# IRISH MATHEMATICAL OLYMPIADS

# 1988 – 2014

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## Preface

This is an unofficial collection of the Irish Mathematical Olympiads. Unofficial in the sense that it probably contains minor typos and has not benefitted from being proofread by the IrMO committee. This annual competition is typically held on a Saturday at the beginning of May. The first paper runs from 10am – 1 pm and the second paper from 2pm – 5 pm.

Predicted FAQs:

Q: Where can I find the solutions?
A: Google and `http://www.mathlinks.ro` are your friends.

Q: I have written a solution to one of the problems, will you check it?
A: Absolutely not!

Q: Are there any books associated with this?
A: You might try

- *Irish Mathematical Olympiad Manual* by O'Farrell et al., Logic Press, Maynooth.
- *Irish Mathematical-Olympiad Problems 1988-1998*, edited by Finbarr Holland of UCC, published by the IMO Irish Participation Committee, 1999.

Q: Who won these competitions?
A: The six highest scoring candidates attend the IMO. They can be found at `www.imo-official.org` → Results → IRL. Note however that some candidates may have pulled out due to illness etc.

Q: Will this file be updated annually?
A: This is the plan but no promises are made.

Q: Can I copy and paste this file to my website?
A: It may be better to link to

`http://www.raunvis.hi.is/~dukes/irmo.html`

so that the latest version is always there.

Q: I've found a typo or what I suspect is a mistake?
A: Pop me an email about it and I will do my very best to check......

Q: How do I find out more about the IrMO competition and training?
A: `http://www.irmo.ie/`

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## 1st Irish Mathematical Olympiad
## 30 April 1988, Paper 1

1. A pyramid with a square base, and all its edges of length 2, is joined to a regular tetrahedron, whose edges are also of length 2, by gluing together two of the triangular faces. Find the sum of the lengths of the edges of the resulting solid.

2. $A, B, C, D$ are the vertices of a square, and $P$ is a point on the arc $CD$ of its circumcircle. Prove that
$$|PA|^2 - |PB|^2 \;=\; |PB|.|PD| - |PA|.|PC|.$$

3. $ABC$ is a triangle inscribed in a circle, and $E$ is the mid-point of the arc subtended by $BC$ on the side remote from $A$. If through $E$ a diameter $ED$ is drawn, show that the measure of the angle $DEA$ is half the magnitude of the difference of the measures of the angles at $B$ and $C$.

4. A mathematical moron is given the values $b, c, A$ for a triangle $ABC$ and is required to find $a$. He does this by using the cosine rule
$$a^2 \;=\; b^2 + c^2 - 2bc\cos A$$
and misapplying the low of the logarithm to this to get
$$\log a^2 \;=\; \log b^2 + \log c^2 - \log(2bc\cos A).$$
He proceeds to evaluate the right-hand side correctly, takes the anti-logarithms and gets the correct answer. What can be said about the triangle $ABC$?

5. A person has seven friends and invites a different subset of three friends to dinner every night for one week (seven days). In how many ways can this be done so that all friends are invited at least once?

6. Suppose you are given $n$ blocks, each of which weighs an integral number of pounds, but less than $n$ pounds. Suppose also that the total weight of the $n$ blocks is less than $2n$ pounds. Prove that the blocks can be divided into two groups, one of which weighs exactly $n$ pounds.

7. A function $f$, defined on the set of real numbers $\mathbb{R}$ is said to have a *horizontal chord* of length $a > 0$ if there is a real number $x$ such that $f(a + x) = f(x)$. Show that the cubic
$$f(x) \;=\; x^3 - x \qquad\qquad (x \in \mathbb{R})$$
has a horizontal chord of length $a$ if, and only if, $0 < a \le 2$.

8. Let $x_1, x_2, x_3, \ldots$ be a sequence of nonzero real numbers satisfying
$$x_n \;=\; \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}, \qquad n = 3, 4, 5, \ldots$$
Establish necessary and sufficient conditions on $x_1, x_2$ for $x_n$ to be an integer for infinitely many values of $n$.

9. The year 1978 was "peculiar" in that the sum of the numbers formed with the first two digits and the last two digits is equal to the number formed with the middle two digits, i.e., $19 + 78 = 97$. What was the last previous peculiar year, and when will the next one occur?

10. Let $0 \le x \le 1$. Show that if $n$ is any positive integer, then
$$(1 + x)^n \;\ge\; (1 - x)^n + 2nx(1 - x^2)^{\frac{n-1}{2}}.$$

11. If facilities for division are not available, it is sometimes convenient in determining the decimal expansion of $1/a$, $a > 0$, to use the iteration

$$x_{k+1} = x_k(2 - ax_k), \qquad k = 0, 1, 2, \ldots,$$

where $x_0$ is a selected "starting" value. Find the limitations, if any, on the starting values $x_0$, in order that the above iteration converges to the desired value $1/a$.

12. Prove that if $n$ is a positive integer, then

$$\sum_{k=1}^{n} \cos^4\left(\frac{k\pi}{2n+1}\right) = \frac{6n-5}{16}.$$

## 1st Irish Mathematical Olympiad
## 30 April 1988, Paper 2

1. The triangles $ABG$ and $AEF$ are in the same plane. Between them the following conditions hold:

   (a) $E$ is the mid-point of $AB$;

   (b) points $A, G$ and $F$ are on the same line;

   (c) there is a point $C$ at which $BG$ and $EF$ intersect;

   (d) $|CE| = 1$ and $|AC| = |AE| = |FG|$.

   Show that if $|AG| = x$, then $|AB| = x^3$.

2. Let $x_1, \ldots, x_n$ be $n$ integers, and let $p$ be a positive integer, with $p < n$. Put

$$\begin{aligned} S_1 &= x_1 + x_2 + \ldots + x_p, \\ T_1 &= x_{p+1} + x_{p+2} + \ldots + x_n, \\ S_2 &= x_2 + x_3 + \ldots + x_{p+1}, \\ T_2 &= x_{p+2} + x_{p+3} + \ldots + x_n + x_1, \\ &\vdots \\ S_n &= x_n + x_1 + x_2 + \ldots + x_{p-1}, \\ T_n &= x_p + x_{p+1} + \ldots + x_{n-1}. \end{aligned}$$

   For $a = 0, 1, 2, 3$, and $b = 0, 1, 2, 3$, let $m(a, b)$ be the number of numbers $i$, $1 \leq i \leq n$, such that $S_i$ leaves remainder $a$ on division by 4 and $T_i$ leaves remainder $b$ on division by 4. Show that $m(1, 3)$ and $m(3, 1)$ leave the same remainder when divided by 4 if, and only if, $m(2, 2)$ is even.

3. A city has a system of bus routes laid out in such a way that

   (a) there are exactly 11 bus stops on each route;

   (b) it is possible to travel between any two bus stops without changing routes;

   (c) any two bus routes have exactly one bus stop in common.

   What is the number of bus routes in the city?

**2nd Irish Mathematical Olympiad**
**29 April 1989, Paper 1**

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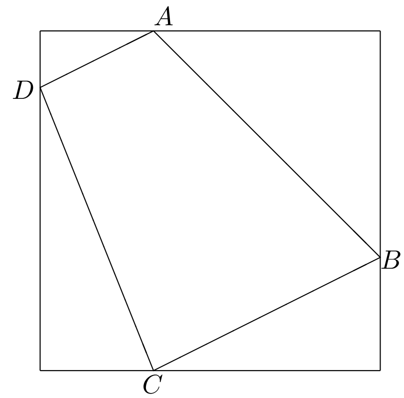
1. A quadrilateral $ABCD$ is inscribed, as shown, in a square of area one unit. Prove that
$$2 \leq |AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 \leq 4.$$




2. A $3 \times 3$ magic square, with magic number $m$, is a $3 \times 3$ matrix such that the entries on each row, each column and each diagonal sum to $m$. Show that if the square has positive integer entries, then $m$ is divisible by 3, and each entry of the square is at most $2n - 1$, where $m = 3n$. [An example of a magic square with $m = 6$ is
$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}.]$$

3. A function $f$ is defined on the natural numbers $\mathbb{N}$ and satisfies the following rules:

   (a) $f(1) = 1$;

   (b) $f(2n) = f(n)$ and $f(2n + 1) = f(2n) + 1$ for all $n \in \mathbb{N}$.

   Calculate the maximum value $m$ of the set $\{f(n) : n \in \mathbb{N}, 1 \leq n \leq 1989\}$, and determine the number of natural numbers $n$, with $1 \leq n \leq 1989$, that satisfy the equation $f(n) = m$.

4. Note that $12^2 = 144$ end in two 4's and $38^2 = 1444$ end in three 4's. Determine the length of the longest string of equal nonzero digits in which the square of an integer can end.

5. Let $x = a_1 a_2 \ldots a_n$ be an $n$-digit number, where $a_1, a_2, \ldots, a_n$ ($a_1 \neq 0$) are the digits. The $n$ numbers
$$x_1 = x = a_1 a_2 \ldots a_n, \quad x_2 = a_n a_1 \ldots a_{n-1}, \quad x_3 = a_{n-1} a_n a_1 \ldots a_{n-2},$$
$$x_4 = a_{n-2} a_{n-1} a_n a_1 \ldots a_{n-3}, \quad \ldots \quad , \quad x_n = a_2 a_3 \ldots a_n a_1$$
are said to be obtained from $x$ by the cyclic permutation of digits. [For example, if $n = 5$ and $x = 37001$, then the numbers are $x_1 = 37001$, $x_2 = 13700$, $x_3 = 01370 (= 1370)$, $x_4 = 00137 (= 137)$, $x_5 = 70013$.]

   Find, with proof, (i) the smallest natural number $n$ for which there exists an $n$-digit number $x$ such that the $n$ numbers obtained from $x$ by the cyclic permutation of digits are all divisible by 1989; and (ii) the smallest natural number $x$ with this property.

## 2nd Irish Mathematical Olympiad
## 29 April 1989, Paper 2

1. Suppose $L$ is a fixed line, and $A$ a fixed point not on $L$. Let $k$ be a fixed nonzero real number. For $P$ a point on $L$, let $Q$ be a point on the line $AP$ with $|AP|.|AQ| = k^2$. Determine the locus of $Q$ as $P$ varies along the line $L$.

2. Each of the $n$ members of a club is given a different item of information. They are allowed to share the information, but, for security reasons, only in the following way: A pair may communicate by telephone. During a telephone call only one member may speak. The member who speaks may tell the other member all the information s(he) knows. Determine the minimal number of phone calls that are required to convey all the information to each other.

3. Suppose $P$ is a point in the interior of a triangle $ABC$, that $x, y, z$ are the distances from $P$ to $A, B, C$, respectively, and that $p, q, r$ are the perpendicular distances from $P$ to the sides $BC, CA, AB$, respectively. Prove that
$$xyz \;\geq\; 8pqr,$$
with equality implying that the triangle $ABC$ is equilateral.

4. Let $a$ be a positive real number, and let
$$b \;=\; \sqrt[3]{a+\sqrt{a^2+1}} + \sqrt[3]{a-\sqrt{a^2+1}}.$$
Prove that $b$ is a positive integer if, and only if, $a$ is a positive integer of the form $\frac{1}{2}n(n^2+3)$, for some positive integer $n$.

5. (i) Prove that if $n$ is a positive integer, then
$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$
is a positive integer that is divisible by all prime numbers $p$ with $n < p \leq 2n$, and that
$$\binom{2n}{n} < 2^{2n}.$$

   (ii) For $x$ a positive real number, let $\pi(x)$ denote the number of prime numbers $p \leq x$. [Thus, $\pi(10) = 4$ since there are 4 primes, viz., 2, 3, 5 and 7, not exceeding 10.]
   Prove that if $n \geq 3$ is an integer, then

   (a) $\pi(2n) \;<\; \pi(n) + \dfrac{2n}{\log_2(n)}$; (b) $\pi(2^n) \;<\; \dfrac{2^{n+1}\log_2(n-1)}{n}$;

   (c) Deduce that, for all real numbers $x \geq 8$,
$$\pi(x) \;<\; \frac{4x\log_2(\log_2(x))}{\log_2(x)}.$$

**3rd Irish Mathematical Olympiad**
**5 May 1990, Paper 1**

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1. Given a natural number $n$, calculate the number of rectangles in the plane, the coordinates of whose vertices are integers in the range 0 to $n$, and whose sides are parallel to the axes.

2. A sequence of primes $a_n$ is defined as follows: $a_1 = 2$, and, for all $n \geq 2$, $a_n$ is the largest prime divisor of $a_1 a_2 \cdots a_{n-1} + 1$. Prove that $a_n \neq 5$ for all $n$.

3. Determine whether there exists a function $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) such that
$$f(n) = f(f(n-1)) + f(f(n+1)),$$
for all natural numbers $n \geq 2$.

4. The real number $x$ satisfies all the inequalities
$$2^k < x^k + x^{k+1} < 2^{k+1}$$
for $k = 1, 2, \ldots, n$. What is the greatest possible value of $n$?

5. Let $ABC$ be a right-angled triangle with right-angle at $A$. Let $X$ be the foot of the perpendicular from $A$ to $BC$, and $Y$ the mid-point of $XC$. Let $AB$ be extended to $D$ so that $|AB| = |BD|$. Prove that $DX$ is perpendicular to $AY$.

6. Let $n$ be a natural number, and suppose that the equation
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + \cdots + x_{n-1}x_n + x_nx_1 = 0$$
has a solution with all the $x_i$'s equal to $\pm 1$. Prove that $n$ is divisible by 4.

**3rd Irish Mathematical Olympiad**
**5 May 1990, Paper 2**

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1. Let $n \geq 3$ be a natural number. Prove that
$$\frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{n^3} < \frac{1}{12}.$$

2. Suppose that $p_1 < p_2 < \ldots < p_{15}$ are prime numbers in arithmetic progression, with common difference $d$. Prove that $d$ is divisible by $2, 3, 5, 7, 11$ and $13$.

3. Let $t$ be a real number, and let
$$a_n = 2\cos\left(\frac{t}{2^n}\right) - 1, \quad n = 1, 2, 3, \ldots$$
Let $b_n$ be the product $a_1 a_2 a_3 \cdots a_n$. Find a formula for $b_n$ that does not involve a product of $n$ terms, and deduce that
$$\lim_{n\to\infty} b_n = \frac{2\cos t + 1}{3}.$$

4. Let $n = 2k - 1$, where $k \geq 6$ is an integer. Let $T$ be the set of all $n$-tuples
$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \text{ where, for } i = 1, 2, \ldots, n, \ x_i \text{ is 0 or 1.}$$
For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $T$, let $d(\mathbf{x}, \mathbf{y})$ denote the number of integers $j$ with $1 \leq j \leq n$ such that $x_j \neq y_j$. (In particular, $d(\mathbf{x}, \mathbf{x}) = 0$).

   Suppose that there exists a subset $S$ of $T$ with $2^k$ elements which has the following property: given any element $\mathbf{x}$ in $T$, there is a unique $\mathbf{y}$ in $S$ with $d(\mathbf{x}, \mathbf{y}) \leq 3$.

   Prove that $n = 23$.

## 4th Irish Mathematical Olympiad
## 4 May 1991, Paper 1

1. Three points $X, Y$ and $Z$ are given that are, respectively, the circumcentre of a triangle $ABC$, the mid-point of $BC$, and the foot of the altitude from $B$ on $AC$. Show how to reconstruct the triangle $ABC$.

2. Find all polynomials
$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$
satisfying the equation
$$f(x^2) = (f(x))^2$$
for all real numbers $x$.

3. Three operations $f, g$ and $h$ are defined on subsets of the natural numbers $\mathbb{N}$ as follows:
$f(n) = 10n$, if $n$ is a positive integer;
$g(n) = 10n + 4$, if $n$ is a positive integer;
$h(n) = \frac{n}{2}$, if $n$ is an *even* positive integer.
Prove that, starting from 4, every natural number can be constructed by performing a finite number of operations $f$, $g$ and $h$ in some order.

   [For example: $35 = h(f(h(g(h(h(4))))))$.]

4. Eight politicians stranded on a desert island on January 1st, 1991, decided to establish a parliment. They decided on the following rules of attendance:

   (a) There should always be at least one person present on each day.

   (b) On no two days should be same subset attend.

   (c) The members present on day $N$ should include for each $K < N$, $(K \geq 1)$ at least one member who was present on day $K$.

   For how many days can the parliment sit before one of the rules is broken?

5. Find all polynomials
$$f(x) \ = \ x^n + a_1 x^{n-1} + \cdots + a_n$$
with the following properties:

   (a) all the coefficients $a_1, a_2, \ldots, a_n$ belong to the set $\{-1, 1\}$

   (b) all the roots of the equation
$$f(x) \ = \ 0$$
are real.

## 4th Irish Mathematical Olympiad
## 4 May 1991, Paper 2

1. The sum of two consecutive squares can be a square: for instance, $3^2 + 4^2 = 5^2$.

   (a) Prove that the sum of $m$ consecutive squares cannot be a square for the cases $m = 3, 4, 5, 6$.

   (b) Find an example of eleven consecutive squares whose sum is a square.

2. Let
$$a_n = \frac{n^2+1}{\sqrt{n^4+4}}, \quad n = 1, 2, 3, \ldots$$
and let $b_n$ be the product of $a_1, a_2, a_3, \ldots, a_n$. Prove that
$$\frac{b_n}{\sqrt{2}} = \frac{\sqrt{n^2+1}}{\sqrt{n^2+2n+2}},$$
and deduce that
$$\frac{1}{n^3+1} < \frac{b_n}{\sqrt{2}} - \frac{n}{n+1} < \frac{1}{n^3}$$
for all positive integers $n$.

3. Let $ABC$ be a triangle and $L$ the line through $C$ parallel to the side $AB$. Let the internal bisector of the angle at $A$ meet the side $BC$ at $D$ and the line $L$ at $E$, and let the internal bisector of the angle at $B$ meet the side $AC$ at $F$ and the line $L$ at $G$. If $|GF| = |DE|$, prove that $|AC| = |BC|$.

4. Let $\mathbb{P}$ be the set of positive rational numbers and let $f : \mathbb{P} \to \mathbb{P}$ be such that
$$f(x) + f\left(\frac{1}{x}\right) = 1$$
and
$$f(2x) = 2f(f(x))$$
for all $x \in \mathbb{P}$.
Find, with proof, an explicit expression for $f(x)$ for all $x \in \mathbb{P}$.

5. Let $\mathbb{Q}$ denote the set of rational numbers. A nonempty subset $S$ of $\mathbb{Q}$ has the following properties:

   (a) 0 is not in $S$;

   (b) for each $s_1, s_2$ in $S$, the rational number $s_1/s_2$ is in $S$; also

   (c) there exists a nonzero number $q \in \mathbb{Q}\backslash S$ that has the property that every nonzero number in $\mathbb{Q}\backslash S$ is of the form $qs$, for some $s$ in $S$.

   Prove that if $x$ belongs to $S$, then there exist elements $y, z$ in $S$ such that $x = y + z$.

## 5th Irish Mathematical Olympiad
## 2 May 1992, Paper 1

1. Describe in geometric terms the set of points $(x, y)$ in the plane such that $x$ and $y$ satisfy the condition $t^2 + yt + x \geq 0$ for all $t$ with $-1 \leq t \leq 1$.

2. How many ordered triples $(x, y, z)$ of real numbers satisfy the system of equations

$$\begin{aligned} x^2 + y^2 + z^2 &= 9, \\ x^4 + y^4 + z^4 &= 33, \\ xyz &= -4\,? \end{aligned}$$

3. Let $A$ be a nonempty set with $n$ elements. Find the number of ways of choosing a pair of subsets $(B, C)$ of $A$ such that $B$ is a nonempty subset of $C$.

4. In a triangle $ABC$, the points $A', B'$ and $C'$ on the sides opposite $A$, $B$ and $C$, respectively, are such that the lines $AA'$, $BB'$ and $CC'$ are concurrent. Prove that the diameter of the circumscribed circle of the triangle $ABC$ equals the product $|AB'|.|BC'|.|CA'|$ divided by the area of the triangle $A'B'C'$.

5. Let $ABC$ be a triangle such that the coordinates of the points $A$ and $B$ are rational numbers. Prove that the coordinates of $C$ are rational if, and only if, $\tan A$, $\tan B$ and $\tan C$, when defined, are all rational numbers.

## 5th Irish Mathematical Olympiad
## 2 May 1992, Paper 2

1. Let $n > 2$ be an integer and let $m = \sum k^3$, where the sum is taken over all integers $k$ with $1 \leq k < n$ that are relatively prime to $n$. Prove that $n$ divides $m$. (Note that two integers are *relatively prime* if, and only if, their greatest common divisor equals 1.)

2. If $a_1$ is a positive integer, form the sequence $a_1, a_2, a_3, \ldots$ by letting $a_2$ be the product of the digits of $a_1$, etc.. If $a_k$ consists of a single digit, for some $k \geq 1$, $a_k$ is called a *digital root* of $a_1$. It is easy to check that every positive integer has a unique digital root. (For example, if $a_1 = 24378$, then $a_2 = 1344$, $a_3 = 48$, $a_4 = 32$, $a_5 = 6$, and thus 6 is the digital root of 24378.) Prove that the digital root of a positive integer $n$ equals 1 if, and only if, all the digits of $n$ equal 1.

3. Let $a, b, c$ and $d$ be real numbers with $a \neq 0$. Prove that if all the roots of the cubic equation
$$az^3 + bz^2 + cz + d = 0$$
lie to the left of the imaginary axis in the complex plane, then
$$ab > 0,\ bc - ad > 0,\ ad > 0.$$

4. A convex pentagon has the property that each of its diagonals cuts off a triangle of unit area. Find the area of the pentagon.

5. If, for $k = 1, 2, \ldots, n$, $a_k$ and $b_k$ are positive real numbers, prove that
$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \leq \sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)};$$
and that equality holds if, and only if,
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

## 6th Irish Mathematical Olympiad
## 8 May 1993, Paper 1

1. The real numbers $\alpha$, $\beta$ satisfy the equations
$$\begin{aligned}\alpha^3 - 3\alpha^2 + 5\alpha - 17 &= 0,\\ \beta^3 - 3\beta^2 + 5\beta + 11 &= 0.\end{aligned}$$
Find $\alpha + \beta$.

2. A natural number $n$ is called **good** if it can be written in a *unique* way simultaneously as the sum $a_1 + a_2 + \ldots + a_k$ and as the product $a_1 a_2 \ldots a_k$ of some $k \geq 2$ natural numbers $a_1, a_2, \ldots, a_k$. (For example 10 is good because $10 = 5 + 2 + 1 + 1 + 1 = 5.2.1.1.1$ and these expressions are unique.) Determine, in terms of prime numbers, which natural numbers are good.

3. The line $l$ is tangent to the circle $S$ at the point $A$; $B$ and $C$ are points on $l$ on opposite sides of $A$ and the other tangents from $B$, $C$ to $S$ intersect at a point $P$. If $B$, $C$ vary along $l$ in such a way that the product $|AB|.|AC|$ is constant, find the locus of $P$.

4. Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, where $n \geq 1$, and let the polynomial
$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$$
be such that $|f(0)| = f(1)$ and each root $\alpha$ of $f$ is real and satisfies $0 < \alpha < 1$. Prove that the product of the roots does not exceed $1/2^n$.

5. Given a complex number $z = x + iy$ ($x, y$ real), we denote by $P(z)$ the corresponding point $(x, y)$ in the plane. Suppose $z_1, z_2, z_3, z_4, z_5, \alpha$ are nonzero complex numbers such that

   (a) $P(z_1)$, $P(z_2)$, $P(z_3)$, $P(z_4)$, $P(z_5)$ are the vertices of a convex pentagon $\mathbf{Q}$ containing the origin 0 in its interior and

   (b) $P(\alpha z_1)$, $P(\alpha z_2)$, $P(\alpha z_3)$, $P(\alpha z_4)$ and $P(\alpha z_5)$ are all inside $\mathbf{Q}$.

   If $\alpha = p + iq$, where $p$ and $q$ are real, prove that $p^2 + q^2 \leq 1$ and that
$$p + q\tan(\pi/5) \leq 1.$$

## 6th Irish Mathematical Olympiad
## 8 May 1993, Paper 2

1. Given five points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ in the plane having integer coordinates, prove that there is at least one pair $(P_i, P_j)$, with $i \neq j$, such that the line $P_iP_j$ contains a point $Q$ having integer coordinates and lying strictly between $P_i$ and $P_j$.

2. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be $2n$ real numbers, where $a_1, a_2, \ldots, a_n$ are distinct, and suppose that there exists a real number $\alpha$ such that the product
$$(a_i + b_1)(a_i + b_2) \ldots (a_i + b_n)$$
has the value $\alpha$ for $i = 1, 2, \ldots, n$. Prove that there exists a real number $\beta$ such that the product
$$(a_1 + b_j)(a_2 + b_j) \ldots (a_n + b_j)$$
has the value $\beta$ for $j = 1, 2, \ldots, n$.

3. For nonnegative integers $n, r$, the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of $n$ objects chosen $r$ at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity
$$\sum_{d=1}^{\infty} \binom{n-r+1}{d} \binom{r-1}{d-1} = \binom{n}{r}$$
for all integers $n$ and $r$, with $1 \leq r \leq n$.

4. Let $x$ be a real number with $0 < x < \pi$. Prove that, for all natural numbers $n$, the sum
$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots + \frac{\sin(2n-1)x}{2n-1}$$
is positive.

5. (a) The rectangle $PQRS$ has $|PQ| = \ell$ and $|QR| = m$, where $\ell$, $m$ are positive integers. It is divided up into $\ell m$ $1 \times 1$ squares by drawing lines parallel to $PQ$ and $QR$. Prove that the diagonal $PR$ intersects $\ell + m - d$ of these squares, where $d$ is the greatest common divisor, $(\ell, m)$, of $\ell$ and $m$.

   (b) A cuboid (or box) with edges of lengths $\ell$, $m$, $n$, where $\ell$, $m$, $n$ are positive integers, is divided into $\ell mn$ $1 \times 1 \times 1$ cubes by planes parallel to its faces. Consider a diagonal joining a vertex of the cuboid to the vertex furthest away from it. How many of the cubes does this diagonal intersect?

## 7th Irish Mathematical Olympiad
## 7 May 1994, Paper 1

1. Let $x, y$ be positive integers, with $y > 3$, and
$$x^2 + y^4 = 2[(x-6)^2 + (y+1)^2].$$
Prove that $x^2 + y^4 = 1994$.

2. Let $A, B, C$ be three collinear points, with $B$ between $A$ and $C$. Equilateral triangles $ABD$, $BCE$, $CAF$ are constructed with $D, E$ on one side of the line $AC$ and $F$ on the opposite side. Prove that the centroids of the triangles are the vertices of an equilateral triangle. Prove that the centroid of this triangle lies on the line $AC$.

3. Determine, with proof, all real polynomials $f$ satisfying the equation
$$f(x^2) = f(x)f(x-1),$$
for all real numbers $x$.

4. Consider the set of $m \times n$ matrices with every entry either 0 or 1. Determine the number of such matrices with the property that the number of "1"s in each row and in each column is even.

5. Let $f(n)$ be defined on the set of positive integers by the rules: $f(1) = 2$ and
$$f(n+1) = (f(n))^2 - f(n) + 1, \qquad n = 1, 2, 3, \ldots$$
Prove that, for all integers $n > 1$,
$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \ldots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}.$$

## 7th Irish Mathematical Olympiad
## 7 May 1994, Paper 2

1. A sequence $x_n$ is defined by the rules: $x_1 = 2$ and
$$nx_n = 2(2n-1)x_{n-1}, \qquad n = 2, 3, \ldots$$
Prove that $x_n$ is an integer for every positive integer $n$.

2. Let $p, q, r$ be distinct real numbers that satisfy the equations
$$\begin{aligned} q &= p(4-p), \\ r &= q(4-q), \\ p &= r(4-r). \end{aligned}$$
Find all possible values of $p + q + r$.

3. Prove that, for every integer $n > 1$,
$$n\left((n+1)^{2/n} - 1\right) < \sum_{i=1}^{n} \frac{2i+1}{i^2} < n\left(1 - n^{-2/(n-1)}\right) + 4.$$

4. Let $w, a, b$ and $c$ be *distinct* real numbers with the property that there exist real numbers $x$, $y$ and $z$ for which the following equations hold:
$$\begin{aligned} x \quad + y \quad + z \quad &= 1, \\ xa^2 + yb^2 + zc^2 &= w^2, \\ xa^3 + yb^3 + zc^3 &= w^3, \\ xa^4 + yb^4 + zc^4 &= w^4. \end{aligned}$$
Express $w$ in terms of $a$, $b$ and $c$.

5. If a square is partitioned into $n$ convex polygons, determine the maximum number of edges present in the resulting figure.

# 8th Irish Mathematical Olympiad
## 6 May 1995, Paper 1

1. There are $n^2$ students in a class. Each week all the students participate in a table quiz. Their teacher arranges them into $n$ teams of $n$ players each. For as many weeks as possible, this arrangement is done in such a way that any pair of students who were members of the same team one week are not on the same team in subsequent weeks. Prove that after at most $n+2$ weeks, it is necessary for some pair of students to have been members of the same team on at least two different weeks.

2. Determine, with proof, all those **integers** $a$ for which the equation
$$x^2 + axy + y^2 \;=\; 1$$
has infinitely many distinct **integer** solutions $x$, $y$.

3. Let $A$, $X$, $D$ be points on a line, with $X$ between $A$ and $D$. Let $B$ be a point in the plane such that $\angle ABX$ is greater than $120°$, and let $C$ be a point on the line between $B$ and $X$. Prove the inequality
$$2|AD| \;\geq\; \sqrt{3}(|AB| + |BC| + |CD|).$$

4. Consider the following one-person game played on the $x$-axis. For each integer $k$, let $X_k$ be the point with coordinates $(k, 0)$. During the game discs are piled at some of the points $X_k$. To perform a move in the game, the player chooses a point $X_j$ at which at least two discs are piled and then takes two discs from the pile at $X_j$ and places one of them at $X_{j-1}$ and one at $X_{j+1}$.
To begin the game, $2n+1$ discs are placed at $X_0$. The player then proceeds to perform moves in the game for as long as possible. Prove that after $n(n+1)(2n+1)/6$ moves no further moves are possible, and that, at this stage, one disc remains at each of the positions
$$X_{-n}, X_{-n+1}, \ldots, X_{-1}, X_0, X_1, \ldots, X_{n-1}, X_n.$$

5. Determine, with proof, all real-valued functions $f$ satisfying the equation
$$xf(x) - yf(y) \;=\; (x-y)f(x+y),$$
for all real numbers $x$, $y$.

## 8th Irish Mathematical Olympiad
## 6 May 1995, Paper 2

1. Prove the inequalities
$$n^n \;\le\; (n!)^2 \;\le\; [(n+1)(n+2)/6]^n,$$
for every positive integer $n$.

2. Suppose that $a$, $b$ and $c$ are complex numbers, and that all three roots $z$ of the equation
$$x^3 + ax^2 + bx + c \;=\; 0$$
satisfy $|z| = 1$ (where $|\;|$ denotes absolute value). Prove that all three roots $w$ of the equation
$$x^3 + |a|x^2 + |b|x + |c| \;=\; 0$$
also satisfy $|w| = 1$.

3. Let $S$ be the square consisting of all points $(x, y)$ in the plane with $0 \le x, y \le 1$. For each real number $t$ with $0 < t < 1$, let $C_t$ denote the set of all points $(x, y) \in S$ such that $(x, y)$ is on or above the line joining $(t, 0)$ to $(0, 1-t)$.
Prove that the points common to all $C_t$ are those points in $S$ that are on or above the curve $\sqrt{x} + \sqrt{y} = 1$.

4. We are given three points $P, Q, R$ in the plane. It is known that there is a triangle $ABC$ such that $P$ is the mid-point of the side $BC$, $Q$ is the point on the side $CA$ with $|CQ|/|QA| = 2$, and $R$ is the point on the side $AB$ with $|AR|/|RB| = 2$. Determine, with proof, how the triangle $ABC$ may be constructed from $P, Q, R$.

5. For each integer $n$ such that $n = p_1p_2p_3p_4$, where $p_1, p_2, p_3, p_4$ are distinct primes, let
$$d_1 = 1 < d_2 < d_3 < \cdots < d_{15} < d_{16} = n$$
be the sixteen positive integers that divide $n$.
Prove that if $n < 1995$, then $d_9 - d_8 \ne 22$.

## 9th Irish Mathematical Olympiad
## 4 May 1996, Paper 1

1. For each positive integer $n$, let $f(n)$ denote the highest common factor of $n!+1$ and $(n+1)!$ (where ! denotes factorial). Find, with proof, a formula for $f(n)$ for each $n$. [Note that "highest common factor" is another name for "greatest common divisor".]

2. For each positive integer $n$, let $S(n)$ denote the sum of the digits of $n$ when $n$ is written in base ten. Prove that, for every positive integer $n$,
$$S(2n) \leq 2S(n) \leq 10S(2n).$$
Prove also that there exists a positive integer $n$ with
$$S(n) = 1996S(3n).$$

3. Let $K$ be the set of all real numbers $x$ such that $0 \leq x \leq 1$. Let $f$ be a function from $K$ to the set of all real numbers $\mathbb{R}$ with the following properties

   (a) $f(1) = 1$;

   (b) $f(x) \geq 0$ for all $x \in K$;

   (c) if $x, y$ and $x+y$ are all in $K$, then
   $$f(x+y) \geq f(x) + f(y).$$
   Prove that $f(x) \leq 2x$, for all $x \in K$.

4. Let $F$ be the mid-point of the side $BC$ of a triangle $ABC$. Isosceles right-angled triangles $ABD$ and $ACE$ are constructed externally on the sides $AB$ and $AC$ with right-angles at $D$ and $E$ respectively. Prove that $DEF$ is an isosceles right-angled triangle.

5. Show, with proof, how to dissect a square into at most five pieces in such a way that the pieces can be re-assembled to form three squares no two of which are the same size.

**9th Irish Mathematical Olympiad**
**4 May 1996, Paper 2**

1. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined as follows: $F_0 = 0$, $F_1 = 1$ and, for all $n \geq 0$,
$$F_{n+2} = F_n + F_{n+1}.$$
(So,
$$F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8 \ldots)$$
Prove that

   (a) The statement "$F_{n+k} - F_n$ is divisible by 10 for all positive integers $n$" is true if $k = 60$, but not true for any positive integer $k < 60$.

   (b) The statement "$F_{n+t} - F_n$ is divisible by 100 for all positive integers $n$" is true if $t = 300$, but not true for any positive integer $t < 300$.

2. Prove that the inequality
$$2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdots (2^n)^{\frac{1}{2^n}} < 4$$
holds for all positive integers $n$.

3. Let $p$ be a prime number, and $a$ and $n$ positive integers. Prove that if
$$2^p + 3^p = a^n,$$
then $n = 1$.

4. Let $ABC$ be an acute-angled triangle and let $D$, $E$, $F$ be the feet of the perpendiculars from $A$, $B$, $C$ onto the sides $BC$, $CA$, $AB$, respectively. Let $P$, $Q$, $R$ be the feet of the perpendiculars from $A$, $B$, $C$ onto the lines $EF$, $FD$, $DE$, respectively. Prove that the lines $AP$, $BQ$, $CR$ (extended) are concurrent.

5. We are given a rectangular board divided into 45 squares so that there are five rows of squares, each row containing nine squares. The following game is played:
Initially, a number of discs are randomly placed on some of the squares, no square being allowed to contain more than one disc. A complete move consists of moving every disc from the square containing it to another square, subject to the following rules:

   (a) each disc may be moved one square up or down, or left or right, of the square it occupies to an adjoining square;

   (b) if a particular disc is moved up or down as part of a complete move, then it must be moved left or right in the next complete move;

   (c) if a particular disc is moved left or right as part of a complete move, then it must be moved up or down in the next complete move;

   (d) at the end of each complete move no square can contain two or more discs.

   The game stops if it becomes impossible to perform a complete move. Prove that if initially 33 discs are placed on the board, then the game must eventually stop. Prove also that it is possible to place 32 discs on the board in such a way that the game could go on forever.

## 10th Irish Mathematical Olympiad
## 10 May 1997, Paper 1

1. Find, with proof, all pairs of integers $(x, y)$ satisfying the equation
$$1 + 1996x + 1998y = xy.$$

2. Let $ABC$ be an equilateral triangle.
For a point $M$ inside $ABC$, let $D$, $E$, $F$ be the feet of the perpendiculars from $M$ onto $BC$, $CA$, $AB$, respectively. Find the locus of all such points $M$ for which $\angle FDE$ is a right-angle.

3. Find all polynomials $p$ satisfying the equation
$$(x - 16)p(2x) = 16(x - 1)p(x)$$
for all $x$.

4. Suppose $a$, $b$ and $c$ are nonnegative real numbers such that $a+b+c \geq abc$. Prove that $a^2+b^2+c^2 \geq abc$.

5. Let $S$ be the set of all odd integers greater than one. For each $x \in S$, denote by $\delta(x)$ the unique integer satisfying the inequality
$$2^{\delta(x)} < x < 2^{\delta(x)+1}.$$
For $a, b \in S$, define
$$a * b = 2^{\delta(a)-1}(b - 3) + a.$$
[For example, to calculate $5*7$, note that $2^2 < 5 < 2^3$, so $\delta(5) = 2$, and hence $5*7 = 2^{2-1}(7-3)+5 = 13$. Also $2^2 < 7 < 2^3$, so $\delta(7) = 2$ and $7 * 5 = 2^{2-1}(5 - 3) + 7 = 11$].

Prove that if $a, b, c \in S$, then

(a) $a * b \in S$ and

(b) $(a * b) * c = a * (b * c)$.

**10th Irish Mathematical Olympiad**
**10 May 1997, Paper 2**

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1. Given a positive integer $n$, denote by $\sigma(n)$ the sum of all positive integers which divide $n$. [For example, $\sigma(3) = 1 + 3 = 4$, $\sigma(6) = 1 + 2 + 3 + 6 = 12$, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$].
We say that $n$ is *abundant* if $\sigma(n) > 2n$. (So, for example, 12 is abundant).
Let $a, b$ be positive integers and suppose that $a$ is abundant. Prove that $ab$ is abundant.

2. $ABCD$ is a quadrilateral which is circumscribed about a circle $\Gamma$ (i.e., each side of the quadrilateral is tangent to $\Gamma$.) If $\angle A = \angle B = 120°$, $\angle D = 90°$ and $BC$ has length 1, find, with proof, the length of $AD$.

3. Let $A$ be a subset of $\{0, 1, 2, 3, \ldots, 1997\}$ containing more than 1000 elements. Prove that either $A$ contains a power of 2 (that is, a number of the form $2^k$, with $k$ a nonnegative integer) or there exist two distinct elements $a, b \in A$ such that $a + b$ is a power of 2.

4. Let $S$ be the set of all natural numbers $n$ satisfying the following conditions:

   (i) $n$ has 1000 digits;

   (ii) all the digits of $n$ are odd, and

   (iii) the absolute value of the difference between adjacent digits of $n$ is 2.

   Determine the number of distinct elements in $S$.

5. Let $p$ be a prime number, $n$ a natural number and $T = \{1, 2, 3, \ldots, n\}$. Then $n$ is called *p-partitionable* if there exist $p$ nonempty subsets $T_1, T_2, \ldots, T_p$ of $T$ such that

   (i) $T = T_1 \cup T_2 \cup \cdots \cup T_p$;

   (ii) $T_1, T_2, \ldots, T_p$ are disjoint (that is $T_i \cap T_j$ is the empty set for all $i, j$ with $i \neq j$), and

   (iii) the sum of the elements in $T_i$ is the same for $i = 1, 2, \ldots, p$.

   [For example, 5 is 3-partitionable since, if we take $T_1 = \{1, 4\}$, $T_2 = \{2, 3\}$, $T_3 = \{5\}$, then (i), (ii) and (iii) are satisfied. Also, 6 is 3-partitionable since, if we take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, then (i), (ii) and (iii) are satisfied.]

   (a) Suppose that $n$ is $p$-partitionable. Prove that $p$ divides $n$ or $n + 1$.

   (b) Suppose that $n$ is divisible by $2p$. Prove that $n$ is $p$-partitionable.

## 11th Irish Mathematical Olympiad
## 9 May 1998, Paper 1

1. Show that if $x$ is a nonzero real number, then
$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0.$$

2. $P$ is a point inside an equilateral triangle such that the distances from $P$ to the three vertices are 3, 4 and 5, respectively. Find the area of the triangle.

3. Show that no integer of the form $xyxy$ in base 10, where $x$ and $y$ are digits, can be the cube of an integer.
Find the smallest base $b > 1$ for which there is a perfect cube of the form $xyxy$ in base $b$.

4. Show that a disc of radius 2 can be covered by seven (possibly overlapping) discs of radius 1.

5. If $x$ is a real number such that $x^2 - x$ is an integer, and, for some $n \geq 3$, $x^n - x$ is also an integer, prove that $x$ is an integer.

## 11th Irish Mathematical Olympiad
## 9 May 1998, Paper 2

1. Find all positive integers $n$ that have exactly 16 positive integral divisors $d_1$, $d_2$, …, $d_{16}$ such that
$$1 = d_1 < d_2 < \cdots < d_{16} = n,$$
$d_6 = 18$ and $d_9 - d_8 = 17$.

2. Prove that if $a$, $b$, $c$ are positive real numbers, then

(a) $$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right),$$

and

(b) $$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

3. Let $\mathbb{N}$ be the set of all natural numbers (i.e., the positive integers).

   (a) Prove that $\mathbb{N}$ can be written as a union of three mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m - n| = 2$ or $5$, then $m$ and $n$ are in different sets.

   (b) Prove that $\mathbb{N}$ can be written as a union of four mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m - n| = 2$, $3$ or $5$, then $m$ and $n$ are in different sets. Show, however, that it is impossible to write $\mathbb{N}$ as a union of three mutually disjoint sets with this property.

4. A sequence of real numbers $x_n$ is defined recursively as follows: $x_0, x_1$ are arbitrary positive real numbers, and
$$x_{n+2} = \frac{1 + x_{n+1}}{x_n},\ n = 0, 1, 2, \ldots$$
Find $x_{1998}$.

5. A triangle $ABC$ has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90^\circ$. Find the minimum length of its perimeter.

## 12th Irish Mathematical Olympiad
## 8 May 1999, Paper 1

1. Find all real values $x$ that satisfy
$$\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}.$$

2. Show that there is a positive number in the Fibonacci sequence that is divisible by 1000.
[The Fibonacci sequence $F_n$ is defined by the conditions:
$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2$$
So, the sequence begins 0, 1, 1, 2, 3, 5, 8, 13, ... ]

3. Let $D$, $E$ and $F$, respectively, be points on the sides $BC$, $CA$ and $AB$, respectively, of a triangle $ABC$ so that $AD$ is perpendicular to $BC$, $BE$ is the angle-bisector of $\angle B$ and $F$ is the mid-point of $AB$. Prove that $AD$, $BE$ and $CF$ are concurrent if, and only if,
$$a^2(a-c) = (b^2-c^2)(a+c),$$
where $a$, $b$ and $c$ are the lengths of the sides $BC$, $CA$ and $AB$, respectively, of the triangle $ABC$.

4. A square floor consists of 10000 squares (100 squares $\times$ 100 squares – like a large chessboard) is to be tiled. The only available tiles are rectangular $1\times3$ tiles, fitting exactly over three squares of the floor.

   (a) If a $2 \times 2$ square is removed from the centre of the floor, prove that the remaining part of the floor can be tiles with the available tiles.

   (b) If, instead, a $2 \times 2$ square is removed from a corner of the floor, prove that the remaining part of the floor cannot be tiled with the available tiles.

   [There are sufficiently many tiles available. To *tile* the floor – or a portion thereof – means to completely cover it with the tiles, each tile covering three squares, and no pair of tiles overlapping. The tiles may not be broken or cut.]

5. Three real numbers $a$, $b$, $c$ with $a < b < c$, are said to be in arithmetic progression if $c - b = b - a$.
Define a sequence $u_n$, $n = 0, 1, 2, 3, \ldots$ as follows: $u_0 = 0$, $u_1 = 1$ and, for each $n \geq 1$, $u_{n+1}$ is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \ldots, u_n, u_{n+1}\}$ contains no three elements that are in arithmetic progression.
Find $u_{100}$.

## 12th Irish Mathematical Olympiad
## 8 May 1999, Paper 2

1. Solve the system of (simultaneous) equations
$$\begin{aligned} y^2 &= (x+8)(x^2+2), \\ y^2 &= (8+4x)y + 5x^2 - 16x - 16. \end{aligned}$$

2. A function $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers) satisfies

   (a) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of $a$ and $b$ is 1,

   (b) $f(p+q) = f(p) + f(q)$ for all prime numbers $p$ and $q$.

   Prove that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

3. Let $a$, $b$, $c$ and $d$ be positive real numbers whose sum is 1. Prove that
$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2},$$
with equality if, and only if, $a = b = c = d = 1/4$.

4. Find all positive integers $m$ with the property that the fourth power of the number of (positive) divisors of $m$ equals $m$.

5. $ABCDEF$ is a convex (not necessarily regular) hexagon with $AB = BC$, $CD = DE$, $EF = FA$ and
$$\angle ABC + \angle CDE + \angle EFA = 360^\circ.$$
Prove that the perpendiculars from $A$, $C$ and $E$ to $FB$, $BD$ and $DF$, respectively, are concurrent.

**13th Irish Mathematical Olympiad**
**6 May 2000, Paper 1**

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1. Let $S$ be the set of all numbers of the form $a(n) = n^2 + n + 1$, where $n$ is a natural number. Prove that the product $a(n)a(n+1)$ is in $S$ for all natural numbers $n$. Give, with proof, an example of a pair of elements $s, t \in S$ such that $st \notin S$.

2. Let $ABCDE$ be a regular pentagon with its sides of length one. Let $F$ be the midpoint of $AB$ and let $G, H$ be points on the sides $CD$ and $DE$, respectively, such that $\langle GFD = \langle HFD = 30°$. Prove that the triangle $GFH$ is equilateral. A square is inscribed in the triangle $GFH$ with one side of the square along $GH$. Prove that $FG$ has length
$$t = \frac{2\cos 18°(\cos 36°)^2}{\cos 6°},$$
and that the square has sides of length
$$\frac{t\sqrt{3}}{2+\sqrt{3}}.$$

3. Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer $a$ such that 65 divides $f(x)$ for every integer $x$.

4. Let
$$a_1 < a_2 < a_3 < \cdots < a_M$$
be real numbers. $\{a_1, a_2, \ldots, a_M\}$ is called a **weak arithmetic progression** of length $M$ if there exist real numbers $x_0, x_1, x_2, \ldots, x_M$ and $d$ such that
$$x_0 \leq a_1 < x_1 \leq a_2 < x_2 \leq a_3 < x_3 \leq \cdots \leq a_M < x_M$$
and for $i = 0, 1, 2, \ldots, M-1$, $x_{i+1} - x_i = d$ i.e. $\{x_0, x_1, x_2, \ldots, x_M\}$ is an arithmetic progression.

   (a) Prove that if $a_1 < a_2 < a_3$, then $\{a_1, a_2, a_3\}$ is a weak arithmetic progression of length 3.

   (b) Let $A$ be a subset of $\{0, 1, 2, 3, \ldots, 999\}$ with at least 730 members. Prove that $A$ contains a weak arithmetic progression of length 10.

5. Consider all parabolas of the form $y = x^2 + 2px + q$ ($p, q$ real) which intersect the $x$- and $y$-axes in three distinct points. For such a pair $p$, $q$ let $C_{p,q}$ be the circle through the points of intersection of the parabola $y = x^2 + 2px + q$ with the axes. Prove that all the circles $C_{p,q}$ have a point in common.

## 13th Irish Mathematical Olympiad
## 6 May 2000, Paper 2

1. Let $x \geq 0$, $y \geq 0$ be real numbers with $x + y = 2$. Prove that
$$x^2y^2(x^2 + y^2) \leq 2.$$

2. Let $ABCD$ be a cyclic quadrilateral and $R$ the radius of the circumcircle. Let $a, b, c, d$ be the lengths of the sides of $ABCD$ and $Q$ its area. Prove that
$$R^2 = \frac{(ab + cd)(ac + bd)(ad + bc)}{16Q^2}.$$
Deduce that
$$R \geq \frac{(abcd)^{3/4}}{Q\sqrt{2}},$$
with equality **if and only if** $ABCD$ is a square.

3. For each positive integer $n$ determine with proof, all positive integers $m$ such that there exist positive integers $x_1 < x_2 < \cdots < x_n$ with
$$\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{n}{x_n} = m.$$

4. Prove that in each set of ten consecutive integers there is one which is coprime with each of the other integers.
For example, taking $114, 115, 116, 117, 118, 119, 120, 121, 122, 123$ the numbers 119 and 121 are each coprime with all the others. [Two integers $a, b$ are coprime if their greatest common divisor is one.]

5. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and that $p(16) = 8$. Prove that $p(8) \leq 4$ and find, with proof, all such polynomials with $p(8) = 4$.

## 14th Irish Mathematical Olympiad
## 12 May 2001, Paper 1

1. Find, with proof, all solutions of the equation
$$2^n = a\,! + b\,! + c\,!$$
in positive integers $a, b, c$ and $n$. (Here, ! means "factorial".)

2. Let $ABC$ be a triangle with sides $BC, CA, AB$ of lengths $a, b, c$, respectively. Let $D, E$ be the midpoints of the sides $AC, AB$, respectively. Prove that $BD$ is perpendicular to $CE$ if, and only if,
$$b^2 + c^2 = 5a^2.$$

3. Prove that if an odd prime number $p$ can be expressed in the form $x^5 - y^5$, for some integers $x, y$, then
$$\sqrt{\frac{4p+1}{5}} = \frac{v^2+1}{2},$$
for some odd integer $v$.

4. Prove that

   (a) $\displaystyle \frac{2n}{3n+1} \leq \sum_{k=n+1}^{2n} \frac{1}{k},$ and

   (b) $\displaystyle \sum_{k=n+1}^{2n} \frac{1}{k} \leq \frac{3n+1}{4(n+1)},$

   for all positive integers $n$.

5. Let $a, b$ be real numbers such that $ab > 0$. Prove that
$$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}} \leq \frac{a^2 + 10ab + b^2}{12}.$$
Determine when equality occurs.

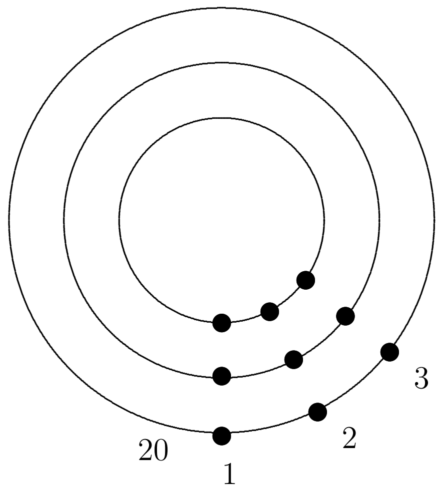
   Hence, or otherwise, prove for *all* real numbers $a, b$ that
$$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}} \leq \frac{a^2 + ab + b^2}{3}.$$
Determine the cases of equality.

## 14th Irish Mathematical Olympiad
## 12 May 2001, Paper 2

1. Find the least positive integer $a$ such that 2001 divides $55^n + a32^n$ for some odd integer $n$.

2. Three hoops are arranged concentrically as in the diagram. Each hoop is threaded with 20 beads, of which 10 are black and 10 are white. On each hoop the positions of the beads are labelled 1 through 20 starting at the bottom and travelling counterclockwise.

   We say there is a *match* at position $i$ if all three beads at position $i$ have the same colour. We are free to slide all of the beads around any hoop (but not to unthread and rethread them).




   Show that it is possible (by sliding) to find a configuration involving at least 5 matches.

3. Let $ABC$ be an acute angled triangle, and let $D$ be the point on the line $BC$ for which $AD$ is perpendicular to $BC$. Let $P$ be a point on the line segment $AD$. The lines $BP$ and $CP$ intersect $AC$ and $AB$ at $E$ and $F$ respectively. Prove that the line $AD$ bisects the angle $EDF$.

4. Determine, with proof, all non-negative real numbers $x$ for which
$$\sqrt[3]{13+\sqrt{x}} + \sqrt[3]{13-\sqrt{x}}$$
is an integer.

5. Determine, with proof, all functions $f$ from the set of positive integers to itself which satisfy
$$f(x+f(y)) = f(x) + y$$
for all positive integers $x, y$.

## 15th Irish Mathematical Olympiad
## 11 May 2002, Paper 1

1. In a triangle $ABC$, $AB = 20$, $AC = 21$ and $BC = 29$. The points $D$ and $E$ lie on the line segment $BC$, with $BD = 8$ and $EC = 9$. Calculate the angle $\angle DAE$.

2. (a) A group of people attends a party. Each person has at most three acquaintances in the group, and if two people do not know each other, then they have a mutual acquaintance in the group. What is the maximum number of people present?

   (b) If, in addition, the group contains three mutual acquaintances (i.e., three people each of whom knows the other two), what is the maximum number of people?

3. Find all triples of positive integers $(p, q, n)$, with $p$ and $q$ primes, satisfying
$$p(p+3) + q(q+3) = n(n+3).$$

4. Let the sequence $a_1, a_2, a_3, a_4, \ldots$ be defined by
$$a_1 = 1,\ a_2 = 1,\ a_3 = 1$$
and
$$a_{n+1}a_{n-2} - a_n a_{n-1} = 2,$$
for all $n \geq 3$. Prove that $a_n$ is a positive *integer* for all $n \geq 1$.

5. Let $0 < a, b, c < 1$. Prove that
$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \geq \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}.$$
Determine the case of equality.

## 15th Irish Mathematical Olympiad
## 11 May 2002, Paper 2

1. A $3 \times n$ grid is filled as follows : the first row consists of the numbers from 1 to $n$ arranged from left to right in ascending order. The second row is a cyclic shift of the top row. Thus the order goes $i, i+1, \ldots, n-1, n, 1, 2, \ldots, i-1$ for some $i$. The third row has the numbers 1 to $n$ in some order, subject to the rule that in each of the $n$ columns, the sum of the three numbers is the same.

   For which values of $n$ is it possible to fill the grid according to the above rules? For an $n$ for which this is possible, determine the number of different ways of filling the grid.

2. Suppose $n$ is a product of four distinct primes $a, b, c, d$ such that

   (a) $a + c = d$;

   (b) $a(a + b + c + d) = c(d - b)$;

   (c) $1 + bc + d = bd$.

   Determine $n$.

3. Denote by $\mathbb{Q}$ the set of rational numbers. Determine all functions $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ such that
$$f(x + f(y)) = y + f(x), \text{ for all } x,\, y \in \mathbb{Q}.$$

4. For each real number $x$, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to $x$.

   Let $\alpha = 2 + \sqrt{3}$. Prove that
$$\alpha^n - \lfloor \alpha^n \rfloor = 1 - \alpha^{-n}, \text{ for } n = 0, 1, 2, \ldots$$

5. Let $ABC$ be a triangle whose side lengths are all integers, and let $D$ and $E$ be the points at which the incircle of $ABC$ touches $BC$ and $AC$ respectively. If $||AD|^2 - |BE|^2| \leq 2$, show that $|AC| = |BC|$.

## 16th Irish Mathematical Olympiad
## 10 May 2003, Paper 1

1. Find all solutions in (not necessarily positive) integers of the equation
$$(m^2 + n)(m + n^2) = (m + n)^3.$$

2. $P, Q, R$ and $S$ are (distinct) points on a circle. $PS$ is a diameter and $QR$ is parallel to the diameter $PS$. $PR$ and $QS$ meet at $A$. Let $O$ be the centre of the circle and let $B$ be chosen so that the quadrilateral $POAB$ is a parallelogram. Prove that $BQ = BP$.

3. For each positive integer $k$, let $a_k$ be the greatest integer not exceeding $\sqrt{k}$ and let $b_k$ be the greatest integer not exceeding $\sqrt[3]{k}$. Calculate
$$\sum_{k=1}^{2003}(a_k - b_k).$$

4. Eight players, Ann, Bob, Con, Dot, Eve, Fay, Guy and Hal compete in a chess tournament. No pair plays together more than once and there is no group of five people in which each one plays against all of the other four.

   (a) Write down an arrangement for a tournament of 24 games satisfying these conditions.

   (b) Show that it is impossible to have a tournament of more than 24 games satisfying these conditions.

5. Show that there is no function $f$ defined on the set of positive real numbers such that
$$f(y) > (y - x)(f(x))^2$$
for all $x, y$ with $y > x > 0$.

## 16th Irish Mathematical Olympiad
## 10 May 2003, Paper 2

1. Let $T$ be a triangle of perimeter 2, and let $a$, $b$ and $c$ be the lengths of the sides of $T$.

   (a) Show that
   $$abc + \frac{28}{27} \geq ab + bc + ac.$$

   (b) Show that
   $$ab + bc + ac \geq abc + 1.$$

2. $ABCD$ is a quadrilateral. $P$ is at the foot of the perpendicular from $D$ to $AB$, $Q$ is at the foot of the perpendicular from $D$ to $BC$, $R$ is at the foot of the perpendicular from $B$ to $AD$ and $S$ is at the foot of the perpendicular from $B$ to $CD$. Suppose that $\angle PSR = \angle SPQ$. Prove that $PR = SQ$.

3. Find all solutions in integers $x, y$ of the equation
   $$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

4. Let $a, b > 0$. Determine the largest number $c$ such that
   $$c \leq \max\left(ax + \frac{1}{ax}, bx + \frac{1}{bx}\right)$$
   for all $x > 0$.

5. (a) In how many ways can 1003 distinct integers be chosen from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10?

   (b) Show that there are $(3(5151) + 7(1700))\,101^7$ ways to choose 1002 distinct integers from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10.

## 17th Irish Mathematical Olympiad
## 8 May 2004, Paper 1

1. (a) For which positive integers $n$, does $2n$ divide the sum of the first $n$ positive integers?

   (b) Determine, with proof, those positive integers $n$ (if any) which have the property that $2n+1$ divides the sum of the first $n$ positive integers.

2. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group $A,\ B,\ C$ of three players for which $A$ beat $B$, $B$ beat $C$ and $C$ beat $A$.

3. $AB$ is a chord of length 6 of a circle centred at $O$ and of radius 5. Let $PQRS$ denote the square inscribed in the sector $OAB$ such that $P$ is on the radius $OA$, $S$ is on the radius $OB$ and $Q$ and $R$ are points on the arc of the circle between $A$ and $B$. Find the area of $PQRS$.

4. Prove that there are only two real numbers $x$ such that
$$(x-1)(x-2)(x-3)(x-4)(x-5)(x-6) \ = \ 720.$$

5. Let $a, b \geq 0$. Prove that
$$\sqrt{2}\left(\sqrt{a(a+b)^3} + b\sqrt{a^2+b^2}\right) \ \leq \ 3(a^2+b^2),$$
with equality if and only if $a = b$.

## 17th Irish Mathematical Olympiad
## 8 May 2004, Paper 2

1. Determine all pairs of prime numbers $(p, q)$, with $2 \leq p, q < 100$, such that $p+6$, $p+10$, $q+4$, $q+10$ and $p+q+1$ are all prime numbers.

2. $A$ and $B$ are distinct points on a circle $T$. $C$ is a point distinct from $B$ such that $|AB| = |AC|$, and such that $BC$ is tangent to $T$ at $B$. Suppose that the bisector of $\angle ABC$ meets $AC$ at a point $D$ inside $T$. Show that $\angle ABC > 72°$.

3. Suppose $n$ is an integer $\geq 2$. Determine the first digit after the decimal point in the decimal expansion of the number
$$\sqrt[3]{n^3 + 2n^2 + n}.$$

4. Define the function $m$ of the three real variables $x, y, z$ by
$$m(x, y, z) \ = \ \max(x^2, y^2, z^2), \ \ x, y, z \in \mathbb{R}.$$
Determine, with proof, the minimum value of $m$ if $x, y, z$ vary in $\mathbb{R}$ subject to the following restrictions:
$$x + y + z \ = \ 0, \quad x^2 + y^2 + z^2 \ = \ 1.$$

5. Suppose $p$, $q$ are distinct primes and $S$ is a subset of $\{1, 2, \ldots, p-1\}$. Let $N(S)$ denote the number of solutions of the equation
$$\sum_{i=1}^{q} x_i \ \equiv \ 0 \quad \bmod p,$$
where $x_i \in S$, $i = 1, 2, \ldots, q$. Prove that $N(S)$ is a multiple of $q$.

## 18th Irish Mathematical Olympiad
## 7 May 2005, Paper 1

1. Prove that $2005^{2005}$ is a sum of two perfect squares, but not the sum of two perfect cubes.

2. Let $ABC$ be a triangle and let $D, E$ and $F$, respectively, be points on the sides $BC, CA$ and $AB$, respectively—none of which coincides with a vertex of the triangle—such that $AD, BE$ and $CF$ meet at a point $G$. Suppose the triangles $AGF, CGE$ and $BGD$ have equal area. Prove that $G$ is the centroid of $ABC$.

3. Prove that the sum of the lengths of the medians of a triangle is at least three quarters of the sum of the lengths of the sides.

4. Determine the number of different arrangements $a_1, a_2, \ldots, a_{10}$ of the integers $1, 2, \ldots, 10$ such that
$$a_i > a_{2i} \text{ for } 1 \leq i \leq 5,$$
and
$$a_i > a_{2i+1} \text{ for } 1 \leq i \leq 4.$$

5. Suppose $a, b$ and $c$ are non-negative real numbers. Prove that
$$\frac{1}{3}[(a-b)^2 + (b-c)^2 + (c-a)^2] \leq a^2 + b^2 + c^2 - 3\sqrt[3]{a^2b^2c^2} \leq (a-b)^2 + (b-c)^2 + (c-a)^2.$$

**18th Irish Mathematical Olympiad**
**7 May 2005, Paper 2**

1. Let $ABC$ be a triangle, and let $X$ be a point on the side $AB$ that is not $A$ or $B$. Let $P$ be the incentre of the triangle $ACX$, $Q$ the incentre of the triangle $BCX$ and $M$ the midpoint of the segment $PQ$. Show that $|MC| > |MX|$.

2. Using only the digits $1, 2, 3, 4$ and $5$, two players $A, B$ compose a 2005-digit number $N$ by selecting one digit at a time as follows: $A$ selects the first digit, $B$ the second, $A$ the third and so on, in that order. The last to play wins if and only if $N$ is divisible by 9. Who will win if both players play as well as possible?

3. Suppose that $x$ is an integer and $y, z, w$ are odd integers. Show that 17 divides $x^{y^{z^w}} - x^{y^z}$.

   [Note: Given a sequence of integers $a_n$, $n = 1, 2, \ldots$, the terms $b_n$, $n = 1, 2, \ldots$, of its sequence of "towers" $a_1$, $a_2^{a_1}$, $a_3^{a_2^{a_1}}$, $a_4^{a_3^{a_2^{a_1}}}$, $\ldots$, are defined recursively as follows: $b_1 = a_1$, $b_{n+1} = a_{n+1}^{b_n}$, $n = 1, 2, \ldots$. ]

4. Find the first digit to the left, and the first digit to the right, of the decimal point in the decimal expansion of $(\sqrt{2} + \sqrt{5})^{2000}$.

5. Let $m, n$ be odd integers such that $m^2 - n^2 + 1$ divides $n^2 - 1$. Prove that $m^2 - n^2 + 1$ is a perfect square.

## 19th Irish Mathematical Olympiad
## 6 May 2006, Paper 1

1. Are there integers $x$, $y$ and $z$ which satisfy the equation
$$z^2 = (x^2+1)(y^2-1)+n$$
when (a) $n = 2006$ (b) $n = 2007$ ?

2. $P$ and $Q$ are points on the equal sides $AB$ and $AC$ respectively of an isosceles triangle $ABC$ such that $AP = CQ$. Moreover, neither $P$ nor $Q$ is a vertex of $ABC$. Prove that the circumcircle of the triangle $APQ$ passes through the circumcentre of the triangle $ABC$.

3. Prove that a square of side 2.1 units can be completely covered by seven squares of side 1 unit.

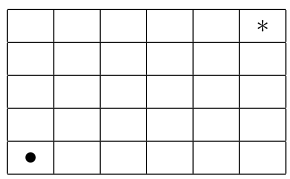
4. Find the greatest value and the least value of $x+y$, where $x$ and $y$ are real numbers, with $x \geq -2$, $y \geq -3$ and
$$x - 2\sqrt{x+2} = 2\sqrt{y+3} - y.$$

5. Determine, with proof, all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$, and
$$f(xy + f(x)) = xf(y) + f(x)$$
for all $x, y \in \mathbb{R}$.

   Notation: $\mathbb{R}$ denotes the set of real numbers.

## 19th Irish Mathematical Olympiad
## 6 May 2006, Paper 2

1. The rooms of a building are arranged in a $m \times n$ rectangular grid (as shown below for the $5 \times 6$ case). Every room is connected by an open door to each adjacent room, but the only access to or from the building is by a door in the top right room. This door is locked with an elaborate system of $mn$ keys, one of which is located in every room of the building. A person is in the bottom left room and can move from there to any adjacent room. However, as soon as the person leaves a room, all the doors of that room are instantly and automatically locked. Find, with proof, all $m$ and $n$ for which it is possible for the person to collect all the keys and escape the building.



$\bullet$ starting position

$*$ room with locked external door

2. $ABC$ is a triangle with points $D$, $E$ on $BC$, with $D$ nearer $B$; $F$, $G$ on $AC$, with $F$ nearer $C$; $H$, $K$ on $AB$, with $H$ nearer $A$. Suppose that $AH = AG = 1$, $BK = BD = 2$, $CE = CF = 4$, $\angle B = 60°$ and that $D$, $E$, $F$, $G$, $H$ and $K$ all lie on a circle. Find the radius of the incircle of the triangle $ABC$.

3. Suppose $x$ and $y$ are positive real numbers such that $x + 2y = 1$. Prove that
$$\frac{1}{x} + \frac{2}{y} \geq \frac{25}{1 + 48xy^2}.$$

4. Let $n$ be a positive integer. Find the greatest common divisor of the numbers
$$\binom{2n}{1}, \binom{2n}{3}, \binom{2n}{5}, \ldots, \binom{2n}{2n-1}.$$
Notation: If $a$ and $b$ are nonnegative integers such that $a \geq b$, then
$$\binom{a}{b} = \frac{a!}{(a-b)!b!}.$$

5. Two positive integers $n$ and $k$ are given, with $n \geq 2$. In the plane there are $n$ circles such that any two of them intersect at two points and all these intersection points are distinct. Each intersection point is coloured with one of $n$ given colours in such a way that all $n$ colours are used. Moreover, on each circle there are precisely $k$ different colours present. Find all possible values for $n$ and $k$ for which such a colouring is possible.

## 20th Irish Mathematical Olympiad
## 12 May 2007, Paper 1

1. Find all prime numbers $p$ and $q$ such that $p$ divides $q + 6$ and $q$ divides $p + 7$.

2. Prove that a triangle $ABC$ is right-angled if and only if
$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

3. The point $P$ is a fixed point on a circle and $Q$ is a fixed point on a line. The point $R$ is a variable point on the circle such that $P$, $Q$ and $R$ are not collinear. The circle through $P$, $Q$ and $R$ meets the line again at $V$. Show that the line $VR$ passes through a fixed point.

4. Air Michael and Air Patrick operate direct flights connecting Belfast, Cork, Dublin, Galway, Limerick and Waterford. For each pair of cities exactly one of the airlines operates the route (in both directions) connecting the cities. Prove that there are four cities for which one of the airlines operates a round trip. (Note that a round trip of four cities $P$, $Q$, $R$ and $S$, is a journey that follows the path $P \to Q \to R \to S \to P$.)

5. Let $r$ and $n$ be nonnegative integers such that $r \leq n$.

   (a) Prove that
   $$\frac{n+1-2r}{n+1-r}\binom{n}{r}$$
   is an integer.

   (b) Prove that
   $$\sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n+1-2r}{n+1-r}\binom{n}{r} < 2^{n-2}$$
   for all $n \geq 9$.

   (Note that $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$. Also, if $x$ is a real number then $\lfloor x \rfloor$ is the unique integer such that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.)

## 20th Irish Mathematical Olympiad
## 12 May 2007, Paper 2

1. Let $r, s$ and $t$ be the roots of the cubic polynomial
$$p(x) = x^3 - 2007x + 2002.$$
Determine the value of
$$\frac{r-1}{r+1} + \frac{s-1}{s+1} + \frac{t-1}{t+1}.$$

2. Suppose $a, b$ and $c$ are positive real numbers. Prove that
$$\frac{a+b+c}{3} \leq \sqrt{\frac{a^2+b^2+c^2}{3}} \leq \frac{\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}}{3}.$$
For each of the inequalities, find conditions on $a, b$ and $c$ such that equality holds.

3. Let $ABC$ be a triangle the lengths of whose sides $BC, CA, AB$, respectively, are denoted by $a, b, c$, respectively. Let the internal bisectors of the angles $\angle BAC, \angle ABC, \angle BCA$, respectively, meet the sides $BC, CA, AB$, respectively, at $D, E, F$, respectively. Denote the lengths of the line segments $AD, BE, CF$, respectively, by $d, e, f$, respectively. Prove that
$$def = \frac{4abc(a+b+c)\Delta}{(a+b)(b+c)(c+a)},$$
where $\Delta$ stands for the area of the triangle $ABC$.

4. Find the number of zeros in which the decimal expansion of the integer 2007! ends. Also find its last non-zero digit.

5. Suppose $a$ and $b$ are real numbers such that the quadratic polynomial
$$f(x) = x^2 + ax + b$$
has no nonnegative real roots. Prove that there exist two polynomials $g, h$, whose coefficients are nonnegative real numbers, such that
$$f(x) = \frac{g(x)}{h(x)},$$
for all real numbers $x$.

## 21st Irish Mathematical Olympiad
## 10 May 2008, Paper 1

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1. Let $p_1$, $p_2$, $p_3$ and $p_4$ be four different prime numbers satisfying the equations
$$\begin{aligned} 2p_1 + 3p_2 + 5p_3 + 7p_4 &= 162, \\ 11p_1 + 7p_2 + 5p_3 + 4p_4 &= 162. \end{aligned}$$
Find all possible values of the product $p_1p_2p_3p_4$.

2. For positive real numbers $a$, $b$, $c$ and $d$ such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that
$$a^2b^2cd + ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \le \frac{3}{32},$$
and determine the cases of equality.

3. Determine, with proof, all integers $x$ for which $x(x + 1)(x + 7)(x + 8)$ is a perfect square.

4. How many sequences $a_1, a_2, \ldots, a_{2008}$ are there such that each of the numbers $1, 2, \ldots, 2008$ occurs once in the sequence, and $i \in \{a_1, a_2, \ldots, a_i\}$ for each $i$ such that $2 \le i \le 2008$?

5. A triangle $ABC$ has an obtuse angle at $B$. The perpendicular at $B$ to $AB$ meets $AC$ at $D$, and $|CD| = |AB|$. Prove that
$$|AD|^2 = |AB|.|BC| \quad \text{if and only if} \quad \angle CBD = 30^\circ.$$

## 21st Irish Mathematical Olympiad
## 10 May 2008, Paper 2

1. Find, with proof, all triples of integers $(a, b, c)$ such that $a$, $b$ and $c$ are the lengths of the sides of a right angled triangle whose area is $a+b+c$.

2. Circles $S$ and $T$ intersect at $P$ and $Q$, with $S$ passing through the centre of $T$. Distinct points $A$ and $B$ lie on $S$, inside $T$, and are equidistant from the centre of $T$. The line $PA$ meets $T$ again at $D$. Prove that $|AD| = |PB|$.

3. Find $a_3, a_4, \ldots, a_{2008}$, such that $a_i = \pm 1$ for $i = 3, \ldots, 2008$ and
$$\sum_{i=3}^{2008} a_i 2^i = 2008,$$
and show that the numbers $a_3, a_4, \ldots, a_{2008}$ are uniquely determined by these conditions.

4. Given $k \in \{0, 1, 2, 3\}$ and a positive integer $n$, let $f_k(n)$ be the number of sequences $x_1, \ldots, x_n$, where $x_i \in \{-1, 0, 1\}$ for $i = 1, \ldots, n$, and
$$x_1 + \cdots + x_n \equiv k \mod 4.$$
   (a) Prove that $f_1(n) = f_3(n)$ for all positive integers $n$.

   (b) Prove that
$$f_0(n) = \frac{3^n + 2 + (-1)^n}{4}$$
   for all positive integers $n$.

5. Suppose that $x$, $y$ and $z$ are positive real numbers such that $xyz \geq 1$.

   (a) Prove that
$$27 \leq (1+x+y)^2 + (1+y+z)^2 + (1+z+x)^2,$$
   with equality if and only if $x = y = z = 1$.

   (b) Prove that
$$(1+x+y)^2 + (1+y+z)^2 + (1+z+x)^2 \leq 3(x+y+z)^2,$$
   with equality if and only if $x = y = z = 1$.

**22nd Irish Mathematical Olympiad**
**9 May 2009, Paper 1**

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1. Hamilton Avenue has eight houses. On one side of the street are the houses numbered 1,3,5,7 and directly opposite are houses 2,4,6,8 respectively. An eccentric postman starts deliveries at house 1 and delivers letters to each of the houses, finally returning to house 1 for a cup of tea. Throughout the entire journey he must observe the following rules. The numbers of the houses delivered to must follow an odd-even-odd-even pattern throughout, each house except house 1 is visited exactly once (house 1 is visited twice) and the postman at no time is allowed to cross the road to the house directly opposite. How many different delivery sequences are possible?

2. Let $ABCD$ be a square. The line segment $AB$ is divided internally at $H$ so that $|AB|.|BH| = |AH|^2$. Let $E$ be the mid point of $AD$ and $X$ be the midpoint of $AH$. Let $Y$ be a point on $EB$ such that $XY$ is perpendicular to $BE$. Prove that $|XY| = |XH|$.

3. Find all positive integers $n$ for which $n^8 + n + 1$ is a prime number.

4. Given an $n$-tuple of numbers $(x_1, x_2, \ldots, x_n)$ where each $x_i = +1$ or $-1$, form a new $n$-tuple
$$(x_1x_2, x_2x_3, x_3x_4, \ldots, x_nx_1),$$
and continue to repeat this operation. Show that if $n = 2^k$ for some integer $k \geq 1$, then after a certain number of repetitions of the operation, we obtain the $n$-tuple
$$(1, 1, 1, \ldots, 1)\ .$$

5. Suppose $a, b, c$ are real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Prove that
$$a^2b^2c^2 \ \leq \ \frac{1}{54},$$
and determine the cases of equality.

## 22nd Irish Mathematical Olympiad
## 9 May 2009, Paper 2

1. Let $p(x)$ be a polynomial with rational coefficients. Prove that there exists a positive integer $n$ such that the polynomial $q(x)$ defined by
$$q(x) \;=\; p(x+n) - p(x)$$
has integer coefficients.

2. For any positive integer $n$ define
$$E(n) \;=\; n(n+1)(2n+1)(3n+1)\cdots(10n+1).$$
Find the greatest common divisor of $E(1)$, $E(2)$, $E(3)$, $\ldots$, $E(2009)$.

3. Find all pairs $(a, b)$ of positive integers, such that $(ab)^2 - 4(a+b)$ is the square of an integer.

4. At a strange party, each person knew exactly 22 others.

   For any pair of people X and Y who knew one another, there was no other person at the party that they both knew.

   For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew.

   How many people were at the party?

5. In the triangle $ABC$ we have $|AB| < |AC|$. The bisectors of the angles at $B$ and $C$ meet $AC$ and $AB$ at $D$ and $E$ respectively. $BD$ and $CE$ intersect at the incentre $I$ of $\triangle ABC$.

   Prove that $\angle BAC = 60^\circ$ if and only if $|IE| = |ID|$.

## 23rd Irish Mathematical Olympiad
## 24 April 2010, Paper 1

1. Find the least $k$ for which the number 2010 can be expressed as the sum of the squares of $k$ integers.

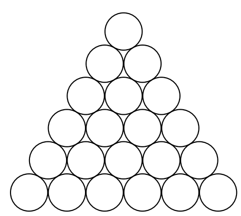
2. Let $ABC$ be a triangle and let $P$ denote the midpoint of the side $BC$. Suppose that there exist two points $M$ and $N$ interior to the sides $AB$ and $AC$ respectively, such that
$$|AD| = |DM| = 2|DN|,$$
where $D$ is the intersection point of the lines $MN$ and $AP$. Show that $|AC| = |BC|$.

3. Suppose $x$, $y$, $z$ are positive numbers such that $x + y + z = 1$. Prove that

   (a) $xy + yz + zx \geq 9xyz$;

   (b) $xy + yz + zx < \frac{1}{4} + 3xyz$.

4. The country of Harpland has three types of coin: *green*, *white* and *orange*. The unit of currency in Harpland is the shilling. Any coin is worth a positive integer number of shillings, but coins of the same colour may be worth different amounts. A set of coins is stacked in the form of an equilateral triangle of side $n$ coins, as shown below for the case of $n = 6$.



   The stacking has the following properties:

   (a) no coin touches another coin of the same colour;

   (b) the total worth, in shillings, of the coins lying on any line parallel to one of the sides of the triangle is divisible by three.

   Prove that the total worth in shillings of the *green* coins in the triangle is divisible by three.

5. Find all polynomials $f(x) = x^3 + bx^2 + cx + d$, where $b, c, d$ are real numbers, such that $f(x^2 - 2) = -f(-x)f(x)$.

## 23rd Irish Mathematical Olympiad
## 24 April 2010, Paper 2

1. There are 14 boys in a class. Each boy is asked how many other boys in the class have his first name, and how many have his last name. It turns out that each number from 0 to 6 occurs among the answers.

   Prove that there are two boys in the class with the same first name and the same last name.

2. For each odd integer $p \geq 3$ find the number of real roots of the polynomial
$$f_p(x) = (x-1)(x-2)\cdots(x-p+1)+1 \ .$$

3. In the triangle $ABC$ we have $|AB| = 1$ and $\angle ABC = 120°$. The perpendicular line to $AB$ at $B$ meets $AC$ at $D$ such that $|DC| = 1$. Find the length of $AD$.

4. Let $n \geq 3$ be an integer and $a_1, a_2, \ldots, a_n$ be a finite sequence of positive integers, such that, for $k = 2, 3, \ldots, n$
$$n(a_k + 1) - (n-1)a_{k-1} = 1 \ .$$
   Prove that $a_n$ is not divisible by $(n-1)^2$.

5. Suppose $a, b, c$ are the side lengths of a triangle $ABC$. Show that
$$x = \sqrt{a(b+c-a)}, \ \ y = \sqrt{b(c+a-b)}, \ \ z = \sqrt{c(a+b-c)}$$
   are the side lengths of an acute-angled triangle $XYZ$, with the same area as $ABC$, but with a smaller perimeter, unless $ABC$ is equilateral.

**24th Irish Mathematical Olympiad**
**7 May 2011, Paper 1**

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1. Suppose $abc \neq 0$. Express in terms of $a$, $b$, and $c$ the solutions $x, y, z, u, v, w$ of the equations
$$x + y = a, \quad z + u = b, \quad v + w = c, \quad ay = bz, \quad ub = cv, \quad wc = ax.$$

2. Let $ABC$ be a triangle whose side lengths are, as usual, denoted by $a = |BC|$, $b = |CA|$, $c = |AB|$. Denote by $m_a, m_b, m_c$, respectively, the lengths of the medians which connect $A, B, C$, respectively, with the centres of the corresponding opposite sides.

   (a) Prove that $2m_a < b + c$. Deduce that $m_a + m_b + m_c < a + b + c$.

   (b) Give an example of

      (i) a triangle in which $m_a > \sqrt{bc}$;

      (ii) a triangle in which $ma \leq \sqrt{bc}$.

3. The integers $a_0, a_1, a_2, a_3, \ldots$ are defined as follows:
$$a_0 = 1, \quad a_1 = 3, \quad \text{and } a_{n+1} = a_n + a_{n-1} \text{ for all } n \geq 1.$$
Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

4. The incircle $\mathcal{C}_1$ of triangle $ABC$ touches the sides $AB$ and $AC$ at the points $D$ and $E$, respectively. The incircle $\mathcal{C}_2$ of the triangle $ADE$ touches the sides $AB$ and $AC$ at the points $P$ and $Q$, and intersects the circle $\mathcal{C}_1$ at the points $M$ and $N$. Prove that

   (a) the centre of the circle $\mathcal{C}_2$ lies on the circle $\mathcal{C}_1$.

   (b) the four points $M$, $N$, $P$, $Q$ in appropriate order form a rectangle if and only if twice the radius of $\mathcal{C}_1$ is three times the radius of $\mathcal{C}_2$.

5. In the mathematical talent show called "The $X^2$-factor" contestants are scored by a panel of 8 judges. Each judge awards a score of 0 ('fail'), $X$ ('pass'), or $X^2$ ('pass with distinction'). Three of the contestants were Ann, Barbara and David. Ann was awarded the same score as Barbara by exactly 4 of the judges. David declares that he obtained different scores to Ann from at least 4 of the judges, and also that he obtained different scores to Barbara from at least 4 judges.

   In how many ways could scores have been allocated to David, assuming he is telling the truth?

## 24th Irish Mathematical Olympiad
## 7 May 2011, Paper 2

1. Prove that
$$\frac{2}{3}+\frac{4}{5}+\cdots+\frac{2010}{2011}$$
is not an integer.

2. In a tournament with $N$ players, $N<10$, each player plays once against each other player scoring 1 point for a win and 0 points for a loss. Draws do not occur. In a particular tournament only one player ended with an odd number of points and was ranked fourth. Determine whether or not this is possible. If so, how many wins did the player have?

3. $ABCD$ is a rectangle. $E$ is a point on $AB$ between $A$ and $B$, and $F$ is a point on $AD$ between $A$ and $D$. The area of the triangle $EBC$ is 16, the area of the triangle $EAF$ is 12 and the area of the triangle $FDC$ is 30. Find the area of the triangle $EFC$.

4. Suppose that $x$, $y$ and $z$ are positive numbers such that
$$1=2xyz+xy+yz+zx. \tag{1}$$
Prove that

(i)
$$\frac{3}{4}\leq xy+yz+zx<1.$$

(ii)
$$xyz\leq\frac{1}{8}.$$

Using (i) or otherwise, deduce that
$$x+y+z\geq\frac{3}{2}, \tag{2}$$
and derive the case of equality in (2).

5. Find with proof all solutions in nonnegative integers $a,b,c,d$ of the equation
$$11^a5^b-3^c2^d=1.$$

## 25th Irish Mathematical Olympiad
## 12 May 2012, Paper 1

1. Let
$$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$
and let
$$S = \{4, 5, 9, 14, 23, 37\}.$$
Find two sets $A$ and $B$ with the properties

   (a) $A \cap B = \emptyset$.

   (b) $A \cup B = C$.

   (c) The sum of two distinct elements of $A$ is not in $S$.

   (d) The sum of two distinct elements of $B$ is not in $S$.

2. $A$, $B$, $C$ and $D$ are four points in that order on the circumference of a circle $K$. $AB$ is perpendicular to $BC$ and $BC$ is perpendicular to $CD$. $X$ is a point on the circumference of the circle between $A$ and $D$. $AX$ extended meets $CD$ extended at $E$ and $DX$ extended meets $BA$ extended at $F$.

   Prove that the circumcircle of triangle $AXF$ is tangent to the circumcircle of triangle $DXE$ and that the common tangent line passes through the centre of the circle $K$.

3. Find, with proof, all polynomials $f$ such that $f$ has nonnegative integer coefficients, $f(1) = 8$ and $f(2) = 2012$.

4. There exists an infinite set of triangles with the following properties:

   (a) the lengths of the sides are integers with no common factors, and

   (b) one and only one angle is $60°$.

   One such triangle has side lengths 5, 7 and 8. Find two more.

5. (a) Show that if $x$ and $y$ are positive real numbers, then
$$(x + y)^5 \geq 12xy(x^3 + y^3).$$

   (b) Prove that the constant 12 is the best possible. In other words, prove that for any $K > 12$ there exist positive real numbers $x$ and $y$ such that
$$(x + y)^5 < Kxy(x^3 + y^3).$$

## 25th Irish Mathematical Olympiad
## 12 May 2012, Paper 2

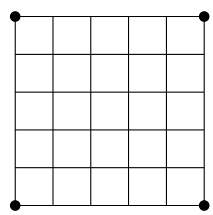
6. Let $S(n)$ be the sum of the decimal digits of $n$. For example, $S(2012) = 2 + 0 + 1 + 2 = 5$. Prove that there is no integer $n > 0$ for which $n - S(n) = 9990$.

7. Consider a triangle $ABC$ with $|AB| \neq |AC|$. The angle bisector of the angle $CAB$ intersects the circumcircle of $\triangle ABC$ at two points $A$ and $D$. The circle of centre $D$ and radius $|DC|$ intersects the line $AC$ at two points $C$ and $B'$. The line $BB'$ intersects the circumcircle of $\triangle ABC$ at $B$ and $E$.
   Prove that $B'$ is the orthocentre of $\triangle AED$.

8. Suppose $a, b, c$ are positive numbers. Prove that
$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1\right)^2 \geq (2a + b + c)\left(\frac{2}{a} + \frac{1}{b} + \frac{1}{c}\right),$$
with equality if and only if $a = b = c$.

9. Let $x > 1$ be an integer. Prove that $x^5 + x + 1$ is divisible by at least two *distinct* prime numbers.

10. Let $n$ be a positive integer. A mouse sits at each corner point of an $n \times n$ square board, which is divided into unit squares as shown below for the example $n = 5$.



The mice then move according to a sequence of *steps*, in the following manner:

(a) In each step, each of the four mice travels a distance of one unit in a horizontal or vertical direction. Each unit distance is called an *edge* of the board, and we say that each mouse *uses* an edge of the board.

(b) An edge of the board may not be used twice in the same direction.

(c) At most two mice may occupy the same point on the board at any time.

The mice wish to collectively organise their movements so that each edge of the board will be used twice (not necessarily by the same mouse), and each mouse will finish up at its starting point. Determine, with proof, the values of $n$ for which the mice may achieve this goal.

## 26th Irish Mathematical Olympiad
## 11 May 2013, Paper 1

1. Find the smallest positive integer $m$ such that $5m$ is an exact $5^{\text{th}}$ power, $6m$ is an exact $6^{\text{th}}$ power, and $7m$ is an exact $7^{\text{th}}$ power.

2. Prove that
$$1 - \frac{1}{2012}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2013}\right) \geq \frac{1}{\sqrt[2012]{2013}}.$$

3. The altitudes of a triangle $ABC$ are used to form the sides of a second triangle $A_1B_1C_1$. The altitudes of $\Delta A_1B_1C_1$ are then used to form the sides of a third triangle $A_2B_2C_2$. Prove that $\Delta A_2B_2C_2$ is similar to $\Delta ABC$.

4. Each of the 36 squares of a $6 \times 6$ table is to be coloured either Red, Yellow or Blue.

   (a) No row or column is contain more than two squares of the same colour.

   (b) In any four squares obtained by intersecting two rows with two columns, no colour is to occur exactly three times.

   In how many different ways can the table be coloured if both of these rules are to be respected?

5. $A$, $B$ and $C$ are points on the circumference of a circle with centre $O$. Tangents are drawn to the circumcircles of triangles $OAB$ and $OAC$ at $P$ and $Q$ respectively, where $P$ and $Q$ are diametrically opposite $O$. The two tangents intersect at $K$. The line $CA$ meets the circumcircle of $\Delta OAB$ at $A$ and $X$. Prove that $X$ lies on the line $KO$.

## 26th Irish Mathematical Olympiad
## 11 May 2013, Paper 2

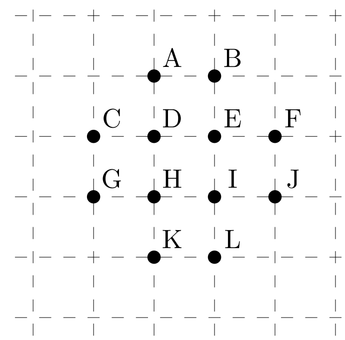
6. The three distinct points $B, C, D$ are collinear with $C$ between $B$ and $D$. Another point $A$ not on the line $BD$ is such that $|AB| = |AC| = |CD|$.

   Prove that $\angle BAC = 36^\circ$ if and only if $\dfrac{1}{|CD|} - \dfrac{1}{|BD|} = \dfrac{1}{|CD| + |BD|}$.

7. Consider the collection of different squares which may be formed by sets of four points chosen from the 12 labelled points in the diagram on the right.

   For each possible area such a square may have, determine the number of squares which have this area.

   Make sure to explain why your list is complete.




8. Find the smallest positive integer $N$ for which the equation
$$(x^2 - 1)(y^2 - 1) = N$$
is satisfied by at least two pairs of integers $(x, y)$ with $1 < x \leq y$.

9. We say that a doubly infinite sequence
$$\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$$
is *subaveraging* if $s_n = (s_{n-1} + s_{n+1})/4$ for all integers $n$.

   (a) Find a subaveraging sequence in which all entries are different from each other. Prove that all entries are indeed distinct.

   (b) Show that if $(s_n)$ is a subaveraging sequence such that there exist distinct integers $m, n$ such that $s_m = s_n$, then there are infinitely many pairs of distinct integers $i, j$ with $s_i = s_j$.

10. Let $a, b, c$ be real numbers and let
$$x = a + b + c,\ y = a^2 + b^2 + c^2,\ z = a^3 + b^3 + c^3 \quad \text{and} \quad S = 2x^3 - 9xy + 9z.$$

   (a) Prove that $S$ is unchanged when $a, b, c$ are replaced by $a + t, b + t, c + t$, respectively, for any real number $t$.

   (b) Prove that $\left(3y - x^2\right)^3 \geq 2S^2$.

## 27th Irish Mathematical Olympiad
## 10 May 2014, Paper 1

1. Given an $8\times 8$ chess board, in how many ways can we select 56 squares on the board while satisfying both of the following requirements:

   (a) All black squares are selected.

   (b) Exactly seven squares are selected in each column and in each row.

2. Prove for all integers $N>1$ that $(N^2)^{2014}-(N^{11})^{106}$ is divisible by $N^6+N^3+1$.

3. In the triangle $ABC$, $D$ is the foot of the altitude from $A$ to $BC$, and $M$ is the midpoint of the line segment $BC$. The three angles $\angle BAD$, $\angle DAM$ and $\angle MAC$ are all equal. Find the angles of the triangle $ABC$.

4. Three different nonzero real numbers $a, b, c$ satisfy the equations
$$a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}=p$$
where $p$ is a real number. Prove that $abc+2p=0$.

5. Suppose $a_1,\dots,a_n>0$, where $n>1$ and $\sum_{i=1}^n a_i=1$. For each $i=1,2,\dots,n$, let $b_i=a_i^2/\sum_{j=1}^n a_j^2$. Prove that
$$\sum_{i=1}^n \frac{a_i}{1-a_i}\le\sum_{i=1}^n\frac{b_i}{1-b_i}\,.$$
When does equality occur?

## 27th Irish Mathematical Olympiad
## 10 May 2014, Paper 2

6. Each of the four positive integers $N, N+1, N+2, N+3$ has exactly six positive divisors. There are exactly 20 different positive numbers which are exact divisors of at least one of the numbers. One of these is 27. Find all possible values of $N$.
(Both 1 and $m$ are counted as divisors of the number $m$.)

7. The square $ABCD$ is inscribed in a circle with centre $O$. Let $E$ be the midpoint of $AD$. The line $CE$ meets the circle again at $F$. The lines $FB$ and $AD$ meet at $H$. Prove $|HD| = 2|AH|$.

8. (a) Let $a_0, a_1, a_2$ be real numbers and consider the polynomial
$$P(x) = a_0 + a_1 x + a_2 x^2 .$$
Assume that $P(-1), P(0)$ and $P(1)$ are integers.
Prove that $P(n)$ is an integer for all integers $n$.

   (b) Let $a_0, a_1, a_2, a_3$ be real numbers and consider the polynomial
$$Q(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 .$$
Assume that there exists an integer $i$ such that $Q(i), Q(i+1), Q(i+2)$ and $Q(i+3)$ are integers.
Prove that $Q(n)$ is an integer for all integers $n$.

9. Let $n$ be a positive integer and $a_1, \ldots, a_n$ be positive real numbers. Let $g(x)$ denote the product
$$(x + a_1) \cdots (x + a_n) .$$
Let $a_0$ be a real number and let
$$\begin{aligned} f(x) &= (x - a_0) g(x) \\ &= x^{n+1} + b_1 x^n + b_2 x^{n-1} + \ldots + b_n x + b_{n+1} . \end{aligned}$$
Prove that all the coefficients $b_1, b_2, \ldots, b_{n+1}$ of the polynomial $f(x)$ are negative if and only if
$$a_0 > a_1 + a_2 + \ldots + a_n .$$

10. Over a period of $k$ consecutive days, a total of 2014 babies were born in a certain city, with at least one baby being born each day. Show that:

    (a) If $1014 < k \leq 2014$, there must be a period of consecutive days during which exactly 100 babies were born.

    (b) By contrast, if $k = 1014$, such a period might not exist.