Russian Mathematical Olympiad

21st Russian 1995 problems

1. A goods train left Moscow at x hrs y mins and arrived in Saratov at y hrs z mins. The journey took z hrs x mins. Find all possible values of x.

2. The chord CD of a circle center O is perpendicular to the diameter AB. The chord AE goes through the midpoint of the radius OC. Prove that the chord DE goes through the midpoint of the chord BC.

3. f(x), g(x), h(x) are quadratic polynomials. Can $f(g(h(x))) = 0$ have roots 1, 2, 3, 4, 5, 6, 7, 8?

4. Can the integers 1 to 81 be arranged in a 9 x 9 array so that the sum of the numbers in every 3 x 3 subarray is the same?

5. Solve $\cos(\cos(\cos(\cos x))) = \sin(\sin(\sin(\sin x)))$.

6. Does there exist a sequence of positive integers such that every positive integer occurs exactly once in the sequence and for each k the sum of the first k terms is divisible by k?

7. A convex polygon has all angles equal. Show that at least two of its sides are not longer than their neighbors.

8. Can we find 12 geometrical progressions whose union includes all the numbers 1, 2, 3, ... , 100?

9. R is the reals. $f: R \to R$ is any function. Show that we can find functions $g: R \to R$ and $h: R \to R$ such that $f(x) = g(x) + h(x)$ and the graphs of g and h both have an axial symmetry.

10. Given two points in a plane a distance 1 apart, one wishes to construct two points a distance n apart using only a compass. One is allowed to draw a circle whose center is any point constructed so far (or given initially) and whose radius is the distance between any two points constructed so far (or given initially). One is also allowed to mark the intersection of any two circles. Let C(n) be the smallest number of circles which must be drawn to get two points a distance n apart. One can also carry out the construction with rule and compass. In this case one is also allowed to draw the line through any two points constructed so far (or given initially) and to mark the intersection of any two lines or of any line and a circle. Let R(n) be the smallest number of circles and lines which must be drawn in this case to get two points a distance n apart (starting with just two points, which are a distance 1 apart). Show that $C(n)/R(n) \to \infty$.

11. Show that we can find positive integers A, B, C such that (1) A, B, C each have 1995 digits, none of them 0, (2) B and C are each formed by permuting the digits of A, and (3) A + B = C.

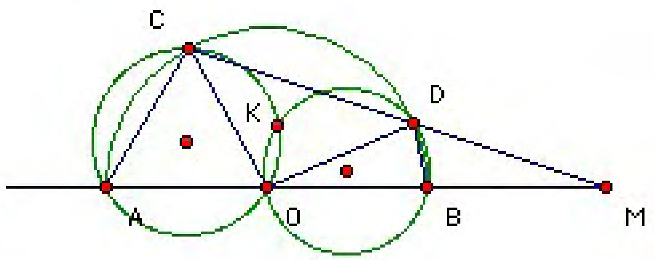
12. ABC is an acute-angled triangle. A_2, B_2, C_2 are the midpoints of the altitudes AA_1, BB_1, CC_1 respectively. Find $\angle B_2A_1C_2 + \angle C_2B_1A_2 + \angle A_2C_1B_2$.

13. There are three heaps of stones. Sisyphus moves stones one at a time. If he takes a stone from one pile, leaving A behind, and adds it to a pile containing B before the move, then Zeus pays him B - A. (If B - A is negative, then Sisyphus pays Zeus A - B.) After some moves the three piles all have the same number of stones that they did originally. What is the maximum net amount that Zeus can have paid Sisyphus?

14. The number 1 or -1 is written in each cell of a 2000 x 2000 array. The sum of all the numbers in the array is non-negative. Show that there are 1000 rows and 1000 columns such that the sum of the numbers at their intersections is at least 1000.

15. A sequence $a_1, a_2, a_3, \ldots$ of positive integers is such that for all $i \neq j$, $\gcd(a_i, a_j) = \gcd(i, j)$. Prove that $a_i = i$ for all i.

16. C, D are points on the semicircle diameter AB, center O. CD meets the line AB at M (with MB < MA, MD < MC). The circumcircles of AOC and DOB meet again at K. Show that $\angle MKO = 90^o$.



17. p(x) and q(x) are non-constant polynomials with leading coefficient 1. Prove that the sum of the squares of the coefficients of the polynomial p(x)q(x) is at least p(0) + q(0).

18. Given any positive integer k, show that we can find $a_1 < a_2 < a_3 < \ldots$ such that $a_1 = k$ and $(a_1^2 + a_2^2 + \ldots + a_n^2)$ is divisible by $(a_1 + a_2 + \ldots + a_n)$ for all n.

19. For which n can we find n-1 numbers $a_1, a_2, \ldots, a_{n-1}$ all non-zero mod n such that 0, a_1, a_1+a_2, $a_1+a_2+a_3$, ... , $a_1+a_2+\ldots+a_{n-1}$ are all distinct mod n.

20. ABCD is a tetrahedron with altitudes AA', BB', CC', DD'. The altitudes all pass through the point X. B" is a point on BB' such that BB"/B"B' = 2. C" and D" are similar points on CC', DD' respectively. Prove that X, A', B", C", D" lie on a sphere.

Problem 18

Given any positive integer k, show that we can find $a_1 < a_2 < a_3 < \ldots$ such that $a_1 = k$ and $(a_1^2 + a_2^2 + \ldots + a_n^2)$ is divisible by $(a_1 + a_2 + \ldots + a_n)$ for all n.

Solution

Induction on n. Suppose $a_1 + a_2 + \ldots + a_n = A$ divides $a_1^2 + a_2^2 + \ldots + a_n^2 = B$. Then put $a_{n+1} = A^2+B-A$. We have $a_1 + a_2 + \ldots + a_{n+1} = A^2+B$, and $a_{n+1}^2 = (A^2+B)^2 - 2A(A^2+B) + A^2$, so $a_1^2 + a_2^2 + \ldots + a_{n+1}^2 = (A^2+B)^2 - 2A(A^2+B) + (A^2+B)$, which is divisible by A^2+B.

22nd Russian 1996 problems

1. Can a majority of the numbers from 1 to a million be represented as the sum of a square and a (non-negative) cube?

2. Non-intersecting circles of equal radius are drawn centered on each vertex of a triangle. From each vertex a tangent is drawn to the other circles which intersects the opposite side of the triangle. The six resulting lines enclose a hexagon. Color alternate sides of the hexagon red and blue. Show that the sum of the blue sides equals the sum of the red sides.

3. $a^n + b^n = p^k$ for positive integers a, b and k, where p is an odd prime and n > 1 is an odd integer. Show that n must be a power of p.

4. The set X has 1600 members. P is a collection of 16000 subsets of X, each having 80 members. Show that there must be two members of P which have 3 or less members in common.

5. Show that the arithmetic progression 1, 730, 1459, 2188, ... contains infinitely many powers of 10.

6. The triangle ABC has CA = CB, circumcenter O and incenter I. The point D on BC is such that DO is perpendicular to BI. Show that DI is parallel to AC.

7. Two piles of coins have equal weight. There are m coins in the first pile and n coins in the second pile. For any $0 < k \le \min(m, n)$, the sum of the weights of the k heaviest coins in the first pile is not more than the sum of the weights of the k heaviest coins in the second pile. Show that if h is a positive integer and we replace every coin (in either pile) whose weight is less than h by a coin of weight h, then the first pile will weigh at least as much as the second.

8. An L is formed from three unit squares, so that it can be joined to a unit square to form a 2 x 2 square. Can a 5 x 7 board be covered with several layers of Ls (each covering 3 unit squares of the board), so that each square is covered by the same number of Ls?

9. ABCD is a convex quadrilateral. Points D and F are on the side BC so that the points on BC are in the order B, E, F, C. $\angle BAE = \angle CDF$ and $\angle EAF = \angle EDF$. Show that $\angle CAF = \angle BDE$.

10. Four pieces A, B, C, D are placed on the plane lattice. A move is to select three pieces and to move the first by the vector between the other two. For example, if A is at (1, 2), B at (-3, 4) and C at (5, 7), then one could move A to (9, 5). Show that one can always make a series of moves which brings A and B onto the same node.

11. Find powers of 3 which can be written as the sum of the kth powers (k > 1) of two relatively prime integers.

12. $a_1, a_2, \dots, a_m$ are non-zero integers such that $a_1 + a_2 2^k + a_3 3^k + \dots + a_m m^k = 0$ for $k = 0, 1, 2, \dots, n$ (where $n < m - 1$). Show that the sequence a_i has at least n+1 pairs of consecutive terms with opposite signs.

13. A different number is placed at each vertex of a cube. Each edge is given the greatest common divisor of the numbers at its two endpoints. Can the sum of the edge numbers equal the sum of the vertex numbers?

14. Three sergeants A, B, C and some soldiers serve in a platoon. The first day A is on duty, the second day B is on duty, the third day C, the fourth day A, the fifth B, the sixth C, the seventh A and so on. There is an infinite list of tasks. The commander gives the following orders: (1) the duty sergeant must issue at least one task to a soldier every day, (2) no soldier may have three or more tasks, (3) no soldier may be given more than one new task on any one day, (4) the set of soldiers receiving tasks must be different every day, (5) the first sergeant to violate any of (1) to (4) will be jailed. Can any of the sergeants be sure to avoid going to jail (strategies that involve collusion are not allowed)?

15. No two sides of a convex polygon are parallel. For each side take the angle subtended by the side at the point whose perpendicular distance from the line containing the side is the largest. Show that these angles add up to 180^o.

16. Two players play a game. The first player writes ten positive real numbers on a board. The second player then writes another ten. All the numbers must be distinct. The first player then arranges the numbers into 10 ordered pairs (a, b). The first player wins iff the ten quadratics $x^2 + ax + b$ have 11 distinct real roots between them. Which player wins?

17. The numbers from 1 to $n > 1$ are written down without a break. Can the resulting number be a palindrome (the same read left to right and right to left)? For example, if n was 4, the result would be 1234, which is not a palindrome.

18. n people move along a road, each at a fixed (but possibly different) speed. Over some period the sum of their pairwise distances decreases. Show that we can find a person such that the sum of his distances to the other people is decreasing throughout the period. [Note that people may pass each other during the period.]

19. $n > 4$. Show that no cross-section of a pyramid whose base is a regular n-gon (and whose apex is directly above the center of the n-gon) can be a regular (n+1)-gon.

20. Do there exist three integers each greater than one such that the square of each less one is divisible by both the others?

21. ABC is a triangle with circumcenter O and AB = AC. The line through O perpendicular to the angle bisector CD meets BC at E. The line through E parallel to the angle bisector meets AB at F. Show that DF = BE.

22. Do there exist two finite sets such that we can find polynomials of arbitrarily large degree with all coefficients in the first set and all roots real and in the second set?

23. The integers from 1 to 100 are permuted in an unknown way. One may ask for the order of any 50 integers. How many such questions are needed to deduce the permutation?

Problem 1

Can a majority of the numbers from 1 to a million be represented as the sum of a square and a (non-negative) cube?

Solution

No. There are 10^3 squares and 10^2 cubes available, so at most 10^5 sums.

Problem 20

Do there exist three integers each greater than one such that the square of each less one is divisible by both the others?

Answer

No.

Solution

Take $x > y > z$. Then x divides $y^2 - 1$, so x and y are coprime. Both divide z^2-1, so their product xy must divide $z^2 - 1$. But $xy > z^2 > z^2-1$. Contradiction.

23rd Russian 1997 problems

1. p(x) is a quadratic polynomial with non-negative coefficients. Show that $p(xy)^2 \le p(x^2)p(y^2)$.

2. A convex polygon is invariant under a 90^o rotation. Show that for some R there is a circle radius R contained in the polygon and a circle radius $R\sqrt{2}$ which contains the polygon.

3. A rectangular box has integral sides a, b, c, with c odd. Its surface is covered with pieces of rectangular cloth. Each piece contains an even number of unit squares and has its sides parallel to edges of the box. The pieces may be bent along box edges length c (but not along the edges length a or b), but there must be no gaps and no part of the box may be covered by more than one thickness of cloth. Prove that the number of possible coverings is even.

4. The members of the Council of the Wise are arranged in a column. The king gives each sage a black or a white cap. Each sage can see the color of the caps of all the sages in front of him, but he cannot see his own or the colors of those behind him. Every minute a sage guesses the color of his cap. The king immediately executes those sages who are wrong. The Council agree on a strategy to minimise the number of executions. What is the best strategy? Suppose there are three colors of cap?

5. Find all integral solutions to $(m^2 - n^2)^2 = 1 + 16n$.

6. An n x n square grid is glued to make a cylinder. Some of its cells are colored black. Show that there are two parallel horizontal, vertical or diagonal lines (of n cells) which contain the same number of black cells.

7. Two circles meet at A and B. A line through A meets the circles again at C and D. M, N are the midpoints of the arcs BC, BD which do not contain A. K is the midpoint of the segment CD. Prove that $\angle MKN = 90^o$.

8. A polygon can be divided into 100 rectangles, but not into 99 rectangles. Prove that it cannot be divided into 100 triangles.

9. A cube side n is divided into unit cubes. A closed broken line without self-intersections is given. Each segment of the broken line connects the centers of two unit cubes with a common face. Show that we can color the edges of the unit cubes with two colors, so that each face of a small cube which is intersected by the broken line has an odd number of edges of each color, and each face which is not intersected by the broken line has an even number of edges of each color.

10. Do there exist reals b, c so that $x^2 + bx + c = 0$ and $x^2 + (b+1)x + (c+1) = 0$ both have two integral roots?

11. There are 33 students in a class. Each is asked how many other students share is first name and how many share his last name. The answers include all numbers from 0 to 10. Show that two students must have the same first name and the same last name.

12. The incircle of ABC touches AB, BC, CA at M, N, K respectively. The line through A parallel to NK meets the line MN at D. The line through A parallel to MN meets the line NK at E. Prove that the line DE bisects AB and AD.

13. The numbers 1, 2, 3, ... , 100 are arranged in the cells of a 10 x 10 square so that given any two cells with a common side the sum of their numbers does not exceed N. Find the smallest possible value of N.

14. The incircle of ABC touches the sides AC, AB, BC at K, M, N respectively. The median BB' meeets MN at D. Prove that the incenter lies on the line DK.

15. Find all solutions in positive integers to $a + b = \gcd(a,b)^2$, $b + c = \gcd(b,c)^2$, $c + a = \gcd(c,a)^2$.

16. Some stones are arranged in an infinite line of pots. The pots are numbered ... -3, -2, -1, 0, 1, 2, 3, ... Two moves are allowed: (1) take a stone from pot n-1 and a stone from pot n and put a stone into pot n+1 (for any n); (2) take two stones from pot n and put one stone into each of pots n+1 and n-2. Show that any sequence of moves must eventually terminate (so that no more moves are possible) and that the final state depends only on the initial state.

17. Consider all quadratic polynomials x2 + ax + b with integral coefficients such that $1 \le a, b \le 1997$. Let A be the number with integral roots and B the number with no real roots. Which of A, B is larger?

18. P is a polygon. L is a line, and X is a point on L, such that the lines containing the sides of P meet L in distinct points different from X. We color a vertex of P red iff its the lines containing its two sides meet L on opposite sides of X. Show that X is inside P iff there are an odd number of red vertices.

19. A sphere is inscribed in a tetrahedron. It touches one face at its incenter, another face at its orthocenter, and a third face at its centroid. Show that the tetrahedron must be regular.

20. 2 x 1 dominos are used to tile an m x n square, except for a single 1 x 1 hole at a corner. A domino which borders the hole along its short side may be slid one unit along its long side to cover the hole and open a new hole. Show that the hole may be moved to any other corner by moves of this type.

Problem 1

p(x) is a quadratic polynomial with non-negative coefficients. Show that $p(xy)^2 \le p(x^2)p(y^2)$.

Solution

lhs = $((\sqrt{a}\, x^2)(\sqrt{a}\, y^2) + (\sqrt{b}\, x)(\sqrt{b}\, y) + \sqrt{c}\,\sqrt{c})^2 \le (ax^4 + bx^2 + c)(ay^4 + by^2 + c)$ = rhs by Cauchy-Schwartz.

Problem 5

Find all integral solutions to $(m^2 - n^2)^2 = 1 + 16n$.

Answer

$(m,n) = (\pm 1,0), (\pm 4,3), (\pm 4,5)$

Solution

Clearly n cannot be negative. On the other hand, if (m,n) is a solution, then so is (-m,n). If n = 0, then $m = \pm 1$. If m = 0, then $n^4 = 1 + 16n$, which has no solutions (because n must divide 1, but 1 is not a solution). So let us assume m and n are positive.

We have $m^2 = n^2 + \sqrt{(1+16n)}$ or $m^2 = n^2 - \sqrt{(1+16n)}$. In the first case, obviously m > n. But $(n+1)^2 = n^2 + 2n + 1$ which is greater than $n^2 + \sqrt{(1+16n)}$ for $(2n+1)^2 > 1 + 16n$ or

$n > 3$. So if $n > 3$, then m^2 lies strictly between n^2 and $(n+1)^2$, which is impossible. It is easy to check that $n = 1, 2$ do not give solutions, but $n = 3$ gives the solution $m = 4$.

Similarly, in the second case we must have $n \le 5$, and it is easy to check that $n = 1, 2, 3, 4$ do not give solutions, but $n = 5$ gives the solution $m = 4$.

Problem 10

Do there exist reals b, c so that $x^2 + bx + c = 0$ and $x^2 + (b+1)x + (c+1) = 0$ both have two integral roots?

Solution

$b = 2$, $c = 1$

Problem 15

Find all solutions in positive integers to $a + b = \gcd(a,b)^2$, $b + c = \gcd(b,c)^2$, $c + a = \gcd(c,a)^2$.

Answer

$(a,b,c) = (2,2,2)$

Solution

Put $d = \gcd(a,b,c)$. Then $a = Ad$, $b = Bd$, $c = Cd$. Put $t = \gcd(A,B)$, $r = \gcd(B,C)$, $s = \gcd(A,C)$. If r, s have a common factor, then it divides all of A, B, C, contradicting the definition of d. So r, s, t are coprime in pairs. Hence $A = Rst$, $B = rSt$, $C = rsT$ for some R, S, T which are relatively prime in pairs. Thus $a = Rstd$, $b = rStd$, $c = rsTd$.

Now $a + b = \gcd(a,b)^2$ implies $A + B = t^2d$. Similarly, $B + C = r2d$, $C + A = s2d$. Adding, $2(A + B + C) = d(r^2 + s^2 + t^2)$. Now if any odd prime divides d, then it must divide $A + B + C$ and $A + B$, $B + C$ and $C + A$ and hence each of A, B, C, which is impossible. Similarly, if 4 divides d, then 2 divides A, B, C, which is impossible. Hence $d = 1$ or 2.

We can also write $a + b = \gcd(a,b)^2$ etc as $Rs + rS = td$, $St + sT = rd$, $Tr + Rt = sd$. Suppose (without loss of generality) that r is the smallest of r, s, t. Then $2r \ge rd = St + sT \ge s + t \ge 2r$. So we must have equality throughout and hence $d = 2$, $S = T = 1$ and $r = s = t$. But r, s, t are coprime, so $r = s = t = 1$. Hence from $Rs + rS = td$, we have $R = 1$ also. Hence $a = b = c = 2$.

24th Russian 1998 problems

1. a and b are such that there are two arcs of the parabola $y = x^2 + ax + b$ lying between the ray $y = x$, $x > 0$ and $y = 2x$, $x > 0$. Show that the projection of the left-hand arc onto the x-axis is smaller than the projection of the right-hand arc by 1.

2. A convex polygon is partitioned into parallelograms, show that at least three vertices of the polygon belong to only one parallelogram.

3. Can you find positive integers a, b, c, so that the sum of any two has digit sum less than 5, but the sum of all three has digit sum more than 50?

4. A maze is a chessboard with walls between some squares. A piece responds to the commands left, right, up, down by moving one square in the indicated direction (parallel to the sides of the board), unless it meets a wall or the edge of the board, in which case it does not move. Is there a universal sequence of moves so that however the maze is constructed and whatever the initial position of the piece, by following the sequence it will visit every square of the board. You should assume that a maze must be constructed, so that some sequence of commands would allow the piece to visit every square.

5. Five watches each have the conventional 12 hour faces. None of them work. You wish to move forward the time on some of the watches so that they all show the same time and so that the sum of the times (in minutes) by which each watch is moved forward is as small as possible. How should the watches be set to maximise this minimum sum?

6. In the triangle ABC, AB > BC, M is the midpoint of AC and BL is the angle bisector of B. The line through L parallel to BC meets BM at E and the line through M parallel to AB meets BL at D. Show that ED is perpendicular to BL.

7. A chain has n > 3 numbered links. A customer asks for the order of the links to be changed to a new order. The jeweller opens the smallest possible number of links, but the customer chooses the new order in order to maximise this number. How many links have to be opened?

8. There are two unequal rational numbers r < s on a blackboard. A move is to replace r by rs/(s - r). The numbers on the board are initially positive integers and a sequence of moves is made, at the end of which the two numbers are equal. Show that the final numbers are positive integers.

9. A, B, C, D, E, F are points on the graph of $y = ax^3 + bx^2 + cx + d$ such that ABC and DEF are both straight lines parallel to the x-axis (with the points in that order from left to right). Show that the length of the projection of BE onto the x-axis equals the sum of the lengths of the projections of AB and CF onto the x-axis.

10. Two polygons are such that the distance between any two vertices of the same polygon is at most 1 and the distance between any vertex of one polygon and any vertex of the other is more than $1/\sqrt{2}$. Show that the interiors of the two polygons are disjoint.

11. The point A' on the incircle of ABC is chosen so that the tangent at A' passes through the foot of the bisector of angle A, but A' does not lie on BC. The line L_A is the line through A' and the midpint of BC. The lines L_B and L_C are defined similarly. Show that L_A, L_B and L_C all pass through a single point on the incircle.

12. X is a set. P is a collection of subsets of X, each of which have exactly 2k elements. Any subset of X with at most $(k+1)^2$ elements either has no subsets in P or is such that all its subsets which are in P have a common element. Show that every subset in P has a common element.

13. The numbers 19 and 98 are written on a blackboard. A move is to take a number n on the blackboard and replace it by n+1 or by n^2. Is it possible to obtain two equal numbers by a series of moves?

14. A binary operation * is defined on the real numbers such that (a * b) * c = a + b + c for all a, b, c. Show that * is the same as +.

15. Given a convex n-gon with no 4 vertices lying on a circle, show that the number of circles through three adjacent vertices of the n-gon such that all the other vertices lie inside the circle exceeds by two the number of circles through three vertices, no two of which are adjacent, such that all other vertices lie inside the circle.

16. Find the number of ways of placing a 1 or -1 into each cell of a $(2^n - 1)$ by $(2^n - 1)$ board, so that the number in each cell is the product the numbers in its neighbours (a neighbour is a cell which shares a side).

17. The incircle of the triangle ABC touches the sides BC, CA, AB at D, E, F respectively. D' is the midpoint of the arc BC that contains A, E' is the midpoint of the arc CA that contains B, and F' is the midpoint of the arc AB that contains C. Show that DD', EE', FF' are concurrent.

18. Given a collection of solid equilateral triangles in the plane, each of which is a translate of the others, such that every two have a common point. Show that there are three points, so that every triangle contains at least one of the points.

19. A connected graph has 1998 points and each point has degree 3. If 200 points, no two of them joined by an edge, are deleted, show that the result is a connected graph.

20. C_1 is the circle center (0, 1/2), diameter 1 which touches the parabola $y = x^2$ at the point (0, 0). The circle C_{n+1} has its center above C_n on the y axis, touches C_n and touches the parabola at two symmetrically placed points. Find the diameter of C_{1998}.

21. Do there exist 1998 different positive integers such that the product of any two is divisible by the square of their difference?

22. The tetrahedron ABCD has all edges less than 100 and contains two disjoint spheres of diameter 1. Show that it contains a sphere of diameter 1.01.

23. A figure is made out of unit squares joined along their sides. It has the property that if the squares of an m x n rectangle are filled with real numbers with positive sum, then the figure can be placed over the rectangle (possibly after being rotated, but with each square of the figure coinciding with a square of the rectangle) so that the sum of the numbers under each square is positive. Prove that a number of copies of the figure can be placed over an m x n rectangle so that each square of the rectangle is covered by the same number of figures.

Problem 3

Can you find positive integers a, b, c, so that the sum of any two has digit sum less than 5, but the sum of all three has digit sum more than 50?

Solution

4554554555
5455455455
5545545545
Sums of pairs 10010010010, 11001001000, 10100100100
Sum of all three 15555555555, total 51.

25th Russian 1999 problems

1. The digits of n strictly increase from left to right. Find the sum of the digits of 9n.

2. Each edge of a finite connected graph is colored with one of N colors in such a way that there is just one edge of each color at each point. One edge of each color but one is deleted. Show that the graph remains connected.

3. ABC is a triangle. A' is the midpoint of the arc BC not containing A, and C' is the midpoint of the arc AB not containing C. S is the circle center A' touching BC and S' is the circle center C' touching AB. Show that the incenter of ABC lies on a common external tangent to S and S'.

4. The numbers from 1 to a million are colored black or white. A move consists of choosing a number and changing the color of the number and every other number which is not coprime to it. If the numbers are initially all black, can they all be changed to white by a series of moves?

5. An equilateral triangle side n is divided into n^2 equilateral triangles of side 1 by lines parallel to its sides, thus giving a network of nodes connected by line segments of length 1. What is the maximum number of segments that can be chosen so that no three chosen segments form a triangle?

6. Let $\{x\}$ denote the fractional part of x. Show that $\{\sqrt{1}\} + \{\sqrt{2}\} + \{\sqrt{3}\} + \dots + \{\sqrt{(n^2)}\} \le (n^2 - 1)/2$.

7. ABC is a triangle. A circle through A and B meets BC again at D, and a circle through B and C meets AB again at E, so that A, E, D, C lie on a circle center O. The two circles meet at B and F. Show that BFO = 90 deg.

8. A graph has 2000 points and every two points are joined by an edge. Two people play a game. The first player always removes one edge. The second player removes one or three edges. The player who first removes the last edge from a point wins. Does the first or second player have a winning strategy?

9. There are three empty bowls X, Y and Z on a table. Three players A, B and C take turns playing a game. A places a piece into bowl Y or Z, B places a piece into bowl Z or X, and C places a piece into bowl X or Y. The first player to place the 1999th piece into a bowl loses. Show that irrespective of who plays first and second (thereafter the order of play is determined) A and B can always conspire to make C lose.

10. The sequence $a_1, a_2, a_3, \dots$ of positive integers is determined by its first two members and the rule $a_{n+2} = (a_{n+1} + a_n)/\gcd(a_n, a_{n+1})$. For which values of a_1 and a_2 is it bounded?

11. The incircle of the triangle ABC touches the sides BC, CA, AB at D, E, F respectively. Each pair from the incircles of AEF, DBF, DEC has two common external tangents, one of which is a side of the triangle ABC. Show that the other three tangents are concurrent.

12. A piece is placed in each unit square of an n x n square on an infinite board of unit squares. A move consists of finding two adjacent pieces (in squares which have a common side) so that one of the pieces can jump over the other onto an empty square. The piece jumped over is removed. Moves are made until no further moves are possible. Show that at least $n^2/3$ moves are made.

13. A number n has sum of digits 100, whilst 44n has sum of digits 800. Find the sum of the digits of 3n.

14. The positive reals x, y satisfy $x^2 + y^3 \ge x^3 + y^4$. Show that $x^3 + y^3 \le 2$.

15. A graph of 12 points is such that every 9 points contain a complete subgraph of 5 points. Show that the graph has a complete subgraph of 6 points. [A complete graph has all possible edges.]

16. Do there exist 19 distinct positive integers whose sum is 1999 and each of which has the same digit sum?

17. The function f assigns an integer to each rational. Show that there are two distinct rationals r and s, such that $f(r) + f(s) \le 2 f(r/2 + s/2)$.

18. A quadrilateral has an inscribed circle C. For each vertex, the incircle is drawn for the triangle formed by the vertex and the two points at which C touches the adjacent sides. Each pair of adjacent incircles has two common external tangents, one a side of the quadrilateral. Show that the other four form a rhombus.

19. Four positive integers have the property that the square of the sum of any two is divisible by the product of the other two. Show that at least three of the integers are equal.

20. Three convex polygons are drawn in the plane. We say that one of the polygons, P, can be separated from the other two if there is a line which meets none of the polygons such that the other two polygons are on the opposite side of the line to P. Show that there is a line which intersects all three polygons iff one of the polygons cannot be separated from the other two.

21. Let A be a vertex of a tetrahedron and let p be the tangent plane at A to the circumsphere of the tetrahedron. Let L, L', L" be the lines in which p intersects the three sides of the tetrahedron through A. Show that the three lines form six angles of 60^o iff the product of each pair of opposite sides of the tetrahedron is equal.

Problem 1

The digits of n strictly increase from left to right. Find the sum of the digits of 9n.

Answer

9

Solution

Suppose the digits of n are $a_1a_2...a_k$. Then the digits of 10n are $a_1a_2...a_k0$. Now subtract n. The units digit is 10-ak and we have carried one from the tens place. But because ai > ai-1, no further carries are needed and the digits of 9n are a_1 (a_2-a_1) (a_3-a_2) ... $(a_k-a_{k-1}-1)$ $(10-a_k)$. Thus the sum of the digits is 9.

Problem 2

Each edge of a finite connected graph is colored with one of N colors in such a way that there is just one edge of each color at each point. One edge of each color but one is deleted. Show that the graph remains connected.

Solution

Induction on N. For N=2, the graph must be a cycle with edges of alternating color, so removing one edge will not disconnect it. Now assume the result is true for N.

Take a graph colored with N+1 colors such that there is just one edge of each color at each point. Remove all the edges of one color X. If the graph remains connected, then it still has an edge of each color at each point, so by induction if we remove an edge of each color but one, then it remains connected. Obviously, if we now put back all but one of the X-colored edges, then it remains connected. So suppose removing all the edges of color X disconnects the graph, giving components $C_1, C_2, \dots , C_r$. By induction removing an edge of each color but one does not disconnect any of the components. Now each component must have an even number of points (because before removing the edges, if there are k points, then there are k/2 edges of any given color). Hence if we now put back the X-colored edges, then each component has an even number of X-colored edges to the other components. So if we regard the components as points, then every point of the resulting graph has even degree. But that means that every edge must be part of a cycle, so removing a single edge cannot disconnect the graph. Hence removing a single X-colored edge must leave all the components connected together and hence leave the graph connected.

Problem 5

An equilateral triangle side n is divided into n^2 equilateral triangles of side 1 by lines parallel to its sides, thus giving a network of nodes connected by line segments of length 1. What is the maximum number of segments that can be chosen so that no three chosen segments form a triangle?

Answer

n(n+1)

Solution

Every segment belongs to just one of the n(n+1)/2 triangles with base horizontal. We can choose at most 2 sides of each of these triangles, or n(n+1) in all. If we choose all the segments that are not horizontal, then we choose n(n+1) segments. Since every triangle has one horizontal segment, no three chosen segments form a triangle.

Problem 6

Let {x} denote the fractional part of x. Show that $\{\sqrt{1}\} + \{\sqrt{2}\} + \{\sqrt{3}\} + \dots + \{\sqrt{(n^2)}\} \le (n^2 - 1)/2$.

Solution

For i = 1, 2, ... , 2n we have $\{\sqrt{(n^2+i)}\} = \sqrt{(n^2+i)} - n < (n+i/2n) - n = i/2n$. So $\{\sqrt{(n^2+1)}\} + \{\sqrt{(n^2+2)}\} + \dots + \{\sqrt{(n^2+2n+1)}\} = \{\sqrt{(n^2+1)}\} + \{\sqrt{(n^2+2)}\} + \dots + \{\sqrt{(n^2+2n)}\} < (1 + 2 + \dots + 2n)/2n = (2n+1)/2$. Hence $\{\sqrt{1}\} + \{\sqrt{2}\} + \{\sqrt{3}\} + \dots + \{\sqrt{(n^2)}\} \le 3/2 + 5/2 + \dots + (2n-1)/2 = (n-1)(2n+2)/2$.

Problem 9

There are three empty bowls X, Y and Z on a table. Three players A, B and C take turns playing a game. A places a piece into bowl Y or Z, B places a piece into bowl Z or X, and C places a piece into bowl X or Y. The first player to place the 1999th piece into a bowl loses. Show that irrespective of who plays first and second (thereafter the order of play is determined) A and B can always conspire to make C lose.

Solution

A always plays into bowl X, and B always plays into bowl Y (if they can). For the first 999 moves each, A, B and C will certainly play into X and Y. If C can still play, then at least one of A and B can still play into X and Y. So for the next 500 moves C and at least one of A, B will play into X and Y. During that time at most 500 pieces go into Z, so A and B are still free to play into Z. After 999 + 500 moves there are at least $3\cdot 999 + 2\cdot 500 = 3997$ pieces in X and Y. After the next piece is played into X and Y, they are both full and C cannot play. But both A and B can still play into Z. If both of A and B play into X/Y beyond the 999th move, then C loses quicker.

Problem 10

The sequence a_1, a_2, a_3, ... of positive integers is determined by its first two members and the rule $a_{n+2} = (a_{n+1} + a_n)/gcd(a_n, a_{n+1})$. For which values of a_1 and a_2 is it bounded?

Answer

$a_1 = a_2 = 2$

Solution

Put $d_n = gcd(a_n, a_{n+1})$. Note that d_{n+1} divides a_{n+1} and a_{n+1} and hence also $d_n a_{n+2} - a_{n+1} = a_n$. So it also divides d_n. Hence, in particular, $d_n \geq d_{n+1}$. Since all d_n are positive integers, we must have $d_n = d$ for all $n \geq$ some N.

If d = 1, then $a_{n+2} = a_{n+1} + a_n > a_{n+1}$ for any n > N. So a_n cannot be bounded.

If d ≥ 3, then $a_{n+2} < (a_{n+1} + a_n)/2$ for all n > N. Hence $a_{n+2} < max(a_{n+1}, a_n)$. Now $a_{n+3} < max(a_{n+2}, a_{n+1}) \leq max(a_{n+1}, a_n)$. Hence $max(a_{n+3}, a_{n+2}) < max(a_{n+1}, a_n)$. So we get an infinite strictly decreasing sequence of positive integers. Contradiction.

So we must have d = 2. Hence $a_{n+2} = (a_{n+1} + a_n)/2$ for all n > N. Hence $a_{n+2} - a_{n+1} = (a_n - a_{n+1})/2$. So if $a_N \neq a_{N+1}$, then for sufficiently large n we get an non-integral. Contradiction. So $a_N = a_{N+1}$. But $gcd(a_N,a_{N+1}) = 2$, so $a_N = a_{N+1} = 2$. So 2 = (2 + aN-1)/$gcd(2,a_{N-1})$. If $gcd(2,a_{N-1}) = 1$, then $a_{N-1} = 0$. Contradiction. So $gcd(2,a_{N-1}) = 2$. Hence $a_{N-1} = 2\ gcd(2,a_{N-1}) - 2 = 2$. Now by a trivial induction all terms are 2. In particular a_1 and a_2 are 2.

Problem 13

A number n has sum of digits 100, whilst 44n has sum of digits 800. Find the sum of the digits of 3n.

Answer

300

Solution

Suppose n has digits $a_1a_2...a_k$ and digit sum $s = \sum a_i$. If we temporarily allow digits larger than 9, then $4n = (4a_1)(4a_2)...(4a_k)$. Each carry of one reduces the digit sum by 9, so after making all necessary carries, the digit sum for 4n is at most 4s. It can only be 4s iff there are no carries. Similarly, 11n has digit sum at most 2s, with equality iff there are no carries.

Since 44n has digit sum 8s, there cannot be any carries. In particular, there are no carries in forming 4n, so each digit of n is at most 2. Hence there are no carries in forming 3n and the digit sum of 3n is 300.

Problem 14

The positive reals x, y satisfy $x^2 + y^3 \geq x^3 + y^4$. Show that $x^3 + y^3 \leq 2$.

Solution

If x, y both ≤ 1, then the inequality is obvious. If x, y both > 1, then they do not satisfy the condition. So either $x \ge 1 \ge y$ or $y \ge 1 \ge x$. We show first that $x^3 + y^3 \le x^2 + y^2$. In the first case we have $x^2 + y^2(1-y) \ge x^2 + y^3(1-y) \ge x^3 + y^4 - y^4 = x^3$, so $x^2 + y^2 \ge x^3 + y^3$, as required. In the second case we have $x^2 \ge x^3 + y^3(y-1) \ge x^3 + y^2(y-1)$, so again $x^2 + y^2 \ge x^3 + y^3$.

Now by Cauchy-Scwartz, $x^2 + y^2 = x^{3/2}x^{1/2} + y^{3/2}y^{1/2} \le \sqrt{(x^3+y^3)}\ \sqrt{(x+y)}$. But $x^2 + y^2 \ge x^3 + y^3$, so $x^2 + y^2 \le x + y$. But $2xy \le (x^2+y^2)$, so $(x + y)^2 \le$ & $2(x^2 + y^2)$ and we have just shown that $2(x^2 + y^2) \le 2(x + y)$, so $x + y \le 2$. Thus we have $x^3 + y^3 \le x^2 + y^2 \le x + y \le 2$.

Problem 16

Do there exist 19 distinct positive integers whose sum is 1999 and each of which has the same digit sum?

Answer

no

Solution

Suppose each of the integers has digit sum k. Then each number = k mod 9 and so their sum = 19k mod 9. But their sum is 1999 = 1 mod 9. Hence k = 1 mod 9. So k = 1 or 10 or k ≥ 19. If k = 1, then the only possible numbers under 1999 are 1, 10, 100, 1000, so we cannot get 19 distinct integers. If k ≥ 19, then each number must have at least 3 digits. So their sum is at least 100 + 101 + ... + 118 = 2071 > 1999. So we must have k = 10.

The 20 smallest integers with digit sum 10 are: 19, 28, 37, 46, 55, 64, 73, 82, 91, 109, 118, 127, 136, 145, 154, 163, 172, 181, 190, 208. The sum of the first 19 is 1990, which is too small. However, the next smallest sum comes from replacing 190 by 208, but that gives 2008, which is too large.

Problem 19

Four positive integers have the property that the square of the sum of any two is divisible by the product of the other two. Show that at least three of the integers are equal.

Solution

Suppose that we can find 4 such integers with no three equal. Let a, b, c, d be the set with the smallest a + b + c + d. Suppose a prime p divides a and b. Since a divides $(b+c)^2$ and $(c+d)^2$, it follows that p must divide b+c and c+d. Hence it divides c and d. But now a/p, b/p, c/p, d/p has the same property and smaller sum. Contradiction.

Now suppose an odd prime p divides a. Then p must divide b+c, c+d and d+b and hence also their sum 2(b+c+d). But p is odd, so it must divide b+c+d and hence b = (b+c+d) - (c+d). Contradiction. So a must be a power of 2. Similarly, b, c, d must be powers of 2. If two of them are > 1, then they have a common factor 2. Contradiction. So three of them must be 1. Contradiction.

26th Russian 2000 problems

1. The equations $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ have a common real root, and the equations $x^2 + x + a = 0$ and $x^2 + cx + b = 0$ have a common real root. Find a + b + c.

2. A chooses a positive integer $X \le 100$. B has to find it. B is allowed to ask 7 questions of the form "What is the greatest common divisor of $X + m$ and n?" for positive integers $m, n < 100$. Show that he can find X.

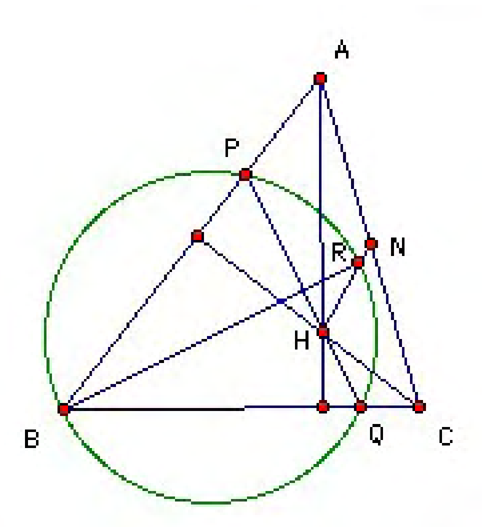
3. O is the circumcenter of the obtuse-angled triangle ABC. K is the circumcenter of AOC. The lines AB, BC meet the circumcircle of AOC again at M, N respectively. L is the reflection of K in the line MN. Show that the lines BL and AC are perpendicular.

4. Some pairs of towns are connected by a road. At least 3 roads leave each town. Show that there is a cycle containing a number of towns which is not a multiple of 3.

5. Find $[1/3] + [2/3] + [2^2/3] + [2^3/3] + \dots + [2^{1000}/3]$.

6. We have $-1 < x_1 < x_2 < \dots < x_n < 1$ and $y_1 < y_2 < \dots < y_n$ such that $x_1 + x_2 + \dots + x_n = x_1^{13} + x_2^{13} + \dots + x_n^{13}$. Show that $x_1^{13}y_1 + x_2^{13}y_2 + \dots + x_n^{13}y_n < x_1y_1 + \dots + x_ny_n$.

7. ABC is acute-angled and is not isosceles. The bisector of the acute angle between the altitudes from A and C meets AB at P and BC at Q. The angle bisector of B meets the line joining HN at R, where H is the orthocenter and N is the midpoint of AC. Show that BPRQ is cyclic.



8. We wish to place 5 stones with distinct weights in increasing order of weight. The stones are indistinguisable (apart from their weights). Nine questions of the form "Is it true that A < B < C?" are allowed (and get a yes/no answer). Is that sufficient?

9. R is the reals. Find all functions $f: R \to R$ which satisfy $f(x+y) + f(y+z) + f(z+x) \ge 3f(x+2y+3z)$ for all x, y, z.

10. Show that it is possible to partition the positive integers into 100 non-empty sets so that if $a + 99b = c$ for integers a, b, c, then a, b, c are not all in different sets.

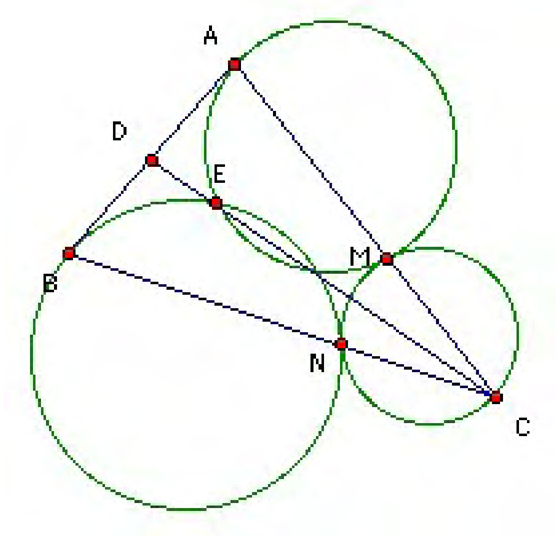
11. ABCDE is a convex pentagon whose vertices are all lattice points. A'B'C'D'E' is the pentagon formed by the diagonals. Show that it must have a lattice point on its boundary or inside it.

12. $a_1, a_2, \dots, a_n$ are non-negative integers not all zero. Put $m_1 = a_1$, $m_2 = \max(a_2, (a_1+a_2)/2)$, $m_3 = \max(a_3, (a_2+a_3)/2 + (a_1+a_2+a_3)/3)$, $m_4 = \max(a_4, (a_3+a_4)/2, (a_2+a_3+a_4)/3, (a_1+a_2+a_3+a_4)/4)$, ... , $m_n = \max(a_n, (a_{n-1}+a_n)/2, (a_{n-2}+a_{n-1}+a_n)/3, \dots, (a_1+a_2+\dots+a_n)/n)$. Show that for any $\alpha > 0$ the number of $m_i > \alpha$ is $< (a_1+a_2+\dots+a_n)/\alpha$.

13. The sequence $a_1, a_2, a_3, \ldots$ is constructed as follows. $a_1 = 1$. $a_{n+1} = a_n - 2$ if $a_n - 2$ is a positive integer which has not yet appeared in the sequence, and $a_n + 3$ otherwise. Show that if a_n is a square, then $a_n > a_{n-1}$.

14. Some cells of a 2n x 2n board contain a white token or a black token. All black tokens which have a white token in the same column are removed. Then all white tokens which have one of the remaining black tokens in the same row are removed. Show that we cannot end up with more than n^2 black tokens and more than n^2 white tokens.

15. ABC is a triangle. E is a point on the median from C. A circle through E touches AB at A and meets AC again at M. Another circle through E touches AB at B and meets BC again at N. Show that the circumcircle of CMN touches the two circles.

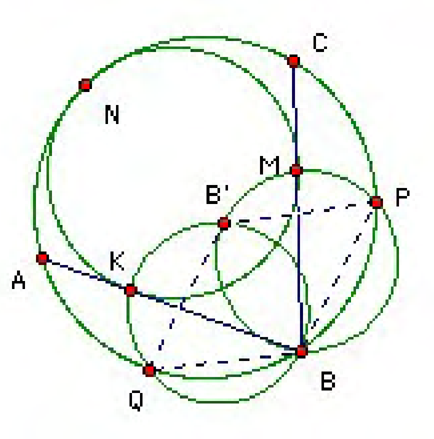


16. 100 positive integers are arranged around a circle. The greatest common divisor of the numbers is 1. An allowed operation is to add to a number the greatest common divisor of its two neighbors. Show that by a sequence of such operations we can get 100 numbers, every two of which are relatively prime.

17. S is a finite set of numbers such that given any three there are two whose sum is in S. What is the largest number of elements that S can have?

18. A perfect number is equal to the sum of all its positive divisors other than itself. Show that if a perfect number > 6 is divisible by 3, then it is divisible by 9. Show that a perfect number > 28 divisible by 7 must be divisible by 49.

19. A larger circle contains a smaller circle and touches it at N. Chords BA, BC of the larger circle touch the smaller circle at K, M respectively. The midpoints of the arcs BC, BA (not containing N) are P, Q respectively. The circumcircles of BPM, BQK meet again at B'. Show that BPB'Q is a parallelogram.



20. Several thin unit cardboard squares are put on a rectangular table with sides parallel to the sides of the table. The squares may overlap. Each square is colored with one of k colors. Given any k squares of different colors, we can find two that overlap. Show that for one of the colors we can nail all the squares of that color to the table with 2k-2 nails.

21. Show that $\sin^n 2x + (\sin^n x - \cos^n x)^2 \le 1$.

22. ABCD has an inscribed circle center O. The lines AB and CD meet at X. The incircle of XAD touches AD at L. The excircle of XBC opposite X touches BC at K. X, K, L are collinear. Show that O lies on the line joining the midpoints of AD and BC.

23. Each cell of a 100 x 100 board is painted with one of four colors, so that each row and each column contains exactly 25 cells of each color. Show that there are two rows and two columns whose four intersections are all different colors.

Problem 1

The equations $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ have a common real root, and the equations $x^2 + x + a = 0$ and $x^2 + cx + b = 0$ have a common real root. Find a + b + c.

Answer

-3

Solution

The common root of $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ must also satisfy $(a-b)x + (1-c) = 0$, so it must be $(c-1)/(a-b)$. Note that the other root of $x^2 + ax + 1 = 0$ must be $(a-b)/(c-1)$, since the product of the roots is 1. Similarly the common root of $x^2 + x + a = 0$ and $x^2 + cx + b = 0$ must satisfy $(c-1)x + (b-a) = 0$, so it is $x = (a-b)/(c-1)$. Hence $x^2 + x + a = 0$ and $x^2 + ax + 1 = 0$ have a common root. Hence it satisfies $(a-1)x + (1-a) = 0$. Now we cannot have a = 1, for then $x^2 + ax + 1$ has no real roots. Hence the common root must be 1. Hence both roots of $x^2 + ax + 1 = 0$ are 1 and so a = -2.

So $x^2 + bx + c = 0$ has one root 1. Then its other root must be c/1 = c. Hence -b = 1 + c, or b + c = -1. Hence a + b + c = -3.

Problem 2

A chooses a positive integer $X \le 100$. B has to find it. B is allowed to ask 7 questions of the form "What is the greatest common divisor of X + m and n?" for positive integers m, n < 100. Show that he can find X.

Solution

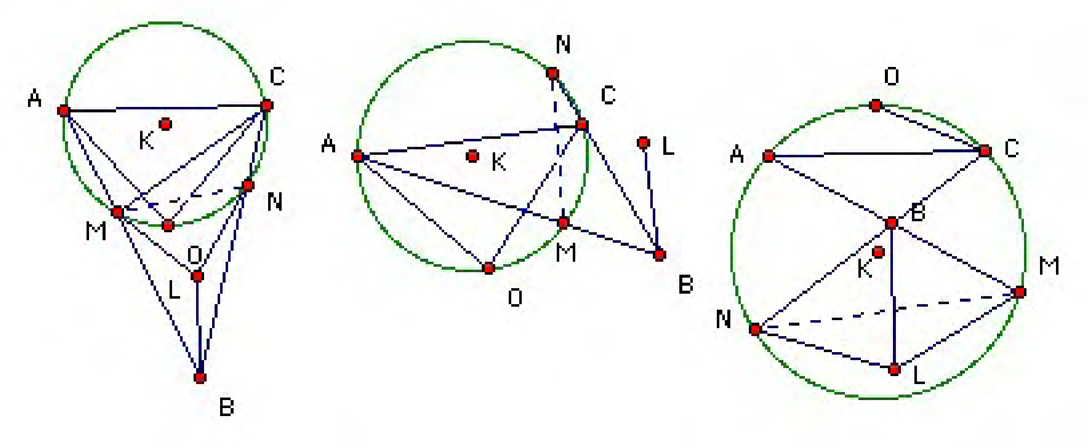
The idea is that X can be regarded as having 7 binary digits $b_6b_5...b_0$ and we can find each digit with one question. If there was no restriction on the numbers m, n, then the procedure would be trivial: having determined the last k binary digits which give a number m, we then ask for $\gcd(X - m, 2^{k+1})$. If it is 2^{k+1}, then $b_k = 0$, if it is 2_k, then $b_k = 1$. Given the restrictions, we have to use a slight variant on this procedure.

For b_0 we can ask for gcd(X+1,2). If it is 1, then $b_0 = 0$. If it is 2, then $b_0 = 1$. Now if we have found $b_0, b_1, ... b_{k-1}$ with $k \le 5$, we want to subtract off $b = b_{k-1}...b_1b_0$. So we add $m = 2_{k+1} - b$. Then the last k+1 digits of $X + 2^{k+1} - b$ are $b_k0...0$, so $\gcd(X+m,2^{k+1})$ is 2^{k+1} if $b_k = 0$, or 2^k if $b_k = 1$. That gives us $b_0, b_1, ... , b^5$ in the first 6 questions. If $b = b_5b_4...b_0$, we know that X = b or b+64. These are different mod 3. So we take m = 1, 2 or 3 to give b + m a multiple of 3 and and take n = 3. Then gcd(X+m,n) distinguishes b and b+64.

Problem 3

O is the circumcenter of the triangle ABC. K is the circumcenter of AOC. The lines AB, BC meet the circumcircle of AOC again at M, N respectively. L is the reflection of K in the line MN. Show that the lines BL and AC are perpendicular.

Solution



We start by showing that $\angle MCB = \angle MBC$. The argument is slightly different in the three different cases. In the first we have: $\angle MCB = \angle MCO + \angle OCB = \angle MAO + \angle OBC = \angle MBO + \angle OBC = \angle MBC$. In the second case we have $\angle MCB = \angle OCB - \angle OCM = \angle OBC - \angle OAB = \angle OBC - \angle OBA = \angle MBC$. In the third case we have $\angle MCB = \angle MCO - \angle OCB = (180^o - \angle OAB) - \angle OBC = 180^o - \angle OBA - \angle OBC = \angle MBC$.

Next we show that L is the circumcenter of MNB. In all cases we have $\angle MLN = \angle MKN$. In the first case, $\angle MKN = 2\angle MCN = 2\angle MBN$. But LM = LN (because KM = KN), so L is the circumcenter. In the second case, $\angle MKN = 2\angle MAN = 2\angle MCB = 2\angle MBN$ and then as before. In the third case, $\angle MKN = 2\angle MCN = 360^o - 2\angle MBN$. So $360^o - \angle MLN = 2\angle MBN$, so L is the circumcenter.

Finally, we have $\angle MLB = 2\angle MNB = 2\angle A$. Hence $\angle MLB = 90^o - \angle A$.

Problem 4

Some pairs of towns are connected by a road. At least 3 roads leave each town. Show that there is a cycle containing a number of towns which is not a multiple of 3.

Solution

Take any town T_1. Now having found $T_1, T_2, \dots , T_i$ take T_{i+1} distinct from $T_1, T_2, \dots , T_i$ such that there is a road from T_i to T_{i+1}. Since there are only finitely many towns, this process must terminate at some T_n. Since T_n has at least 3 roads and the chain cannot be extended there must be at least two towns T_a and T_b with roads to T_n in the chain (apart from T_{n-1}). wlog a < b. Now consider the cycles:
$T_a, T_{a+1}, \dots , T_n$ length n-a+1
$T_b, T_{b+1}, \dots , T_n$ length n-b+1
$T_a, T_{a+1}, \dots , T_b, T_n$ length b-a+2
Now (n-a+1) - (n-b+1) - (b-a+2) = -2, which is not divisible by 3, so at least one of the cycles must have length not divisible by 3.

Problem 5

Find $[1/3] + [2/3] + [2^2/3] + [2^3/3] + \dots + [2^{1000}/3]$.

Answer

$(2/3)(2^{1000}-1)-500$

Solution

Let $f(n) = [2^n/3]$. Note that $f(0) = f(1) = 0$. We have $2^n = (3-1)^n = (-1)^n \bmod 3$, so if n is even, then $2^n/3 = f(n) + 1/3$ and hence $2^{n+1}/3 = 2f(n) + 2/3$. So $f(n+1) = 2f(n)$. Similarly, if n is odd, $f(n+1) = 2f(n) + 1$. Thus $f(2n) = 2f(2n-1)+1 = 4f(2n-2)+1$. Put $u_n = f(2n) + 1/3$, then $u_n = 4u_{n-1}$. But $u_1 = 4/3$, so $u_n = 4^n/3$.

Hence $u_1 + u_2 + \dots + u_n = (4/3)(1 + \dots + 4^{n-1}) = (4/9)(4^n-1)$. So $f(2) + f(4) + \dots + f(2n) = (4/9)(4^n-1) - n/3$. We have $f(2n+1) = f(2n)$, so $f(3) + f(5) + \dots + f(2n-1) = (8/9)(4^{n-1}-1) - (2/3)(n-1)$. Hence $f(2) + f(3) + \dots + f(2n) = (2/3)(4^n-1) - n$.

Problem 6

We have $-1 < x_1 < x_2 < \dots < x_n < 1$ and $y_1 < y_2 < \dots < y_n$ such that $x_1 + x_2 + \dots + x_n = x_1^{13} + x_2^{13} + \dots + x_n^{13}$. Show that $x_1^{13}y_1 + x_2^{13}y_2 + \dots + x_n^{13}y_n < x_1y_1 + \dots + x_ny_n$.

Solution

Note that if $x_i > 0$, then $x_i^{13} < x_i$ and if $x_i < 0$, then $x_i^{13} > x_i$. So not all the x_i can be non-negative and not all can be non-positive. Hence $x_1 < 0 < x_n$.

Put $z_i = x_i - x_i^{13}$. Then $z_1 < z_2 < z_3 < \dots < z_n$. So if $z_i + z_{i+1} + \dots + z_n \le 0$ for $i > 1$, then z_i must be negative and hence all of $z_1, z_2, \dots , z_{i-1}$ must be negative and hence $z_1 + z_2 + \dots + z_n < 0$. Contradiction. hence $z_i + z_{i+1} + \dots + z_n > 0$ for $i > 1$.

Now we have $\sum x_iy_i - \sum x_iy_i^{13} = \sum y_iz_i = y_1(z_1 + z_2 + \dots + z_n) + (y_2 - y_1)(z_2 + \dots + z_n) + (y_3 - y_2)(z_3 + \dots + z_n) + \dots + (y_n - y_{n-1})z_n > 0$.

Problem 8

We wish to place 5 stones with distinct weights in increasing order of weight. The stones are indistinguisable (apart from their weights). Nine questions of the form "Is it true that A < B < C?" are allowed (and get a yes/no answer). Is that sufficient?

Answer

no

Solution

There are 120 possible orders for the weights. 20 will fit any given A < B < C. So a "No" answer can exclude at most 20 possible orders. Thus if the answer to the first 5 questions is "No", then there are still 20 possible orders after them. Now if a "No" to the next question leaves k possibilities, then a "Yes" will leave at least 20-k, so one answer must leave at least 10 possibilities. Similarly, one answer to the 7th question must leave at least 5 possibilities, one answer to the 8th question must leave at least 3 possibilities, and one answer to the 9th question must leave at least 2 possibilities.

Problem 9

R is the reals. Find all functions f: R R which satisfy f(x+y) + f(y+z) + f(z+x) ≥ 3f(x+2y+3z) for all x, y, z.

Answer

f constant.

Solution

Put x = a, y = z = 0, then 2f(a) + f(0) ≥ 3f(a), so f(0) ≥ f(a). Put x = a/2, y = a/2, z = -a/2. Then f(a) + f(0) + f(0) ≥ 3f(0), so f(a) ≥ f(0). Hence f(a) = f(0) for all a. But any constant function obviously satisfies the given relation.

Problem 10

Show that it is possible to partition the positive integers into 100 non-empty sets so that if a + 99b = c for integers a, b, c, then a, b, c are not all in different sets.

Solution

Let t(n) be the largest k such that 2^k divides n. Put A_i = {n : t(n) = i mod 100} for i = 0, 1, 2, ... , 99.

Suppose a + 99b = c and t(a) < t(b). Then 99b is divisible by a higher power of 2, than a. Hence t(c) = t(a). Similarly if t(a) > t(b), then t(c) = t(b). So at least two of t(a), t(b), t(c) must be the same.

Problem 13

The sequence a_1, a_2, a_3, ... is constructed as follows. a_1 = 1. a_{n+1} = a_n - 2 if a_n - 2 is a positive integer which has not yet appeared in the sequence, and a_n + 3 otherwise. Show that if a_n is a square, then a_n > a_{n-1}.

Solution

We show by induction that a_{5n+1} = 5n+1, a_{5n+2} = 5n+4, a_{5n+3} = 5n+2, a_{5n+4} = 5n+5, a_{5n+5} = 5n+3. It is easy to check that a_1 = 1, a_2 = 4, a_3 = 2, a_4 = 5, a_5 = 3, which establishes n = 1.

So now suppose the result is true for n and smaller. Then in particular a_{5n+5} - 2 = 5n+1 is already in the sequence. Hence a_{5n+6} = a_{5n+5} + 3 = 5(n+1)+1. Similarly a_{5n+6} - 2 = 5n+4 is already in the sequence, so a_{5n+7} = a_{5n+6} + 3 = 5(n+1)+4. But 5(n+1)+2 = a_{5n+7} - 2 is not yet in the sequence, so a_{5n+8} = 5(n+1)+2. 5(n+1) is already in the sequence, so a_{5n+9} = a_{5n+8} + 3 = 5(n+1)+5. 5(n+1)+3 is not yet in the sequence, so a_{5n+10} = a_{5n+9} - 2 = 5(n+1)+3. So we have established the result for n+1 and hence for all n.

Now any square must be 0, 1 or 4 mod 5, so it must be a_{5n+4}, a_{5n+1} or a_{5n+2}, which all satisfy the required condition.

Problem 14

Some cells of a 2n x 2n board contain a white token or a black token. All black tokens which have a white token in the same column are removed. Then all white tokens which have one of the remaining black tokens in the same row are removed. Show that we cannot end up with more than n^2 black tokens and more than n^2 white tokens.

Solution

After the first move every column contains nothing but black tokens (and empty cells) or nothing but white tokens (and empty cells). After the second move the same applies to the rows. So let B be the number of rows containing a black token, and W be the number of rows with no black token. Similarly, let b be the number of columns containing a black token, and w the number of columns with no black token. Then $B+W+b+w = 4n$, so $BWbw \le n^4$ (by AM/GM). Hence either $Bb \le n^2$ or $Ww \le n^2$. But the number of black tokens $\le Bb$ (because only rows contain black tokens, and in each of those columns there are at most b black tokens). Similarly, the no. of white tokens is $\le Ww$.

Problem 17

S is a finite set of numbers such that given any three there are two whose sum is in S. What is the largest number of elements that S can have?

Answer

7

Solution

Consider $S = \{-3, -2, -1, 0, 1, 2, 3\}$. Suppose $\{a, b, c\} \in S$. If $a = 0$, then $a+b \in S$. If a and b have opposite sign, then $|a+b| < \max(|a|,|b|)$, so $a+b \in S$. The only remaining possibilities are $\{1,2,3\}$ and $\{-3,-2,-1\}$ and $1+2$, $-1-2 \in S$. So there is a set with 7 elements having the required property.

Now suppose S has 4 positive elements. Take the largest four: $a < b < c < d$. Consider $\{b,c,d\}$. Clearly $c+d$, $b+d \in S$, so $b+c \in S$ and hence $b+c = d$. But the same argument shows that $a+c = d$, so $a = b$. Contradiction. Hence S has at most 3 positive elements. Similarly, it has at most 3 negative elements, and hence at most 7 elements in all.

Problem 18

A perfect number is equal to the sum of all its positive divisors other than itself. Show that if a perfect number > 6 is divisible by 3, then it is divisible by 9. Show that a perfect number > 28 divisible by 7 must be divisible by 49.

Solution

Suppose n is perfect and divisible by 3 but not 9. Put $n = 3m$. Write the sum of the positive divisors of k as $s(k)$. Then $6m = s(n) = s(3m) = s(3)s(m) = 4s(m)$, so $s(m) = 3m/2$. Hence, in particular m is even. So it certainly has divisors 1, m, m/2 (which are all distinct for $n > 6$, but not for $n = 6$, when $m/2 = 1$) with sum $> 3m/2$. Contradiction.

Similarly, if n is divisible by 7 but not 49, but $n = 7k$. Then $14k = s(n) = s(7k) = s(7)s(k) = 8s(k)$, so $s(k) = 7/4\ k$. Hence k must be divisible by 4, so it certainly has factors 1, k, k/2, k/4 (which are all distinct for $n > 28$, but not for $n = 28$, when $k/4 = 1$) with sum $7k/4 + 1$. Contradiction.

Problem 21

Show that $\sin^n 2x + (\sin^n x - \cos^n x)^2 \le 1$.

Solution

Put $s = \sin x$, $c = \cos x$, so $s^2 + c^2 = 1$. Then lhs $= 2^n s^{2n} c^{2n} + (s^n - c^n)^2 = (2^n-2)s^n c^n + s^{2n} + c^{2n}$ and rhs $= 1 = (s^2 + c^2)^n = s^{2n} + c^{2n} + \sum_1^{n-1} nCi\, s^{2n-2i}c^{2i}$. So we have to show that $(2^n-2)s^n c^n \le \sum_1^{n-1} nCi\, s^{2n-2i}c^{2i}$. But that is immediate from AM/GM applied to the 2^n-2 terms $s^h c^k$. (Note that there are the same number of terms $s^{2n-2i}c^{2i}$ and $s^{2i}c^{2n-2i}$ and the product of each pair is $s^{2n}c^{2n}$. Hence the geometric mean is $s^n c^n$.)

27th Russian 2001 problems

1. Are there more positive integers under a million for which the nearest square is odd or for which it is even?

2. A monic quartic and a monic quadratic both have real coefficients. The quartic is negative iff the quadratic is negative and the set of values for which they are negative is an interval of length more than 2. Show that at some point the quartic has a smaller value than the quadratic.

3. ABCD is parallelogram and P a point inside, such that the midpoint of AD is equidistant from P and C, and the midpoint of CD is equidistant from P and A. Let Q be the midpoint of PB. Show that $\angle PAQ = \angle PCQ$.

4. No three diagonals of a convex 2000-gon meet at a point. The diagonals (but not the sides) are each colored with one of 999 colors. Show that there is a triangle whose sides are on three diagonals of the same color.

5. 2001 coins, each value 1, 2 or 3 are arranged in a row. Between any two coins of value 1 there is at least one coin, between any two of value 2 there are at least two coins, and between any two of value 3 there are at least three coins. What is the largest number of value 3 coins that could be in the row?

6. Given a graph of $2n+1$ points, given any set of n points, there is another point joined to each point in the set. Show that there is a point joined to all the other points.

7. N is any point on AC is the longest side of the triangle ABC, such that the perpendicular bisector of AN meets the side AB at K and the perpendicular bisector of NC meets the side BC at M. Prove that BKOM is cyclic, where O is the circumcenter of ABC.

8. Find all odd positive integers $n > 1$ such that if a and b are relatively prime divisors of n, then $a + b - 1$ divides n.

9. Let $A_1, A_2, \dots , A_{100}$ be subsets of a line, each a union of 100 disjoint closed segments. Prove that the intersection of all hundred sets is a union of at most 9901 disjoint closed segments. [A single point is considered to be a closed segment.]

10. The circle C' is inside the circle C and touches it at N. A tangent at the point X of C' meets C at A and B. M is the midpoint of the arc AB which does not contain N. Show that the circumradius of BMX is independent of the position of X.

11. Some pairs of towns in a country are joined by roads, so that there is a unique route from any town to another which does not pass through any town twice. Exactly 100 of the towns have only one road. Show that it is possible to construct 50 new roads so that there will still be a route between any two towns even if any one of the roads (old or new) is closed for maintenance.

12. $x^3 + ax^2 + bx + c$ has three distinct real roots, but $(x^2 + x + 2001)^3 + a(x^2 + x + 2001)^2 + b(x^2 + x + 2001) + c$ has no real roots. Show that $2001^3 + a\, 2001^2 + b\, 2001 + c > 1/64$.

13. An n x n *Latin square* has the numbers from 1 to n^2 arranged in its cells (one per cell) so that the sum of every row and column is the same. For every pair of cells in a Latin square the centers of the cells are joined by an arrow pointing to the cell with the larger number. Show that the sum of these vectors is zero.

14. The altitudes AD, BE, CF of the triangle ABC meet at H. Points P, Q, R are taken on the segments AD, BE, CF respectively, so that the sum of the areas of triangles ABR, AQC and PBC equals the area of ABC. Show that P, Q, R, H are cyclic.

15. S is a set of 100 stones. f(S) is the set of integers n such that we can find n stones in the collection weighing half the total weight of the set. What is the maximum possible number of integers in f(S)?

16. There are two families of convex polygons in the plane. Each family has a pair of disjoint polygons. Any polygon from one family intersects any polygon from the other family. Show that there is a line which intersects all the polygons.

17. N contestants answered an exam with n questions. a_i points are awarded for a correct answer to question i and nil for an incorrect answer. After the questions had been marked it was noticed that by a suitable choice of positive numbers a_i any desired ranking of the contestants could be achieved. What is the largest possible value of N?

18. The quadratics $x^2 + ax + b$ and $x^2 + cx + d$ have real coefficients and take negative values on disjoint intervals. Show that there are real numbers h, k such that $h(x^2 + ax + b) + k(x^2 + cx + d) > 0$ for all x.

19. $m > n$ are positive integers such that $m^2 + mn + n^2$ divides $mn(m + n)$. Show that $(m - n)^3 > mn$.

20. A country has 2001 towns. Each town has a road to at least one other town. If a subset of the towns is such that any other town has a road to at least one member of the subset, then it has at least $k > 1$ towns. Show that the country may be partitioned into $2001 - k$ republics so that no two towns in the same republic are joined by a road.

21. ABCD is a tetrahedron. O is the circumcenter of ABC. The sphere center O through A, B, C meets the edges DA, DB, DC again at A', B', C'. Show that the tangent planes to the sphere at A', B', C' pass through the center of the sphere through A', B', C', D.

Problem 1

Are there more positive integers under a million for which the nearest square is odd or for which it is even?

Answer

odd

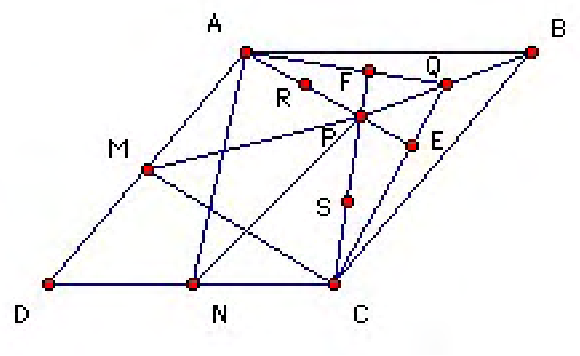
Solution

There are 2n integers for which the closest integer is n^2, namely $n^2-n+1, n^2-n+2, \dots, n^2+n$. So there are $2(1 + 3 + 5 + \dots + 999) = 500\cdot1000$ integers under a million for which the nearest square is odd. Hence there are 499999 integers for which the nearest square is even.

Problem 3

ABCD is parallelogram and P a point inside, such that the midpoint of AD is equidistant from P and C, and the midpoint of CD is equidistant from P and A. Let Q be the midpoint of PB. Show that ∠PAQ = ∠PCQ.

Solution



Let M, N be the midpoints of AD, CD respectively, and let R, S be the midpoints of AP, CP respectively. Since Q, R are the midpoints of PB, PA, we have QR parallel to AB and half its length. Hence QR = CN and is parallel to it. So QRNC is a parallelogram. Hence NR is parallel to CQ. But NR is perpendicular to AP (because NA = NP), so CQ is perpendicular to AP. Suppose they meet at E, so that ∠PEC = 90^o.

Similarly, QS is equal and parallel to AM, so AMSQ is a parallelogram, so AQ is parallel to MS and hence perpendicular to CP. Suppose they meet at F, so that ∠AFP = 90^o.

But now triangles AFP and CEP are similar, so ∠PAF = ∠PCE, as required.

Problem 5

2001 coins, each value 1, 2 or 3 are arranged in a row. Between any two coins of value 1 there is at least one coin, between any two of value 2 there are at least two coins, and between any two of value 3 there are at least three coins. What is the largest number of value 3 coins that could be in the row?

Answer

501

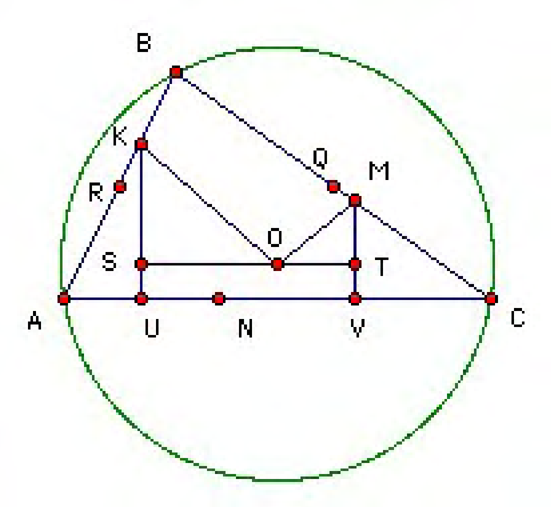
Solution

The only way we can have two value 3 coins the minimum distance apart is ...31213... . If we repeat that pattern then the conditions are all satisfied. So the optimal configuration is 312131213... . Since 2001 = 1 mod 4 that will give the last coin as 3 and a total of 1 + 2000/4 = 501 value 3 coins.

Problem 7

N is any point on AC is the longest side of the triangle ABC, such that the perpendicular bisector of AN meets the side AB at K and the perpendicular bisector of NC meets the side BC at M. Prove that BKOM is cyclic, where O is the circumcenter of ABC.

Solution



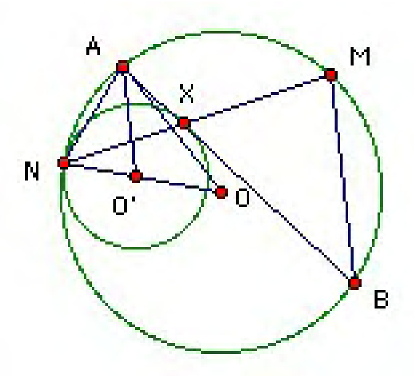
Let the midpoints of BC, BA be Q, R respectively. Then OQ is perpendicular to BC and OR is perpendicular to AB, so OQBR is cyclic. Hence ∠B + ∠QOR = 180^o. So it is sufficient to show that ∠MOQ = ∠KOR, because then ∠MOK = ∠QOR.

Let the line through O parallel to BC meet the perpendiculars from K and M to AC at S and T respectively. Then ∠OQM = ∠OTM = 90^o, so MQOT is cyclic, so ∠MOQ = ∠MTQ. Similarly, ∠KOR = ∠KSR. But QR and ST are both half the length of AC and parallel to it, so QRST is a parallelogram. Hence ∠MTQ = ∠KSR. Hence ∠MOQ = ∠KOR as required.

Problem 10

The circle C' is inside the circle C and touches it at N. A tangent at the point X of C' meets C at A and B. M is the midpoint of the arc AB which does not contain N. Show that the circumradius of BMX is independent of the position of X.

Solution



Let C, C' have centers O, O' respectively. An expansion center N takes C' to C. Suppose it takes X to M'. Then OM' is parallel to O'X and hence perpendicular to AB, so M' = M.

Let the radii of C, C' be R, R' respectively. Let the circumradius of BMX be r. Then MX = 2r sin B, but MA = 2R sin B, so r/R = MX/MA. Triangles XMB, XAN are similar, so MX/MA = MX/MB = AX/AN.

Put AON = k. Then $AN^2 = 2R^2(1 - \cos k)$ and $AX^2 = O'A^2 - O'X^2 = R^2 + (R-R')^2 - 2(R-R')\cos k = 2R(R-R')(1 - \cos k)$. Hence $(AX/AN)^2 = (R-R')/R$. So $r^2 = R(R-R')$, which is independent of k.

Problem 12

$x^3 + ax^2 + bx + c$ has three distinct real roots, but $(x^2 + x + 2001)^3 + a(x^2 + x + 2001)^2 + b(x^2 + x + 2001) + c$ has no real roots. Show that $2001^3 + a\,2001^2 + b\,2001 + c > 1/64$.

Solution

We can write $x^2 + x + 2001 = (x + 1/2)^2 + 2000 - 1/4$, so it takes any value $\geq 2000 - 1/4$. Hence the roots of $x^3 + ax^2 + bx + c$ must all be $< 2000 - 1/4$. Suppose the roots are p, q, r. We have $p, q, r < 2000 - 1/4$, hence $2000-p > 1/4$. Similarly, $2000-q > 1/4$ and $2000-r > 1/4$. Multiplying we get $2001^3 + a\,2001^2 + b\,2001 + c > 1/64$.

28th Russian 2002 problems

1. Can the cells of a 2002 x 2002 table be filled with the numbers from 1 to 2002^2 (one per cell) so that for any cell we can find three numbers a, b, c in the same row or column (or the cell itself) with a = bc?

2. ABC is a triangle. D is a point on the side BC. A is equidistant from the incenter of ABD and the excenter of ABC which lies on the internal angle bisector of B. Show that AC = AD.

3. Given 18 points in the plane, no three collinear, so that they form 816 triangles. The sum of the area of these triangles is A. Six are colored red, six green and six blue. Show that the sum of the areas of the triangles whose vertices are the same color does not exceed A/4.

4. A graph has n points and 100 edges. A move is to pick a point, remove all its edges and join it to any points which it was not joined to immediately before the move. What is the smallest number of moves required to get a graph which has two points with no path between them?

5. The real polynomials p(x), q(x), r(x) have degree 2, 3, 3 respectively and satisfy $p(x)^2 + q(x)^2 = r(x)^2$. Show that either q(x) or r(x) has all its roots real.

6. ABCD is a cyclic quadrilateral. The tangent at A meets the ray CB at K, and the tangent at B meets the ray DA at M, so that BK = BC and AM = AD. Show that the quadrilateral has two sides parallel.

7. Show that for any integer n > 10000, there are integers a, b such that $n < a^2 + b^2 < n + 3\,n^{1/4}$.

8. A graph has 2002 points. Given any three distinct points A, B, C there is a path from A to B that does not involve C. A move is to take any cycle (a set of distinct points $P_1, P_2, \dots, P_n$ such that P_1 is joined to P_2, P_2 is joined to P_3, ... , P_{n-1} is joined to P_n, and P_n is joined to P_1) remove its edges and add a new point X and join it to each point of the cycle. After a series of moves the graph has no cycles. Show that at least 2002 points have only one edge.

9. n points in the plane are such that for any three points we can find a cartesian coordinate system in which the points have integral coordinates. Show that there is a cartesian coordinate system in which all n points have integral coordinates.

10. Show that for n > m > 0 and $0 < x < \pi/2$ we have $| \sin^n x - \cos^n x | \leq 3/2\, | \sin^m x - \cos^m x |$.

11. [unclear]

12. Eight rooks are placed on an 8 x 8 chessboard, so that there is just one rook in each row and column. Show that we can find four rooks, A, B, C, D, so that the distance between the centers of the squares containing A and B equals the distance between the centers of the squares containing C and D.

13. Given k+1 cells. A stack of 2n cards, numbered from 1 to 2n, is in arbitrary order on one of the cells. A move is to take the top card from any cell and place it either on an unoccupied cell or on top of the top card of another cell. The latter is only allowed if the card being moved has number m and it is placed on top of card m+1. What is the largest n for which it is always possible to make a series of moves which result in the cards ending up in a single stack on a different cell.

14. O is the circumcenter of ABC. Points M, N are taken on the sides AB, BC respectively so that $\angle MON = \angle B$. Show that the perimeter of MBN is at least AC.

15. 2^{2n-1} odd numbers are chosen from $\{2^{2n} + 1, 2^{2n} + 2, 2^{2n} + 3, \dots , 2^{3n}\}$. Show that we can find two of them such that neither has its square divisible by any of the other chosen numbers.

16. Show that $\sqrt{x} + \sqrt{y} + \sqrt{z} \ge xy + yz + zx$ for positive reals x, y, z with sum 3.

17. In the triangle ABC, the excircle touches the side BC at A' and a line is drawn through A' parallel to the internal bisector of angle A. Similar lines are drawn for the other two sides. Show that the three lines are concurrent.

18. There are a finite number of red and blue lines in the plane, no two parallel. There is always a third line of the opposite color through the point of intersection of two lines of the same color. Show that all the lines have a common point.

19. Find the smallest positive integer which can be represented both as a sum of 2002 positive integers each with the same sum of digits, and as a sum of 2003 positive integers each with the same sum of digits.

20. ABCD is a cyclic quadrilateral. The diagonals AC and BD meet at X. The circumcircles of ABX and CDX meet again at Y. Z is taken so that the triangles BZC and AYD are similar. Show that if BZCY is convex, then it has an inscribed circle.

21. Show that for infinitely many n the if 1 + 1/2 + 1/3 + ... + 1/n = r/s in lowest terms, then r is not a prime power.

Problem 16

Show that $\sqrt{x} + \sqrt{y} + \sqrt{z} \ge xy + yz + zx$ for positive reals x, y, z with sum 3.

Solution

$x^2 + \sqrt{x} + \sqrt{x} \ge 3x$ by AM/GM. Adding similar inequalities for y, z, we get $x^2 + y^2 + z^2 + 2(\sqrt{x} + \sqrt{y} + \sqrt{z}) \ge 3(x + y + z) = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$.