Britische Mathematikolympiade

1st BMO 1965

1. Sketch $f(x) = (x^2 + 1)/(x + 1)$. Find all points where $f'(x) = 0$ and describe the behaviour when x or f(x) is large.

2. X, at the centre a circular pond. Y, at the edge, cannot swim, but can run at speed 4v. X can run faster than 4v and can swim at speed v. Can X escape?

3. Show that $n^p - n$ is divisible by p for p = 3, 7, 13 and any integer n.

7. What is the remainder on dividing $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ by $x - 1$? By $x^2 - 1$?

8. For what real b can we find x satisfying: $x^2 + bx + 1 = x^2 + x + b = 0$?

9. Show that for any real, positive x, y, z, not all equal, we have:
$(x + y)(y + z)(z + x) > 8xyz$.

10. A chord length $\sqrt{3}$ divides a circle C into two arcs. R is the region bounded by the chord and the shorter arc. What is the largest area of rectangle than can be drawn in R?

Problem 4
What is the largest power of 10 dividing 100 x 99 x 98 x ... x 1?

Solution
Answer: 24.
There are 20 multiples of 5: $1\cdot5, 2\cdot5, \dots, 20\cdot5$. Of these 4 are multiples of 25: $1\cdot25$, $2\cdot25$, $3\cdot25$, $4\cdot25$. None are multiples of 125. Hence the highest power of 5 dividing 100! is 24. The highest power of 2 is obviously higher.

Problem 5
Show that $n(n + 1)(n + 2)(n + 3) + 1$ is a square for n = 1, 2, 3, ...

Solution
$n(n + 1)(n + 2)(n + 3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1 = (n^2 + 3n + 1)^2$.

Problem 6
The fractional part of a real is the real less the largest integer not exceeding it. Show that we can find n such that the fractional part of $(2 + \sqrt{2})^n > 0.999$.

Solution
$(2 + \sqrt{2})^n + (2 - \sqrt{2})^n$ is an integer. So the fractional part of $(2 + \sqrt{2})^n$ is $1 - (2 - \sqrt{2})^n$. But $(2 - \sqrt{2})$ lies between 0 and 1, so $(2 - \sqrt{2})^n$ becomes arbitarily small for large n.

Problem 7
What is the remainder on dividing $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$ by $x - 1$? By $x^2 - 1$?

Solution
Let $f(x) = x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$. The remainder on dividing by $x - 1$ is k, satisfying: $f(x) = (x - 1) g(x) + k$, for some polynomial g(x). Setting $x = 1$, $k = f(1) = 6$. Similarly, the remainder on dividing by $x^2 - 1$ is $ax + b$, where $f(x) = (x^2 - 1) h(x) + ax + b$, for some polynomial h(x). Hence $6 = f(1) = a + b$, $-6 = f(-1) = -a + b$. So $a = 6$, $b = 0$, and the remainder is 6x.

Problem 8
For what real b can we find x satisfying: $x^2 + bx + 1 = x^2 + x + b = 0$?

Solution
Answer: $b = -2$.
If b and x satisfy $x^2 + bx + 1 = x^2 + x + b$, then $(b - 1)(x - 1) = 0$, so either $b = 1$ or $x = 1$. If $b = 1$, then $x^2 + x + 1 = 0$, which has no real roots. If $x = 1$, then $b = -2$, which is a solution.

Problem 9
Show that for any real, positive x, y, z, not all equal, we have:
$(x + y)(y + z)(z + x) > 8xyz$.

Solution
$(x + y)/2 \geq \sqrt{xy}$ with equality only if $x = y$.

Problem 10
A chord length $\sqrt{3}$ divides a circle C radius 1 into two arcs. R is the region bounded by the chord and the shorter arc. What is the largest area of rectangle than can be drawn in R?

Solution
Answer:
Let the ends of the chord be A and B and its midpoint M. Let O be the centre of the circle. $OM = 1/2$ (recognize a $1/2$, $\sqrt{3}/2$, 1 triangle or use Pythagoras). So if we extend OM to meet the arc at N, them $MN = 1/2$. Angles AOB and ANB = 120^o.
First, we have to show that 2 vertices of a maximal rectangle lie on the chord. Although that is "obvious", proving it is the hardest part.
If P, Q are vertices of a maximal rectangle on the chord, then they must be equidistant from M. Otherwise we could enlarge the rectangle by moving the point close to M outwards to until it was equidistant with the other point.

2nd BMO 1966

1. Find the greatest and least values of $f(x) = (x^4 + x^2 + 5)/(x^2 + 1)^2$ for real x.

2. For which distinct, real a, b, c are all the roots of $\pm\sqrt{x - a} \pm\sqrt{x - b} \pm\sqrt{x - c} = 0$ real?

3. Sketch $y^2 = x^2(x + 1)/(x - 1)$. Find all stationary values and describe the behaviour for large x.

4. A_1, A_2, A_3, A_4 are consecutive vertices of a regular n-gon. $1/A_1A_2 = 1/A_1A_3 + 1/A_1A_4$. What are the possible values of n?

5. A spanner has an enclosed hole which is a regular hexagon side 1. For what values of s can it turn a square nut side s?

6. Find the largest interval over which $f(x) = \sqrt{x - 1} + \sqrt{x + 24 - 10\sqrt{x - 1}}$ is real and constant.

7. Prove that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ cannot be terms in an arithmetic progression.

a different colour? Show that given 8 different colours, we can colour a regular octahedron in 1680 ways so that each face has a different colour.

9. The angles of a triangle are A, B, C. Find the smallest possible value of tan A/2 + tan B/2 + tan C/2 and the largest possible value of tan A/2 tan B/2 tan C/2.

10. One hundred people of different heights are arranged in a 10 x 10 array. X, the shortest of the 10 people who are the tallest in their row, is a different height from Y, the tallest of the 10 people who are the shortest in their column. Is X taller or shorter than Y?

11. (a) Show that given any 52 integers we can always find two whose sum or difference is a multiple of 100.
(b) Show that given any set 100 integers, we can find a non-empty subset whose sum is a multiple of 100.

Problem 1
Find the greatest and least values of $f(x) = (x^4 + x^2 + 5)/(x^2 + 1)^2$ for real x.

Solution
Answer: 5, 0.95.
$f(x) = 5 - (4x^4 + 9x^2)/(x^4 + 2x^2 + 1) <= 5$, with equality iff $x = 0$.
Put $z = (1 + x^2)$, then $f(x) = (z(z - 1) + 5)/z^2$. Put $w = 1/z = 1/(1 + x^2)$, then $f(x) = 5w^2 - w + 1 = 5(w - 1/10)^2 + 95/100 \geq 0.95$ with equality iff $w = 1/10$ or $x = \pm 3$.

Problem 3
Sketch $y^2 = x^2(x + 1)/(x - 1)$. Find all stationary values and describe the behaviour for large x.

Solution
It is not easy to incorporate sketches on this web page, and anyway this is rather trivial bookwork, so I will just give a brief description.
The obvious point is that $x^2(x + 1)/(x - 1)$ is negative for $-1 < x < 1$, so there are no points on the graph for these values of x. Also the graph is symmetrical about the x-axis. $x = 1$ is an asymptote. Focussing on the positive y-part, y tends to infinity as x tends to 1 from above. It has a minimum at $(1 + \sqrt{5})/2$ and tends to infinity as x^2 as x tends to infinity. Obviously, the reflection in the x-axis has a maximum at the same value of x. Differentiating, the gradient is infinite at $x = -1$. So for large negative x, the curve is similar to $y = \pm x^2$. It there are two symmetrically placed inflections and the curve is vertical as it cuts the x-axis.

Problem 4
A_1, A_2, A_3, A_4 are consecutive vertices of a regular n-gon. $1/A_1A_2 = 1/A_1A_3 + 1/A_1A_4$. What are the possible values of n?

Solution
Answer: n = 7.
It is a nice question whether one has to show that n = 7 *is* a solution! [Does the question give as a fact that there *exists* a solution?] Clearly, there are no other possible solutions. Suppose the n-gon has side 1. Then A_1A_3 and A_1A_4 are obviously strictly increasing functions of n. So $1/A_1A_3 + 1/A_1A_4$ is a strictly decreasing function of n, whereas $1/A_1A_2$ is always 1. So there can be at most one solution. Moreover, for n = 6, the lhs is $1/\sqrt{3} + 1/2 > 1$, whereas for n = 8, $1/\sqrt{(2 + \sqrt{2})} + (2 - 1) < 1$. So if there is a solution it is 7.
If we are not allowed to assume that there is a solution, then we have to prove equality for n = 7.

A kludgy (but easy) proof is as follows. $A_1A_3 = 2 \cos \pi/7$, $A_1A_4 = (1 + 2 \cos 2\pi/7)$. Put $c = \cos \pi/7$. The $\cos 2\pi/7 = 2c^2 - 1$. So the result is true if $8c^3 - 4c^2 - 4c + 1 = 0$. Put $s = \sin \pi/7$. Then $(c + is)^7 = - 1$. Expanding and comparing real parts, we get: $c^7 - 21c^5(1 - c^2) + 35c^3(1 - c^2)^2 - 7c(1 - c^2)^3 + 1 = 0$, or $64c^7 - 112c^5 + 56c^3 - 7c + 1 = 0$. But we know that $c = -1$ is a solution (not the one we want), so we can factor that out to get: $64c^6 - 64c^5 - 48c^4 + 48c^3 + 8c^2 - 8c + 1 = 0$. But we know that this has three repeated roots ($\cos \pi/7 = \cos (-\pi/7)$ etc), so it should factorize as the square of a cubic and indeed it does: $(8c^3 - 4c^2 - 4c + 1)^2 = 0$, so $8c^3 - 4c^2 - 4c + 1 = 0$ as required.

Problem 5
A spanner has an enclosed hole which is a regular hexagon side 1. For what values of s can it turn a square nut side s?

Solution
Answer: $3 - \sqrt{3} >= s > \sqrt{(3/2)}$. Note that $3 - \sqrt{3} = 1.27$, $\sqrt{(3/2)} = 1.22$, so the range is fairly narrow.
The square must be small enough to fit into the hole and big enough to stick.
The sticking condition is easy. Inscribe a circle in the hexagon. If the square will fit into the circle, then it obviously does not stick. So we require the diagonal of the square to be $> \sqrt{3}$, or $s > \sqrt{(3/2)}$. On the other hand if the diagonal exceeds $\sqrt{3}$, then it obviously does stick because the parallel opposite sides of the hexagon are only a distance $\sqrt{3}$ apart.
It seems fairly clear that the maximum square has sides parallel to a pair of opposite sides and each vertex a distance x from on endpoint of those sides. For the square to have equal sides, we then need $\sqrt{3}\,(1 - x) = 1 + x$, or $x = 2 - \sqrt{3}$, and hence $s = 3 - \sqrt{3}$. But how do we prove it?

Problem 7
Prove that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ cannot be terms in an arithmetic progression.

Solution
If they are, then for some non-zero rational R, $(\sqrt{5} - \sqrt{2}) = R(\sqrt{3} - \sqrt{2})$. Squaring and rearranging: $\sqrt{10} = (7 - 5R^2)/2 + R^2\sqrt{6}$. If $7 - 5R^2 = 0$, then squaring we have an obvious contradiction: $10 = (49/25)\ 6$. So $(7 - 5R^2)/2$ is non-zero and squaring again gives that $\sqrt{6}$ is rational. Contradiction by the standard argument. [If $\sqrt{6} = m/n$ in lowest terms, then $m^2 = 6\ n^2$. 2 divides 6, so it must divide m, hence also n.]

Problem 8
Given 6 different colours, how many ways can we colour a cube so that each face has a different colour? Show that given 8 different colours, we can colour a regular octahedron in 1680 ways so that each face has a different colour.

Solution
Answer: 30.
Take the colours as $C_1, C_2, \dots , C_6$. We may always rotate the cube so that C_1 is on the top face. There are then 5 choices for the opposite face. The lowest remaining colour may then be rotated into a fixed position (say north). The three remaining colours may then be distributed over the three remaining faces in 6 ways. Total 5 x 6 = 30.
Similarly, for the octahedron. Take C_1 on the top face. There are 7 possibilities for the bottom face. There are now 20 (3 chosen from 6) choices for the colours of the three sides adjacent to the top face. Rotate the lowest colour into a fixed position. There are then 2 choices for the other two faces. Finally, the remaining 3 colours may be assigned to the remaining 3 faces in 6 ways. Total: 7 x 20 x 2 x 6 = 1680.

Problem 9
The angles of a triangle are A, B, C. Find the smallest possible value of $\tan A/2 + \tan B/2 + \tan C/2$ and the largest possible value of $\tan A/2 \tan B/2 \tan C/2$.

Solution
Answer: equilateral in both cases, $\sqrt{3}$, $1/(3\sqrt{3})$.
Using the familiar $\tan(a + x) = (\tan a + \tan x)/(1 - \tan a \tan x)$, we have that for a and x in the interval $[0, \pi/4]$, $\tan(a + x) + \tan(a - x) = 2 \tan a (1 + \tan^2 x)/(1 - \tan^2 a \tan^2 x) >= 2 \tan a$, with equality iff $x = 0$. Hence the minimum value of $\tan A/2 + \tan B/2 + \tan C/2$ occurs when $A = B = C$ (if any pair of A, B, C are unequal then we can lower the value by replacing the pair by two angles equal to their mean).
Similarly, we show that for a and x in the interval $(0, \pi/4)$, $\tan(a - x) \tan(a + x) < \tan^2 a$. The lhs $= (\tan^2 a - \tan^2 x)/(1 - \tan^2 a \tan^2 x) = \tan^2 a - \tan^2 x (1 - \tan^4 a)/(1 - \tan^2 a \tan^2 x) < \tan^2 a$.
This proves that the maximum value of $\tan A/2 \tan B/2 \tan C/2$ occurs for $A = B = C$. For if $A \neq B$, then we can increase the value by replacing A and B by their mean (and similarly for any other unequal pair). This does not deal with the degenerate case of a flat triangle. But if the angles are $A = 2\delta$, $B = 2\varepsilon$, $C = \pi - (2\delta + 2\varepsilon)$, where δ and ε are small, then $\tan A/2 \tan B/2 \tan C/2 = \delta \varepsilon/(\delta + \varepsilon)$ which tends to zero as δ and ε tend to zero. So the flat triangle gives a smaller value.

Problem 10
One hundred people of different heights are arranged in a 10 x 10 array. X, the shortest of the 10 people who are the tallest in their row, is a different height from Y, the tallest of the 10 people who are the shortest in their column. Is X taller or shorter than Y?

Solution
Answer: taller.
If X and Y are in the same row or column, then the result is immediate (by definition, X is taller than others in the row, and Y is shorter than others in the column). So suppose they are not. Take Z in the same row as X and the same column as Y. Then Z is shorter than X and taller than Y.

Problem
For which distinct, real a, b, c are all the roots of $\pm\sqrt{(x - a)} \pm\sqrt{(x - b)} \pm\sqrt{(x - c)} = 0$ real?

Solution
Answer: always true.
This is slightly tricky.
Suppose x, a, b, c satisfy the equation for some combination of signs. We show that they must also satisfy a quadratic, which is independent of the choice of sign.
$\pm\sqrt{(x - a)} = \pm\sqrt{(x - b)} \pm\sqrt{(x - c)}$. Squaring: $x - a = 2x - b - c \pm 2 \sqrt{(x - b)}\sqrt{(x - c)}$. So $x + a - b - c = \pm 2 \sqrt{(x - b)}\sqrt{(x - c)}$. Squaring: $x^2 + 2(a - b - c)x + a^2 + b^2 + c^2 - 2(ab - bc + ca) = 4x^2 - 4(b + c)x + 4bc$, so $3x^2 - 2(a + b + c)x - (a^2 + b^2 + c^2) + 2(ab + bc + ca) = 0$. Set $f(x) = 3x^2 - 2(a + b + c)x - (a^2 + b^2 + c^2) + 2(ab + bc + ca)$. Clearly $f(x) > 0$ for $|x|$ large. But $f(a) = 3a^2 - 2a^2 - 2ab - 2ca - a^2 - b^2 - c^2 + 2ab + 2bc + 2ca = -(b - c)^2 < 0$. So $f(x) = 0$ always has two distinct real roots, one $< a$ and one $> a$. The same is true for b and c. So if $a > b > c$, we have two distinct real roots, one $> a$ and one $< c$.
Reading the question strictly, that gives us the answer: if x is a root of the original equations, then it is real. But that does not tell us whether the original equations have 0, 1 or 2 roots. The status of the root $< c$ is slightly problematical. It is in fact a real root of the original equation, but we need complex numbers to see it. For example, let $a = 8$, $b = 5$, $c = 0$. The roots are 9 (giving $+3 - 2 - 1 = -3 + 2 + 1 = 0$), which is straightforward, and $-1/3$ (giving $i/\sqrt{3} (1 + 4 - 5) = i/\sqrt{3} (-1 - 4 + 5) = 0$). But we still have to prove this is true in general.
Let $g(x) = \sqrt{(x - c)} - \sqrt{(x - b)} - \sqrt{(x - a)}$. For $x > a$, $g(x)$ is clearly real and continuous. But $g(a) = \sqrt{(a - c)} - \sqrt{(a - b)} > 0$. Whereas for x large, $g(x) = \sqrt{x} < 0$, so $g(x)$ has at least one real root $> a$. Similarly, if $x < c$, then (with a suitable choice of signs) $g(x) = i (\sqrt{(a - x)} - \sqrt{(b - x)} - \sqrt{(c - x)})$. So $g(c) > 0$, but $g(x) < 0$ for x large and negative, so $g(x)$ has at least one real root $< c$. But we already know that the equations have at most two real roots, so they have exactly two. If we choose to regard only values of $x >= a$ as admissible, then they have one real root (and no imaginary roots).

Problem 6
Find the largest interval over which $f(x) = \sqrt{x - 1} + \sqrt{x + 24 - 10\sqrt{x - 1}}$ is real and constant.

Solution
Answer: [1, 26].
$f(x)$ is not real for $x < 1$. For $1 <= x <= 26$, we have $0 < \sqrt{x - 1} <= 5$. Now $(5 - \sqrt{x - 1})^2 = x + 24 - 10\sqrt{x - 1}$. So for $1 <= x <= 26$, $f(x) = 5$. For $x > 26$, $f(x) = 2\sqrt{x - 1} - 5$.

3rd BMO 1967

1. a, b are the roots of $x^2 + Ax + 1 = 0$, and c, d are the roots of $x^2 + Bx + 1 = 0$. Prove that $(a - c)(b - c)(a + d)(b + d) = B^2 - A^2$.

2. Graph $x^8 + xy + y^8 = 0$, showing stationary values and behaviour for large values. [Hint: put $z = y/x$.]

3. (a) The triangle ABC has altitudes AP, BQ, CR and AB > BC. Prove that AB + CR ≥ BC + AP. When do we have equality?
(b) Prove that if the inscribed and circumscribed circles have the same centre, then the triangle is equilateral.

4. We are given two distinct points A, B and a line l in the plane. Can we find points (in the plane) equidistant from A, B and l? How do we construct them?

5. Show that $(x - \sin x)(\pi - x - \sin x)$ is increasing in the interval $(0, \pi/2)$.

6. Find all x in $[0, 2\pi]$ for which $2 \cos x \le |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}| \le \sqrt{2}$.

7. Find all reals a, b, c, d such that $abc + d = bcd + a = cda + b = dab + c = 2$.

8. For which positive integers n does 61 divide $5^n - 4^n$?

9. None of the angles in the triangle ABC are zero. Find the greatest and least values of $\cos^2 A + \cos^2 B + \cos^2 C$ and the values of A, B, C for which they occur.

10. A collects pre-1900 British stamps and foreign stamps. B collects post-1900 British stamps and foreign special issues. C collects pre-1900 foreign stamps and British special issues. D collects post-1900 foreign stamps and British special issues. What stamps are collected by (1) no one, (2) everyone, (3) A and D, but not B?

11. The streets for a rectangular grid. B is h blocks north and k blocks east of A. How many shortest paths are there from A to B?

Problem 1
a, b are the roots of $x^2 + Ax + 1 = 0$, and c, d are the roots of $x^2 + Bx + 1 = 0$. Prove that $(a - c)(b - c)(a + d)(b + d) = B^2 - A^2$.

Solution
We use $ab = cd = 1$, $a + b = -A$, $c + d = -B$ repeatedly.
$(a - c)(b - c) = 1 + Ac + c^2$, $(a + d)(b + d) = 1 - Ad + d^2$. So $(a - c)(b - c)(a + d)(b + d) = 1 - Ad + d^2 + Ac - A^2 + Ad + c^2 - Ac + 1 = 2 + (c^2 + d^2) - A^2 = 2 + (c + d)^2 - 2 - A^2 = B^2 - A^2$.

Problem 2
Graph $x^8 + xy + y^8 = 0$, showing stationary values and behaviour for large values. [Hint: put $z = y/x$.]

Solution
Answer: a figure of 8, with the crossing point at the origin, touching both axes at the origin, extending to $x, y = \pm 0.93$, with $y = \pm x$ as axes of symmetry.

Problem 4
We are given two distinct points A, B and a line *l* in the plane. Can we find points (in the plane) equidistant from A, B and *l*? How do we construct them?

Solution
Provided A does not lie on *l*, the locus of points equidistant from A and *l* is (one branch of) a hyperbola, on the same side of *l* as A. So if A and B are on opposite sides of *l*, then there are no equidistant points.
Assume A and B are on the same side of *l*. The locus of points equidistant from A and B is the perpendicular bisector *m* of the line joining them. This will intersect the other locus in two points unless *m* is parallel to *l*, in which case it only intersects the other locus in one point.
The only remaining possibility is if A (or B) lies on *l*. In this case the locus of points equidistant from A and *l* is the line perpendicular to *l* at A. So there is just one point equidistant from A, B and *l*, unless both A and B lie on *l*, in which case there are none.

Problem 7
Find all reals a, b, c, d such that $abc + d = bcd + a = cda + b = dab + c = 2$.

Solution
Answer: 1, 1, 1, 1 (one solution); 3, -1, -1, -1 (four solutions).
Either all a, b, c, d are equal, which gives the first solution, or there is an unequal pair. Suppose $a \neq b$. But $bcd + a = cda + b$, so $(a - b)(cd - 1) = 0$, and hence $cd = 1$. Hence $a + b = 2$. It follows that $ab \neq 1$ (because the only solution to $a + b = 2$, $ab = 1$ is $a = b = 1$). Hence $c = d$ (since $(c - d)(ab - 1) = 0$). So either $c = d = 1$, or $c = d = -1$. But the first case gives $ab = 1$, $a + b = 2$ (contradiction, since the only possibility then is $a = b = 1$, whereas we are assuming $a \neq b$). So $c = d = -1$. Hence $ab = -3$, $a + b = 2$, so $a = 3$, $b = -1$ (or vice versa).

Problem 8
For which positive integers n does 61 divide $5^n - 4^n$?

Solution
Answer: multiples of 3.
Clearly $5 - 4 = 1 \pmod{61}$, $5^2 - 4^2 = 9 \pmod{61}$ and $5^3 - 4^3 = 0 \pmod{61}$. So it follows that $5^{3m} = 4^{3m} \pmod{61}$. In other words 61 does divide $5^n - 4^n$ if n is a multiple of 3. It remains to show that it does not divide $5^n - 4^n$ if $n = 3m + 1$ or $3m + 2$.
$5^{3m+1} = 5\ 5^{3m} = 5\ 4^{3m} = 4^{3m+1} + 4^{3m} \pmod{61}$. But 61 does not divide 4^{3m}, and hence it does not divide $5^{3m+1} - 4^{3m+1}$. Similarly, $5^{3m+2} = 25\ 5^{3m} = 25\ 4^{3m} = 4^{3m+2} + 9\ 4^{3m}$ (mod 61). But 61 does not divide 9 or 4^{3m}, so it does not divide $5^{3m+2} - 4^{3m+2}$.

Problem 10
A collects pre-1900 British stamps and foreign stamps. B collects post-1900 British stamps and foreign special issues. C collects pre-1900 foreign stamps and British special issues. D collects post-1900 foreign stamps and British special issues. What stamps are collected by (1) no one, (2) everyone, (3) A and D, but not B?

Solution
Answer: (1) 1900, non-special, British; (2) none; (3) pre-1900, special, British, and post-1900, non-special, foreign.

Problem 11
The streets for a rectangular grid. B is h blocks north and k blocks east of A. How many shortest paths are there from A to B?

Solution
Answer: (h + k)!/(h! k!).
A shortest path must involve h moves north and k moves east, a total of h + k moves. So there is a (1, 1) correspondence between shortests paths and subsets size h taken from the set 1, 2, 3, ... , (h + k). [Move i is north if i is in the subset and east otherwise.] But there are (h + k) C h = (h + k)! /(h! k!) such subsets.

4th BMO 1968

1. C is the circle center the origin and radius 2. Another circle radius 1 touches C at (2, 0) and then rolls around C. Find equations for the locus of the point P of the second circle which is initially at (2, 0) and sketch the locus.

2. Cows are put in a field when the grass has reached a fixed height, any cow eats the same amount of grass a day. The grass continues to grow as the cows eat it. If 15 cows clear 3 acres in 4 days and 32 cows clear 4 acres in 2 days, how many cows are needed to clear 6 acres in 3 days?

3. The distance between two points (x, y) and (x', y') is defined as |x - x'| + |y - y'|. Find the locus of all points with non-negative x and y which are equidistant from the origin and the point (a, b) where a > b.

4. Two balls radius a and b rest on a table touching each other. What is the radius of the largest sphere which can pass between them?

5. If reals x, y, z satisfy sin x + sin y + sin z = cos x + cos y + cos z = 0. Show that they also satisfy sin 2x + sin 2y + sin 2z = cos 2x + cos 2y + cos 2z = 0.

6. Given integers a_1, a_2, ... , a_7 and a permutation of them $a_{f(1)}$, $a_{f(2)}$, ... , $a_{f(7)}$, show that the product $(a_1 - a_{f(1)})(a_2 - a_{f(2)}) \dots (a_7 - a_{f(7)})$ is always even.

7. How many games are there in a knock-out tournament amongst n people?

8. C is a fixed circle of radius r. L is a variable chord. D is one of the two areas bounded by C and L. A circle C' of maximal radius is inscribed in D. A is the area of D outside C'. Show that A is greatest when D is the larger of the two areas and the length of L is $16\pi r/(16 + \pi^2)$.

9. The altitudes of a triangle are 3, 4, 6. What are its sides?

10. The faces of the tetrahedron ABCD are all congruent. The angle between the edges AB and CD is x. Show that cos x = sin(∠ABC - ∠BAC)/sin(∠ABC + ∠BAC).

11. The sum of the reciprocals of n distinct positive integers is 1. Show that there is a unique set of such integers for n = 3. Given an example of such a set for every n > 3.

12. What is the largest number of points that can be placed on a spherical shell of radius

number such that the distance is > $\sqrt{2}$?

5th BMO 1969

1. Find the condition on the distinct real numbers a, b, c such that $(x - a)(x - b)/(x - c)$ takes all real values. Sketch a graph where the condition is satisfied and another where it is not.

2. Find all real solutions to $\cos x + \cos^5 x + \cos 7x = 3$.

3. For which positive integers n can we find distinct integers a, b, c, d, a', b', c', d' greater than 1 such that $n^2 - 1 = aa' + bb' + cc' + dd'$? Give the solution for the smallest n.

4. Find all integral solutions to $a^2 - 3ab - a + b = 0$.

5. A long corridor has unit width and a right-angle corner. You wish to move a pipe along the corridor and round the corner. The pipe may have any shape, but every point must remain in contact with the floor. What is the longest possible distance between the two ends of the pipe?

6. If a, b, c, d, e are positive integers, show that any divisor of both ae + b and ce + d also divides ad - bc.

7. (1) f is a real-valued function on the reals, not identically zero, and differentiable at x = 0. It satisfies $f(x) f(y) = f(x+y)$ for all x, y. Show that f(x) is differentiable arbitrarily many times for all x and that if $f(1) < 1$, then $f(0) + f(1) + f(2) + \dots = 1/(1 - f(1))$.
(2) Find the real-valued function f on the reals, not identically zero, and differentiable at x = 0 which satisfies $f(x) f(y) = f(x-y)$ for all x, y.

8. A square side x has its vertices on the sides of a triangle with inradius r. Show that $2r > x > r\sqrt{2}$.

9. Let A_n be an n x n array of lattice points ($n > 3$). Is there a polygon with n^2 sides whose vertices are the points of A_n such that no two sides intersect except adjacent sides at a vertex? You should prove the result for n = 4 and 5, but merely state why it is plausible for $n > 5$.

10. Given a triangle, construct an equilateral triangle with the same area using ruler and compasses.

6th BMO 1970

1. (1) Find $1/\log_2 a + 1/\log_3 a + \dots + 1/\log_n a$ as a quotient of two logs to base 2.
(2) Find the sum of the coefficients of $(1 + x - x^2)^3(1 - 3x + x^2)^2$ and the sum of the coefficients of its derivative.

2. Sketch the curve $x^2 + 3xy + 2y^2 + 6x + 12y + 4$. Where is the center of symmetry?

3. Morley's theorem is as follows. ABC is a triangle. C' is the point of intersection of the trisector of angle A closer to AB and the trisector of angle B closer to AB. A' and B' are defined similarly. Then A'B'C' is equilateral. What is the largest possible value of area A'B'C'/area ABC? Is there a minimum value?

4. Prove that any subset of a set of n positive integers has a non-empty subset whose sum is divisible by n.

5. What is the minimum number of planes required to divide a cube into at least 300 pieces?

6. $y(x)$ is defined by $y' = f(x)$ in the region $|x| \le a$, where f is an even, continuous function. Show that (1) $y(-a) + y(a) = 2\, y(0)$ and (2) $\int_{-a}^{a} y(x)\, dx = 2a\, y(0)$. If you integrate numerically from (-a, 0) using 2N equal steps δ using $g(x_{n+1}) = g(x_n) + \delta \times g'(x_n)$, then the resulting solution does not satisfy (1). Suggest a modified method which ensures that (1) is satisfied.

7. ABC is a triangle with $\angle B = \angle C = 50^o$. D is a point on BC and E a point on AC such that $\angle BAD = 50^o$ and $\angle ABE = 30^o$. Find $\angle BED$.

8. 8 light bulbs can each be switched on or off by its own switch. State the total number of possible states for the 8 bulbs. What is the smallest number of switch changes required to cycle through all the states and return to the initial state?

9. Find rationals r and s such that $\sqrt{(2\sqrt{3} - 3)} = r^{1/4} - s^{1/4}$.

10. Find "some kind of 'formula' for" the number f(n) of incongruent right-angled triangles with shortest side n? Show that f(n) is unbounded. Does it tend to infinity?

7th BMO 1971

1. Factorise $(a + b)^7 - a^7 - b^7$. Show that $2n^3 + 2n^2 + 2n + 1$ is never a multiple of 3.

2. Let $a = 9^9$, $b = 9^a$, $c = 9^b$. Show that the last two digits of b and c are equal. What are they?

3. A and B are two vertices of a regular 2n-gon. The n longest diameters subtend angles $a_1, a_2, \dots , a_n$ and $b_1, b_2, \dots , b_n$ at A and B respectively. Show that $\tan^2 a_1 + \tan^2 a_2 + \dots + \tan^2 a_n = \tan^2 b_1 + \tan^2 b_2 + \dots + \tan^2 b_n$.

4. Given any n+1 distinct integers all less than 2n+1, show that there must be one which divides another.

5. The triangle ABC has circumradius R. $\angle A \ge \angle B \ge \angle C$. What is the upper limit for the radius of circles which intersect all three sides of the triangle?

6. (1) Let $I(x) = \int_c^x f(x, u)\, du$. Show that $I'(x) = f(x, x) + \int_c^x \partial f/\partial x\, du$.
(2) Find $\lim_{\theta\ 0} \cot \theta \sin(t \sin \theta)$.
(3) Let $G(t) = \int_0^t \cot \theta \sin(t \sin \theta)\, d\theta$. Prove that $G'(\pi/2) = 2/\pi$.

7. Find the probability that two points chosen at random on a segment of length h are a distance less than k apart.

8. A is a 3 x 2 real matrix, B is a 2 x 3 real matrix. AB = M where det M = 0 BA = det N where det N is non-zero, and $M^2 = kM$. Find det N in terms of k.

9. A solid spheres is fixed to a table. Another sphere of equal radius is placed on top of it at rest. The top sphere rolls off. Show that slipping occurs then the line of centers makes an angle θ to the vertical, where 2 sin θ = μ(17 cos θ - 10). Assume that the top sphere has moment of inertia 2/5 Mr^2 about a diameter, where r is its radius.

8th BMO 1972

1. The relation R is defined on the set X. It has the following two properties: if aRb and bRc then cRa for distinct elements a, b, c; for distinct elements a, b either aRb or bRa but not both. What is the largest possible number of elements in X?

2. Show that there can be at most four lattice points on the hyperbola (x + ay + c)(x + by + d) = 2, where a, b, c, d are integers. Find necessary and sufficient conditions for there to be four lattice points.

3. C and C' are two unequal circles which intersect at A and B. P is an arbitrary point in the plane. What region must P lie in for there to exist a line L through P which contains chords of C and C' of equal length. Show how to construct such a line if it exists by considering distances from its point of intersection with AB or otherwise.

4. P is a point on a curve through A and B such that PA = a, PB = b, AB = c, and ∠APB = θ. As usual, $c^2 = a^2 + b^2 - 2ab \cos \theta$. Show that $\sin^2\theta \, ds^2 = da^2 + db^2 - 2 \, da \, db \cos \theta$, where s is distance along the curve. P moves so that for time t in the interval T/2 < t < T, PA = h cos(t/T), PB = k sin(t/T). Show that the speed of P varies as cosec θ.

5. A cube C has four of its vertices on the base and four of its vertices on the curved surface of a right circular cone R with semi-vertical angle x. Show that if x is varied the maximum value of vol C/vol R is at sin x = 1/3.

6. Define the sequence a_n, by $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, and $a_{2n} = a_{2n-5} + 2^n$, $a_{2n+1} = a_{2n} + 2^{n-1}$. Show that $a_{2n} = [17/7 \, 2^{n-1}] - 1$, $a_{2n-1} = [12/7 \, 2^{n-1}] - 1$.

7. Define sequences of integers by $p_1 = 2$, $q_1 = 1$, $r_1 = 5$, $s_1 = 3$, $p_{n+1} = p_n^2 + 3 q_n^2$, $q_{n+1} = 2 p_n q_n$, $r_n = p_n + 3 q_n$, $s_n = p_n + q_n$. Show that $p_n/q_n > \sqrt{3} > r_n/s_n$ and that p_n/q_n differs from $\sqrt{3}$ by less than $s_n/(2 r_n q_n^2)$.

8. Three children throw stones at each other every minute. A child who is hit is out of the game. The surviving player wins. At each throw each child chooses at random which of his two opponents to aim at. A has probability 3/4 of hitting the child he aims at, B has probability 2/3 and C has probability 1/2. No one ever hits a child he is not aiming at. What is the probability that A is eliminated in the first round and C wins.

9. A rocket, free of external forces, accelerates in a straight line. Its mass is M, the mass of its fuel is m exp(-kt) and its fuel is expelled at velociy v exp(-kt). If m is small compared to M, show that its terminal velocity is mv/(2M) times its initial velocity.

9th BMO 1973

1. A variable circle touches two fixed circles at P and Q. Show that the line PQ passes through one of two fixed points. State a generalisation to ellipses or conics.

2. Given any nine points in the interior of a unit square, show that we can choose 3 which form a triangle of area at most 1/8.

3. The curve C is the quarter circle $x^2 + y^2 = r^2$, $x \geq 0$, $y \geq 0$ and the line segment $x = r$, $0 \geq y \geq -h$. C is rotated about the y-axis for form a surface of revolution which is a hemisphere capping a cylinder. An elastic string is stretched over the surface between $(x, y, z) = (r \sin \theta, r \cos \theta, 0)$ and $(-r, -h, 0)$. Show that if $\tan \theta > r/h$, then the string does not lie in the xy plane. You may assume spherical triangle formulae such as $\cos a = \cos b \cos c + \sin b \sin c \cos A$, or $\sin A \cot B = \sin c \cot b - \cos c \cos A$.

4. n equilateral triangles side 1 can be fitted together to form a convex equiangular hexagon. The three smallest possible values of n are 6, 10 or 13. Find all possible n.

5. Show that there is an infinite set of positive integers of the form $2^n - 7$ no two of which have a common factor.

6. The probability that a teacher will answer a random question correctly is p. The probability that randomly chosen boy in the class will answer correctly is q and the probability that a randomly chosen girl in the class will answer correctly is r. The probability that a randomly chosen pupil's answer is the same as the teacher's answer is 1/2. Find the proportion of boys in the class.

7. From each 10000 live births, tables show that y will still be alive x years later. $y(60) = 4820$ and $y(80) = 3205$, and for some A, B the curve $Ax(100-x) + B/(x-40)^2$ fits the data well for $60 \leq x \leq 100$. Anyone still alive at 100 is killed. Find the life expectancy in years to the nearest 0.1 year of someone aged 70.

8. $T: z \to (az + b)/(cz + d)$ is a map. M is the associated matrix

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

Show that if M is associated with T and M' with T' then the matrix MM' is associated with the map TT'. Find conditions on a, b, c, d for T^4 to be the identity map, but T^2 not to be the identity map.

9. Let $L(\theta)$ be the determinant:

$$\begin{vmatrix} x & y & 1 \\ a + c\cos\theta & b + c\sin\theta & 1 \\ l + n\cos\theta & m + n\sin\theta & 1 \end{vmatrix}$$

Show that the lines are concurrent and find their point of intersection.

10. Write a computer program to print out all positive integers up to 100 of the form $a^2 - b^2 - c^2$ where a, b, c are positive integers and $a \geq b + c$.

11. (1) A uniform rough cylinder with radius a, mass M, moment of inertia $Ma^2/2$ about its axis, lies on a rough horizontal table. Another rough cylinder radius b, mass m, moment of inertia $mb^2/2$ about its axis, rests on top of the first with its axis parallel. The cylinders start to roll. The plane containing the axes makes the angle θ with the vertical. Show the forces during the period when there is no slipping. Write down equations, which will give on elimination a differential equation, but you do not need to find the differential

(2) Such a differential equation is $\theta_2(4 + 2\cos\theta - 2\cos^2\theta + 9k/2) + \theta_1^2 \sin\theta\,(2\cos\theta - 1) = 3g(1 + k)(\sin\theta/(a + b)$, where $k = M/m$. Find θ_1 in terms of θ. Here θ_1 denotes $d\theta/dt$ and θ_2 denotes the second derivative.

10th BMO 1974

1. C is the curve $y = 4x^2/3$ for $x \geq 0$ and C' is the curve $y = 3x^2/8$ for $x \geq 0$. Find curve C" which lies between them such that for each point P on C" the area bounded by C, C" and a horizontal line through P equals the area bounded by C", C and a vertical line through P.

2. S is the set of all 15 dominoes (m, n) with $1 \leq m \leq n \leq 5$. Each domino (m, n) may be reversed to (n, m). How many ways can S be partitioned into three sets of 5 dominoes, so that the dominoes in each set can be arranged in a closed chain: (a, b), (b, c), (c, d), (d, e), (e, a)?

3. Show that there is no convex polyhedron with all faces hexagons.

4. A is the 16 x 16 matrix $(a_{i,j})$. $a_{1,1} = a_{2,2} = \dots = a_{16,16} = a_{16,1} = a_{16,2} = \dots = a_{16,15} = 1$ and all other entries are 1/2. Find A^{-1}.

5. In a standard pack of cards every card is different and there are 13 cards in each of 4 suits. If the cards are divided randomly between 4 players, so that each gets 13 cards, what is the probability that each player gets cards of only one suit?

6. ABC is a triangle. P is equidistant from the lines CA and BC. The feet of the perpendiculars from P to CA and BC are at X and Y. The perpendicular from P to the line AB meets the line XY at Z. Show that the line CZ passes through the midpoint of AB.

7. b and c are non-zero. $x^3 = bx + c$ has real roots α, β, γ. Find a condition which ensures that there are real p, q, r such that $\beta = p\alpha^2 + q\alpha + r$, $\gamma = p\beta^2\ q\beta + r$, $\alpha = p\gamma^2 + q\gamma + r$.

8. p is an odd prime. The product $(x + 1)(x + 2) \dots (x + p - 1)$ is expanded to give $a_{p-1}x^{p-1} + \dots + a_1x + a_0$. Show that $a_{p-1} = 1$, $a_{p-2} = p(p-1)/2!$, $2a_{p-3} = p(p-1)(p-2)/3! + a_{p-2}(p-1)(p-2)/2!$, ... , $(p-2)a_1 = p + a_{p-2}(p-1) + a_{p-3}(p-2) + \dots + 3a_2$, $(p-1)a_0 = 1 + a_{p-2} + \dots + a_1$. Show that $a_1, a_2, \dots, a_{p-2}$ are divisible by p and $(a_0 + 1)$ is divisible by p. Show that for any integer x, $(x+1)(x+2) \dots (x+p-1) - x^{p-1} + 1$ is divisible by p. Deduce Wilson's theorem that p divides $(p-1)! + 1$ and Fermat's theorem that p divides $x^{p-1} - 1$ for x not a multiple of p.

9. A uniform rod is attached by a frictionless joint to a horizontal table. At time zero it is almost vertical and starts to fall. How long does it take to reach the table? You may assume that $\int \operatorname{cosec} x\, dx = \log |\tan x/2|$.

10. A long solid right circular cone has uniform density, semi-vertical angle x and vertex V. All points except those whose distance from V lie in the range a to b are removed. The resulting solid has mass M. Show that the gravitational attraction of the solid on a point of unit mass at V is $3/2\ GM(1 + \cos x)/(a^2 + ab + b^2)$.

11th BMO 1975

1. Find all positive integer solutions to $[1^{1/3}] + [2^{1/3}] + \dots + [(n^3 - 1)^{1/3}] = 400$.

2. The first k primes are divided into two groups. n is the product of the first group and n is the product of the second group. M is any positive integer divisible only by primes in the first group and N is any positive integer divisible only by primes in the second group. If d > 1 divides Mm - Nn, show that d exceeds the kth prime.

3. Show that if a disk radius 1 contains 7 points such that the distance between any two is at least 1, then one of the points must be at the center of the disk. [You may wish to use the pigeonhole principle.]

4. ABC is a triangle. Parallel lines are drawn through A, B, C meeting the lines BC, CA, AB at D, E, F respectively. Collinear points P, Q, R are taken on the segments AD, BE, CF respectively such that AP/PD = BQ/CE = CR/RF = k. Find k.

5. Let nCr represent the binomial coefficient n!/((n-r)! r!). Define f(x) = (2m)C0 + (2m)C1 cos x + (2m)C2 cos 2x + (2m)C3 cos 3x + ... + (2m)C(2m) cos 2mx. Let g(x) = (2m)C0 + (2m)C2 cos 2x + (2m)C4 cos 4x + ... + (2m)C(2m) cos 2mx. Find all x such that x/π is irrational and $\lim_{m\ \infty}$ g(x)/f(x) = 1/2. You may use the identity: f(x) = $(2 \cos(x/2))^{2m}$ cos mx.

6. Show that for n > 1 and real numbers x > y > 1, $(x^{n+1} - 1)/(x^n - x) > (y^{n+1} - 1)/(y^n - y)$.

7. Show that for each n > 0 there is a unique set of real numbers $x_1, x_2, \dots , x_n$ such that $(1 - x_1)^2 + (x_1 - x_2)^2 + \dots + (x_{n-1} - x_n)^2 + x_n^2 = 1/(n + 1)$.

8. A wine glass has the shape of a right circular cone. It is partially filled with water so that when tilted the water just touches the lip at one end and extends halfway up at the other end. What proportion of the glass is filled with water?

12th BMO 1976

1. ABC is a triangle area k. Let d be the length of the shortest line segment which bisects the area of the triangle. Find d. Give an example of a curve which bisects the area and has length < d.

2. Prove that x/(y + z) + y/(z + x) + z/(x + y) ≥ 3/2 for any positive reals x, y, z.

3. Given 50 distinct subsets of a finite set X, each containing more than | X |/2 elements, show that there is a subset of X with 5 elements which has at least one element in common with each of the 50 subsets.

4. Show that $8^n19 + 17$ is not prime for any non-negative integer n.

5. aCb represents the binomial coefficient a!/((a - b)! b!). Show that for n a positive integer, r ≤ n and odd, r' = (r - 1)/2 and x, y reals we have: $\sum_0^{r'}$ nC(r-i) nCi $(x^{r-i}y^i + x^iy^{r-i}) = \sum_0^{r'}$ nC(r-i) (r-i)Ci $x^iy^i(x + y)^{r-2i}$.

6. A sphere has center O and radius r. A plane p, a distance r/2 from O, intersects the sphere in a circle C center O'. The part of the sphere on the opposite side of p to O is removed. V lies on the ray OO' a distance 2r from O'. A cone has vertex V and base C, so with the remaining part of the sphere it forms a surface S. XY is a diameter of C. Q is a

p. P is a point on VY such that the shortest path from P to Q along the surface S cuts C at 45 deg. Show that $VP = r\sqrt{3} / \sqrt{1 + 1/\sqrt{5}}$.

Problem 2

Prove that $x/(y + z) + y/(z + x) + z/(x + y) \geq 3/2$ for any positive reals x, y, z.

Solution

If $x + y + z = k$, then $(1/(k-x) + 1/(k-y) + 1/(k-z))(k-x + k-y + k-z) \geq 9$ by Cauchy Schwartz. Hence $1/(y+z) + 1/(z+x) + 1/(x+y) \geq 9/(2k)$. So $k/(y+z) + k/(z+x) + k/(x+y) \geq 9/2$. Subtracting $(y+z)/(y+z) + (z+x)/(z+x) + (x+y)/(x+y) = 3$ gives the result.

13th BMO 1977

1. f(n) is a function on the positive integers with non-negative integer values such that: (1) $f(mn) = f(m) + f(n)$ for all m, n; (2) $f(n) = 0$ if the last digit of n is 3; (3) $f(10) = 0$. Show that $f(n) = 0$ for all n.

2. S is either the incircle or one of the excircles of the triangle ABC. It touches the line BC at X. M is the midpoint of BC and N is the midpoint of AX. Show that the center of S lies on the line MN.

3. (1) Show that $x(x - y)(x - z) + y(y - z)(y - x) + z(z - x)(z - y) \geq 0$ for any non-negative reals x, y, z.
(2) Hence or otherwise show that $x^6 + y^6 + z^6 + 3x^2y^2z^2 \geq 2(y^3z^3 + z^3x^3 + x^3y^3)$ for all real x, y, z.

4. $x^3 + qx + r = 0$, where r is non-zero, has roots u, v, w. Find the roots of $r^2x^3 + q^3x + q^3 = 0$ (*) in terms of u, v, w. Show that if u, v, w are all real, then (*) has no real root x satisfying $-1 < x < 3$.

5. Five spheres radius a all touch externally two spheres S and S' of radius a. We can find five points, one on each of the first five spheres, which form the vertices of a regular pentagon side 2a. Do the spheres S and S' intersect?

6. Find all n > 1 for which we can write $26(x + x^2 + x^3 + \dots + x^n)$ as a sum of polynomials of degree n, each of which has coefficients which are a permutation of 1, 2, 3, ... , n.

14th BMO 1978

1. Find the point inside a triangle which has the largest product of the distances to the three sides.

2. Show that there is no rational number m/n with $0 < m < n < 101$ whose decimal expansion has the consecutive digits 1, 6, 7 (in that order).

3. Show that there is a unique sequence $a_1, a_2, a_3, \dots$ such that $a_1 = 1$, $a_2 > 1$, $a_{n+1}a_{n-1} = a_n^3 + 1$, and all terms are integral.

4. An altitude of a tetrahedron is a perpendicular from a vertex to the opposite face. Show that the four altitudes are concurrent iff each pair of opposite edges is perpendicular.

5. There are 11000 points inside a cube side 15. Show that there is a sphere radius 1 which contains at least 6 of the points.

6. Show that 2 cos nx is a polynomial of degree n in (2 cos x). Hence or otherwise show that if k is rational then cos kπ is 0, ±1/2, ±1 or irrational.

15th BMO 1979

1. Find all triangles ABC such that AB + AC = 2 and AD + BD = √5, where AD is the altitude.

2. Three rays in space have endpoints at O. The angles between the pairs are α, β, γ, where $0 < \alpha < \beta < \gamma$. Show that there are unique points A, B, C, one on each ray, so that the triangles OAB, OBC, OCA all have perimeter 2s. Find their distances from O.

3. Show that the sum of any n distinct positive odd integers whose pairs all have different differences is at least $n(n^2 + 2)/3$.

4. f(x) is defined on the rationals and takes rational values. f(x + f(y)) = f(x) f(y) for all x, y. Show that f must be constant.

5. Let p(n) be the number of partitions of n. For example, p(4) = 5: 1 + 1 + 1 + 1, 1 + 1 + 2, 2 + 2, 1 + 3, 4. Show that $p(n+1) \ge 2p(n) - p(n-1)$.

6. Show that the number $1 + 10^4 + 10^8 + \dots + 10^{4n}$ is not prime for n > 0.

16th BMO 1980

1. Show that there are no solutions to $a^n + b^n = c^n$, with n > 1 is an integer, and a, b, c are positive integers with a and b not exceeding n.

2. Find a set of seven consecutive positive integers and a polynomial p(x) of degree 5 with integer coefficients such that p(n) = n for five numbers n in the set including the smallest and largest, and p(n) = 0 for another number in the set.

3. AB is a diameter of a circle. P, Q are points on the diameter and R, S are points on the same arc AB such that PQRS is a square. C is a point on the same arc such that the triangle ABC has the same area as the square. Show that the incenter I of the triangle ABC lies on one of the sides of the square and on the line joining A or B to R or S.

4. Find all real a_0 such that the sequence $a_0, a_1, a_2, \dots$ defined by $a_{n+1} = 2^n - 3a_n$ has $a_{n+1} > a_n$ for all $n \ge 0$.

5. A graph has 10 points and no triangles. Show that there are 4 points with no edges between them.

17th BMO 1981 - Further International Selection Test

1. ABC is a triangle. Three lines divide the triangle into four triangles and three pentagons. One of the triangle has its three sides along the new lines, the others each have just two sides along the new lines. If all four triangles are congruent, find the area of each in terms of the area of ABC.

2. An axis of a solid is a straight line joining two points on its boundary such that a rotation about the line through an angle greater than 0 deg and less than 360 deg brings the solid into coincidence with itself. How many such axes does a cube have? For each axis indicate the minimum angle of rotation and how the vertices are permuted.

3. Find all real solutions to $x^2y^2 + x^2z^2 = axyz$, $y^2z^2 + y^2x^2 = bxyz$, $z^2x^2 + z^2y^2 = cxyz$, where a, b, c are fixed reals.

4. Find the remainder on dividing $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

5. The sequence $u_0, u_1, u_2, \dots$ is defined by $u_0 = 2$, $u_1 = 5$, $u_{n+1}u_{n-1} - u_n^2 = 6^{n-1}$. Show that all terms of the sequence are integral.

6. Show that for rational c, the equation $x^3 - 3cx^2 - 3x + c = 0$ has at most one rational root.

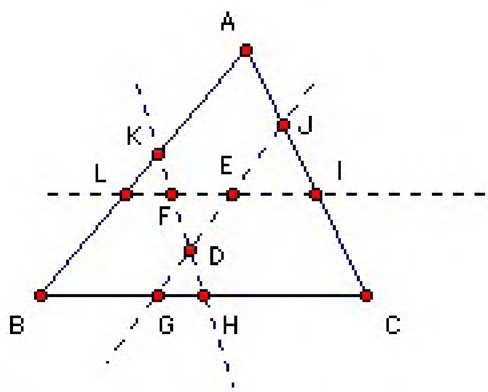
7. If x and y are non-negative integers, show that there are non-negative integers a, b, c, d such that $x = a + 2b + 3c + 7d$, $y = b + 2c + 5d$ iff $5x \geq 7y$.

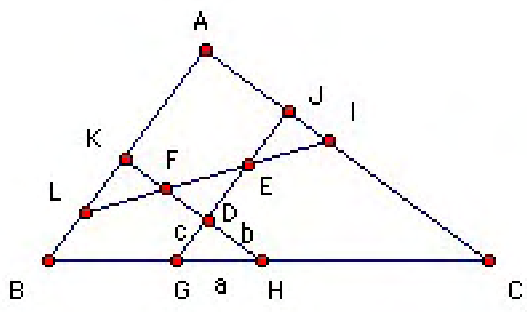
Problem 1

ABC is a triangle. Three lines divide the triangle into four triangles and three pentagons. One of the triangle has its three sides along the new lines, the others each have just two sides along the new lines. If all four triangles are congruent, find the area of each in terms of the area of ABC.

Solution

The obvious configuration has $\angle EDF = \angle EJI$, so KH is parallel to AC and similarly LI parallel to BC, and GJ parallel to AB. Then to get equal area we need E,F to trisect LI and similarly for the other two lines. Suppose GH = kBC. Then GJ = 3kAB, so CG = 3kBC. Hence CH = 2kBC and similarly BG = 2kBC, so k = 1/5 and the





area of each small triangle is 1/25 area ABC.

However, there are other configurations. For example, suppose we take GJ parallel to AB and KH parallel to AC, but LI not parallel to BC. Take GH = a, DH = b, GD = c. Then GDH is evidently similar to BAC because of the parallel lines. We find DE = b, EJ = b, so GJ = 2b+c. GJC is similar to GDH, so GC = a(2b+c)/c. Similarly, DF = KF = c, so KH = b+2c. Hence BH = a(b+2c)/b. Hence BC = $a(bc+2b^2+2c^2)/bc$. So BC/GH = $(bc+2b^2+2c^2)/bc$ and hence the area of a small triangle is $1/(1 + 2b/c + 2c/b)^2$ times the area of ABC. Here the ratio can take any value less than 1/25 depending on the ratio b/c.

Problem 2

An *axis* of a solid is a straight line joining two points on its boundary such that a rotation about the line through an angle greater than 0 deg and less than 360 deg brings the solid into coincidence with itself. How many such axes does a cube have? For each axis indicate the minimum angle of rotation and how the vertices are permuted.

Answer
3 through the centers of two opposite faces, minimum angle 90^o
6 through the midpoints of two opposite edges, minimum angle 180^o
4 through opposite vertices, minimum angle 120^o

Problem 3
Find all real solutions to $x^2y^2 + x^2z^2 = axyz$, $y^2z^2 + y^2x^2 = bxyz$, $z^2x^2 + z^2y^2 = cxyz$, where a, b, c are fixed reals.

Answer
(x,y,z) = (t,0,0), (0,t,0), (0,0,t) are solutions for any t and any a,b,c.
If a, b, c are the sides of a triangle (so a,b,c > 0 and a+b > c etc), or -a, -b, -c are the sides of a triangle, then $x = \pm\sqrt{(s-b)(s-c)}$, $y = \pm\sqrt{(s-a)(s-c)}$, $z = \pm\sqrt{(s-a)(s-b)}$, where we have either 0 or 2 minus signs, are solutions.

Solution
If x = 0, then $y^2z^2 = 0$, so y or z = 0. If x = y = 0, then z can be arbitrary. Similarly for the other pairs. Thus we get solutions (x,y,z) = (0,0,t), (0,t,0), (t,0,0) for any t. So suppose $xyz \neq 0$.
Subtracting the second equation from the first, $x^2z^2-y^2z^2 = (a-b)xyz$. Adding to the last equation, $x^2z^2 = (a-b+c)xyz/2$. Similarly $y^2z^2 = (-a+b+c)xyz/2$. Hence $x^2y^2z^4 = (a-b+c)(-a+b+c)x^2y^2z^2/4$, so $z^2 = (a-b+c)(-a+b+c)/4 = (s-a)(s-b)$, where $s = (a+b+c)/2$. Similarly, $x^2 = (s-b)(s-c)$, $y^2 = (s-a)(s-c)$. However, this solution is only possible if a, b, c satisfy certain conditions.
If s-a > 0, then $z^2 > 0$ implies s-b > 0 and $x^2 > 0$ implies s-c > 0. Adding s-a > 0 and s-b > 0 gives c > 0. Similarly, a > 0 and b > 0. So a, b, c must be the sides of a triangle.
Similarly, if s-a < 0, then s-b < 0 and s-c < 0, and hence a, b, c < 0. In this case -a, -b, -c are the sides of a triangle.

Problem 4
Find the remainder on dividing $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

Solution
Put $p(x) = x^{81} + x^{49} + x^{25} + x^9 + x = q(x)(x^3 - x) + ax^2 + bx + c$. Putting x = 0 gives c = 0. Putting x = 1 gives 5 = a + b, putting x = -1 gives -5 = a - b, so a = 0, b = 5. Hence the remainder is 5x.

Problem 5
The sequence $u_0, u_1, u_2, \ldots$ is defined by $u_0 = 2$, $u_1 = 5$, $u_{n+1}u_{n-1} - u_n^2 = 6^{n-1}$. Show that all terms of the sequence are integral.

Solution
We show by induction that $u_n = 2^n + 3^n$. True for n = 0 and 1. Suppose it is true for n and n-1, then $u_{n+1}(2^{n-1}+3^{n-1}) - (2^n+3^n)^2 = 6^{n-1}$. Hence $u_{n+1}(2^{n-1}+3^{n-1}) = 2^{2n} + 3^{2n} + 13\cdot6^{n-1} = (2^{n+1}+3^{n+1})(2^{n-1}+3^{n-1})$. Hence $u_{n+1} = 2^{n+1}+3^{n+1}$, so the result is true for n+1 and hence for all n.

Problem 6
Show that for rational c, the equation $x^3 - 3cx^2 - 3x + c = 0$ has at most one rational root.

Solution

Suppose it has a rational root k. Then $c = (k^3-3k)/(3k^2-1)$, so the equation is $(3k^2-1)x^3 - 3(k^3-3k)x^2 - 3(3k^2-1)x + k^3-3k = 0$, which factorises as $(x-k)((3k^2-1)x^2 + 8kx - (k^2-3)) = 0$. The roots of the quadratic are $-4k/(3k^2-1) \pm (\sqrt{3})(k^2+1)/(3k^2-1)$, which are irrational.

Problem 7

If x and y are non-negative integers, show that there are non-negative integers a, b, c, d such that $x = a + 2b + 3c + 7d$, $y = b + 2c + 5d$ iff $5x \geq 7y$.

Solution

If $5x = 7y$, then $y = 5d$ for some non-negative integer d, and we can take $a = b = c = 0$. If $5x = 7y+1$, then $x = 7d+3$ for some non-negative integer d, so $y = 5d+2$, $c = 1$, $a = b = 0$. If $5x = 7y+2$, then $x = 7d+6$, so $y = 5d+4$, $c = 2$, $a = b = 0$. If $5x = 7y+3$, then $x = 7d+2$, $y = 5d+1$, $b = 1$, $a = c = 0$. If $5x = 7y+4$, then $x = 7d+5$, $y = 5d+3$, $a = 0$, $b = c = 1$. If $5x = 7y + n$, for $n > 4$, then we can take m as the residue of n mod 5, $a = n-m$ and b,c,d as above.

The implication the other way is trivial.