first player picks any real number, then the second player makes it the coefficient of a cubic, except that the coefficient of x^3 is already fixed at 1. Can the first player make his choices so that the final cubic has three distinct integer roots?

Problem 24

Given 2n genuine coins and 2n fake coins. The fake coins look the same as genuine coins but weigh less (but all fake coins have the same weight). Show how to identify each coin as genuine or fake using a balance at most 3n times.

25th ASU 1991

Problem 1

Find all integers a, b, c, d such that $ab - 2cd = 3$, $ac + bd = 1$.

Answer

$(a,b,c,d) = (1,3,1,0), (-1,-3,-1,0), (3,1,0,1), (-3,-1,0,-1)$

Solution

$11 = (ab - 2cd)^2 + 2(ac + bd)^2 = (a^2 + 2d^2)(b^2 + 2c^2)$, so we must have either (1) $a^2 + 2d^2 = 1$, $b^2 + 2c^2 = 11$, or (2) $a^2 + 2d^2 = 11$, $b^2 + 2c^2 = 1$.

(1) gives $a = \pm 1$, $d = 0$, $b = \pm 3$, $c = \pm 1$. If $a = 1$ and $d = 0$, then $ac + bd = 1$ implies $c = 1$, and $ab - 2cd = 3$ implies $b = 3$. Similarly, if $a = -1$, then $c = -1$, and $b = -3$. Similarly, (2) gives $(a,b,c,d) = (3,1,0,1), (-3,-1,0,-1)$.

Problem 2

n numbers are written on a blackboard. Someone then repeatedly erases two numbers and writes half their arithmetic mean instead, until only a single number remains. If all the original numbers were 1, show that the final number is not less than $1/n$.

Solution

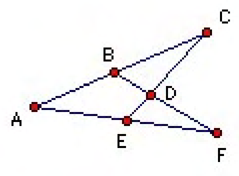
Put $c = (a+b)/4$. We have $1/c = 4/(a+b) \le 1/a + 1/b$, so each move does not increase the sum of the reciprocals of the numbers. If the final number is k, then the final sum of reciprocals is $1/k$. The initial sum is n, so $1/k \le n$, or $k \ge 1/n$.

Problem 3

Four lines in the plane intersect in six points. Each line is thus divided into two segments and two rays. Is it possible for the eight segments to have lengths 1, 2, 3, ... , 8? Can the lengths of the eight segments be eight distinct integers?

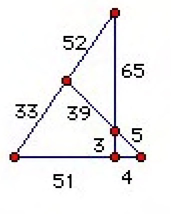
Answer no, yes

Solution



If a triangle has integer sides, one of which is 1, then it must be isosceles. So the only candidates for the segment length 1 are AB and AE. wlog $AB = 1$, so $BF = AF$. Hence cos

DFE = 1 - 1/(2 AF^2). Hence $ED^2 = DF^2 + EF^2 + 2DF\cdot EF(1 - 1/2AF^2) = DF^2 + EF^2 + 2DF\cdot EF - DF\cdot EF/AF^2$. But the first three terms are integers and the last term is < 1. Contradiction. (Careful, looking at the figure one is tempted to conclude that ED < AB, but a more realistic figure shows that is false.).



Building on the 3,4,5 triangle we get the figure above.

Problem 4

A lottery ticket has 50 cells into which one must put a permutation of 1, 2, 3, ... , 50. Any ticket with at least one cell matching the winning permutation wins a prize. How many tickets are needed to be sure of winning a prize?

Answer 26

Solution

Take the tickets:

```
 1  2  3  ... 25 26 27 ... 50
 2  3  4  ... 26  1 27 ... 50
 3  4  5  ...  1  2 27 ... 50
 ...
26  1  2  ... 24 25 27 ... 50
```

Each of the numbers 1, 2, ... , 26 occurs in each of the places 1, 2, ... , 26, but the winning ticket cannot have all these numbers in the last 24 places. So there must be at least one match. So 26 tickets suffice.

Now given any 25 tickets we show that they could all fail to match the winning permutation. In other words, we construct a permutation which fails to match any of the 25 tickets in any cell. We place the numbers 1, 2, 3, ... , 50 in turn. We start by placing 1. Clearly at most 25 places are ruled out, so we can place the 1. Now suppose we have placed 1, 2, ... , a. There must be at least 25 places where a+1 is not ruled out. If any of them are still unoccupied, then we are done. If not, they must be occupied by numbers $x_1, x_2, \ldots , x_{25}$ already placed. Take any empty place. 26 numbers cannot be ruled out for it, and we know that a+1 is ruled out, so at least one of the x_i is not ruled out. So we can move that x_i to it and then place a+1 where the x_i came from.

Problem 5

Find unequal integers m, n such that mn + n and mn + m are both squares. Can you find such integers between 988 and 1991?

Answer no

Solution

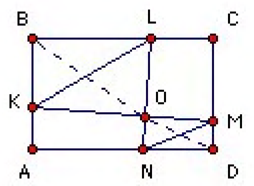
For example, $49 = 7^2$, $50 = 2\cdot 5^2$, $8 = 2\cdot 2^2$, $9 = 3^2$, so $49\cdot 8 + 8 = 20^2$, $49\cdot 8 + 49 = 21^2$.

wlog $m < n$. Then $mn + m = (m+h)^2$, $mn + n = (m+k)^2$, with $k > h$. So $n - m = (m+k)^2 - (m+h)^2 = (k-h)(2m+k+h) > 2m$, so $n > 3m$. Hence we cannot have m and n between 988 and 1991.

Problem 6

ABCD is a rectangle. Points K, L, M, N are chosen on AB, BC, CD, DA respectively so that KL is parallel to MN, and KM is perpendicular to LN. Show that the intersection of KM and LN lies on BD.

Solution



Let LN and KM meet at O. $\angle NOM = \angle NDM = 90^o$, so OMDN is cyclic. Hence $\angle NOD = \angle NMD$. Similarly, BLOK is cyclic and $\angle LOB = \angle LKB$. But NM is parallel to LK and AB is parallel to CD, so $\angle LKB = \angle NMD$. Hence $\angle NOD = \angle LOB$, so DOB is a straight line.

Problem 7

An investigator works out that he needs to ask at most 91 questions on the basis that all the answers will be yes or no and all will be true. The questions may depend upon the earlier answers. Show that he can make do with 105 questions if at most one answer could be a lie.

Solution

Suppose he asks n questions as usual, and then asks "did you lie to any of the last n questions?" If the reply is a truthful no, then the n answers were correct. If the reply is a lying no, then the n answers were still correct. On the other hand if the answer is yes, then the n answers might have been correct and might not. However, a lie has certainly been told, so all future answers must be truthful and so he could ask the n questions again.

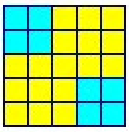
$91 = 7\cdot 13$, so the obvious candidates for n are 7 and 13. If we take $n = 7$, then the worst case is 13 check questions and 7 repeat questions. That does not work because he needs 20 extra questions and only has 14. A little thought suggests reducing n each time. So the first batch of questions is 13, followed by a check question. If the check answer is yes, then he knows a lie has been told and asks the 13 questions again. No further check questions are needed, and he has used exactly 14 extra questions. If the check answer is no, then the lie may not have been told, so the next batch of questions is 12, followed by a check question, and so on. That allows him to ask $13+12+...+1 = 91$ questions. If he gets a yes to the check question after the batch of i, then he ignores the answers to that batch and asks them again, thus asking a total of 14 extra questions, but thereafter asks no check questions.

Problem 8

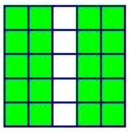
A minus sign is placed on one square of a 5 x 5 board and plus signs are placed on the remaining squares. A move is to select a 2 x 2, 3 x 3, 4 x 4 or 5 x 5 square and change all the signs in it. Which initial positions allow a series of moves to change all the signs to plus?

Answer only the central square

Solution



We take the 5x5 square, the two yellow 3x3 squares, which overlap at the center, and the two blue 2x2 squares. Then every square except the center square is changed an even number of times. So this works if the central square was selected.



It is easy to check that any 2x2, 3x3, 4x4 or 5x5 square has an even number of green squares, so if the selected square was green, and we change it an odd number of times, then some other green square must also be changed an odd number of times and hence end up with a minus. So if all the squares end up plus, then the selected square was not green, so it must belong to the central white column. Similarly, it must belong to the central row and hence must be the center square.

Problem 9

Show that $(x + y + z)^2/3 \geq x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)}$ for all non-negative reals x, y, z.

Solution

By AM/GM $xy + yz \geq 2x\sqrt{(yz)}$. Adding the similar results gives $2(xy + yz + zx) \geq 2(x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)}\ )$.

By AM/GM $x^2 + x^2 + y^2 + z^2 \geq 4x\sqrt{(yz)}$. Adding the similar results gives $x^2 + y^2 + z^2 \geq x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)}$. Adding the first result gives $(x + y + z)^2/3 \geq x\sqrt{(yz)} + y\sqrt{(zx)} + z\sqrt{(xy)}$.

Problem 10

Does there exist a triangle in which two sides are integer multiples of the median to that side? Does there exist a triangle in which every side is an integer multiple of the median to that side?

Answer yes, no

Solution

The obvious approach is to make the triangle isosceles. So suppose the sides are a, b, b. Then the length m of a median to one of the sides length b satisfies: $a^2 + b^2 = 2m^2 + b^2/2$. The simplest possbility is to take m = b, so $a^2 = 3b^2/2$. Thus if b = 2, a = √6.

Suppose we have a triangle ABC, with medians AD, BE, CF, and BC/AD, CA/BE, AB/CF all integers. If AD = BC/2, then ∠A = 90°. If AD < BC/2, then ∠A is obtuse, so at least two of the medians must be equal to the corresponding sides. So wlog we have $b^2 + c^2 = 5a^2/2$, $c^2 + a^2 = 5b^2/2$. Subtracting, $b^2 - a^2 = (5/2)(a^2 - b^2)$, so a = b. Hence c/a = √(3/2). So the third median has length m where $a^2 + a^2 = (3/4)a^2 + 2m^2$, so a/m = √(8/5), which is not integral. Contradiction.

Problem 11

The numbers 1, 2, 3, ... , n are written on a blackboard (where n ≥ 3). A move is to replace two numbers by their sum and non-negative difference. A series of moves makes all the numbers equal k. Find all possible k.

Answer all powers of 2 ≥ n

Solution

If a prime p divides a+b and a-b, then it divides 2a and 2b, so if p is odd, it divides a and b. Thus if an odd prime p divides k, then it must divide all the original numbers including 1. So k must be a power of 2. Note that k, k → 0, 2k → 2k, 2k and k, k, k → 0, k, 2k → k, k, 2k → 0, 2k, 2k → 2k, 2k, 2k. So (by a trivial induction) if we get all the numbers equal to k, then we can get them all to equal 2k. Finally, note that we can never decrease the largest number on the board, so the answer must be all powers of 2 greater than some minimum, which must be at least n.

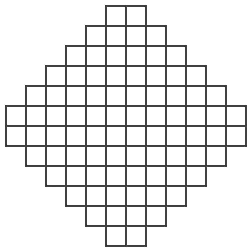
We use induction to show that if 2^m is the smallest power of 2 which is ≥ n, then we can get all numbers equal to 2^m. Note that 0, k → k, k → 0, 2k, so with a zero we can double each member of any set of numbers as often as we wish and finally convert the zero. For example, we could convert 0, 2, 4 to 8, 8, 8. It is convenient to take the induction hypothesis as S_n: we can convert 1, 2, ... , n to 0, 2^k, 2^k, ... , 2^k, where 2^k is the smallest power of 2 which is ≥ n.

We show first that S_n is true for n ≤ 8. For n = 3, we take 1,3 → 2,4, then 2,2 → 0,4. For n = 4, we ignore the 4 and use the case n = 3. For n = 5, we take 3,5 → 2,8. Then 2,2 → 0,4. Then we use the 0 to convert the remaining powers of 2 (1,4,4) to 8. For n = 6, we take 2,6 → 4,8 and 3,5 → 2,8, then 4,4 → 0,8. Finally, we use the 0 to convert 1 and 2 to 8. For n = 7, we take 1,7 → 6,8, then 2,6 → 4,8, then 3,5 → 2,8, then 4,4 → 0,8, then 2,6 → 4,8 and finally use the 0 to convert the remaining 4 to 8.

Let $n = 2^a + b$, where $0 < b \le 2^a$ and assume S_m is true for all m < n. If b = 1, we convert the pair 2^a-1, 2^a+1 to 2, 2^{a+1}. We have 2^a-2 > 2, so by induction we can convert 1, 2, ... , 2^a-2 to 0, 2^a, ... , 2^a. Now all the numbers except 0 are powers of 2 and we can use the 0 to convert them each to 2^{a+1}. Similarly, if b = 2, we convert 2^a-1, 2^a+1 to 2, 2^{a+1} and 2^a-2, 2^a+2 to 4, 2^{a+1} and then proceed as in the previous case. If $3 \le b < 2^a$, then we start by converting the pairs (2^a + b, 2^a - b), (2^a + b-1, 2^a - b+1), (2^a + b-2, 2^a - b+2), ... , (2^a + 1, 2^a - 1). That gives some 2^{a+1}s and 2, 4, ... , 2b. Now by S_b we can convert 2, 4, ... , 2b to 0, 2^{a+1}, ... , 2^{a+1}. The remaining numbers 1, 2, ... , 2^a-b-1 can either be converted to powers of 2 by S^{2a-b-1} (if 2^a-b-1 ≥ 3) or are already powers of 2. Finally we use the 0 to bring all powers of 2 up to 2^{a+1}. In the case b = 2^a, we ignore 2^a + b (= 2^{a+1}) and use the case b-1 to convert the others.

Problem 12

The figure below is cut along the lines into polygons (which need not be convex). No polygon contains a 2 x 2 square. What is the smallest possible number of polygons?



Answer 12

Solution

We can clearly cut the polygon into 12 strips width 1, so the smallest number is ≤ 12.

There are 84 unit squares in the figure. Each cut along the edge of a unit square not already cut (and not on the boundary) increases the number of pieces by at most 1. So it is sufficient to show that at most 72 edges remain uncut (after cutting into polygons). Because then cutting the remaining edges would increase the total number of pieces by at most 72. But the final number of pieces is 84, so we would have to start with at least 12.

Initially, there are 144 edges, so we have to show that at least 72 of them are cut to make the polygons. An interior vertex has 4 edges. At least two of them must be cut, or the vertex would be the center of an uncut 2 x 2 square. If we take alternate interior vertices (36 in total, as shown below), then each has at least two cut edges, so in total at least 72 edges are cut to make the polygons.

Problem 13

ABC is an acute-angled triangle with circumcenter O. The circumcircle of ABO intersects AC and BC at M and N. Show that the circumradii of ABO and MNC are the same.

Solution

It is sufficient to show that ∠MBN = ∠C. But ∠MBN = ∠MBO + ∠OBN = ∠MAO + ∠OBN = ∠MCO + ∠OCN = ∠C.

Problem 14

A polygon can be transformed into a new polygon by making a straight cut, which creates two new pieces each with a new edge. One piece is then turned over and the two new edges are reattached. Can repeated transformations of this type turn a square into a triangle?

Problem 15

An *h x k minor* of an n x n table is the hk cells which lie in h rows and k columns. The *semiperimeter* of the minor is h + k. A number of minors each with semiperimeter at least n together include all the cells on the main diagonal. Show that they include at least half the cells in the table.

Problem 16

(1) $r_1, r_2, \dots , r_{100}, c_1, c_2, \dots , c_{100}$ are distinct reals. The number $r_i + c_j$ is written in position i, j of a 100 x 100 array. The product of the numbers in each column is 1. Show that the product of the numbers in each row is -1. (2) $r_1, r_2, \dots , r_{2n}, c_1, c_2, \dots , c_{2n}$ are distinct reals. The number $r_i + c_j$ is written in position i, j of a 2n x 2n array. The product of the numbers in each column is the same. Show that the product of the numbers in each row is also the same.

Problem 17

A sequence of positive integers is constructed as follows. If the last digit of a_n is greater than 5, then a_{n+1} is $9a_n$. If the last digit of a_n is 5 or less and a_n has more than one digit, then a_{n+1} is obtained from a_n by deleting the last digit. If a_n has only one digit, which is 5 or less, then the sequence terminates. Can we choose the first member of the sequence so that it does not terminate?

Problem 18

p(x) is the cubic $x^3 - 3x^2 + 5x$. If h is a real root of p(x) = 1 and k is a real root of p(x) = 5, find h + k.

Solution

Put y = 2-h, where p(h) = 1, then $(2-y)^3 - 3(2-y)^2 + 5(2-y) - 1 = 0$, so $8-12y+6y^2-y^3 - 12+12y-3y^2 + 10-5y - 1 = 0$, or $y^3 - 3y^2 + 5y = 5$, or p(y) = 5. So if h is a root of p(h) = 1, then there is a root k of p(k) = 5 such that h+k = 2. To complete the proof we have to show that p(x) = 5 has only one real root.

But $x^3 - 3x^2 + 5x = (x-1)^3 + 2(x-1) + 3$ which is a strictly increasing function of x-1 and hence of x. So p(x) = k has only one real root.

Problem 19

The chords AB and CD of a sphere intersect at X. A, C and X are equidistant from a point Y on the sphere. Show that BD and XY are perpendicular.

Problem 20

Do there exist 4 vectors in the plane so that none is a multiple of another, but the sum of each pair is perpendicular to the sum of the other two? Do there exist 91 non-zero vectors in the plane such that the sum of any 19 is perpendicular to the sum of the others?

Problem 21

ABCD is a square. The points X on the side AB and Y on the side AD are such that AX· AY = 2 BX· DY. The lines CX and CY meet the diagonal BD in two points. Show that these points lie on the circumcircle of AXY.

Problem 22

X is a set with 100 members. What is the smallest number of subsets of X such that every pair of elements belongs to at least one subset and no subset has more than 50 members? What is the smallest number if we also require that the union of any two subsets has at most 80 members?

Problem 23

The real numbers $x_1, x_2, \ldots, x_{1991}$ satisfy $|x_1 - x_2| + |x_2 - x_3| + \ldots + |x_{1990} - x_{1991}| = 1991$. What is the maximum possible value of $|s_1 - s_2| + |s_2 - s_3| + \ldots + |s_{1990} - s_{1991}|$, where $s_n = (x_1 + x_2 + \ldots + x_n)/n$?

1st CIS 1992

Problem 1

Show that $x^4 + y^4 + z^2 \geq xyz\sqrt{8}$ for all positive reals x, y, z.

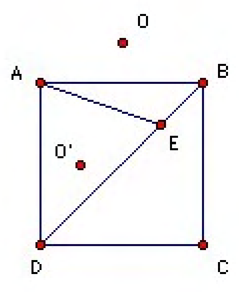
Solution

By AM/GM $x^4 + y^4 \geq 2x^2y^2$. Then by AM/GM again $2x^2y^2 + z^2 \geq (\sqrt{8})xyz$.

Problem 2

E is a point on the diagonal BD of the square ABCD. Show that the points A, E and the circumcenters of ABE and ADE form a square.

Solution



Let O, O' be the circumcenters of ABE, ADE respectively. Then OA = OE and $\angle AOB = 2\angle ABE = 90^o$. Similarly, O'A = O'E and $\angle AO'E = 2\angle ADE = 90^o$. Hence AOEO' is a square.

Problem 3

A country contains n cities and some towns. There is at most one road between each pair of towns and at most one road between each town and each city, but all the towns and cities are connected, directly or indirectly. We call a route between a city and a town a gold route if there is no other route between them which passes through fewer towns. Show that we can divide the towns and cities between n republics, so that each belongs to just one republic, each republic has just one city, and each republic contains all the towns on at least one of the gold routes between each of its towns and its city.

Solution

Let the cities be $C_1, C_2, \dots, C_n$. For each town T take the shortest path from T to a city. If there are shortest paths to more than one city, then take one to the city with the smallest index. We assign T to that city. Now suppose T is assigned to C_i. Let G be a gold route from T to C_i length n. Suppose T' is another town on G. Note that T' cannot be a city, or T would be assigned to T', not C_i. Suppose the part of G between T' and Ci has length k. There cannot be a path from T' to any city length < k, otherwise we would have a path from T to a city shorter than n. Nor can there be a path length k from T' to C_j with j < i, otherwise there would be a path length n from T to C_j and T should be assigned to C_j. Hence T' is also assigned to C_i, as required.

Problem 4

Given an infinite sheet of square ruled paper. Some of the squares contain a piece. A move consists of a piece jumping over a piece on a neighbouring square (which shares a side) onto an empty square and removing the piece jumped over. Initially, there are no pieces except in an m x n rectangle (m, n > 1) which has a piece on each square. What is the smallest number of pieces that can be left after a series of moves?

Answer

2 if mn is a multiple of 3, 1 otherwise

Solution

Obviously 1 x 2 and 2 x 2 can be reduced to 1. Obviously 3 x 2 can be reduced to 2. Note that pieces on the four X squares can be reduced to a single X provided that the square Y is empty (call this the L move):

```
. . Y   . . X   . . X   . . .
X X X   X X .   . . X   . . .
. . X   . . .   . . .   . . X
```

Now given 4 x 2 we can reduce it to 1 x 2:

```
. . . .   X . . .   X X . .   . . X . then the L move
X X X X   . X X X   . . X X   . . X X
X X X X   . X X X   . . X X   . . X X
```

Thus given m x 2 with m > 3 we can reduce it to (m-3) x 2 and hence to one of 1 x 2, 2 x 2, 3 x 2. Note also that we are removing 3 pieces at each stage so we end up with 1 piece unless m is a multiple of 3.

Given m x 3 with m > 1 we can use the L move to reduce it to (m-1) x 3. Hence by a series of L moves we get to 3 x 1 and hence to 2 pieces.

Now given m x n with m ≥ 4 and n ≥ 3, we can treat it as a 3 x n rectangle adjacent to an (m-3) x n rectangle. We can now reduce the 3 x n to 3 x 3 using L moves (with the L upright). We can then eliminate the 3 x 3 using L moves (with the L horizontal). Note that we have not changed mn mod 3.

This deals with all cases, except that we do not reduce 4 x 4 to 1 x 4. Instead we use L moves as follows:

```
X X X X   X X X X   X X X X   X X X .   X X . .   X . . .
X X X X   . X X X   . X X X   . X X .   . X . .   . . . .
X X X X   . X X X   . X X X   . X X .   . X . .   . . . .
X X X X   . X X X   . . . .   . . . .   . . . .   . . . .
```

So we have shown that if mn is not a multiple of 3 we can always reduce to a single piece. Clearly we cannot do better than that. We have also shown how to reduce to two pieces if mn is a multiple of 3. It remains to show that we cannot do better. Color the board with 3 colors in the usual way:

```
. . .
...1 2 3 1 2 3 ...
```

```
...2 3 1 2 3 1 ...
...3 1 2 3 1 2 ...
...
```

Then any move changes the parity of the number of pieces on each color. If mn is a multiple of three, then these three numbers start off equal and hence with equal parity. But a single piece has one number odd and the other two even. So we cannot get to a single piece.

Problem 5

Does there exist a 4-digit integer which cannot be changed into a multiple of 1992 by changing 3 of its digits?

Answer Yes, eg 2251

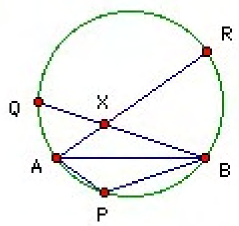
Solution

The only 4 digit multiples of 1992 are: 1992, 3984, 5976, 7968, 9960. All have first digit odd, second digit 9, third digit > 5 and last digit even, so it is easy to find a number which has all digits different from all of them.

Problem 6

A and B lie on a circle. P lies on the minor arc AB. Q and R (distinct from P) also lie on the circle, so that P and Q are equidistant from A, and P and R are equidistant from B. Show that the intersection of AR and BQ is the reflection of P in AB.

Solution



Let AR and BQ meet at X. Since arcs QA and AP are equal, we have ∠ABX = ∠ABP. Similarly, ∠BAX = ∠BAP. Side AB is common, so triangles ABX and ABP are congruent. Hence X is the reflection of P.

Problem 7

Find all real x, y such that $(1 + x)(1 + x^2)(1 + x^4) = 1+ y^7$, $(1 + y)(1 + y^2)(1 + y^4) = 1+ x^7$?

Answer $(x,y) = (0,0)$ or $(-1,-1)$

Solution

If $x = y$, then clearly $x \neq 1$, so we have $(1-x^8) = (1-x)(1+x^7) = 1-x^8-x+x^7$, so $x = 0$ or $x^6 = 1$, whose only real root (apart from the $x = 1$ we have discarded) is $x = -1$. That gives the two solutions above. So assume $x \neq y$. wlog $x > y$.

So $(1+x) > (1+y)$ and $(1+x^7) > (1+y^7)$. So we must have $(1+x^2)(1+x^4) < (1+y^2)(1+y^4)$ and hence $y < 0$. If $x > 0$, then $(1 + x)(1 + x^2)(1 + x^4) > 1 > 1+y^7$, so $x < 0$ also.

Multiplying the first equ by $(1-x)$ and the second by $(1-y)$ and subtracting: $y^8 - x^8 = (y-x) + (y^7 - x^7) + xy(x^6 - y^6)$. But lhs > 0 and each term on rhs < 0. Contradiction. So there are no more solutions.

Problem 8

An m x n rectangle is divided into mn unit squares by lines parallel to its sides. A gnomon is the figure of three unit squares formed by deleting one unit square from a 2 x 2 square. For what m, n can we divide the rectangle into gnomons so that no two gnomons form a rectangle and no vertex is in four gnomons?

Answer None

Solution

Suppose an m x n rectangle could be tiled as described. We will establish a contradiction by counting gnomon vertices.

A gnomon cannot touch a side of the rectangle along a length 1, because then the gnomon that fitted under the overhang would form a rectangle with the first. So each gnomon along a sides of the rectangle touches it along a length 2. So m and n must be even. Put $m = 2M$, $n = 2N$. There are $(2M-1)(2N-1)$ gridpoints inside the rectangle. None of these points can have 4 gnomon vertices. But it is easy to see that they cannot have 3, because the angle inside a gnomon at vertex is either 90^o or 270^o. So they have at most 2 gnomon vertices each, or $2(2M-1)(2N-1)$ in total. There is only one gnomon at each of the 4 corners, or 4 gnomon vertices in total. Along the sides there are alternately 2 and 0, so at most $4(M-1) + 4(N-1)$ in total, giving a grand total of $\le 2(2M-1)(2N-1) + 4(M-1) + 4(N-1) + 4 = 8MN - 2 < 8MN$. On the other hand, there are $4MN/3$ gnomons each with 6 vertices, a total of $8MN$. Contradiction.

Problem 9

Show that for any real numbers $x, y > 1$, we have $x^2/(y - 1) + y^2/(x - 1) \ge 8$.

Solution

We have $(x-2)2 \ge 0$, so $x2 \ge 4(x-1)$. Hence $x/\sqrt{(x-1)} \ge 2$. Now by AM/GM, $x^2/(y - 1) + y^2/(x - 1) \ge 2xy/\sqrt{((x-1)(y-1))}$. But rhs $\ge 2\cdot\ 2\cdot\ 2$.

Problem 10

Show that if 15 numbers lie between 2 and 1992 and each pair is coprime, then at least one is prime.

Solution

Suppose not. Then since $452 = 2025 > 1992$, each of the numbers must have a prime factor $\le$ 43. But there are only 14 such primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43. Hence there must be two numbers with the same prime factor and they cannot be coprime. Contradiction.

Problem 11

A cinema has its seats arranged in n rows x m columns. It sold mn tickets but sold some seats more than once. The usher managed to allocate seats so that every ticket holder was in the correct row or column. Show that he could have allocated seats so that every ticket holder was in the correct row or column and at least one person was in the correct seat. What is the maximum k such that he could have always put every ticket holder in the correct row or column and at least k people in the correct seat?

Answer 1

Solution

Suppose it is not possible. Take any person, label him P_1. Suppose he should be in seat S_1. If seat S_1 is vacant, then we can just move him to S_1, so S_1 must be occupied by someone. Call him P_2. Continue, so that we get a sequence P_1, P_2, P_3, ... where P_i should be in the seat occupied by P_{i+1}. Since there are only finitely many people, we must get a repetition. Suppose the first repetition is $P_i = P_{i+j}$. Then we can move P_i to P_{i+1}, P_{i+2} to P_{i+3}, ... , P_{i+j-1} to P_i and then these j people will all be in their correct seats. Contradiction. So it is possible.

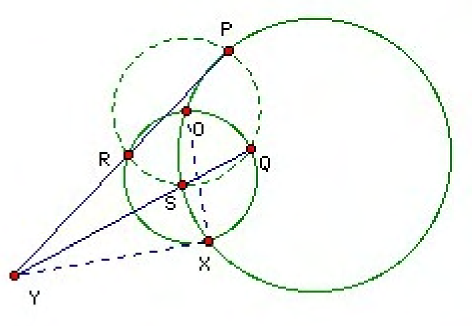
Suppose that m+n-1 tickets to seat (1,1) have been sold and n-1 seats to each of (2,1), (3,1), ... , (m,1). Then to comply with the conditions the people with tickets to (1,1) must occupy the whole of the first row and first column. Hence those with tickets to (k,1) for k > 1 must occupy the whole of row k apart from (k,1). Thus the seating is completely determined and only one person is in the correct seat - namely the person in (1,1).

Problem 12

Circles C and C' intersect at O and X. A circle center O meets C at Q and R and meets C' at P and S. PR and QS meet at Y distinct from X. Show that $\angle YXO = 90^o$.

Solution

We show first that YRSX is cyclic.



It is sufficient to show that $\angle RYS = \angle RXS$. We have $\angle RYS = \angle PRQ - \angle RQY = \angle PSQ - \angle RQY = \angle OSQ - \angle OSP - \angle RQY = \angle OQS - \angle RQY - \angle OSP = \angle OQR - \angle OSP$. But $\angle RXS$

$= \angle RXO - \angle OXS = \angle OQR - \angle OXS = \angle OQR - \angle OPS = \angle OQR - \angle OSP$. So YRSX is cyclic.

Hence $\angle YXO = \angle YXS + \angle SXO = \angle PRS + \angle SXO = \angle PRS + \angle SPO = ½\angle POS + \angle SPO = 90^o$.

Problem 13

Define the sequence $a_1 = 1, a_2, a_3, \ldots$ by $a_{n+1} = a_1^2 + a_2^{\ 2} + a_3^2 + \ldots + a_n^2 + n$. Show that 1 is the only square in the sequence.

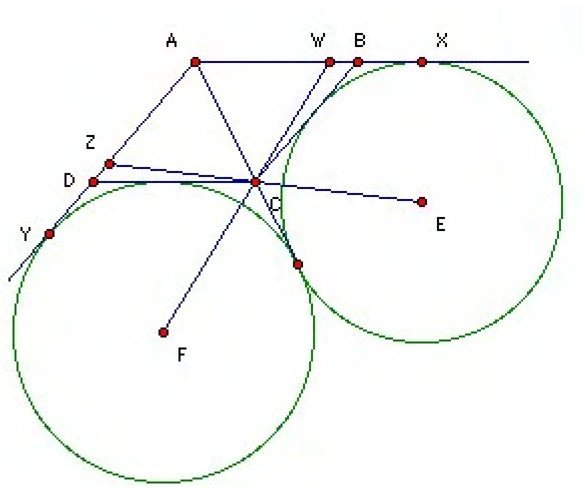
Solution

Obviously all a_n are positive integers. So we have $a_n^2 < a_{n+1} = a_n^2 + (a_1^2 + \ldots + a_{n-1}^2 + n-1) + 1 = a_n^2 + a_n + 1 < (a_n + 1)^2$. So a_{n+1} lies between two consecutive squares and hence cannot be a square

Problem 14

ABCD is a parallelogram. The excircle of ABC opposite A has center E and touches the line AB at X. The excircle of ADC opposite A has center F and touches the line AD at Y. The line FC meets the line AB at W, and the line EC meets the line AD at Z. Show that WX = YZ.

Solution



We have the familiar result that AY is perimeter ADC (chase round using the fact that the two tangents from the same point have the same length). Similarly, AX = perimeter ABC = perimeter ADC. So AX = AY (*)

AE is parallel to the bisector of ACD, which is perpendicular to CF. So CW is perpendicular to AE. Hence AW = AC. Similarly AZ = AC. Hence AW = AZ. Subtracting from (*) gives result.

Problem 15

Half the cells of a 2m x n board are colored black and the other half are colored white. The cells at the opposite ends of the main diagonal are different colors. The center of each black cell is connected to the center of every other black cell by a straight line segment, and similarly for the white cells. Show that we can place an arrow on each segment so that it becomes a vector and the vectors sum to zero.

Solution

Suppose we have an odd number of arbitrary points A_1, A_2, ... , A_{2k+1} then we claim that if we take the vector A_iA_j for $i < j$ and j-i odd and the vector A_jA_i for $i < j$ and j-i even, then we get the sum of the vectors zero. We prove the claim by induction. It is true for k=3 because we have $A_1A_2 + A_2A_3 + A_3A_1 = 0$. So suppose it is true for 2k-1. The additional vectors when we move to 2k+1 are $A_{2k}A_{2k+1}$, $\sum A_{2i+1}A_{2k}$, $\sum A_{2i}A_{2k+1}$, $\sum A_{2k}A_{2i}$ and $\sum A_{2k+1}A_{2i+1}$. But $A_{2i+1}A_{2k} + A_{2k}A_{2i} + A_{2i}A_{2k+1} + A_{2k+1}A_{2i+1} = A_{2i+1}A_{2i} + A_{2i}A_{2i+1} = 0$, leaving the three terms $A_{2k+1}A_1$, A_1A_{2k} and $A_{2k}A_{2k+1}$ which also sum to zero. Hence the result is true for 2k+1 and hence for all odd numbers. Thus for mn odd we can number the centers of the black squares in an arbitrary fashion, use the rule given for the arrow directions and then the vectors for the black squares will sum to zero. Similarly for the white squares.

However, the same general result is not true for an even number of points. So we need something else for mn even. Let B be the center of the black square at one end of the main diagonal and W be the center of the white square at the other end. Let B_1, B_2, ... , B_{2k+1} be the centers of the other black squares and W_1, W_2, ... , W_{2k+1} the centers of the other white squares. Take vectors BB_i, WW_i and for B_iB_j and W_iW_j take the same rule as before (if $i < j$ take B_iB_j if j-i is odd and B_jB_i if j-i is even, similarly for W_iW_j). Now consider square centers X, Y which are symmetric wrt the center of the rectangle (in other words the center of the rectangle is the midpoint of XY). The pairs (X,Y) are of three types: opposite colors, both white and both black. The number of both white pairs must equal the number of both black pairs since the number of white and black squares is equal. For the first type we have BX + WY = 0. For the second type we have WX + WY = WB and for the third type we have BX + BY = BW. Since they are equal in number the second and third type sum to zero.

Problem 16

A graph has 17 points and each point has 4 edges. Show that there are two points which are not joined and which are not both joined to the same point.

Solution

Suppose not. We will obtain a contradiction.

Take any point A. Suppose the four edges at A are BA, CA, DA, EA. If there is any other point X not joined to any of A, B, C, D, E then with A it forms the required pair of points. Suppose the three other points joined to B (apart from A) are B_1, B_2, B_3. Similarly C_i, D_i and E_i. Then all 12 points B_i, C_i, D_i, E_i must be distinct from each other and from A, B, C, D, E or there would be a point X. Thus, in particular, A is not part of a triangle. But A was arbitrary, so the graph has no triangles. Hence there cannot be an edge B_iB_j (or C_iC_j, D_iD_j, E_iE_j).

We have 4 edges AX, 12 edges BX, CX etc, and $17\cdot 4/2 = 34$ edges in all, so there must be 18 edges B_iC_j etc. Each gives a different cycle length 5 through A (eg ABB_iC_jC). The same argument shows that every point must lie on 18 cycles length 5. Hence there must be a total of $17\cdot 18/5$ such cycles. Contradiction.

Problem 17

Let $f(x) = a\cos(x + 1) + b\cos(x + 2) + c\cos(x + 3)$, where a, b, c are real. Given that f(x) has at least two zeros in the interval $(0, \pi)$, find all its real zeros.

Answer f(x) must be identically zero.

Solution

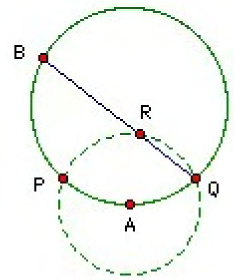
We have $f(x) = (a\cos 1 + b\cos 2 + c\cos 3)\cos x - (a\sin 1 + b\sin 2 + c\sin 3)\sin x$. This can be written as $d\cos(x + \theta)$ for some d, θ. But if $d \neq 0$, then this has only one zero in the interval $(0, \pi)$. Hence $d = 0$.

Problem 18

A plane intersects a sphere in a circle C. The points A and B lie on the sphere on opposite sides of the plane. The line joining A to the center of the sphere is normal to the plane. Another plane p intersects the segment AB and meets C at P and Q. Show that $BP\cdot BQ$ is independent of the choice of p.

Solution

All points of the circle C are equidistant from A. The plane p also meets the sphere in a circle C'. Let C'' be the circle center A radius AP. Provided that AB is not a diameter of C' one of the lines BP, BQ will meet C'' again at some point R (see diagram). Now since arcs AP, AQ are equal, so are the angles ABP, ABQ. Hence triangles ABP, ABQ are congruent and so BP = BR. Hence $BP\cdot BQ = BR\cdot BQ$. But the square of the tangent from B to C'' is $AB^2 - AP^2$, so $BR\cdot BQ = AB^2 - AP^2$, which is independent of the position of p.



Problem 19

If you have an algorithm for finding all the real zeros of any cubic polynomial, how do you find the real solutions to $x = p(y)$, $y = p(x)$, where p is a cubic polynomial?

Solution

Let $p(x) \equiv ax^3 + bx^2 + cx + d$. Finding the solutions with x = y is obvious, just solve the cubic $ax^3 + bx^2 + (c-1)x + d = 0$. For $x \neq y$, we have $x - y = a(y^3 - x^3) + b(y^2 - x^2) + c(y - x)$.

Dividing by y - x gives $a(x^2 + xy + y^2) + b(x + y) + c + 1 = 0$. Put $s = x + y$, $t = xy$ and this becomes $as^2 - at + bs + c + 1 = 0$ (*).

We also have $x + y = a(x + y)(x^2 - xy + y^2) + b(x^2 + y^2) + c(x + y) + 2d$, or $s = as(s^2 - 3t) + b(s^2 - 2t) + cs + 3d$. Substituting for t from (*) we get a cubic in s. Solving, we then recover t from (*) and then solve a quadratic to get x,y from s,t.

Problem 20

Find all integers $k > 1$ such that for some distinct positive integers a, b, the number $k^a + 1$ can be obtained from $k^b + 1$ by reversing the order of its (decimal) digits.

Answer

$k = 3$, $3^3+1 = 28$, $3^4+1 = 82$.

Solution

$k = 10$ does not work because $k^a + 1$ is a palindrome. If $k > 10$, then for $a < b$ we have $k^b + 1 \ge k^{a+1} + 1 \ge 11k^a + 1 \ge 10(k^a + 1) + k^a - 9 > 10(k^a + 1)$. So $k^b + 1$ has more digits than $k^a + 1$. So we only need to consider $k = 2, 3, \dots, 9$.

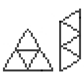
wlog $a < b$. Suppose $2a < b$. Then $k^b + 1 > k^a k^{a+1} > k^a(k^a + 1)$, so $k^a < 10$. But $k^a + 1 \ge 10$ (or reversing its digits would not change it). Hence $k^a + 1 = 10$, which obviously does not work. Hence $2a \ge b$. So $a \ge b - a$. Hence $k^b + 1 > k^b - 1 \ge k^b - k^a + k^{b-a} - 1 = (k^a + 1)(k^{b-a} - 1)$. So $k^{b-a} - 1 < 10$. If $k^{b-a} - 1 = 9$, then $k^{b-a} = 10$, so $k = 10$, which we already know does not work. Hence $k^{b-a} - 1 < 9$.

But $(k^b + 1) - (k^a + 1) = (k^{b-a} - 1)k^a$. This must be divisible by 9, because $k^b + 1$ and $k^a + 1$ have the same digit sum and hence are the same mod 9. Hence k^a must be divisible by 9, so k must be 3, 6 or 9.

If k = 6 or 9, then kb-a - 1 ≥ 5, so the first digit of $k^a + 1$ must be 1 or $(k^{b-a} - 1)(k^a + 1)$ would have more digits than $k^a + 1$. But that means the last digit of $k^b + 1$ is 1 and hence the last digit of k^b is 0, which is impossible. So k = 3. It is easy to check that there is a solution for k = 3.

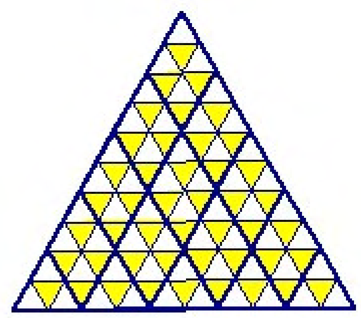
Problem 21

An equilateral triangle side 10 is divided into 100 equilateral triangles of side 1 by lines parallel to its sides. There are m equilateral tiles of 4 unit triangles and 25 - m straight tiles of 4 unit triangles (as shown below). For which values of m can they be used to tile the original triangle. [The straight tiles may be turned over.]



Answer m = 5, 7, 9, ... or 25.

Solution



There are 45 upside down triangles (yellow in the diagram) and 55 right way up. A straight tile always covers two of each. A triangular tile covers 3 of one and 1 of the other. So suppose there are t1 triangular tiles placed the right way up and t2 triangular tiles placed upside down. Then we have $3t_1 + t_2 + 2(25-t_1-t_2) = 55$, so $t_1 - t_2 = 5$. So t_1 and t_2 have opposite parity and hence $m = t_1 + t_2$ must be odd. Also $m \ge t_1 \ge 5$.
Now each rhombus shown may be filled with two triangular tiles or two straight tiles. So any tiling with m odd and ≥ 5 is possible.

Problem 22

1992 vectors are given in the plane. Two players pick unpicked vectors alternately. The winner is the one whose vectors sum to a vector with larger magnitude (or they draw if the magnitudes are the same). Can the first player always avoid losing?

Answer Yes

Solution

Suppose the vectors sum to **s**. Take the x-axis along **s** (or in any direction if **s** = 0). At each move the first player picks the vector with biggest x-coordinate. Each player makes 996 moves and the x-coordinate the first player picks on any move is larger than the x-coordinate the second player picks on the following move. So the sum of the first player's choices has larger x-coordinate than the sum of the second player's. Since the sum of all the x-coordinates is non-negative, the sum of the first player's choices must also have larger absolute value. The sum of the y-coordinates of all the vectors is zero, so the sum must be the same for the first and second players. Hence the first player's sum has larger magnitude than the second player's.

Problem 23

If $a > b > c > d > 0$ are integers such that $ad = bc$, show that $(a - d)^2 \ge 4d + 8$.

Solution
We need first that $a + d > b + c$. Put $a = m + h$, $d = m - h$, $b = m' + k$, $c = m' - k$. Then since $a - d > b - c$, we have $h > k$. But $m^2 - h^2 = ad = bc = m'^2 - k^2$, so $m > m'$ and hence $a + d > b + c$. Since a, b, c, d are integers it follows that $(a + d - b - c) \ge 1$.

Now $(a - d)^2 = (a + d)^2 - 4ad = (a + d)^2 - 4bc > (a + d)^2 - (b + c)^2$ (AM/GM) $= (a + b + c + d)(a + d - b - c) \ge (a + b + c + d)$. But $a \ge d + 3$, $b \ge d + 2$, $c \ge d + 1$, so $(a - d)^2 \ge 4d + 6$. But a square cannot = 2 or 3 mod 4, so $(a - d)^2 \ge 4d + 8$.