22nd IMO 1981 shortlisted problems

1. A 3 x 3 cube is assembled from 27 white unit cubes. The large cube is then painted black on the outside and then disassembled. If it is reassembled at random, what is the probability that the large cube is still completely black on the outside?

2. Let Fn be the Fibonacci sequence defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Find all pairs (a, b) of real numbers such that for each n, $aF_n + bF_{n+1}$ is a member of the sequence. Find all pairs (u, v) of real numbers such that for each n, $uF_n^2 + vF_{n+1}^2$ is a member of the sequence.

3. A sequence a_n is defined as follows, $a_0 = 1$, $a_{n+1} = (1 + 4a_n + \sqrt{(1 + 24a_n)})/16$ for $n \geq 0$. Find an explicit formula for a_n.

Answer $(1/3)(1 + 1/2^n)(1 + 1/2^{n+1})$

Solution

Put $b_n = 2^{2n+1}a_n$. Then the relation becomes $b_{n+1} = 2^{2n-1} + b_n + 2^n \sqrt{(2^{2n-2} + 3b_n)}$. Put $c_n = \sqrt{(2^{2n-2} + 3b_n)}$ and this becomes $c_{n+1}^2 = c_n^2 + 3 \cdot 2^n c_n + 9 \cdot 2^{2n-2} = (c_n + 3 \cdot 2^{n-1})^2$. Hence $c_{n+1} = c_n + 3 \cdot 2^{n-1}$. Iterating, $c_{n+1} = 3 \cdot 2^{n-1} + 3 \cdot 2^{n-2} + \ldots + 3 \cdot 2^0 + c_1 = 3(2^n - 1) + c_1 = 3 \cdot 2^n + 1$ (we have $a_1 = 5/8$, so $b_1 = 5$, $c_1 = 4$). Hence $b_n = (2^{2n+1} + 3 \cdot 2^n + 1)/3$, $a_n = 1/3 + 1/2^{n+1} + 1/(3 \cdot 2^{2n+1}) = (1/3)(1 + 1/2^n)(1 + 1/2^{n+1})$.

4. A real sequence u_n is defined by u_1 and $4u_{n+1} = (64u_n + 15)^{1/3}$. Describe the behavior of the sequence as $n \to \infty$.

5. a+b+c+d+e+f+g = 1 and a, b, c, d, e, f, g are non-negative. Find the minimum value of max(a+b+c, b+c+d, c+d+e, d+e+f, e+f+g).

6. p(k) is a polynomial of degree n such that p(k) = (n+1-k)!k!/(n+1)! for k = 0, 1, ... , n. Find p(n+1).

7. AB, BC, CD, DE are consecutive chords on a semicircle of unit radius with lengths a, b, c, d. Prove that $a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$.

8. A convex pentagon ABCDE has equal sides and $\angle A \geq \angle B \geq \angle C \geq \angle D \geq \angle E$. Prove that it is regular.

9. Find the smallest positive integer n such that for every integer $m \geq n$, it is possible to partition a given square into m squares, not necessarily of the same size.

10. A finite set of unit circular disks is given in a plane, such that the area of their union is S. Prove that there exists a subset of mutually disjoint disks whose union has area > 2S/9.

11. Several equal spherical planets are in outer space. On the surface of each planet is a set of points which is invisible from any of the remaining planets. Prove that the sum of the areas of all these sets equals the surface area of one planet.

12. A sphere S is tangent to the edges AB, BC, CD, DA of a tetrahedron ABCD at the points E, F, G, H respectively. If EFGH is a square, prove that the sphere is tangent to the edge AC iff it is tangent to the edge BD.

23rd IMO 1982 shortlisted problems

1. An urn contains w white balls and b black balls. In each move, two balls are drawn at random and removed from the urn, and one ball is added. If the two balls drawn have the same color, then a black ball is added. If they are opposite colors, then a white ball is added. Eventually, only one ball is left. What is the probability that it is white?

2. $p(x)$ is a cubic polynomial with integer coefficients and leading coefficient 1. One of its roots is the product of the other two. Prove that $2p(-1)$ is a multiple of $p(1) + p(-1) - 2(1 + p(0))$.

3. Let $a_1, a_2, \dots , a_n$ be a permutation of $1, 2, \dots , n$. Find the permutation which maximises $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n + a_na_1$ and the permutation which minimises it.

4. Let M be the set of real numbers of the form $(m+n)/\sqrt{(m^2+n^2)}$, where m and n are positive integers. Show that if $x < y$ are two elements of M, then there is an element z of M such that $x < z < y$.

5. Prove that $(1 - s^a)/(1 - s) \le (1 + s)^a/(1 + s)$ for every positive real $s \ne 1$ and every positive rational $a \le 1$.

6. Find all real numbers a such that the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four distinct real roots which form a geometric progression.

7. P is a point inside the triangle ABC such that $\angle PAC = \angle PBC$. The perpendiculars from P meet the lines BC, CA at L, M respectively. Prove that DL = DM where D is the midpoint of AB.

8. The triangles ABC and AB'C' have opposite orientation. $\angle BCA = \angle B'C'A = 90^o$. BC' and B'C intersect at M. Prove that if the lines AM and CC' are well defined, then they are perpendicular.

9. ABCD is a convex quadrilateral. A_1, B_1, C_1, D_1 are the circumcenters of BCD, CDA, DAB, ABC respectively. Prove that if two of A_1, B_1, C_1, D_1 coincide, then they all coincide. Prove that if they are distinct, then $A_1B_1C_1D_1$ is convex. In this case let A_2, B_2, C_2, D_2 be the circumcenters of $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ respectively. Show that $A_2B_22C_2D_2$ is similar to ABCD.

10. ABCD is a convex quadrilateral. ABM and CDP are equilateral triangles on the outside of the sides AB, CD. BCN and DAQ are equilateral triangles on the inside of sides BC and DA. Prove that MN = AC. What can be said about MNPQ?

11. A convex figure lies inside a circle. The points on the boundary of the figure are considered to be in the figure. For each point on the circle, draw the smallest angle with this point as the vertex which contains the figure. If the angle is always a right angle, prove that the center of the circle is a center of symmetry of the figure.

12. Exactly one quarter of the area of a convex polygon in the coordinate plane lies in each quadrant. If (0,0) is the only lattice point in or on the polygon, prove that its area is less than 4.

13. S is a unit sphere with center at the origin. For any point P on S, the unit sphere with center P intersects the x-axis at O and X, the y-axis at O and Y and the z-axis at O and Z (where X, Y or Z may coincide with O). What is the locus of the centroid of XYZ as P varies over the sphere.

24th IMO 1983 shortlisted problems

1. 1983 cities are served by ten airlines. All services are both ways. There is a direct service between any two cities. Show that at least one of the airlines can offer a round trip with an odd number of landings.

Solution

We show by induction that if n airlines serve more than 2^n cities then at least one offers an odd round trip. For n = 1, this is trivial (there are at least 3 cities, all served by a single airline, so we have a round trip around 3 cities).

Suppose the result is true for n. For n+1, suppose airline X does not offer an odd round trip. We claim that we can divide the cities into two sets A and B, such that X does not fly between any two cities in A, and X does not fly between any two cities in B. For start by putting any city C into A, then put into B all the cities to which X flies direct from a city in A. Then put into A all the cities to which X flies direct from a city in B. Repeat until we stop adding cities. If this does not exhaust the cities, then put one of the remaining cities into A and repeat, and so on.

Now at least one of A, B must have > 2^n cities. Also it must be served entirely by the remaining n airlines. So the result follows by induction.

Since 1983 > 2^{10}, the required result follows.

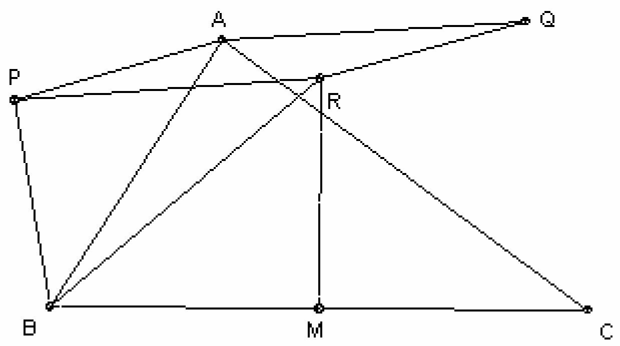
2. Let s(n) be the sum of the positive divisors of n. For example, s(6) = 1 + 2 + 3 + 6 = 12. Show that there are infinitely many n such that s(n)/n > s(m)/m for all m < n.

Solution

The sequence s(n)/n is unbounded. For example take n = product of first m primes $p_1p_2 \dots p_m$. Then $s(n)/n = (1 + 1/p_1) \dots (1 + 1/p_m) > 1 + 1/p_1 + \dots + 1/p_m$, which diverges. So take the sequence $n_1, n_2, \dots$ where $n_1 = 1$ and n_i is the *first* i such that $s(n_i)/n_i$ exceeds $s(n_{i-1})/n_{i-1}$. Then each n_i has the desired property.

4. ABC is a triangle with AB not equal to AC. P is taken on the opposite side of AB to C such that PA = PB. Q is taken on the opposite side of AC to B such that QA = QC and $\angle Q = \angle P$. R is taken on the *same* side of BC as A such that RB = RC and $\angle R = \angle P$. Show that APRQ is a parallelogram.

Solution



Put $\angle RBC = x$. Then RB = a/(2 cos x). Since angle P = angle R, we have $\angle PBA$ = angle RBC = x, and so $\angle PBR = \angle B$. Also PB = c/(2 cos x). Similarly, AQ = b/(2 cos x).

Applying the cosine rule to triangle PBR, we get $PR^2 = PB^2 + BR^2 - 2\, PB\cdot BR \cos B$. Hence $(2 \cos x)^2PR^2 = c^2 + a^2 - 2ac \cos B = (2 \cos x)^2b^2$. Hence PR = b/(2 cos x) = AQ. Similarly, QR = AP. Hence result.

5. Let S_n be the set of all strictly decreasing sequences of n positive integers such that no term divides any other term. Given two sequences A = (a_i) and B = (b_j) we say that A < B if for some k, $a_k < b_k$ and $a_i = b_i$ for i < k. For example, (7, 5, 3) < (9, 7, 2) < (9, 8, 7). Find the sequence A in S_n such that A < B for any other B in S_n.

Solution

Note that any integer can be written (uniquely) as the product of an odd integer (the *odd part*) and a power of 2. Two members of a sequence in S_n must have different odd part, otherwise one would divide the other. So we cannot do better than start the minimal sequence with 2n-1, 2n-3, ... However, we cannot go further than 2k+1, where k = [(n+1)/3] in this way, or we would get one term 3x another. To check this we need to consider three cases: (1) n=3m, so k = m and 3(2k-1) = 2n-3; (2) n=3m+1, so k=m, and 3(2k-1) = 2n-5; (3) n=3m+2, so k=m+1 and 3(2k-1)=2n-1. So for the final k terms we need to stop them dividing any of the preceding terms. The way to do this whilst keeping them as small as possible is to take 2 x the terms in A_k. (Note that we clearly need to multiply the odd terms in A_k by 2, but could we get away without doubling the even terms? They already do not divide the early odd terms in A_n, but if we fail to double them they will divide one of the newly doubled terms from A_k.)

Thus we get:
A_1 = {1}
A_2 = {3,2}
A_3 = {5,3.2}
A_4 = {7,5,3,2}
A_5 = {9,7,6,5,4}
A_6 = {11,9,7,6,5,4}

6. The n positive integers $a_1, a_2, \dots, a_n$ have sum 2n+2. Show that we can find an integer r such that:
$a_{r+1} \le 3$
$a_{r+1} + a_{r+2} \le 5$
...
$a_{r+1} + a_{r+2} + \dots + a_n + a_1 + \dots + a_{r-1} \le 2n-1$
Note that a_r does not appear in the inequalities. Show that there are either one or two r for which the inequalities hold, and that if there is only one, then all the inequalities are strict.

For example, suppose n = 3, $a_1 = 1$, $a_2 = 6$, $a_3 = 1$. Then r = 1 and r = 3 do not work, but r = 2 does work: $a_3 = 1 < 3$, $a_3 + a_1 = 2 < 5$. So r is unique and the inequalities are strict.

Solution

Let $s_i = a_1 + a_2 + \dots + a_i - 2i$. Let max s_i be attained for the first time at i = r. Suppose first that r = n. Then since $\sum a_i = 2n+2$, the maximum is 2. We have $s_i < 2$ for i < n, so $a_1 + a_2 + \dots + a_i \le 2i+1$ for i = 1, 2, ... , n-1, which are the required inequalities. So suppose r < n. Then for k = r+1, ... , n we have $a_{r+1} + a_{r+2} + \dots + a_k = (a_1 + \dots + a_k) - (a_1 + \dots + a_r) = s_k - s_r + 2(k-r) \le 2(k-r) < 2(k-r)+1$, as required. We also have $a_{r+1} + \dots + a_n + s_i < a_{r+1} + \dots + a_n + s_r = 2n+2-2r$, for i = 1, 2, ... , r-1. Hence $a_{r+1} + \dots + a_n + a_1 + a_2 + \dots + a_i < 2n+2-2r+2i$. So $a_{r+1} + \dots + a_n + a_1 + a_2 + \dots + a_i \le 2n-2r+2i+1 = 2(n+i-r)+1$, as required. That completes the proof that the inequalities hold.

Suppose that for some r the inequalities are all strict. wlog r = n, so that we have $a_1 + ... + a_i \le 2i$ for i = 1, 2, ... , n-1. Then for any k < n we have $a_1 + ... + a_k + a_{k+1} + ... + a_n \le 2k + (a_{k+1} + ... + a_n)$, so $(a_{k+1} + ... + a_n) \ge 2n+2-2k > 2(n-k)+1$ and so the inequalities do not all hold for k. Thus there is only one r for which the inequalities hold. Conversely, if the inequalities hold for r=n and k, then we have $a_1 + ... + a_k = 2n+2 - (a_{k+1} + ... + a_n) \ge 2n+2 - (2(n-k)+1) = 2k+1$. Hence (at least) one of the inequalities is not strict for r.

Suppose the inequalities hold for r < s < t. The same argument as above shows that $a_{r+1} + a_{r+2} + ... + a_s = 2(s-r)+1$, $a_{s+1} + ... + a_t = 2(t-s)+1$ and $a_{t+1} + ... + a_r = 2(n-t+r)+1$, so the sum of all the terms is 2n+3, not 2n+2. Contradiction.

7. Let N be a positive integer. Define the sequence $a_0, a_1, a_2, ...$ by $a_0 = 0$, $a_{n+1} = N(a_n + 1) + (N + 1)a_n + 2(N(N+1)a_n(a_n+1))^{1/2}$. Show that all terms are positive integers.

Solution

Induction on n. Note first that $\sqrt{a_{n+1}} = (\sqrt{(N+1)}\sqrt{a_n} + \sqrt{N}\sqrt{(a_n+1)}$ and $\sqrt{(a_{n+1}+1)} = \sqrt{N}\sqrt{a_n} + \sqrt{(N+1)}\sqrt{(a_n+1)}$. So $\sqrt{N}\sqrt{(N+1)}\sqrt{a_{n+1}}\sqrt{(a_{n+1}+1)} = N(N+1)(a_n+a_n+1) + (2N+1)\sqrt{N}\sqrt{(N+1)}\sqrt{a_n}\sqrt{(a_n+1)}$ (*).

If a_n and a_{n+1} are integers it follows from the definition that $\sqrt{N}\sqrt{(N+1)}\sqrt{a_n}\sqrt{(a_n+1)}$ is an integer, and hence (*) implies that $\sqrt{N}\sqrt{(N+1)}\sqrt{a_{n+1}}\sqrt{(a_{n+1}+1)}$ is an integer. Hence a_{n+2} is an integer. But it is easy to check that $a_0 = 0$ and $a_1 = N$ are integers, so a_n is an integer for all n.

8. 3n students are sitting in 3 rows of n. The students leave one at a time. All leaving orders are equally likely. Find the probability that there are never two rows where the number of students remaining differs by 2 or more.

Solution

Initially, we have n, n, n. It does not matter which student leaves first. We get n, n, n-1 (in some order). Now one of the students in the fuller rows must leave, which has probability 2n/(3n-1), giving n, n-1, n-1 (in some order). Now one of the students in the fullest row must leave, which has probability n/(3n-2) and gives n-1, n-1, n-1. Thus we have probability $6n^3/(3n(3n-1)(3n-2)$ of getting from n, n, n to n-1, n-1, n-1. Continuing we have probability of $6^n(n!)^3/(3n)!$ of completing the job.

9. Let m be any integer and n any positive integer. Show that there is a polynomial p(x) with integral coefficients such that $| p(x) - m/n | < 1/n^2$ for all x in some interval of length 1/n.

Solution

Take $p(x) = (m/n) (1 + (nx-1)^{2k+1})$ and the interval to be [1/2n, 3/2n]. If we expand p(x) by the binomial theorem it is obvious that p(x) has integral coefficients. Also $|p(x) - m/n| = |m/n| |nx - 1|^{2k+1} \le |m/n| 1/2^{2k+1}$ which $< 1/n^2$ for k sufficiently large.

10. c is a positive real constant and b = (1 + c)/(2 + c). f is a real-valued function defined on the interval [0, 1] such that f(2x) = f(x)/b for $0 \le x \le 1/2$ and f(x) = b - (1 - b) f(2x - 1) for $1/2 \le x \le 1$. Show that 0 < f(x) - x < c for all 0 < x < 1.

12. Let S be the set of lattice points (a, b, c) in space such that $0 \le a, b, c \le 1982$. Find the number of ways of coloring each point red or blue so that the number of red vertices of each rectangular parallelepiped (with sides parallel to the axes) is 0, 4 or 8.

14. Does there exist a set S of positive integers such that given any integer $n > 1$ we can find a, b in S such that $a + b = n$, and if a, b, c, d are all in S and all < 10 and $a + b = c + d$, then $a = c$ or d.

15. Let S be the set of polynomials $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ with non-negative real coefficients such that $a_0 = a_n \leq a_1 = a_{n-1} \leq a_2 = a_{n-2} \leq \dots$. For example, $x^3 + 2.1 x^2 + 2.1 x + 1$ or $0.1 x^2 + 15 x + 0.1$. Show that the product of any two members of S belongs to S.

16. Given n distinct points in the plane, let D be the greatest distance between two points and d the shortest distance between two points. Show that $D \geq d \sqrt{3} (\sqrt{n} - 1)/2$.

18. Let $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ be the Fibonacci sequence. Let p(x) be a polynomial of degree 990 such that $p(k) = F_k$ for $k = 992, 993, \dots , 1982$. Show that $p(1983) = F_{1983} - 1$.

Solution

We show first that $mC0\ F_n + mC1\ F_{n+1} + mC2\ F_{n+2} + \dots + mCm\ F_{n+m} = F_{n+2m}$, where aCb is the binomial coefficient. This is a simple induction on m. For m = 1 it is immediate. Suppose it is true for m. Then $\sum_0^{m+1} m{+}1Ci\ F_{n+i} = F_n + \sum_1^m (mCi + mCi{-}1)F_{n+i} + F_{n+m+1} = \sum mCi\ F_{n+i} + \sum mCi\ F_{n+i+1} = \sum mCi\ F_{n+2+i} = F_{n+2+2m} = F_{n+2(m+1)}$, so it is true for m+1.

Now we claim that $p(x) = F_{992} + F_{991} (x-992) + (1/2!) F_{990} (x-992)(x-993) + (1/3!) F_{989} (x-992)(x-993)(x-994) + \dots + (1/990!) F_2 (x-992)(x-993)\dots(x-1981)$. For we have $p(992+n) = F_{992} + nC1\ F_{991} + nC2\ F_{990} + \dots + nCn\ F_{992-n} = F_{992+n}$ for $n = 0, 1, \dots , 990$.

If we add the term $1/991!\ F_1 (x-992)(x-993)\dots(x-1982)$, then we keep the same values at $x = 992, 993, \dots , 1982$ and we get F_{1983} at $x = 1983$. Hence $p(1983) = F_{1983}$ - the value of the extra term at $1983 = F_{1983} - 1$.

19. k is a positive real. Solve the equations:
$x_1 | x_1 | = x_2 | x_2| + (x_1 - k) | x_1 - k |$
$x_2 | x_2 | = x_3 | x_3| + (x_2 - k) | x_2 - k |$
...
$x_n | x_n | = x_1 | x_1| + (x_n - k) | x_n - k |$.

Solution

Adding we get &sum $(x_i-k)|x_i-k| = 0$ (*), so for some m we must have $xm \geq k > 0$ (otherwise all terms would be negative). Now the equation for m gives $x_{m+1}|x_{m+1}| = 2kx_m - k^2 = k(2x_m-k) \geq k^2$. Hence $x_{m+1} \geq k$ also. Hence $x_i \geq k$ for all i. So all terms in (*) are non-negative and hence all must be zero. So the unique solution is $x_i = k$ for all i.

20. Find the greatest integer not exceeding $1 + 1/2^k + 1/3^k + \dots + 1/N^k$, where $k = 1982/1983$ and $N = 2^{1983}$.

Solution

We have $(a^{1983} - b^{1983}) = (a - b)(a^{1982} + a^{1981}b + a^{1980}b^2 + \dots + b^{1982})$. So if $a > b$, we have $(a^{1983} - b^{1983}) < (a - b)\ 1983\ a^{1982}$. Take $a = m^{1/1983}$, $b = (m-1)^{1/1983}$ and this becomes $(1/m)^{1982/1983} < 1983 (m^{1/1983} - (m-1)^{1/1983})$. Summing we get $1/2^k + 1/3^k + \dots + 1/N^k < 1983(N^{1/1983} - 1) = 1983 (2 - 1)$. So $1 + 1/2^k + 1/3^k + \dots + 1/N^k < 1984$.

Similarly, $(a^{1983} - b^{1983}) > (a - b)\ 1983\ b^{1982}$, so $(1/(m-1))^k > 1983\ (m^{1/1983} - (m-1)^{1/1983})$. Summing gives $1 + 1/2^k + 1/3^k + \dots + 1/(N-1)^k > 1983(2 - 1) = 1983$. Hence $1 + 1/2^k + 1/3^k + \dots + 1/N^k$ lies strictly between 1983 and 1984. So the integer part is 1983.

21. Let n be a positive integer which is not a prime power. Show that there is a permutation $a_1, a_2, \dots, a_n$ of $1, 2, \dots, n$ such that $\cos(2\pi a_1/n) + 2\cos(2\pi a_2/n) + 3\cos(2\pi a_3/n) + \dots + n\cos(2\pi a_n/n) = 0$.

Solution

We show that a slightly stronger result is true: we can find such a permutation for n not a power of 2.

Put $c_k = \cos(2\pi k/n)$. We show first that for odd $n = 2m+1$ we have $c_1 + 2c_2 + 3c_3 + \dots + mc_m + (m+1)c_{2m+1} + (m+2)c_{m+1} + (m+3)c_{m+2} + \dots + (2m+1)c_{2m} = 0$ (*). This follows almost immediately from $c_1 + c_2 + \dots + c_n = 0$ (1), $c_k = c_{n-k}$ (2) and $c_n = 1$ (3). (1) just says that the sum of the complex roots of 1 has zero real part (obvious from the polynomial $z^n = 1$), (2) follows from $\cos(-x) = \cos x$, and (3) is immediate. So we have $c_1 + c_2 + \dots + c_m = -1/2$. Hence $(2m+2)(c_1 + c_2 + \dots + c_m) = -(m+1) = -(m+1)\ c_{2m+1}$, which is (*).

Next we show that if the result is true for n, then it is also true for 2n. We show first by a similar argument to the above that $c_1 + 3c_3 + 5c_5 + 7c_7 + \dots + (2n-1)c_{2n-1} = 0$ (**). Note first that $c_2 + c_4 + c_6 + \dots + c_{2n}$ is just $\sum \cos(2\pi k/n)$ and so is zero by (1). Subtracting from $c_1 + c_2 + c_3 + \dots + c_{2n} = 0$ gives that $c_1 + c_3 + c_5 + c_7 + \dots + c_{2n-1} = 0$. But $c_i = c_{2n-i}$, so $c_1 + 3c_3 + 5c_5 + 7c_7 + \dots + (2n-1)c_{2n-1} = (2n-1)c_1 + (2n-3)c_3 + \dots + c_{2n-1}$. Adding the two sides gives $2n \times (c_1 + c_3 + \dots + c_{2n-1}) = 0$, so each side is zero, giving (**).

Now by assumption we have $\sum a_k c_k = 0$ where $c_k = \cos(2\pi k/n)$ for some permutation a_k of $1,2,3, \dots, n$. So $\sum (2a_k)c_{2k} = 0$, where $c_{2k} = \cos(2\pi 2k/2n)$. Putting this together with (**) gives the result for 2n.

24. d_n is the last non-zero digit of n! . Show that there is no k such that $d_n = d_{n+k}$ for all sufficiently large n.

Solution

Let L(n) be the last non-zero digit of n. The basic idea is that if L(N) and L(N') are both even and L(NN') = L(N), then L(N') must be 6.

Note first that L(n!) is always even for n > 1 because n! has more 2 factors than 5 factors. Suppose there is a period T. For k even, we always have $L(k) \neq L(2k)$, so T is not 1. But if n has last digit 1, then L(n!) = L(n-1!) so if the sequence had period 2, it would also have period 1, which is impossible. So T is not 2. If n has last digit 2, then n+1 has last digit 3 and n(n+1) has last digit 6. But L(k) = L(6k) for k even, so L(n+1!) = L(n-1!). However, $L(k) \neq L(4k)$ for k even, so $L(n+2!) \neq L(n-1!)$. Hence T is not 3. So T > 3. Hence (T-1)! has last non-zero digit even.

Now take m sufficiently large that $10^m > (T-1)!$. Put $n = 10^m - 1$. Then $N' = (n+1)(n+1) \dots (n+T) = 10^m(10^m+1)\dots(10^m+T-1) = 10^m(T-1)! \bmod 10^{2m}$. So L(N') = L(T-1!) and is therefore even. By assumption L(n+T!) = L(n!) and is even. So L(N') = 6.

Now take $n' = 2\cdot10^m - 1$. Again L(n'+T!) = L(n'!) and is even. We have $(n'+T)!/n'! = 2\cdot10^m(2\cdot10^m + 1) \dots (2\cdot10^m + T-1) = 2(T-1)! \bmod 10^{2m}$. So L((n'+T)!/n'!) = 2L(T-1!) = 2. But that implies $L(n'+T!) \neq L(n'!)$. Contradiction.

25th IMO 1984 shortlisted problems

1. Given real A, find all solutions $(x_1, x_2, \dots, x_n)$ to the n equations $(i = 1, 2, \dots, n)$:
$x_i |x_i| - (x_i - A) |(x_i - A)| = x_{i+1} |x_{i+1}|$, where we take x_{n+1} to mean x_1. *(France 1)*

2. Prove that there are infinitely many triples of positive integers (m, n, p) satisfying $4mn - m - n = p^2 - 1$, but none satisfying $4mn - m - n = p^2$. *(Canada 2)*

Solution

We can take $(m,n,p) = (3k^2,1,3k)$, or $(k(k-1)/2, k(k+1)/2, k^2-1)$ as solutions of $4mn-m-n = p^2-1$.

Suppose m, n, p satisfy $4mn-m-n = p^2$, so $(4m-1)(4n-1) = 4p^2+1 = (2p)^2 + 1$. Now suppose q is any prime dividing the lhs. Then $(2p)^2 = -1 \bmod q$. But $(2p)^2 = 1 \bmod q$ (Fermat), so $q = 1 \bmod 4$. But $4m-1$ must have at least one prime factor $= 3 \bmod 4$. Contradiction.

3. Find all positive integers n such that $n = d_6^2 + d_7^2 - 1$, where $1 = d_1 < d_2 < \dots < d_k = n$ are all the positive divisors of n. *(USSR 3)*

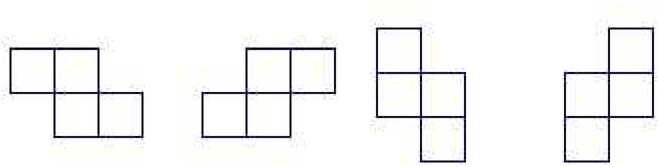
6. c is a positive integer. The sequence $f_1, f_2, f_3, \dots$ is defined by $f_1 = 1$, $f_2 = c$, $f_{n+1} = 2 f_n - f_{n-1} + 2$. Show that for each k there is an r such that $f_k f_{k+1} = f_r$. *(Canada 3)*

Answer $k^2 + (c-3)k + (c-4)$

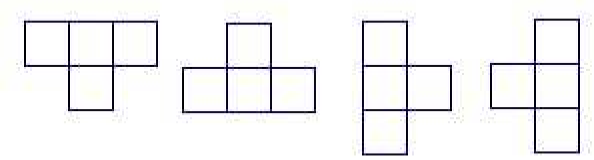
Solution

We have $f_{n+1} - f_n = f_n - f_{n-1} + 2$, so $f_{n+1} - f_n = 2n+c-3$. Hence $f_n = (n-2)^2 + (n-1)c = n^2+bn-b$, where $b = c-4$. Hence $f_kf_{k+1} = (k^2+bk-b)(k^2+bk+2k+1)$. Put $r = k^2+(b+1)k-b$. Then $f_kf_{k+1} = (r-k)(r+k+b+1) = r^2+br-b = f_r$. So $r = k^2+(b+1)k-b = k^2 + (c-3)k + (c-4)$.

7. Can we number the squares of an 8 x 8 board with the numbers 1, 2, ... , 64 so that any four squares with any of the following shapes



have sum = 0 mod 4? Can we do it for the following shapes?



(German Federal Republic 5)

9. Let a, b, c be positive reals such that $\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{3}/2$. Show that the equations:
$\sqrt{y - a} + \sqrt{z - a} = 1$
$\sqrt{z - b} + \sqrt{x - b} = 1$
$\sqrt{x - c} + \sqrt{y - c} = 1$
have exactly one solution in reals x, y, z. *(Poland 2)*

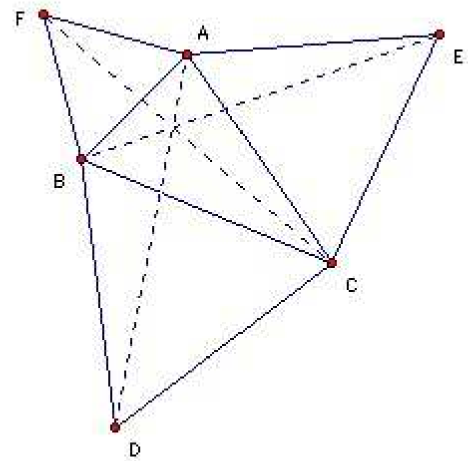
10. Prove that the product of five consecutive positive integers cannot be the square of an integer. *(Great Britain 1)*

Solution

One of the numbers is not divisible by 2 or 3. If it is divisible by a prime p, then $p \geq 5$, so p does not divide any of the other numbers, so the number must be a square. So one of the numbers is $b^2 = (6c \pm 1)^2 = 12c(3c \pm 1)+1 = 1 \bmod 24$, since $c(3c \pm 1)$ is even. The only 5 consecutive positive integers which include *two* squares are 1,2,3,4,5, whose product is not a square (because the difference between any two squares other than 1,4 exceeds 4). So none of the other four numbers (apart from b^2) is a square and hence none of the others is ±1 mod 6. So the numbers must be 24k, $24k+1 = b^2$, 24k+2, 24k+3, 24k+4. But now 24k+4 = 4(6k+1), so 6k+1 must be a square and hence 24k+4 is a square. Contradiction.

11. $a_1, a_2, \dots, a_{2n}$ are distinct integers. Find all integers x which satisfy $(x - a_1)(x - a_2) \dots (x - a_{2n}) = (-1)^n(n!)^2$. *(Canada 1)*

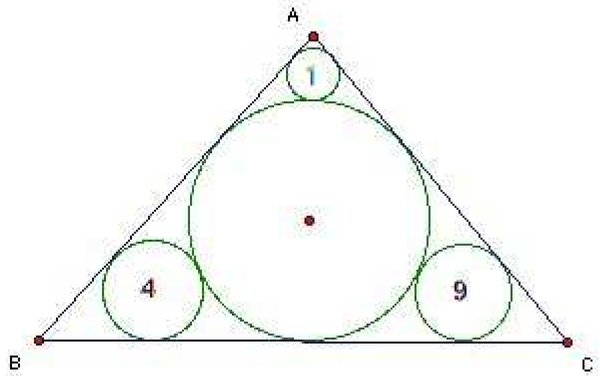
13. A tetrahedron is inscribed in a straight circular cylinder of volume 1. Show that its volume cannot exceed $2/(3\pi)$. *(Bulgaria 5)*



15. The angles of the triangle ABC are all < 120^o. Equilateral triangles are constructed on the outside of each side as shown. Show that the three lines AD, BE, CF are concurrent. Suppose they meet at S. Show that SD + SE + SF = 2(SA + SB + SC). *(Luxembourg 2)*

17. If $(x_1, x_2, \dots, x_n)$ is a permutation of (1, 2, ... , n) we call the pair (x_i, x_j) discordant if $i < j$ and $x_i > x_j$. Let d(n, k) be the number of permutations of (1, 2, ... , n) with just k discordant pairs. Find d(n, 2) and d(n, 3). *(German Federal Republic 3)*

18. ABC is a triangle. A circles with the radii shown are drawn inside the triangle each touching two sides and the incircle. Find the radius of the incircle. *(USA 5)*



19. The harmonic table is a triangular array:

1
1/2 1/2
1/3 1/6 1/3
1/4 1/12 1/12 1/4
...

where $a_{n,1} = 1/n$ and $a_{n,k+1} = a_{n-1,k} - a_{n,k}$. Find the harmonic mean of the 1985th row. *(Canada 5)*

20. Find all pairs of positive reals (a, b) with a not 1 such that $\log_a b < \log_{a+1}(b+1)$. *(USA 2)*

Note: This list does not incude the problems used in the Olympiad (4, 5, 8, 12, 14, 16 on the shortlist, which were 5, 1, 3, 2, 4, 6 in the Olympiad).

26th IMO 1985 shortlisted problems

1. Show that if n is a positive integer and a and b are integers, then n! divides a(a + b)(a + 2b) ... (a + (n-1)b) b^{n-1}.

Solution

If p^k is the highest power of a prime p dividing n!, then k = [n/p] + [n/p^2] + [n/p^3] + ... We have p ≥ 2, so [n/p] + [n/p^2] + [n/p^3] + ... ≤ [n/2] + [n/4] + [n/8] + ... < n/2 + n/4 + n/8 + ... = n. Note that the inequality is strict, because [n/2] + [n/4] + [n/8] + ... has only finitely many non-zero terms. But k is integral, so k ≤ n-1.

If p divides b, then p^{n-1} divides b^{n-1} and hence p^k divides b^{n-1}. So suppose p does not divide b.

The set S = {a, a+b, a+2b + ... + a+(n-1)b} contains [n/p^r] complete sets of residues mod p^r, so it contains at least [n/p^r] numbers divisible by p^r. Suppose it has exactly a_r numbers divisible by p^r. Then it has a_r - a_{r+1} numbers divisible by p^r but not p^{r+1}, so the product of its elements is divisible by (a_1 - a_2) + 2(a_2 - a_3) + 3(a_3 - a_4) + ... = a_1 + a_2 + a_3 + ... powers of p. But a_1 + a_2 + a_3 + ... ≥ [n/p] + [n/p^2] + [n/p^3] + ... = k. So p^k divides a(a+b)(a+2b) ... (a+(n-1)b).

2. A convex quadrilateral ABCD is inscribed in a circle radius 1. Show that 0 < |AB + BC + CD + DA - AC - BD| < 2.

Solution

The triangle inequality gives: AB + BC > AC, BC + CD > BD, CD + DA > AC, DA + AB > BD. Adding and dividing by 2 gives AB + BC + CD + DA - AC - BC > 0, which gives the first part.

Let O be the center of the circle. Let angle AOB = 2a, angle BOC = 2b, angle COD = 2c, angle DOA = 2d. Then AB = 2 sin a, BC = 2 sin b, CD = 2 sin c, DA = 2 sin d, AC = 2 sin(a+b), BD = 2 sin(b+c). So it remains to show that sin a + sin b + sin c + sin d - sin(a+b) - sin(b+c) < 1. We have d = 180^o - (a+b+c), so this is equivalent to: sin a + sin b + sin c + sin(a+b+c) < sin(a+b) + sin(b+c) + 1. We claim that the stronger inequality sin a + sin b + sin c + sin(a+b+c) < sin(a+b) + sin(b+c) + sin(c+a) (*) holds.

We have sin b + sin c = sin((b+c)/2 + (b-c)/2) + sin((b+c)/2 - (b-c)/2) = 2 sin((b+c)/2) cos((b-c)/2), sin a + sin(a+b+c) = sin(a + (b+c)/2 - (b+c)/2) + sin(a + (b+c)/2 + (b+c)/2) = 2 sin(a + (b+c)/2) cos((b+c)/2), sin(a+b) + sin(a+c) = sin(a + (b+c)/2 + (b-c)/2) + sin(a + (b+c)/2 - (b-c)/2) = 2 sin(a + (b+c)/2) cos((b-c)/2), sin(b+c) = 2 sin((b+c)/2) cos((b+c)/2). So (*) is equivalent to: sin((b+c)/2) cos((b-c)/2) + sin(a + (b+c)/2) cos((b+c)/2) < sin(a + (b+c)/2) cos((b-c)/2) + sin((b+c)/2) cos((b+c)/2), and hence to: (sin(a + (b+c)/2) - sin((b+c)/2))(cos((b-c)/2) - cos((b+c)/2) > 0.

There is no loss of generality in taking a to be the smallest of a, b, c, d. So a + (b+c)/2 ≤ (a+d)/2 + (b+c)/2 <= 90^o. But sin x is an increasing function over 0, 90^o, so sin(a + (b+c)/2) - sin((b+c)/2) > 0. Since b/2 and c/2 are both positive with sum less than 90^o, we have |(b-c)/2| < |(b+c)/2| and hence cos((b-c)/2) - cos((b+c)/2) > 0. So (*) is established.

Note that the inequalities are the best possible. If we take A, B, C, D almost coincident, then we can get AB + BC + CD + DA - AC - BD as close as we like to 0. If we take A, B, D almost coincident and AC a diameter, then we can get AB + BC + CD + DA - AC - BD as close as we like to 0 + 2 + 2 + 0 - 2 - 0 = 2.

3. Given $n > 1$, find the maximum value of $\sin^2x_1 + \sin^2x_2 + ... + \sin^2x_n$, where x_i are non-negative and have sum π.

Solution

Answer: 2 for $n = 2$; 9/4 for $n > 2$.

The result for $n = 2$ is obvious. So assume $n > 2$. The expression is the sum of the squares of the sides of an n-gon inscribed in a circle radius 1/2 whose sides subtend angles $2x_1, 2x_2, ... , 2x_n$ at the center. If $n > 3$, then this n-gon must have an angle $\geq 90^o$. Suppose it is the angle at B between the sides AB and AC. Then by the cosine rule $AC^2 \geq AB^2 + BC^2$, so the expression is not reduced if we drop one of the vertices to get an (n-1)-gon. Hence the maximum is achieved by taking at least n-3 angles $x_i = 0$. So we need only consider the case $n = 3$.

So let ABC be a triangle with circumcenter O, centroid G and circumradius R. Take G as the origin and P as any point. Using vectors it is immediate that $PA^2 + PB^2 + PC^2 = 3\ GP^2 + GA^2 + GB^2 + GC^2$. Putting $P = O$ gives $3R^2 + 3OG^2 = GA^2 + GB^2 + GC^2$. Putting $P = A$ gives $AB^2 + AC^2 = 4GA^2 + GB^2 + GC^2$. Putting $P = B$ and $P = C$ and adding the three equations gives: $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$. Hence $AB^2 + BC^2 + CA^2 = 9R^2 - 9\ OG^2 \leq 9R^2 = 9/4$ for $R = 1/2$. [Note that equality is certainly achieved by $x_1 = x_2 = x_3 = 60^o$ and other $x_i = 0$.]

4. Show that $x_1^2/(x_1^2 + x_2x_3) + x_2^2/(x_2^2 + x_3x_4) + ... + x_{n-1}^2/(x_{n-1}^2 + x_nx_1) + x_n^2/(x_n^2 + x_1x_2) \leq n-1$ for all positive reals x_i.

Solution

The inequality would not have any obvious meaning if $n = 1$. For $n = 2$ it presumably becomes: $a^2/(a^2 + ba) + b^2/(b^2 + ab) \leq 1$. But the lhs is $a/(a+b) + b/(a+b) = 1$, so the result is true. Assume then that $n > 2$.

Put $a_i = x_{i+1}x_{i+2}/x_i^2$ (using cyclic subscripts). We have to show that $\sum 1/(1 + a_i) \leq n-1$, or equivalently that $\sum a_i/(1 + a_i) \geq 1$. Also we have $a_1a_2 ... a_n = 1$. We may assume that $a_1 \geq a_2 \geq ... \geq a_n$.

$f(x) = x/(1+x)$ is an increasing function of x and $f(1) = 1/2$. So if there are two or more $a_i \geq 1$, then $\sum a_i \geq 1$. Not all the a_i can be less than 1, because their product is 1, so we must have exactly one $a_i \geq 1$ and that must be a_1. Since the others are < 1 and the product is 1, we must have $a_1 > 1$.

Now by the AM/GM inequality we have $a_2/(1 + a_2) + ... + a_n/(1 + a_n) \geq (n-1) (\prod_2^n a_i/(1 + a_i))^{1/n-1}$. But $a_2a_3 ... a_n = 1/a_1 > 1/(1 + a_1)$ and $(1 + a_i) < (1 + a_1)$ for $i = 2, 3, ... , n-1$, so $(n-1) (\prod_2^n a_i/(1 + a_i))^{1/n-1} > (n-1)\ 1/(1 + a_n)^{1/n-1}\ 1/(1 + a_1)$. Now $a_n < 1$, so $(n-1)/(1 + a_n)^{n-1} > 2/2^{1/n-1} > 1$. Hence $a_2/(1 + a_2) + ... + a_n/(1 + a_n) > 1/(1 + a_1)$. Hence $a_1/(1 + a_1) + ... + a_n/(1 + a_n) > 1$.

5. T is the set of all lattice points in space. Two lattice points are *neighbors* if they have two coordinates the same and the third differs by 1. Show that there is a subset S of T such that if a lattice point x belongs to S then none of its neighbors belong to S, and if x does not belong to S, then exactly one of its neighbors belongs to S.

Solution

Let S be the set of points (x, y, z) with $2x + 4y + 6z = 0 \bmod 7$. If $2x + 4y + 6z$

```
= 0 mod 7, then (x, y, z) is in S
```

```
= 1 mod 7, then (x, y, z+1) is in S
= 2 mod 7, then (x-1, y, z) is in S
= 3 mod 7, then (x, y+1, z) is in S
= 4 mod 7, then (x, y-1, z) is in S
= 5 mod 7, then (x+1, y, z) is in S
= 6 mod 7, then (x, y, z-1) is in S
```

So if x is in T - S, then at least one of its neighbours belongs to S. Also if (x, y, z) is in S, then (x±1, y, z), (x, y±1, z), (x, y, z±1) are not in S. In other words, if a point is in S, then none of its neighbours are in S. Finally, if (x, y, z) is in S, then none of (x±2, y, z), (x, y±2, z), (x, y, z±2), (x±1, y±1, z), (x±1, y, z±1), (x, y±1, z±1) are in S, so no point has two of its neighbours in S. That establishes that S has the required properties.

Comment

2x + 4y + 6z may seem fairly obvious, but I got there by a tortuous route. I started, of course, by looking at 1D. That is obvious: ... 0 0 X 0 0 X 0 0 X 0 ... , where X denotes the members of S. 1 point in 3 belongs to S.

2D is not quite so obvious. But with a little experimentation one can get:

```
0  0  0  0  X  0
0  0  X  0  0  0
X  0  0  0  0  X
0  0  0  X  0  0
0  X  0  0  0  0
```

Obviously 1 point in 5 belongs to S and the pattern repeats with period 5 in both directions. At this point I started looking for a suitable pattern for 3D. I needed 1 point in 7. It did not take long to find:

```
0  0  0  X  0  0  0
X  0  0  0  0  0  0
0  0  0  0  X  0  0
0  X  0  0  0  0  0
0  0  0  0  0  X  0
0  0  X  0  0  0  0
0  0  0  0  0  0  X
0  0  0  X  0  0  0
```

I then tried to think what the next plane up would look like. After some more fiddling around I hit on:

```
2  6  3  0  4  1  5
0  4  1  5  2  6  3
5  2  6  3  0  4  1
3  0  4  1  5  2  6
1  5  2  6  3  0  4
6  3  0  4  1  5  2
4  1  5  2  6  3  0
```

The number denotes the z-coordinate. So the pattern represents a 7 x 7 x 7 cube of lattice points which repeats.

At this point it occurred to me that since the pattern in each line was the same, the points belonging to S satisified some equation like 4x + 2y - z = 0 (or some constant, but 0 if we take the origin in the right place) mod 7. I then junked all the scaffolding and wrote out the solution above.

6. Let A be a set of positive integers such that $|m - n| \geq mn/25$ for any m, n in A. Show that A cannot have more than 9 elements. Give an example of such a set with 9 elements.

Solution

It is easy to check that {1, 2, 3, 4, 6, 8, 12, 24, 600} satisfies the conditions. Clearly if $m > n > k$ and $mn/25 \leq m - n$, then $mk/25 < mn/25 \leq m - n < m - k$. So it is sufficient to check the pairs (m, n) = (2, 1), (3, 2), (4, 3), (6, 4), (8, 6), (12, 8), (24, 12), (600, 24). The first four obviously work because $mn < 25$. For (8, 6), we have $mn/25 = 48/25 < 2 = 8 - 6$. For (12, 8) we have $mn/25 = 96/25 < 4 = 12 - 8$. For (24, 12) we have $mn/25 = 288/25 < 12 = 24 - 12$. For (600, 24) we have $mn/25 = 24^2 = 600 - 24$.

Now consider $A = \{n_1, n_2, \dots , n_k\}$. Obviously all n_i must be distinct, so we may assume $n_1 > n_2 > \dots > n_k$. We have $n_1 - n_2 = n_1n_2/25$. But $n_1 - n_2 < n_1$, so $n_2/25 < 1$ and hence $n_2 \leq 24$.

Similarly, $n_2 - n_3 = n_2n_3/25$, so $n_3 = n_2(1 - n_3/25) \leq 24(1 - n_3/25)$. Hence $n_3 \leq 24 \cdot 25/49$. So $n_3 \leq 12$.
Similarly, $n_4 \leq 12(1 - n_4/25)$, so $n_4 \leq 12 \cdot 25/37 = 8.1$, so $n_4 \leq 8$.
Similarly, $n_5 \leq 8(1 - n_5/25)$, so $n_5 \leq 8 \cdot 25/33 = 6.1$, so $n_5 \leq 6$.
Similarly, $n_6 \leq 6(1 - n_6/25)$, so $n_6 \leq 6 \cdot 25/31 = 4.8$, so $n_6 \leq 4$. Hence $n_7 \leq 3$, $n_8 \leq 2$, $n_9 \leq 1$. All elements are required to be positive, so 9 is the largest possible number.

7. Do there exist 100 distinct lines in the plane having just 1985 distinct points of intersection?

Solution

Answer: yes.

Take 73 lines parallel to the x-axis and 26 lines parallel to the y-axis. These meet in 73.26 = 1898 points. Now take the 73 lines as y = 1, 2, ... , 73 and the 26 lines as x = 1, 2, 3, 4, 5, 6 and 101, 102, ... , 120. Take the 100th line as y = x. It passes through 6 of the existing points of intersection (namely (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6)). So it only creates (73 + 26 - 12) = 87 new points of intersection, giving 1985 in total.

8. Find 8 positive integers $n_1, n_2, \dots , n_8$ such that we can express every integer n with $|n| < 1986$ as $a_1n_1 + \dots + a_8n_8$ with each $a_i = 0, \pm 1$.

Solution

Answer: 1, 3, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7.

Put $f(n) = 1 + 3 + 3^2 + \dots + 3^n$. We can prove by induction that any number in the range [-f(n), f(n)] can be represented as $a_0 + a_13 + \dots + a_n3^n$ with each $a_i = 0$ or ± 1. It is evidently true for n = 1. So suppose it is true for n. Note that $f(n) = (3^{n+1} - 1)/2$, so $f(n) + 1 - 3^{n+1} > - (3^{n+1} - 1)/2 = - f(n)$. So if $f(n) < m \leq f(n+1)$, then $-f(n) < m - 3^{n+1} \leq f(n)$. By induction $m - 3^{n+1} = a_0 + \dots + a_n3^n$, and hence $m = a_0 + \dots + a_{n+1}3^{n+1}$. Similarly for $-f(n+1) \leq m < -f(n)$, whilst for $-f(n) <= m \leq f(n)$, the result follows immediately. That completes the induction.

[Another way of looking at it is that there are n+1 coefficients each of which can take 3 values, so there are 3^{n+1} possibilities. No two possibilities can give the same number, or, equating them and moving negative coefficients to the opposite side, we would have two different representations for the same number in base 3. But obviously all the values lie in the range [-f(n), f(n)], and that range contains just 3^{n+1}values.]

9. The points A, B, C are not collinear. There are three ellipses, each pair of which intersects. One has foci A and B, the second has foci B and C and the third has foci C and A. Show that the common chords of each pair intersect.

Solution

We show first that each pair of ellipses intersects in just two points. Clearly, it is possible for them not to intersect at all, but we are obviously meant to assume that each pair intersects in *at least* two points. We have to show that they do not intersect in more than two points.

Let the point P be a distance a from A, b from B and c from C. The ellipse with foci A and B can be taken to be the set of points P such that $a + b = k_3$ (a constant). Similarly, the ellipse with foci A and C can be taken to be the set of points P such that $a + c = k_2$ (another constant). So if P belongs to both ellipses, then (subtracting) $b - c = k_3 - k_2$. But this is the equation of *one branch* of a hyperbola, which intersects one of the ellipses in at most two points. So the two ellipses intersect in two points.

Now consider the equation $(k_2 - k_3)a^2 - k_2b^2 + k_3c^2 = k_2k_3(k_2 - k_3)$, (*). If P lies on the ellipse foci A, B and on the ellipse foci A, C, then we have: $b = k_3 - a$, $c = k_2 - a$. So substituting in the lhs of (*) we get: $(k_2 - k_3)a^2 - k_2(k_3 - a)^2 + k_3(k_2 - a)^2 = k_2k_3(k_2 - k_3) =$ rhs. So the two points on both ellipses satisfy the equation (*). But suppose P is (x_1, x_2), A is (a_1, a_2), B is (b_1, b_2), and C is (c_1, c_2). Then we have $a^2 = (x_1- a_1)^2 + (x_2 - a_2)^2$ etc and hence we may write (*) as: $(k_2 - k_3)((x_1 - a_1)^2 + (x_2 - a_2)^2) - k_2((x_1 - b_1)^2 + (x_2 - b_2)^2) + k_3((x_1 - c_1)^2 + (x_2 - c_2)^2) = k_2k_3(k_2 - k_3)$. But we see that the terms in x_1^2 and x_2^2 cancel, so that (*) is a linear equation and hence represents a straight line. So it is the equation of the common chord of the two ellipses.

So if we take the third ellipse to be $b + c = k_1$, then the equations of the three common chords are:
$(k_2 - k_3) a^2 - k_2 b^2 + k_3 c^2 = k_2k_3(k_2 - k_3)$, (1)
$k_1 a^2 + (k_3 - k_1) b^2 - k_3 c^2 = k_3k_1(k_3 - k_1)$, (2)
$-k_1 a^2 + k_2 b^2 + (k_1 - k_2) c^2 = k_1k_2(k_1 - k_2)$, (3).

But now we find that the linear combination $k_1(k_2 + k_3 - k_1)$ (1) + $k_2(k_3 + k_1 - k_2)$ (2) + $k_3(k_1 + k_2 - k_3)$ (3) is zero. So the three lines are concurrent.

10. The polynomials $p_0(x, y, z)$, $p_1(x, y, z)$, $p_2(x, y, z)$, ... are defined by $p_0(x, y, z) = 1$ and $p_{n+1}(x, y, z) = (x + z)(y + z) p_n(x, y, z+1) - z^2p_n(x, y, z)$. Show that each polynomial is symmetric in x, y, z.

Solution

We use induction on n.

Let S_n be the statement that (1) $p_n(x, y, z) = p_n(y, x, z)$, (2) $p_n(x, y, z) = p_n(z, y, x)$, and (3) $(y+z) p_n(x, y, z+1) - (x+y) p_n(z, y, x+1) = (z-x) p_n(x, y, z)$. Clearly S_0 is true ((1) and (2) are trivial, and both sides of (3) are (z-x)). So suppose S_n is true.

(1) for n+1 follows immediately. We have $p_{n+1}(x, y, z) - p_{n+1}(z, y, x)$
$= (x+z)(y+z) p_n(x, y, z+1) - (x+z)(x+y) p_n(z, y, x+1) - z^2 p_n(x, y, z) + x^2 p_n(z, y, x)$, by the identity in the question
$= (x+z) (z-x) p_n(x, y, z) + (x^2 p_n(z, y, x) - z^2 p_n(x, y, z))$, by (3) for n
$= (x+z) (z-x) p_n(x, y, z) + (x^2 - z^2) p_n(x, y, z))$, by (2) for n
= 0. So (2) holds for n+1.

We have $(y+z)\, p_{n+1}(x, y, z+1) - (x+y)\, p_{n+1}(z, y, x+1)$
$= (y+z)\, p_{n+1}(z+1, x, y) - (x+y)\, p_{n+1}(x+1, z, y)$, using (1) and (2) for n+1
$= ((y+z)(y'+z)(x+y)\, p_n(z+1, x, y') - (y+z)y^2 p_n(z+1, x, y)) - ((x+y)(x+y')(y+z)\, p_n(x+1, z, y') - (x+y)y^2 p_n(x+1, z, y)$, where $y' = y+1$ and we have used the equality in the question
$= (x+y)(y+z)((y'+z)\, p_n(x, y', z+1) - (x+y')\, p_n(x+1, y', z)) - y^2((y+z)\, p_n(x, y, z+1) - (x+y)\, p_n(z, y, x+1))$, using (1) and (2) for n
$= (x+y)(y+z)(z-x)\, p_n(x, y', z) - y^2(z-x)\, p_n(x, y, z)$, using (3) for n
$= (z-x)((x+y)(z+y)\, p_n(x, z, y+1) - y^2 p_n(x, z, y))$
$= (z-x)\, p_{n+1}(x, z, y)$, using the identity in the question
$= (z-x)\, p_{n+1}(x, y, z)$, which establishes (3) for n+1.

So S_n is true for all n. (1) and (2) together establish that $p_n(x, y, z)$ is symmetric.

11. Show that if there are $a_i = \pm 1$ such that $a_1a_2a_3a_4 + a_2a_3a_4a_5 + \dots + a_na_1a_2a_3 = 0$, then n is divisible by 4.

Solution

Suppose we change the sign of a_i. Then exactly four terms change sign. If they are initially + + + +, then the net change in the sum is - 8. Similarly, if they are initially + + + - (in some order), the net change is -4, if they are + + - -, the net change is nil, if they are - - - +, the net change is +4, and if they are - - - -, the net change is +8. So the change is always 0 mod 4. But by changing the sign of at most n terms we get all $a_i$positive and hence sum n. So if the sum was initially 0, then we must have n = 0 mod 4.

12. Given 1985 points inside a unit cube, show that we can always choose 32 such that any polygon with these points as vertices has perimeter less than $8\sqrt{3}$.

Solution

64 x 31 = 1984. So if we divide the cube into 64 cubes side 1/4, then one of the small cubes must contain 32 points. Each pair of points in the cube is at most a distance $(\sqrt{3})/4$ apart, so the 32-gon has perimeter *less* than $8\sqrt{3}$ (if all pairs had distance $(\sqrt{3})/4$ then some of the points would be coincident).

13. A die is tossed repeatedly. A wins if it is 1 or 2 on two consecutive tosses. B wins if it is 3 - 6 on two consecutive tosses. Find the probability of each player winning if the die is tossed at most 5 times. Find the probability of each player winning if the die is tossed until a player wins.

Solution

Answer: 5 throws or less: A 55/243, B 176/243; arbitrarily many throws: A 5/21, B 16/21.

Denote an outcome of 1 or 2 as L and an outcome of 3, 4, 5, or 6 as H. For 5 tosses or less we have:

```
LL     A wins prob. 27/243
LHLL   A wins prob 6/243
LHLHL  draw prob 4/243
LHLHH  B wins prob 8/243
LHH    B wins prob 36/243
HLL    A wins prob 18/243
HLHLL  A wins prob 4/243
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```
HLHLH  draw prob 8/243
HLHH   B wins prob 24/243
HH     B wins prob 108/243
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Summarising, the prob of A winning is 55/243, of B winning is 176/243 and of a draw is 12/243.

With an arbitrary number of throws A wins with LL, LHLL, LHLHLL, ... or with HLL, HLHLL, HLHLHLL, ... Probability $(1 + 2/3)\ 1/9\ (1 + 2/9 + (2/9)^2 + \dots) = (5/3)(1/7) = 5/21$. But no draw is possible, so B wins with prob 16/21.

14. At time $t = 0$ a point starts to move clockwise around a regular n-gon from each vertex. Each of the n points moves at constant speed. At time T all the points reach vertex A simultaneously. Show that they will never all be simultaneously at any other vertex. Can they be together again at vertex A?

Solution

Assume the sides of the n-gon are length 1. So in time T each point moves an integral distance. So in time nT each point returns to its original position. Suppose that T' is the first time (after the start) at which all points are at vertices. If all the points are at vertices at time T'', then we may put $T'' = kT' + t$, for some positive integer k and some $0 \le t < T'$. But then each point travels an integral distance in time $t < T'$, so we must have $t = 0$. Hence, in particular, $T = mT'$ for some positive integer m. Clearly $m \le n$.

Label the points $P_0 = A, P_1, P_2, \dots, P_{n-1}$ moving anticlockwise around the n-gon. Suppose that at time T', point P_i has moved a distance a_i mod n. Then we have $m\ a_i = i$ mod n. In particular, $m\ a_1 = 1$ mod n, so m must be coprime to n. But $m\ a_0 = 0$ mod n, so $a_0 = 0$ mod n. Hence at any time T'' at which all the points are at vertices, P_0 is at vertex A, so they can never be together at some vertex distinct from A.

The motion is periodic with period nT, so they are regularly together at vertex A.

15. On each edge of a regular tetrahedron of side 1 there is a sphere with that edge as diameter. S be the intersection of the spheres (so it is all points whose distance from the midpoint of every edge is at most 1/2). Show that the distance between any two points of S is at most $1/\sqrt{6}$.

Solution

The midpoints of the edges are the vertices of a regular octahedron side 1/2. The sphere center a vertex of the octahedron and radius 1/2 passes through four of the other five vertices of the octahedron. Less obviously, it passes through the center of each of the four faces not adjacent to the vertex.

For suppose ABC and ABD are faces of the octahedron. CD is the diagonal of a square side 1/2, so it has length $1/\sqrt{2}$. Let M be the midpoint of AB and G the center of ABD. Using the cosine rule we find that $\cos CMD = (CM^2 + DM^2 - CD^2)/(2\ CM.DM) = (3/16 + 3/16 - 1/2)/(3/8) = -1/3$. Hence $CG^2 = CM^2 + GM^2 - 2.CM.GM \cos CMD = (3/16 + 1/48 + 1/24) = (9 + 1 + 2)/48 = 1/4$. So CG = CA as claimed.

The centers of the faces form the vertices of a cube side $(\sqrt{2})/6$. Each sphere encloses the cube and passes through the four vertices furthest from the sphere's center. Thus the intersection of the six spheres is the cube, but with curved faces which bulge out slightly because they are each part of a sphere.

Consider the curved face GHIJ of the cube. It is part of the surface of the sphere center C. This lies inside the circumsphere of the cube, because the circumcenter also lies on the normal to the square GHIJ but closer to it than C, so the circumsphere bulges more and hence contains the part of the sphere center C which is bounded by GHIJ. This is true for each face of the cube. So the intersection S lies inside the circumsphere of the cube. The diameter of the circumsphere is the long diagonal of the cube, which has length $(\sqrt{3}) \times (\sqrt{2})/6 = 1/\sqrt{6}$. Obviously the distance between any two points of the sphere is at most the diameter of the sphere. So any two points of S are at most a distance $1/\sqrt{6}$ apart. The vertices of the cube all belong to S, so this maximum is realised.

16. Let $x_2 = 2^{1/2}$, $x_3 = (2 + 3^{1/3})^{1/2}$, $x_4 = (2 + (3 + 4^{1/4})^{1/3})^{1/2}$, ... , $x_n = (2 + (3 + ... + n^{1/n} ...)^{1/3})^{1/2}$ (where the positive root is taken in every case). Show that $x_{n+1} - x_n < 1/n!$.

Solution

We have $(a^{1/n} - b^{1/n})(a^{(n-1)/n} + a^{(n-2)/n}b^{1/n} + ... + a^{1/n}b^{(n-2)/n} + b^{(n-1)/n}) = a - b$. So if $a > b \geq 1$, then the long parenthesis is $\geq n$ and hence $(a^{1/n} - b^{1/n}) \leq (a - b)/n$.

Applying this repeatedly, we get $x_{n+1} - x_n \leq ((3 + ... + (n+1)^{1/(n+1)})^{1/3} - (3 + ... + n^{1/n})^{1/3})/2! \leq ((4 + ... + (n+1)^{1/(n+1)})^{1/4} - (4 + ... + n^{1/n})^{1/4})/3! \leq ... \leq (n+1)^{1/n+1}\ 1/n!$. So that almost gets us there, but not quite.

Note that $(n + 1/n)^n < 3$, so $(n + 1)^n < 3.n^n \leq n^{n+1}$ for $n \geq 3$. Taking the n(n+1)th root, we get $(n + 1)^{1/(n+1)} < n^{1/n}$ for $n \geq 3$. Also $4^{1/4} = 2^{1/2}$, so $(n + 1)^{1/(n+1)} \leq 2^{1/2}$ for $n \geq 3$.

We now make the first inequality in the chain slightly stronger. We have $a^{1/2} - b^{1/2} = (a - b)/(a^{1/2} + b^{1/2}) < (a - b)/(2^{1/2} + 2^{1/2}) = (a - b)/(2\ 2^{1/2})$ for $a > b \geq 2$. So the chain gives $x_{n+1} - x_n < (n+1)^{1/(n+1)}\ (1/n!)\ (1/2^{1/2}) \leq 1/n!$ for $n \geq 3$.

It remains to show that $x_3 - x_2 < 1/2$. We have $x_3 - x_2 = 3^{1/3}/((2 + 3^{1/3})^{1/2} + 2^{1/2}) < 1.5/(3^{1/2} + 2^{1/2}) < 1.5/3 < 1/2$.

17. p is a prime. For which k can the set {1, 2, ... , k} be partitioned into p subsets such that each subset has the same sum?

18. a, b, c, ... , k are positive integers such that a divides $2^b - 1$, b divides $2^c - 1$, ... , k divides $2^a - 1$. Show that a = b = c = ... = k = 1.

Solution

Suppose a prime p divides $2^b - 1$. Then $2^b = 1$ mod p and (by Fermat's theorem) $2^{p-1} = 1$ mod p. Hence if d is the greatest common divisor of b and p-1, then $2^d = 1$ mod p. Evidently $d > 1$ (since 2 is not 1 mod p). Since d divides p-1, we have $d < p$. But d divides b, so there is a prime q smaller than p which divides b.

But if $b > 1$, then we may take p to be the *smallest* prime which divides $2^b - 1$. Then we find $q < p$ which divides b and hence $2^c - 1$. Continuing in this way we find a prime strictly smaller than p which divides $2^b - 1$. Contradiction. So b must be 1. Similarly for the other integers.

19. Show that the sequence $[n \sqrt{2}]$ for n = 1, 2, 3, ... contains infinitely many powers of 2.

Solution

Suppose not, so that the highest power of 2 in the sequence is 2^{m-1}. Since $\sqrt{2} < 2$, the difference between $[n\sqrt{2}]$ and $[(n+1)\sqrt{2}]$ is at most 2. Since $\sqrt{2} > 1$, the difference is at least 1. So it is either 1 or 2. So for some N we must have $[N\sqrt{2}] = 2^m - 1$, $[(N+1)\sqrt{2}] = 2^m + 1$.

We claim that for k = 0, 1, 2, ... we have $[2^kN\sqrt{2}] = 2^{m+k} - 1$, $[(2^kN+1)\sqrt{2}] = 2^{m+k} + 1$. We use induction on k. It is true for k = 0. So suppose it is true for k. Then $2^kN\sqrt{2} < 2^{m+k}$, so $2^{k+1}N\sqrt{2} < 2^{m+k+1}$. Also $(2^kN+1)\sqrt{2} > 2^{m+k} + 1$, so $2^kN\sqrt{2} > 2^{m+k} - (\sqrt{2} - 1)$ and hence $2^{k+1}N\sqrt{2} > 2^{m+k+1} - 2(\sqrt{2} - 1) > 2^{m+k+1} - 1$. Hence $[2^{k+1}N\sqrt{2}] = 2^{m+k+1} - 1$. So $[(2^{k+1}N+1)\sqrt{2}] = 2^{m+k+1}$ or $2^{m+k+1} + 1$. By assumption it is not 2^{m+k+1}, so it must be $2^{m+k+1} + 1$. That establishes the result for k+1 and hence for all k.

But $\sqrt{2}$ is irrational, so $N\sqrt{2} = 2^m - h$ for some positive real h. Take k such that $h > 1/2^k$, then $2^kN\sqrt{2} = 2^{m+k} - h2^k < 2^{m+k} - 1$, so $[2^kN\sqrt{2}] < 2^{m+k} - 1$. Contradiction. So the assumption must be wrong and there must be infinitiely many powers of 2.

20. Two equilateral triangles are inscribed in a circle radius r. Show that the area common to both triangles is at least $r^2(\sqrt{3})/2$.

Solution

Let the circle have center O. Suppose the triangles are ABC and PQR and that PQ and AB meet at X, PR and AB meet at Y, PQ and AC meet at Z. If we reflect in the line OX, then PXY becomes AXZ, so PX = AX. Similarly, reflecting in OY, we find that PY = BY. So the triangle PXY has perimeter AX + XY + YB = AB = $r\sqrt{3}$. If we rotate through 120^o twice, we see that the part of PQR which lies outside ABC is three triangles congruent to PXY. So the area common to both triangles is area ABC - 3 area PXY.

But of all triangles with a given perimeter, the equilateral triangle has largest area (if we keep X and Y fixed and vary P subject to PX + XY + YP constant, then we maximise the area by taking P as far as possible from the line XY and hence taking PX = PY. Similarly for the other pairs of sides). So area PXY $\le$ 1/9 area ABC. Hence the common area >= 2/3 area ABC = $(2/3)\ r^2\ (3\sqrt{3})/4 = r^2\ (\sqrt{3})/2$.

21. Show that if the real numbers x, y, z satisfy $1/(yz - x^2) + 1/(zx - y^2) + 1/(xy - z^2) = 0$, then $x/(yz - x^2)^2 + y/(zx - y^2)^2 + z/(xy - z^2)^2 = 0$.

Solution

Put $yz - x^2 = k$ (1), $zx - y^2 = h$ (2). Then the given equality implies $xy - z^2 = -hk/(h + k)$. Evidently h and k are non-zero (or the given equality would not hold).

y (1) + z (2) + x(3) gives: $ky + hz - hkx/(h + k) = 0$. Hence $hkx = hky + k^2y + h^2z + hkz$. Similarly, z (1) + x (2) + y (3) gives: $hky = h^2x + hkx + hkz + k^2z$. Adding, we get $0 = h^2x + k^2y + (h + k)^2z$. Dividing by h^2k^2 gives the required result.

22. Show how to construct the triangle ABC given the distance between the circumcenter O and the orthocenter H, the fact that OH is parallel to the side AB, and the length of the side AB.

Solution

Let A be the point (-1, 0), B the point (1, 0), O the point (0, b) and H the point (3a, b). Then G is the point (a, b) (using the Euler line) and the midpoint M of AB is (0, 0). Hence C is the point (3a, 3b). We are given a but not b. Since OC = OA, we have $1 + b^2 = 9a^2 + 4b^2$, so $3b^2 = 1 - 9a^2$.

Construct A, M, B. Then construct D (1, 1), E (1, 3a) and F(1-3a, 0). Take the line through E parallel to DF. Suppose it meets AB at K. Then BK/BF = BE/BD = 3a, so BK = $9a^2$, so K is ($3b^2$, 0). Take L as ($9b^2$, 0). Take the circle diameter AL. If it meets the y-axis at (0, ±k), then k^2 = AM.ML = $9b^2$, so the have the point (0, 3b). The rest is now trivial.

23. Find all positive integers a, b, c such that 1/a + 1/b + 1/c = 4/5.

24. Factorise $5^{1985} - 1$ as a product of three integers, each greater than 5^{100}.

Solution

Answer: $(5^{397} - 1)(5^{794} - 5^{596} + 3\cdot 5^{397} - 5^{199} + 1)(5^{794} + 5^{596} + 3\cdot 5^{397} + 5^{199} + 1)$.

Put $A = 5^{397}$. Then $5^{1985} - 1 = (A - 1)(A^4 + A^3 + A^2 + A + 1)$. Clearly (A - 1) is one of the desired factors.

We have $(A^2 + 3A + 1)^2 = A^4 + 6A^3 + 11A^2 + 6A + 1$, so $A^4 + A^3 + A^2 + A + 1 = (A^2 + 3A + 1)^2 - 5A(A + 1)^2$. But $5A = B^2$, where $B = 5^{199}$. So $A^4 + A^3 + A^2 + A + 1 = (A^2 + 3A + 1 + BA + B)(A^2 + 3A + 1 - BA - B)$.

It remains to check that $A^2 + 3A + 1 - BA - B > 5^{100}$. But obviously $5^{794} > 5^{596}$ and $5^{397} > 5^{199}$. So the factor is $> 2\cdot 5^{397}$.

25. 34 countries each sent a leader and a deputy leader to a meeting. Some of the participants shook hands before the meeting, but no leader shook hands with his deputy. Let S be the set of all 68 participants except the leader of country X. Every member of S shook hands with a different number of people (possibly zero). How many people shook hands with the leader or deputy leader of X?

Solution

Answer: 33.

There are 68 delegates. Each shakes hands with at most 66 others, so we can label the 67 members of S as $a_0, a_1, \dots , a_{66}$, where a_n shakes hands with n delegates. Let L be the leader of country X. We claim that: (1) a_{66} shakes hands with $a_1, a_2, \dots , a_{65}$ and L, and is from the same country as a_0; (2) for n = 1, 2, ... , 32, delegate a_n shakes hands with $a_{67-n}, a_{68-n}, \dots , a_{66}$ and is the partner of a_{66-n} who shakes hands with , $a_{n+1}, a_{n+2}, \dots , a_{66}$ (except himself) and L, (3) the deputy leader is a_{33}, (4) L shakes hands with $a_{34}, a_{35}, \dots , a_{66}$.

Note that a_{66} shakes hands with everyone except himself and the other delegate from his country. a_0 does not shake hands with anyone, and hence in particular does not shake hands with a_{66}, so they must be from the same country. That establishes (1).

We prove (2) by induction. (1) is effectively the case n = 0. Suppose it is true for all m < n < 32. Then a_{66-m} shakes hands with a_n (by induction) for m = 0, 1, ... , n-1. That accounts for all n people who shake hands with a_n. Similarly, a_m does not shake hands with a_{66-n}, which gives n people who do not shake hands with a_{66-n}. We have just established that a_n does not and a_{66-n} himself does not, so we have n+2 people who do not, and hence at most 68-(n+2) = 66-n who do. So the n+2 people are the *only* people who do not. The only candidate for the partner of a_{66-n} is a_n. That establishes the case n. So by induction it is true for n = 1, 2, ... , 32.

The only person unaccounted for, who could be L's partner is a_{33}. That establishes (3). We establish (4) by inspection of (1) and (2). Finally, we note that $a_0, a_1, \ldots, a_{32}$ do not shake hands with L or a_{33}, whereas $a_{34}, a_{35}, \ldots, a_{66}$ shake hands with both a_{33} and L.

26. Find the smallest positive integer n such that n has exactly 144 positive divisors including 10 consecutive integers.

Solution

Answer: $110880 = 2^5 3^2 5.7.11$.

If $n = p^a q^b \ldots$, then n has exactly $(a + 1)(b + 1) \ldots$ distinct positive divisors. $144 = 2\cdot2\cdot2\cdot2\cdot3\cdot3$, so n can have at most 6 prime divisors. It is divisible by 10 consecutive integers. These must include a multiple of 5, a multiple of 7, a multiple of 9 and a multiple of 8. So n must be divisible by 2, 3, 5, 7. Also if n the power of 2 dividing n is 2^m, then m >= 3, so $(m + 1) \geq 4$, which must account for at least two of the factors $2\cdot2\cdot2\cdot2\cdot3\cdot3$. So n has either 4 or 5 prime factors. If it has 5, then clearly we must take the fifth to be as small as possible and hence to be 11. Thus we must have $n = 2^a 3^b 5^c 7^d 11^e$, with $a \geq 3, b \geq 2, c \geq 1, d \geq 1, e \geq 0$ and $(a + 1)(b + 1)(c + 1)(d + 1)(e + 1) = 144$. So (a+1, b+1, c+1, d+1, e+1) must be a permutation of one of the following: (12, 3, 2, 2, 1), (9, 4, 2, 2, 1), (8, 3, 3, 2, 1), (6, 6, 2, 2, 1), (6, 4, 3, 2, 1), (6, 3, 2, 2, 2), (4, 3, 3, 2, 2). Where possible the larger exponents should go with the smaller primes, so that gives the following possibilities: $2^{11}3^2 5.7$, $2^8 3^3 5.7$, $2^7 3^2 5^2 7$, $2^5 3^5 5.7$, $2^5 3^3 5^2 7$, $2^5 3^2 5.7.11$, $2^3 3^2 5^2 7.11$.

Taking out the common factor $2^3 3^2 5\cdot7$ we are comparing: $2^8 = 256$, $2^5 3 = 96$, $2^4 3^3 = 432$, $2^2 3\cdot5 = 60$, $2^2 11 = 44$, $5\cdot11 = 55$. So the smallest is $2^5 3^2 5\cdot7\cdot11$ (corresponding to 44).

27. Find the largest and smallest values of $w(w + x)(w + y)(w + z)$ for reals w, x, y, z such that $w + x + y + z = 0$ and $w^7 + x^7 + y^7 + z^7 = 0$.

Solution

Answer: both 0.

We have that $(w+x)^7 - 7wx(w+x)(w^2+wx+x^2)^2 = w^7 + x^7$. Similarly for y and z. So adding gives $wx(w+x)(w^2+wx+x^2)^2 + yz(y+z)(y^2+yz+z^2)^2 = 0$. Dividing by $w+x = -(y+z)$, we get $wx(w^2+wx+x^2)^2 = yz(y^2+yz+z^2)^2$.

We have $w^2 + wx + x^2 = (w+x)^2 - wx$, and $y^2 + yz + z^2 = (y+z)^2 - yz = (w+x)^2 - yz$. Hence $(wx - yz)(w+x)^4 - 2(wx)^2(w+x)^2 + 2(yz)^2(w+x)^2 + (wx)^3 - (yz)^3 = 0$. So either $(wx - yz) = 0$ or $(w+x)^4 - 2(wx + yz)(w+x)^2 + ((wx)^2 + wxyz + (yz)^2) = 0$.

If $wx = yz$, then since also $w + x = -(y + z)$, we have that w and x are the roots of the quadratic $t^2 + (y+z) t + yz$. Solving, w, x = -y, -z in some order. So $(w + y)(w + z) = 0$.

So suppose $wx - yz$ is non-zero. Then $(w+x)^4 - 2(wx + yz)(w+x)^2 + ((wx)^2 + wxyz + (yz)^2) = 0$, or $((w+x)^2 - (wx + yz))^2 = wxyz$ (*). Since $(w+x)^2 = (y+z)^2$, we have $2((w+x)^2 - (wx + yz)) = ((w+x)^2 - 2wx + (y+z)^2 - 2yz) = (w^2 + x^2 + y^2 + z^2)$. So (*) becomes $(w^2 + x^2 + y^2 + z^2)^2 = 4wxyz$. But by AM/GM we have $(w^2 + x^2 + y^2 + z^2) > 4(w^2x^2y^2z^2)^{1/4}$ and hence $(w^2 + x^2 + y^2 + z^2)^2 >= 16wxyz$. So $4wxyz >= 16wxyz$. Also $4wxyz$ is non-negative, since it is a square. So we must have $wxyz = 0$. Hence also $(w^2 + x^2 + y^2 + z^2) = 0$, so $w = x = y = z = 0$ and we have a special case of $wx = yz$ after all.

So the *only possible* value of $w(w + x)(w + y)(w + z)$ is zero.

28. X is the set $\{1, 2, \ldots , n\}$. $P_1, P_2, \ldots , P_n$ are distinct pairs of elements of X. P_i and P_j have an element in common iff $\{i, j\}$ is one of the pairs. Show that every element of X belongs to exactly two of the pairs.

Solution

Suppose that i appears in a_i of the pairs. Counting pairs, we have $2n = a_1 + a_2 + \ldots + a_n$. If two pairs both contain i, then they do not have any other element in common. So the a_i pairs containing i give rise to $a_i(a_i- 1)/2$ pairs with i as common element. Thus the number of pairs is also $\sum a_i(a_i - 1)/2$. So $\sum a_i(a_i - 1) = 2n$. Hence $\sum a_i^2 = 4n$. But, by Cauchy Schwartz, $4n^2 = (\sum 1\ a_i)^2 \le (\sum 1^2)(\sum a_i^2) = 4n^2$ with equality iff all $a_i$are equal and hence iff all $a_i = 2$.

29. Show that for any point P on the surface of a regular tetrahedron we can find another point Q such that there are at least three different paths of minimal length from P to Q.

Solution

Let the tetrahedron be ABCD. Open the sides out flat. We get the large triangle AA'A" (the label on the top vertex should be A). For convenience we show two other copies of the face ABC: A"BC" and A'B'C. P is a point on ABC. Without loss of generality it is closer to A than to B or C. P' and P" are its images in A'B'C and A"BC". Note that we obtain P' by rotating P through 180^o about C, and we obtain P" by rotating P through 180^o about B. Let the perpendicular to PP' at C and the perpendicular to PP" at B meet at Q. Note that angle ABP $\le 30^o$, so angle QBC $\le 60^o$ and similarly angle QCB $\le 60^o$, so Q lies in the face BCD. Also Q lies on the perpendicular bisector of PP", so QP = QP". Similarly QP = QP'. So we have three equal paths from P to Q, corresponding to PQ, P'Q and P"Q. Since they are straight lines, they are also all of minimal length.

30. C is a circle and L a line not meeting it. M and N are variable points on L such that the circle diameter MN touches C but does not contain it. Show that there is a fixed point P such that the $\angle$ MPN is constant.

Solution

Answer: let O be the center of C, let C have radius 1, let D be the foot of the perpendicular from O to L and let OD = a. Then P is the point on the segment OD a distance $\sqrt{(a^2 - 1)}$ from D.

Let MN = 2x. Let Q be the midpoint of MN. The circles touch, so OQ = x + 1. Suppose first that D is between M and N and that N is closer to D. We have $QD^2 = (x+1)^2 - a^2$, $QP^2 = QD^2 + DP^2 = (x+1)^2 - a^2 + a^2- 1$. Now tan MPD = (x + QD)/PD, tan NPD = (x - QD)/PD, so tan MPN = tan(MPD + NPD) = $2x\ PD/(PD^2 + QD^2 - x^2) = \sqrt{(a^2 - 1)}$, which is independent of x.

Similarly, if D is not between M and N, but is closer to M, then we have tan MPD = (QD - x)/PD, tan NPD = (QD + x)/PD, tan MPD = tan(NPD - MPD) and we get the same result.

27th IMO 1986 shortlisted problems

7. Let $A_1 = 0.12345678910111213\ldots$, $A_2 = 0.14916253649\ldots$, $A_3 = 0.182764125216\ldots$, $A_4 = 0.11681256625\ldots$, and so on. The decimal expansion of A_n is obtained by writing out the *n*th powers of the integers one after the other. Are any of the A_n rational?

8. A, B, C are three points on the edge of a circular pond and form an equilateral triangle with side 86 and B due west of C. A boy swims from A directly towards B. After a distance x he turns and swims due west a distance y to reach the edge of the pond. Given that x and y are positive integers, find y.

9. For any positive integer n and any prime p > 3, find at least 3(n+1) sets of positive integers $a \le b < c$ such that $abc = p^n(a + b + c)$.

10. Find four positive integers < 70000 each with more than 100 positive divisors.

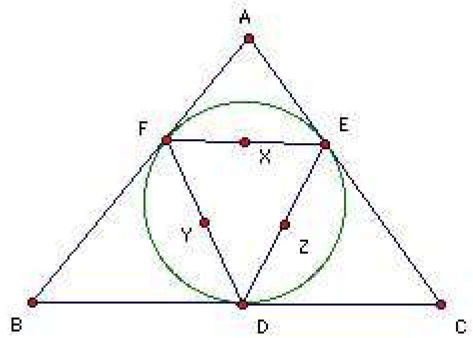
11. The real numbers $x_0, x_1, \dots , x_{n+1}$ satisfy $0 = x_0 < x_1 < \dots < x_n < x_{n+1} = 1$ and $1/(x_i - x_0) + 1/(x_i - x_1) + \dots + 1/(x_i - x_{i-1}) + 1/(x_i - x_{i+1}) + 1/(x_i - x_{i+2}) + \dots + 1/(x_i - x_{n+1}) = 0$ for i = 1, 2, ... , n. Show that $x_i + x_{n+1-i} = 1$ for all i.

12. A graph has n points and q edges. The edges are labeled 1, 2, ... , q. Show that we can always find a sequence of at least 2q/n edges such that the labels increase monotonically and adjacent edges have a common vertex.

13. A k-element subset is chosen at random from {1, 2, ... , 1986}. For which k is there an equal chance that the sum of the elements in the subset will be 0, 1 or 2 mod 3.

14. Find an explicit expression for f(n), the least number of distinct points in the plane such that for each k = 1, 2, ... , n there is a straight line containing just k points.

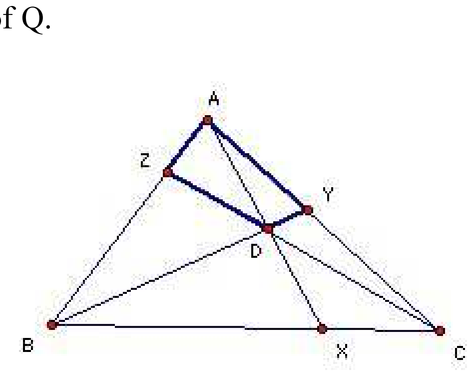
15. A particle starts at (0, 0). A fair coin is tossed repeatedly. For each head the x-coordinate is increased by 1 unless x = n, when it does not move. For each tail the y-coordinate is increased by 1 unless y = n, when it does not move. Find the probability that just 2n+k tosses are needed to reach (n, n).



16. The incircle of ABC touches the sides at DEF and the midpoints of the sides of DEF are X, Y, Z. Show that the incenter of ABC, and the circumcenters of ABC and XYZ are collinear.

17. Q is a convex quadrilateral which is not cyclic. f(Q) is the quadrilateral formed by the circumcenters of the four triangles whose vertices are vertices of Q. Show that f(f(Q)) is similar to Q and find the ratio of similitude in terms of the angles of Q.

18. X, Y, Z are points on the sides of the triangle ABC, such that AX, BY, CZ meet at D inside the triangle. Show that if AYDZ and BZDX are cyclic, so is CXDY.



19. The tetrahedron ABCD has AD = BC = a, AC = BD = b and $AB \cdot CD = c^2$. Find the smallest value of AP + BP + CP + DP for any point P in space.

20. Show that the sum of the face angles at each vertex of a tetrahedron is 180° iff the faces are all congruent triangles.

21. A tetrahedron has each sum of opposite edges equal to 1. Show that the sum of the four inradii of the faces is at most $1/\sqrt{3}$ with equality iff the tetrahedron is regular.

28th IMO 1987 shortlisted problems

1. f is a real-valued function on the reals such that:
(1) if $x \geq y$ and $f(y) - y \geq v \geq f(x) - x$, then $f(z) = v + z$ for some z between x and y;
(2) for some k, $f(k) = 0$ and if $f(h) = 0$, then $h <= k$;
(3) $f(0) = 1$;
(4) $f(1987) \leq 1988$;
(5) $f(x) f(y) = f(x f(y) + y f(x) - xy)$ for all x, y.
Find f(1987). *(Australia 6)*

2. $S = \{a_1, a_2, \dots , a_n, b_1, b_2, \dots , b_n\}$. There are subsets $C_1, C_2, \dots , C_k$, such that (1) for no i, j do both a_i and b_i both belong to C_j, (2) for any pair of distinct elements of S, not of the form a_i, b_i, there is just one C_j containing both elements. Show that if $n > 3$, then $k \geq 2n$. *(USA 3)*

3. Does there exist a polynomial p(x, y) of degree 2 such that, for each non-negative integer n, we have $n = p(a, b)$ for just one pair (a, b) of non-negative integers? *(Finland 3)*

4. ABCDA'B'C'D' is any parallelepiped (with ABCD, A'B'C'D' faces and AA', BB', CC', DD' edges). Show that AC + AB'+ AD' <= AB + AD + AA' + AC' (the sum of the thre short diagonals from A is less than the sum of the three edges from A plus the long diagonal from A). *(France 5)*

5. Find the smallest real c such that $x_1^{1/2} + x_2^{1/2} + \dots + x_n^{1/2} \leq c (x_1 + x_2 + \dots + x_n)^{1/2}$ for all n and all real sequences $x_1, x_2, x_3, \dots$ which satisfy $x_1 + x_2 + \dots + x_n \leq x_{n+1}$. *(United Kingdom 6)*

6. Show that $a^n/(b+c) + b^n/(c+a) + c^n/(a+b) \geq (2/3)^{n-2} s^{n-1}$ for all $n \geq 1$, where a, b, c are the sides of a triangle and $s = (a + b + c)/2$. *(Greece 4)*

7. Given any 5 real numbers u_0, u_1, u_2, u_3, u_4, show that we can always find 5 real numbers v_0, v_1, v_2, v_3, v_4 such that each $u_i - v_i$ is integral and $\sum_{i<j}(v_i - v_j)^2 < 4$. *(Netherlands 1)*

8. Does there exist a subset M of Euclidean space such that any plane meets M in a finite non-empty set? *(Hungary 1)*

9. Show that for any relatively prime positive integers m, n we can find integers $a_1, a_2, \dots , a_m$ and $b_1, b_2, \dots , b_n$ such that each product a_ib_j gives a different residue mod mn. *(Hungary 2)*

10. Two spheres S and S' touch externally and lie inside a cone C. Each sphere touches the cone in a full circle. n solid spheres are arranged in the cone in a ring so that each touches S and S' externally, touches the cone, and touches its two neighbouring solid spheres. What are the possible values of n? *(Iceland 3)*

11. Find the number of ways of partitioning $\{1, 2, 3, \dots , n\}$ into three (possibly empty) subsets A, B, C such that (1) for each subset, if the elements are written in ascending order, then they alternate in parity, and (2) if all three subsets are non-empty, then just one of them has its smallest element even. *(Poland 1)*

12. ABC is a non-equilateral triangle. Find the locus of the centroid of all equilateral triangles A'B'C' such that A, B', C' are collinear, A', B, C' are collinear, A', B', C are collinear, and both ABC and A'B'C' have their vertices anti-clockwise. *(Poland 5)*

14. How many n-digit words can be formed from the alphabet {0, 1, 2, 3, 4} if neighboring digits must differ by exactly 1? *(German Federal Republic 1)*

17. Show that we can color the elements of the set {1, 2, ... , 1987} with 4 colors so that any arithmetic progression with 10 terms in the set is not monochromatic. *(Romania 1)*

18. For any positive integer r find the smallest positive integer h(r) such that for any partition of {1, 2, ... , h(r) } into r parts, there are integers $a \geq 0$ and $1 \leq x \leq y$ such that a + x, a + y and a + x + y all belong to the same part. *(Romania 4)*

19. Given angles A, B, C such that $A + B + C < 180^o$, show that there is a triangle with sides sin A, sin B, sin C and that its area is less than (sin 2A + sin 2B + sin 2C)/8. *(USSR 2)*

23. Show that for any integer k > 1, there is an irrational number r such that $[r^m] = -1 \bmod k$ for every natural number m. *(Yugoslavia 2)*

29th IMO 1988 shortlisted problems

1. The sequence $a_0, a_1, a_2, \dots$ is defined by $a_0 = 0$, $a_1 = 1$, $a_{n+2} = 2a_{n+1} + a_n$. Show that 2^k divides a_n iff 2^k divides n.

2. Find the number of odd coefficients of the polynomial $(x^2 + x + 1)^n$.

3. The angle bisectors of the triangle ABC meet the circumcircle again at A', B', C'. Show that area A'B'C' ≥ area ABC.

4. The squares of an n x n chessboard are numbered in an arbitrary manner from 1 to n^2 (every square has a different number). Show that there are two squares with a common side whose numbers differ by at least n.

6. ABCD is a tetrahedron. Show that any plane through the midpoints of AB and CD divides the tetrahedron into two parts of equal volume.

7. c is the largest positive root of $x^3 - 3x^2 + 1$. Show that $[c^{1788}]$ and $[c^{1988}]$ are multiples of 17.

8. $u_1, u_2, \dots, u_n$ are vectors in the plane, each with length at most 1, and with sum zero. Show that one can rearrange them as $v_1, v_2, \dots, v_n$ so that all the partial sums $v_1, v_1 + v_2, v_1 + v_2 + v_3, \dots, v_1 + v_2 + \dots + v_n$ have length at most $\sqrt{5}$.

10. Let X = {1, 2, ... , n}. Find the smallest number of subsets f(n) of X with union X such that given any distinct a, b in X, one of the subsets contains just one of a, b.

11. The lock on a safe has three wheels, each of which has 8 possible positions. It is known that the lock is defective and will open if any two wheels are in the correct position. How many combinations must be tried to guarantee opening the safe?

12. ABC is a triangle. K, L, M are points on the sides BC, CA, AB respectively. D, E, F are points on the sides LM, MK, KL respectively. Show that (area AME)(area CKE)(area BKF)(area ALF)(area BDM)(area CLD) ≤ $(1/8 \text{ area ABC})^6$.

14. For what values of n, does there exist an n x n array of entries 0, ±1 such that all rows and columns have different sums?

15. ABC is an acute-angled triangle. H is the foot of the perpendicular from A to BC. M and N are the feet of the perpendiculars from H to AB and AC. L_A is the line through A perpendicular to MN. L_B and L_C are defined similarly. Show that L_A, L_B and L_C are concurrent.

17. ABCDE is a convex pentagon such that BC, CD and DE are equal and each diagonal is parallel to a side (AC is parallel to DE, BD is parallel to AE etc). Show that the pentagon is regular.

19. f has positive integer values and is defined on the positive integers. It satisfies $f(f(m) + f(n)) = m + n$ for all m, n. Find all possible values for f(1988).

20. Find the smallest n such that if $\{1, 2, \dots , n\}$ is divided into two disjoint subsets then we can always find three distinct numbers a, b, c in the same subset with $ab = c$.

21. 49 students solve a set of 3 problems. Each problem is marked from 0 to 7. Show that there are two students A and B such that A scores at least as many as B for each problem.

22. Show that there are only two values of N for which $(4N+1)(x_1^2 + x_2^2 + \dots + x_N^2)= 4(x_1 + x_2 + \dots + x_N)^2 + 4N + 1$ has an integer solution x_i.

23. I is the incenter of the triangle ABC. Show that for any point P, $BC\cdot PA^2 + CA\cdot PB^2 + AB\cdot PC^2 = BC\cdot IA^2 + CA\cdot IB^2 + AB\cdot IC^2 + (AB + BC + CA)\cdot IP^2$.

24. $x_1, x_2, x_3, \dots$ is a sequence of non-negative reals such that $x_{n+2} = 2x_{n+1} - x_n$ and $x_1 + x_2 + \dots + x_n \le 1$ for all $n > 0$. Show that $x_{n+1} \le x_n$ and $x_{n+1} > x_n - 2/n^2$ for all $n > 0$.

25. A *double* number has an even number of digits and the first half of its digits are the same as the second half. For example, 360360 is a double number, but 36036 is not. Show that infinitely many double numbers are squares.

27. ABC is an acute-angled triangle area S. L is a line. The lengths of the perpendiculars from A, B, C to L are u, v, w respectively. Show that $u^2\tan A + v^2\tan B + w^2\tan C \ge 2S$. For which L does equality occur?

28. The sequence of integers $a_1, a_2, a_3, \dots$ is defined by $a_1 = 2$, $a_2 = 7$, and $-1/2 < a_{n+2} - a_{n+1}^2/a_n \le 1/2$. Show that a_n is odd for $n > 1$.

29. n signals are equally spaced along a rail track. No train is allowed to leave a signal whilst there is a moving train between that signal and the next. Any number of trains can wait at a signal. At time 0, k trains are waiting at the first signal. Except when waiting at a signal each train travels at a constant speed, but each train has a different speed. Show that the last train reaches signal n at the same time irrespective of the order in which the trains are arranged at the first signal.

30. ABC is a triangle. M is a point on the side AC such that the inradii of ABM and CBM are the same. Show that $BM^2 = \cot(B/2)$ area ABC.

31. An even number of people have a discussion sitting at a circular table. After a break they sit down again in a different order. Show that there must be two people with the same number of people sitting between them before and after the break.