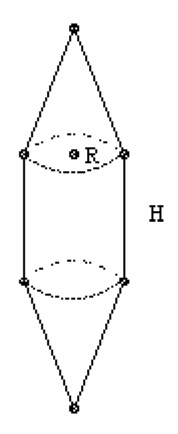
1st Putnam 1938

Problem A1
A solid in Euclidean 3-space extends from $z = -h/2$ to $z = +h/2$ and the area of the section $z = k$ is a polynomial in k of degree at most 3. Show that the volume of the solid is $h(B + 4M + T)/6$, where B is the area of the bottom ($z = -h/2$), M is the area of the middle section ($z = 0$), and T is the area of the top ($z = h/2$). Derive the formulae for the volumes of a cone and a sphere.

Solution
Let the polynomial be $az^3 + bz^2 + cz + d$. Then the volume is $\int_{-h/2}^{h/2} (az^3 + bz^2 + cz + d)\, dz = bh^3/12 + dh$. But $B + T = bh^2/2 + 2d$, $M = d$, so $h(B + 4M + T)/6 = bh^3/12 + dh$, which proves the formula. For a sphere radius R, we have $h = 2R$, $B + T = 0$ and $M = \pi R^2$, so the formula gives $4/3\ \pi R^3$, as usual. For a cone height h, base area A, we have $B = A$, $T = 0$, $M = A/4$, so the volume is $hA/3$, as usual.

Problem A2
A solid has a cylindrical middle with a conical cap at each end. The height of each cap equals the length of the middle. For a given surface area, what shape maximizes the volume?



Solution
Let the radius be R and the height H. The area is $2\pi RH + 2\pi R\sqrt{R^2+H^2}$. The volume is $5/3\ \pi R^2 H$.

The area is fixed, so for some fixed k, we have $R (H + \sqrt{R^2+H^2}) = k$. This gives $H = (k^2 - R^4)/(2kR)$. We must now choose R to maximise $f(R) = R^2H = R (k^2 - R^4)/2k$. Evidently the allowed range for R is from $R = 0$ up to $\sqrt{k}$ (corresponding to $H = 0$). But $f(0) = 0$ and $f(\sqrt{k}) = 0$, so the maximum is at some interior point of the interval. Differentiating, we find it is at $R_{max} = (k^2/5)^{1/4}$. In terms of the area A, we have $A = 2\pi k$, so $R_{max} = (A/(\pi\ 2\sqrt{5}))^{1/2}$.

Problem A3
A particle moves in the Euclidean plane. At time t (taking all real values) its coordinates are $x = t^3 - t$ and $y = t^4 + t$. Show that its velocity has a maximum at $t = 0$, and that its path has an inflection at $t = 0$.

Solution
The speed squared is $(dx/dt)^2 + (dy/dt)^2 = 16t^6 + 9t^4 + 8t^3 - 6t^2 + 2$. Let this be f(t). We have $f'(t) = 12t(8t^4 + 3t^2 + 2t - 1)$. So $f'(t) = 0$ at $t = 0$. Also for t small (positive or negative), $8t^4 + 3t^2 + 2t - 1$ is close to -1 and hence negative, so $f'(t)$ is positive for t just less than 0 and negative for t just greater than 0. Hence f(t) has a maximum at $t = 0$. Hence the speed does also.

The gradient $dy/dx = (4t^3 + 1)/(3t^2 - 1)$. Let this be g(t). Then $g'(t) = 6t(2t^3 - 2t - 1)/(3t^2 - 1)^2$. Hence $g'(0) = 0$. Also $g'(t)$ is positive for t just less than 0 and negative for t just greater than 0, so it is a point of inflection.

Problem A4
A notch is cut in a cylindrical vertical tree trunk. The notch penetrates to the axis of the cylinder and is bounded by two half-planes. Each half-plane is bounded by a horizontal line passing through the axis of the cylinder. The angle between the two half-planes is θ. Prove that the volume of the notch is minimized (for given tree and θ) by taking the bounding planes at equal angles to the horizontal plane.

Solution
We find the volume of the notch above the horizontal plane. Suppose that the upper bounding half-plane is at an angle φ to the horizontal. We may take the radius of the tree to be 1. A vertical section through the notch at a distance x from its widest extent is a right-angled triangle with base $\sqrt{1 - x^2}$ and area $1/2\ (1 - x^2) \tan\varphi$. Hence the volume is $2/3 \tan\varphi$. So the total volume of the notch is $2/3 (\tan\varphi + \tan(\theta-\varphi))$. So we have to find the angle φ which minimises $(\tan\varphi + \tan(\theta - \varphi)$. Differentiating, or otherwise, we easily find that the minimum is at $\varphi/2$.

Problem A5
(1) Find $\lim_{x\to\inf} x^2/e^x$.
(2) Find $\lim_{k\to 0} 1/k \int_0^k (1 + \sin 2x)^{1/x}\, dx$.

Solution
(1) Let $f(x) = x^3e^{-x}$. Then $f'(x) = (3x^2 - x^3) e^{-x} < 0$ for $x > 3$. Hence $f(x) < f(3)$ for $x > 3$, so $x^2 e^{-x} < f(3)/x$ for $x > 3$. Hence $x^2 e^{-x}$ tends to zero.

(2) We use L'Hôpital's rule $\lim f(x)/g(x) = \lim f'(x)/g'(x)$. Applied to the expression given it gives $\lim (1 + \sin 2x)^{1/x}$. Write $(1 + \sin 2x)^{1/x} = \exp(1/x \ln(1 + \sin 2x))$. So apply the rule again to $1/x \ln(1 + \sin 2x)$ to get $2 \cos 2x/(1 + \sin 2x)$ which tends to 2. Hence $(1 + \sin 2x)^{1/x}$ tends to e^2 and so does the original expression.

Problem A6
A swimmer is standing at a corner of a square swimming pool. She swims at a fixed speed and runs at a fixed speed (possibly different). No time is taken entering or leaving the pool. What path should she follow to reach the opposite corner of the pool in the shortest possible time?

Solution
Answer: let k be the running speed divided by the swimming speed. For $k > \sqrt{2}$, the unique solution is to run round the outside. For $k < \sqrt{2}$, the unique solution is to swim direct. For $k = \sqrt{2}$ there is no unique solution. Run along a side to X, swim to Y equidistant from the corner between X and Y, then run from Y. The time taken is independent of X.

We may take the side of the square to be 1, the swimming speed to be 1 and the running speed to be k. Let the square be ABCD. Suppose the start is at A and the finish at C. Possible routes are (1) run to X on AB, swim to Y on BC, run to C, (2) run to X on AD, swim to Y on CD, run to C, (3) run to X on AB, swim to Y on CD, run to C. We start by considering case (1). Take BX to be x, BY to be y. Then the time taken is $(2 - x - y)/k + \sqrt{x^2 + y^2}$. Note that this includes the extreme cases of running all the way ($x = y = 0$) and swimming all the way ($x = y = 1$).

Now $(x - y)^2 >= 0$, with equality iff $x = y$, so $(x + y)^2 <= 2(x^2 + y^2)$ and hence $(x + y) \le \sqrt{2} \sqrt{x^2 + y^2}$, with equality iff $x = y$. So if $k > \sqrt{2}$, then $(x + y) < k \sqrt{x^2 + y^2}$ and hence $2/k < (2 - x - y)/k + \sqrt{x^2 + y^2}$ unless $x = y = 0$ (when we have equality). Hence for $k > \sqrt{2}$, the unique solution is to run all the way.

If $k < \sqrt{2}$, then $(x + y) \le \sqrt{2} \sqrt{x^2 + y^2}$ implies $\sqrt{2} (\sqrt{2} - \sqrt{x^2 + y^2}) \le 2 - x - y$ and hence $k (\sqrt{2} - \sqrt{x^2 + y^2}) < 2 - x - y$ unless $x = y = 1$ (when we have equality). So $\sqrt{2} < (2 - x - y)/k + \sqrt{x^2 + y^2}$ unless $x = y = 1$. In other words, for $k < \sqrt{2}$ the unique solution is to swim all the way.

For $k = \sqrt{2}$ we have equality (in both the previous paragraphs) iff $x = y$. So any solution with $x = y$ is optimal in this case.

Problem A7
Do either (1) or (2)
(1) S is a thin spherical shell of constant thickness and density with total mass M and center O. P is a point outside S. Prove that the gravitational attraction of S at P is the same as the gravitational attraction of a point mass M at O.

(2) K is the surface $z = xy$ in Euclidean 3-space. Find all straight lines lying in S. Draw a diagram to illustrate them.

Solution
(1) Let Q be a point on S. The obvious coordinate is the angle θ = angle QOP. The density is $\rho = M/4\pi r^2$. By symmetry the attraction on P is towards O. Let the distance PO be d and the radius of the sphere be r. Let the gravitational constant be G. The component of the attraction towards O (per unit mass at P) is $G \int_0^\pi r\, d\theta\, 2 \pi r \sin \theta\, \rho\, (d - r \cos \theta) (d^2 + r^2 - 2dr \cos \theta)^{-3/2}$. Note that the factor $(d - r \cos \theta) (d^2 + r^2 - 2dr \cos \theta)^{-1/2}$ is needed to resolve the force in the direction PO. Writing $x = \cos \theta$, this becomes $2G \pi r^2\rho \int_{-1}^1 (d - r x)/(d^2 + r^2 - 2dr x)^{3/2} dx$.

This is not as bad as it looks. It is just the sum of a $(1 - x)^{-1/2}$ and a $(1 - x)^{-3/2}$ integral, both of which are straightforward. Moreover, when we come to substitute $x = \pm 1$, the factor $(d^2 + r^2 - 2dr x)^{1/2}$ becomes just $d - r$ or $d + r$. So we get after a little simplification $4 \pi r^2\rho/d^2 = MG/d^2$, which is the same result as if all the mass was concentrated at O.

(2) We can write a general line as $x = at + b$, $y = ct + d$, $z = et + f$, for some constants a, b, c, d, e, f and a parameter t which takes all real values. If this lies in $z = xy$, then $et + f = ac\, t^2 + (bc + ad) t + bd$ for all t. Hence a or c must be

zero. If a is 0, then z = by, so the line can be written as x = b, z = by. Similarly, if c = 0, then the line can be written as y = d, z = dx. Conversely, it is easy to see that these two families of lines lie in the surface.

Problem B1

Do either (1) or (2)

(1) Let A be matrix (a_{ij}), $1 \le i,j \le 4$. Let d = det(A), and let A_{ij} be the cofactor of a_{ij}, that is, the determinant of the 3 x 3 matrix formed from A by deleting a_{ij} and other elements in the same row and column. Let B be the 4 x 4 matrix (A_{ij}) and let D be det B. Prove $D = d^3$.

(2) Let P(x) be the quadratic $Ax^2 + Bx + C$. Suppose that P(x) = x has unequal real roots. Show that the roots are also roots of P(P(x)) = x. Find a quadratic equation for the other two roots of this equation. Hence solve $(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0$.

Solution

Answer: (2) The quadratic is $A^2x^2 + (AB + A)x + (AC + B + 1) = 0$. The quartic in y has roots 0, 1, 2, 2.

(1) We have $a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + a_{i4}A_{i4} = d$. But $a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} + a_{i4}A_{j4} = 0$ for i not equal to j (because it can be considered as an expansion of the determinant for the matrix derived from A by replacing row i by row j - the resulting matrix has two identical rows and hence zero determinant). So if we multiply the transpose of A by the matrix (A_{ij}) then we get d down the diagonal and zeros elsewhere. Hence $d D = d^4$, so $D = d^3$.

(2) It is obvious that if P(x) = x, then P(P(x)) = x.

P(P(x)) = x is $A(Ax^2 + Bx + C)^2 + B(Ax^2 + Bx + C) + C = x$. This is evidently a quartic and two of its roots are those of $Ax^2 + (B - 1)x + C = 0$. We could obtain the quadratic for the other two roots by multiplying out P(P(x)) - x and factorising it. But it is sufficient to obtain the coefficients of x^4, x^3 and x^0. This gives us the sum of the four roots as -2B/A and their product as $(AC + B + 1)C/A^3$. The sum and product of the two known roots are -B/A - 1/A and C/A. Hence the sum and product of the other two roots are -B/A + 1/A and $(AC + B + 1)/A^2$, so the roots are the roots of the quadratic $A^2x^2 + (AB + A)x + (AC + B + 1) = 0$.

$y^2 - 3y + 2 = 0$ has roots 1 and 2. So these values are also roots of $(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0$. The other two roots are also the roots of $x^2 + (-3 + 1)x + (2 - 3 + 1) = 0$. These are obviously 0 and 2.

Problem B2

Find all solutions of the differential equation zz'' - 2z'z' = 0 which pass through the point x=1, z=1.

Solution

Answer: z = 1/(A(x - 1) + 1).

We have z''/z' = 2 z'/z. Integrating, ln z' = 2 ln z + const, so $z' = - A/z^2$. Integrating again: 1/z = Ax + B. But z(1) = 1, so B = 1 - A.

Problem B3

A horizontal disk diameter 3 inches rotates once every 15 seconds. An insect starts at the southernmost point of the disk facing due north. Always facing due north, it crawls over the disk at 1 inch per second. Where does it again reach the edge of the disk?

Solution

Answer: at the northernmost point of the disk.

Take polar coordinates with r = 3/2, θ = 0 at the start. The equations of motion are dr/dt = - cos θ, r dθ/dt = 2rπ/15 + sin θ.

Differentiating the second equation: $(dr/dt)(d\theta/dt) + r\, d^2\theta/dt^2 = (2\pi/15)\, dr/dt + (d\theta/dt)\cos\theta$. Substituting from the first equation, $2\,(dr/dt)(d\theta/dt) + r\, d^2\theta/dt^2 = (2\pi/15)\, dr/dt$. Multiplying through by r and integrating wrt t, we get $r^2\, d\theta/dt = (\pi/15)\, r^2 + C$, for some constant C. At t = 0, r = 3/2 and dθ/dt = 2π/15, so C = 3π/20. Thus $r^2\, d\theta/dt = (\pi/15)\, r^2 + 3\pi/20$. But the original equation gives $r^2\, d\theta/dt = 2r^2\pi/15 + r \sin\theta$. Hence $\pi r^2/15 + r\sin\theta = 3\pi/20$ (**). Hence if r = ±3/2, we have sin θ = 0 and hence θ = 0 or π.

That is not quite enough to show that the insect reaches the edge again at $\theta = \pi$. But we can treat (**) as a quadratic in r and solve to get $r^2 = (\sin^2\theta + (\pi/5)^2)^{1/2} - \sin\theta$. This shows that r first decreases, but then increases again to ± 1 at $\theta = \pi$. We can rule out -1 because r^2 is always positive, and r starts positive. So by continuity it must always remain positive. Thus the insect next reaches the edge at the northenmost point of the disk.

Problem B4
The parabola P has focus a distance m from the directrix. The chord AB is normal to P at A. What is the minimum length for AB?

Solution
Answer: $3\sqrt{3}$ m.

We may take the equation of P as $2my = x^2$. The gradient at the point A $(a, a^2/2m)$ is a/m, so the normal at (a, b) is $(y - a^2/2m) = -m/a\ (x - a)$. Substituting in $2my = x^2$, it meets P at (x, y) where $x^2 + 2m^2/a\ x - (2m^2 + a^2) = 0$, so the other point B has $x = -(2m^2/a + a)$.

Thus $AB^2 = (2a + 2m^2/a)^2 + 4m^2(1 + m^2/a^2)^2 = 4a^2(1 + m^2/a^2)^3$. Differentiating, we find the minimum is at $a^2 = 2m^2$ and is $AB^2 = 27m^2$.

Problem B5
Find the locus of the foot of the perpendicular from the center of a rectangular hyperbola to a tangent. Obtain its equation in polar coordinates and sketch it.

Solution
Answer: $r^2 = 2k^2 \sin 2\theta$. It is a figure of 8 with its axis along the line y = x and touching the x-axis and y-axis at the origin.

Take the hyperbola as $xy = k^2$. Then the tangent at $(a, k^2/a)$ is $(y - k^2/a) = - k^2/a^2\ (x - a)$. The perpendicular line through the origin is $y = a^2/k^2\ x$. They intersect at $x = 2k^2/(a\ (a^2/k^2 + k^2/a^2)\ )$, $y = 2a/(a^2/k^2 + k^2/a^2)$. So the polar coordinates r, θ satisfy $\tan\theta = y/x = a^2/k^2$, $r^2 = x^2 + y^2 = 4a^2(k^4/a^4 + 1)/(k^2/a^2 + a^2/k^2) = 4a^2(\cot^2\theta + 1)/(\tan\theta + \cot\theta)^2$ $= 4\ a^2 \cos^2\theta/(\sin^2\theta + \cos^2\theta)^2 = 4a^2\cos^2\theta = 4k^2\sin\theta\cos\theta = 2k^2 \sin 2\theta$. Thus the polar equation of the locus is $r^2 = 2k^2 \sin 2\theta$.

Problem B6
What is the shortest distance between the plane $Ax + By + Cz + 1 = 0$ and the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. You may find it convenient to use the notation $h = (A^2 + B^2 + C^2)^{-1/2}$, $m = (a^2A^2 + b^2B^2 + c^2C^2)^{1/2}$. What is the algebraic condition for the plane not to intersect the ellipsoid?

Solution
The tangent plane to the ellipsoid at (X, Y, Z) is $Xx/a^2 + Yy/b^2 + Zz/c^2 = 1$. It is parallel to $Ax + By + Cz + 1 = 0$ iff $X/a^2 = kA$, $Y/b^2 = kB$, $Z/c^2 = kC$ for some k. But $1 = X^2/a^2 + Y^2/b^2 + Z^2/c^2 = k^2(a^2A^2 + b^2B^2 + c^2C^2) = k^2m^2$, so $k = \pm 1/m$. There are two values corresponding to two parallel tangent planes (one on either side of the ellipse). The equation of the tangent plane is $k(Ax + By + Cz) = 1$.

The distance of the origin from the plane $Ax + By + Cz + 1 = 0$ is $1/(A^2 + B^2 + C^2)^{1/2} = h$. The distance of the origin from the tangent plane $k(Ax + By + Cz) = 1$ is $h/|k| = hm$. So if $m \geq 1$, the plane $Ax + By + Cz + 1 = 0$ lies between the two tangent planes and hence intersects the ellipse. So in this case the minimum distance is zero. If $m < 1$, then the distance between the plane $Ax + By + Cz + 1 = 0$ and the nearer tangent plane is $h(1 - m)$ and that is the required shortest distance.

2nd Putnam 1939

Problem A1
Let C be the curve $y^2 = x^3$ (where x takes all non-negative real values). Let O be the origin, and A be the point where the gradient is 1. Find the length of the curve from O to A.

Solution
Ans: $8/27 (2\sqrt{2} - 1)$. Trivial integration.

Problem A2
Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. Prove that the gradient at B is 4 times the gradient at A.

Solution
Trivial. [Take the point as (a,a^3). Write down the equation of the tangent. Write down its point of intersection with the curve: $(x^3 - a^3) = 3a^2(x - a)$. We know this has a repeated root $x = a$. The sum of the roots is zero, so the third root is $x = - 2a$. Finally, $3(-2a)^2 = 4$ times $3 a^2$.]

Problem A3
The roots of $x^3 + a x^2 + b x + c = 0$ are α, β and γ. Find the cubic whose roots are α^3, β^3, g^3.

Solution
$x^3 + (a^3 - 3ab + 3c) x^2 + (b^3 - 3abc + 3c^2)x + c^3 = 0.$

A routine manipulation. Suppose the roots are α, β, γ. Then $\alpha + \beta + \gamma = - a$, $\alpha\beta + \beta\gamma + \gamma\alpha = b$, $\alpha\beta\gamma = -c$. So to get the coefficients of the desired polynomial we have to find the corresponding expressions in the cubes: $\alpha^3 + \beta^3 + \gamma^3$ etc. You obviously start with $(\alpha + \beta + \gamma)^3$ etc and then add additional terms to get the desired expressions.

Problem A4
Given 4 lines in Euclidean 3-space:

L_1: $x = 1, y = 0$; L_2: $y = 1, z = 0$; L_3: $x = 0, z = 1$; L_4: $x = y, y = -6z$.

Find the equations of the two lines which both meet all of the L_i.

Solution
A routine computation. Assume the line meets L_1 at (1,0,a) and L_2 at (b,1,0). Then it is $(x - 1) = t(x - b)$, $y = t(y - 1)$, $(z - a) = tz$. So it can only cut L_3 if $1/b = 1 - a$, and L_4 if $6a = 6ab - 1$. This gives a quadratic for a, which we can solve to get $a = -1/2$ or $1/3$. Hence the possible lines are $(1,0,-1/2) + t(-1/3,1,1/2)$ and $(1,0,1/3) + t(1/2,1,-1/3)$.

Problem A5
Do either (1) or (2)
(1) x and y are functions of t. Solve $x' = x + y - 3$, $y' = -2x + 3y + 1$, given that $x(0) = y(0) = 0$.

(2) A weightless rod is hinged at O so that it can rotate without friction in a vertical plane. A mass m is attached to the end of the rod A, which is balanced vertically above O. At time $t = 0$, the rod moves away from the vertical with negligible initial angular velocity. Prove that the mass first reaches the position under O at $t = \sqrt{(OA/g)} \ln (1 + \sqrt{2})$.

Solution
(1) Differentiate the first equation, use the second equation to eliminate y', and then the (undifferentiated) first equation to eliminate y, giving:
$x'' - 4 x' + 5x = 10$. Solving: $x = 2 + A e^{2t} \sin t + B e^{2t} \cos t$. But $x(0) = 0$, so $B = -2$. The first equation now gives $y = x' - x + 3 = 1 + (A + 2) e^{2t} \sin t + (A - 2) e^{2t} \cos t$. But $y(0) = 1$, so $A = 1$. The final solution is thus: $x = 2 + e^{2t} \sin t - 2 e^{2t} \cos t$; $y = 1 + 3 e^{2t} \sin t - e^{2t} \cos t$.

(2) Trivial, except for the integral, which is moderately hard, unless you happen to know it.

Let the angle the rod makes with the (upward) vertical be θ. Conservation of energy gives immediately: 1/2 $OA^2(d\theta/dt)^2 = OA.g(1 - \cos\theta)$.
Now for the first time in this exam we come up against something that is not completely obvious. How do we do the integral?

You need the half-angle formulae, eg $(1 - \cos\theta) = 2\sin^2\theta/2$. Now if you can remember the integral for 1/sin z (eg ln sin z - ln(1 + cos z), or equivalently ln(cosec z - cot z), then you are home.

If not, use the half-angle formulae again: sin θ/2 = 2 sin θ/4 cos θ/4. Putting c = cos θ/4, we have to integrate $1/(c(1 - c^2))$. Expand using partial fractions and now the integral is just a sum of logs.

Problem A6
Do either (1) or (2):
(1) A circle radius r rolls around the inside of a circle radius 3r, so that a point on its circumference traces out a curvilinear triangle. Find the area inside this figure.

(2) A frictionless shell is fired from the ground with speed *v* at an unknown angle to the vertical. It hits a plane at a height *h*. Show that the gun must be sited within a radius $v/g\,(v^2 - 2gh)^{1/2}$ of the point directly below the point of impact.

Solution
(1) This is moderately difficult. It is not immediately obvious what coordinates to use (or at least, after meeting an integral I did not immediately recognize, I started worrying that there might be a better choice of coordinates), and it is not immediately obvious how to do the resulting integral.

Let C be the center of the large circle and let O be the initial point of contact between the two circles. Take O as the origin and OC as the x-axis, take the y-axis so that P the point of contact gets a positive y-coordinate just after rolling starts. The easiest parameter is to take angle OCP = θ. Then it is not hard to see that x/r = 3 - 2 cos θ - cos 2θ, y/r = 2 sin θ - sin 2θ.

Evidently we need something like ∫ y dx. We need a little care on the limits of integration. Let A, B be the other vertices of the curvilinear triangle (A corresponding to θ = 2π/3, B to θ = 4π/3). Let X be the point where the curve AB cuts the x-axis and Y the point where the line AB cuts the x-axis. $\int_O^A$ gives the area under the curve OA, in other words the area OAX plus the area AXY. Then $\int_A^X$ gives minus the area AXY (because x is decreasing, so dx is negative). $\int_X^B$ gives minus area BXY (x increasing, but y negative), and $\int_B^O$ gives plus area BXY plus area OBX (x decreasing and y negative). So the entire integral $\int_{\theta=0}^{2\pi}$ gives the required area inside the curvilinear triangle OAB. In other words we need:

$$2r^2 \int_{\theta=0}^{2\pi} (2\sin\theta - \sin 2\theta)(\sin\theta + \sin 2\theta)\, d\theta.$$

This is the point at which we are likely to get stuck. Changing variable to z = cos θ does not apparently help. The trick is to put things in terms of sin nx or cos nx. We may remember that $\sin^2 z = (1 - \cos 2z)/2$, and cos(w +/- z) = cos w cos z -/+ sin w sin z, so that sin z sin 2z = (cos z - cos 3z)/2.

Expanding the integrand gives: $2\sin^2\theta + \sin\theta\sin 2\theta - \sin^2 2\theta$. Using the two formulae above transforms this to: 1 - cos 2θ + (cos θ - cos 3θ)/2 - 1/2 + 1/2 cos 4θ. The cos terms all integrate to zero and the constant term 1/2 integrates to π, so the final answer is $2\pi r^2$.

(2) There is a slight trap here. We may be tempted to argue that the extreme case is where the shell reaches the plane with zero vertical velocity. The horizontal velocity does not change during the trajectory, so taking u as the horizontal velocity and w as the vertical, we can write down immediately that $w^2 = 2gh$ (energy) and hence the radius $r = w/g\,(v^2 - 2gh)^{1/2}$, which is unfortunately wrong. Notice that it is smaller than the answer sought. The reason is that impact at zero vertical velocity is not the extreme case. We can usually do better by having the shell peak before the plane, so that it hits it on the way down - the extra time to travel horizontally outweighs the loss of horizontal velocity.

So we have to write down the equations: r = tu, $h = tw - gt^2/2$. Squaring to eliminate u, w in favor of v, gives:

$$g^2t^4/4 + (gh - v^2)t^2 + (h^2 + r^2) = 0. \quad (*)$$

For this to have a real root we require $(gh - v^2)^2 \geq g^2(h^2 + r^2)$ and hence $r \leq v/g\, (v^2 - 2gh)^{1/2}$.

We are not asked to prove that the plane can be hit from anywhere within this radius, so we could stop here. But $v^2 \geq 2gh > gh$, so (*) is a quadratic of the form $a z^2 - b z + c = 0$, with a, b, c positive. Hence if r satisfies the condition, it has two *positive* roots for z and (*) has two real (positive) roots for t. Having solved for t, we can then solve for the angle (or equivalently for u, w), which shows that we can hit the plane from anywhere inside the radius.

Problem A7
Do either (1) or (2):
(1) Let C_a be the curve $(y - a^2)^2 = x^2(a^2 - x^2)$. Find the curve which touches all C_a for $a > 0$. Sketch the solution and at least two of the C_a.

(2) Given that $(1 - hx)^{-1}(1 - kx)^{-1} = \sum_{i\geq 0} a_i x^i$, prove that $(1 + hkx)(1 - hkx)^{-1}(1 - h^2x)^{-1}(1 - k^2x)^{-1} = \sum_{i\geq 0} a_i^2 x^i$.

Solution
(1) It is not hard to see that C_1 is a figure 8 lying on its side with the double point at (1,0), vertical tangents at (1,1) and (-1,1) and horizontal tangents at $(1/\sqrt{2}, 1/2)$, $(1/\sqrt{2}, 3/2)$, $(-1/\sqrt{2},1/2)$, $(-1/\sqrt{2},3/2)$. C_a is obtained from C_1 by transforming (x,y) to (ax,a^2y) (so stretching by a factor a in the x-direction and a factor a^2 in the y-direction). So we get a sequence of ever-larger horizontal 8s centered ever-further up the y-axis. This suggests that the envelope is something like a parabola $y = k x^2$.

At this point it is distinctly helpful to know more classical differential geometry than today's undergraduate. The classical method for finding the envelope for a 1-parameter family of curves (which *usually* works) is to differentiate wrt the parameter. Thus we eliminate a from:

$(y - a^2)^2 - x^2(a^2 - x^2) = 0$, and $4a(y - a^2) + 2a x^2 = 0$

giving $y = 3/4\, x^2$ (or the y-axis, which is presumably a spurious solution arising from the double points).

Alternatively, we might guess that that the envelope is a curve $y = k x^2$ for some k. The intersection of this with C_a is given by:
$(k^2 + 1)x^4 - (2k + 1) a^2x^2 + a^4 = 0$, or $x^2 = (2k + 1)a^2/(2k^2 + 2) +/- a^2(4k - 3)^{1/2}/(2k^2 + 2)$.

For this to be a tangent we need double roots and hence k = 3/4. It is now easy to check that this parabola meets C_a at $(2a/\sqrt{5},3a^2/5)$, $(-2a/\sqrt{5},3a^2/5)$ and to check that the gradients match.

(2) Fairly easy.

This is obviously not a general result (whereby we derive $\sum a_i^2x^i$ from $\sum a_ix^i$), so we need to evaluate the a_i. In fact, it is easily seen that $a_i = (h^{i+1} - k^{i+1})/(h - k)$. [For example, multiply the expansions of $(1 - hx)^{-1}$ and $(1 - kx)^{-1}$ to get $a_i = h^i + h^{i-1}k + ... + h k^{i-1} + k^i = (h^{i+1} - k^{i+1})/(h - k)$.]

Multiplying across to try to show that higher powers of $(1 - hkx)(1 - h^2x)(1 - k^2x) \sum a_i^2x^i$ have zero coefficients is a mistake (doable, but much algebra). It is better to evaluate $\sum a_i^2x^i$ directly. After substituting for a_i, we are going to get terms of the form $\sum z^i$ which evaluates immediately to $(1 - z)^{-1}$, giving the expression in the question in partial fraction form.

Problem B1
The points P(a,b) and Q(0,c) are on the curve y/c = cosh (x/c). The line through Q parallel to the normal at P cuts the x-axis at R. Prove that QR = b.

Solution
Trivial. [Let O be the origin. Then OR/OQ = sinh a/c, so $QR^2 = c^2(1 + \sinh^2 a/c)$, so QR = b.]

Problem B2
Evaluate $\int_1^3 ((x - 1)(3 - x))^{-1/2} dx$ and $\int_1^{inf} (e^{x+1} + e^{3-x})^{-1} dx$.

Solution
(1) π. Trivial. [Write $(x - 1)(3 - x) = 1 - (x - 2)^2$. So we have a standard $\sin^{-1}z$ integral.]

(2) $\pi/(4 e^2)$. Trivial. [Multiply top and bottom by e^x. Change variable to $y = e^{x-1}$. We now have a standard $\tan^{-1}z$ integral.]

Problem B3
Given $a_n = (n^2 + 1) 3^n$, find a recurrence relation $a_n + p a_{n+1} + q a_{n+2} + r a_{n+3} = 0$. Hence evaluate $\sum_{n\geq 0} a_n x^n$

Solution
We can solve formally to get the recurrence relation, but it is quicker to get there informally. We look for a relation between $b_n = a_n$, $b_{n+1} = a_{n+1}/3$, $b_{n+2} = a_{n+2}/9$, $b_{n+3} = a_{n+3}/27$, because that takes care of the powers of 3. So, ignoring the 3^n, we are looking at:

$n^2 + 1$
$n^2 + 2n + 2$
$n^2 + 4n + 5$
$n^2 + 6n + 10$

We try to get a linear combination of the first three which is constant. But that is easy: subtracting twice the second from the third gets rid of the n term, then adding the first gets rid of the n^2 term. So, $b_{n+2} - 2b_{n+1} + b_n = 2.3^n$. But $b_{n+3} - 2b_{n+2} + b_{n+1}$ has the *same* value, so subtracting:

$a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n = 0$, which is the required recurrence relation.

Let the power series sum to y. Then taking $y - 9x + 27x^2y - 27x^3y$ will give $a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n$ as the coefficient of x^{n+3}, so we need only worry about the early terms: $a_0 + (a_1 - 9a_0)x + (a_2 - 9a_1 + 27a_0)x^2 = (1 - 3x + 18x^2)$. Hence $y = (1 - 3x + 18x^2)/(1 - 9x + 27x^2 - 27x^3)$.

Using the ratio test, the original series evidently converges for $|x| < 1/3$, which may prompt us to notice that $1 - 9x + 27x^2 - 27x^3 = (1 - 3x)^3$.

That in turn may prompt us to try solving the problem backwards. We know that:

$1/(1 - z) = \sum z^n$; $1/(1 - z)^2 = \sum (n+1)z^n$; $1/(1 - z)^3 = \sum (n+1)(n+2)/2\ z^n$.

Hence $2/(1 - z)^3 - 3/(1 - z)^2 + 2/(1 - z) = \sum (n^2 + 1)z^n$. Replacing z by 3x gives $\sum a_n x^n = (1 - 3x + 18x^2)/(1 - 3x)^3$. Multiplying across by $(1 - 3x)^3$ now gives the required recurrence relation.

Problem B4
The *axis* of a parabola is its axis of symmetry and its *vertex* is its point of intersection with its axis. Find: the equation of the parabola which touches y = 0 at (1,0) and x = 0 at (0,2); the equation of its axis; and its vertex.

Solution
The general equation of the a parabola is: $(ax + by)^2 + cx + dy + e = 0$. Its intersection with y = 0 is given by $a^2x^2 + cx + e = 0$. This must have a double root, so $c = -2a^2$, $e = -a^2$. Considering the other tangent, we find: $d = -4b^2$, $e = 4b^2$. So (up to an irrelevant constant factor) we have: a = 2, b = 1, c = -8, d = -4, e = 4; or a = 2, b = -1, c = -8, d = -4, e = 4. But in the first case the equation can be written as $(2x + y - 2)^2 = 0$, which is a double line. It is debatable whether this qualifies as a parabola, but it would not normally be said to *touch* the points (1,0) and (0,2). So we are left with the parabola: $(2x - y)^2 - 8x - 4y + 4 = 0$.

We want to put this in the form $u = kv^2$. The line x + 2y = 0 is perpendicular to the line 2x - y = 0, so we change variables to X = 2x - y, Y = x + 2y, giving: $X^2 - 16/5\ Y - 12/5\ X + 4 = 0$, or $16/5\ (Y - 4/5) = (X - 6/5)^2$, which is the equation of a parabola with vertex X = 6/5, Y = 4/5, axis X = 6/5. Changing back to the original coordinates, x = (2X + Y)/5, y = (2Y - X)/5, the vertex is (16/25, 2/25) and the axis is 10x - 5y = 6.

Problem B5
Do either (1) or (2):

(1) Prove that $\int_1^k [x] f'(x) dx = [k] f(k) - \sum_1^{[k]} f(n)$, where $k > 1$, and $[z]$ denotes the greatest integer $\leq z$. Find a similar expression for: $\int_1^k [x^2] f'(x) dx$.

(2) A particle moves freely in a straight line except for a resistive force proportional to its speed. Its speed falls from 1,000 ft/s to 900 ft/s over 1,200 ft. Find the time taken to the nearest 0.01 s. [No calculators or log tables allowed!]

Solution

(1) $[x]$ is constant over the interval $[i, i+1)$ for i an integer, so we split the range of integration to get $\int_1^k = \int_{[k]}^k + \int_1^2 + \int_2^3 + \dots + \int_{[k]-1}^{[k]}$. We can write down each of these integrals, collect terms and get the result.

The same idea works the second integral, except that we divide at $\sqrt{2}, \sqrt{3}, \sqrt{4}, \dots, \sqrt{[k^2]}$, giving the result: $[k^2] f(k) - (f(1) + f(\sqrt{2}) + f(\sqrt{3}) + \dots + f(\sqrt{[k^2]}))$.

(2) Easy. 1.26 s.

The equation of motion is $x'' = - k x'$. Integrating: $x' = k A e^{-kt}$. Integrating again, and putting $x(0) = 0$, $x = A(1 - e^{-kt})$. Suppose T is the required time. Then from the speed $e^{-kT} = 0.9$. $x(T) = 1200$, so $A = 12000$. The initial speed is 1000, so $k = 1/12$ and $T = -12 \ln 0.9$. The only slight snag is that in the exam calculators did not exist and log tables were not allowed. So we have to use the expansion $\ln(1 + x) = x - x^2/2 + x^3/3 - \dots$, or more usefully: $- \ln(1 - x) = x + x^2/2 + x^3/3 + \dots$, giving $T = 1.2 + 0.06 + 0.004 + 0.0003 + \dots$ or 1.26 s.

Problem B6

Do either (1) or (2):

(1) f is continuous on the closed interval $[a, b]$ and twice differentiable on the open interval (a, b). Given $x_0 \in (a, b)$, prove that we can find $\xi \in (a, b)$ such that $((f(x_0) - f(a))/(x_0 - a) - (f(b) - f(a))/(b - a))/(x_0 - b) = f''(\xi)/2$.

(2) AB and CD are identical uniform rods, each with mass *m* and length 2*a*. They are placed a distance *b* apart, so that ABCD is a rectangle. Calculate the gravitational attraction between them. What is the limiting value as *a* tends to zero?

Solution

(1) We obviously have to use the mean value theorem. So we need to construct a suitable auxiliary function to apply it to. It is usually easiest to apply the MVT in cases where the function has equal values at the two ends of the interval. So we want to find some function g, such that g' has equal values at two different points. Let the value of the expression given, $((f(x_0) - f(a))/(x_0 - a) - (f(b) - f(a))/(b - a))/(x_0 - b)$, be y_0. Then we are looking for $g''(x)$ to be something like $1/2 f''(x) - y_0$.

Let us start by rearranging the expression for y_0 to give $f(x_0) = f(a) + (f(b) - f(a))(x_0 - a)/(b - a) + y_0(x_0 - a)(x_0 - b)$. After a little experimentation we may try looking at:

$$g(x) = f(x) - f(a) - (f(b) - f(a))(x - a)/(b - a) - y_0(x - a)(x - b).$$

We notice that $g(a) = 0$, $g(b) = 0$, $g(x_0) = 0$, $g'(x) = f'(x) - (f(b) - f(a))/(b - a) - y_0(2x - a - b)$, $g''(x) = f''(x) - 2 y_0$. At this point we should realize that we are home, because we have to show that we can find ξ such that $g''(\xi) = 0$. But the mean value theorem gives us a value in the interval (a, x_0) at which g' is zero and another in (x_0, b). Hence there must be a value between the two (and *a fortiori* in (a, b)) at which g'' is zero.

(2) Straightforward, apart from an awkward integral. Answer: $Gm^2(1 - (1 + 4a^2/b^2)^{1/2})/(2a^2)$, which tends to Gm^2/b^2 as the rods shorten to become point masses.

By symmetry the net force must be perpendicular to the rods, so we just calculate the perpendicular component. Take coordinates x along one rod and y along the other. Then we can write down immediately that the perpendicular component is:

$$Gm^2/(4a^2) \int (b^2 + (x - y)^2)^{-1} b (b^2 + (x - y)^2)^{-1/2} dx\, dy.$$

The integrand is of the form $(1 + z^2)^{-3/2}$. It helps to know that this integrates to $z/(1 + z^2)^{1/2}$ (as is easily checked). People used to mess around with trigonometric substitutions (eg tan θ reduces it to cos θ, which integrates immediately to sin θ, which one has to remember to think of as tan θ/sec θ). But it was always easier simply to learn a large number of integrals by rote. Nowadays, of course, one tends to look them up or use Mathematica/Maple.

So integrating first by y, we get $Gm^2/(4a^2) \int dx/b \{ (x - 2a)/b (1 + (x - 2a)^2/b^2)^{-1/2} - x/b (1 + x^2/b^2)^{-1/2} \}$. This is much less horrible than it looks, because we are just faced with integrating $z/(1 + z^2)^{1/2}$ which is obviously $(1 + z^2)^{1/2}$. So we get the expression above. For the limit, just expand the square root as a power series $1 + 2a^2/b^2 - a^4/(4b^4) + \dots$.

Problem B7

Do either (1) or (2):

(1) Let $a_i = \sum_{n=0}^{\infty} x^{3n+i}/(3n+i)!$ Prove that $a_0^3 + a_1^3 + a_2^3 - 3 a_0a_1a_2 = 1$.

(2) Let O be the origin, λ a positive real number, C be the conic $ax^2 + by^2 + cx + dy + e = 0$, and C_λ the conic $ax^2 + by^2 + \lambda cx + \lambda dy + \lambda^2 e = 0$. Given a point P and a non-zero real number k, define the transformation D(P,k) as follows. Take coordinates (x',y') with P as the origin. Then D(P,k) takes (x',y') to (kx',ky'). Show that D(O,λ) and D(A,-λ) both take C into C_λ, where A is the point (-cλ/(a(1 + λ)), -dλ/(b(1 + λ))). Comment on the case λ = 1.

Solution

(1) Moderately hard, unless differentiating is a reflex, in which case it is easy.

The series are all absolutely convergent for all x, so we can carry out whatever operations we want. But it is not at all obvious what to do. Multiplying out the power series to get a complicated sum for the coefficient of x^n is offputting. Much scope for algebraic error, and no guarantee that the eventual simplification will be obvious. So we look for some trick. The a_i are closely related to the exponential series, eg $a_0 + a_1 + a_2 = e^x$, and the prevalence of 3 in the question may eventually suggest looking at ω, the cube root of 1. Indeed, $a_0 + \omega a_1 + \omega^2 a_2 = e^{\omega x}$. Since ω^2 is also a root, we also have $a_0 + \omega^2 a_1 + \omega a_2 = e^{\omega 2x}$. Remembering that $1 + \omega + \omega^2 = 0$ might prompt us to multiply these three expressions together, but it helps to remember the product of the three left hand sides: $(a_0 + a_1 + a_2)(a_0 + \omega a_1 + \omega^2 a_2)(a_0 + \omega^2 a_1 + \omega a_2) = a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2$. Otherwise, you are faced with multiplying this out the hard way: after collecting terms, you get $(a_0^3 + a_1^3 + a_2^3)$, then 6 expressions of the type $a_0^2(1 + \omega + \omega^2)$, which are all zero, and $3a_0a_1a_2(\omega + \omega^2)$, which is $-3a_0a_1a_2$.

A more general approach, which is more likely to work, is to differentiate the expression $a_0^3 + a_1^3 + a_2^3 - 3 a_0a_1a_2$. Provided you notice that $a_0' = a_2$ etc, this gives the result almost immediately (the derivative is zero, so the expression must be constant, but its value for x = 0 is 1).

(2) Trivial. [D(O,λ) takes (x,y) to (λx,λy). So if (x,y) satisfies the equation for C, just check that (λx,λy) satisfies the equation for C_λ. Similarly, D(A,-λ) takes (x,y) to (-λx - λc/a, -λy - λd/b). Again, just check by substituting this into the equation for C_λ and using the fact that (x,y) satisfies the equation for C. If λ = 1, then C_λ = C, D(O,λ) is the identity transformation, and D(A,-λ) the central symmetry.]

3rd Putnam 1940

Problem A1
p(x) is a polynomial with integer coefficients. For some positive integer c, none of p(1), p(2), ... , p(c) are divisible by c. Prove that p(b) is not zero for any integer b.

Solution
Suppose p(b) = 0. Then p(x) = (x - b)q(x), where q is a polynomial with integer coefficients. Put b = cd + r, where $1 \le r \le c$ (note that this is different from the conventional $0 \le r < c$, but still possible because 1, 2, ... c are a complete set of residues mod c). Then p(r) = p(b - cd) = - cd q(r) which is divisible by c. Contradiction.

Problem A2
y = f(x) is continuous with continuous derivative. The arc PQ is concave to the chord PQ. X is a point on the arc PQ for which PX + XQ is a maximum. Prove that XP and XQ are equally inclined to the tangent at X.

Solution
Take s to be the arc-length PX and z to be PX + XQ. Suppose the tangent is WXY, so that angle WXP = θ and angle YXQ = φ. Then to first order if we vary X, the change $\delta z = \delta s (\cos \theta - \cos \varphi)$, which is can be made positive (by choosing the sign of δs appropriately), thus contradicting the maximality of X, unless $\theta = \varphi$.

The official solution. If X is a point of S such that PX + XQ ≥ PY + YQ for all points Y of S, then we can easily show that S lies in the half-plane bounded by external angle bisector of PXQ. If X lies on PQ, then S must be a subset of the segment PQ and the result is trivial. So assume it does not. Now reflect Q in the bisector to get Q', with PXQ' a straight line. Then if Y is any point in the same half-plane as Q', we have YQ' < YQ and hence PY + YQ > PY + YQ' ≥ PQ' = PX + XQ' = PX + XQ, so Y is not in S.

This argument depends upon X achieving a global maximum. The original wording of the question, "a point ... for which ... is a maximum" (which on this point I quoted exactly), is somewhat ambiguous. Does it mean a local or a global maximum? If it means a local, then we have to take S to be a small arc for which the maximum is global. Then that arc lies on one side of the line. Since it also has a point in common (X) and is differentiable, the line must be the tangent to the arc at X.

This solution is somewhat harder and less obvious (unless you have seen it before), so I prefer the simpler solution.

Problem A3
α is a fixed real number. Find all functions f: R → R (where R is the reals) which are continuous, have a continuous derivative, and satisfy $\int_b^y f^\alpha(x)\,dx = (\int_b^y f(x)\,dx)^\alpha$ for all y and some b.

Solution
Straightforward to get the basic idea, but care is needed with the details. It is quite hard to get the answer exactly right (the official solution gives a spurious solution for the case (2) - overlooking the problem with the integration limits).

Answer:
(1) No solutions for $\alpha = 0$;
(2) No solutions for $\alpha < 0$;
(3) Any continuous f with continuous derivative is a solution for $\alpha = 1$;
(4) For $\alpha > 0$ and not of the form p/q, with p and q odd positive integers, $f = A e^{kx}$, where A is any real, and k is the positive real value of $\alpha^{1/(\alpha-1)}$;
(5) For $\alpha = p/q$, with p and q odd positive integers (but not both 1), $f = A e^{kx}$ or $A e^{-kx}$, where A is any real, and k is the positive real value of $\alpha^{1/(\alpha-1)}$.

Case (3) is obvious. Case (1) is almost obvious: if $\alpha = 0$, then the lhs varies with y (= y - b), but the rhs does not (= 1), so there are no solutions. Assume now that α is not 0 or 1.

Differentiate wrt y and put $g(y) = \int_b^y f(x)\,dx$. Then f = g' and we have $f^\alpha = \alpha g^{\alpha-1} f$. So f = k g, where $k = \alpha^{1/(\alpha-1)}$. Since f = g', we can integrate immediately to get $f(x) = A e^{kx}$ (*).

However, we have to consider how many real values k can have. If $\alpha > 0$, then k certainly has a positive value, but we can also take the corresponding negative value if $1/(\alpha - 1)$ involves an even root, in other words if $\alpha = p/q$ with p and q both odd. Finally, (*) is clearly necessary, but not necessarily sufficient, so we have to subsitute (*) back into the original equation. For $k > 0$, we find the solution works provided $b = -\infty$. If $k < 0$, then it works with $b = \infty$. This gives cases (4) and (5). [Note, however, that we can only allow A negative for α not of the form p/q with p and q odd, provided we are content for both sides of the original equation to have complex values (even though f is real valued).]

For $\alpha < 0$, all values of $\alpha^{1/(\alpha -1)}$ are complex unless $\alpha = - p/q$ with p an even positive integer and q an odd positive integer. If α has that form, then we may take $k = - 1/|\alpha|^{1/(|\alpha|+1)}$. But now there is a problem with b. Taking A = 1 for simplicity, the $f^\alpha(x) = e^{|\alpha k|x}$, so the lhs = const $(e^{|\alpha k|y} - e^{|\alpha k|b})$ and the rhs = const $/(e^{-|k|y} - e^{-|k|b})^{|\alpha|}$. The constants are the same, but we need $b = -\infty$ to get rid of the b term on the lhs and $b = \infty$ to get rid of the term on the rhs. So there are no solutions in case (2).

Problem A4
p is a positive constant. Let R is the curve $y^2 = 4px$. Let S be the mirror image of R in the y-axis ($y^2 = - 4px$). R remains fixed and S rolls around it without slipping. O is the point of S initially at the origin. Find the equation for the locus of O as S rolls.

Solution
Answer: $x(x^2 + y^2) + 2p\, y^2 = 0$.

Take the point of contact as (X, Y), so $Y^2 = 4pX$. The tangent at to R at this point has gradient 2p/Y, and hence has equation $(y - Y) = 2p/Y (x - X)$. The perpendicular to the tangent through the origin has equation $x = - 2p/Y\, y$. If (x, y) is their point of intersection, then (2x, 2y) is the point O (since S in its new position is the reflection of R in the tangent).

Solving for x, y: $y = Y^3/(2(Y^2 + 4p^2))$, $x = - 2p/Y = - pY/(Y^2 + 4p^2)$. So the point O is $(-2pY^2/(Y^2 + 4p^2), Y^3/(Y^2 + 4p^2))$ (*). Using $Y = -2py/x$, we get $x = -2py^2/(x^2 + y^2)$, or $x(x^2 + y^2) + 2p\, y^2 = 0$ (**). We have shown that the locus is given by (*) and that all points on (*) are on (**). However, we must check that (**) does not include additional points. Writing (**) as $y^2 = - x^3/(x + 2p)$, shows that for each value of x in the range (-2p, 0) there are exactly two possible values of y, and the only other point on (**) is the origin. Inspection shows that the same is true for (*), so the two expressions are equivalent.

Problem A5
Prove that the set of points satisfying $x^4 - x^2 = y^4 - y^2 = z^4 - z^2$ is the union of 4 straight lines and 6 ellipses.

Solution
$x^4 - x^2 = (x^2 - 1/2)^2 - 1/4$, so $x^4 - x^2 = y^4 - y^2$ is equivalent to $(x^2 - 1/2)^2 = (y^2 - 1/2)^2$ and hence to $x^2 - 1/2 = +/- (y^2 - 1/2)$, which is equivalent to $x = y$, or $x = -y$, or $x^2 + y^2 = 1$. Similarly, for $y^4 - y^2 = z^4 - z^2$. So the equation given is equivalent to: (1) $x = y$ and $y = z$, or (2) $x = y$ and $y = -z$, or (3) $x = -y$ and $y = z$, or (4) $x = -y$ and $y = -z$, or (5) $x = y$ and $y^2 + z^2 = 1$, or (6) $x = -y$ and $y^2 + z^2 = 1$, or (7) $x^2 + y^2 = 1$ and $y = z$, or (8) $x^2 + y^2 = 1$ and $y = -z$, or (9) $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

Clearly (1) - (4) are straight lines. (5) is the intersection of a plane and a cylinder, which is an ellipse. Similarly (6), (7) and (8). (9) is slightly harder to see. If one's visualization is good, then one can see that the intersection of two cylinders with the same radius and axes intersecting at right angles is two perpendicular ellipses with a common minor axis. Otherwise, subtracting the two equations we see that x = z or -z and the intersection is also given by the intersection of a cylinder with two planes.

Problem A6
p(x) is a polynomial with real coefficients and derivative $r(x) = p'(x)$. For some positive integers a, b, $r^a(x)$ divides $p^b(x)$. Prove that for some real numbers A and α and for some integer n, we have $p(x) = A(x - \alpha)^n$.

Solution
Write $p(x) = A \prod (x - \alpha_i)^{n_i}$.
Then $r(x) = p(x) \sum n_i/(x - \alpha_i) = (A \prod (x - \alpha_i)^{n_i-1})\, q(x)$, where no α_i is a root of q(x). This is easily seen, because q(x)

is a sum of terms, all but one of which has a factor $(x - \alpha_i)$. But $q(x)$ divides $p(x)^b$ which has no roots except the α_i. Hence $q(x)$ must be a constant. But now the degree of $r(x)$ is wrong unless there is just one α_i.

Problem A7

a_i and b_i are real, and $\sum_1^\infty a_i^2$ and $\sum_1^\infty b_i^2$ converge. Prove that $\sum_1^\infty (a_i - b_i)^p$ converges for $p \ge 2$.

Solution

Notice first that it is sufficient to prove the result for p = 2. For that is equivalent to the statement that $\sum |a_i - b_i|^2$ converges. Hence for sufficiently large i, $|a_i - b_i| < 1$, and hence $|a_i - b_i|^p \le |a_i - b_i|^2$. So $\sum (a_i - b_i)^p$ is absolutely convergent and hence convergent.

$(a_i - b_i)^2 = a_i^2 - 2a_ib_i + b_i^2$. The only tricky part is the middle term. It may be positive, so we cannot simply argue that $0 \le (a_i - b_i)^2 \le a_i^2 + b_i^2$. However, it is true that $0 \le (a_i - b_i)^2 = 2a_i^2 + 2b_i^2 - (a_i + b_i)^2 \le 2a_i^2 + 2b_i^2$. That suffices, since $\sum a_i^2$ and $\sum b_i^2$ are absolutely convergent, hence also $\sum (2a_i^2 + 2b_i^2)$.

Problem A8

Show that the area of the triangle bounded by the lines $a_ix + b_iy + c_i = 0$ (i = 1, 2, 3) is $\Delta^2/|2(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)|$, where Δ is the 3 x 3 determinant with columns a_i, b_i, c_i.

Solution

Fairly easy if you remember some formulae for determinants (which people did in those days). Of course, you can just slog through the expression (*) below in terms of a_i, b_i, c_i. That is doable, but completely mindless, the only required skill is doing elementary algebra fast without mistakes. I suppose I am fairly out of sympathy with the rather common Putnam style of problem where the basic idea is obvious, but you have to be skilful at evaluating integrals, determinants etc, often using tricks.

Take A_i be the cofactor of a_i in Δ. Similarly, B_i and C_i. [So, for example, $A_1 = b_2c_3 - b_3c_2$, $A_2 = b_3c_1 - b_1c_3$, $A_3 = b_1c_2 - b_2c_1$.]

The lines $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ intersect at $(A_1/C_1, B_1/C_1)$. Similarly, the other two points of intersection are $(A_2/C_2, B_2/C_2)$ and $(A_3/C_3, B_3/C_3)$.

The area of the triangle is therefore (the absolute value of) the determinant K with rows $A_1/C_1, B_1/C_1, 1; A_2/C_2, B_2/C_2, 1; A_3/C_3, B_3/C_3, 1$. (*) [For example, take a z-coordinate perpendicular to the plane, and take the cross product of the vectors along two sides.] But $K |C_1C_2C_3|$ is the determinant whose elements are the cofactors of the original determinant. This has value equal to Δ^2. For example, on multiplying it by Δ, we get Δ down the diagonal and zeros elsewhere, and hence Δ^3.

Problem B1

A stone is thrown from the ground with speed v at an angle θ to the horizontal. There is no friction and the ground is flat. Find the total distance it travels before hitting the ground. Show that the distance is greatest when $\sin \theta \ln (\sec \theta + \tan \theta) = 1$.

Solution

Take coordinates (time and distance) zeroed on the peak of the trajectory. After time t, the stone travels a distance $x = v t \cos \theta$ horizontally and a distance $1/2\ g t^2$ vertically. So the trajectory is a parabola $2y = k x^2$, where $k = g/(v \cos \theta)^2$. The stone is on the ground at $t = +/- v/g \sin \theta$, at a horizontal distance +/- a from the peak, where $a = v^2/g \sin \theta \cos \theta$.

The length of the parabola $y = k/2\ x^2$ between x = -a and x = a is $2 \int_0^a (1 + k^2x^2)^{1/2} dx$.

To do the integral it helps to remember that $1 + \sinh^2 z = \cosh^2 z$. So substituting $kx = \sinh z$, will essentially give us the integral of $\cosh^2 z$. That is doable, using the analog of the double angle formulae. So setting $I = \int (1 + k^2x^2)^{1/2} dx$, and substituting $x = \sinh z$, we have $I = 1/k \int \cosh^2 z\, dz = 1/(2k) \int (\cosh 2z + 1) dz = 1/(4k) \sinh 2z + 1/(2k) z = 1/(2k) \sinh z \cosh z + 1/2 z = x/2 (1 + k^2x^2)^{1/2} + 1/(2k) \sinh^{-1}(kx)$. We have $k = g/(v \cos \theta)^2$, and $a = v^2/g \sin \theta \cos \theta$, so $ka = \tan \theta$. So the required path length is $p(\theta) = a(1 + k^2a^2)^{1/2} + 1/k \sinh^{-1}a = v^2/g (\sin \theta + \cos^2\theta \sinh^{-1} \tan \theta)$.

To find the maximum, we differentiate, getting $p'(\theta) = \cos\theta - 2\cos\theta\sin\theta\sinh^{-1}\tan\theta + \cos^2\theta(1 + \tan^2\theta)^{-1/2}\sec^2\theta = 2\cos\theta(1 - \sin\theta\sinh^{-1}\tan\theta)$. In the range $[0, \pi/2]$, $\tan\theta$ is monotone increasing. $\sinh^{-1}z$ is strictly monotone increasing for positive z, so $\sinh^{-1}\tan\theta$ is strictly monotone increasing on $(0, \pi/2)$. Indeed it evidently tends to ∞ as θ tends to $\pi/2$. Hence $(1 - \sin\theta\sinh^{-1}\tan\theta)$ is strictly monotone decreasing on $(0, \pi/2)$ and crosses zero once. $\cos\theta$ is monotone decreasing and positive, so $p'(\theta)$ is strictly monotone decreasing and crosses zero once. Hence $p(\theta)$ has a single maximum on $[0, \pi/2]$, which is achieved for the value φ in $(0, \pi/2)$ for which $(1 - \sin\varphi\sinh^{-1}\tan\varphi) = 0$. Rearranging, $\sinh(1/\sin\varphi) = \tan\varphi$. Squaring, adding 1, and taking the square root: $\cosh(1/\sin\varphi) = \sec\varphi$. Adding the last two equations: $e^{1/\sin\varphi} = \tan\varphi + \sec\varphi$, or $1/\sin\varphi = \ln(\tan\varphi + \sec\varphi)$, or $\sin\varphi\ln(\tan\varphi + \sec\varphi) = 1$.

Problem B2

C_1, C_2 are cylindrical surfaces with radii r_1, r_2 respectively. The axes of the two surfaces intersect at right angles and $r_1 > r_2$. Let S be the area of C_1 which is enclosed within C_2. Prove that $S = 8r_2^2A = 8r_1^2C - 8(r_1^2 - r_2^2)B$, where $A = \int_0^1 (1 - x^2)^{1/2}(1 - k^2x^2)^{-1/2}\,dx$, $B = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{-1/2}\,dx$, and $C = \int_0^1 (1 - x^2)^{-1/2}(1 - k^2x^2)^{1/2}\,dx$, and $k = r_2/r_1$.

Solution

It is hard to see the point of the second half [$8r_2^2A = 8r_1^2C - 8(r_1^2 - r_2^2)B$], which is trivial. Multiply top and bottom of the integrand in A by $(1 - x^2)^{1/2}$, so that we get $(1 - x^2)$ on the top. Then note that $k^2(1 - x^2) = 1 - k^2x^2 + (k^2 - 1)$.

For the first half, the part of C_1 enclosed inside C_2 comprises two bent ovals, one at each end. It is tempting to think that when rolled flat, each piece is an ellipse. If that were true, then the problem would be trivial - the semi-axes are r_2 and $r_1\sin^{-1}k$, so the total area would be $2\pi\, r_1r_2\sin^{-1}k$. But conics only remain conics when projected onto flat surfaces and here we are projecting onto a curved surface.

So we have to use integration. Take C_1 to be $x^2 + y^2 = r_1^2$, and C_2 to be $y^2 + z^2 = r_2^2$. We calculate the area of the quarter of the piece with $x > 0$ having $y, z > 0$. We may divide it into strips parallel to the z-axis. Take the angle θ to be the axial angle - the angle by which the strip has to be rotated about the z-axis from the $y = 0$ position (keeping in the surface C_1). [Points on the strip have the same x and y coordinates, but varying z-coordinates.] The strip has y-coordinate $r_1\sin\theta$ and hence length $(r_2^2 - r_1^2\sin^2\theta)^{1/2}$ and width $r_1 d\theta$. Thus the area of the quarter-oval is $\int_0^{\sin-1k} (r_2^2 - r_1^2\sin^2\theta)^{1/2}\, r_1\, d\theta$. Put $t = 1/k \sin\theta$, and the integral becomes $r_2^2 \int_0^1 (1 - t^2)^{1/2} (1 - k^2t^2)^{-1/2}\,dt$. Hence the complete oval is 4 times this, and the total area 8 times.

Comments. The original question was badly worded, because at first sight it appeared to be about volumes, not areas, which may have confused some. It is important to get a clear picture of the geometry - the easiest way to do this is to roll a piece of paper into a cylinder.

Problem B3

Let p be a positive real, let S be the parabola $y^2 = 4px$, and let P be a point with coordinates (a, b). Show that there are 1, 2 or 3 normals from P to S according as $4(2p - a)^2 + 27\,pb^2 >, = \text{or} < 0$.

Solution

The general point on the parabola is $(pt^2, 2pt)$. The slope of the tangent is $2p/y = 1/t$, and so the slope of the normal is $-t$. Hence the equation of the normal is $(y - 2pt) = -t(x - pt^2)$. This passes through (a, b) iff $pt^3 + (2p - a)t - b = 0$ (*).

This is a cubic, so it has 1, 2 or 3 real roots. We have to decide which. If $(2p - a) > 0$, then $pt^3 + (2p - a)t$ is strictly increasing and takes all values in $(-\infty, \infty)$, so (*) has just one real root. The same is true if $(2p - a) = 0$, unless $b = 0$, in which case there are three coincident roots.

If $(2p - a) < 0$, then $pt^3 + (2p - a)t$ has a maximum and a minimum. Differentiating, we find that these are at $t = \pm\sqrt{(-(2p - a)/(3p))}$, with values $\pm 2(2p - a)/3\sqrt{(-(2p - a)/(3p))}$. So there are 1, 2 or 3 real roots according as $b^2 >, = \text{or} < 4(2p - a)^2/9\ -(2p - a)/3p$, which is the condition in the question. Note that if $(2p - a) \ge 0$, then this expression is certainly > 0, so the same rule applies.

Problem B4

Let S be the surface $ax^2 + by^2 + cz^2 = 1$ (a, b, c all non-zero), and let K be the sphere $x^2 + y^2 + z^2 = 1/a + 1/b + 1/c$ (known as the *director* sphere). Prove that if a point P lies on 3 mutually perpendicular planes, each of which is tangent to S, then P lies on K.

Solution
Let $f(x, y, z) = ax^2 + by^2 + cz^2$. Then the normal vector is the vector grad f, so the tangent plane at (u, v, w) is a.u.x + b.v.y + c.w.z = 1.

Note that this does not pass through the origin. The general plane not through the origin has equation $\mathbf{p}.(\mathbf{x} - \lambda\mathbf{p}) = 0$ or $\mathbf{p.x} = \lambda$, where $\mathbf{x}$ is the vector (x, y, z) representing a general point on the plane, $\mathbf{p}$ = (p, q, r) is a unit vector normal to the plane, and $\lambda > 0$ is the distance of the plane from the origin. If this is a tangent plane at some point of the quadric, then $a(p/\lambda a)^2 + b(q/\lambda b)^2 + c(r/\lambda c)^2 = 1$, or $p^2/a + q^2/b + r^2/c = \lambda^2$.

So suppose P is the point of intersection of three perpendicular tangent planes $p_ix + q_iy + r_iz = \lambda_i$, i = 1, 2, 3, where $\mathbf{p}_i$ are orthonormal vectors. The squared distance of P from the origin is $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = (p_1^2 + p_2^2 + p_3^2)/a + (q_1^2 + q_2^2 + q_3^2)/b + (r_1^2 + r_2^2 + r_3^2)/c = 1/a + 1/b + 1/c$, since $\mathbf{p}_i$ are orthonormal. [This is the key trick: if the rows of a matrix are orthonormal vectors, then the matrix is orthogonal and hence its *columns* are also orthonormal vectors.].

Problem B5
Find all rational triples (a, b, c) for which a, b, c are the roots of $x^3 + ax^2 + bx + c = 0$.

Solution
Answer: (0, 0, 0); (1, -1, -1), (1, -2, 0).

We require (1) a + b + c = -a, (2) ab + bc + ca = b, and (3) abc = -c.
From (3), either c = 0, or ab = -1. If c = 0, then (1) becomes b = -2a, and (2) becomes b(a - 1) = 0. Hence either a = b = 0, or a = 1, b = -2.
So assume $c \neq 0$, and ab = -1. (1) becomes c = - b - 2a. Substituting in (2), we get: -1 - (2a + b)(a + b) = b, so $-a^2 - 2a^4 + 3a^2 - 1 = -a$, or $2a^4 - 2a^2 - a + 1 = 0$. So a = 1, or $2a^3 + 2a^2 - 1 = 0$ (*). The first possibility gives a = 1, b = -1, c = -1. Suppose a = m/n is a root of (*) with m, n relatively prime integers. Then $2m^3 + 2m^2n - n^3 = 0$. So any prime factor of n must divide 2 and any prime factor of m must divide 1. Hence the only possibilities are a = 1, -1, 1/2, -1/2, and we easily check that these are not solutions. So (*) has no rational roots.

Problem B6
The n x n matrix (m_{ij}) is defined as $m_{ij} = a_ia_j$ for $i \neq j$, and $a_i^2 + k$ for i = j. Show that $\det(m_{ij})$ is divisible by k^{n-1} and find its other factor.

Solution
Answer: $\det(m_{ij}) = k^{n-1}(k + \Sigma a_i^2)$.

Induction on n. Clearly true for n = 1.
Expanding by the first row we get $k.\ k^{n-2}(k + \Sigma_{i>1}a_i^2) + \det(m'_{ij})$, where m'_{ij} is the same as m_{ij} except that $m'_{11} = a_1^2$. Subtracting appropriate multiples of the first row from the others we zero all the elements outside the first row except those on the diagonal, which become k. Hence $\det(m'_{ij}) = k^{n-2}a_1^2$.

Problem B7
Given n > 8, let $a = \sqrt{n}$ and $b = \sqrt{(n+1)}$. Which is greater a^b or b^a?

Solution
Answer: a^b is greater.

$a^b = e^{b \ln a}$ and $b^a = e^{a \ln b}$. So we have to decide which of b ln a and a ln b is greater, or, equivalently, which of (ln a)/a and (ln b)/b is greater. The latter is clearly more promising. So set f(x) = (ln x)/x. Then $f'(x) = 1/x^2 - (\ln x)/x^2$ which is negative for x > e. Obviously b > a, so provided a > e, (ln a)/a > (ln b)/b and hence b ln a > a ln b and $a^b > b^a$. But $e^2 < 9$, so the result is certainly true for $n \geq 9$.

4th Putnam 1941

Problem A1
Prove that $(a - x)^6 - 3a(a - x)^5 + 5/2\ a^2(a - x)^4 - 1/2\ a^4(a - x)^2 < 0$ for $0 < x < a$.

Solution
Change variables to $t = 1 - x/a$ and the polynomial becomes $a^6(t^6 - 3t^5 + 5/2\ t^4 - 1/2\ t^2)$. This obviously has a factor t^2, and almost obviously $(t - 1)$. Dividing these out, we see that the resulting cubic has another factor $(t - 1)$. So we can write the original as $a^6t^2(t - 1)^2(t(t - 1) - 1/2)$, which evidently has the same sign as $t(t - 1) - 1/2$. But that is clearly negative for t between 0 and 1.

Problem A2
Define $f(x) = \int_0^x \sum_{i=0}^{n-1} (x - t)^i / i!\ dt$. Find the nth derivative $f^{(n)}(x)$.

Solution
Note that x appears both in the integrand and in the limits, so a little care is needed. Write $g_r(x, t) = \sum_{i=0}^{r-1} (x - t)^i / i!$ so that $f(x) = \int_0^x g_n(x, t) e^{nt}\ dt$. By definition $f'(x) = \lim_{\delta x \to 0} (\int_0^{x+\delta x} g_n(x+\delta x, t) e^{nt}\ dt - \int_0^x g_n(x, t) e^{nt}\ dt) / \delta x = e^{nx} g_n(x, x) + \int_0^x g_n'(x,t) e^{nt}\ dt$, where the ' denotes the partial derivative wrt the first variable. But $g_n(x, x) = 1$, and $g_r'(x, t) = g_{r-1}(x, t)$, so $f'(x) = e^{nx} + \int_0^x g_{n-1}(x, t) e^{nt}\ dt$. Hence by an easy induction $f^{(n)}(x) = e^{nx}(1 + n + n^2 + ... + n^{n-1})$.

Problem A3
A circle radius a rolls in the plane along the x-axis the envelope of a diameter is the curve C. Show that we can find a point on the circumference of a circle radius a/2, also rolling along the x-axis, which traces out the curve C.

Solution
Consider a circle radius a/2 with center initially at (0,a/2) rolling along the x-axis. After rolling through an angle θ, the point initially at (0,a) is at $a/2 \sin \theta$, $a/2 \cos \theta$ relative to the center and hence at $P\ (a\theta/2 + (a/2) \sin \theta, a/2 + (a/2) \cos \theta)$. The tangent at P is $(y - (a/2)(1 + \cos \theta)/(x - (a/2)(\theta + \sin \theta) = (-\sin \theta)/(1 + \cos \theta)$, or $x \sin \theta + y(1 - \cos \theta) = (a/2)(\theta \sin \theta + 2 \cos \theta + 2)$.

If we put $\theta = 2\varphi$, then $\sin \theta = 2 \sin \varphi \cos \varphi$, $1 + \cos \theta = 2 \cos^2\varphi$, so the tangent at P has equation, $x \sin \varphi + y \cos \varphi = a(\varphi \sin \varphi + \cos \varphi)$.

Now consider the circle radius a with center initially at (0,a). When it has rolled through an angle φ, its center is at $(a\varphi, a)$, so the diameter which is initially horizontal lies on the line $(y-a)/(x-a\varphi) = -\tan \varphi$, or $x \sin \varphi + y \cos \varphi = a(\varphi \sin \varphi + \cos \varphi)$. In other words, the diameter is tangent to the point P of the curve traced out by the point on the circumference of the circle radius a/2. Hence the envelope of the diameter is that curve.

Problem A4
The real polynomial $x^3 + px^2 + qx + r$ has real roots $a \le b \le c$. Prove that f ' has a root in the interval $[b/2 + c/2, b/3 + 2c/3]$. What can we say about f if the root is at one of the endpoints?

Solution
$p(x) = (x - a)(x - b)(x - c)$, so $p'(x) = (x - a)(x - b) + (x - b)(x - c) + (x - a)(x - c)$.

We can write $p'(x) = (x - b)(x - c) + (x - a)(2x - b - c)$, so $p'(b/2 + c/2) = -1/4\ (c - b)^2 \le 0$, with equality iff $b = c$.

$p'(b/3 + 2c/3) = -2/9\ (c - b)^2 + (x - a)\ 1/3\ (c - b) = 1/3\ (c - b)\ (b - a) \ge 0$ with equality iff $a = b$ or $b = c$.

If $b = c$, then b is a repeated root and $p'(b) = 0$. If $a = b$, then $p'(b/3 + 2c/3) = 0$. Otherwise, $p'(x)$ is negative at $b/2 + c/2$ and positive at $b/3 + 2c/3$, so it has a zero in the interior of the interval.

Problem A6
f is defined for the non-negative reals and takes positive real values. The centroid of the area lying under the curve $y = f(x)$ between $x = 0$ and $x = a$ has x-coordinate $g(a)$. Prove that for some positive constant k, $f(x) = k\ g'(x)/(x - g(x))^2\ e^{\int 1/(t - g(t))\ dt}$.

Problem A7
Do either (1) or (2):

(1) *Do either (1) or (2):*

(1) Let A be the 3 x 3 matrix

$1+x^2-y^2-z^2$	$2(xy+z)$	$2(zx-y)$
$2(xy-z)$	$1+y^2-z^2-x^2$	$2(yz+x)$
$2(zx+y)$	$2(yz-x)$	$1+z^2-x^2-y^2$

Show that det A = $(1 + x^2 + y^2 + z^2)^3$.

(2) A solid is formed by rotating about the x-axis the first quadrant of the ellipse $x^2/a^2 + y^2b^2 = 1$. Prove that this solid can rest in stable equilibrium on its vertex (corresponding to x = a, y = 0 on the ellipse) iff $a/b \le \sqrt{(8/5)}$.

Solution
(1) subtract z times row 2 from row 1 and add y times row 3 to row 1. After taking out the common factor $1+x^2+y^2+z^2$ from row 1 we get:

1	z	-y
$2(xy-z)$	$1+y^2-z^2-x^2$	$2(yz+x)$
$2(zx+y)$	$2(yz-x)$	$1+z^2-x^2-y^2$

Subtract z times col 1 from col 2 and add y times col 1 to col 3. We get:

1	0	0
$2(xy-z)$	$1+y^2+z^2-x^2-2xyz$	$2x(1+y^2)$
$2(zx+y)$	$-2x(1+z^2)$	$1+z^2-x^2+y^2+2xyz$

Multiplying this out, we get $(1-x^2+y^2+z^2)^2 - 4x^2y^2z^2 + 4x^2(1+y^2+z^2+y^2z^2) = (1+x^2+y^2+z^2)^2$. Hence with the additional factor we took out, we get the result.

(2) We first have to find the position of the centre of mass on the axis. The moment about the y-axis of the solid is $\int_0^a \pi y^2 x\, dx = \pi b^2 \int_0^a (x - x^3/a^2)\, dx = \pi b^2a^2/4$. The volume is $\int_0^a \pi y^2\, dx = \pi b^2 \int_0^a (1 - x^2/a^2)\, dx = 2/3\ \pi b^2 a$. Hence the centre of mass is a distance 3a/8 from the flat surface or 5a/8 from the point of contact.

Now suppose the point of contact is at (a cos t, b sin t). The tangent has gradient - b/a cot t, so the normal has gradient a/b tan t. So the equation of the normal is y - b sint = a/b tan t (x - a cos t). This meets the x-axis at $a(1 - b^2/a^2)\cos t$. For stability we want this to be closer to the origin than the centre of mass, in other words we want $(1 - b^2/a^2)\cos t < 3/8$. The point of contact is at cos t = 1, so we require $(1 - b^2/a^2) < 3/8$ or $b/a > \sqrt{(5/8)}$.

Problem B1
A particle moves in the plane so that its angular velocity about the point (1, 0) equals minus its angular velocity about the point (-1, 0). Show that its trajectory satisfies the differential equation $y'\, x(x^2 + y^2 - 1) = y(x^2 + y^2 + 1)$. Verify that this has as solutions the rectangular hyperbolae with center at the origin and passing through (±1, 0).

Solution
The angular momentum (for unit mass) about (1, 0) is (x - 1) dy/dt - y dx/dt. Hence the angular velocity is $((x - 1)\, dy/dt - y\, dx/dt)/((x - 1)^2 + y^2)$. Similarly, the angular velocity about (-1, 0) is $((x + 1)\, dy/dt - y\, dx/dt)/((x + 1)^2 + y^2)$. Hence the trajectory satisfies: $((x - 1) y' - y)(x^2 + y^2 + 1 + 2x) + ((x + 1) y' - y)(x^2 + y^2 + 1 - 2x) = 0$ or $2x y' (x^2 + y^2 + 1) - 4x y' - 2y(x^2 + y^2 + 1) = 0$, or $x y' (x^2 + y^2 - 1) = y (x^2 + y^2 + 1)$.

We see immediately that $xy = 0$ is a solution and almost immediately that $x^2 - y^2 = 1$ is a solution. Hence also all linear combinations $x^2 + k\,xy - y^2 = 1$. These are rectancular hyperbolae because the sum of the coefficients of x^2 and y^2 are zero.

Problem B2
Find:
(1) $\lim_{n\to\infty} \sum_{1\le i\le n} 1/\sqrt{n^2 + i^2}$;
(2) $\lim_{n\to\infty} \sum_{1\le i\le n} 1/\sqrt{n^2 + i}$;
(3) $\lim_{n\to\infty} \sum_{1\le i\le n^2} 1/\sqrt{n^2 + i}$;

Solution
(1) $1/\sqrt{n^2 + i^2} = (1/n) / \sqrt{1 + (i/n)^2}$. So the sum is just a Riemann sum for the integral $\int_0^1 dx/\sqrt{1 + x^2} = \sinh^{-1}1 = \ln(1 + \sqrt{2}) = 0.8814$.

(2) $1/\sqrt{n^2 + i} = (1/n) / \sqrt{1 + i/n^2}$. Each term is less than $1/n$, so the (finite) sum is less than 1. But each term is at least $(1/n) / \sqrt{1 + 1/n}$. So the sum is at least $1/\sqrt{1 + 1/n}$, which tends to 1. Hence the limit of the sum is 1.

(3) $1/\sqrt{n^2 + i} = n\,(1/n^2) / \sqrt{1 + i/n^2}$. Now $\sum (1/n^2) / \sqrt{1 + i/n^2}$ is just a Riemann sum for $\int_0^1 dx/\sqrt{1 + x} = 2\sqrt{1 + x}\,|_0^1 = 2(\sqrt{2} - 1)$. So the sum given tends to $2(\sqrt{2} - 1)\,n$, which diverges to infinity. [Or simpler, there are n^2 terms, each at least $1/\sqrt{2n^2} = 1/(n\sqrt{2})$, so the sum is at least $n/\sqrt{2}$ which diverges.]

Problem B3
Let y and z be any two linearly independent solutions of the differential equation $y'' + p(x)\,y' + q(x)\,y = 0$. Let $w = y\,z$. Find the differential equation satisfied by w.

Solution
We have $w = yz$, $w' = y'z + yz'$, $w'' = y''z + 2y'z' + yz''$. Hence $w'' + pw' + 2qw = 2y'z'$ (1). Now $(y'z')' = y''z' + y'z''$, $2py'z' = py'z' + py'z'$, $qw' = qyz' + qy'z$, so $(y'z')' + 2py'z' + qw' = 0$ (2).

Differentiating (1) we get: $w''' + p'w' + pw'' + 2q'w + 2qw' = 2(y'z')' = -4py'z' - 2qw'$ (using (2)), $= -2p(w'' + pw' + 2qw) - 2qw'$. Rearranging, this gives: $w''' + 3p\,w'' + (p' + 2p^2 + 4q)\,w' + (4pq + 2q')w = 0$.

Problem B4
Given an ellipse center O, take two perpendicular diameters AOB and COD. Take the diameter A'OB' parallel to the tangents to the ellipse at A and B (this is said to be *conjugate* to the diameter AOB). Similarly, take C'OD' conjugate to COD. Prove that the rectangular hyperbola through A'B'C'D' passes through the foci of the ellipse.

Solution
Take the ellipse as $x^2/a^2 + y^2/b^2 = 1$ and the point A as $(a \cos t, b \sin t)$. Then AB has slope $b/a \tan t$, so CD has slope $-a/b \cot t$. The tangent at A has slope $-b/a \cot t$. Suppose C is $(a \cos u, b \sin u)$, then $b/a \tan u = -a/b \cot t$ and the tangent at C has slope $-b/a \cot u = b^3/a^3 \tan t$. Hence the line pair A'B', C'D' has equation $(y + x\, b/a \cot t)(y - x\, b^3/a^3 \tan t)$. Now we have the equations for two distinct conics through A'B'C'D': the original ellipse and the line pair A'B',C'D'. The equation of any other conic through these four points must be a linear combination of the equations of these two, in other words, $(x^2/a^2 + y^2/b^2 - 1) + k(y + x\, b/a \cot t)(y - x\, b^3/a^3 \tan t) = 0$ for some k.

The criterion for a rectangular hyperbola is that the coefficients of x^2 and y^2 should have sum zero, or that $1/a^2 + 1/b^2 + k - k\, b^4/a^4 = 0$. Hence $k = a^2/(b^4-b^2a^2)$ and the equation of the rectangular hyperbola is $x^2 - y^2 + (b/a \tan t - a/b \cot t)\, xy = a^2 - b^2$. But the foci are at $(\pm(a^2 - b^2)^{1/2}, 0)$, so they lie on the rectangular hyperbola.

Problem B5
A wheel radius r is traveling along a road without slipping with angular velocity $\omega > \sqrt{g/r}$. A particle is thrown off the rim of the wheel. Show that it can reach a maximum height above the road of $(r\omega + g/\omega)^2/(2g)$. [Ignore air resistance.]

Solution
Suppose the pebble leaves the wheel from a point on the rim which is at an angle θ to the vertical. Its point of departure is a distance $r + r \cos\theta$ above the road. Its upward velocity is $r\,\omega \sin\theta$, so it ascends a further $(r\,\omega \sin\theta)^2/2g$. Thus the total height is $r + r^2\omega^2/2g + r \cos\theta - r^2\omega^2/2g \cos^2\theta = r + g/2\omega^2 + r^2\omega^2/2g - r^2\omega^2/2g\,(\cos\theta - g/r\omega^2)$

(*). We are given that $g/r\omega^2 < 1$, so (*) has a maximum when $\cos\theta = g/r\omega^2$ and the maximum value is $1/2g\ (r^2\omega^2 + 2gr + g^2/\omega^2) = (r\omega + g/\omega)^2/2g$.

Problem B6
f is a real valued function on [0, 1], continuous on (0, 1). Prove that $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x)\ f(y)\ f(z)\ dz\ dy\ dx = 1/6\ (\int_{x=0}^{x=1} f(x)\ dx\ )^3$.

Solution
Let S_{xyz} be the points in the cube for which $x \le y \le z$, let S_{yxz} be the points for which $y \le x \le z$ and so on. Then the union of the six sets is the cube and the intersection of any two has measure zero. Also by changing the variables of integration we see that the integral of f(x) f(y) f(z) over each set is the same. Hence the integral over S_{xzy} is 1/6 of the integral over the cube. But the integral over S_{xzy} is just $\int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x)\ f(y)\ f(z)\ dz\ dy\ dx$ and the integral over the cube is $(\int_{x=0}^{x=1} f(x)\ dx\ )^3$.

Problem B7
Do either (1) or (2):

(1) f is a real-valued function defined on the reals with a continuous second derivative and satisfies $f(x + y)\ f(x - y) = f(x)^2 + f(y)^2 - 1$ for all x, y. Show that for some constant k we have $f''(x) = \pm k^2 f(x)$. Deduce that f(x) is one of ±cos kx, ±cosh kx.

(2) a_i and b_i are constants. Let A be the (n+1) x (n+1) matrix A_{ij}, defined as follows: $A_{i1} = 1$; $A_{1j} = x^{j-1}$ for $j \le n$; $A_{1\,(n+1)} = p(x)$; $A_{ij} = a_{i-1}^{j-1}$ for $i > 1, j \le n$; $A_{i\,(n+1)} = b_{i-1}$ for $i > 1$. We use the identity det A = 0 to define the polynomial p(x). Now given any polynomial f(x), replace b_i by $f(b_i)$ and p(x) by q(x), so that det A = 0 now defines a polynomial q(x). Prove that f(p(x)) is a multiple of &prodc; $(x - a_i)$ plus q(x).

Solution
(1) Putting y = 0 gives $f(0)^2 = 1$, so $f(0) = \pm 1$. Differentiating wrt y gives $f'(x+y)\ f(x-y) - f(x+y)\ f'(x-y) = 2\ f(y)\ f'(y)$. Putting y = 0 gives $f'(0) = 0$. Differentiating wrt x gives $f''(x+y)\ f(x-y) = f(x+y)\ f''(x-y)$. Putting $x = y = z/2$ gives $f''(z) = h\ f(z)$, where $h = \pm f''(0)$. If h is positive, we may put $h = k^2$ and integrate, using $f(0) = \pm 1$, $f'(0) = 0$ to get $f(x) = \pm \cos kx$. If h is negative, we may put $h = - k^2$ and integrate to get $f(x) = \pm \cosh kx$.

(2)

5th Putnam 1942

Problem A1
ABCD is a square side 2a with vertices in that order. It rotates in the first quadrant with A remaining on the positive x-axis and B on the positive y-axis. Find the locus of its center.

Solution
Answer: the segment (a, a) to $(a\sqrt{2}, a\sqrt{2})$.

Let AB make an angle θ with the x-axis. Then we find that the coordinates of the center to be $x = y = a \cos \theta + a \sin \theta$. But $a \cos \theta + a \sin \theta = a \sqrt{2} \sin(\theta + \pi/4)$.

Problem A2
a and b are unequal reals. What is the remainder when the polynomial p(x) is divided $(x - a)^2(x - b)$.

Solution
Suppose the remainder is $cx^2 + dx + e$. We have $p(a) = ca^2 + da + e$, $p(b) = cb^2 + db + e$. Also, differentiating, we get $p'(a) = 2ca + d$. Solving, $c = p'(a)/(a - b) - p(a)/(a - b)^2 + p(b)/(a - b)^2$, $d = (2a\, p(a) - 2a\, p(b) - (a^2 - b^2)p'(a))/(a - b)^2$, $e = p(a) - a^2(p(a) - p(b))/(a - b)^2 + ab\, p'(a) /(a - b)$.

Problem A3
Does $\sum_{n\geq 0} n!\, k^n/(n + 1)^n$ converge or diverge for k = 19/7?

Solution
The nth term divided by the n-1th term is $k\, n\, n^{n-1}/(n+1)^n = k/(1 + 1/n)^n$ which tends to k/e. But k/e < 1, so the series converges by the ratio test.

Problem A4
Let C be the family of conics $(2y + x)^2 = a(y + x)$. Find C', the family of conics which are orthogonal to C. At what angle do the curves of the two families meet at the origin?

Solution
For most points P in the plane we can find a unique conic in the family passing through the point. Thus we should be able to find the gradient of members of the family at (x, y) in a formula which is independent of a. We then use this to get a formula for the gradient of the orthogonal family and solve the resulting first-order differential equation to get the orthogonal family.

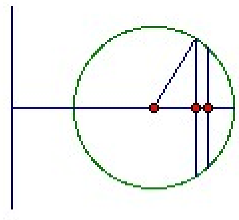
Thus we have $8y y' + 4x y' + 4y + 2x = ay' + a = (y' + 1)(2y + x)^2/(y + x)$. So $y'(2y+x)(4(x+y) - (2y+x)) = (2y+x)^2 - (x+2y)(x+y)$, or $y'(2y+x)(2y+3x) = -2x(2y+x)$, so $y' = -x/(2y+3x)$. Hence the orthogonal family satisfies $y' = (2y+3x)/x$. So $y'/x^2 - 2y/x^3 = 3/x^2$. Integrating $y = bx^2 - 3x$. These are all parabolas.

All members of both families pass through the origin. Changing coordinates to X = x + 2y, Y = y - 2x, the equation of a member of the first family becomes $X^2 = a(3X-Y)/5$ or $Y = - 5/a\, (X - 3a/10)^2 + 9a/20$. This has gradient 3 (in the new system) at the origin. In the old system the tangent is y = -x. The orthogonal set obviously has gradient -3 at the origin. If the angle between them is k, then tan k = (-1 +3)/(1+3) = 1/2. So $k = \tan^{-1} 1/2$.

Problem A5
C is a circle radius a whose center lies a distance b from the coplanar line L. C is rotated through π about L to form a solid whose center of gravity lies on its surface. Find b/a.

Answer
$(\pi + \sqrt{(\pi^2+2\pi-4)})/(2\pi-4)$ = about 2.9028

Solution
The solid is half a torus. We can divide it into a large number of thin disks. Each disk has variable thickness, with thickness proportional to the distance from L. So we must integrate to find the distance of the centroid of the disk from L. Take the density to be kd, where d is the distance from L.



Take x to be distance along the line perpendicular to x, and θ to be the angle between the radius vector and the x-axis. We have $x = a \cos\theta$, so $dx = - a \sin\theta\, d\theta$. The mass is $\int_0^\pi 2a \sin\theta\, (a \sin\theta\, d\theta)\, k(b + a \cos\theta) = 2a^2bk \int_0^\pi \sin^2\theta\, d\theta + 2a^3k \int_0^\pi \sin^2\theta \cos\theta\, d\theta = a^2bk\pi + 0$. So the mass times the centroid distance is $\int_0^\pi 2a^2k \sin^2\theta\, (a \cos\theta + b)^2\, d\theta = 2a^4k \int_0^\pi \sin^2\theta \cos^2\theta\, d\theta + 4a^3bk \int_0^\pi \sin^2\theta \cos\theta\, d\theta + 2a^2b^2k \int_0^\pi \sin^2\theta\, d\theta = ½a^4k \int_0^\pi \sin^2 2\theta\, d\theta + 0 + a^2b^2k\pi = ka^2\pi(a^2/4 + b^2)$. So the centroid distance is $b + a^2/4b$. Thus we can regard the mass as uniformly spread over a semicircle radius $b + a^2/4b$.

We need another integration to find the distance of the mass of a semicircle radius r from its center. It is $(1/\pi r) \int_0^\pi r^2 \sin\theta\, d\theta = 2r/\pi$. Thus the cm of the half-torus is a distance $(2/\pi)(b + a^2/4b)$ from L. We want it to be a distance b-a from L so that it lies on the surface. Thus $(2/\pi)(b + a^2/4b) = b - a$, so $(2\pi-4)b^2 - 2\pi ab - a^2 = 0$. Hence $b/a = (\pi + \sqrt{\pi^2+2\pi-4})/(2\pi-4)$ = about 2.9028.

Problem A6

P is a plane and H is the half-space on one side of P. K is a fixed circle in P. C is a circle in P which cuts K at an angle α. Let C have center O and radius r. f(C) is the point in H on the normal to P through O and a distance r from O. Show that the locus of f(C) is a one-sheet hyperboloid and that it has two families of rulings in it.

Problem B1

S is a solid square side 2a. It lies in the quadrant $x \ge 0$, $y \ge 0$, and it is free to move around provided a vertex remains on the x-axis and an adjacent vertex on the y-axis. P is a point of S. Show that the locus of P is part of a conic. For what P does the locus degenerate?

Solution

Let A be the vertex that moves along the x-axis and B the vertex that moves along the y-axis. Suppose that when AB is horizontal P has coordinates b, c. In the general configuration let be the angle BAO be θ. Then P has coordinates $x = (2a - b) \cos\theta + c \sin\theta$, $y = b \sin\theta + c \cos\theta$. Hence $cx - (2a - b)y = (b^2 + c^2 - 2ab) \sin\theta$, $bx - cy = (2ab - b^2 - c^2) \cos\theta$. Squaring and adding we eliminate θ to get: $(b^2 + c^2) x^2 - 4ac\, xy + (4a^2 + b^2 + c^2 - 4ab) y^2 = (b^2 + c^2 - 2ab)^2$, which is the equation of a conic. So the locus of P must form part of this conic.

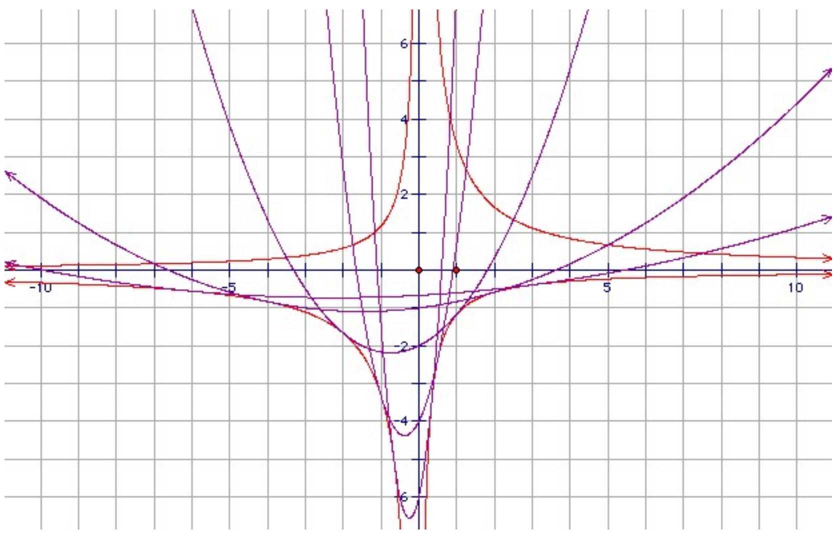
The conic degenerates if $b^2 + c^2 = 2ab$. In this case, the equation becomes $2ab\, x^2 - 4ac\, xy + (4a^2 - 2ab)y^2 = 0$, or $bx^2 - 2c\, xy + (2a - b)y^2 = 0$, or $b^2x^2 - 2bc\, xy + c^2 y^2 = 0$, or $bx = cy$. So in this case the locus lies on a straight line. We may write the condition $b^2 + c^2 = 2ab$ as $(a - b)^2 + c^2 = a^2$, which shows that such P lie on the semicircle diameter AB.

Problem B2

Let P_a be the parabola $y = a^3x^2/3 + a^2x/2 - 2a$. Find the locus of the vertices of P_a, and the envelope of P_a. Sketch the envelope and two P_a.

Solution

We can write the equation of P_a as $(y + 35a/16) = (a^3/3)(x + 3/4a)^2$, so the vertex is $x = -3/4a$, $y = -35a/16$. The locus of the vertex is $xy = 105/64$.



The graph shows P_3, P_2, P_1, $P_{1/2}$, $P_{1/3}$ and the two hyperbolae yx = -7/6, yx = 10/3. It shows that for positive a, the parabolas touch the lower branches of the hyperbolae. For negative a they touch the upper branches.

That is not hard to verify. We claim that P_a and xy = 10/3 touch at x = -2/a, y = -5a/3. The point obviously lies on xy = 10/3. We have $(a^3/3)(x + 3/4a)^2 = (1/3)(-2 + 3/4)^2\ a = 25a/48 = (-5a/3 + 35a/16)$, so the point also lies on P_a. The gradient of xy = 10/3 at the point is $-10/(3x^2) = -5a^2/6$. The gradient of P_a at the point is $2a^3x/3 + a^2/2 = -5a^2/6$.

Similarly, we claim that P_a and xy = -7/6 touch at x = 1/a, y = -7a/6. The point obviously lies on xy = -7/6. We have $(a^3/3)(x + 3/4a)^2 = (a/3)(1 + 3/4)^2 = 49a/48 = (-7a/6 + 35a/16)$, so the point also lies on P_a. The gradient of xy = -7/6 at the point is $7/(6x^2) = 7a^2/6$. The gradient of P_a at the point is $2a^3x/3 + a^2/2 = a^2(2/3 + 1/2) = 7a^2/6$.

It is less clear how you get the hyperbolas. One standard approach is to look for the singular points of the mapping $f(a,t) = (t, a^3t^2/3 + a^2t/2 - 2a)$. The matrix for the derivative is:

$$\begin{matrix} 1 & 0 \\ 2a^3t/3+a^2/2 & a^2t^2+at-2 \end{matrix}$$

which has zero determinant when at = 1 or -2, so xy = 7/6 or -10/3.

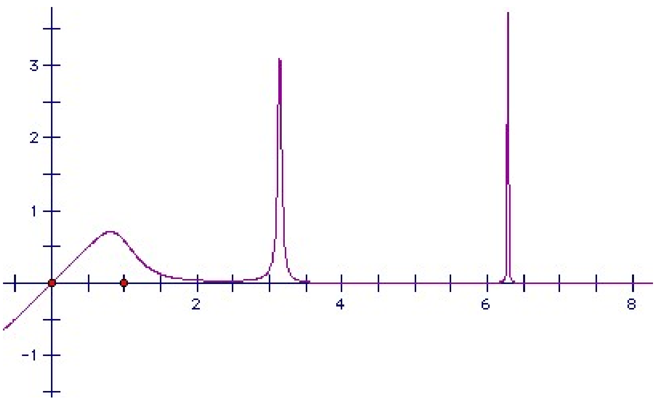
Problem B3
f(x, y) and g(x, y) satisfy the differential equation $f_1(x, y)\ g_2(x, y) - f_2(x, y)\ g_1(x, y) = 1$ (*). Taking r = f(x, y) and y as independent variables, and x = h(r, y), g(x, y) = k(r, y), show that $k_2(r, y) = h_1(r, y)$. Integrate and hence obtain a solution to (*). What other solutions does (*) have?

Problem B4
A particle moves in a circle through the origin under the influence of a force a/r^k towards the origin (where r is its distance from the origin). Find k.

Solution
The equations of motion are $r(\theta')^2 - r'' = a/r^k$, $r^2\theta' = A$ (conservation of angular momentum).

If the particle moves in a circle as described, then we can write its orbit as $r = B\cos\theta$. Differentiating, $r' = -B\theta'\sin\theta = -AB/r^2 \sin\theta$. Differentiating again, $r'' = -AB\cos\theta\ A/r^2 + 2AB/r^3 \sin\theta\ r' = -A^2/r^3 - 2A^2B^2/r^5 \sin^2\theta = -A^2/r^3 - 2A^2B^2/r^5 (1 - r^2B^2) = A^2/r^3 - 2A^2B^2/r^5$. So substituting back in the equation of motion we get: $2A^2B^2/r^5 = a/r^k$. Hence k = 5.
Note that this is unphysical, since we require infinite velocity as we reach the origin.

Problem B5
Let $f(x) = x/(1 + x^6\sin^2x)$. Sketch the curve y = f(x) and show that $\int_0^\infty f(x)\,dx$ exists.



Solution
Obviously f(x) is positive for positive x. But it has an infinite number of spikes at x = nπ. The spike at nπ is height nπ, so we have to show that the integral is bounded above.

We have sin x > ½x near x = 0 (certainly for x < π/3). So $|\sin x| > 1/(n\pi)^k$ except possibly for $|x| < 2/(n\pi)^k$. Let I_n be the interval centered on nπ width $4/(n\pi)^k$. For $x \in I_n$ we have f(x) < 2nπ, so the integral of f(x) over the interval is less than $8/(n\pi)^{k-1}$. The total integral over all such intervals is bounded provided that k > 2. Outside such intervals, $x^6 \sin^2x > ½\, x^6/x^{2k}$, so $f(x) < 2/x^{2k-5}$. Hence the interval of f(x) over 0 to ∞ excluding the intervals I_n is bounded provided k > 5/2. By taking k = 2 1/4, for example, we get that the whole integral is bounded.