Iberoamerican Mathematical Olympiads

1st Iberoamerican 1985

Problem A1

Find all integer solutions to: $a + b + c = 24$, $a^2 + b^2 + c^2 = 210$, $abc = 440$.

Solution

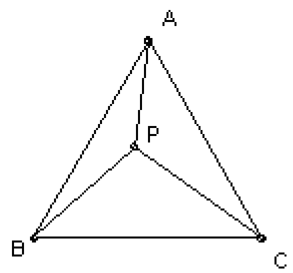
$ab + bc + ca = ((a + b + c)^2 - (a^2 + b^2 + c^2))/2 = 183$, so a, b, c are roots of the cubic $x^3 - 24x^2 + 183x - 440 = 0$. But it easily factorises as $(x - 5)(x - 8)(x - 11) = 0$, so the only solutions are permutations of (5, 8, 11).

Problem A2

P is a point inside the equilateral triangle ABC such that PA = 5, PB = 7, PC = 8. Find AB.

Solution

Answer: $\sqrt{129}$.



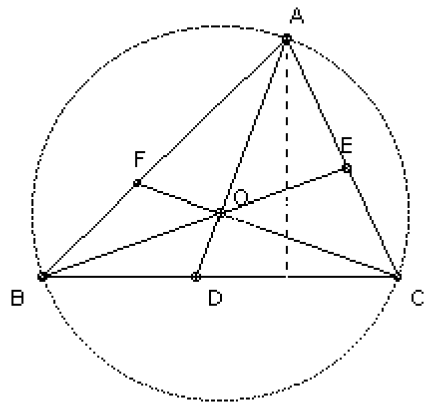
Let the side length be x. Using the cosine formula, we have $\cos APB = (74 - x^2)/70$, $\cos APC = (89 - x^2)/80$, $\cos BPC = (113 - x^2)/112$. But $\cos BPC = \cos APC \cos BPC - \sin APC \sin BPC$, so $(113 - x^2)/112 = (74 - x^2)/79\ (89 - x^2)/80 - \sqrt{((1 - (74 - x^2)^2/70^2) (1 - (89 - x^2)^2/80^2))}$.

We isolate the square root term, then square. We multiply through by 25.256.49 and, after some simplification, we get $x^6 - 138x^4 + 1161x^2 = 0$. Hence $x = 0, \pm 3, \pm\sqrt{129}$. We discard the zero and negative solutions. $x = 3$ corresponds to a point P outside the triangle. So the unique solution for a point P inside the triangle is $x = \sqrt{129}$.

Alternative solution by Johannes Tang

Rotate the triangle about C through 60^o. Let P go to P'. We have AP' = 7, CP' = 8 and angle PCP' = 60^o, so PP'C is equilateral. Hence angle CPP' = 60^o. Also PP' = 8. Using the cosine formula on triangle APP' we find angle APP' = 60^o. Hence angle APC = 120^o. Now applying cosine formula to triangle APC, we get result.

Problem A3



Find the roots r_1, r_2, r_3, r_4 of the equation $4x^4 - ax^3 + bx^2 - cx + 5 = 0$, given that they are positive reals satisfying $r_1/2 + r_2/4 + r_3/5 + r_4/8 = 1$.

Solution

We have $r_1r_2r_3r_4 = 5/4$ and hence $(r_1/2) (r_2/4) (r_3/5) (r_4/8) = 1/4^4$. But AM/GM gives that $(r_1/2) (r_2/4) (r_3/5) (r_4/8) \le ((r_1/2 + r_2/4 + r_3/5 + r_4/8)/4)^4 = 1/4^4$ with equality iff $r_1/2 = r_2/4 = r_3/5 = r_4/8$. Hence we must have $r_1 = 1/2$, $r_2 = 1$, $r_3 = 5/4$, $r_4 = 2$.

Problem B1

The reals x, y, z satisfy $x \ne 1$, $y \ne 1$, $x \ne y$, and $(yz - x^2)/(1 - x) = (xz - y^2)/(1 - y)$. Show that $(yx - x^2)/(1 - x) = x + y + z$.

Solution

We have $yz - x^2 - y^2z + yx^2 = xz - y^2 - x^2z + xy^2$. Hence $z(y - x - y^2 + x^2) = -y^2 + xy^2 - x^2y + x^2$. Hence $z = (x + y - xy)/(x + y - 1)$.

So $yz = x + y + z - xy - xz$, so $yz - x^2 = x + y + z - x^2 - xy - xz = (x + y + z)(1 - x)$, so $(yz - x^2)/(1 - x) = (x + y + z)$.

Problem B2

The function f(n) is defined on the positive integers and takes non-negative integer values. It satisfies (1) f(mn) = f(m) + f(n), (2) f(n) = 0 if the last digit of n is 3, (3) f(10) = 0. Find f(1985).

Solution

If f(mn) = 0, then f(m) + f(n) = 0 (by (1)). But f(m) and f(n) are non-negative, so f(m) = f(n) = 0. Thus f(10) = 0 implies f(5) = 0. Similarly f(3573) = 0 by (2), so f(397) = 0. Hence f(1985) = f(5) + f(397) = 0.

Problem B3

O is the circumcenter of the triangle ABC. The lines AO, BO, CO meet the opposite sides at D, E, F respectively. Show that 1/AD + 1/BE + 1/CF = 2/AO.

Solution

Projecting onto the altitude from A, we have AD cos(C - B) = AC sin C = 2R sin B sin C, so 2R/AD = cos(C - B)/(sin B sin C).

Hence 2R/AD + 2R/BE + 2R/CF =cos(C - B)/(sin B sin C) + cos(A - C)/(sin C sin A) + cos(B - A)/(sin A sin B). So 2R sin A sin B sin C (1/AD + 1/BE + 1/CF) = sin A cos(B - C) + sin B cos(C - A) + sin C cos(A - B) = 3 sin A sin B sin C + sin A cos B cos C + sin B cos A cos C +

sin C cos A cos B = 3 sin A sin B sin C + sin(A + B) cos C + sin C cos A cos B = 3 sin A sin B sin C + sin C (cos C + cos A cos B) = 3 sin A sin B sin C + sin C (-cos(A + B) + cos A cos B) = 4 sin A sin B sin C. Hence 1/AD + 1/BE + 1/CF = 2/R.

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Problem A1

Find f(x) such that $f(x)^2 f((1-x)/(1+x)) = 64x$ for x not 0, ±1.

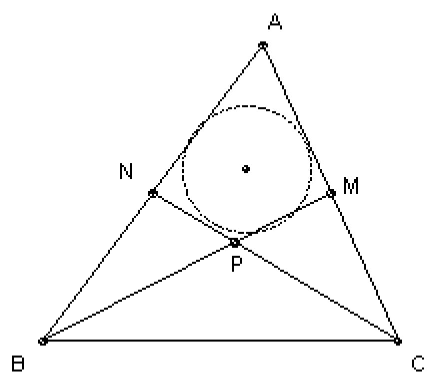
Solution

Put x = (1-y)/(1+y), then (1--x)/(1+x) = y, so $f((1-y)/(1+y))^2 f(y) = 64(1-y)/(1+y)$. Hence $f((1-x)/(1+x))^2 f(x) = 64(1-x)/(1+x)$. But $f(x)^4 f((1-x)/(1+x))^2 = 64^2x^2$, so $f(x)^3 = 64 x^2(1 + x)/(1 - x)$. Hence $f(x) = 4 (x^2(1 + x)/(1 - x))^{1/3}$.

Problem A2

In the triangle ABC, the midpoints of AC and AB are M and N respectively. BM and CN meet at P. Show that if it is possible to inscribe a circle in the quadrilateral AMPN (touching every side), then ABC is isosceles.

Solution



If the quadrilateral has an inscribed circle then AM + PN = AN + PM (consider the tangents to the circle from A, M, P, N). But if AB > AC, then BM > CN (see below). We have AN = AB/2, PM = BM/3, AM = AC/2, PN = CN/3, so it follows that AM + PN < AN + PM. Similarly, AB < AC implies AM + PN > AN + PM, so the triangle must be isosceles.

To prove the result about the medians, note that $BM^2 = BC^2 + CM^2 - 2\ BC.CM \cos C = (BC - CM \cos C)^2 + (CM \sin C)^2$. Similarly, $CN^2 = (BC - BN \cos B)^2 + (BN \sin B)^2$. But MN is parallel to BC, so CM sin C = BN sin B. But AB > AC, so BN > CM and B < C, so cos B > cos C, hence BN cos B > CM cos C and BC - CM cos C > BC - BN cos B. So BM > CN.

Problem A3

Show that if $(2 + \sqrt{3})^k = 1 + m + n\sqrt{3}$, for positive integers m, n, k with k odd, then m is a perfect square.

Solution

We have $(2 + \sqrt{3})^4 = 97 + 56\sqrt{3} = 14 (7 + 4\sqrt{3}) - 1 = 14 (2 + \sqrt{3})^2 - 1$. Hence $(2 + \sqrt{3})^{k+2} = 14 (2 + \sqrt{3})^k - (2 + \sqrt{3})^{k-2}$. Thus if $(2 + \sqrt{3})^k = a_k + b_k\sqrt{3}$, then $a_{k+2} = 14 a_k - a_{k-2}$.

Now suppose the sequence c_k satisfies $c_1 = 1$, $c_2 = 5$, $c_{k+1} = 4 c_k - c_{k-1}$. We claim that $c_k^2 - c_{k-1}c_{k+1} = 6$. Induction on k. We have $c_3 = 19$, so $c_2^2 - c_1c_3 = 25 - 19 = 6$. Thus the result is true for k = 2. Suppose it is true for k. Then $c_{k+1} = 4 c_k - c_{k-1}$, so $c_{k+1}^2 = 4 c_kc_{k+1} - c_{k-1}c_{k+1} = 4 c_kc_{k+1} - c_k^2 + 6 = c_k(4 c_{k+1} - c_k) + 6 = c_kc_{k+2} + 6$, so the result is true for k+1.

Now put $d_k = c_k^2 + 1$. We show that $d_{k+2} = 14 d_{k+1} - d_k$. Induction on k. We have $d_1 = 2$, $d_2 = 26$, $d_3 = 362 = 14 d_2 - d_1$, so the result is true for k = 1. Suppose it is true for k. We have $c_{k+3} - 4 c_{k+2} + c_{k+1} = 0$. Hence $12 + 2 c_{k+3}c_{k+1} - 8 c_{k+2}c_{k+1} + 2 c_{k+1}^2 = 12$. Hence $2 c_{k+2}^2 - 8 c_{k+2}c_{k+1} + 2 c_{k+1}^2 = 12$. Hence $16 c_{k+2}^2 - 8 c_{k+2}c_{k+1} + c_{k+1}^2 + 1 = 14 c_{k+2}^2 + 14 - c_{k+1}^2 - 1$, or $(4 c_{k+2} - c_{k+1})^2 + 1 = 14 (c_{k+2}^2 + 1) - (c_{k+1}^2 + 1)$, or $c_{k+3}^2 + 1 = 14 (c_{k+2}^2 + 1) - (c_{k+1}^2 + 1)$, or $d_{k+3} = 14 d_{k+2} - d_{k+1}$. So the result is true for all k.

But $a_1 = 2$, $a_3 = 26$ and $a_{2k+3} = 14 a_{2k+1} - a_{2k-1}$, and $d_1 = 2$, $d_2 = 26$ and $d_{k+1} = 14 d_k - d_{k-1}$. Hence $a_{2k-1} = d_k = c_k^2 + 1$.

Problem B1

Define the sequence $p_1, p_2, p_3, \ldots$ as follows. $p_1 = 2$, and p_n is the largest prime divisor of $p_1p_2 \ldots p_{n-1} + 1$. Prove that 5 does not occur in the sequence.

Problem B2

Show that the roots r, s, t of the equation $x(x - 2)(3x - 7) = 2$ are real and positive. Find $\tan^{-1}r + \tan^{-1}s + \tan^{-1}t$.

Solution

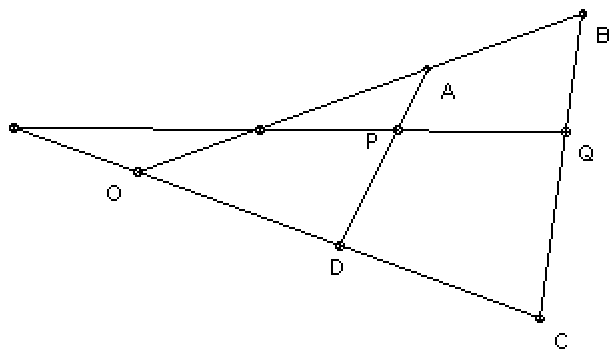
Put $f(x) = x(x - 2)(3x - 7) - 2 = 3x^3 - 13x^2 + 14x - 2$. Then $f(0) = -2$, $f(1) = 2$, so there is a root between 0 and 1. $f(2) = -2$, so there is another root between 1 and 2. $f(3) = 4$, so the third root is between 2 and 3. $f(x) = 0$ has three roots, so they are all real and positive.

We have $\tan(a + b + c) = (\tan a + \tan b + \tan c - \tan a \tan b \tan c)/(1 - (\tan a \tan b + \tan b \tan c + \tan c \tan a))$. So putting $a = \tan^{-1}r$, $b = \tan^{-1}s$, $c = \tan^{-1}t$, we have, $\tan(a + b + c) = ((r + s + t) - rst)/(1 - (rs + st + tr)) = (13/3 - 2/3)/(1 - 14/3) = -1$. So $a + b + c = -\pi/4 + k\pi$. But we know that each of r, s, t is real and positive, so $a + b + c$ lies in the range 0 to $3\pi/2$. Hence $a + b + c = 3\pi/4$.

Problem B3

ABCD is a convex quadrilateral. P, Q are points on the sides AD, BC respectively such that AP/PD = BQ/QC = AB/CD. Show that the angle between the lines PQ and AB equals the angle between the lines PQ and CD.

Solution



If AB is parallel to CD, then it is obvious that PQ is parallel to both. So assume AB and CD meet at O. Take O as the origin for vectors. Let **e** be a unit vector in the direction OA and **f** a unit vector in the direction OC. Take the vector OA to be a**e**, OB to be b**e**, OC to be c**f**, and OD to be d**f**. Then OP is ((d - c)a**e** + (a - b)d**f**)/(d - c + a - b) and OQ is ((d - c)b**e** + (a - b)c**f**)/(d - c + a - b). Hence PQ is (c - d)(a - b)(**e** + **f**)/(d - c + a - b). But **e** and **f** are unit vectors, so **e**+ **f** makes the same angle with each of them and hence PQ makes the same angle with AB and CD.

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Problem A1

The sides of a triangle form an arithmetic progression. The altitudes also form an arithmetic progression. Show that the triangle must be equilateral.

Solution

Let the sides be a, a+d, a+2d with d >= 0. Then the altitudes are $k/a \ge k/(a+d) \ge k/(a+2d)$, where k is twice the area. We claim that $k/a + k/(a+2d) > 2k/(a + d)$ unless d = 0. This is equivalent to $(a + d)(a + 2d) + a(a + d) > 2a(a + 2d)$ or $2d^2 > 0$, which is obviously true. So the altitudes can only form an arithmetic progression if d = 0 and hence the triangle is equilateral.

Problem A2

The positive integers a, b, c, d, p, q satisfy ad - bc = 1 and a/b > p/q > c/d. Show that q >= b + d and that if q = b + d, then p = a + c.

Solution

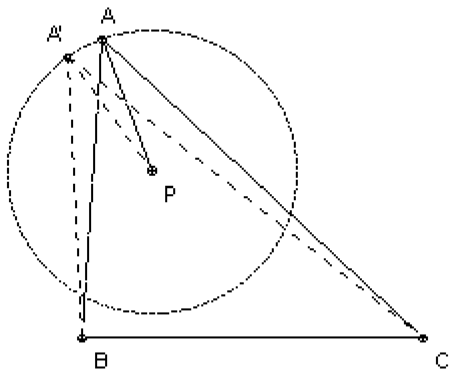
p/q > c/d implies pd > cq and hence pd >= cq + 1, so $p/q \ge c/d + 1/(qd)$. Similarly, a/b > p/q implies $a/b \ge p/q + 1/(bq)$. So $a/b - c/d \ge 1/(qd) + 1/(qb) = (b + d)/(qbd)$. But a/b - c/d = 1/bd. Hence $q \ge b + d$.

Now assume q = b + d. We have $ad - bc = 1 \le d$, so $ad + cd - d \le bc + cd$ and hence $(a+c-1)/(b+d) \le c/d$. So $p \ge a + c$. Similarly $ad - bc \le b$, so $bc + b + ab \ge ad + ab$, so $(a+c+1)/(b+d) \ge a/b$. So $p \le a + c$. Hence p = a + c.

Problem A3

P is a fixed point in the plane. Show that amongst triangles ABC such that PA = 3, PB = 5, PC = 7, those with the largest perimeter have P as incenter.

Solution



Given points P, B, C and a fixed circle center P, we show that the point A on the circle which maximises AB + AC is such that PA bisects angle BAC. Consider a point A' close to A. Then the change in AB + AC as we move A to A' is AA'(sin PAC - sin PAB) + O(AA'^2). So for a maximal configuration we must have sin PAC = sin PAB, otherwise we could get a larger sum by taking A' on one side or the other. This applies to each vertex of the triangle, so P must be the incenter.

Problem B1

Points $A_1, A_2, \dots , A_n$ are equally spaced on the side BC of the triangle ABC (so that $BA_1 = A_1A_2 = \dots = A_{n-1}A_n = A_nC$). Similarly, points $B_1, B_2, \dots , B_n$ are equally spaced on the side CA, and points $C_1, C_2, \dots , C_n$ are equally spaced on the side AB. Show that $(AA_1^2 + AA_2^2 + \dots + AA_n^2 + BB_1^2 + BB_2^2 + \dots + BB_n^2 + C_1^2 + \dots + CC_n^2)$ is a rational multiple of $(AB^2 + BC^2 + CA^2)$.

Solution

Using the cosine formula, $AA_k^2 = AB^2 + k^2BC^2/(n+1)^2 - 2k\ AB.BC/(n+1) \cos B$. So $\sum AA_k^2 = n\ AB^2 + BC^2/(n+1)^2 (1^2 + 2^2 + \dots + n^2) - 2\ AB.BC \cos B\ (1 + 2 + \dots + n)/(n+1)$. Similarly for the other two sides.

Thus the total sum is $n\ (AB^2 + BC^2 + CA^2) + n(2n+1)/(6(n+1)\ )\ (AB^2 + BC^2 + CA^2) - n\ (AB.BC \cos B + BC.CA \cos C + CA.AB \cos A)$. But $AB.BC \cos B = (AB^2 + BC^2 - CA^2)/2$, so $AB.BC \cos B + BC.CA \cos C + CA.AB \cos A = (AB^2 + BC^2 + CA^2)/2$. Thus the sum is rational multiple of $(AB^2 + BC^2 + CA^2)$.

Problem B2

Let $k^3 = 2$ and let x, y, z be any rational numbers such that $x + y\,k + z\,k^2$ is non-zero. Show that there are rational numbers u, v, w such that $(x + y\,k + z\,k^2)(u + v\,k + w\,k^2) = 1$.

Solution

We need $xu + 2zv + 2yw = 1$, $yu + xv + 2zw = 0$, $zu + yv + xw = 0$. This is just a straightforward set of linear equations. Solving, we get: $u = (x^2 - 2yz)/d$, $v = (2z^2 - xy)/d$, $w = (y^2 - xz)/d$, were $d = x^3 + 2y^3 + 4z^3 - 6xyz$.

This would fail if $d = 0$. But if $d = 0$, then multiplying through by a suitable integer we have $6mnr = m^3 + 2n^3 + 4r^3$ for some integers m, n, r. But we can divide by any common factor of m, n, r to get them without any common factor. But $6mnr$, $2n^3$, $4r^3$ are all even, so m must be even. Put $m = 2M$. Then $12Mnr = 8M^3 + 2n^3 + 4r^3$, so $6Mnr = 4M^3 + n^3 + 2r^3$. But $6Mnr$, $4M^3$ and $2r^3$ are all even, so n must be even. Put $n = 2N$. Then $12MNr = 4M^3 + 8N^3 + 2r^3$, so $6MNr = 2M^3 + 4N^3 + r^3$, so r must be even. So m, n, r had a common factor 2. Contradiction. So d cannot be zero.

Problem B3

Let S be the collection of all sets of n distinct positive integers, with no three in arithmetic progression. Show that there is a member of S which has the largest sum of the inverses of its elements (you do not have to find it or to show that it is unique).

Solution

Induction on n. For $n = 1$, $\{1\}$ is obviously maximal. Now suppose $a_1 < a_2 < ... < a_n$ is a maximal set for n. Take a_{n+1} to be the smallest integer $> a_n$ such that $\{a_1, a_2, ... , a_{n+1}\}$ has no three members in AP. Now consider the sequences $b_1 < b_2 < ... < b_n$ which have no three in AP and $b_{n+1} \le a_{n+1}$. There are only *finitely* many such sequences. So we can find one which is maximal. Suppose it is $c_1 < c_2 < ... < c_{n+1}$. Now take whichever of a_i, c_i has the larger sum of inverses. It is clearly maximal with respect to sequences whose largest member is $\le a_{n+1}$. Suppose we have a sequence $x_1 < x_2 < ... < x_{n+1}$ with no three in AP and $x_{n+1} > a_{n+1}$. Then we have $1/x_{n+1} < 1/a_{n+1}$ and, by induction, $1/x_1 + ... + 1/x_n \le 1/a_1 + ... + 1/a_n$, so $1/x_1 + ... + 1/x_{n+1} < 1/a_1 + ... + 1/a_{n+1}$, so it is worse than the sequence we have chosen.

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Problem A1

Find all real solutions to: $x + y - z = -1$; $x^2 - y^2 + z^2 = 1$, $-x^3 + y^3 + z^3 = -1$.

Solution

Answer: (x, y, z) = (1, -1, 1) or (-1, -1, -1).

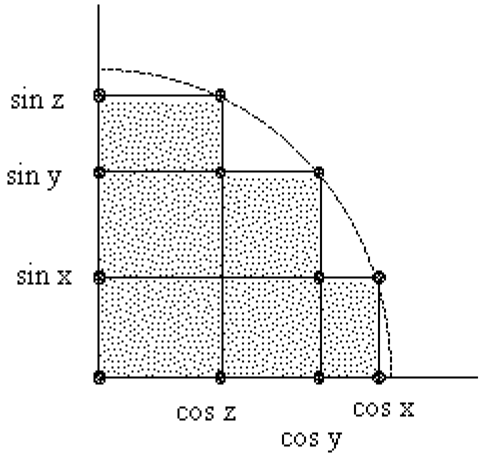
From the first equation $x = z - y - 1$. Substituting in the second equation: $2z^2 - 2yz + 2y - 2z = 0$, so $(z - 1)(z - y) = 0$. Hence $z = 1$ or $y = z$. If $z = 1$, then from the first equation $x + y = 0$, and hence from the last equation, $x = 1$, $y = -1$. If $y = z$, then $x = -1$, and hence from the last equation $y = z = -1$.

Problem A2

Given positive real numbers x, y, z each less than $\pi/2$, show that $\pi/2 + 2 \sin x \cos y + 2 \sin y \cos z > \sin 2x + \sin 2y + \sin 2z$.

Solution

We have sin 2x + sin 2y + sin 2z - 2 sin x cos y - 2 sin y cos z = 2 sin x(cos x - cos y) + 2 sin y(cos y - cos z) + 2 sin z cos z, so we wish to show that sin x(cos x - cos y) + sin y(cos y - cos z) + sin z cos z < $\pi/2$ (*).



We have to consider six cases: (1) $x \le y \le z$; (2) $x \le z \le y$; (3) $y \le x \le z$; (4) $y \le z \le x$; (5) $z \le x \le y$; (6) $z \le y \le x$. The first case is obvious from the diagram, because the lhs represents the shaded area, and the rhs represents the whole quarter circle.

In cases (2) and (5) the second term is negative, and - sin y < - sin x, so the sum of the first two terms is less than sin x (cos x - cos y) + sin x (cos y - cos z) = sin x (cos x - cos z). But by the same argument as the first case the two rectangles represented by sin x(cos x - cos z) and sin z cos z are disjoint and fit inside the quarter circle. So we have proved (2) and (5).

In cases (3) and (4), the first term is negative. The remaining two terms represent disjoint rectangles lying inside the quarter circle, so again the inequality holds.

In case (6) the first two terms are negative. The last term is ½ sin 2z ≤ ½ < $\pi/2$, so the inequality certainly holds.

Problem A3

If a, b, c, are the sides of a triangle, show that (a - b)/(a + b) + (b - c)/(b + c) + (c - a)/(a + c) < 1/16.

Solution

Put f(a, b, c) = (a - b)/(a + b) + (b - c)/(b + c) + (c - a)/(a + c). Let A, B, C be the permutation of a, b, c, with A <= B <= C. If (A, B, C) = (b, a, c), (a, c, b) or (c, b, a), then f(a, b, c) = X, where X = (B - A)/(B + A) + (C - B)/(C + B) - (C - A)/(A + C). If (A, B, C) = (a, b, c), (b, c, a) or (c, a, b), then f(a, b, c) = -X.

Put B = A + h, C = B + k = A + h + k, where h, k ≥ 0. Since A, B, C are the sides of a triangle, we also have A + B > C or A > k. So X = h/(2A + h) + k/(2A + 2h + k) - (h + k)/(2A + h + k) = hk(h + k)/((2A + h)(2A + h + k)(2A + 2h + k)). This is obviously non-negative. We claim also that it is < 1/20. That is equivalent to: $20h^2k + 20hk^2 < (2A + h)(2A + h + k)(2A + 2h + k)$. Since k < A it is sufficient to show that $20h^2k + 20hk^2 \le (2k + h)(2k + h + k)(2k + 2h + k)$

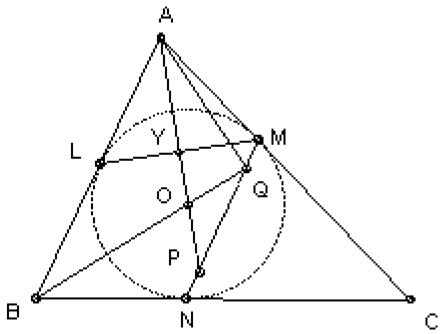
$= 18k^3 + 27hk^2 + 13h^2k + 2h^3$ or $18k^3 + 7hk^2 - 7h^2k + 2h^3 \geq 0$. But $7k^2 - 7hk + 2h^2 = 7(k - h/2)^2 + h^2/4 \geq 0$ and h and k are non-negative, so $18k^3 + h(7k^2 - 7hk + 2h^2) \geq 0$.

Thus we have established that 0 <= X < 1/20, which shows that f(a, b, c) < 1/20, which is slightly stronger than the required result.

Problem B1

The incircle of the triangle ABC touches AC at M and BC at N and has center O. AO meets MN at P and BO meets MN at Q. Show that MP.OA = BC.OQ.

Solution



The key to getting started is to notice that angle AQB = 90^o.

Angle BAQ = 90^o - B/2, so angle OAQ = 90^o - B/2 - A/2 = C/2. So OQ = AO sin C/2. Thus we have to show that MP = BC sin C/2.

Let the incircle touch AB at L and let Y be the midpoint of ML (also the intersection of ML with AO). Angle NMC = 90^o - C/2. It is also A/2 + angle MPY, so angle MPY = 90 - C/2 - A/2 = B/2. Hence MP = MY/sin B/2. We have MY = MO sin MOA = r cos A/2 (where r is the inradius, as usual). So MP = (r cos A/2)/sin B/2. We have BC = BN + NC = r (cot B/2 + cot C/2), so MP/BC = (cos A/2)/(sin B/2 (cot B/2 + cot C/2)). Hence MP/(BC sin C/2) = (cos A/2)/(cos B/2 sin C/2 + sin B/2 cos C/2) = cos A/2 /sin(B/2 + C/2) = 1.

Problem B2

The function f on the positive integers satisfies f(1) = 1, f(2n + 1) = f(2n) + 1 and f(2n) = 3 f(n). Find the set of all m such that m = f(n) for some n.

Solution

We show that to obtain f(n), one writes n in base 2 and then reads it in base 3. For example 12 = 1100_2, so f(12) = 1100_3 = 36. Let g(n) be defined in this way. Then certainly g(1) = 1. Now 2n+1 has the same binary expansion as 2n except for a final 1, so g(2n+1) = g(2n) + 1. Similarly, 2n has the same binary expansion as n with the addition of a final zero. Hence g(2n) = 3 g(n). So g is the same as f. Hence the set of all m such that m = f(n) for some n is the the set of all m which can be written in base 3 without a digit 2.

Problem B3

Show that there are infinitely many solutions in positive integers to $2a^2 - 3a + 1= 3b^2 + b$.

Solution

Put $A = a - 1$ and the equation becomes $A(2A + 1) = b(3b + 1)$. Let d be the greatest common divisor of A and b. Put $A = dx$, $b = dy$. Then $x(2dx + 1) = y(3dy + 1)$. Since x and y are coprime, x must divide $3dy + 1$. So put $3dy + 1 = nx$. Then $2dx + 1 = ny$. Solving for x and y in terms of n and d we get $x = (n + 3d)/(n^2 - 6d^2)$, $y = (n + 2d)/(n^2 - 6d^2)$.

So we would certainly be home if we could show that there were infinitely many solutions to $n^2 - 6d^2 = 1$. It is not hard to find the first few: $1^2 - 6.0^2 = 1$, $5^2 - 6.2^2 = 1$, $49^2 - 6.20^2 = 1$. We notice that $49^2 = 2.5^2 - 1$, so we wonder whether $n = 2.49^2 - 1$ might be another solution and indeed we find it gives $d = 1960 = 2.49.20$. This suggests we try $(2 n^2 - 1)^2 - 6(2nd)^2 = 4n^4 - 4n^2 + 1 - 24n^2d^2 = 4n^2(n^2 - 6d^2 - 1) + 1 = 1$. So there are indeed infinitely many solutions to $n^2 - 6d^2 = 1$ and we are done.

5th Iberoamerican 1990

Problem A1

The function f is defined on the non-negative integers. $f(2^n - 1) = 0$ for n = 0, 1, 2, ... If m is not of the form $2^n - 1$, then $f(m) = f(m+1) + 1$. Show that $f(n) + n = 2^k - 1$ for some k, and find $f(2^{1990})$.

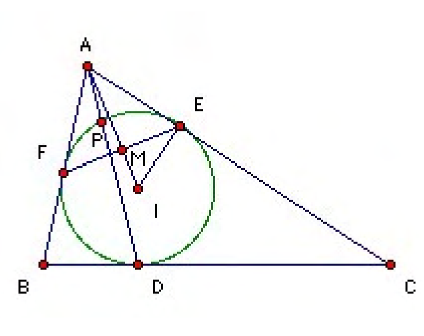
Solution

We claim that if $2^m <= n < 2^{m+1}$, then $f(n) = 2^{m+1} - n - 1$. Put $r = 2^{m+1} - n$. Then the claim follows by induction on r. Hence $f(2^{1990}) = 2^{1990} - 1$.

Problem A2

I is the incenter of the triangle ABC and the incircle touches BC, CA, AB at D, E, F respectively. AD meets the incircle again at P. M is the midpoint of EF. Show that PMID is cyclic (or the points are collinear).

Solution



∠AEI = ∠AME = 90°, so AEI and AME are similar. Hence AM/AE = AE/AI or AM· AI = AE^2. AE is tangent to the incircle, so AE2 = AP· AD. Hence AM· AI = AP· AD, so if P,M,I,D are not collinear, then they are cyclic.

Problem A3

$f(x) = (x + b)^2 + c$, where b and c are integers. If the prime p divides c, but p^2 does not divide c, show that f(n) is not divisible by p^2 for any integer n. If an odd prime q does not divide c, but divides f(n) for some n, show that for any r, we can find N such that q^r divides f(N).

Solution

The first part is trivial. If p does not divide (x+b), then it does not divide $(x+b)^2$, so it does not divide $(x+b)^2+c$. On the other hand, if p does divide x+b, then p^2 divides $(x+b)^2$, so p^2 does not divide $(x+b)^2+c$.

For the second part, we use induction on r. For r = 1, we are given that q divides f(n). Now suppose that q^r divides f(N) for some N. If q^{r+1} divides f(N), then we are done. So suppose q^{r+1} does not divide f(N), so $f(N) = q^r h$ where q does not divide h. We have $f(N+kq^r) = f(N) + q^r(2N+2b)k = q^r h + q^r(2N+2b)k$. Now q divides $(N+b)^2+c$, and does not divide c, so it does not divide $(N+b)^2$ and hence does not divide N+b. It is odd, so it does not divide 2N+2b. Hence we can find k such that $k(2N+2b) = -h \bmod q$. Then we have q^{r+1} divides $f(N+kq^r)$, which completes the induction.

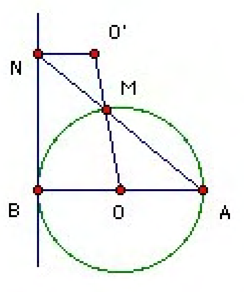
Problem B1

The circle C has diameter AB. The tangent at B is T. For each point M (not equal to A) on C there is a circle C' which touches T and touches C at M. Find the point at which C' touches T and find the locus of the center of C' as M varies. Show that there is a circle orthogonal to all the circles C'.

Answer

C' touches T at the intersection of T and the line AM
the locus of the center is a parabola vertex B
the circle center A radius AB is orthogonal to all circles C'

Solution



Let O be the center of C. Let the line AM meet T at N. Let the perpendicular to T at N meet the line OM at O'. Then ∠O'NM = ∠MAB (O'N parallel to AB, because both perpendicular to T) = ∠OMA (OM = OA) = ∠O'MN. So O'M = O'N. Hence O' is the center of C'.

Take B to be the origin and A to be the point (2a,0), so O is (a,0) and C has radius a. If O' is (x,y), then we require that O'O = x+a or $(x-a)^2+y^2 = (x+a)^2$, or $y^2 = 4ax$, which is a parabola with vertex B and axis the x-axis.

Triangles AMB, ABN are similar ($\angle AMB = \angle ABN = 90^o$), so AM/AB = AB/AN and hence $AM \cdot AN = AB^2$. Now consider the circle center A radius AB. It must meet the circle C', because it contains the point M. Suppose it meets at X. Then $AX^2 = AB^2 = AM \cdot AN$, so AX is tangent to C' and hence the circles are orthogonal.

Problem B2

A and B are opposite corners of an n x n board, divided into n^2 squares by lines parallel to the sides. In each square the diagonal parallel to AB is drawn, so that the board is divided into $2n^2$ small triangles. The board has $(n + 1)^2$ nodes and large number of line segments, each of length 1 or $\sqrt{2}$. A piece moves from A to B along the line segments. It never moves along the same segment twice and its path includes exactly two sides of every small triangle on the board. For which n is this possible?

Answer

n=2 only

Solution

The diagram above shows that n=2 is possible (the path is AHEFGHCDIHB). Now suppose n > 2.

Note that if X is any vertex except A or B, then an even number of segments with endpoint X must be in the path.

Let F be the bottom left-hand vertex. Two sides of the triangle EFG are in the path, so at least one of EF and FG is. But EF and EG are the only segments with endpoint F, so an even number of them must be in the path, so both are in the path. Hence, again considering EFG, EG is not in the path. Hence, considering EHG, EH and HG are in the path.

E has an even number of segments on the path, so CE is not on the path. Hence (considering CEH) CH is on the path. Similarly, GJ is not on the path and HJ is on the path. An even number of segments at H are on the path, so DH and HI are either both on the path or neither is on the path. But (considering DHI) at least one must be, so they both are. Hence DI is not, and CD is not.

Since n > 2, C is not the top left vertex. Considering MCD, MC and MD are both on the path. Considering DLI, DL is on the path. There must be an even number of segments at D, so DP is on the path. Hence MP is not. Now M cannot be the top left vertex (with n = 3) because then it should have an odd number of segments, whereas it would have two (MC and MD). So there must be a vertex N above M. Considering NMP, MN must be in the path. But now M has an odd number of segments. Contradiction.

Problem B3

f(x) is a polynomial of degree 3 with rational coefficients. If its graph touches the x-axis, show that it has three rational roots.

Solution

Without loss of generality, $f(x) = x^3 - ax^2 + bx - c$, where a, b, c are rational. Since the graph touches the x-axis, there is a repeated root, so we may take the roots to be h, h, k. Hence $2h + k = a$, $2hk + k^2 = b$, $h^2k = c$. Hence $a^2 - 3b = (h - k)^2$. Put $r = \pm\sqrt{(a^2 - 3b)}$, where the sign is chosen so that $h = a/3 + r/3$, $k = a/3 - 2r/3$. We need to show that r is rational. If r is zero there is nothing to prove, so assume r is non-zero.

We have $9h^2 = 2a^2 - 3b + 2ar$. Hence $27h^2k = -2a^3 + 9ab + (6b - 2a^2)r$. But $27h^2k = 27c$. So $r = (27c + 2a^3 - 9ab)/(2(3b - a^2))$. Note that $3b - 2a^2$ is non-zero because r is non-zero. So r is a rational combination of a, b, c and hence is rational.

6th Iberoamerican 1991

Problem 1

The number 1 or the number -1 is assigned to each vertex of a cube. Then each face is given the product of its four vertices. What are the possible totals for the resulting 14 numbers?

Solution

Answer: 14, 6, 2, -2, -6, -10.

If every vertex is 1, we get 14 and that is clearly the highest possible total. The lowest possible total cannot be lower than -14, but we cannot even achieve that because if all the vertices are -1, then all the faces are 1.

If we change a vertex, then we also change three faces. If the vertex and the three faces are all initially the same, then we make a change of ±8. If three are of one kind and one the opposite, then we make a change of ±4. If two are of one kind and two the opposite, then we make no change. Thus any sequence of changes must take us to 14 + 4n for some integer n. But we have already shown that the total is greater than -14 and at most 14, so the only possibilities are -10, -6, -2, 2, 6, 10 and 14.

We show that 10 is not possible. If more than 2 vertices are -1, then the vertex total is at most 2, there are only 6 faces, so the total is less than 10. If all vertices are 1, then the total is 14. If all but one vertex is 1, then the total is 6. So the only possibility for 10 is just two vertices -1. But however we choose any two vertices, there is always a face containing only one of them, so at least one face is -1, so the face total is at most 4 and the vertex total is 4, so the total is less than 10. The other totals are possible, for example:

```
14: all vertices 1

6: one vertex -1, rest 1

2: three vertices of one face -1, rest 1

-2: all vertices -1

-6: all vertices but one -1

-10: two opposite corners 1, rest -1
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Problem A2

Two perpendicular lines divide a square into four parts, three of which have area 1. Show that the fourth part also has area 1.

Problem A3

f is a function defined on all reals in the interval [0, 1] and satisfies $f(0) = 0$, $f(x/3) = f(x)/2$, $f(1 - x) = 1 - f(x)$. Find $f(18/1991)$.

Problem B1

Find a number N with five digits, all different and none zero, which equals the sum of all distinct three digit numbers whose digits are all different and are all digits of N.

Solution

Answer: 35964

There are 4.3 = 12 numbers with a given digit of n in the units place. Similarly, there are 12 with it in the tens place and 12 with it in the hundreds place. So the sum of the 3 digit numbers is 12.111 (a + b + c + d + e), where n = abcde. So 8668a = 332b + 1232c + 1322d + 1331e. We can easily see that a = 1 is too small and a = 4 is too big, so a = 2 or 3. Obviously e must be even. 0 is too small, so e = 2, 4, 6 or 8. Working mod 11, we see that 0 = 2b + 2d, so b + d = 11. Working mod 7, we see that 2a = 3b + 6d + e. Using the mod 11 result, b = 2, d = 9 or b = 3, d = 8 or b = 4, d = 7 or b = 5, d = 6 or b = 6, d = 5 or b = 7, d = 4 or b = 8, d = 3 or b = 9, d = 2. Putting each of these into the mod 7 result gives 2a - e = 4, 1, 5, 2, 6, 3, 0, 4 mod 7. So putting a = 2 and remembering that e must be 2, 4, 6, 8 and that all digits must be different gives a, b, d, e = 2,4, 7, 6 or 2, 7, 4, 8 or 2, 8, 3, 4 as the only possibilities. It is then straightforward but tiresome to check that none of these give a solution for c. Similarly putting a = 4, gives a, b, d, e = 3, 4, 7, 8 or 3, 5, 6, 4 as the only possibilities. Checking, we find the solution above and no others.

Problem B2

Let p(m, n) be the polynomial $2m^2 - 6mn + 5n^2$. The range of p is the set of all integers k such that k = p(m, n) for some integers m, n. Find which members of {1, 2, ... , 100} are in the range of p. Show that if h and k are in the range of p, then so is hk.

Answer

1,2,4,5,8,9,10,13,16,17,18, 20,25,26,29,32,34,36,37, 40,41,45,49,50,52,53,58, 61,64,65,68,72,73,74, 80,81,82,85,89,90,97,98,100

Solution

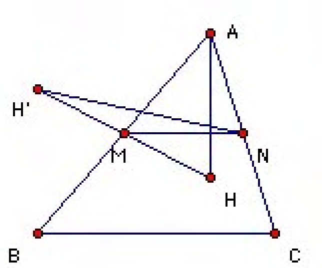
We have $p(m,n) = (m-2n)^2 + (m-n)^2$, so $p(2a-b,a-b) = a^2 + b^2$. Hence the range of p is just the sums of two squares.

$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$, which establishes that if h and k are in the range, then so is hk.

Problem B3

Given three non-collinear points M, N, H show how to construct a triangle which has H as orthocenter and M and N as the midpoints of two sides.

Solution



Take H' so that M is the midpoint of HH'. The circle diameter NH' meets the line through H perpendicular to MN in two points (in general), either of which we may take as A. Then B is the reflection of A in M, and C is the reflection of A in N.

To see that this works, note that M is the midpoint of HH' and AB, so AHBH' is a parallelogram. Hence AH' is parallel to BH and hence perpendicular to AC. In other words $\angle NAH' = 90^o$, so A lies on the circle diameter NH'. MN is parallel to BC, so A lies on the perpendicular to MN through H.

7th Iberoamerican 1992

Problem A1

a_n is the last digit of $1 + 2 + ... + n$. Find $a_1 + a_2 + ... + a_{1992}$.

Solution

It is easy to compile the following table, from which we see that a_n is periodic with period 20, and indeed the sum for each decade (from 0 to 9) is 35. Thus the sum for 1992 is $199 \cdot 35 + 5 + 6 + 8 = 6984$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a_n	0	1	3	6	0	5	1	8	6	5	5	6	8	1	5	0	6	3	1	0	0
sum	0	1	4	10	10	15	16	24	30	35	40	46	54	55	60	60	66	69	70	70	70

Problem A2

Let $f(x) = a_1/(x + a_1) + a_2/(x + a_2) + ... + a_n/(x + a_n)$, where a_i are unequal positive reals. Find the sum of the lengths of the intervals in which $f(x) \geq 1$.

Answer $\sum a_i$

Solution

wlog $a_1 > a_2 > ... > a_n$. The graph of each $a_i/(x + a_i)$ is a rectangular hyberbola with asymptotes $x = -a_i$ and $y = 0$. So it is not hard to see that the graph of $f(x)$ is made up of $n+1$ strictly decreasing parts. For $x < -a_1$, $f(x)$ is negative. For $x \in (-a_i, -a_{i+1})$, $f(x)$ decreases from ∞ to $-\infty$. Finally, for $x > -a_n$, $f(x)$ decreases from ∞ to 0. Thus $f(x) = 1$ at n values $b_1 < b_2 < ... < b_n$, and $f(x) \geq 1$ on the n intervals $(-a_1,b_1)$, $(-a_2,b_2)$, ... , $(-a_n,b_n)$. So the sum of the lengths of these intervals is $\sum (a_i + b_i)$. We show that $\sum b_i = 0$.

Multiplying $f(x) = 1$ by $\prod(x + a_j)$ we get a polynomial of degree n:

$$\prod(x + a_j) - \sum_i (a_i \prod_{j\neq i}(x + a_j)) = 0.$$

The coefficient of x^n is 1 and the coefficient of x^{n-1} is $\sum a_j - \sum a_i = 0$. Hence the sum of the roots, which is $\sum b_i$, is zero.

Problem A3

ABC is an equilateral triangle with side 2. Show that any point P on the incircle satisfies $PA^2 + PB^2 + PC^2 = 5$. Show also that the triangle with side lengths PA, PB, PC has area $(\sqrt{3})/4$.

Solution

Take vectors centered at the center O of the triangle. Write the vector OA as **A** etc. Then $PA^2 + PB^2 + PC^2 = (\mathbf{P} - \mathbf{A})^2 + (\mathbf{P} - \mathbf{B})^2 + (\mathbf{P} - \mathbf{C})^2 = 3P^2 + (A^2 + B^2 + C^2) - 2\mathbf{P}.(\mathbf{A} + \mathbf{B} + \mathbf{C}) = 15P^2$, since $A^2 = B^2 = C^2 = 4P^2$ and $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$. Finally the side is 2, so an altitude is $\sqrt{3}$ and the inradius is $(\sqrt{3})/3 = 1/\sqrt{3}$, so $PA^2 + PB^2 + PC^2 = 15/3 = 5$.

Take Q outside the triangle so that BQ = BP and CQ = AP. Then BQC and BPA are congruent, so $\angle ABP = \angle CBQ$ and hence $\angle PBQ = 60^o$, so PBQ is equilateral. Hence PQ is PB and PQC has sides equal to PA, PB, PC. If we construct two similar points outside the other two sides then we get a figure with total area equal to 2 area ABC and to 3 area PQC + area of three equilateral triangles sides PA, PB, PC. Hence 3 area PQC = 2 area ABC - area ABC $(PA^2+PB^2+PC^2)/PA^2 = (3/4)$ area ABC $= (3\sqrt{3})/4$. So area PQC $= (\sqrt{3})/4$.

Problem B1

Let a_n, b_n be two sequences of integers such that: (1) $a_0 = 0$, $b_0 = 8$; (2) $a_{n+2} = 2 a_{n+1} - a_n + 2$, $b_{n+2} = 2 b_{n+1} - b_n$, (3) $a_n^2 + b_n^2$ is a square for $n > 0$. Find at least two possible values for (a_{1992}, b_{1992}).

Answer

$(1992\cdot 1996, 4\cdot 1992+8)$, $(1992\cdot 1988, -4\cdot 1992+8)$

Solution

a_n satisfies a standard linear recurrence relation with general solution $a_n = n^2 + An + k$. But $a_n = 0$, so $k = 0$. Hence $a_n = n^2 + An$. If you are not familiar with the general solution, then you can guess this solution and prove it by induction.

Similarly, $b_n = Bn + 8$. Hence $a_n^2+b_n^2 = n^4 + 2An^3 + (A^2+B^2)n^2 + 16Bn + 64$. If this is a square, then looking at the constant and n^3 terms, it must be $(n^2 + An + 8)$. Comparing the other terms, $A = B = \pm 4$.

Problem B2

Construct a cyclic trapezium ABCD with AB parallel to CD, perpendicular distance h between AB and CD, and AB + CD = m.

Problem B3

Given a triangle ABC, take A' on the ray BA (on the opposite side of A to B) so that AA' = BC, and take A" on the ray CA (on the opposite side of A to C) so that AA" = BC. Similarly take B', B" on the rays CB, AB respectively with BB' = BB" = CA, and C', C" on the rays AB, CB. Show that the area of the hexagon A"A'B"B'C"C' is at least 13 times the area of the triangle ABC.

8th Iberoamerican 1993

Problem 1

A palindrome is a positive integers which is unchanged if you reverse the order of its digits. For example, 23432. If all palindromes are written in increasing order, what possible prime values can the difference between successive palindromes take?

Solution

Answer: 2, 11.

Let x be a palindrome and x' the next highest palindrome. If x < 101, then it is easy to see by inspection that x' - x = 1, 2 or 11, so the only prime differences are 2 and 11.

So assume x > 100. If x and x' have the same final digit, then their difference is divisible by 10 and hence not prime. So they must have different digits. Thus either x = d9...9d and x' = d'0...0d', where d < 9 and d' = d+1, or x' has one more digit than x and d = 9, d' = 1. In the first case x' - x = 11. In the second case x' - x = 2. So again the only prime differences are 2 and 11.

Problem 2

Show that any convex polygon of area 1 is contained in some parallelogram of area 2.

Solution

Let the vertices X, Y of the polygon be the two which are furthest apart. The polygon must lie between the lines through X and Y perpendicular to XY (for if a vertex Z lay outside the line through Y, then ZY > XY). Take two sides of a rectangle along these lines and the other two sides as close together as possible. There must be a vertices U and V on each of the other two sides. But now the area of the rectangle is twice the area of XUYV, which is at most the area of the polygon. [In the case of a triangle one side of the rectangle will be XY.]

Problem A3

Find all functions f on the positive integers with positive integer values such that (1) if $x < y$, then $f(x) < f(y)$, and (2) $f(y f(x)) = x^2 f(xy)$.

Solution

Answer: $f(x) = x^2$.

Note that (1) implies f is (1, 1).

Put $y = 1$. Then $f(f(x)) = x^2 f(x)$.

Put $y = f(z)$, then $f(f(z) f(x)) = x^2 f(x f(z)) = x^2 z^2 f(xz) = f(f(xz))$. But f is (1, 1) so $f(xz) = f(x) f(z)$.

Now suppose $f(m) > m^2$ for some m. Then by (1), $f(f(m)) > f(m^2 = f(m.m) = f(m)^2$. But $f(f(m)) = m^2 f(m)$, so $m^2 > f(m)$. Contradiction.

Similarly, suppose $f(m) < m^2$. Then $m^2 f(m) = f(f(m)) < f(m^2) = f(m)^2$, so $m^2 < f(m)$. Contradiction. So we must have $f(m) = m^2$.

Problem B1

ABC is an equilateral triangle. D is on the side AB and E is on the side AC such that DE touches the incircle. Show that $AD/DB + AE/EC = 1$.

Solution

Put $BD = x$, $CE = y$, $BC = a$. Then since the two tangents from B to the incircle are of equal length, and similarly the two tangents from D and E, we have $ED + BC = BD + CE$, or $ED = x + y - a$. By the cosine law, $ED^2 = AE^2 + AD^2 - AE.AD$. Substituting and simplifying, we get $a = 3xy/(x + y)$. Hence $AD/DB = (2y - x)/(x + y)$ and $AE/EC = (2x - y)/(x + y)$ with sum 1.

Problem B2

If P and Q are two points in the plane, let m(PQ) be the perpendicular bisector of PQ. S is a finite set of $n > 1$ points such that: (1) if P and Q belong to S, then some point of m(PQ) belongs to S, (2) if PQ, P'Q', P"Q" are three distinct segments, whose endpoints are all in S, then if there is a point in all of m(PQ), m(P'Q'), m(P"Q") it does not belong to S. What are the possible values of n?

Answer

n = 3 (equilateral triangle), 5 (regular pentagon).

Solution

There are $n(n-1)/2$ pairs of points. Each has a point of S on its bisector. But each point of S is on at most two bisectors, so $2n \geq n(n-1)/2$. Hence $n \leq 5$.

The equilateral triangle and regular pentagon show that n = 3, 5 are possible.

Consider n = 4. There are 6 pairs of points, so at least one point of S must be on two bisectors. wlog A is on the bisectors of BC and BD. But then it is also on the bisector of CD. Contradiction.

Problem B3

We say that two non-negative integers are *related* if their sum uses only the digits 0 and 1. For example 22 and 79 are related. Let A and B be two infinite sets of non-negative integers such that: (1) if a □ A and b □ B, then a and b are related, (2) if c is related to every member of A, then it belongs to B, (3) if c is related to every member of B, then it belongs to A. Show that in one of the sets A, B we can find an infinite number of pairs of consecutive numbers.

Solution

Suppose there is a member of A with last digit d. Then every member of B must have one of two possible last digits. Suppose there are members of B with both possibilities. Then every member of A must have last digit d. So either every member of A has the same last digit or every member of B has the same last digit (or both). Suppose every member of A has the same last digit d.

But now if n belongs to B and n + d has last digit 0, then n+1 + d has last digit 1. Moreover, if m is any member of A, then m+n has last digit 0 and other digits all 0 or 1. Hence m+n+1 last last digit 1 and other digits all 0 or 1, so n+1 must also belong to B. Similarly, if n is in B and n+d has last digit 1, then n-1 must also belong to B. So in either case there are infinitely many pairs of consecutive numbers in B.

9th Iberoamerican 1994

Problem A1

Show that there is a number 1 < b < 1993 such that if 1994 is written in base b then all its digits are the same. Show that there is no number 1 < b < 1992 such that if 1993 is written in base b then all its digits are the same.

Solution

Any even number 2n can be written as 22 in base n-1. In particular $1994 = 22_{996}$.

We have to show that we cannot write $1993 = aaa \ldots a_b$. If the number has n digits, then $1993 = a(1 + b + \ldots + b^{n-1}) = a(b^n - 1)/(b - 1)$. But 1993 is prime, so a must be 1. Hence $b^{n-1} + \ldots + b - 1992 = 0$. So b must divide $1992 = 2^3 3.83$. We cannot have n = 2, for then b = 1992 and we require b < 1992. So n > 2. But $83^2 = 6889 > 1993$, so b must divide 24. Hence b = 2, 3, 4, 6, 8, 12, or 24. But we can easily check that none of these work:

$1 + 2 + 2^2 + \ldots + 2^9 = 1023,\ 1 + \ldots + 2^{10} = 2047.$

$1 + 3 + \ldots + 3^6 = 1093,\ 1 + \ldots + 3^7 = 3280$

$1 + 4 + \ldots + 4^5 = 1365,\ 1 + \ldots + 4^6 = 5461$

$1 + 6 + \dots 6^4 = 1555$, $1 + \dots + 6^5 = 9331$

$1 + 8 + 8^2 + 8^3 = 585$, $1 + \dots + 8^4 = 4681$

$1 + 12 + 12^2 + 12^3 = 1885$, $1 + \dots + 12^4 = 22621$

$1 + 24 + 24^2 = 601$, $1 + \dots + 24^3 = 14425$

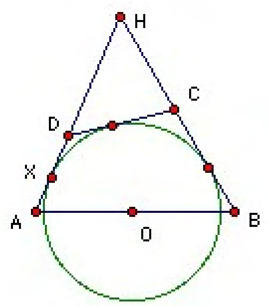
Problem A2

ABCD is a cyclic quadrilateral. A circle whose center is on the side AB touches the other three sides. Show that AB = AD + BC. What is the maximum possible area of ABCD in terms of |AB| and |CD|?

Answer

$(h/2 + k/2) \sqrt{(hk/2 - h^2/4)}$, where $h = |CD|$, $k = |AB|$

Solution



Let the circle have center O on AB and radius r. Let $\angle OAD = \theta$, $\angle OBC = \varphi$. Since ABCD is cyclic, $\angle ADC = 180^o - \varphi$, so $\angle ODA = 90^o - \varphi/2$. If AD touches the circle at X, then $AD = AX + XD = r \cot \theta + r \tan(\varphi/2)$. Similarly, $BC = r \cot \varphi + r \tan(\theta/2)$. Put $t = \tan(\theta/2)$. Then $\cot \theta = (1-t^2)/2t$, so $\cot \theta + \tan(\theta/2) = (1+t^2)/2t = 1/\sin \theta$. Similarly for φ, so $AD + BC = r/\sin \theta + r/\sin \varphi = AO + OB = AB$.

Suppose AD and BC meet at H (we deal below with the case where they are parallel). Then HCD and HAB are similar, so area HCD = (CD^2/AB^2) area HAB and area ABCD = $(1 - CD^2/AB^2)$ area HAB. Also AB/CD = HA/HC = HB/HD = (HA+HB)/(HC+HD) = (HA+HB)/(HB-BC+HA-DA) = (HA+HB)/(HA+HB-AB). Hence $HA+HB = AB^2/(AB-CD)$, which is fixed. Now for fixed HA+HB we maximise the area of HAB by taking HA = HB and hence AD = BC.

Put h = CD, k = AB. So $k \cos \theta + h = k$. Hence $\cos \theta = (1-h/k)$. Hence $\sin \theta = \sqrt{(2h/k - h^2/k^2)}$. So area ABCD = $½(h+k)\ ½ k \sin \theta = (h/2 + k/2) \sqrt{(hk/2 - h^2/4)}$ (*).

If AD and BC are parallel then A and B must lie on the circle, so that $\angle DAB = \angle ABC = 90^o$. But ABCD is cyclic, so it must be a rectangle. Hence AB = CD and area ABCD = $k^2/2$. In this case (*) still gives the correct answer.

Problem A3

There is a bulb in each cell of an n x n board. Initially all the bulbs are off. If a bulb is touched, that bulb and all the bulbs in the same row and column change state (those that are on, turn off, and those that are off, turn on). Show that it is possible by touching m bulbs to turn all the bulbs on. What is the minimum possible value of m?

Answer

n odd, n is minimum
n even, n^2 is minimum

Solution

If n is odd, touch each bulb in the first column. Then bulbs in the first column are each switched n times, which is odd and so end up on. All other bulbs are switched just once, and so end up on. n is obviously minimal, because if m < n, then there is a bulb which is not switched at all (there must be a column with no bulb touched and a row with no bulb touched, so the bulb in that column and row is not switched).

In n is even, touch each bulb. Then each bulb is switched 2n-1 times, so ends up on. We show that it is not possible to do better.

Note first that there is no benefit in touching a bulb more than once, so each must be touched zero of one times. Thus we can represent the scheme as an array of 0s and 1s, where 0 means that the corresponding bulb is not touched, and 1 means that it is touched.

Let A, B, C, D be four values at the corners of a rectangle. We claim that A+B has the same parity as C+D. Let L_{AB} be the number of 1s in the row AB are touched, similarly L_{BC} (the number of 1s in the column BC), L_{CD}, L_{DA}. Since bulb A is switched we must have $L_{AB} + L_{DA}$ + A odd (note that $L_{AB} + L_{DA}$ double-counts the no. of touches of A). Similarly, $L_{BC} + L_{CD}$ + C is odd, so A + C + ($L_{AB} + L_{BC} + L_{CD} + L_{DA}$) is even. Similarly, considering B and D, we find that B + D + ($L_{AB} + L_{BC} + L_{CD} + L_{DA}$) is even, so A+C and B+D have the same parity. Adding B+C to both, we get that A+B and C+D have the same parity. It follows that either A = D and B = C, or A ≠ D and B ≠ D.

Keeping A and B fixed, we can now vary C (and hence D). It follows that either the row through B matches that through A, or it has every cell different (to the corresponding cell in row A). Similarly for the other rows. So we have k rows of one type and n-k rows which are equal to its "complement". Suppose first that k = n, so that all rows are the same. If we have all 1s, then we have a solution. If we have all 0s, we obviously do not have a solution. So suppose there is a 0 and a 1 in each row. Then the total count at a 1 is n-1 higher than at a 0 (because of the extra n-1 1s in the same column). So they cannot both be odd (because n is even). Contradiction.

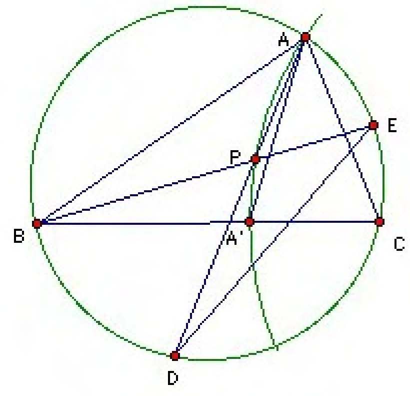
Finally suppose there is a row and a complement row. So position A in one is 1, then position B in the same column in the other has 0. If a row has h 1s, then a complement row has n-h 1s. The column has z 1s, so A has z+h-1 or z+n-h-1 1s, and B has z+h or z+n-h 1s. But since n is even, z+h and z+n-h have the same parity, so A and B have opposite parity. Contradiction. So the only solution for n even is all 1s.

Problem B1

ABC is an acute-angled triangle. P is a point inside its circumcircle. The rays AP, BP, CP intersect the circle again at D, E, F. Find P so that DEF is equilateral.

Solution

Let the angle bisector of A meet BC at A'. Let the perpendicular bisector of AA' meet the line BC at X. Take the circle center X through A and A'. Similarly, let the angle bisector of B meet AC at B' and let the perpendicular bisector of BB' meet the line AC at Y. Take the circle center Y through B and B'. The two circles meet at a point P inside the triangle, which is the desired point.



PAB and PED are similar, so DE/AB = PD/PB. Similarly, DF/AC = PD/PC, so DE/DF = (AB/AC)(PC/PB). Thus we need PB/PC = AB/AC. So P must lie on the circle of Apollonius, which is the circle we constructed with center X. Similarly, it must lie on the circle of Apollonius with center Y and hence be one of their points of intersection. It also lies on the third circle and hence we choose the point of intersection inside the triangle.

Problem B2

n and r are positive integers. Find the smallest k for which we can construct r subsets A_1, A_2, ... , A_r of {0, 1, 2, ... , n-1} each with k elements such that each integer $0 \le m < n$ can be written as a sum of one element from each of the r subsets.

Answer

smallest integer such that $k^r \ge n$.

Solution

We can form at most k^r distinct sums, so k^r must be $\ge n$.

Now consider $A_1 = \{0, 1, 2, \dots , k-1\}$, $A_2 = \{0, k, 2k, \dots , (k-1)k\}$, $A_3 = \{0, k^2, 2k^2, \dots , (k-1)k^2\}$, ... , $A_r = \{0, k^{r-1}, 2k^{r-1}, \dots , (k-1)k^{r-1}\}$. Then for any non-negative integer $m < k^r$, we can write m with r digits in base k (using leading zeros as necessary) and hence as a sum of one element from each A_i. This subset works for $(k-1)k^{r-1} < n \le k^r$. For smaller n above $(k-1)^r$ we cannot use all the elements given above, but we do not need them, so we just replace the elements which are too large by arbitrary elements under n.

For example, suppose $n = 17$, $r = 4$. We need $k = 3$. So we form $A_1 = \{0, 1, 2\}$, $A_2 = \{0, 3, 6\}$, $A_3 = \{0, 9, 18\}$, $A_4 = \{0, 27, 54\}$. Now 18, 27, 54 are unnecessary, so we pad out A_3 and A_4 with other elements. We could take $A_3 = \{0, 1, 9\}$, $A_4 = \{0, 1, 2\}$.

Problem B3

Show that given any integer $0 < n \le 2^{1000000}$ we can find at set S of at most 1100000 positive integers such that S includes 1 and n and every element of S except 1 is a sum of two (possibly equal) smaller elements of S.

10th Iberoamerican 1995

Problem A1

Find all possible values for the sum of the digits of a square.

Solution

Answer: any non-negative integer = 0, 1, 4 or 7 mod 9.

$0^2 = 0$, $(\pm 1)^2 = 1$, $(\pm 2)^2 = 4$, $(\pm 3)^2 = 0$, $(\pm 4)^2 = 7$ mod 9, so the condition is necessary.

We exhibit squares which give these values.

0 mod 9. Obviously $0^2 = 0$. We have $9^2 = 81$, $99^2 = 9801$ and in general $9...9^2 = (10^n - 1)^2 = 10^{2n} - 2.10^n + 1 = 9...980...01$, with digit sum 9n.

1 mod 9. Obviously $1^2 = 1$ with digit sum 1, and $8^2 = 64$ with digit sum 10. We also have $98^2 = 9604$, $998^2 = 996004$, and in general $9...98^2 = (10^n - 2)^2 = 10^{2n} - 4.10^n + 4 = 9...960...04$, with digit sum 9n+1.

4 mod 9 Obviously $2^2 = 4$ with digit sum 4, and $7^2 = 49$ with digit sum 13. Also $97^2 = 9409$ with digit sum 22, $997^2 = 994009$ with digit sum 31, and in general $9...97^2 = (10^n - 3)^2 = 10^{2n} - 6.10^n + 9 = 9...940...09$, with digit sum 9n+4.

7 mod 9 Obviously $4^2 = 16$, with digit sum 7. Also $95^2 = 9025$, digit sum 16, $995^2 = 990025$ with digit sum 25, and in general $9...95^2 = (10^n - 5)^2 = 10^{2n} - 10^{n+1} + 25 = 9...90...025$, with digit sum 9n-2.

Problem A2

Find all solutions in real numbers $x_1, x_2, \dots , x_{n+1}$ all at least 1 such that: (1) $x_1^{1/2} + x_2^{1/3} + x_3^{1/4} + \dots + x_n^{1/(n+1)} = n\, x_{n+1}^{1/2}$; and (2) $(x_1 + x_2 + \dots + x_n)/n = x_{n+1}$.

Answer

The only solution is the obvious, all $x_i = 1$.

Solution

By Cauchy-Schwartz, $(\sum x_i^{1/2})^2 \le (\sum 1)$(&sum x_i), with equality iff all x_i equal. In other words, if we put $x_{n+1} = (x_1 + x_2 + \dots + x_n)/n$, then $\sum x_i^{1/2} \le n\, x_{n+1}^{1/2}$. But since all $x_i \ge 1$, we have $x_1^{1/2} + x_2^{1/3} + x_3^{1/4} + \dots + x_n^{1/(n+1)} \le \sum x_i^{1/2}$ with equality iff $x_2 = x_3 = \dots = x_n = 1$. Hence $x_1^{1/2} + x_2^{1/3} + x_3^{1/4} + \dots + x_n^{1/(n+1)} \le x_{n+1}^{1/2}$ with equality iff all $x_i = 1$.

Problem A3

L and L' are two perpendicular lines not in the same plane. AA' is perpendicular to both lines, where A belongs to L and A' belongs to L'. S is the sphere with diameter AA'. For which points P on S can we find points X on L and X' on L' such that XX' touches S at P?

Problem B1

ABCD is an n x n board. We call a diagonal row of cells a positive diagonal if it is parallel to AC. How many coins must be placed on an n x n board such that every cell either has a coin or is in the same row, column or positive diagonal as a coin?

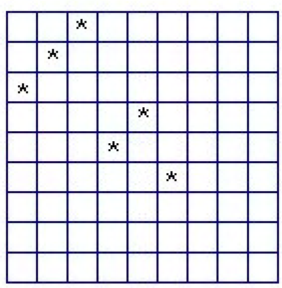
Answer

smallest integer $\ge (2n-1)/3$
[so 2m-1 for n = 3m-1, 2m for n = 3m, 2m+1 for n = 3m+1]

Solution

There must be at least n-k rows without a coin and at least n-k columns without a coin. Let $r_1, r_2, \dots, r_{n-k}$ be cells in the top row without a coin which are also in a column without a coin. Let $r_1, c_2, c_3, \dots, c_{n-k}$ be cells in the first column without a coin which are also in a row without a coin. Each of the 2n-2k-1 r_i and c_j are on a different positive diagonal, so we must have $k \ge 2n-2k-1$ and hence $k \ge (2n-1)/3$.

Let (i,j) denote the cell in row i, col j. For n = 3m-1, put coins in (m,1), (m-1,2), (m-2,3), ... , (1,m) and in (2m-1,m+1), (2m-2,m+2), ... , (m+1,2m-1). It is easy to check that this works. For n = 3m, put an additional coin in (2m,2m), it is easy to check that works. For n = 3m+1 we can use the same arrangement as for 3m+2.



Problem B2

The incircle of the triangle ABC touches the sides BC, CA, AB at D, E, F respectively. AD meets the circle again at X and AX = XD. BX meets the circle again at Y and CX meets the circle again at Z. Show that EY = FZ.

Problem B3

f is a function defined on the positive integers with positive integer values. Use $f^m(n)$ to mean $f(f(\dots f(n) \dots)) = n$ where f is taken m times, so that $f^2(n) = f(f(n))$, for example. Find the largest possible $0 < k < 1$ such that for some function f, we have $f^m(n) \neq n$ for $m = 1, 2, \dots , [kn]$, but $f^m(n) = n$ for some m (which may depend on n).

Answer

we can get k arbitrarily close to 1

Solution

The basic idea is to take a block of integers m+1, m+2, ... , M and to define f(m+1) = m+2, f(m+2) = m+3, ... , f(M-1) = M, f(M) = m+1. Then for any integer h in the block we have $f^n(h) \neq h$ for n = 1, 2, ... , M-m-1 and $f^{M-m}(h) = h$. Note that the ratio (M-m)/h is worst (smallest) for h = M.

For example, take the first block to be 1, 2, ... , N, the second block to be N+1, ... , N^2, the third block, $N^2+1, \dots , N^3$ and so on. Then for any integer n we have $f^m(n) \neq n$ for $m < kn$ where $k = 1 - 1/N$.

11th Iberoamerican 1996

Problem A1

Find the smallest positive integer n so that a cube with side n can be divided into 1996 cubes each with side a positive integer.

Solution

Answer: 13.

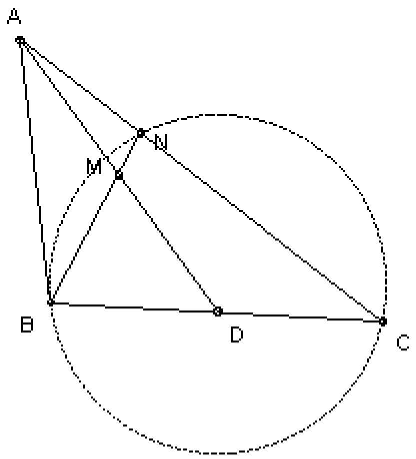
Divide all the cubes into unit cubes. Then the 1996 cubes must each contain at least one unit cube, so the large cube contains at least 1996 unit cubes. But $12^3 = 1728 < 1996 < 2197 = 13^3$, so it is certainly not possible for n < 13.

It can be achieved with 13 by $1.5^3 + 11.2^3 + 1984.1^3 = 13^3$ (actually packing the cubes together to form a 13 x 13 x 13 cube is trivial since there are so many unit cubes).

Problem 2

M is the midpoint of the median AD of the triangle ABC. The ray BM meets AC at N. Show that AB is tangent to the circumcircle of NBC iff $BM/BN = (BC/BN)^2$.

Solution



Applying Menelaus to the triangle ADC, we have (AM/MD)(BD/DC)(CN/NA) = 1, so (CN/NA) = 2. Hence AN/AC = 1/3. Applying Menelaus to the triangle BNC, we have (BM/MN)(AN/AC)(CD/DB) = 1, so BM/MN = 3. That is true irrespective of whether AB is tangent to the circle NBC.

If AB is tangent, then $AB^2 = AN.AC = 1/3\ AC^2$. Also angle ABN = angle BCN, so triangles ANB and ABC are similar. Hence BC/BN = AC/AB. Hence $(BC/BN)^2 = 3 = BM/BN$.

Conversely, if $(BC/BN)^2 = BM/BN$, then $(BC/BN)^2 = 3$.

Now applying the cosine formula to AMN and AMB and using cos AMN + cos AMB = 0, we have $(3AN^2 - 3AM^2 - 3MN^2) + (AB^2 - AM^2 - BM^2) = 0$, so $AB^2 + AC^2/3 = AD^2 + 3/4\ BN^2$. Similarly from triangles ADC and ADB we get $AB^2 + AC^2 = 2AD^2 + BC^2/2$. So using $BN^2 = BC^2/3$ we get $2AB^2 + 2/3\ AC^2 = AB^2 + AC^2$ and hence $(AC/AB)^2 = 3 = (BC/BN)^2$. So AC/AB = BC/BN. Note that is not enough to conclude that triangles ABC and BNC are similar, because the common angle C is not between AC and AB. However, we have AN/AB = (1/3) AC/AB = AB/AC, so $AB^2 = AN.AC$, so AB is tangent to the circle NBC.

Problem A3

$n = k^2 - k + 1$, where k is a prime plus one. Show that we can color some squares of an n x n board black so that each row and column has exactly k black squares, but there is no rectangle with sides parallel to the sides of the board which has its four corner squares black.

Solution

We can regard the rows as lines and the columns as points. Black squares denote incidence. So line 3 contains point 4 iff square (3, 4) is black. The condition about rectangles then means that there is at most one line through two distinct points.

Suppose we take the points to be (a, b, c), where a, b, c are residues mod p, not all zero, and the coordinates are homogeneous, so that we regard (a, b, c), (2a, 2b, 2c), ... , ((p-1)a, (p-1)b, (p-1)c) as the same point. That gives $(p^3 - 1)/(p-1) = p^2 + p + 1$ points, which is the correct number.

We can take lines to be $lx + my + nz = 0$, where the point is (x, y, z). In other words, the lines are also triples (l, m, n), with l, m, n residues mod p, not all zero and (l, m, n), $(2l, 2m, 2n)$, ... , $((p-1)l, (p-1)m, (p-1)n)$ representing the same line.

One way of writing the points is p^2 of the form (a, b, 1), p of the form (a, 1, 0) and lastly (1, 0, 0). Similarly for the lines. We must show that (1) each point is on p+1 lines (so each column has p+1 black squares), (2) each line has p+1 points (so each row has p+1 black squares, (3) two lines meet in just one point (so no rectangles).

(1): Consider the point P (a, b, 1) with a non-zero. Then for any m, there is a unique l such that $la + mb + 1.1 = 0$, so there are p lines of the form $(l, m, 1)$ which contain P. Similarly, there is a unique l such that $la + 1b + 0.1 = 0$, so one line of the form $(l, 1, 0)$ contains P. The line (1, 0, 0) does not contain P. So P lies on just p+1 lines. Similarly for (a, b, 1) with b non-zero. The point (0, 0, 1) does not lie on any lines $(l, m,, 1)$, but lies on $(l, 1, 0)$ and (1, 0, 0), so again it lies on p+1 lines.

Consider the point Q (a, 1, 0) with a non-zero. For any m, there is a unique l such that Q lies on $(l, m, 0)$. There is also a unique l such that Q lies on $(l, 1, 0)$. Q does not lie on (1, 0, 0), so it lies on just p+1 lines. Similarly, the point (0, 1, 0) lies on the p lines $(l, 0, 0)$ and on (1, 0, 0), but no others.

Finally, the point (1, 0, 0) lies on the p lines $(0, m, 1)$, the line (0, 1, 0) and no others. Thus in all cases a point lies on just p+1 lines. The proof of (2) is identical.

(3). Suppose the lines are (l, m, n) and (L, M, N). If l and L are non-zero, then we can take the lines as $(1, m', n')$ and $(1, M', N')$. So any point (x, y, z) on both satisfies $x + m'y + n'z = 0$ (*) and $x + M'y + N'z = 0$. Subtracting, $(m' - M')y + (n' - N')z = 0$. The coefficients cannot both be zero, since the lines are distinct. So the ratio y : z is fixed. Then (*) gives the ratio x : y. So the point is uniquely determined. If just one of l, L is non-zero, then we can take the lines as $(0, m', n')$, $(1, M', N')$. We cannot have both m' and n' zero, so the ratio y : z is determined, then the other line determines the ratio x : y. So again the point is uniquely determined. Finally, suppose l and L are both zero. Then since the lines are distinct y and z must both be zero. So the unique point on both lines is (1, 0, 0).

Problem B1

n > 2 is an integer. Consider the pairs (a, b) of relatively prime positive integers, such that $a < b \le n$ and $a + b > n$. Show that the sum of 1/ab taken over all such pairs is 1/2.

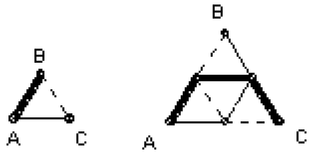
Solution

Induction on n. It is obvious for n = 3, because the only pairs are (1, 3) and (2, 3), and 1/3 + 1/6 = 1/2. Now suppose it is true for n. As we move to n+1, we introduce the new pairs (a, n+1) with a relatively prime to n+1 and we lose the pairs (a, n+1-a) with a relatively prime to n+1-a and hence to n+1. So for each a relatively prime to n+1 and < (n+1)/2 we gain (a, n+1) and (n+1-a, n+1) and lose (a, n+1-a). But 1/a(n+1) + 1/((n+1-a)(n+1)) = (n+1-a + a)/(a(n+1-a)(n+1)) = 1/(a(n+1-a)).

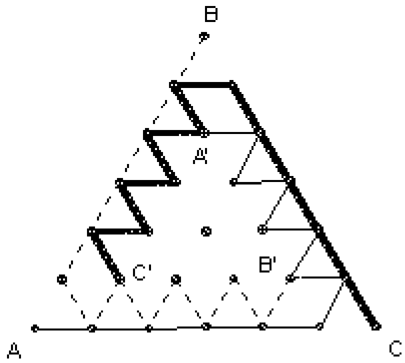
Problem B2

An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1 by lines parallel to the sides. Initially, all the sides of all the small triangles are painted blue. Three coins A, B, C are placed at vertices of the small triangles. Each coin in turn is moved a distance 1 along a blue side to an adjacent vertex. The side it moves along is painted red, so once a coin has moved along a side, the side cannot be used again. More than one coin is allowed to occupy the same vertex. The coins are moved repeatedly in the order A, B, C, A, B, C, ... Show that it is possible to paint all the sides red in this way.

Solution



We use induction. It is obvious for n = 1 and 2 - see diagram above. Note that A, B, C start and end at vertices of the large triangle.



Now assume that for n we can find a solution with A, B, C starting and ending at the vertices of the large triangle. Take n+1. We start with the paths shown which bring A, B, C to A', B', C' at the vertices of a triangle side n-1. Now by induction we can continue the paths so that we bring A, B, C, back to the vertices of that triangle after tracing out all its edges. Finally, note that for each of the points A', B', C' there is a path length 2 over untraced segments to a vertex of the large triangle. So we get a solution for n+1 and hence for all n.

Problem B3

$A_1, A_2, \dots, A_n$ are points in the plane. A non-zero real number k_i is assigned to each point, so that the square of the distance between A_i and A_j (for $i \neq j$) is $k_i + k_j$. Show that n is at most 4 and that if n = 4, then $1/k_1 + 1/k_2 + 1/k_3 + 1/k_4 = 0$.

Solution

Suppose we have four points A, B, C, D with associated numbers a, b, c, d. Then $AB^2 = a + b$, $AC^2 = a + c$, so $AB^2 - AC^2 = b - c$. Similarly, $DB^2 - DC^2 = b - c$, so $AB^2 - AC^2 = DB^2 - DC^2$. Let X be the foot of the perpendicular from A to BC, and Y the foot of the perpendicular from D to BC. Then $AB^2 - AC^2 = (AX^2 + XB^2) - (AX^2 + XC^2) = XB^2 - XC^2$. Similarly for D, so $XB^2 - XC^2 = YB^2 - YC^2$. Hence X = Y, so AD is perpendicular to BC. Similarly, BD is perpendicular to AC, and CD is perpendicular to AB. Hence D is the (unique) orthocenter of ABC. So $n \le 4$.

Suppose n = 4, so we have four points A, B, C, D with associated numbers a, b, c, d. We have $AB^2 + AC^2 - BC^2 = (a + b) + (a + c) - (b + c) = 2a$. But by the cosine formula it is also 2 AB AC cos BAC. Hence a = AB AC cos BAC. Similarly for A, B, D etc. Hence ab/cd = (AB AC cos BAC)(BA BD cos ABD)/((CA.CD cos ACD)(DB DC cos BDC)) = (AB^2/CD^2) (cos BAC/cos BDC) (cos ABD/cos ACD).

Take ABC to be acute with D inside. Then angle ABD = angle ACD (= 90^o - angle BAC), and angle BDC = 90^o + angle ACD = 180^o - angle BAC. So cos BAC/cos BDC = -1. Thus $ab/cd = - AB^2/CD^2 = - (a + b)/(c + d)$. Hence ab(c + d) + cd(a + b) = 0, so 1/a + 1/b + 1/c + 1/d = 0.

12th Iberoamerican 1997

Problem A1

k >= 1 is a real number such that if m is a multiple of n, then [mk] is a multiple of [nk]. Show that k is an integer.

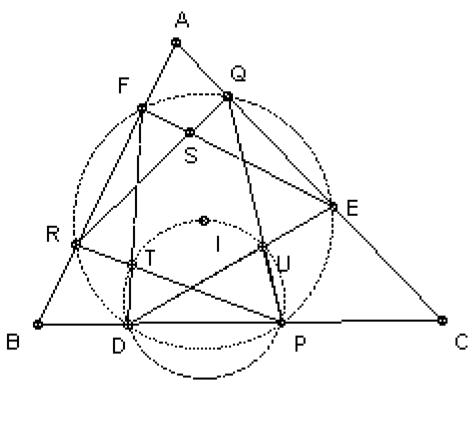
Solution

Suppose k is not an integer. Take an integer n such that nk > 1, but nk is not an integer. Now take a positive integer c such that 1/(c+1) <= nk - [nk] < 1/c. Then 1 <= (c+1)nk - (c+1) [nk] < 1 + 1/c. Hence [(c+1)nk] = (c+1) [nk] + 1. Put m = (c+1)n. Then m is a multiple of n. But if [mk] is a multiple of [nk], then [mk] - (c+1) [nk] = 1 is a multiple of [nk], which is impossible since nk > 1. So we have a contradiction. So k must be an integer.

Problem 2

I is the incenter of the triangle ABC. A circle with center I meets the side BC at D and P, with D nearer to B. Similarly, it meets the side CA at E and Q, with E nearer to C, and it meets AB at F and R, with F nearer to A. The lines EF and QR meet at S, the lines FD and RP meet at T, and the lines DE and PQ meet at U. Show that the circumcircles of DUP, ESQ and FTR have a single point in common.

Solution



D and P are the reflections of Q and E respectively in the line CI. Hence PQ and DE meet at a point on CI. So U lies on CI. So ∠PIU = 1/2 ∠PIE = ∠PDE (I is center of circle through D, P,

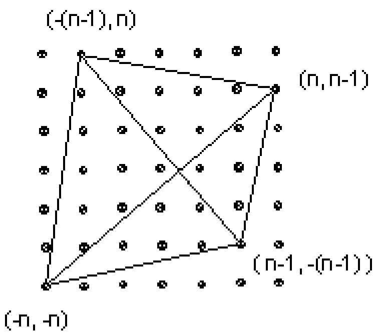
E) = ∠PDU (same angle). Hence PDIU is cyclic. In other words, I lies on the circumcircle of DUP. Similarly, it lies on the circumcircles of ESQ and FTR.

But the same argument shows that ∠DPT = ∠DIT, so DPIT is cyclic. So T lies on the circle through D, P and I and hence on the circumcircle of DUP. Similarly, for the other circles. So the circumcircles of CUP and FTR meet at T and I. Similarly, the circumcircles of FTR and ESQ meet at S and I, and the circumcircles of ESQ and DUP meet at U and I. So the three circumcircles have just one point in common, namely I.

Problem A3

$n > 1$ is an integer. D_n is the set of lattice points (x, y) with $|x|, |y| \le n$. If the points of D_n are colored with three colors (one for each point), show that there are always two points with the same color such that the line containing them does not contain any other points of D_n. Show that it is possible to color the points of D_n with four colors (one for each point) so that if any line contains just two points of D_n then those two points have different colors.

Solution



Consider the 4 points shown in the diagram. In each case the segment joining them is the diagonal of an m x 1 parallelogram or rectangle, so it cannot contain any other lattice points. The next points along each line are obviously outside set D_n. That proves the first part.

The second part is the standard parity argument. Color (x, y) with color 1 if x and y are both even, 2 if x is even and y is odd, 3 if x is odd and y is even, and 4 if x and y are both odd. Then if two points are the same color, that means the first coordinates are the same parity and their second coordinates are the same parity. Hence the midpoint of the segment joining them is also a lattice point and they are not the only two points of D_n on the line.

Problem B1

Let o(n) be the number of 2n-tuples $(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$ such that each $a_i, b_j = 0$ or 1 and $a_1b_1 + a_2b_2 + \dots + a_nb_n$ is odd. Similarly, let e(n) be the number for which the sum is even. Show that $o(n)/e(n) = (2^n - 1)/(2^n + 1)$.

Solution

We prove by induction that $o(n) = 2^{2n-1} - 2^{n-1}$. For n = 1, this reads $o(1) = 2^1 - 2^0 = 1$, which is obviously true - the only such 2-tuple is (1, 1). Suppose it is true for n.

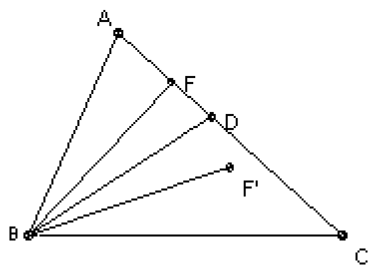
If $(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$ gives an odd sum, then we can take (a_{n+1}, b_{n+1}) to be any of (0, 0), (0, 1), (1, 0) and still get an odd sum for $(a_1, a_2, \dots, a_{n+1}, b_1, b_2, \dots, b_{n+1})$. On the other hand if $(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$ is even, then we must have $a_{n+1} = b_{n+1} = 1$ to get an odd sum. Thus $o(n+1) = 3\,o(n) + e(n)$. But $o(n) = 2_{2n-1} - 2_{n-1}$ and $e(n) = (o(n) + e(n)) - o(n) = 2^{2n} - 2^{2n-1} + 2^{n-1} = 2^{2n-1} + 2^{n-1}$. So $o(n+1) = 4.2^{2n-1} - 2.2^{n-1} = 2^{2(n+1)-1} - 2^{(n+1)-1}$, which establishes the result for n+1 and hence for all n.

Hence $e(n) = 2^{2n} - o(n) = 2^{2n-1} + 2^{n-1}$ and $o(n)/e(n) = (2^n - 1)/(2^n + 1)$.

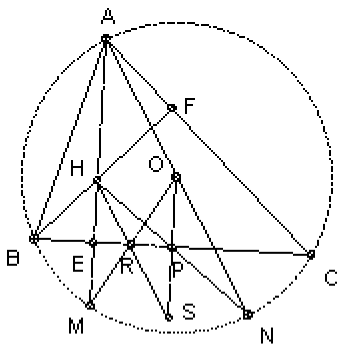
Problem B2

ABC is an acute-angled triangle with orthocenter H. AE and BF are altitudes. AE is reflected in the angle bisector of angle A and BF is reflected in the angle bisector of angle B. The two reflections intersect at O. The rays AE and AO meet the circumcircle of ABC at M and N respectively. P is the intersection of BC and HN, R is the intersection of BC and OM, and S is the intersection of HR and OP. Show that AHSO is a parallelogram.

Solution



We show first that O is the circumcenter of ABC. $\angle ABF = 90^o - A$. The line BC is the reflection in BD of the line BA and the line BF' is the refection of BF, so angle $CBF' = 90^o - A$. But if O' is the circumcenter, then $\angle BO'C = 2\angle BAC = 2A$, so $\angle O'BC = 90^o - A$. Hence O' lies on BF'. Similarly, it lies on AE' (the reflection of AE in the angle bisector of A). Hence O = O'.



$\angle MBC = \angle MAC = 90^o - C$ (since AH is an altitude) $= \angle FBC$ (since BF is an altitude) $= \angle HBC$ (same angle). So triangles HBE and MBE are congruent and HE = EM. [Note: this should be a familiar result.].

AN is a diameter, so angle AMN = 90^o = angle AEC, so BC and MN are parallel. Hence P is the midpoint of HN and of BC. So OP is perpendicular to BC. So AH and OS are parallel.

Since R lies on BC, triangles HER and MER are congruent, so $\angle$EHR = $\angle$EMR = $\angle$AMO (same angle) = $\angle$MAO. Hence HS and AO are parallel. So AHSO is a parallelogram.

Problem B3

Given 1997 points inside a circle of radius 1, one of them the center of the circle. For each point take the distance to the closest (distinct) point. Show that the sum of the squares of the resulting distances is at most 9.

Solution

Let the points be P_i for i = 1, 2, ... , 1997. Take P_1 to be the center of the given unit circle. Let x_i be the distance from P_i to the closest of the other 1996 points. Let C_i be the circle center P_i radius $x_i/2$. Then C_i and C_j cannot overlap by more than one point because x_i and $x_j \le P_iP_j$. Also $x_i \le 1$, since $P_1P_i \le 1$. Thus C_i is entirely contained in the circle center P_1 radius 3/2. Since the circles C_i do not overlap, their total area cannot exceed the area of the circle radius 3/2. Hence $(x_1^2 + x_2^2 + ... + x_{1997}^2)/4 \le 9/4$.

13th Iberoamerican 1998

Problem A1

There are 98 points on a circle. Two players play alternately as follows. Each player joins two points which are not already joined. The game ends when every point has been joined to at least one other. The winner is the last player to play. Does the first or second player have a winning strategy?

Solution

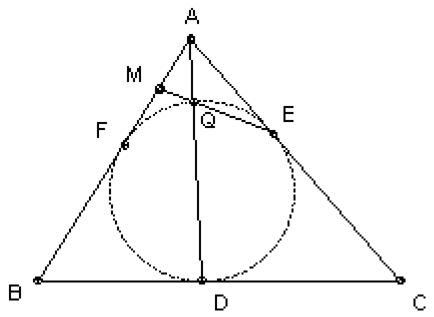
Answer: the first player has a winning strategy.

Assume there are n points. The first to play so that n-2 points each have at least one segment loses, because the other player simply joins the last two points and the game ends. But there are N = (n-3)(n-4)/2 possible plays amongst the first n-3 points to get a segment. For n = 1 or 2 mod 4, N is odd and for n = 0 or 3 it is even. So the first player wins for n = 1 or 2 mod 4 (and in particular for n = 98) and the second player for n = 0 or 3 mod 4.

Problem A2

The incircle of the triangle ABC touches BC, CA, AB at D, E, F respectively. AD meets the circle again at Q. Show that the line EQ passes through the midpoint of AF iff AC = BC.

Solution



$\angle AQM = \angle EQD$ (opposite angle) $= \angle EDC$ (CD tangent to circle EQD) $= (180^o - \angle C)/2 = \angle A/2 + \angle B/2$ (*).

$MF^2 = MQ.ME$ (MF tangent to circle FQE). So AM = AF is equivalent to $AM^2 = MQ.ME$ or AM/MQ = ME/AM. But since triangles AMQ and EMA have a common angle M, AM/MQ = ME/AM iff they are similar, and hence iff $\angle AQM = \angle A$. Using (*) AM = AF iff $\angle A = \angle B$.

Problem A3

Find the smallest number n such that given any n distinct numbers from {1, 2, 3, ... , 999}, one can choose four different numbers a, b, c, d such that a + 2b + 3c = d.

Solution

Answer: n = 835.

Consider the set S = {166, 167, ... , 999}. The smallest possible value for a + 2b + 3c, for distinct a, b, c in S is 168 + 2.167 + 3.166 = 1000. So we cannot find distinct a, b, c, d in S with a + 2b + 3c = d. So the smallest n > 834.

Now suppose S is any subset of 835 elements which satisfies the condition. Take it elements to be $m = a_1 < a_2 < ... < a_{835} = M$. Obviously $M \geq m + 834 \geq 835$, so $-3m \geq 3.834 - 3M$ and hence $M - 3m \geq 2502 - 2M \geq 2502 - 2.999 = 504$. Put k = M - 3m.

There are at least 167 disjoint pairs (a, b) of numbers taken from {1, 2, ... , 999} with a + 2b = k, namely

```
(k - 2, 1)

(k - 4, 2)

(k - 6, 3)

...

(k-334, 167) - note that in the extreme case k = 504 this is (170, 167)
```

At least one number from each pair must either (1) be M or m or (2) not belong to S - or otherwise we would have a + 2b + 3m = M for distinct elements a, b, m and M in S. None of the numbers can be M and at most one of them can be m, so we have at least 166 numbers which are not in S. That means S contains at most 999 - 166 = 833 numbers. Contradiction.

So S cannot have 835 elements. Nor can it have more than 835 elements (or we just take a subset of 835 elements, which must also satisfy the condition, and get a contradiction).

Problem B1

Representatives from n > 1 different countries sit around a table. If two people are from the same country then their respective right hand neighbors are from different countries. Find the maximum number of people who can sit at the table for each n.

Solution

Answer: n^2.

Obviously there cannot be more than n^2 people. For if there were, then at least one country would have more than n representatives. But there are only n different countries to choose their right-hand neighbours from. Contradiction.

Represent someone from country i by i. Then for n = 2, the arrangement 1122 works. [It wraps round, so that the second 2 is adjacent to the first 1.] Suppose we have an arrangement for n. Then each of 11, 22, ... , nn must occur just once in the arrangement. Replace 11 by 1(n+1)11, 22 by 2(n+1)22, ... , and (n-1)(n-1) by (n-1)(n+1)(n-1)(n-1). Finally replace nn by n(n+1)(n+1)nn. It is easy to check that we now have an arrangement for n+1. We have added one additional representative for each of the countries 1 to n and n+1 representatives for country n+1, so we have indeed got $(n+1)^2$ people in all. We have also given a representative of each country 1 to n a neighbour from country n+1 on his right and we have given the (n+1) representatives from country n+1 neighbours (on their right) from each of the other countries. Otherwise we have left the seating unchanged.

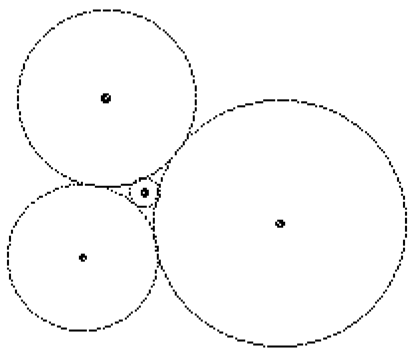
Problem B2

$P_1, P_2, \ldots, P_n$ are points in the plane and $r_1, r_2, \ldots, r_n$ are real numbers such that the distance between P_i and P_j is $r_i + r_j$ (for i not equal to j). Find the largest n for which this is possible.

Solution

Answer: n = 4.

Draw a circle radius r_i at P_i. Then each pair of circles must touch. But that is possible iff $n \leq 4$.



Problem B3

k is the positive root of the equation $x^2 - 1998x - 1 = 0$. Define the sequence $x_0, x_1, x_2, \ldots$ by $x_0 = 1$, $x_{n+1} = [k\, x_n]$. Find the remainder when x_{1998} is divided by 1998.

Solution

Put $p(x) = x^2 - 1998x - 1$. Then $p(1998) = -1$, $p(1999) = 1998$, so $1998 < k < 1999$. Also k is irrational (using the formula for the root of a quadratic). We have $x_n = [k\, x_{n-1}]$, so $x_n < k\, x_{n-1}$ and $> k\, x_{n-1} - 1$. Hence $x_n/k < x_{n-1} < x_n/k + 1/k$, so $[x_n/k] = x_{n-1} - 1$.

$k = (1998k + 1)/k = 1998 + 1/k$. Hence $kx_n = 1998x_n + x/k$. Hence $x_{n+1} = [kx_n] = 1998x_n + [x_n/k] = 1998x_n + x_{n-1} - 1$. Hence $x_{n+1} = x_{n-1} - 1 \bmod 1998$. So $x_{1998} = 1 - 999 = 1000 \bmod 1998$.

14th Iberoamerican 1999

Problem A1

Find all positive integers $n < 1000$ such that the cube of the sum of the digits of n equals n^2.

Solution

$n < 1000$, so the sum of the digits is at most 27, so n^2 is a cube not exceeding 27^3. So we are looking for m^3 which is also a square. That implies m is a square. So the only possibilities are $m = 1, 4, 9, 16, 25$. Giving $n = 1, 8, 27, 64, 125$. The corresponding cubes of the digit sums are 1, 512, 729, 1000, 512, whereas the corresponding squares are 1, 64, 729, 4096, 15625. Thus the only solutions are $n = 1, 27$.

Problem A2

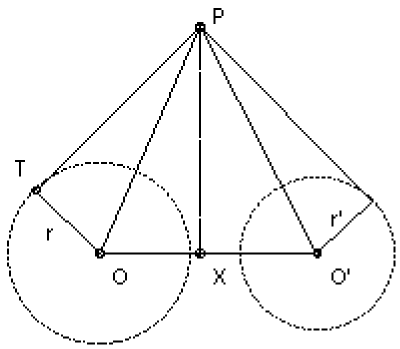
Given two circles C and C' we say that C *bisects* C' if their common chord is a diameter of C'. Show that for any two circles which are not concentric, there are infinitely many circles which bisect them both. Find the locus of the centers of the bisecting circles.

Solution

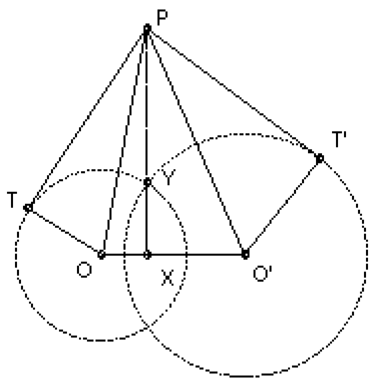
Let C, C' have center O, O' respectively and radius r, r' respectively. Let a circle center P bisect C. Suppose it meets C at A and B. Then AB is perpendicular to OP and is a diameter of C. Hence $PA^2 = OP^2 + r^2$. Conversely, the circle center P, radius $\sqrt{(OP^2 + r^2)}$ bisects C. So P will bisect C and C' iff $OP^2 + r^2 = OP'^2 + r'^2$.

It is well-known that the locus of points P' with equal tangents to C and C' is the *radical axis*. Call the radical axis R. For a point P' on the radical axis we have $P'O^2 - r^2 = P'O'^2 - r'^2$. If we reflect P' in the perpendicular bisector of OO' to get P, then PO = P'O' and PO' = P'O, so $PO'^2 - r^2 = PO^{2 - r'2}$ and hence $PO^2 + r^2$. Call the reflection of the R in the perpendicular bisector of OO' the line R'. We have established that points on R' form part of the locus. Conversely, if P' is such that there is a circle center P' bisecting both circles, then $OP'^2 + r^2 = O'P'^2 + r'^2$, so if P is the reflection of P' then $OP^2 - r^2 = OP'^2 - r'^2$ and hence P lies on the radical axis R. Hence P' must lie on R'.

Radical axis



We have $PT^2 = PO^2 - r^2 = PX^2 + OX^2 - r^2$, and similarly $PT'^2 = PX^2 + O'X^2 - r'^2$. So PT = PT' iff $OX^2 - r^2 = O'X^2 - r'^2$. There is evidently a unique point X for which that is true, so the locus of such P is the line through X perpendicular to OO'



If the circles intersect, then the point X evidently lies on the line joining the two common points, because $OX^2 - r^2 = -XY^2 = O'X^2 - r'^2$. In any case the midpoint of each common tangent evidently lies on the line, so that provides a way of constructing it.

Problem A3

Given points $P_1, P_2, \dots , P_n$ on a line we construct a circle on diameter P_iP_j for each pair i, j and we color the circle with one of k colors. For each k, find all n for which we can always find two circles of the same color with a common external tangent.

Solution

Answer: $n > k+1$.

There are n-1 circles with diameter P_iP_{i+1}. Obviously, each pair has a common tangent. If $n-1 > k$, then two of them must have the same color.

If $n-1 \leq k$, then color all circles with diameter P_iP_j and $i < j$ with color i. Then if two circles have the same color, then both have a tangent at one of the points. Hence one lies inside the other and they do not have a common external tangent.

Problem B1

Show that any integer greater than 10 whose digits are all members of {1, 3, 7, 9} has a prime factor ≥ 11.

Solution

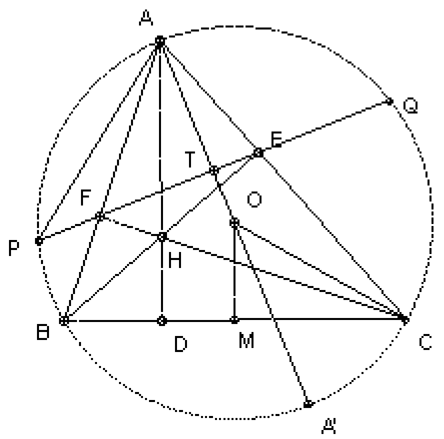
Such a number cannot be divisible by 2 (or its last digit would be even) or by 5 (or its last digit would be 0 or 5). So if the result is false then the number must be of the form 3^m7^n for non-negative integers m, n. But we claim that a number of this form must have even 10s digit.

It is easy to prove the claim by induction. It is true for 3 and 7 (the digit is 0 in both cases). But if we multiply such a number by 3 or 7, then the new 10s digit has the same parity as the carry from the units digit. But multiplying 1, 3, 7, 9 by 3 gives a carry of 0, 0, 2, 6 respectively, which is always even, and multiplying by 7 gives a carry of 0, 2, 4, 6, which is also always even. So the new number also has an even 10s digit.

Problem B2

O is the circumcenter of the acute-angled triangle ABC. The altitudes are AD, BE and CF. The line EF cuts the circumcircle at P and Q. Show that OA is perpendicular to PQ. If M is the midpoint of BC, show that $AP^2 = 2\ AD\cdot\ OM$.

Solution



Let OA and PQ meet at T. $\angle AEH = \angle AFH = 90^o$, so AEHF is cyclic, so $\angle AFT = \angle AFE$ (same angle) $= \angle AHE = 90^o - \angle HAE = 90^o - \angle DAC$ (same angle) $= \angle C$. But $\angle TAF = \angle OAF$ (same angle) $= 90^o - (1/2)\angle AOB = 90^o - \angle C$. Hence $\angle AFT = 90^o$, which establishes that OA and PQ are perpendicular.

Let the circumradius be R and let AA' be a diameter. We have AF = AC cos A = 2R sin B cos A. Hence AT = AF cos OAB = AF sin C = 2R cos A sin B sin C. Now $PT^2 = PT\cdot\ TQ = AT.A'T = AT(2R - AT)$. Hence $AP^2 = 2R\cdot\ AT = 4R^2 \cos A \sin B \sin C$.

We have AD = AC sin C = 2R sin B sin C, and OM = OC cos COM = R cos A. Hence 2 $AD\cdot\ OM = AP^2$.

Problem B3

Given two points A and B, take C on the perpendicular bisector of AB. Define the sequence $C_1, C_2, C_3, \dots$ as follows. $C_1 = C$. If C_n is not on AB, then C_{n+1} is the circumcenter of the triangle ABC_n. If C_n lies on AB, then C_{n+1} is not defined and the sequence terminates. Find all points C such that the sequence is periodic from some point on.

Solution

Answer: any C such that $\angle ACB = 180^o$ r/s, with r and s relatively prime integers and s not a power of 2.

Let $\angle AC_nB = x_n$, where the angle is measured clockwise, so that x_n is positive on one side of AB and negative on the other side. Then x_n uniquely identifies C_n on the perpendicular bisector.

We have $x_{n+1} = 2x_n$. To make this work in all cases we have to take it mod 180^o (so that if AC_nB is obtuse, then C_{n+1} lies on the other side of AB). If x_n is eventually periodic then $x_{m+1} = x_{n+1}$, for some $n > m$, so $(2^n - 2^m)x_1 = 0 \bmod 180$. Hence $x_1 = 180$ r/s for some relatively prime integers r, s. Also s cannot be a power of 2 for then we would have $x_k = 180r$ for some k, in which case the sequence would terminate rather than be periodic.

Conversely, suppose $x_1 = 180$ r/s, with r and s relatively prime and s not a power of 2. Then $x_{n+1} = 180\ 2^n r/s$ cannot be 0 mod 180, so the sequence does not terminate. Put $s = 2^b c$, with c odd. Let $d = \varphi(c)$, where $\varphi(m)$ is Euler's phi function, so that $2^d = 1 \bmod c$. Then $x_{b+1} = 180$ r/c mod 180 and $2^{b+d} = 2^b \bmod c$, so $x_{b+d+1} = 180$ r/c mod 180. Hence the sequence is periodic.

15th Iberoamerican 2000

Problem A1

Label the vertices of a regular n-gon from 1 to n > 3. Draw all the diagonals. Show that if n is odd then we can label each side and diagonal with a number from 1 to n different from the labels of its endpoints so that at each vertex the sides and diagonals all have different labels.

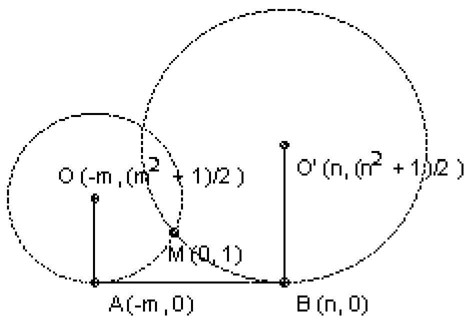
Solution

Labeling the diagonal/side between i and j as i+j (reduced if necessary mod n) almost works. The labels for all the lines at a given vertex will be different. But the line between i and n will have label i, the same as one endpoint. However, we are not using the label 2i for the lines from vertex i. So for the line between i and n we use 2i instead of i+n. The only points that need checking are (1) whether a line from i to n has a label different from n, and (2) whether all the lines at n have different labels. Both points are ok because n is odd.

Problem A2

Two circles C and C' have centers O and O' and meet at M and N. The common tangent closer to M touches C at A and C' at B. The line through B perpendicular to AM meets the line OO' at D. BO'B' is a diameter of C'. Show that M, D and B' are collinear.

Solution



A neat coordinate solution by Massaki Yamamoto (a competitor) is as follows.

Take AB as the x-axis and the perpendicular line through M as the y-axis. Choose the unit of length so that M has coordinates (0, 1). Let A be (-m, 0) and B be (n, 0). Then considering the right-angled triangle O'MK, where K is (n, 1) we find that O' is (n, $(n^2+1)/2$). Similarly, O is (-m, $(m^2+1)/2$)).

The gradient of the lie AM is 1/m, so the gradient of the line BD is -m and hence its equation is mx + y = mn. The gradient of the line OO' is (n-m)/2, so its equation is 2y - x(n-m) = mn+1. These intersect at ((mn-1)/(m+n), $(mn^2+m)/(m+n)$). B' is (n, n^2+1). It is now easy to check that the lines MB' and MD both have gradient n, so M, D, B' are collinear.

Problem A3

Find all solutions to $(m + 1)^a = m^b + 1$ in integers greater than 1.

Answer

(m, a, b) = (2, 2, 3).

Solution

Taking equation mod m+1 we get $(-1)^b = -1$, so b is odd. Hence we can divide the rhs by m+1 to get $m^{b-1} - m^{b-2} + ... - m + 1$. This has an odd number of terms. If m is odd, then each term is odd and so the total is odd, but $(m+1)^{a-1}$ is even (note that a > 1). Contradicton, so m is even.

We have $m^b = (m+1)^a - 1$. Expanding the rhs by the binomial theorem, and using b > 1, we see that m must divide a. So a is even also. Put a = 2A, m = 2M. We can factorise $(m+1)^a - 1$ as ($(m+1)^A + 1$) ($(m+1)^A - 1$). The two factors have difference 2, so their gcd divides 2, but both factors are even, so their gcd is exactly 2.

If M = 1 or a power of 2, then the smaller factor $3^A - 1$ must be 2, so A = 1 and we have $3^A + 1 = 4$, so $(2M)^b = 8$. Hence M = 1 and b = 3 and we have the solution (m, a, b) = (2, 2, 3).

If M is not a power of 2, then $M^b > 2^b$, so we must have the larger factor $2\cdot M^b$ and the smaller factor 2^{b-1}. But the larger factor is now $> 2^{b+1}$, so the difference between the factors is at least $3\cdot 2^{b-1} > 2$. Contradiction.

Problem B1

Some terms are deleted from an infinite arithmetic progression 1, x, y, ... of real numbers to leave an infinite geometric progression 1, a, b, ... Find all possible values of a.

Solution

Answer: the positive integers.

If a is negative, then the terms in the GP are alternately positive and negative, whereas either all terms in the AP from a certain point on are positive or all terms from a certain point on are negative. So a cannot be negative. If a is zero, then all terms in the GP except the first are zero, but at most one term of the AP is zero, so a cannot be zero. Thus a must be positive, so the AP must have infinitely many positive terms and hence $x \geq 1$.

Let $d = x - 1$, so all terms of the AP have the form $1+nd$ for some positive integer n. Suppose $a = 1 + md$, $a^2 = 1 + nd$, then $(1 + md)^2 = 1 + nd$, so $d = (n - 2m)/m^2$, which is rational. Hence a is rational. Suppose $a = b/c$, where b and c are relatively prime positive integers and $c > 1$. Then the denominator of the nth term of the GP is c^n, which becomes arbitrarily large as n increases. But if $d = h/k$, then all terms of the AP have denominator at most k. So we cannot have $c > 1$. So a must be a positive integer.

On the other hand, it is easy to see that any positive integer works. Take $x = 2$, then the AP includes all positive integers and hence includes any GP with positive integer terms.

Problem B2

Given a pile of 2000 stones, two players take turns in taking stones from the pile. Each player must remove 1, 2, 3, 4, or 5 stones from the pile at each turn, but may not take the same number as his opponent took on his last move. The player who takes the last stone wins. Does the first or second player have a winning strategy?

Solution

The first player has a winning strategy. He takes 4 on his first move leaving 7 mod 13 (2000 = 153.13 + 7 + 4). Now we claim that the first player can always leave: (1) 0 mod 13, (2) 3 mod 13 by taking away 3, (3) 5 mod 13 by taking away 5, or (4) 7 mod 13, and that the second player can never leave 0 mod 13.

Let us look at each of these in turn. If the first player leaves 0 mod 13, then the second player can take 3 and leave 10. In that case the first player takes 5 (a type (3) move). If the second player takes 1, 2, 4 or 5, leaving 12, 11, 9 or 8 mod 13, then the first player takes 5, 4, 2, 1 (respectively) and leaves 7 mod 13 (a type (4) move).

If the first player leaves 3 mod 13 by taking away 3, then the second player cannot leave 0 mod 13, because he cannot take 3 stones. If he takes 1, 2 leaving 2, 1 mod 13 respectively, then the first player takes 2, 1 leaving 0 mod 13 (a type (1) move). If the second player takes 4, 5 leaving 12, 11 mod 13, then the first player takes 5, 4 leaving 7 mod 13 (a type (4) move).

If the first player leaves 5 mod 13 by taking 5, then the second player cannot leave 0 mod 13, because he cannot take 5 stones. If he takes 1, 2, 3, 4 stones, leaving 4, 3, 2, 1 mod 13, then the first player takes 4, 3, 2, 1 stones leaving 0 mod 13 (a type (1) move).

Finally, if the first player leaves 7 mod 13, and the second player takes 1 stone, then the first player takes 3 stones leaving 3 mod 13 (a type (2) move). If the second player takes 2, 3, 4, or 5 stones leaving 5, 4, 3, 2 mod 13, then the first player takes 5, 4, 3, 2 stones leaving 0 mod 13 (a type (1) move).

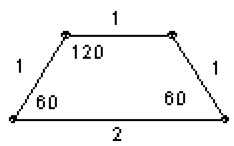
So the second player can never leave 0 mod 13 and hence, in particular, can never take the last stone. But we have shown that the first player can always make a move of one of the four types, so can always move and hence must win (since after less than 2000 moves there will be no stones left).

Problem B3

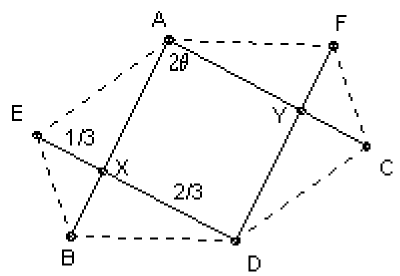
A convex hexagon is called a *unit* if it has four diagonals of length 1, whose endpoints include all the vertices of the hexagon. Show that there is a unit of area k for any $0 < k \le 1$. What is the largest possible area for a unit?

Solution

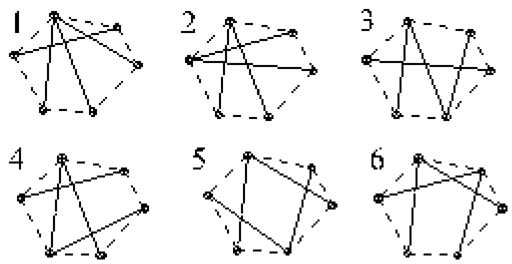
Answer: We can get arbitrarily close to (but not achieve) $(3\sqrt{3})/4$ (approx 1.3) by:



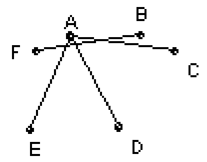
To prove the first part, consider the diagram below. Take AB = AC = 1 and angle BAC = 2θ. Take DE = DF = 1 and take the points of intersection X and Y such that AX = DX = AY = DY = 2/3. It is easy to check that the area of the hexagon is $\sin 2\theta$. So by taking θ in the interval $(0, \pi/4]$ we can get any area $0 < k \le 1$.



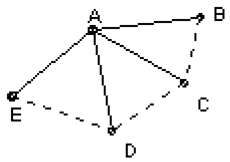
It is easy to check that there are six possible configurations for the unit diagonals, as shown in the diagram below.



Consider case 1.

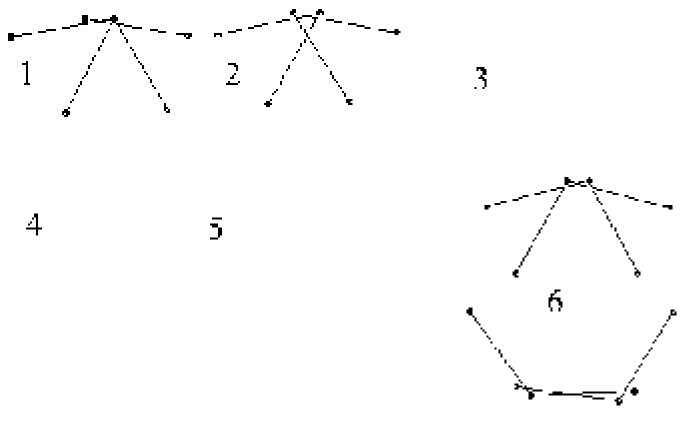


The area of the hexagon is area AEDC + area AFE + area BAC. The part of the segment BF that lies inside AEDC is wasted. The rest goes to provide height for the triangles on bases AE and AC. So area AFE + area BAC can be maximised by taking F close to A and ∠BAC as close to a right angle as possible, so that the height of the triangle BAC (on the base AC) is as large as possible. We can then get arbitrarily close to the area of:



We obviously make AEB a straight line. Now area ADE + area ADC = area ACE + area CDE. So if we regard every point except D as fixed, then we maximise the area by taking ∠EAD = ∠CAD, so that D is the maximum distance from CE. Thus a maximal configuration must have ∠AED = ∠CAD. Similarly, it must have ∠CAD = ∠CAB, so all three angles must be equal. That disposes of case 1.

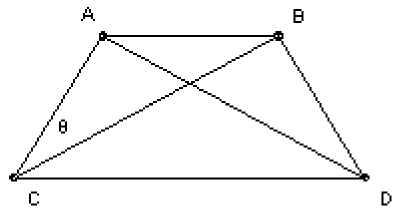
In cases 2 and 6 we find by a similar (but more tedious argument) the same maximum, although in one case we have to use the argument at the end for the final optimisation. In the other cases the maximum is smaller.



However, all these details would take an already long solution way over length. Does anyone have a better approach?

No. 6 (second case) can be made arbitrarily close to the figure below (with AB = AC = BD = 1). To optimise it, suppose ∠ACB = θ. Area ABDC = area ABC + area BCD. If we fix θ, then BC is fixed, so to maximise area BCD we must take ∠CBD = 90°. But θ cannot be optimal unless also ∠CAD = 90°. We have BA = BD and hence ∠BAD = ∠BDA = 45° - θ/2. Hence

$90^o = \angle CAD = \angle BAC - \angle BAD = (180^o - 2\theta) - (45^o - \theta/2)$. Hence $\theta = 30^o$. So $\angle ACD = \angle BDC = 60^o$ and $\angle CAB = \angle ABD = 120^o$. It is easy to check that this has area $(3\sqrt{3})/4$.



16th Iberoamerican 2001

Problem A1

Show that there are arbitrarily large numbers n such that: (1) all its digits are 2 or more; and (2) the product of any four of its digits divides n.

Solution

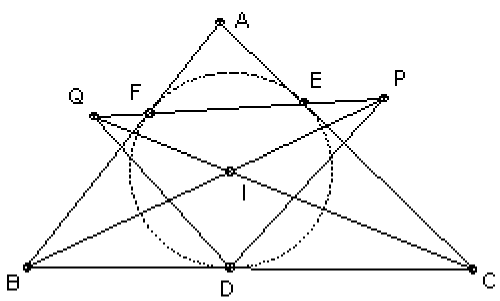
3232 = 16 x 202 and 10000 = 16 x 625. So any number with 3232 as its last 4 digits is divisible by 16. So consider N = 22223232. Its sum of digits is 18, so it is divisible by 9. Hence it is divisible by 9.16 = 144. But any four digits have at most four 2s and at most two 3s, so the product of any four digits divides 144 and hence N. But now we can extend N by inserting an additional 9m 2s at the front. Its digit sum is increased by 18m, so it remains divisible by 144 and it is still divisible by the product of any four digits.

Alternative solution

The number 111111111 with nine 1s is divisible by 9. Hence the number with twenty-seven 1s which equals 111111111 x 1000000001000000001 is divisible by 27. So N, the number with twenty-seven 3s, is divisible by 3^4. Now the number with 27n 3s is divisible by N and hence by 3^4.

Problem A2

ABC is a triangle. The incircle has center I and touches the sides BC, CA, AB at D, E, F respectively. The rays BI and CI meet the line EF at P and Q respectively. Show that if DPQ is isosceles, then ABC is isosceles.



Solution

AF = AE, so ∠AFE = 90^o - A/2. Hence ∠BFP = 90^o + A/2. But ∠FBP = B/2, so ∠FPB = C/2. But BFP and BDP are congruent (BF = BD, BP common, ∠FBP = ∠FDP), so ∠DPB = C/2 and ∠DPQ = C. Similarly, ∠DQP = B. Hence ∠PDQ = A. So DQP and ABC are similar. So if one is isosceles, so is the other.

Problem A3

Let X be a set with n elements. Given k > 2 subsets of X, each with at least r elements, show that we can always find two of them whose intersection has at least r - nk/(4k - 4) elements.

Problem B1

Call a set of 3 distinct elements which are in arithmetic progression a *trio*. What is the largest number of trios that can be subsets of a set of n distinct real numbers?

Answer

(m-1)m for n = 2m
m^2 for n = 2m+1

Solution

Let X be one of the elements. What is the largest number of trios that can have X as middle element? Obviously, at most max(b,a), where b is the number of elements smaller than X and a is the number larger. Thus if n = 2m, the no. of trios is at most 0 + 1 + 2 + ... + m-1 + m-1 + m-2 + ... + 1 + 0 = (m-1)m. If n = 2m+1, then the no. is at most 0 + 1 + 2 + ... + m-1 + m + m-1 + ... + 1 + 0 = m^2.

These maxima can be achieved by taking the numbers 1, 2, 3, ... , n.

Problem B2

Two players play a game on a 2000 x 2001 board. Each has one piece and the players move their pieces alternately. A *short move* is one square in any direction (including diagonally) or no move at all. On his first turn each player makes a short move. On subsequent turns a player must make the same move as on his previous turn followed by a short move. This is treated as a single move. The board is assumed to wrap in both directions so a player on the edge of the board can move to the opposite edge. The first player wins if he can move his piece onto the same square as his opponent's piece. For example, suppose we label the squares from (0, 0) to (1999, 2000), and the first player's piece is initially at (0, 0) and the second player's at (1996, 3). The first player could move to (1999, 2000), then the second player to (1996, 2). Then the first player could move to (1998, 1998), then the second player to (1995, 1). Can the first player always win irrespective of the initial positions of the two pieces?

Problem B3

Show that a square with side 1 cannot be covered by five squares with side less than 1/2.

17th Iberoamerican 2002

Problem A1

The numbers 1, 2, ... , 2002 are written in order on a blackboard. Then the 1st, 4th, 7th, ... , 3k+1th, ... numbers in the list are erased. Then the 1st, 4th, 7th, ... 3k+1th numbers in the remaining list are erased (leaving 3, 5, 8, 9, 12, ...). This process is carried out repeatedly until there are no numbers left. What is the last number to be erased?

Solution

Answer: 1598.

Let a_n be the first number remaining after n iterations, so $a_0 = 1$, $a_1 = 2$, $a_3 = 3$, $a_4 = 5$ etc. We claim that:

$$a_{n+1} = \begin{cases} 3/2\ a_n & \text{if } a_n \text{ is even, and} \\ 3/2\ (a_n + 1) - 1 & \text{if } a_n \text{ is odd.} \end{cases}$$

We use induction on n. Suppose $a_n = 2N$. Consider the number 3N. There are initially N smaller numbers = 1 mod 3. So after the first iteration, it will lie in 2Nth place. Hence, it will lie in first place after n+1 iterations. Similarly, suppose $a_n = 2N+1$. Consider 3N+2. There are initially N+1 smaller numbers = 1 mod 3. So after the first iteration, it will lie in 2N+1st place. Hence, it will lie in first place after n+1 iterations. That completes the induction.

We may now calculate successively the members of the sequence: 1, 2, 3, 5, 8, 12, 18, 27, 41, 62, 93, 140, 210, 315, 473, 710, 1065, 1598, 2397. Hence 1598 is the last surviving number from 1, 2, ... , 2002.

Problem A2

Given a set of 9 points in the plane, no three collinear, show that for each point P in the set, the number of triangles containing P formed from the other 8 points in the set must be even.

Solution

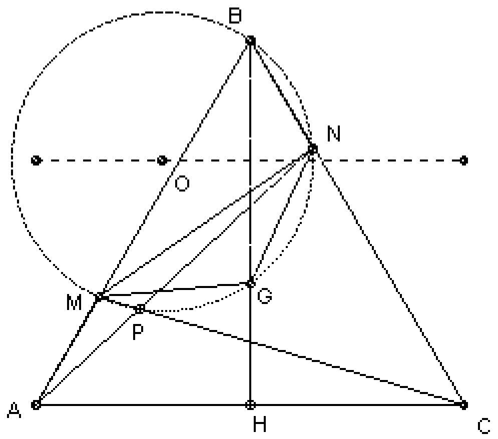
Join each pair of points, thus dividing the plane into polygonal regions. If a point P moves around within one of the regions then the number of triangles it belongs to does not change. But if it crosses one of the lines then it leaves some triangles and enters others. Suppose the line is part of the segment joining the points Q and R of the set. Then it can only enter or leave a triangle QRX for some X in the set. Suppose x points in the set lie on the same side of the line QR as P. Then there are 6 - x points on the other side of the line QR. So P leaves x triangles and enters 6-x. Thus the net change is even. Thus if we move P until it is in the outer infinite region (outside the convex hull of the other 8 points), then we change the number of triangles by an even number. But in the outside region it belongs to no triangles.

Problem A3

ABC is an equilateral triangle. P is a variable interior point such that $\angle APC = 120^o$. The ray CP meets AB at M, and the ray AP meets BC at N. What is the locus of the circumcenter of the triangle MBN as P varies?

Solution

Answer: the segment of the perpendicular bisector of BG (where G is the center of the triangle) which forms a rectangle with AC.



$\angle MPN = \angle APC = 120^o$ and $\angle MBN = 60^o$, so MBNP is cyclic, in other words, P lies on the circumcircle of BMN.

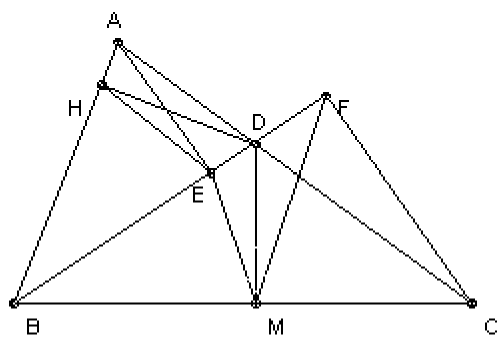
P also lies on the circle AGC, so $\angle CPG = \angle CAG$ (if P is on the same side of AG as A) $= 30^o = \angle MBG$. So PMBG is cyclic. In other words, G also lies on the circumcircle of BMN. If P lies on the other side, the same conclusion follows from considering $\angle APG$.

Since B and G lie on the circumcircle, the center O must lie on the perpendicular bisector of BG. But it is clear that the extreme positions of O occur when P is at A and B and that these are the feet of the perpendiculars from A and B to the perpendicular bisector.

Problem B1

ABC is a triangle. BD is the an angle bisector. E, F are the feet of the perpendiculars from A, C respectively to the line BD. M is the foot of the perpendicular from D to the line BC. Show that $\angle DME = \angle DMF$.

Solution



Let H be the foot of the perpendicular from D to AB. ∠AHD = ∠AED = 90^o, so AHED is cyclic. Hence ∠DAE = ∠DHE. But M is the reflection of H is the line BD, so ∠DME = ∠DAE.

AE is parallel to CD, so ∠DAE = ∠DCF. ∠DFC = ∠DMC, so DMCF is cyclic. Hence ∠DCF = ∠DMF. Hence ∠DME = ∠DMF.

Problem B2

The sequence a_n is defined as follows: $a_1 = 56$, $a_{n+1} = a_n - 1/a_n$. Show that $a_n < 0$ for some n such that $0 < n < 2002$.

Solution

Note that whilst a_n remains positive we have $a_1 > a_2 > a_3 > ... > a_n$. Hence if a_m and a_{m+n} are in this part of the sequence, then $a_{m+1} = a_m - 1/a_m$, $a_{m+2} = a_{m+1} - 1/a_{m+1} < a_{m+1} - 1/a_m = a_m - 2/a_m$. By a trivial induction $a_{m+n} < a_m - n/a_m$.

If we use one step then we need $56^2 = 3136$ terms to get $a_{1+3136} < 56 - 56^2/56 = 0$, which is not good enough. So we try several steps.

Thus suppose that $a_n > 0$ for all n<= 2002. Then we get successively:
$a_{337} < 56 - 336/56 = 50$
$a_{837} < 50 - 500/50 = 40$
$a_{1237} < 40 - 400/40 = 30$
$a_{1537} < 30 - 300/30 = 20$
$a_{1737} < 20 - 200/20 = 10$
$a_{1837} < 10 - 100/10 = 0$.

Contradiction. So we must have $a_n < 0$ for some $n < 2002$.

Problem B3

A game is played on a 2001 x 2001 board as follows. The first player's piece is the policeman, the second player's piece is the robber. Each piece can move one square south, one square east or one square northwest. In addition, the policeman (but not the robber) can move from the bottom right to the top left square in a single move. The policeman starts in the central square, and the robber starts one square diagonally northeast of the policeman. If the policeman moves onto the same square as the robber, then the robber is captured and the first player wins. However, the robber may move onto the same square as the policeman without being captured (and play continues). Show that the robber can avoid capture for at least 10000 moves, but that the policeman can ultimately capture the robber.

Solution

Color the squares with three colors as follows:

```
0  1  2  0  1  2  0  ...  2

1  2  0  1  2  0  1  ...  0

2  0  1  2  0  1  2  ...  1
```

```
0   1   2   0   1   2   0   ...   2

1   2   0   1   2   0   1   ...   0

...

2   0   1   2   0   1   2   ...   1
```

The middle square is color 2 (moving 999+1 squares E from the top left increases the color by 1, then moving 999+1 S increases it by another 1) and the square immediately NE of it is also 2. So both P and R start on color 2. Note that any move increases the color by 1 mod 3, *except* for P's special move which changes the color from 1 to 0.

Until P has made this move, after each move of P, P's color is always 1 more than R's color (mod 3), so P cannot win (irrespective of the moves made by either player). Immediately after he makes the special move for the first time, P is on color 0 and R is on color 1, so immediately after his move P's color is now 1 less than R's color mod 3. Again P cannot win. But after P has made the special move for the second time, P's color is the same as R's (mod 3) immediately after P's move.

Note that it takes P at least 2001 moves to complete his special move for the first time and at least 6002 moves (in total) to complete his special move for the second time. This solves the first part of the question. Suppose R just moves down to the bottom right and then moves in small circles (one move NW, one move S, one move E) waiting for P. It takes P at least 6002 + 3999 (moving from top left to the capture square, one square short of the bottom right) = 10001 to capture him, so R makes at least 10000 moves before being captured.

We claim that P wins if he can get into any of the positions shown below relative to R, with R to move (*):

```
x   P   x   x   x

P   x   x   P   x

x   x   R   x   x

x   P   x   x   P

x   x   x   P   x
```

If follows that P can also win from the four positions below (**):

```
x   x   x   P   x   x   x

x   x   x   x   x   x   x

x   x   x   x   x   x   x

P   x   x   R   x   x   P

x   x   x   x   x   x   x

x   x   x   x   x   x   x

x   x   x   P   x   x   x
```

For in each case at least one of R's possible moves allow P to move immediately into one of the winning positions at (*). But R can only make the other moves a limited number of times before running into the border. [That is obvious if the other two moves are E and S. If they are NW and E, then every NW move takes R closer to the top border, but his total number of E moves can never exceed his total number of NW moves by more than 2000 because of the right border. Similarly, for NW and S.]

Now let d be the number of rows plus the number of columns that R and P are apart. It is easy to check that the positions in (*) and (**) represent the only possibilities for d = 2 and 3. We show that P can always get to d = 2 or 3. For P can always copy R's move, so he can certainly move so that d never increases. But one of R's moves will always allow P to decrease d by 1 or 2. There are three cases to consider:
Case 1. If P is east of R and R moves E, then P moving NW will decrease d by 1 or 2. That is not possible if P is in the top row, but then moving S will decrease d by 2 unless R is also in the top row. If both are in the top row, then P moves S. Now after R's next move, P moves NW which reduces d by 2.
Case 2. If P is south of R and R moves S, then a similar argument, shows that P can always decrease d by 1 or 2 in one or two moves.
Case 3. If P is not south or east or R, and R moves NW, then P can always decrease d by 1 or 2 by moving S or E.

But repeated decreases by 1 or 2 must bring d ultimately to 2 or 3 and hence to one of (*) or (**). So P can always win.

It remains to prove the claim that (*) are winning positions. The reason is that in each case R has one move blocked off, so must make one of the other two. P then copies R's move, so next turn R has the *same* move blocked off. Repeated use of the other two moves will bring him ultimately to one of the sides.

We start with the easiest case: in the two following positions. R cannot move to z, so he must move east or south on each move. Hence he will (after at most 4000 moves) reach the bottom right corner. He then loses moving out of it.

```
x   P   x

P   z   x

x   x   R
```

The other cases of (*) are slightly more complicated. Starting from either of the two positions below, we show that R must eventually reach the extreme left column.

```
w   x   P   x

x   R   z   x

x   y   x   P
```

R cannot move to z, so he can only make NW and S moves. But his total number of S moves can never exceed his total number of NW moves by more than 2000 because he cannot move off the bottom of the board, so he must eventually reach the extreme left column. [If he reaches the bottom row at y, then P can always move to z to preserve the configuration. If R reaches the top row by moving to w, then P can always move to z to preserve the configuration.]

Having reached the extreme left column he is forced to move south. Eventually moving to y will take him to the corner. P then moves to z and R is captured on his next move.

The final case to consider is the two positions below. R cannot move to z, so must move E or NW. A similar argument to the previous case shows that he must eventually reach the top row. Having reached it at w, P moves to z. So R is forced to move right along the top row. When he reaches the corner at y, P moves to z and R is captured when he moves out of the corner.

```
w  x  x

x  R  y

P  z  x

x  x  P
```

18th Iberoamerican 2003

Problem A1

Let A, B be two sets of N consecutive integers. If N = 2003, can we form N pairs (a, b) with a $\angle$ A, b $\angle$ B such that the sums of the pairs are N consecutive integers? What about N = 2004?

Answer

Yes, no.

Solution

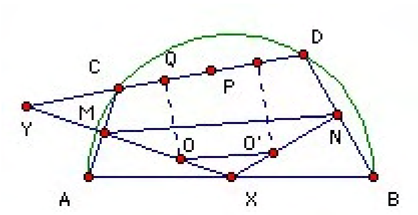
wlog A = B = {1, 2, ... , N} - if we have a solution for A = {a+1, a+2, ... , a+N} and B = {b+1, b+2, ... , b+N}, then subtracting a from every element of A and b from every element of b gives a solution for A = B = {1, 2, ... , N}. Suppose the sum set is (m+1), (m+2), ... , (m+N). It has sum N(2m+N+1)/2 and A and B each have sum N(N+1)/2, so we must have 2m = N+1, hence N must be odd. So we cannot do it for N = 2004.

Suppose N = 2M+1, take the pairs (1,M+1), (3,M), (5,M-1), ... , (2M+1,1), (2,2M+1), (4, 2M), ... , (2M, M+2).

Problem A2

C is a point on the semicircle with diameter AB. D is a point on the arc BC. M, P, N are the midpoints of AC, CD and BD. The circumcenters of ACP and BDP are O, O'. Show that MN and OO' are parallel.

Solution



Let the center of the circle be X and the radius r. Let $\angle AXM = \theta$, $\angle BXN = \varphi$. Note that O is the intersection of XM and the perpendicular to CD at Q, the midpoint of CP. We have $XM = r \cos \theta$. Let CD and XM meet at Y. Then $\angle PYX = 90^o - \angle PXY = 90^o - \angle PXC - \angle CXM = \theta + \varphi - \varphi = \theta$. Hence $OX = PQ \sec \varphi$, so $OX/XM = PQ/(r \cos \theta \cos \varphi)$. Similarly, $O'X/ON = PQ/(r \cos \theta \cos \varphi)$, so OO' and MN are parallel.

Problem A3

Pablo was trying to solve the following problem: find the sequence $x_0, x_1, x_2, \dots, x_{2003}$ which satisfies $x_0 = 1$, $0 \le x_i \le 2 x_{i-1}$ for $1 \le i \le 2003$ and which maximises S. Unfortunately he could not remember the expression for S, but he knew that it had the form $S = \pm x_1 \pm x_2 \pm \dots \pm x_{2002} + x_{2003}$. Show that he can still solve the problem.

Solution

For any combination of signs the maximum is obtained by taking all x_i as large as possible. Suppose we have a different set of x_i. Then for some k we must have $x_k < 2x_{k-1}$ and $x_i = 2x_{i-1}$ for all $i > k$. Suppose $2x^{k-1} - x^k = h > 0$. Then we can increase x_k by h, x_{k+1} by 2h, x_{k+2} by 4h, ... So the sum will be increased by $h(\pm 1 \pm 2 \pm \dots \pm 2^{m-1} + 2^m)$ for some $m \ge 0$. But $\pm 1 \pm 2 \pm \dots \pm 2^{m-1} \ge -(1 + 2 + \dots + 2^{m-1}) = -2^m + 1$, so the overall sum will be increased by at least 1. So the set of x_i was not maximal.

Problem B1

$A \subseteq \{1, 2, 3, \dots, 49\}$ does not contain six consecutive integers. Find the largest possible value of $|A|$. How many such subsets are there (of the maximum size)?

Answer

max = 41; no. ways 495

Solution

We must exclude at least one element of each of the 8 sets $\{1, 2, \dots, 6\}$, $\{7, \dots, 12\}$, $\{13, \dots, 18\}$, ... , $\{43, \dots, 48\}$. So $|A| \le 41$. But a value of 41 is certainly possible, for example, exclude 2, 8, 14, ... , 44.

The largest excluded element must be at least 44 (or we have the 6 consecutive elements 44, 45, 46, 47, 48, 49). The smallest excluded element must be at most 6. If we exclude 2 and 44, then the difference between them is $7 \cdot 6$ and so the other 6 excluded elements are fixed. But if we exclude 3 and 44, for example, then there are several possible choices for the other elements.

There are 5 ways of choosing the smallest and largest excluded element to get a difference of $7\cdot 6$ between them (2 and 44, 3 and 45, 4 and 46, 5 and 47, 6 and 48). There are 4 ways to get a difference of $7\cdot 6 - 1$ (3 and 44, 4 and 45, 5 and 46, 6 and 47). There are 3 ways to get a difference of $7\cdot 6 - 2$ (4 and 44, 5 and 45, 6 and 46), 2 ways to get a difference of $7\cdot 6 - 3$ (5 and 44, 6 and 45), and 1 way to get a difference of $7\cdot 6 - 4$ (6 and 44).

If the difference is $7\cdot 6 - 1$, then we can shorten any of the 7 gaps, so there are 7 possibilities. For example, with 3 and 44, we could shorten the first gap, so excluding 3, 8, 14, 20, 26, 32, 38 and 44, or the second gap, so excluding 3, 9, 14, 20, 26, 32, 38 and 44, and so on.

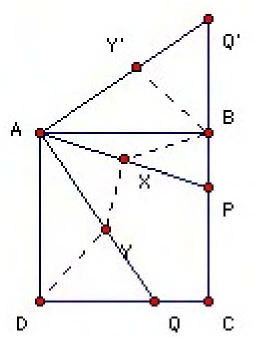
If the difference is $7\cdot 6 - 2$, then we can shorten one gap by two (7 possibilities) or two gaps by one (21 possibilities), total 28. If the difference is $7\cdot 6 - 3$, then we can shorten on gap by three (7), one by two and one by one (42) or three by one (35), total 84. Finally, if the difference is $7\cdot 6 - 4$, we can shorten one by four (7), one by three and one by 1 (42), two by two (21), one by two and two by one (105), or four by one (35), total 210.

So the total number of possibilities is $5\cdot 1 + 4\cdot 7 + 3\cdot 28 + 2\cdot 84 + 1\cdot 210 = 495$.

Problem B2

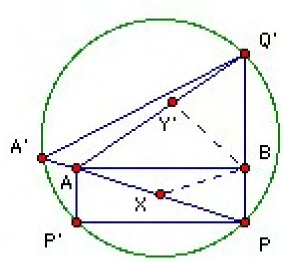
ABCD is a square. P, Q are points on the sides BC, CD respectively, distinct from the endpoints such that BP = CQ. X, Y are points on AP, AQ respectively. Show that there is a triangle with side lengths BX, XY, YD.

Solution



We have DY < BY ≤ BX + XY (this is almost obvious, but to prove formally use the cosine formula for BAY and DAY and notice that ∠BAY > ∠DAY). Similarly, BX < DX ≤ DY + YX. So it remains to show that XY < BX + DY.

Take Q' on the extension of BC so that BQ' = DQ, as shown in the diagram. Take Y' on AQ' so that AY' = AY. Then XY' ≤ BX + BY' = BX + DY. Now we claim that ∠PAQ' > ∠ PAQ, so it follows by the same observation as above that XY' > XY. But the claim is almost obvious. Note that PQ' = AB



So take P' on AD with $\angle P'PQ' = 90^o$. Then A lies inside the circle P'PQ', so extend PA to meet it again at A'. Then $\angle PA'Q' = \angle PP'Q' = 45^o$, so $\angle PAQ' = \angle PA'Q' + \angle AQ'Q' > 45^o$. But $\angle PAQ' + \angle PAQ = 90^o$, so $\angle PAQ' > \angle PAQ$ as claimed.

Problem B3

The sequences $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ are defined by $a_0 = 1$, $b_0 = 4$, $a_{n+1} = a_n^{2001} + b_n$, $b_{n+1} = b_n^{2001} + a_n$. Show that no member of either sequence is divisible by 2003.

Solution

2003 is prime, so $a^{2002} = 1$ mod 2003 for any a not divisible by 2003. Thus $a_{n+1} = a_n^{-1} + b_n$ mod 2003, $b_{n+1} = b_n^{-1} + a_n$ mod 2003. Put $c_n = a_nb_n$. Then $c_{n+1} = c_n + 1/c_n + 2 = (c_n + 1)_2/c_n$ mod 2003. So if $c_n \neq 0$ mod 2003, then $c_{n+1} \neq 0$ mod 2003 unless $c_n = -1$ mod 2003. Then if $c_{n+1} = -1$ mod 2003, we must have $(c_n^2 + 3c_n + 1)/c_n = 0$ mod 2003, so $c_n^2 + 3c_n + 1 = 0$ mod 2003. Note that $c_0 = 4$. So it is sufficient to show that there are no solutions to $x^2 + 3x + 1 = 0$ mod 2003, or equivalently to $(x - 1000)^2 = 1000^2 - 1 = 502$ mod 2003. In other words, we have to show that 502 is a quadratic non-residue mod 2003.

The easiest way to do that is to use the law of quadratic reciprocity, but that is almost certainly outside the syllabus. We note that $4\cdot 502 = 5$ mod 2003, so 502 is a square iff 5 is a square. It is sufficient to show that $5^{1001} = -1$ mod 2003, for then if we had $x^2 = 5$, we would have $x^{2002} = -1$ mod 2003, whereas we know that $x^{2002} = 1$ mod 2003. We note that $1001 = 7\cdot 11\cdot 13$. We start by showing that $5^7 = 8$ mod 2003. We have $5^5 = 3125 = 1122$ mod 2003, so $5^6 = 5610 = 1604$ mod 2003, so $5^7 = 8020 = 8$ mod 2003.

We calculate successively $2^{11} = 2048 = 45$ mod 2003, so $2^{22} = 2025 = 22$ mod 2003. Multiplying by 22 is relatively easy, so $2^{44} = 484$, $2^{66} = 10648 = 633$, $2^{88} = 13926 = -95$, $2^{110} = -2090 = -87$, $2^{132} = -1914 = 89$, $2^{143} = 4005 = -1$ all mod 2003. Hence $8^{11\cdot 13} = -1$ mod 2003, so $5^{1001} = -1$ mod 2003, as required, and we are done.