Indian National Mathematical Olympiad

INMO 1995

1. ABC is an acute-angled triangle with $\angle A = 30^o$. H is the orthocenter and M is the midpoint of BC. T is a point on HM such that HM = MT. Show that AT = 2 BC.

2. Show that there are infinitely many pairs (a,b) of coprime integers (which may be negative, but not zero) such that $x^2 + ax + b = 0$ and $x^2 + 2ax + b$ have integral roots.

3. Show that more 3 element subsets of {1, 2, 3, ... , 63} have sum greater than 95 than have sum less than 95.

4. ABC is a triangle with incircle K, radius r. A circle K', radius r', lies inside ABC and touches AB and AC and touches K externally. Show that $r'/r = \tan^2((\pi\text{-}A)/4)$.

5. $x_1, x_2, \dots, x_n$ are reals > 1 such that $|x_i - x_{i+1}| < 1$ for $i < n$. Show that $x_1/x_2 + x_2/x_3 + \dots + x_{n-1}/x_n + x_n/x_1 < 2n-1$.

6. Find all primes p for which $(2^{p-1} - 1)/p$ is a square.

INMO 1996

1. Given any positive integer n, show that there are distinct positive integers a, b such that a + k divides b + k for k = 1, 2, ... , n. If a, b are positive integers such that a + k divides b + k for all positive integers k, show that a = b.

2. C, C' are concentric circles with radii R, 3R respectively. Show that the orthocenter of any triangle inscribed in C must lie inside the circle C'. Conversely, show that any point inside C' is the orthocenter of some circle inscribed in C.

3. Find reals a, b, c, d, e such that $3a = (b + c + d)^3$, $3b = (c + d + e)^3$, $3c = (d + e + a)^3$, $3d = (e + a + b)^3$, $3e = (a + b + c)^3$.

4. X is a set with n elements. Find the number of triples (A, B, C) such that A, B, C are subsets of X, A is a subset of B, and B is a proper subset of C.

5. The sequence $a_1, a_2, a_3, \dots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 2a_{n+1} - a_n + 2$. Show that for any m, $a_m a_{m+1}$ is also a term of the sequence.

6. A 2n x 2n array has each entry 0 or 1. There are just 3n 0s. Show that it is possible to remove all the 0s by deleting n rows and n columns.

INMO 1997

1. ABCD is a parallelogram. A line through C does not pass through the interior of ABCD and meets the lines AB, AD at E, F respectively. Show that $AC^2 + CE \cdot CF = AB \cdot AE + AD \cdot AF$.

2. Show that there do not exist positive integers m, n such that m/n + (n+1)/m = 4.

3. a, b, c are distinct reals such that $a + 1/b = b + 1/c = c + 1/a = t$ for some real t. Show that $t = -abc$.

4. In a unit square, 100 segments are drawn from the center to the perimeter, dividing the square into 100 parts. If all parts have equal perimeter p, show that $1.4 < p < 1.5$.

5. Find the number of 4 x 4 arrays with entries from {0, 1, 2, 3} such that the sum of each row is divisible by 4, and the sum of each column is divisible by 4.

6. a, b are positive reals such that the cubic $x^3 - ax + b = 0$ has all its roots real. α is the root with smallest absolute value. Show that $b/a < \alpha \leq 3b/2a$.

INMO 1998

1. C is a circle with center O. AB is a chord not passing through O. M is the midpoint of AB. C' is the circle diameter OM. T is a point on C'. The tangent to C' at T meets C at P. Show that $PA^2 + PB^2 = 4\ PT^2$.

2. a, b are positive rationals such that $a^{1/3} + b^{1/3}$ is also a rational. Show that $a^{1/3}$ and $b^{1/3}$ are rational.

3. p, q, r, s are integers and s is not a multiple of 5. If there is an integer a such that $pa^3 + qa^2 + ra + s$ is a multiple of 5, show that there is an integer b such that $sb^3 + rb^2 + qb + p$ is a multiple of 5.

4. ABCD is a cyclic quadrilateral inscribed in a circle radius 1. If $AB\cdot BC\cdot CD\cdot DA \geq 4$, show that ABCD is a square.

5. The quadratic $x^2 - (a+b+c)x + (ab+bc+ca) = 0$ has non-real roots. Show that a, b, c, are all positive and that there is a triangle with sides $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.

6. $a_1, a_2, \ldots, a_{2n}$ is a sequence with two copies each of 0, 1, 2, ... , n-1. A subsequence of n elements is chosen so that its arithmetic mean is integral and as small as possible. Find this minimum value.

INMO 1999

1. ABC is an acute-angled triangle. AD is an altitude, BE a median, and CF an angle bisector. CF meets AD at M, and DE at N. FM = 2, MN = 1, NC = 3. Find the perimeter of ABC.

2. A rectangular field with integer sides and perimeter 3996 is divided into 1998 equal parts, each with integral area. Find the dimensions of the field.

3. Show that $x5 + 2x + 1$ cannot be factorised into two polynomials with integer coefficients (and degree ≥ 1).

4. X, X' are concentric circles. ABC, A'B'C' are equilateral triangles inscribed in X, X' respectively. P, P' are points on the perimeters of X, X' respectively. Show that $P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2$.

5. Given any four distinct reals, show that we can always choose three A, b, C, such that the equations $ax^2 + x + b = 0$, $bx^2 + x + c = 0$, $cx^2 + x + a = 0$ either all have real roots, or all have non-real roots.

6. For which n can {1, 2, 3, ... , 4n} be divided into n disjoint 4-element subsets such that for each subset one element is the arithmetic mean of the other three?

INMO 2000

1. The incircle of ABC touches BC, CA, AB at K, L, M respectively. The line through A parallel to LK meets MK at P, and the line through A parallel to MK meets LK at Q. Show that the line PQ bisects AB and bisects AC.

2. Find the integer solutions to $a + b = 1 - c$, $a^3 + b^3 = 1 - c^2$.

3. a, b, c are non-zero reals, and x is real and satisfies $[bx + c(1-x)]/a = [cx + a(1-x)]/b = [ax + b(1-x)]/b$. Show that $a = b = c$.

4. In a convex quadrilateral PQRS, PQ = RS, SP = $(\sqrt{3} + 1)$QR, and $\angle RSP - \angle SQP = 30^o$. Show that $\angle PQR - \angle QRS = 90^o$.

5. a, b, c are reals such that $0 \le c \le b \le a \le 1$. Show that if α is a root of $z^3 + az^2 + bz + c = 0$, then $|\alpha| \le 1$.

6. Let f(n) be the number of incongruent triangles with integral sides and perimeter n, eg f(3) = 1, f(4) = 0, f(7) = 2. Show that f(1999) > f(1996) and f(2000) = f(1997).

INMO 2001

1. ABC is a triangle which is not right-angled. P is a point in the plane. A', B', C' are the reflections of P in BC, CA, AB. Show that [incomplete].

2. Show that $a^2 + b^2 + c^2 = (a-b)(b-c)(c-a)$ has infinitely many integral solutions.

3. a, b, c are positive reals with product 1. Show that $a^{b+c}b^{c+a}c^{a+b} \le 1$.

4. Show that given any nine integers, we can find four, a, b, c, d such that a + b - c - d is divisible by 20. Show that this is not always true for eight integers.

5. ABC is a triangle. M is the midpoint of BC. $\angle MAB = \angle C$, and $\angle MAC = 15^o$. Show that $\angle AMC$ is obtuse. If O is the circumcenter of ADC, show that AOD is equilateral.

6. Find all real-valued functions f on the reals such that f(x+y) = f(x) f(y) f(xy) for all x, y.

INMO 2002

1. ABCDEF is a convex hexagon. Consider the following statements. (1) AB is parallel to DE, (2) BC is parallel to EF, (3) CD is parallel to FA, (4) AE = BD, (5) BF = CE, (6) CA = DF. Show that if any five of these statements are true then the hexagon is cyclic.

2. Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c. Find all a, b, c which give the smallest value.

3. x, y are positive reals such that x + y = 2. Show that $x^3y^3(x^3 + y^3) \le 2$.

4. Do there exist 100 lines in the plane, no three concurrent, such that they intersect in exactly 2002 points?

5. Do there exist distinct positive integers a, b, c such that a, b, c, -a+b+c, a-b+c, a+b-c, a+b+c form an arithmetic progression (in some order).

6. The numbers 1, 2, 3, ... , n^2 are arranged in an n x n array, so that the numbers in each row increase from left to right, and the numbers in each column increase from top to bottom. Let a_{ij} be the number in position i, j. Let b_j be the number of possible value for a_{jj}. Show that $b_1 + b_2 + ... + b_n = n(n^2-3n+5)/3$.

INMO 2003

1. ABC is acute-angled. P is an interior point. The line BP meets AC at E, and the line CP meets AB at F. AP meets EF at D. K is the foot of the perpendicular from D to BC. Show that KD bisects $\angle$ EKF.

2. Find all primes p, q and even n > 2 such that $p^n + p^{n-1} + ... + p + 1 = q^2 + q + 1$.

3. Show that $8x^4 - 16x^3 + 16x^2 - 8x + k = 0$ has at least one real root for all real k. Find the sum of the non-real roots.

4. Find all 7-digit numbers which use only the digits 5 and 7 and are divisible by 35.

5. ABC has sides a, b, c. The triangle A'B'C' has sides a + b/2, b + c/2, c + a/2. Show that its area is at least (9/4) area ABC.

6. Each lottery ticket has a 9-digit numbers, which uses only the digits 1, 2, 3. Each ticket is colored red, blue or green. If two tickets have numbers which differ in all nine places, then the tickets have different colors. Ticket 122222222 is red, and ticket 222222222 is green. What color is ticket 123123123?

INMO 2004

1. ABCD is a convex quadrilateral. K, L, M, N are the midpoints of the sides AB, BC, CD, DA. BD bisects KM at Q. QA = QB = QC = QD, and LK/LM = CD/CB. Prove that ABCD is a square.

2. p > 3 is a prime. Find all integers a, b, such that $a^2 + 3ab + 2p(a+b) + p^2 = 0$.

3. If α is a real root of $x^5 - x^3 + x - 2 = 0$, show that $[\alpha^6] = 3$.

4. ABC is a triangle, with sides a, b, c (as usual), circumradius R, and exradii r_a, r_b, r_c. If $2R \le r_a$, show that a > b, a > c, $2R > r_b$, and $2R > r_c$.

5. S is the set of all (a, b, c, d, e, f) where a, b, c, d, e, f are integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Find the largest k which divides abcdef for all members of S.

6. Show that the number of 5-tuples (a, b, c, d, e) such that abcde = 5(bcde + acde + abde + abce + abcd) is odd.